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A STUDY OF THE EFFECT OF A TANGIBLE AND CONCEPTUALIZED PRESENTATION OF ARITHMETIC ON ACHIEVEMENT IN THE FIFTH AND SIXTH GRADES

BY

Patricia McNitt Spross

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A DISSERTATION

Submitted to the School of Graduate Studies
of Michigan State University in partial
fulfillment for the degree of

DOCTOR OF EDUCATION

School of Education

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DOCTOR OF EDUCATION

Final examination: May 7, 1962, 1:00 P.M., 304 Education Building

Dissertation: A Study of the Effect of a Tangible and Conceptualized Presentation of Arithmetic on Achievement in the Fifth and Sixth Grades

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A STUDY OF THE EFFECT OF A TANGIBLE AND CONCEPTUALIZED PRESENTATION OF ARITHMETIC ON ACHIEVEMENT IN THE FIFTH AND SIXTH GRADES

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AN ABSTRACT

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This study was concerned with the effect of a tangible and conceptualized presentation of arithmetic to children in the fifth and sixth grades as compared to a routine classroom presentation.

The regular arithmetic course of study of the school system was divided into concepts or mathematical ideas. To these were added enough additional concepts taken from pure mathematics theory to explain the arithmetic processes regularly taught in these grades. Tangible manipulative items that had cultural significance were used in developing the concepts whenever possible. The regular arithmetic text was used as a reference book only. In addition the children were supplied with other reference materials as they requested them.

Only one concept was presented each week in a discussion period in which the teacher acted as a resource person. Major points of interest relating to the topic were identified, and the means of developing these were planned by the students. The students reacted to the concept by producing some tangible item of their own choosing that they felt would represent their understanding of the particular mathematical idea for the week. The students were not given an assignment of problems to work in any text.

Class periods were limited to thirty-five minutes per day. There was no ability grouping, no individual help, no homework, or work at any other time. The lessons included no "drill".

Once a week students were given the opportunity to work as far as possible in a series of problems of gradual difficulty. These were corrected by the teacher and returned to them. This was done in order to keep the children and the patrons of the school informed of the class' progresa

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progress. No student was ever required to complete these problems, only to do as many as possible in the class time.

The tangible and conceptualized method was compared to the routine self-contained classroom presentation in which the arithmetic text structured the program and was worked through as a method of completing the arithmetic course. The self-contained rooms were not prohibited from ability grouping, individual help, drill and rote learning methods, home work, direct correlation with other subjects, or any other method the teacher chose. These classrooms were informed that they were acting as control groups for the study.

There were 166 students in the study comprising 8 heterogeneous classes in two elementary schools. There were two experimental fifth grades and two fifth grade controls; likewise, two experimental sixth grades, and two sixth grade controls. The grades selected were deemed to be typical of the school population.

The relative effectiveness of the conceptualized and tangible presentation and the typical routine of the self-contained classroom was determined by students performance on standardized achievement tests. These were administered at the beginning, the mid-point and the end of the school year. Comparisons were made on the students' performance on achievement tests administered as part of the school routine. These were:

STEP Test - Form A

California Achievement Test - Elementary Arithmetic Form BB - Reasoning and Fundamentals

Tests given the students were treated by using the analysis of co-variance model. F ratio showed that there existed a significant difference between experimental and control groups when measured by the STEP and California Achievement Test-Elementary Arithmetic-Form BB, reasoning. There was no significant difference found in California Achievement, fundamentals.

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Conclusions: It would appear as a result of the study that it would be possible for students to achieve essentially the same progress in arithmetic with an understanding of mathematical concepts and their use as they can in a routine classroom procedure in which rote methods of learning are employed. It would further appear that this can be accomplished in 35 minutes per day in a heterogeneous group in which there is no ability grouping, no individual help, and no additional work outside of the 35 minute class time when the burden of proof of learning rests with the student. It also would appear that it is possible for fifth and sixth grade children to understand sufficient mathematical theory as would give meaning to the arithmetic content at the fifth and sixth grade level.

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CHAPTER I

INTRODUCTION

Improvement in the elementary curriculum has long been a part of educational research. Since 1900 much emphasis has been given to a consideration of methods for improving arithmetic programs.[1] Leaders in the field of education and in national mathematical organizations have given much thought to: (1) factors contributing to a need for an improved program; and, (2) the kind of program that will supply these needs. Price lists the factors which contribute to the need for change in mathematics curriculum as: (1) Research in mathematics; (2) Automation; and, (3) Automatic digital computing machines.[2] Advances in mathematics have created more profound mathematics, greater quantity of content, introduced new subject areas, and created additional and more extensive uses of mathematics. The sequential nature of mathematics makes it imperative that students and teachers possess a thorough knowledge of these continuing developments.[3]

Cultural implications for mathematics have arisen in the application of mathematics to the development of automation, which has in turn created the need for the production of high speed computers. In order to carry out the mechanization of automation, it has become

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[1] G. T. Buswell, "Arithmetic". Review of Educational Research, Chester W. Harris, Editor. New York: The Macmillan Company. 1960. P 62.

[2] G. Baily Price, "Progress in Mathematics and Its Implications for the Schools." The Revolution in School Mathematics. Washington: National Council of Teachers of Mathematics. 1961. Pp. 3-5.

[3] Ibid., p. 2.

necessary to design machines capable of computation at rates far beyond human abilities. The design, maintenance, and operation of such machines require the close association of mathematics with the fields of engineering and mechanics. Development of such processes have had a great influence on the way of life.[4]

Not only has content in mathematics changed, but areas of emphasis have changed. While it is still necessary to teach the traditional subject matter areas, they now receive less emphasis. Specific application of mathematics toward increased environmental control make a thorough understanding of the properties of functions just as necessary as a memorization of these processes. This is brought out by the Rockefeller report.

> . . . we are moving with headlong speed into a new phase of man's long struggle to control his environment, a phase beside which the industrial revolution may appear a modest alteration of human affairs.[5]

Some idea of the scope and recency of the change may be obtained by considering a statement from the Fiftieth Yearbook of the National Society for the Study of Education which was published just nine years ago.

> Arithmetic exhibits some marked contrasts when compared with some of the other content areas of education. Unlike chemistry, physics, and the social sciences, its content is not subject to radical changes due to discoveries. . . .[6]

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[4]Ibid., Pp. 4-5.

[5]Rockefeller Report. The Pursuit of Excellence, Panel Report V of the Special Studies Project. Garden City: Doubleday. 1958. P. 28.

[6]G. T. Bushnell, "Introduction." Fiftieth Yearbook of the National Society for the Study of Education, Part II. Chicago: University of Chicago Press. 1951 P. 1.

Further support indicating the importance of a changing mathematical program to the cultural development is offered by Schaaf.

> The creative language of today is science, and mathematics is the alphabet of science. . . . Contemporary mathematics is to be distinguished from all previous mathematics in two vital respects: (1) the intentional study of abstractions, where the important considerations are not the things related, but the relations themselves; and, (2) the relentless experimentation of the very foundations - the foundations of ideas upon which the elaborate superstructure of mathematics is based. . . .[7]

A consideration of the arithmetic program needed in view of these changes implies the utilization of mathematics by the culture and, as a consequence, a change in methods of classroom presentation. Suelts in emphasizing the pressure to apply research results to classroom methodology lists three components of an arithmetic program that will tend to emphasize the social significance of mathematics:

1. The essence of mathematics.

2. Computations and procedures.

3. Application to the social, physical, cultural, and economic world.[8]

The importance of number systems (arithmetic) to the cultural is also emphasized by Sueltz:

> The fact (well accepted) that modern science and art could not well function without the number system in its present state of development tends to support the concept that the invention of it is, as it has been called, the greatest invention of the human mind.[9]

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[7] William L. Schaaf, "Mathematics as a Cultural Heritage." The Arithmetic Teacher, 8:3-5, January 1961.

[8] Ben A. Sueltz, "A Time for Decision." The Arithmetic Teacher, 8:280, October, 1961.

[9] Ibid., P. 13.

The cautious planning that is needed to implement a mathematics program that will serve and support: (1) the abstractness of mathematical invention; (2) the rapid development of automation; and (3) the resulting construction of rapid mechanistic computation is brought out by Craig:

> In the already crowded curriculum any subject should be most carefully delineated. Although arithmetic can claim a share of curriculum time as its cultural right, the delineation of that share of the curriculum should be most carefully considered. The question appears to be, "How can mathematics be treated in the curriculum to be of the most use?"
>
> Although mathematics as such is a part of the cultural heritage, in the crowded curriculum of today, time spent on mathematics needs to be justified.[10]

It would seem that we need to make a choice as to what course the arithmetic should pursue: (1) Shall we continue in the way of abstraction only; (2) can abstraction have utility as a means of enhancing understanding; (3) is it possible for children in the elementary school to understand functions and processes as well as the memorization of facts; and (4) can this be accomplished in a reasonable amount of class time?

The arithmetic program of the future is pictured by Clark:

> . . . arithmetic programs in the next decade will: (1) be directed by teachers of high scholarship; (2) be better supported by the lay public; (3) give greater emphasis to learning by thinking; (4) make better provision for the range of abilities in heterogeneous groups of pupils; (5) incorporate topics not generally found in school curricula; (6) include experiments with, and evaluation of, newer tools of learning.[11]

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[10]Gerald S. Craig, Science for the Elementary School Teacher. New York: Ginn and Company, 1958. P. 35.

[11]John R. Clark, "Looking Ahead at Instruction in Arithmetic." The Arithmetic Teacher, 8:394, December, 1961.

It is with these implications in mind that this study originated.

Need for the study. It has been pointed out that: (1) mathematics content is rapidly changing and developing; (2) mathematics content is rapidly increasing; (3) there is an urgent necessity for the utilization of mathematics in the control of a rapidly developing technological environment; and (4) these factors imply a change in the content presented to students and the methods by which this is done.

Studies involving mathematical research are numerous and attack many facets of the total mathematical problem situation.[12] However it would seem that some attempt might be made to inaugurate an arithmetic program in a typical classroom situation that would incorporate the implications of such studies.

The sequential nature of mathematics makes a thorough understanding of previous content necessary in order to master developing mathematical innovations.[13] It has been pointed out that the need to utilize mathematics in the control of the environment implies understanding on the part of the student rather than memorization. It would seem then that the elementary curriculum as the sequential antecedent to this developing mathematical program, could not advantageously be continued as a memorizing, or rote type, program.

It was the purpose of this study to attempt to design an elementary course of study in grades five and six that would: (1) be based on

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[12]Buswell, op. cit., p. 63.

[13]Price, op. cit., p. 2

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a thorough understanding of mathematical theory; as it relates to arithmetical processes; (2) be consistent as a sequential antecedent to the new content that is developing at the secondary levels; (3) provide academically talented students with an arithmetic program equal to their abilities; (4) provide the slow learners with arithmetic experiences compatable with their abilities; (5) provide experiences in the application of arithmetic to real-life situations; (7) be taught in a realistic manner consistant with principles of learning; and, (8) be accomplished in thirty-five minutes per day of class time.

In order to implement a program of this type an attempt was made to: (1) identify and analyze methods of improving the elementary mathematics program; (2) investigate methods of reducing the classroom teacher's total load as it relates to areas of specialization; (3) investigate a method of teaching elementary mathematics that would emphasize the understandings of concepts and their application to a social situation with tangible manipulative materials; and, (4) investigate the relationship of such teaching-learning procedures to the development of skill in the manipulation of abstract mathematical symbols.

Statement of the problem. The purpose of this study was to compare two methods of teaching arithmetic in the fifth and sixth grades. The two methods were designated as those practiced in: (1) the self-contained (or routine) classroom procedure; and, (2) a tangible and conceptualized presentation of arithmetic as mathematical theory to be applied in a social situation. In the self-contained classroom, arithmetic was taught in the routine manner prescribed by the curriculum of the school system in which the text structured the arithmetic program. In experimental

rooms the mathematics program was presented as a cultural tool. The two methods were compared as to the performance of students on selected standardized achievement tests. These were administered as a part of the arithmetic program.

In addition the study was concerned with incorporating into the content of the elementary arithmetic curriculum sufficient mathematical theory to serve as an adequate explanation of the arithmetical processes heretofore taught by rote - or memory.

It was the hypothesis of the experimental program that fifth and sixth grade mathematics presented as a mathematical concept to be used as a cultural tool will result in: (1) an improved understanding of mathematical principles; (2) improved ability to manipulate abstract mathematical symbols and to apply these in a social situation; and, (3) that these abilities would be exhibited on standardized achievement tests given as a part of the regular routine of the school.

<u>Basic considerations</u>. The basic considerations pertinent to the study were as follows:

1. Is it possible to increase understandings of mathematical concepts by the manipulation of tangible items?

2. Will an increased understanding of mathematical concepts result in greater mathematical achievement by students as measured by standardized achievement tests?

3. What is the extent which mathematics as a symbolic system may evolve skill in the manipulation of symbols?

4. What emphasis should be placed on the understanding of mathematical concepts as the manipulation of abstract symbols?

Definition of terms. Terms used are defined in appropriate places in the body of the thesis. However, in order to make clear the initial presentation, certain terms are defined here. The terms defined are: arithmetic, mathematical concepts, routine classroom procedure, and conceptualized presentation.

Arithmetic in this study was used interchangeably with mathematics and elementary mathematics. It also refers to the program used in the school to develop the part of the curriculum usually referred to as "arithmetic". The connotations given to arithmetic in the experimental program are defined in Chapter III.

Mathematical concepts was used to refer to the ideas and generalisations of mathematics which were taught in connection with the study of arithmetic. This term was also used to refer to the branches of arithmetic dealing with denominate numbers, and the areas of mathematics having to do with measurement. As used in the study, the term would include ideas, generalizations, principles and symbols referring to the various areas of mathematics.

Routine classroom procedure was the term applied to the process of arithmetic instruction carried on in the self-contained classroom. In this study it was referred to as "the way arithmetic was usually taught." This implies a presentation involving: (1) regular assignments to be completed in the text; (2) following the sequence of the text; (3) requiring the students to complete most of the practice and drill work in the text; (4) direct correlation with other classes; (5) "extra" help for slow learners; (6) additional time other than in class to complete assignments; (7) drill on "facts"; and, (8) homework when necessary

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necessary. This is further developed in the chapter on the teaching procedures used in the study.

Conceptualized presentation was the term arbitrarily chosen to represent the presentation of a broad general idea to students as a teaching process. This was in contrast to a presentation involving a single operation. Conceptualized presentation included many aspects of an arithmetic concept and its application in a social situation.

Limitations of the study. The study was an evaluation of a method of teaching arithmetic to fifth and sixth grade students and was not an attempt to prove that one method was superior to another. The study was limited to the content, and the understanding of the content, of the mathematics contained in the course of study of the school system.

Specifically the study was not concerned with:

1. The evaluation of the existing routine classroom presentation.

2. The evaluation of textbooks.

3. The evaluation of the curriculum.

4. The evaluation of grade placement of mathematical concepts and/or materials.

The study was specifically concerned with ascertaining if it was possible to teach the mathematics contained in the curriculum, as appropriate to grades five and six, as an understanding of concepts rather than the memorization of facts and processes.

Structure of the thesis. Chapter I contained a review of some of the developments in mathematics and society that have given rise to the need for the study. Basic assumptions and terms have been described.

Chapter II will be concerned with a review of the literature that is pertinent to a conceptualized presentation of arithmetic, literature pertaining to learning theory that is particularly applicable to the study, and a review of current literature regarding present practices of teaching arithmetic in the elementary school. Chapter III will contain a detail of the two methods used in the study. Chapter IV will contain descriptions of: (1) the organization of the investigation; (2) the persons involved in the study; and, (3) the cultural and geographic areas participating in the study. Chapter V presents the data and results of the study. Chapter VI contains a summary and implications for further study.

CHAPTER II

REVIEW OF THE LITERATURE

It was indicated in Chapter I that the literature in regard to research in the methods and content of the arithmetic program is very extensive. It would have been beyond the scope of this paper to completely review all these productions. Consequently this chapter will concern itself with a review of literature pertinent to: (1) concepts in arithmetic curricula and methods of teaching them; (2) literature pertaining to theories of learning that are significant to this study; and, (3) literature regarding current practices of teaching arithmetic.

<u>Review of literature pertaining to concepts of arithmetic curricula and methods of teaching them.</u> It should be emphasized that the teaching of arithmetical concepts or ideas does not mean an incidental type of learning situation. This is brought out by Clark:

> Changing conceptions and functions of education, of the philosophy of education, have been clearly reflected during the past several years both in curricula and teaching methods. . . . There is a middle ground between the two extremes of the now outmoded formalism of an over-burdened curriculum, and the established freedom of wholly incidental learning.[14]

The relation of teaching mathematical concepts with cultural application is further emphasized by Bredeoake and Groves:

> . . . to supply children with those experiences which lead to real comprehension of number . . . children who failed to achieve success in "sum lessons" showed considerable grasp of

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[14]John R. Clark, Arthur S. Otis, and Caroline Hatton, <u>Primary Arithmetic Through Experience</u>. Yonkers-on-Hudson: World Book Company. 1939. P. 6.

> the subject when shopping. . . . a scheme which brought to the classroom some of the numerical activities found in daily life. . . .[15]

Many, if not most, references dealing with arithmetic instruction are chiefly concerned with mathematical content, with much less emphasis on method of teaching and learning. Banks specifically illustrates teaching individual and isolated content areas. He does indicate however that the choice of material in texts has made a real effort to "reflect interests, activities, and experiences of children."[16]

The need of structuring the arithmetic curriculum so that it contains mathematical concepts is emphasized by Hartung.

> . . . more than a set of things it might be interesting to try, or a collection of enrichment materials, or drill materials . . . It must provide for a new and deeper understanding of arithmetic. . . . A few simple ideas provide the foundation stones upon which arithmetic is built. . . . The ultimate goal of a modern arithmetic program is that the child be able to solve problems involving quantitative ideas. To attain this goal, it is necessary for him to acquire certain essential concepts and techniques.[17]

The development of arithmetic programs has closely followed the needs of society. It has progressed from a primitive need for an expression of quantity, such as a one-to-one relationship, to the complicated processes demanded by current technology.[18] Evidence has been presented

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[15]E. Bredeoake and I.D. Groves, Arithmetic in Action. London: University of London Press. 1956. P. 128.

[16]J. Houston Banks, Learning and Teaching Arithmetic. Boston: Allyn and Bacon, Incorporated. 1959. P. 5.

[17]Maurice L. Hartung, et. al., Charting the Course for Arithmetic. Chicago: Scott, Foresman and Company. 1960. Pp. 4-19.

[18]David Eugene Smith, History of Mathematics. New York: Ginn and Company. 1951. P. 6.

in Chapter I to indicate this development. Of particular concern to this paper are the changes in content and methodology during the past century.

Mathematics during the colonial period showed little ingenuity.[19] Up to the year 1800 texts were of English origin and related largely to the needs of commerce.[20] Methodology consisted largely in the dictation of practical problems by the teacher which were copied by the students. Texts gave rules for the solution of problems, these were memorized by the students and applied with a minimum of emphasis on understanding. About the first significant change in arithmetic teaching came in 1821, when Colburn, under the influence of Pestalozzi, broke away from the deductive method of memorizing rules and applying them. As arithmetic content continued to incorporate additional mathematics (as it was needed by the culture) the bulk became so heavy it was deemed necessary that it be abridged and enriched. This was undertaken by the Committee of Ten (1893) and the Committee of Fifteen (1895).[21]

In order to reduce the amount of content the theory of "utility" was applied. In about 1920 arithmetic was organized as a succession of unit skills. This was followed by drill (the "law" of exercise) as a law of learning. In 1930 the National Society for the Study of Education

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[19] Walter S. Monroe, Development of Arithmetic as a School Subject, U.S. Bureau of Education Bulletin, No. 10. Washington: Government Printing Office. 1917. p. 8.

[20] Buswell, Op. cit., p. 63.

[21] Ibid.

Yearbook paved the way to teaching for meaning.[22]

As the development of arithmetic programs followed a pattern of increasing content, it began to be opposed by arithmetic curricula which emphasized in turn: (1) mental discipline; (2) social utility; (3)real-life relationships; and, (4) social aims.[23] Wise made some attempt to show that much of the arithmetic taught during the early part of the century did not function in life.[24] Consideration of arithmetic as social utility helped to streamline much of the outmoded content. In addition to theories of social utility much emphasis has been placed on an attempt to make it meaningful.[25] It is now rather well accepted that both aims are necessary[26] and since content (up to the last few years) has been pretty well founded, research has dwelt on arrangement.[27]

This thesis would be incomplete without some reference to "activity" programs that have been inaugurated. Harap and Mapes attempted the learning of decimals in an activity program, and found that the upper and lower extremes of the class (according to I.Q.) acquired some mastery, and that the number of repetitions had nothing

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[22] Ibid., p. 64.

[23] Ibid., p. 65.

[24] Carl L. Wise, "A Survey of Arithmetic Problems Arising in Various Occupations", Elementary School Journal, 20:118-36. 1949.

[25] Buswell, op. cit., p. 65.

[26] Henry Van Engen, "An Analysis of Meaning in Arithmetic." Elementary School Journal. 49:321-29, 395-340; 1949.

[27] Herbert F. Spitzer and Robert L. Burch, "Methods and Materials in the Teaching of Mathematics." Review of Educational Research. 18:337-49; 1948.

to do with degree of mastery.[28] In as much as there was no control these findings might indicate further study. Willey reported a study of social expression intended to lead a child to understandings.[29] In a report of an experience curriculum, Williams showed satisfactory gains, but had no control group.[30]

Buswell reported in 1935.

> . . . on the basis of evidence now available, the incidental, experience approach has not produced a superior substitute for the more systematic organization of content, but it may well provide some insight into supplements to the systematic program . . . (in an activity program) there are many uncontrolable variables . . . it is also practically impossible to duplicate a situation.[31]

Since the method of arithmetic instruction described in this study is concerned with "drill" teaching techniques, the work of Phillips,[32] Brown,[33] and Brownell[34] are significant in which there was some

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[28] Henry Harap and Charlotte E. Mapes, "The Learning of Decimals in an Arithmetic Activity Program". Journal of Educational Research, 29:686-93, 1936.

[29] Ray D. Willey, "A Study in the Use of Arithmetic in the Schools of Santa Clara County, California." Journal of Educational Research, 36:353-65, 1943.

[30] Catherine M. Williams, "Arithmetic Learning in an Experience Curriculum," Educational Research Bulletin, 28:154-62, 167-68; 1919.

[31] Buswell, op. cit., p. 67

[32] L. M. Phillips, "Value of Daily Drill in Arithmetic." Journal of Educational Psychology. 4:159-63, 1913.

[33] J. C. Brown, "An Investigation of the Value of Drill Work in the Fundamental Operation of Arithmetic," Journal of Educational Psychology, 3:485-92, 561-70; 1912.

[34] William A. Brownell and Charlotte B. Chezal, "The Effects of Premature Drill in Third Grade Arithmetic." Journal of Educational Research. 29:17-28, 1935.

indication that drill without understanding may result in poor learning and learning habits.

Review of learning theories of significance to this study. It is not our purpose to investigate all the literature published relative to theories of learning, but rather to confine this discussion to basic theories of learning, (as presented by accepted authorities) that are pertinent to this study. Neither is it the purpose of this dissertation to evaluate various learning theories, only to investigate such aspects as would tend to support the techniques which were used in the teaching of mathematics in this study. In other words, it is our intent to show that the teaching methods employed were based on some accepted theories of the learning process. It is further our purpose to point out that a consideration of some commonly accepted theories of learning would indicate that a sense-perception (tangible), and conceptualized presentation of arithmetic in a social situation would tend to enhance meaning.

The importance of the process of learning to the tremendous task of achievement in arithmetic is emphasized by Judd.

> . . . children are not born with a number system as part of their physical inheritance . . . the school puts them in contact with a system of number symbols which is one of the most perfect creations of the human mind . . . in the short span of a few years, the child becomes expert in the use of a method of expressing ideas of quantity which cost the race centuries of effort to invent and perfect.

If arithmetic education is to accomplish such a task as this it would appear that every effort should be made to do it as effectually as

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[35]Charles H. Judd, Educational Psychology, New York: Houghton Mifflin. 1939. p. 270.

possible. The fact that mathematics might be considered as an inherent or instinctive process should not be totally disregarded, however, there is some support to the theory that behavior in man is a learned process, or, learning is important to education. Instinctive and/or inherent determinants have been shown to have somewhat less significance, though not to be completely discounted.[36]

An investigation of much of the literature relating to the subject matter and methods of teaching, approaches the learning process and its accompanying problems in a symptomatic rather than a casual manner.[37] That is, reported research tends to analyze isolated problem areas and suggest methods of attack, but puts less emphasis on underlying principles of the thinking and learning activities of children. Basic then, to this paper is a consideration not only of learning theory, but the use to which it could be put in teaching and learning mathematics. It has been our assumption that learning theories, or generalizations, might be employed in predicting or anticipating desirable behavior on the part of the student in the area of mathematics.

Such a treatment of learning theory seems to involve three aspects: (1) the antecedents within the students background; (2) the results to be expected from the application of the "teaching process" on the student, and (3) resultant behavior. The learning theory employed in this study

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[36]N. Tenbergen, The Study of Instinct. London: Oxford University Press. 1951. As cited by Ernest R. Hilgard, Theories of Learning. New York: Appleton-Century-Crafts, Incorporated. 1958. p. 3.

[37]Buswell, op. cit., p. 63-67.

would need to deal with all of these areas. All three are difficult to describe in that there appears to be little scientific method which can be used to treat them.

The first of these has to do with what is already in the students mind when exposed to school learning situations. Children think not necessarily in relationship to the momentary classroom situation, but in relationship to this, plus the mental residue contained in their minds of what has gone before. This mental development "includes the sum total of memories, images, percepts, concepts, and attitudes built up over the years."[38]

The conclusions of Piaget are relevant to a methodology that considers the thinking process of children in the learning situation. This work analyses a situation in which children often "think" they understand when the particular understanding is the result of a "syncretistic perception."[39] In other words, children are apt to jump to conclusions when presented with instructional materials that are to be learned. The supposed understanding is no more than a quick relating of the new idea with the child's apperceptive mass. This leads to Piaget's conclusion:

> Syncretistic understanding consists in . . . (a thought process) that the whole is understood before the parts are analyzed and that the understanding of the details takes place only as a function of the general scheme.[40]

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[38]David H. Russell, Children's Thinking. New York: Ginn and Company, 1956. p. 34.

[39]Jean Piaget, The Language and Thought of the Child. New York: Meridian Books, 1955. p. 140.

[40]Ibid., p. 162.

In view of this it would seem illogical to teach isolated "facts" in subject matter as a prerequisite to either understanding or assimilation of knowledge. This does not intimate that facts or mathematical ideas, in and of themselves, are unimportant, but rather that they should be viewed in their totality and their interrelationships. Carpenter's work is significant in this regard:

> We cannot always assume that the perception of the "whole" gestalt can somehow suddenly be grasped without making sure that the pupil knows the meaning of key concepts that compose the gestalt.[41]

This need for an understanding of parts as they relate to wholes is also brought out by Russell:

> Not until eleven years can children understand how three objects are alike, and not until adolesence can they detect similiarities in abstract words. This one example of mental development suggests that elementary children should have many concrete experiences.[42]

The importance of percepts and sensations contributing to a child's total concept of a situation are further emphasized by Russell who supports Piaget's conclusion and gives it application.

> A child's thinking is based on his experience. His immediate interpretation of events in his external or internal environment are his percepts. . . . A percept, an image, or a memory seldom exist in isolation. . . . The older distinction that sensory impression is a lower form of activity than the higher "mental processes" of abstraction or problem solving is now considered to be largely a metaphysical one. Modern research suggests not a sharp break, but a continimum in the cognitive process from

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[41] Finley Carpenter, "Conceptualization as a Function of Differential Reinforcement." Social Science and Mathematics. 38:284-294. 1954.

[42] Russell, op. cit., p. 59.

> relatively simple sensations and perceptions to elaborate aesthetic or creative experiment.[43]

To apply the foregoing to methodology in arithmetic it needs only to be remembered that the subject matter as it is presented to the child becomes a part of this environment, or the subject matter is interpreted by the child in his percepts.

Basic also to this discussion is the function of repetition or "drill" methods, commonly referred to as "rote" learning. A method by which content is presented to the students by the teacher, either orally or written, and which the students repeat a sufficient number of times until they can reproduce it. Thorndike's work is significant to this kind of methodology that involves the teacher telling the pupils "how" rather than building an understanding for them that tells "why". There is some indication in Thorndike's work to show that "response to a command situation will not result in a waning of the initially most frequent connection at the expense of the initially less frequent."[44]

If this were applied to a multiple choice problem situation in which a student may make either a correct or several incorrect answers, continued repetition of incorrect answers would not necessarily enhance the correct responses at the expenses of the incorrect answers.[45]

Further support for a conceptualized, learning and the use of broad ideas is offered by Carpenter:

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[44]Edward L. Thorndike, The Fundamentals of Learning. New York: Bureau of Publications, Teachers College, Columbia University. 1937. p. 18

[45]*Ibid.*, p. 18

> Functional learning of concepts is more efficient than rote learning when measured by retention and ability to verbalize meanings of learned concepts.[46]

In view of the evidence presented in Chapter I indicating that elementary mathematics education needs now to teach structure and concepts rather than isolated facts, the work of Hargrove which indicates a structured, rather than an incidental learning program is significant.[47]

The importance of understanding a number system and its application to social situations is a concern of teachers in instructing children in ways of solving story problems. The use of a thorough understanding of a number system, as it applies to the culture is brought out by Hamilton:

> . . . A problem seldom if ever has any numbers associated with it until we recognize the situation and apply numbers to it. . . . We have to be able to think well enough to be able to arrange the elements of a situation and adapt a strategy using models from reality, concepts, mental imagery, . . .also know enough about numbers to recognize one or several properties that fit the situation, thus choosing an abstract model in the best mathematical sense.[48]

Support for learning in a social situation rather than in isolation is also offered by McHugh.[49]

The learning or behavior to be expected as a result of arithmetic instruction is basic to this paper. This consideration involves the

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[46]Fenley Carpenter, "The Effect of Different Learning Methods on Concept Formation." School Science and Mathematics, 40:282-285, 1956.

[47]W. Richard Hargrove, "Proper Emphasis on Science and Mathematics in the Elementary School." School Science and Mathematics, 44:89-91,1960.

[48]E. W. Hamilton, "About the Articles", The Arithmetic Teacher, 8:49, 1961.

[49]Walter Joseph McHugh, "Pupil Team Learning in Skill Subjects in Intermediate Grades." Dissertation Abstracts,21:1461, Dec. 1960.

relationship of mind to the learning process; or, as suggested on page 17, the use to which the learning (resulting from the instructional process) shall be put. Guthrie makes such a distinction:

> Growth, reproduction, and defense mechanizations are life but they are not mind. Mind is something more; it is growth and reproductions and reactions serving these ends plus something else that common sense might call profiting by experience. . . . The ability to learn, that is to respond differently to a situation because of past response to the situation is what distinguishes those living creatures which common sense endows with minds . . . training leaves no observable changes . . . mind must be a mode of behavior which changes with use or practice.[50]

While it might be desirable to find a teaching method that would assure all the best possible results, such a possibility would seem improbable. In this light Guthrie continues:

> So far as I am aware the only suggestions toward the description or explanation of the circumstances under which specific changes in behavior will or will not occur have been made in the form of association or conditioning . . . when the past of the individual is used for predicting behavior, we find our predictions are always in terms of associative learning . . . we can never understand him (man) but we can understand something of him and know something of him in terms of what we know of human nature in general and in terms of what we know of his past history and the nature of associative learning.

In contrast to this concept of learning as it relates to the total past and present experiences of the individual DeMay offers a brief summary of some early aspects of arithmetic instruction:

> When number work was first thought to be a necessary part of instruction of the first two grades, teachers did not know what to do except to drill their pupils in the abstract combinations. Investigation later showed most of the time spent this way had been wasted . . . (the need is) to provide teachers with

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[50]E. R. Guthrie, The Psychology of Learning. New York: Harper and Brothers. 1935. Pp. 2-4.

[51]Ibid., Pp. 243-245.

sequential materials that will guide the development of exercises through which children will learn the significance of numbers rather than the manipulation of abstract symbols.[52]

This importance of learning theory to the teaching of arithmetic is emphasized by Lankford:

> . . . the effective teacher of mathematics encourages creativity by helping pupils discover the basic laws, or principles of mathematics; he aims for understanding ahead of skills of operation; and he seeks to give students the stimulation that comes from accepting and realizing worthwhile goals.[53]

In setting forth the important elements in an effective arithmetic program Grossnickle lists:

> 1. The nature of the subject is such that it has a cultural value; it is structured; properly taught, it leads to unique quantitative ways of thinking; and it is basic to the further study of mathematics.
>
> 2. A program for the learning of arithmetic should recognize such factors as the mental hygiene of the classroom, adequate records for guidance, provision for optimum individual growth, use of materials, and ability to read quantitative statements.
>
> 3. A specific course in the background of mathematics is recommended as essential in the training of teachers of arithmetic.

The basic understandings needed, opportunities to create, and emphasis on meanings rather than drill are also advocated by Jones:

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[52]Amy J. DeMay, Guiding Beginners in Arithmetic. Evanston: Row, Peterson and Company. 1957. Pp. IX-X.

[53]Francis G. Lankford, Jr., "Implications of the Psychology of Learning for the Teaching of Mathematics," The Growth of Mathematical Ideas, Twenty-Fourth Yearbook. National Council of Teachers of Mathematics. Washington: The Council. 1959. p. 405.

[54]Foster E. Grossnickle, "Introduction." Instruction in Arithmetic, Twenty-Fifth Yearbook of the National Council of Teachers of Mathematics. Washington: The Council. 1960. p. 3.

1. The best learning is that in which the learned facts, concepts, and processes are meaningful to and understood by the learner.

2. Understanding and meaningfulness are rarely if "all or none" insights in either the sense of being achieved instantaneously or in the sense of embracing the whole concept and its implications at any one time.[55]

The foregoing has been presented in support of a method of teaching arithmetic which would emphasize understandings, insights, creativity, and cultural utility; and which would give no emphasis to drill methods, repetitious problem solving, and meaningless memorization.

Current literature regarding present practices in teaching arithmetic. To review all the studies regarding present practices in arithmetic instruction would be far beyond the scope of this study. We have selected those that appear to be of significance and that are representative of the general trends in teaching for meaning and understanding. Glennon has stated the present condition of arithmetic research rather well as he points out that there is much research about what can be taught and little concern about what should be taught.[56] He states:

Prior to 1900 the curriculum was determined by:

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[55]Phillip S. Jones, "The Growth and Development of Mathematical Ideas in Children," The Growth of Mathematical Ideas, Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics. Washington: The Council, 1959. p. 1.

[56]Vincent J. Glennon, "Editorial," Educational Leadership. 19:354-56, 1962.

1. The need for mathematical training by society.

2. The need for the subject to be taught as a system of related ideas.

3. The present emphasis is on the needs of the child.[57]

The author continues that all these emphases have drawbacks, there is a need for balance in the curriculum and that " . . . modernizing the school program is one way of changing the method as well as the content."[58]

Further support as to the disorganized state of arithmetic researeh is offered by Hartung:

> Instructional programs in arithmetic include a layer of ideas and practices accumulated through the years. Examined critically . . . from the point of view of learning theory . . . they are seen to be questionable either on mathematical grounds or on psychological grounds or both . . . much confusion arises between means and ends. Computation for its own sake is fruitless . . . but essential to problem solving. . . . I see great possibilities for improvement in arithmetic instruction if we are willing to acknowledge that some of our teaching has been superficial at best. The remedy is not more drill, but deeper insight as to what is really basic arithmetic.[59]

The need for a culturally significant and useful arithmetic curriculum is brought out by Cook:

> Arithmetic is undoubtedly one of the more poorly taught subjects in the elementary school. Educators may question whether present demands in mathematics are in focus with other areas of the curriculum, but we must recognize the needs of our youth in the world in which they will live. A reappraisal of our approach to arithmetic in the elememtary schools, of the coordination with the mathematics program in the secondary schools, of the preparation of teachers, and

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[57] Ibid., p. 54.

[58] Ibid., p. 55.

[59] Maurice L. Hartung, "Distinguishing Between Basic and Superficial Ideas in Arithmetic Instruction." The Arithmetic Teacher, 6:65-7, 1959.

> of the place of inservice programs in the school system is imperative if we are to meet the demands facing us at the present time.[60]

The need of a program designed to build meaningful mathematical concepts of cultural and social significance is emphasized by Sueltz,[61] Madden,[62] Flournay,[63] Breuckner,[64] and Brownell.[65]

Sueltz offers a fine "time chart" of the development of arithmetic instruction and the factors influencing the development of arithmetic texts:

1850 - 1890	"Faculty" psychology. "Training" of the mind.
1900 - 1920	Principles of learning. Motivation and readiness. Exercise, effect, threshold of learning, over-learning.
1920 - 1935	Readiness. Exercise. Drill.
1930 - 1950	Progressive education. Satisfyingness, peer group. Whole child.

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[60] Raymond F. Cook, "Improving Arithmetic Instruction," National Elementary Principal. 38:37-39; 1959.

[61] Ben A. Sueltz, "Arithmetic in Historical Perspective." National Elementary Principal, 39:12-16; 1959.

[62] Richard Madden, "Major Issues in Teaching Arithmetic." National Elementary School Principal, 39:174; 1959.

[63] Frances Flournay, "Relating Arithmetic to Everyday Life." National Elementary Principal, 39:294; 1959.

[64] Leo J. Breuckner, "Testing, Diagnosis, and Follow-Up in Arithmetic." National Elementary Principal, 39:334; 1959.

[65] William A. Brownell, "Arithmetic in 1970." National Elementary Principal, 39:42-44; 1959.

1950 Child development understandings.
Meaning, action research.
Discovery learning.
Multisensory learning.[66]

Sueltz continues:

> Arithmetic will advance as teachers develop an insight into the subject matter and its significance and as they learn more and more about the boys and girls they teach and how the human minds, bodies, and emotions combine in the behavorial learning situation.[67]

In considering the characteristics of a good arithmetic program Sueltz advises:

> A teaching process through an inductive-deductive cycle (in which) . . . new concepts are introduced in a socially significant situation, developed in their mathematical sequence, and returned to socially significant application.[68]

Methods which would teach arithmetic for understanding make it necessary to present the structure of the number system. This is emphasised by Brueckner.

> It is generally agreed that children must understand the number system. They must also understand how the number system operates in the performance of number operations.[69]

The emphasis on sound mathematics as basic to arithmetic programs is supported by Lundberg[70] and Wilsberg.[71]

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[66]Sueltz, op. cit., Pp. 15-16.

[67]Ibid.

[68]Ibid.

[69]Breuckner, loc. cit.

[70]Hazel Lundberg, "Mathematics in Elementary School." Educational Leadership, 19:364-68;1962.

[71]Mary E. Wilsberg, "Freeing Children in Primary Arithmetic." Educational Leadership, 19:352-6; 1962.

There are many isolated studies which support teaching methods employing meaningful application of mathematics, as pointed out at the beginning of this section. Some of the more outstanding are those of Stone,[72] DeLong,[73] Langer,[74] and Miller,[75] in which meaningful methods generally gave favorable results. Langer emphasized the dependence of significant technological advances on mathematics.

There have been a few studies recently reported showing the relationship of time in class to achievement. Daugherty, in a study conducted in the DesMoines Public Schools, found a fifty minute class favorable.[76] Denny[77] also showed an increased time allotment as contributing to greater achievement.

Changing methods of teaching arithmetic have prompted some studies in regard to evaluation procedures. Breuckner offers six

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[72] Marshall H. Stone, "Fundamental Issues in the Teaching of Elementary School Mathematics." The Arithmetic Teacher, 6:177-79; 1959.

[73] Arthur R. DeLong and Richard M. Cl rk, "Developing Creativity in Arithmetic." The Arithmetic Teacher, 6:208; 1959.

[74] Rudolph E. Langer, "To Hold As't Were the Mirror up to Nature; to Show the Very Age and Body of the Time." The Arithmetic Teacher, 6:289-94; 1959.

[75] G.H. Miller, "How Effective Is the Meaning Method?" The Arithmetic Teacher, 4:45-49; 1957.

[76] James Lewis Daugherty, "A Study of Achievements in Sixth Grade Arithmetic in DesMoines Public Schools," doctoral dissertation as reported in Analysis of Research in the Teaching of Mathematics. United States Department of Health Education and Welfare, Bulletin Number 8. Washington: U.S. Office of Education, 1900.

[77] Robert Ray Denny, "A Two-Year Study of an Increased Time Allotment Upon Achievement in Arithmetic in the Intermediate Grades, doctoral dissertation as reported in Analysis of Research in the Teaching of Mathematics, United States Department of Health Education and Welfare, Bulletin No. 8. Washington: U.S. Office of Education, 1900.

criteria to be considered in a testing program to be used in connection with meaningful teaching procedures:

> 1. The selection and clarification of objectives.
>
> 2. The determination of the rate of growth.
>
> 3. Provision of a basis by which teachers can set up educational experiences adopted to the needs and ability of the learners.
>
> 4. Motivation and guidance of learning, especially by helping children to evaluate their own responses and behavior.
>
> 5. The location diagnosis and treatment of learning difficulties.
>
> 6. Bases for coordinating improvement programs in related fields such as arithmetic, reading, science, and social studies.[78]

He adds five basic steps to aid in evaluation:

> 1. Formulating general and specific objectives.
>
> 2. Defining objectives in terms of pupil behavior.
>
> 3. Designing and selecting suitable means of appraisal.
>
> 4. Securing a record of behavior and performance.
>
> 5. Interpreting and evaluating the information secured.[79]

Grouping has received some attention as offering a possible solution to differences of ability. Lerch reports:

> Ability grouping as it has ordinarily been used does not solve the problem of providing for individual differences in arithmetic. The mere grouping of pupils with similar characteristics does little toward improving the teaching-learning situation.[80]

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[78] Breuckner, op. cit., p. 334.

[79] Ibid.

[80] Harold Lerch, "Inter-Class Grouping for Arithmetic Instruction-Critique and Criteria." The Arithmetic Teacher, 8:406; 1961.

A program of grouping based on individualized textbooks in which students corrected their own work, progressed at their own speed, and were instructed during individual teacher-pupil conference periods was tried by Whitaker. Success seems to have been measured by the amount of text covered.[81]

This review of present practices would be incomplete without references to the studies being conducted on content, or _what_ it is possible to teach rather than why it should be taught, as pointed out previously. The School Mathematics Study Group has prepared materials for grades four to twelve relating to pure mathematics under the support of the National Science Foundation. These materials are being tried in some areas on an experimental basis. Preliminary reports seem to indicate that students accomplish about the same achievement in S.M.S.G. classes as regular classes. There was no indication given as to time students spent in class, teaching methodology, or as to whether these materials should comprise the total arithmetic program of the curriculum.[82]

Other current experiments include that of Suppes[83] in teaching "Sets and Numbers", Stanford University, California, "Geometry in the

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[81] Walter L. Whitaker, "Why Not Individualize Arithmetic?" _The Arithmetic Teacher_, 8:402; 1960.

[82] "School Mathematics Study Group," Newsletter No. 10. Leland Stanford University, 1961. See also, Fred J. Weaver, "The School Mathematics Study Group on Elementary School Mathematics," _The Arithmetic Teacher_, 8:32-35; 1959.

[83] "Improvement Projects Related to Elementary School Mathematics-A Selected Listing." Unpublished mimeographed material. Lansing: Michigan Department of Public Instruction, 1962.

Primary Grades" being conducted by Hawley also at Stanford University,[84] the University of Illinois Arithmetic Project"[85] and the Syracuse University "Madison Project."[86] These studies are concerned with teaching pure mathematical content and not with a total arithmetic curriculum. They are examples of the "what-can-be-taught" studies pointed out by Glennon.

In contrast to the studies listed above, it was the purpose of this investigation to design an arithmetic program that would take into consideration the problems reviewed in this section: (1) offering a meaningful presentation of mathematics; (2) providing for individual differences; (3) structuring so that it could be accomplished in a minimum of class time; (4) teaching only the mathematical theory that was needed to adequately explain the arithmetic content of grades five and six; and, (5) enabling students to show significant achievement on standardized achievement tests at these grade levels.

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[84] Ibid.

[85] Ibid.

[86] Ibid.

CHAPTER III

DESCRIPTION OF THE TWO METHODS USED IN THE STUDY

It is not the purpose of this thesis to present in detail all the materials used in the teaching of the arithmetic program during the year of this study. Representative lesson plans appear in the Appendix. This chapter will be concerned with a descriptive comparison of the experimental and routine classroom procedures. The experimental description will contain: (1) the method of presentation of the mathematical concepts taught; and (2) the developmental processes used. The routine classroom procedure will contain an analysis of interviews with the teachers of the self-contained, or routine, classroom situations.

The experimental method was concerned with concepts to be presented to the students as important mathematical ideas of social significance. These were selected arbitrarily by the experimental teacher as representing the arithmetical materials which the students were required to cover during the course of the year as prescribed by the existing curriculum of the school system. Topics from pure mathematics that would offer sound mathematical explanation for this material were added. One concept was presented to the students each week in a discussion period. Students were not told how to solve problems, but were encouraged to create and investigate ways of doing this. A list of the concepts presented and the approximate time spent on each appears in Table I. The same concepts were presented to both fifth and sixth grades at the same time during the year in the sequence indicated on the table.

TABLE I

CONCEPTS TAUGHT IN GRADES 5 AND 6

TOPIC	APPROXIMATE TIME
My Own Number System	1 week
Egyptian Number System	1 week
Greek Number System	2 days
Roman Number System	3 days
Hindu Arabic Number System	1 week
Linear Measurement	1 week
Square Measurement (area)	1 week
Dry Measurement (volume)	1 week
Weight	1 week
How We Can Add	1 week
How We Can Subtract	1 week
How We Can Multiply	1 week
How We Can Divide	1 week
Place Value as Powers of a Base	1 week
Other Bases (7 and 12)	1 to 2 weeks
Time (and Base 60)	1 week
Modular Arithmetic (calendar)	1 week
Centuries	1 week
Temperature	1 week
Averages	1 week
Data	1 week
Graphs (line)	1 week
Graphs (bar)	1 week
Graphs (area)	1 week
Decimals (as extension of place value)	1 week
Rounding and Estimating	1 week
Big Numbers (as powers of 10)	1 week
How to Add, Subtract, Multiply and Divide Fractions)	About 2 to 4 weeks. Taught at one time all four processes.
Cancellation	1 week
Accounting	1 week
Finding Wholes from Parts	1 week

Table II indicates the general structure of each weekly lesson. The teacher acted as a resource person during the class. The text was used as a reference book. In addition, other reference materials were supplied as students requested them. When children asked for suggestions about locating materials, the teacher helped students to find these. Since much of the pure mathematics needed by the students was not found in published materials, these were written for the students by the teacher. These appear in the Appendix to this study.

At all times the student chose the method he would use in presenting evidence of his understanding of the concept for the week. Students were not given assignments to complete. The only requirement was that each would demonstrate his understanding of the concept under consideration.

TABLE II

WEEKLY ROUTINE OF EXPERIMENTAL CLASS

MONDAY:	Presentation of Concept by teacher in discussion period. Children discuss: (1) what they already know regarding concept; (2) how to find additional materials; (3) what each individual can do to produce tangible evidence of his understanding of concept.
TUESDAY) WEDNESDAY) THURSDAY)	Children are free to work in groups or individually in the area under investigation for the week. The experimental teacher acts as resource person.
FRIDAY:	Children are presented problems in a series of increasing difficulty. These contain selections from the four arithmetical processes. Children are requested to work as many as they can during the class time. These are taken up, corrected by the teacher and generalizations discussed with the group as a whole.

The tangible and conceptualized presentation was intentionally designed as a methodology which would attempt to accomplish the existing arithmetic curriculum in a minimum of time without the use of rote learning techniques. The intents was to increase understanding and to decrease the amount of paper-and-pencil work by both students and teachers. This seemed desirable because of the recent additions of content courses of elementary curricula. Chapter II pointed out the relationships between such a methodology and learning theory.

It was also felt that the elementary arithmetic curriculum should meet the needs of: (1) the student who may leave school at age sixteen, and; (2) the student who will pursue a career in mathematics. The former needs a mathematics program to provide him with a cultural tool for daily living; the later, needs a sound theoretical foundation that will furnish a sequential background for secondary mathematics. The tangible and conceptional presentation was designed to present mathematical theory in a simple manner to the whole group. Each child then took the concept and produced tangible evidence of his understanding. In this manner, slow children were permitted to produce tangible evidence at their own level; academically talented students were permitted to produce evidences representative of their intellectual level. The burden of proof of understanding rested at all times with the student. Details of the mathematical theory taught are presented in the Appendix.

The procedure for teaching the experimental class followed a very definite pattern. Since the classes were conducted in two buildings, a careful time schedule had to be followed. The experimental teacher entered the building, taught the class, or classes, and left. The regular classroom teacher stayed in the room and observed the experimental class. This teacher was prohibited from teaching any additional arithmetic at any other time, and from assigning arithmetic as "seat work" or for "discipline".

At the end of the thirty-five minute experimental class period, the experimental teacher left the room and travelled to the next school. This made it impossible to give any extra help to slow learners. It also prohibited correlation of the arithmetic program with other subject matter areas. Students were requested to stop working at the end of the thirty-five minute class. Table III contains a sample lesson.

TABLE III

DESIGN OF SAMPLE LESSON

YOUR OWN NUMBER SYSTEM

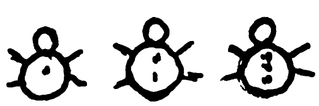
This lesson is designed to help children understand that a number system is an orderly sequence of symbols used to express quantity. Often students become so accustomed to counting and doing arithmetic by rote that they do not realize how a number system functions. Say to the children:

> "Forget everything that you ever knew about arithmetic. Forget how to make numbers and how to do problems. Now pretend you are living many thousands of years ago and that you are in need of keeping track of something. This is really one of the most important uses of arithmetic. . .writing down amounts of things by using symbols. We call these numbers and numerals."

(Do not go into the "number and numeral" question at this point. Use terms the children already understand. This may be introduced as an outcome of the lesson, however. Point out that it is easier to manipulate numerals rather than <u>things</u> to figure something. For instance, if a farmer has a large flock of sheep and buys and sells some, he can simply use a numeral to represent the total flock and add and subtract other numerals to represent the amount he buys and sells. This is easier than bringing them all together everytime and counting them.) After developing these or similar ideas continue:

> "After you have forgotten all the arithmetic you ever knew, pretend that you want to figure something out or keep track of something. Make up a number system and do this in your system."

The children will probably make rather simple symbols such as:

1. I II III — Just marks are typical of a 1-1 relationship.
2.  This type of symbol is really 1-1 relationship also.
3.  Sometimes children will use ideas like those in the text.
4.  Symbols like these are vague.

Children need help in seeing that symbols, or a system of numeration should be:

1. simple
2. convenient
3. useful

After they have invented a few symbols, let them work some problems in their own system. Help them to simplify and improve their system.

TABLE III - (Continued)

YOUR OWN NUMBER SYSTEM - RÉSUMÉ OF LESSON

1. Discuss arithmetic as a way of keeping track of things.

2. Introduce the idea that our numbers are a system of symbols that express amounts.

3. Have children make up their own number system.

4. Let each child work problems in his number system.

5. Discuss how the number systems of the children work. Lead them to conclude that a number system should be:

1. simple
2. useful
3. convenient

(It is best to have children "forget everything that they ever knew about how to do arithmetic" in order to do this)

REFERENCES

Smith, David E., and Jekuthiel Ginsburg, Numbers and Numerals. Washington: National Council of Teachers of Mathematics, 1959.

Smith, David E., History of Mathematics. New York: Gin and Company, 1951.

Batchelor, Julie Forsyth, Communication: from Cave Writing to Television. New York: Harcourt Brace and World, Inc. 1953.

Hogben, Lancelot, Mathematics in the Making. Garden City: Doubleday and Company, 1960.

__________, The Wonderful World of Mathematics. Garden City: Doubleday and Company, 1955.

Mueller, Frances J., Arithmetic, Its Structure and Concepts. Englewood Cliffs: Prentice Hall, Inc., 1956.

Specific procedures of the experimental class are listed in Table IV.

TABLE IV

PROCEDURES USED IN EXPERIMENTAL CLASSES

1. Class time limited to 35 minutes per day.
2. No ability grouping.
3. No home work.
4. No individual help for "slow learners."
5. No assignments were given that required students to finish.
6. No direct coordination with other subjects.
7. No time spent outside of class on arithmetic such as recess or after school.
8. No arithmetic given as "discipline" or "busy work".
9. No drill.
10. No "story problems" from text worked.
11. No attempt was made to work any problems in the text; other than those used as reference and resource.
12. Students worked problems and handed them in once a week only.

In general the class routine was one which: (1) presented an idea or concept to the students; (2) gave them opportunity to investigate it; (3) gave them opportunity to react to it by producing some tangible evidence of their understanding of the concept; and, (4) gave them an idea of the progress that they would be making were they in a routine text. This was done by giving them a set of problems once a week in the four arithmetical processes which were structured in a series of increasing difficulty. This was felt necessary for the following reasons.

1. Patrons of the school wished to know "what page in the text their children were on." It was felt necessary to show them that the students were able to do the routine text materials although they were not using the text as such.

2. The children wanted to know if the conceptualized program would retard their regular progress through the grade. They seemed to wish some evidence of their progress. Doing some problems in the routine fashion seemed to help them feel they were making progress in the grade.

3. The problems presented once a week differed from the type contained in the text in that: (1) the four arithmetical processes were represented; (2) the problems progressed from very simple to very difficult ones. Each child did only the problems of which he felt capable; (3) these problems always contained representative examples from all preceeding lessons. Samples of these weekly lessons appear in the Appendix.

As each concept was presented, every effort was made to correlate and illustrate it with culturally significant items. Fractions were taught with measures. All four arithmetical processes involving fractions were taught at the same time. These were presented as experiences with fractions within the culture rather than as the manipulation of abstract symbols. After the understanding was built by using tangible materials, abstract symbols were manipulated in problem situations. Pages 41 to 45 show a typical lesson in fractions.

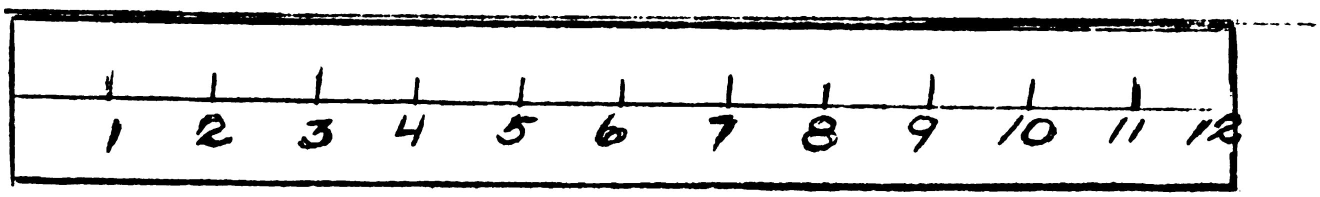
LINEAR MEASUREMENT

GRADE 5

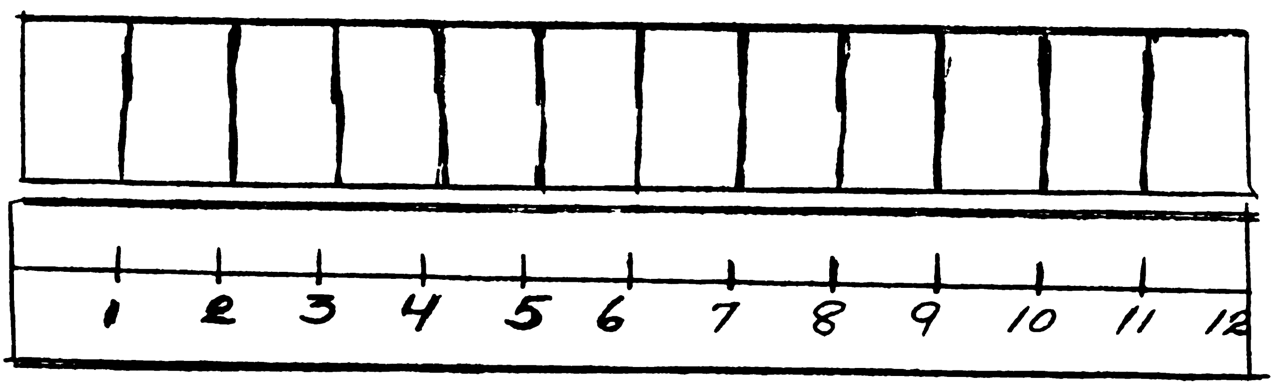
Linear (lĭn e er) measurement is a measurement of line. We say a line is 6 inches long. We also measure line, or length, in feet, miles, rods, and yards. See how many different kinds of linear measurement you can find and list.

We can use linear measurement to help us understand fractions. This is a good thing to do because very few of the things that we measure come in whole inches, whole feet, whole yards, etc.

Take one of the strips of paper that have been given you, place your ruler on it so that you can draw a line down the middle of the strip. On the line mark off the inch divisions so that you have a little oak tag ruler of your own. Like this:



Do the same thing with the other strip, but cut this one up into one-inch sections. This will give you a ruler that you can divide up into fractionals parts. Place the whole ruler beside the one that is cut up into inches on your desk like this:



Do the following things and answer the questions.

1. Divide the cut up inches into two groups. 1/2 foot = ? inches?
2. Divide the cut up inches into four groups. 1/4 foot = ? inches?
3. Divide the cut up inches into three groups. Each of these is a third of a foot. 1/3 foot = ? inches?

****

There are 12 inches in a foot. Each inch is 1/12 of the whole foot. By moving the inch pieces about, answer the following questions.

	A	B
1.	1/4 ft. = ?/12 ft.	3/12 ft. = ?/4 ft.
2.	1/2 ft. = ?/12 ft.	3/4 ft. = ?/2 ft.
3.	1/2 ft. = ?/4 ft.	2/4 ft. = ?/2 ft.

Can you make a list of other statements that are true about the fractional parts of a foot?

****

Answer these questions using your cut up inch pieces to help you.

	A	B	C	D
1.	1/2 = ?/4	1/3 = ?/6	3/4 = ?/12	2/2 = ?/12
2.	1/4 = ?/12	1/2 = ?/4	1/6 = ?/12	2/3 = ?/12

Make as many additional true statements as you can.

****

You have found that there are three-twelfths of a foot in one-fourth of a foot. We can write it this way: 3/12 foot = 1/4 foot, or just 3/12 = 1/4. Use your inches and see if you can complete these statements so that they are true.

1. If 1/4 = 3/12, then 3/4 = ?/12
2. If 2/3 = 8/12, then 1/3 = ?/12
3. If 1/2 = 6/12, then 2/2 = ?/12

These are more difficult. See if you can do them by thinking very carefully.

1. If 1/2 = 2/4, then $\frac{1\ 1/2}{2}$ = ?/4. We would read this this way: "If one half equals two fourths, then one and one half halves equal how many fourths?" The answer would be three fourths, or 3/4. Do this one.

   If 1/2 = 2/4, then $\frac{2\ 1/2}{2}$ = ?/4.

2. If 1/3 = 4/12, then $\frac{1\ 1/2}{3}$ = ?/12.
3. If 6/8 = 3/4, then 1/8 = ?/4.
4. If 4/12 = 1/3, then 5/12 is how many twelfths more than 1/3?

We have a sign in mathematics that means "more than." It is: . "Less than" is this sign: . You can remember them by thinking that more than is pointed in the direction that you write. Less than is pointed backwards to the way you write. Now write the answer to your problem like this:

If 4/12 = 1/3, then 5/12 is 1/12 1/3.

5. If 1/6 = 2/12, then 3/12 is how many twelfths 1/6?
6. Is 3/12 or 1/6?

****

You will need to think very carefully to do these. Use your cut up inches and see if they are true.

1. 3/4 foot = 2 1/2 thirds of a foot. (Hint: change both fractions to twelfths of a foot).
2. 1/2 foot = 1 1/2 thirds of a foot.
3. 2/3 foot = 2 1/4 fourths of a foot.

****

You have seen that you can write 2 1/2 thirds like this: $\frac{2\ 1/2}{3}$.

Write 1 1/2 fourths. If you add like this, then is it true?

$$\begin{array}{c} \frac{1\ 1/2}{4} \\ \\ \frac{1\ 1/2}{4} \\ \hline 3/4 \end{array}$$

Add:

$$\begin{array}{c} \frac{1\ 1/2}{3} \\ \\ \frac{1\ 1/2}{3} \\ \hline \end{array} \qquad \begin{array}{c} \frac{2\ 1/4}{2} \\ \\ \frac{2\ 1/4}{2} \\ \hline \end{array} \qquad \begin{array}{c} \frac{3\ 1/2}{4} \\ \\ \frac{3\ 1/2}{4} \\ \hline \end{array}$$

Divide:

Label your answer <u>and</u> your remainder if you have one.

A	B	C
2 in. ⟌ 12 in.	3 in. ⟌ 18 in.	2 in. ⟌ 24 in.

D	E	F	G
6 in. ⟌ 54 in.	7 in. ⟌ 1 ft. 4 in.	3 In. ⟌ 2 ft.	6 in. ⟌ 3 ft.6 in.

The routine or control classroom procedures were established by interviews with the teachers of these rooms. Pages 46 to 50 contain a summary of these interviews. The replies of the teachers of the routine (control) classrooms have been generalized and listed. A summary appears in the conclusion.

1. How were assignments made?

Control teacher No. 1

Assignments in texts at several levels.

Students required to finish assignment.

Memorizing tables, mandatory - drilled.

Homework about 3 times a week.

Text sequence covered.

Talented pupils worked extra problems in advanced texts.

Repetitive problem work for slow learners.

Control teacher No. 2

Assignments in text every day.

Some children "worked ahead".

Assigned extra work as "discipline".

Assigned homework.

Generally worked through text.

Control teacher No. 3

Assignments in text every day.

Generally covered text.

<u>Control</u> <u>Teacher</u> <u>No. 3</u> - continued

Students required to finish given amount determined by teacher.

Tables memorised.

<u>Control</u> <u>teacher</u> <u>No. 4</u>

Assignments in text every day.

Teacher determined amount to be completed.

Generally covered text.

Followed text sequence.

2. Were story problems in text covered?

<u>Control</u> <u>teacher</u> <u>No. 1</u>

Generally covered except for slow learners.

<u>Control</u> <u>teacher</u> <u>No. 2</u>

Covered at least 60% of those in text and made up others.

<u>Control</u> <u>teacher</u> <u>No. 3</u>

Did most of the story problems in the text.

<u>Control</u> <u>teacher</u> <u>No. 4</u>

Worked most of the story problems in the text.

3. Did you give individual help?

<u>Control</u> <u>teacher</u> <u>No. 1</u>

During odd times all during the day.

Set aside 15 minutes twice a week for "extra" help.

Worked with slow learners.

<u>Control</u> <u>teacher</u> <u>No. 2</u>

Gave extra help to talented and slow students.

<u>Control</u> <u>teacher</u> <u>No. 3</u>

Helped individuals all day through.

<u>Control</u> <u>teacher</u> <u>No. 4</u>

Gave individual help during the day and set aside 15 minutes per day extra for help.

4. How often were papers sent home?

<u>Control</u> <u>teacher</u> <u>No. 1</u>

At marking periods and also other times.

Sent homework assignments home.

<u>Control</u> <u>teacher</u> <u>No. 2</u>

Two or three times a year.

<u>Control</u> <u>teacher</u> <u>No. 3</u>

Twice per week.

<u>Control</u> <u>teacher</u> <u>No. 4</u>

Each week.

5. How were students grouped as to ability?

<u>Control</u> <u>teacher</u> <u>No. 1</u>

Students grouped themselves: below grade level, at grade level, and above grade level.

<u>Control</u> <u>teacher</u> <u>No. 2</u>

Some individual cases.

Control teacher No. 3

Grouped according to amount of text covered.

Control teacher No. 4

All stayed in same text but progressed at different levels.

6. What materials did students cover?

Control teacher No. 1

Used texts at various ability levels.

Control teacher No. 2

Covered text sequence.

Control teacher No. 3

Covered texts at various ability levels.

Control teacher No. 4

Followed text sequence - 2/3 to all of it.

7. How often did you correct papers?

Control teacher No. 1

At least 3 times a week.

Control teacher No. 2

More or less every day.

Control teacher No. 3

Children corrected own papers every day - teacher once a week.

Control teacher No. 4

Two times a week, but had children rework problems missed and then corrected these.

8. What other learning activities did you use?

Control teacher No. 1

Abacus, flash cards, games, extra papers, teamwork, outside of class productions, pictures.

Control teacher No. 2

Memorized tables by repetitious problems.

Control teacher No. 3

Correlated with science.

Had drill problems.

Control teacher No. 4

Some games, tables (drill)

9. How much instructional time was used per day for arithmetic?

Control teacher No. 1

45 minutes, plus extra help during the day and homework.

Control teacher No. 2

35 minutes, plus some homework and "discipline" problems.

Control teacher No. 3

35 minutes, plus extra help at odd times.

Control teacher No. 4

Given extra time to finish.

TABLE V

SUMMARY OF TEACHING TECHNIQUES USED IN EXPERIMENTAL AND ROUTINE CLASSES

EXPERIMENTAL	ROUTINE
Assignments were student determined	Assignments teacher determined
Did not work story problems in text	Worked most of story problems in text
Received no individual help	Received individual help
Worked problems to hand in once a week	Worked problems to hand in 2 - 5 times a week
Were not grouped	Were partially ability grouped
Text did not structure program	Text structured program
Had many other learning activities	Learning activities limited to "drill" games
Spent 35 minutes per day in class	Spent over 35 minutes per day in class

CHAPTER IV

ORGANIZATION OF INVESTIGATION

<u>Design of this investigation</u>. This dissertation was carried out during the course of the "Study in the Use of Special Teachers in Science and Mathematics in Grades Five and Six a part of the Science Teaching Program of the American Association for the Advancement of Science". Cooperating school systems were: Lansing, Michigan; Cedar Rapids, Iowa; Washington, D.C.; and Woodford County, Kentucky. Each school system hired a teacher who would teach only science or only mathematics. The purpose of the over-all study was to compare the two methods: (1) special teachers; or, (2) self-contained classroom teachers.

The use of a conceptualized and tangible presentation of arithmetic was devised as the special method of teaching which would be used in the Lansing, Michigan experimental center. As each of the four experimental centers were free to specialize in any way they chose, it was felt that this in no way either detract from or enhance the study of the American Association for the Advancement of Science.

<u>Students in the study</u>. The students chosen to participate in the study were considered as typical of those in the city. Two schools, each of which contained two fifth and two sixth grades were selected. One fifth grade and one sixth grade were taught by the special, or experimental teacher, and the other fifth and sixth grades were taught by the self-contained, or control room teacher in the routine manner of the school system.

There were, therefore, eight groups in the study. They were arbitrarily numbered as groups I to VIII, inclusive. Odd numbered groups were experimental, even numbers designated the control (self-contained) groups. No effort was made to match the members of the experimental and control classes. They were selected just as they existed in the school during the regular routine of grade placement. In the four fifth and four sixth grades, there were a total of 89 students in the experimental groups and 77 students in the control groups that completed the study. They have been treated as a total universe of experimental and control groups rather than separately.

No effort was made to match the members of the control and experimental classes. The classes were left as they existed in the normal routine of the school. Table VI compares the mean scores of the groups on the Otis Quick Scoring Test of Mental Ability given at the beginning of the study.

TABLE VI

COMPARISON OF MEAN TEST SCORES ON OTIS QUICK SCORING TEST OF MENTAL ABILITY GIVEN AT BEGINNING OF STUDY

	EXPERIMENTAL		CONTROL
Group I	101.1	Group II	103.1
Group III	107.9	Group IV	106.4
Group V	104.4	Group VI	103.9
Group VII	103.9	Group VIII	103.2
MEAN OF THE MEAN SCORES ON OTIS OF EXPERIMENTAL GROUPS			104.3
MEAN OF THE MEAN SCORES ON OTIS OF CONTROL GROUPS			104.1

Table VII shows the composition of the experimental and control groups.

TABLE VII

COMPOSITION OF GROUPS PARTICIPATING IN STUDY

GROUP	GRADE LEVEL	NUMBER OF BOYS	NUMBER OF GIRLS	AGE RANGE IN MONTHS	TOTAL IN GROUP
			EXPERIMENTAL		
I	6	14	9	130 - 142	23
III	5	11	11	119 - 136	22
V	6	11	13	130 - 143	24
VII	5	8	12	118 - 137	20
TOTAL EXPERIMENTAL GROUPS		44	45		89
			CONTROL (SELF-CONTAINED)		
II	6	8	10	131 - 149	18
IV	5	5	12	118 - 129	17
VI	6	10	13	128 - 142	23
VIII	5	8	11	113 - 130	19
TOTAL CONTROL GROUPS		31	46		77

Other personnel in the study. The regular classroom teacher remained in the room and observed the experimental lesson at most of the sessions. It was felt that this would have a certain inservice training effect. Although no study was made of the attitudes of these teachers, their feelings of anxiety were often expressed to the author. They indicated that they felt that the conceptualized presentation, the freedom which the students enjoyed in planning their own lessons, the fact that the sequence of the text was not followed, and that no regular assignments of problems to be completed and handed in would cause the children to fall behind in their grade level of achievement. It seemed difficult for them to disengage themselves from the arithmetic program as taught by the special teacher in their rooms. Pressure and anxiety became so great in some cases that the regular classroom teacher was requested to leave the room during the experimental class.

At the beginning of the study, the patrons of the school exhibited great anxiety about the experimental program. They frequently asked: (1) what page of the text the class was on; (2) why there were no papers worked and brought home by the children; (3) if their children could not be removed from the experimental class because "they were learning nothing". In order to answer such questions, group meetings were held in which the total experimental program was explained to the parents. They were invited to visit the classes and often did so. Frequently they asked to come and learn some of the "new" mathematics with the children. As the study progressed, the support of the parents appeared to increase. In many cases they expressed a desire to have the program continue as part of the regular curriculum. As an outgrowth of the experimental

arithmetic, an adult education class in "Arithmetic for Parents" was inaugurated. This class was taught by the author. The parents showed keen interest and often expressed the feeling that they could now "understand what their children were doing."

The geographic and cultural areas involved in the study were chosen mainly because of their accessability, since it was necessary for the experimental teacher to travel from one school to the next in less than thirty minutes. One of the schools was in an area of the city which contained factories for manufacturing heavy machinery, the other participating school was in a residential district containing only local stores and offices. Table VIII indicates the types of employment listed on the Form CA 39's in the schools. These should be regarded as an indication only as these records are not detailed. When both parents were listed as employed, both occupations were counted.

TABLE VIII

OCCUPATIONS OF PARENTS OF STUDENTS IN STUDY AS INDICATED ON FORM CA 39 FILED IN THE TWO PARTICIPATING SCHOOLS

	SELF EMPLOYED	LABORERS	SKILLED AND ADMINISTRATIVE	CLERICAL	UNKNOWN
"FACTORY" DISTRICT - GROUPS I, II, III, IV					
Experimental	2	33	6	1	2
Control	2	21	11	3	1
"RESIDENTIAL" DISTRICT - GROUPS V, VI, VII, VIII					
Experimental	6	11	23	0	1
Control	4	4	31	2	1

Table IX indicates the general academic background of the parents in the two geographic areas. These listings were taken from the CA 39 records in the participating schools and should be regarded as an indication only. They do however offer some idea as to the composition of the area.

TABLE IX

ACADEMIC BACKGROUND OF PARENTS OF STUDENTS PARTICIPATING IN THE STUDY

"FACTORY" DISTRICT

EDUCATION OF PARENTS IN YEARS - GROUPS I, II, III, IV

	0 - 8	9 - 12	12 - 16	OVER 16
Experimental	10%	82%	5%	1%
Control	8%	81%	7%	1%

"RESIDENTIAL" DISTRICT

EDUCATION OF PARENTS IN YEARS - GROUPS V, VI, VII, VIII

	0 - 8	9 - 12	12 - 16	OVER 16
Experimental	3%	66%	29%	1%
Control	7%	61%	24%	3%

CHAPTER V

DATA AND RESULTS

The achievement of the students in this study was evaluated on the basis of the regular testing program of the school. It was felt that this would offer some indication as to whether or not the children had progressed satisfactorily in the regular school program. This was due to the fact that it was an integral part of the study to ascertain if the arithmetic content required by the curriculum of the school system could be accomplished: (1) in a decreased class time; (2) without drill or rote learning processes; and, (3) with a decreased quantity of "paper and pencil" problem solving.

In addition, the achievement of the students in this study as measured by the regular testing program of the school, could be used to test the assumption that the "experiemental" teaching method was equal to or better than the routine (control) teaching method.

As pointed out above, all students were, at the inception of the study, given the STEP Test Form A. The California Achievement Test - Arithmetic Form BB - Reasoning and Fundamentals had been given in the school routine testing program. They were likewise tested at the conclusion (the end of the academic year) of the study. These standard tests were used to measure the relative gain in achievement of students after the treatment (tangible and conceptualized method) as opposed to routine teaching methods (control group).

Analysis of the Data. The "before-after" comparison was made by the use of analysis of Covariance. Lindquist, Taves, and others have

demonstrated the superiority of the analysis of variance model over most other statistical techniques. This model is especially superior to a standard "differences in the means" model which is based on the (t) distribution and hence "breaks down for small samples primarily because the standard deviations of small samples are not normally distributed."[86] When testing the hypothesis that the samples were drawn from equally variable populations, rather than dealing with the difference between the observed standard deviations (t) we deal with the ratio between corresponding estimates of the true variances as represented by the "variance ratio" of F or t^2. The power and versatility of the analysis of variance model is demonstrated by the fact that it enables us to deal with the mean of the variances around the mean within the groups as well as the variance of the group means.[87]

When dealing with the analysis of "before-after" data normally the experimental design demands "matched" or equated groups to secure increased precision. "This of course means a loss of valuable information, and loss may sometimes offset any advantage gained by the use of equated groups."[88] This "loss" can be prevented by the use of analysis of covariance, an extension of Fisher's analysis of variance model to include regression techniques. Therefore, the analysis of covariance, in addition to the advantage of the use of "statistical

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[86] E.F. Lindquist. Statistical Analysis in Educational Research. Boston: Houghton Mifflin Company. P. 60.

[87] Ibid., P. 87

[88] Ibid., P. 180

control" over "matched" or equated group experimental control functions to: (1) utilize regression techniques to statistically matched groups by cancelling out the effect of initial score differences on final scores; (2) applies analysis of variance to those adjusted final scores to determine the significance of the difference between groups after allowing for initial score differences; and hence, (3) makes assumptions, as to the characteristics of the data, similar to those implied in the use of analysis of variance and regression techniques.[89]

Using the analysis of covariance model the data were analyzed to test the general hypothesis that there are no real differences between the teaching methods (experimental versus control or routine) and that any differences in final mean scores (when initial scores are taken into account) of the methods groups, are due entirely to chance fluctuations in sampling.

Three specific hypotheses can be tested with this data:

Hypothesis I. No significant difference exists between the teaching methods (experimental versus control) indicated by the achievement gain as measured by the STEP Test Form A.

Hypothesis II. No significant difference exists between the teaching methods (experimental versus control) as measured by the California Achievement Test - Elementary Arithmetic Form BB, Reasoning.

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[89] Marvin J. Taves, "The Application of Analysis of Covariance in Social Science Research," Reprinted from American Sociological Review. 15:373-381, 1950.

Hypothesis III. No significant difference exists between the teaching methods (experimental versus control) as measured by California Achievement Test - Elementary Arithmetic Form BB, Fundamentals.

TABLE X

ANALYSIS OF COVARIANCE OF STEP TEST - FORM A ACHIEVEMENT SCORES OF GROUPS TAUGHT BY THE EXPERIMENTAL AND CONTROL METHODS

Source of Variation	Degrees of Freedom	SUMS OF SQUARES AND PRODUCTS Σx^2	Σxy	Σy^2	ERRORS OF ESTIMATE Sum of Squares	Degrees of Freedom	Mean Square[2]
Total	166	14186	12216	14443	3924	165	
Between	1	0	0	0			
Within	165	14186	12217	14432	3665	164	22
DIFFERENCE FOR TESTING MEANS					259	1	259

$$F = \frac{\text{Mean square difference (between)}}{\text{Mean square within}} \quad \frac{259}{22} = 11.77 \text{ significance} > .01$$

$$\text{Degrees of freedom} = \frac{k - 1}{k(N-1)}$$

Table X shows that the F ratio equals 11.77. This ratio is significant at the $>$ .01 level. Hypothesis I can therefore be rejected and interpreted to mean that the achievement gain of the experimental group over the control group as measured by STEP test can be attributed to teaching method.

TABLE XI

ANALYSIS OF COVARIANCE OF CALIFORNIA ARITHMETIC TEST (REASONING) - ACHIEVEMENT SCORES OF GROUPS TAUGHT BY EXPERIMENTAL AND CONTROL METHODS

Source of Variation	Degrees of Freedom	SUMS OF SQUARES AND PRODUCTS $\sum x^2$	$\sum xy$	$\sum y^2$	ERRORS OF ESTIMATE Sum of Squares	Degrees of Freedom	Mean Square²
Total	165	13492	11541	24234	14362	165	
Between	1	48	119	295			
Within	164	13444	11660	23939	11660	164	71
DIFFERENCE FOR TESTING ADJUSTED MEANS					2702	164	71

$F = \frac{\text{Mean square difference (between)}}{\text{Mean square within}} = \frac{2702}{71} = 38.05$ significance $> .01$

Degree of freedom $= \frac{k - 1}{k(N-1}$

The F ratio, as presented in Table XI, is 38.05 and is significant at the $>$ .01 level. Hypothesis II can be rejected and when based on the California Arithmetic Test of achievement, the experimental group shows a significant achievement gain over the control group.

TABLE XII

THE ANALYSIS OF COVARIANCE OF CALIFORNIA ARITHMETIC TEST (FUNDAMENTALS) - ACHIEVEMENT SCORES OF GROUPS TAUGHT BY EXPERIMENTAL AND CONTROL METHOD

Source of Variation	Degrees of Freedom	SUMS OF SQUARES AND PRODUCTS $\sum x^2$	$\sum xy$	$\sum y^2$	ERRORS OF ESTIMATE Sum of Squared	Degrees of Freedom	Mean Square[2]
Total	165	5661	3852	12800	4554	165	
Between	1	104	83	66			
Within	164	5557	3769	12734	10178	164	62
DIFFERENCE FOR TESTING MEANS					0	164	62

$$F = \frac{\text{Mean square difference (between)}}{\text{Mean square within}} = \frac{0}{62} = 0$$

$$\text{Degrees of Freedom} = \frac{k - 1}{k(N-1)}$$

The F ratio, as shown in Table XII is 0; therefore, Hypothesis III must be accepted. When measuring achievement with the California Arithmetic Test - Fundamentals, the experimental teaching method is not superior to the control method.

In general, it can be stated that, when conceptualized and tangible method is used in comparison with routine teaching methods, students show a statistically significant achievement gain as measured by the STEP-Form A and the California Achievement Test - Elementary Arithmetic Fundamentals -BB - (Reasoning). This supports the assumption underlying

the experimental teaching method and indicates that a real learning gain can be accomplished even under the handicaps of limited time allotment and paper and pencil work over routine teaching.

CHAPTER VI

SUMMARY, CONCLUSIONS AND IMPLICATIONS FOR FURTHER STUDY

<u>Summary</u>. It has been the purpose of this dissertation to compare the effect of a tangible and conceptualized method of teaching arithmetic on achievement in fifth and sixth grades with a self-contained classroom method. The tangible and conceptualized method was devised to teach the existing arithmetic curriculum by presenting arithmetical theory within a culturally significant situation. This was done in order to replace rote learning processes with a sound mathematical foundation which would furnish a meaningful arithmetic experience at all levels of academic ability. It was additionally the purpose of the thesis to show that this could be accomplished in thirty-five minutes per day. The burden of proof of understanding rested entirely with the students, who were required, not to complete an assignment, but to produce some tangible evidence of their understanding of the concept.

<u>Conclusions</u>. Analysis of the data revealed a significant difference in favor of the achievement of those students who were taught with the tangible and conceptualized method in: (1) STEP Arithmetic Test - Form A; (2) California Achievement Test - Elementary Arithmetic Form BB. Reasoning. There was found to be no significant difference between the experimental and control groups in performance on California Achievement Test - Elementary Arithmetic Form BB - Fundamentals.

On the basis of these results, it may be assumed that it is possible to meet the arithmetic curriculum requirements in grades five and six by using a tangible and conceptualized presentation which explains arithmetic rather than by rote-teaching methods. It may further be assumed that this can be accomplished in 35 minutes of class time per day without ability grouping, home work, or drill.

<u>Continuation of the study</u>. The limitations of this study became apparent during the progress of the research. The arithmetic program was presented with a method that did not involve the use of any existing text-book materials. It would appear that the preparation of such a text would be indicated so that the study might be repeated by a teacher other than the author of the materials. These materials were experimental in nature and developed with the cooperation of the students themselves. There is

a need for further investigation of the contribution which students are able to make in the structuring of an arithmetic program.

The time spent in class and amount of "paper and pencil" work were very limited. It is the intent of the author to follow the progress of the students involved in the study through the twelfth grade. This seems indicated in order to investigate the effect of such a limited mathematical program on the students performance in the areas of secondary mathematics.

Although it was not a part of the study, the teaching method employed theories of learning involving the understandings of broad concepts and the application of these concepts by the students to social problem situations. Further investigation is needed to evaluate the worth of such a methodology.

Uses of the materials developed. For this study, the content of the arithmetic curriculum was divided into approximately forty concepts or generalizations. These furnished the bases for a year's program in arithmetic. Such a program could be adapted to a television presentation of mathematics. The procedures involved would be:

1. Initial presentation of the arithmetic concept by the television teacher.

2. Development and investigation of the concept by the students and the classroom teacher together.

3. Production by the students of some tangible evidence of their understanding of the concept.

4. Evaluation and application of the concept to a social problem situation by the students.

In addition, it would appear that there is a need to:

1. Investigate other methods of evaluating a study of this nature.

2. Development of instruments to evaluate the understandings of concepts and their application by students to problem situations.

3. Development of instruments with which to evaluate the understanding of pure mathematical theory that may be possessed by fifth and sixth grade students.

4. Re-evaluation and re-interpretation of the purposes and goals of arithmetic programs and their relationship to the culture.

LITERATURE CITED

BOOKS

Banks, J. Houston. Learning and Teaching Arithmetic. Boston: Allyn and Bacon, Incorporated, 1959.

Batchelor, Julie Forsyth, Communication: from Cave Writing to Television. New York: Harcourt Brace and World, Incorporated, 1953.

Bushnell, G.T. "Introduction." Fiftieth Yearbook of the National Society for the Study of Education, Part II. Chicago: University of Chicago Press, 1951.

Buswell, G.T. "Arithmetic", Encyclopedia of Educational Research. Editor, Chester W. Harris. New York: The Macmillan Company, 1960.

Clark, John R., Arthur S. Otis, and Caroline Hutton. Primary Arithmetic Through Experience. Yonkers-on-Hudson: World Book Company, 1939.

Craig, Gerald S. Science for the Elementary School Teacher. New York: Ginn and Company, 1958.

Daugherty, James Lewis. "A Study of Achievements in Sixth Grade Arithmetic in DesMoines Public Schools," Analysis of Research in the Teaching of Mathematics, United States Department of Health, Education and Welfare Bulletin No. 8. Washington: U.S. Office of Education, 1900.

DeMay, Amy J. Guiding Beginners in Arithmetic. Evanston: Row Peterson and Company, 1957.

Denny, Robert Ray. "A Two-Year Study of an Increased Time Allotment Upon Achievement in the Intermediate Grades," Analysis of Research in the Teaching of Mathematics, United States Department of Health, Education and Welfare, Bulletin No. 8. Washington: U.S. Office of Education, 1900.

Grossnickle, Foster E. "Introduction," Instruction in Arithmetic, Twenty-Fifth Yearbook of the National Council of Teachers of Mathematics. Washington: The Council, 1960.

Guthrie, E.R. The Psychology of Learning. New York: Harper and Brothers, 1935.

Hartung, Maurice L., et al. Charting the Course for Arithmetic. Chicago: Scott, Foresman and Company, 1960.

Hilgard, Ernest R. Theories of Learning. New York: Appleton-Century-Crofts, 1958.

Hogben, Lancelot. Mathematics in the Making. Garden City: Doubleday and Company, 1960.

Hogben, Lancelot. The Wonderful World of Mathematics. Garden City: Doubleday and Company, 1960.

Jones, Phillip S. "The Growth and Development of Mathematical Ideas in Children," The Growth of Mathematical Ideas, Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics. Washington: The Council, 1959.

Judd, Charles H. Educational Psychology. New York: Houghton Mifflin, 1939.

Lankford, Francis J. "Implications of the Psychology of Learning for the Teaching of Mathematics," The Growth of Mathematical Ideas, Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics. Washington: The Council, 1959.

Lindquist, E.L. Statistical Analysis in Educational Research. Boston: Houghton Mifflin Company, 1940.

McHugh, Walter Joseph. "Pupil Team Learning in Skill Subjects in Intermediate Grades," Dissertation Abstracts. Ann Arbor: University Microfilms, Incorporated. 21:1461, 1960.

Monroe, Walter S. Development of Arithmetic as a School Subject. United States Bureau of Education Bulletin, No. 10. Washington: Government Printing Office, 1917.

Mueller, Frances J. Arithmetic Its Structure and Concepts. Englwood Cliffs: Prentice Hall Incorporated, 1956.

Piaget, Jean. The Language and Thought of the Child. New York: Meridian Books, 1955.

Price, G. Bailey. "Progress in Mathematics and Its Implications for the Schools," The Revolution in School Mathematics. Washington: National Council of Teachers of Mathematics, 1961.

Rockefeller Report. The Pursuit of Excellence, Panel Report V of the Special Studies Project. Garden City: Doubleday, 1958.

Russell, David H. Children's Thinking. New York: Ginn and Company, 1956.

Smith, David Eugene. History of Mathematics. New York: Ginn and Company, 1951.

Tenbergen, N. The Study of Instinct. London: Oxford University Press, 1958.

Thorndike, Edward L. Theories of Learning. New York: Bureau of Publications, Teachers College Columbia University, 1937.

Smith, David E. Numbers and Numerals. Washington: National Council of Teachers of Mathematics, 1959.

PERIODICAL LITERATURE

Breuckner, Leo J. "Testing Diagnosing and Follow-Up in Arithmetic," National Elementary Principal. 39:334, 1959.

Brown, J.C. "An Investigation of the Value of Drill Work in the Fundamental Operations of Arithmetic," Journal of Educational Psychology. 3:485-492, 561-570; 1912.

Brownell, William A. "Arithmetic in 1970," National Elementary Principal. 39:42-44, 1959.

Brownell, William A. and Charlotte B. Chezal. "The Effects of Premature Drill in Third Grade Arithmetic," Journal of Educational Research. 29:17-28, 1935.

Carpenter, Fenley. "Conceptualization as a Function of Differential Reinforcement," School Science and Mathematics. 38:284-294, 1954.

Carpenter, Fenley. "The Effect of Different Learning Methods on Concept Formation," School Science and Review. 40:282-285, 1956.

Clark, John R. "Looking Ahead at Instruction in Arithmetic," The Arithmetic Teacher. 8:394, 1961.

Cook, Raymond F. "Improving Arithmetic Instruction," The National Elementary Principal. 39:37-39, 1959.

DeLong, Arthur R. and Richard M. Clark. "Developing Creativity in Arithmetic," The Arithmetic Teacher. 6:208, 1959.

Flournay, Frances. "Relating Arithmetic to Everyday Life," National Elementary Principal. 39:294, 1959.

Glennon, Vincent J. "Editorial," Educational Leadership. 19;354-56, 1962.

Hamilton, E.W. "About the Articles," The Arithmetic Teacher. 8:49, 1961.

Harap, Henry and Charlotte E. Mapes. "The Learning of Decimals in an Arithmetic Activity Program," Journal of Educational Research. 29:686-693, 1936.

Hargrove, W. Richard, "Proper Emphasis on Science and Mathematics in the Elementary School," School Science and Mathematics. 44:89-91, 1960.

Hartung, Maurice L. "Distinguishing Between Basic and Superficial Ideas in Arithmetic Instruction," The Arithmetic Teacher. 6:65-67, 1959.

Langer, Rudolph E. "To Hold As't Were the Mirror Up to Nature; to Show the Very Age and Body of the Time," The Arithmetic Teacher. 6:289-294, 1959.

Lerch, Harold. "Inter-Class Grouping for Arithmetic Instruction - Critique and Criteria," The Arithmetic Teacher. 8:406, 1961.

Lundberg, Hazel. "Mathematics in Elementary School," Educational Leadership. 19:364-368, 1962.

Madden, Richard. "Major Issues in Teaching Arithmetic," National Elementary Principal. 39:174, 1959.

Miller, G.H. "How Effective is the Meaning Method?" The Arithmetic Teacher, 4:45-49, 1957.

Phillips, L.M. "Value of Daily Drill in Arithmetic," Journal of Educational Psychology. 3:485-492, 561-570; 1912.

Schaaf, William L. "Mathematics as a Cultural Heritage," The Arithmetic Teacher. 8:35, 1961.

Spitzer, Herbert F. and Robert L. Burch. "Methods and Materials in the Teaching of Mathematics," Review of Educational Research. 18:337-349, 1948.

Stone, Marshall H. "Fundamental Issues in the Teaching of Elementary School Mathematics," The Arithmetic Teacher. 6:177-179, 1959.

Sueltz, Ben A. "A Time for Decision," The Arithmetic Teacher. 8:280, 1961.

Sueltz, Ben A. "Arithmetic in Historical Perspective," National Elementary Principal. 39:12-16, 1959.

Van Engen, Henry. "An Analysis of Meaning in Arithmetic," Elementary School Journal. 49:321-329, 395-340; 1949.

Whitaker, Walter L. "Why Not Individualize Arithmetic?" The Arithmetic Teacher. 8:402, 1960.

Willey, Ray D. "A Study in the Use of Arithmetic in the Schools of Santa Clara County, California," Journal of Educational Research. 36:353-365, 1943.

Williams, Catherine M. "Arithmetic Learning in an Experience Curriculum," Educational Research Bulletin. 28:154-162, 167-168; 1919.

Wilsberg, Mary E. "Freeing Children in Learning Arithmetic," Educational Leadership. 19:352-356, 1962.

Wise, Carl L. "A Survey of Arithmetic Problems Arising in Various Occupations," Elementary School Journal. 20:118-136, 1919.

PAMPHLETS

American Sociological Review Reprint. "The Application of Analysis of Covariance in Social Science Research", Marvin J. Taves. Pullman: State College of Washington, 1950.

Michigan Department of Public Instruction, "Improvement Projects Related to Elementary School Mathematics, A Selected Listing." Unpublished mimeographed material. Lansing: Michigan Department of Public Instruction.

"School Mathematics Study Group. Newsletter No. 10." Leland Stanford University, 1961.

APPENDIX

TABLE XIII

SAMPLE OF WEEKLY PROBLEMS GIVEN IN GRADE FIVE

ADD:

627	45	1/3		1 1/2	4 1/3	16 4/5	829 5/8
356	27	1/3	4/5	23	3 2/3	7 1/5	72 6/8
429	31	1/3	2/5				
31	45						
502							
6							

SUBTRACT:

45	307	4009	726	11/12	4/5	1 5/8	45 6/8
21	86	999	87	3/12	1/5	1 3/8	13 5/8

MULTIPLY:

32	48	126	592	48	79	489	723
3	9	4	7	22	36	23	45

DIVIDE:

16 / 35   24 / 483   75 / 4896

TABLE XIV

SAMPLE OF WEEKLY PROBLEMS GIVEN IN GRADE SIX

ADD:

1.
$$\begin{array}{r} 65 \\ 22 \\ 39 \\ \underline{146} \end{array}$$

2. $4.50 + $.25 + $1.30 =

3. 1/2 + 1/4 =

4. 6/8 + 3/4 =

5. 4 1/2 + 1/4 =

6. 1/9 + 1/8 =

7. 6 1/5 + 4 1/3 =

8. 45 6/15 + 2 12/15 =

9. 7/8 + 4 1/3 =

10. $10_7 + 15_7 =$

MULTIPLY:

1.
$$\begin{array}{r} 62 \\ \underline{5} \end{array}$$

2.
$$\begin{array}{r} 3\ 1/2 \\ \underline{1} \end{array}$$

3. 1/2 X 1/6 =

4. 2/5 X 1/6 =

5. 7/8 X 16/49 =

6. 2 X 4 1/2 =

7. 64 X 42 1/4 =

8. 25 X 1 1/5 =

9. 3/8 X 24/12 =

10. 14/16 X 16/14 =

SUBTRACT:

1. $4.39 - $.25=

2. 4/5 - 1/5 =

3. 2/3 - 1/6 =

4. 3 1/2 - 1/4 =

5. 7/16 - 3/8 =

6. 3/4/9 - 1/9 =

7. 62 1/3 - 3/4 =

8. 27 1/2 - 1/16 =

9. 45 - 6/8 =

DIVIDE:

1. $3\overline{)27}$

2. $12\overline{)48}$

3. $34\overline{)728}$

4. 6/8 ÷ 1/8 =

5. 4/5 ÷ 1/5

6. $5\overline{)\$7.25}$

7. $.05\overline{)\$7.25}$

8. 4 1/3 ÷ 1/3 =

9. 4 1/3 ÷ 1/6 =

10. 4 1/3 ÷ 4 =

TABLE XV

SCORES OF STUDENTS IN GROUP I

	CALIFORNIA ACHIEVEMENT TEST ELEMENTARY ARITHMETIC FORM BB				STEP ARITHMETIC FORM A	
Code No.	Reasoning Before	Fundamentals Before	Reasoning After	Fundamentals After	Before	After
26	4.9	5.4	6.4	5.9	37	39
27	3.6	3.9	4.6	4.5	25	26
28	6.4	6.0	7.4	6.1	36	39
29	4.7	5.0	6.5	5.7	33	35
30	6.0	4.8	6.2	5.6	29	26
31	5.1	5.2	5.7	5.8	29	35
32	4.9	5.4	6.5	5.8	32	38
34	4.5	5.1	5.2	5.8	19	28
35	5.1	5.4	7.9	7.0	45	47
36	4.5	3.8	5.9	4.9	17	25
37	5.8	5.0	6.0	5.3	34	38
39	5.3	4.8	6.5	6.2	40	46
40	5.9	5.4	6.7	6.2	31	41
41	4.6	4.8	4.9	4.6	15	19
42	5.8	5.4	6.4	5.9	34	43
43	3.9	4.4	4.6	5.0	12	11
44	6.1	5.7	7.2	6.7	43	45
46	4.2	4.2	4.7	5.6	19	16
47	4.4	5.0	7.2	6.5	34	33
48	4.2	5.4	5.2	4.8	26	28
49	5.4	5.1	6.7	6.0	25	37
50	4.3	4.9	4.9	5.7	28	33
51	5.9	5.4	7.4	5.8	42	41

SCORES OF STUDENTS IN GROUP II

1	6.1	5.4	6.9	6.2	33	38
5	6.1	4.8	6.4	5.5	33	31
6	4.1	4.8	3.7	4.7	16	11
8	5.3	4.5	4.9	5.7	35	28
10	4.4	5.2	6.4	4.8	13	22
11	5.9	5.2	5.7	5.4	30	35
12	5.6	5.6	6.5	6.2	28	33
13	5.9	4.9	5.6	5.7	33	39
14	4.1	4.5	4.9	5.5	23	29
15	5.8	5.6	6.7	6.1	36	37
16	4.2	4.5	5.0	4.7	21	17
19	4.8	5.1	5.9	6.1	39	46
20	4.0	4.5	4.7	5.4	32	29
21	5.6	5.4	7.2	6.0	40	46
22	4.7	5.7	6.2	6.0	33	36
23	4.8	4.9	6.0	5.9	27	34
24	3.7	3.5	4.9	4.0	16	17
25	4.9	5.1	5.5	5.0	24	29

TABLE XV (Continued)

SCORES OF STUDENTS IN GROUP III

Code No.	California Achievement Test Elementary Arithmetic Form BB: Reasoning Before	Fundamentals Before	Reasoning After	Fundamentals After	STEP Arithmetic Form A: Before	After
53	5.4	5.4	8.4	7.2	36	38
55	4.9	5.1	6.7	5.7	21	32
57	5.5	4.5	5.7	5.1	29	30
59	6.5	5.7	7.7	6.0	27	38
60	5.5	5.1	7.3	7.6	33	35
61	4.8	5.1	6.3	5.1	24	28
62	4.6	5.5	6.1	6.3	26	35
64	4.4	4.1	6.3	5.9	13	28
65	5.9	4.7	6.5	6.7	25	36
67	3.3	4.3	5.3	5.8	10	28
69	4.8	4.5	6.3	4.7	21	22
70	5.9	5.4	7.4	6.7	26	25
72	3.6	4.8	4.2	4.5	13	16
73	5.5	3.8	5.9	5.5	24	28
74	4.2	4.9	5.1	4.2	17	24
75	4.7	4.8	6.5	6.1	19	17
76	5.0	5.4	6.7	7.3	27	27
77	5.5	5.6	7.1	6.5	32	36
78	6.2	6.4	6.9	6.4	32	44
79	5.0	4.9	7.4	5.1	9	21
80	5.2	5.2	7.4	8.4	23	27
81	4.7	5.0	6.1	5.4	15	26

SCORES OF STUDENTS IN GROUP IV

82	7.1	6.3	5.9	6.1	29	39
83	5.4	5.4	6.1	5.7	28	32
87	5.5	5.1	6.1	6.7	33	36
88	5.2	5.8	4.5	5.6	15	19
89	6.4	6.2	9.3	9.4	44	46
90	4.9	5.2	6.5	6.0	22	32
91	5.5	5.6	6.9	6.6	34	40
92	4.1	5.3	6.1	5.7	9	18
93	3.5	4.0	4.8	5.1	11	15
94	4.7	5.6	4.8	5.2	10	16
97	4.2	4.7	5.5	5.8	32	16
98	4.1	3.5	4.4	4.2	12	11
102	4.7	5.4	4.8	5.1	15	19
103	4.9	5.0	6.5	5.6	21	21
104	6.0	5.5	7.1	6.0	35	41
105	4.4	5.7	4.1	5.1	13	16
109	6.9	5.6	9.6	6.7	36	42

TABLE XV (Continued)

SCORES OF STUDENTS IN GROUP V

	CALIFORNIA ACHIEVEMENT TEST ELEMENTARY ARITHMETIC FORM BB				STEP ARITHMETIC FORM A	
Code No.	Reasoning Before	Fundamentals Before	Reasoning After	Fundamentals After	Before	After
111	3.0	4.8	5.7	6.4	19	14
112	4.7	4.6	6.5	5.8	35	45
113	4.7	4.6	6.5	5.8	25	33
114	4.6	4.3	6.3	6.7	20	34
115	5.6	5.0	7.1	6.5	30	37
116	6.1	5.1	8.7	6.9	44	46
117	5.9	5.1	8.4	6.7	44	40
118	5.9	5.4	7.6	6.3	32	38
120	4.4	4.3	5.5	5.6	17	24
122	5.1	4.5	8.1	6.3	36	41
123	5.6	5.5	8.4	6.7	38	41
124	4.9	4.9	7.7	6.3	24	33
125	2.7	3.5	5.3	5.3	15	18
126	4.2	3.9	6.5	6.4	22	29
127	4.8	5.1	6.5	5.9	19	22
128	6.1	5.1	9.3	7.7	44	43
129	5.3	4.3	6.9	6.6	35	42
132	4.8	4.7	6.9	6.9	34	36
133	4.8	4.3	7.4	6.8	32	37
134	4.4	4.9	6.5	5.9	26	33
135	4.9	4.7	6.7	6.1	30	38
136	5.4	5.4	7.1	7.2	30	43
138	4.5	4.5	6.3	6.5	34	33
139	5.9	5.0	5.1	7.1	38	46

SCORES OF STUDENTS IN GROUP VI

141	5.7	5.0	8.1	6.4	37	35
145	5.4	4.9	7.7	6.9	37	45
146	4.7	5.4	6.9	7.6	39	42
147	4.3	4.5	6.5	6.6	21	31
148	5.3	5.4	8.1	6.9	41	38
149	5.9	5.6	7.7	7.3	41	44
150	6.4	5.8	8.1	7.4	38	45
151	5.8	4.9	7.1	7.4	32	38
152	3.9	3.4	4.5	5.2	11	10
153	6.6	5.8	8.7	8.1	39	41
154	6.6	5.3	8.4	7.4	40	37
155	6.1	5.6	6.3	6.5	38	39
156	4.6	4.7	7.1	6.6	25	28
158	5.3	4.7	6.7	6.6	24	24
159	5.1	5.5	8.7	7.2	39	47
160	5.9	5.8	7.7	7.4	41	47
161	4.5	4.8	5.0	5.9	21	26
162	4.4	4.9	6.9	6.2	29	39
163	5.1	4.4	5.5	6.5	29	35
165	5.9	5.1	6.9	7.9	41	41
166	4.8	5.1	6.9	6.4	27	33
167	5.9	5.7	8.7	7.8	37	42
169	5.6	6.2	7.4	6.3	31	36

TABLE XV (Continued)

SCORES OF STUDENTS IN GROUP VII

Code No.	CALIFORNIA ACHIEVEMENT TEST ELEMENTARY ARITHMETIC FORM BB Reasoning Before	Fundamentals Before	Reasoning After	Fundamentals After	STEP ARITHMETIC FORM A Before	After
170	3.6	4.6	5.5	5.4	7	15
171	5.7	5.3	7.4	6.1	25	34
174	7.2	5.6	7.4	5.8	26	29
177	4.4	4.4	6.1	5.7	20	28
179	5.2	5.6	6.1	5.5	17	17
180	7.0	4.7	7.7	6.9	33	35
181	5.0	4.6	6.2	5.2	26	27
183	4.9	5.7	6.5	6.3	24	23
185	5.9	4.1	6.7	5.6	24	24
187	5.4	5.0	7.4	6.2	29	36
189	6.6	5.6	8.4	6.8	36	41
190	6.1	5.5	5.9	5.6	23	27
192	5.0	5.1	7.4	6.0	25	27
193	6.8	5.8	9.3	6.8	35	38
194	5.0	4.8	6.5	5.8	25	31
195	5.7	4.0	6.7	5.8	25	23
196	6.2	5.3	5.3	5.9	28	36
197	5.2	5.6	6.7	5.6	20	28
198	3.8	4.4	4.4	5.2	14	12
199	6.6	5.0	6.7	6.4	27	33

SCORES OF STUDENTS IN GROUP VIII

202	6.6	5.5	8.1	8.1	30	37
203	6.2	4.6	5.5	6.4	18	28
205	4.5	4.2	7.1	5.5	22	37
206	4.7	3.9	4.7	3.9	17	30
207	6.1	5.4	6.9	6.0	16	37
209	5.9	5.3	6.9	6.2	27	28
210	7.8	5.6	8.7	7.6	38	41
211	5.6	4.8	5.0	5.0	11	24
213	4.2	4.4	5.0	6.1	10	18
214	3.3	4.0	5.3	5.8	14	21
216	4.9	5.7	7.7	6.5	24	33
218	4.7	4.7	6.3	6.1	20	17
219	4.1	5.1	4.7	5.6	17	12
220	6.8	5.5	5.9	5.8	28	30
221	5.0	5.3	6.7	6.0	18	26
222	5.6	4.8	5.7	5.6	13	27
223	6.0	6.0	6.1	6.3	27	30
227	6.2	5.8	7.1	6.7	39	41
229	3.5	4.6	3.3	4.7	13	24

ARITHMETIC THEORY TAUGHT TO GRADES
FIVE AND SIX AS EXPLANATION FOR
CONTENT AREAS FOUND IN
CURRICULUM AT THESE LEVELS

I. NUMERATION

Arithmetic as a Symbolic Language

Our Own Number System

Suggested Activities

ARITHMETIC AS A SYMBOLIC LANGUAGE

Numeration is a way of expressing numbers in words. It is difficult to figure in written words alone; so over the years, different peoples have invented different ways of expressing numbers (or quantities) with symbols. Many types of symbols have been used throughout the history of mathematics. It would appear that man has always tried to make whatever number system he used more simple and convenient. In this sense, one could say that arithmetic just "grew". It has grown for some six or eight thousand years. Many number systems might not be recognized as such today.

Counting is the foundation of arithmetic. People count in many ways. For the most part, one can <u>count</u> to the answer in an arithmetic problem - if one has time enough. Since our mathematics should be simple and convenient in order to be of the most use, we try to count in bunches (or groups) rather than by naming every different quantity that we wish to express. We count in bunches (or groups) or tens. We call this a <u>base 10</u> number system. There are other people who count differently. Some systems are very simple. There are records of one like this:

one	(1)
two	(2)
some	(3)
many	(any amount of more than 3)

It makes little difference what these symbols may be or how they are named. We can see that in this kind of a system four could be either <u>many</u>, or <u>some plus one</u>, or <u>one plus some</u>, or perhaps just <u>onesome</u> as one word. This would be something like our number twenty-one. When we say "twenty-one" we think of an amount, and seldom stop to think that it is equal to 2 tens plus 1 one. Other people use other words to express amount.

There are said to be some Eskimos who use a system of naming their fingers and toes and combining these names into a counting system. In this way, they count in groups of five instead of ten as we do. They can do this until they get to twenty and (having run out of fingers and toes) they are forced to count in groups of twenty. They might figure their number system something like this:

1	atauseq	1 finger
2	machdlug	2 fingers
3	pinasut	3 fingers
4	sisamat	4 fingers (hand less 1 finger)
5	tadlimat	5 fingers (hand)
6	achfineq-atauseq	1 hand and 1 finger
7	achfineq-machdlug	1 hand and 2 fingers
8	achfineq-pinasut	1 hand and 3 fingers
9	achfineq-sisamat	1 hand and 4 fingers
10	qulit	2 hands
11	achqaneq-atauseq	2 hands and 1 toe
12	achquaneq-machdlug	2 hands and 2 toes
13	achqaneq-pinasut	2 hands and 3 toes
14	achqaneq-sisamat	2 hands and 4 toes
15	achfechsaneq	2 hands and 1 foot
16	achfechsaneq-atauseq	2 hands, 1 foot, and 1 toe
17	achfechsaneq-machdlug	2 hands, 1 foot, and 2 toes
18	achfechsaneq-pinasut	2 hands, 1 foot, and 3 toes
19	achfechsaneq-sisamat	2 hands, 1 foot, and 4 toes
20	inuk navdlucho	2 hands and 2 feet

You could make your own number system and do problems in it. You would need to make up some symbols to stand for amounts, and then show how they could be manipulated in order to keep track of things. The way the Eskimos have done should give you some ideas.

In the history of our number system, many things have been used to stand for different quantities. People have used letters like the Romans and the Greeks, and pictures like the Egyptians. Usually, the symbols had some meaning or relationship to the quantity that it was to represent, as in the Eskimo system above.

For instance: a nose could stand for one, etc. We might have a system like this:

Nose stands for 1. (Almost everyone has just one of these.)

Eyes come in pairs. So, eyes have stood for two.

Sometimes wings were used.

Stopping to draw 2 eyes every time is a bother,

so one eye might come to mean 2 all by itself.

Nose-eye combined could stand for 3.

Since this is rather a nuisance to have to stop

to make these two separately . . . . . .

The symbol for 3 might come to be no more than

something like this. 

Symbols have grown and changed over hundreds of years in much this same way. They have grown and changed over the centuries to meet the needs of a society. As one studies the history of ancient societies and civilizations, we find a vast variation of the symbols used; but they all served a particular need for some particular civilization. Many of these were adapted by other civilizations and were changed and fitted to their needs much as we have done with our own Hinku-Arabic system. As we discover and create new forms of energy along with space exploration, we will probably be adding new symbols to our system to meet the needs of mathematics. A good illustration of this is the light year. If we figure that light travels about 186,000 miles a second, we can calculate the speed that light would travel in a year by:

$$186{,}000 \times 60 \times 60 \times 24 \times 365 = 1 \text{ light year}$$

OUR OWN NUMBER SYSTEM

The exact beginnings of our own number system are vague. It is believed by some historians that it has been in use since the fifth or sixth century, B.C. It appears to have been in use for some time by the nomadic or "heathen" tribes of Arabia and India before it was recognized as being superior over the old Greek and Roman systems.

Historians disagree as to the exact time and place of its origin. Certain it is that the system has changed a great deal over the centuries. The symbols are of Hindu origin. The nomadic Arabians may have picked them up in India and carried them to Europe without using them much. It probably appeared first without the zero. This would have given it little advantage over the other older systems. It is thought that the zero may have come into use about the ninth century. The Hindus hit upon the ingenious idea of place value, or grouping of numbers. In this way, any amount can be expressed with only 9 digits and the zero to hold an empty place. It should be kept in mind that this so-called zero could be any symbol that would hold the place.

This Hindu-Arabic system was known in Europe as early as the thirteenth century. It was not much used until developments in science and trade made its computational advantages preferred over the old existing systems. This was possibly about the sixteenth century.

It is important to remember that over the past 2,000 years or so our number system has changed. These changes have been influenced by the changing needs of science and society. The Arabic and Hindu influences had much to do with its early formation. Our system has grown and changed with the centuries and it might be expected to continue to do so.

SUGGESTED ACTIVITIES

I

Have the children imagine that they have need to "keep track of something"; but that they know of no number system to use. Ask them to make up symbols to stand for amounts and then do a problem or two in their system.

II

Make a panarama to show how a shepherd might keep track of a flock of sheep by using 1 pebble to stand for each sheep.

Do the same thing, but show a 1 to 5 relationship or a 1 to 10. (One pebble for 5 sheep, or 1 pebble for 10 sheep.)

III

Dramatize a situation of buying and sellingin which characters use primitive ways of counting.

IV

Look up measures used during Biblical times and produce charts, booklets, or posters describing them.

V

Make a sign language for quantities and dramatize a situation involving a sale or exchange in which bargaining takes place.

VI

Look up the signs used by auctioneers to indicate amounts.

VII

Make a time line showing the development of different number systems. Use white wrapping paper and illustrate.

VIII

Make a simple abacus or counter and demonstrate it to the class. A good one can be made with a shirt box and buttons on strings for counters. Others can be made with clay. Use the clay for grooves and small stones for counters. Wonderful World of Mathematics by Hogaben has some good material in it. This book should be in your library or office

II. PLACE VALUE

Pattern of Expressing a Numeral

Use of Abacus in Place Value

The Powers of a Base (Exponents)

PLACE VALUE

We count in bunches or groups of numbers. This makes it possible to do mathematics with only a few symbols. In our number system we count in groups of tens. We call this a BASE ten, or we say we have a number system in Base Ten. We have nine symbols to stand for the quantities from one through nine: 1, 2, 3, 4, 5, 6, 7, 8, 9, and a zero, 0. We need the zero because we give numbers a value according to the place in which they stand: 3 is equal to 3 ones. If we put the 3 in another place and make 30, we use the zero to literally push the 3 into the second place. It now stands for 3 tens, or thirty. We can hold the ten's place and put the three in hundred's place: 300. In each case, we have used the same symbol to stand for three different amounts. We could go on doing this without end.

We do not really have a symbol to stand for ten. We simply use a 1 and put it in ten's place, using the zero to hold the empty one's place. We read this number: 10, as "ten". This stands for one group of ten. Ten is our base. The 1 in the numeral 10 stands for 1 times the base of ten. If the base were something else, this one would stand for that amount. This is also called the model group of a number system.

If we think about counting it helps us to understand about how the base, or model group functions in our number system. It is really unfortunate that we learn to count before we understand what it means, because then we are so familiar with the process that we may fail to see the importance of understanding it. Counting is the very foundation of arithmetic. Counting with understanding can be a great help in learning arithmetical processes. It is absolutely necessary to understand it in order to master arithmetic processes.

We use counting numbers to stand for different quantities. Counting is adding by ones. Each number has a position in relation to the others. That is, six always follows five, and comes before seven. We think of cardinal numbers as the numbers one, two, three, etc. We think of the ordinal numbers as first, second, third, etc. The number 3, for example, has a face value of three things. We could put it into a one-to-one relationship with a group of X X X things. We could count these and the last number we name "three" is the name of the number of members of the group. The number 3 is also the third number we name. This is the ordinal way of thinking of three. One good way to remember is that "third" expresses the order or arrangement of the series of numbers.

Counting numbers can be expressed as they relate to the place in which they stand. This is place value. A number has two values ... a face value and a place value. The number 3 has a value that makes it equal to X X X things. It also can be used to stand for 3 tens, 3 hundreds, or any place that we may choose to use. We express this as though the number were multiplying the value of the particular place it stands in. For instance, 333 can be expressed as (3 x 100) + (3 x 10) + (3 x 1). This is what 333 really means. It is necessary to think about counting numbers in this way to understand how a number system really functions. When it is thought of in this way, it is possible to understand arithmetical functions in any base. This also furnishes a basis for the process we call "borrowing" and "carrying".

The number 333 is simply easier to write than (3 x 100) + (3 x 10) + (3 x 1). In writing 333, it is also easy to lose sight of what the number really is. In order to see this more easily, it is well to indicate how numbers could be taken apart.

We will use parentheses to indicate the multiplication, $3(1) = 3 \times 1$, etc.

$$
\begin{aligned}
1 &= 1(1)\\
2 &= 2(1)\\
3 &= 3(1)\\
4 &= 4(1)\\
5 &= 5(1)\\
6 &= 6(1)\\
7 &= 7(1)\\
8 &= 8(1)\\
9 &= 9(1)\\[1em]
10 &= 1(10) + 0(1)\\
11 &= 1(10) + 1(1)\\[1em]
20 &= 2(10) + 0(1)\\
26 &= 2(10) + 6(1)\\[1em]
45 &= 4(10) + 5(1)\\
46 &= 4(10) + 6(1)\\[1em]
99 &= 9(10) + 9(1)\\
100 &= 1(100) + 0(10) + 0(1)\\
150 &= 1(100) + 5(10) + 0(1)\\
268 &= 2(100) + 6(10) + 8(1)\\[1em]
1000 &= 1(1000) + 0(100) + 0(10) + 0(1)\\
1004 &= 1(1000) + 0(100) + 0(10) + 4(1)\\[1em]
4579 &= 4(1000) + 5(100) + 7(10) + 9(1)\\
&\quad \ldots\ldots \text{etc.}
\end{aligned}
$$

It is easy to see when we look at numbers in this fachion that: (1) the use of place value makes our number system much more convenient; and, (2) we can change the model group or base very easily if we change the <u>value of the places</u>. This will become more apparent as we investigate the structure of other number systems.

USING AN ABACUS TO SHOW PLACE VALUE

Since 10 is our model group, we need to have only nine digits if we use place value. On the other hand, if we have only nine digits, we are forced to use some other way to express quantities over ten. An abacus can be used to demonstrate this very well.

Push up ten ones on the ones column, counting and writing the numbers as you do: 1, 2, 3, 4, 5, 6, 7, 8, 9, ? ......

Now, after we get to 9, we have no more symbols to express additional quantities, so we use place value. Push down the 10 ones beads and push up 1 ten bead on the tens column. Now write 10. This is: 1(10) + 0(1). The zero indicates and empty column.

Do this again: push up ten ones. We have no symbol for tens so push down the ten ones and push up another ten bead. We now have 20, or 2(10) + 0(1).

Continue doing this until you have pushed up all the tens beads. Now push these down and use 1 one hundred bead to stand for 1(100) + 0(10) + 0(1). Again, the zeros stand for empty columns.

You could continue until you have used up all of the hundred beads. After using 9 hundred beads, it is easy to see that we have no symbol for 10 hundred unless we go to the next place and, using place value, write 1,000. This is: 1(1000)+0(10) + 0(1).

If we think carefully about place value, and remember that our model group, or BASE, is ten, we can make a place value chart that shows this mathematically rather than by using an abacus. Think first about place value as we have learned the names for these places We will express these by using numbers that show how many times ten is multiplied by itself in order to equal each place.

THE POWERS OF THE BASE

ten-millions	millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	ones

Each place is ten times the one to the right of it. For example, tens are equal to 10 x 1, and hundreds are equal to 10 x 10, etc.

Each place is also 1/10 of the place to the left of it. For example, tens are 1/10 of 100, and hundreds are 1/10 of thousands.

Another way to understand this is to make a chart like the following one in which the value for each place is expressed as the number of times ten is multiplied by itself.

1	(ones)	
10	(tens)	10 x 1
100	(hundreds)	10 x 10
1,000	(thousands)	10 x 10 x 10
10,000	(ten-thousands)	10 x 10 x 10 x 10
100,000	(hundred-thousands)	10 x 10 x 10 x 10 x 10
1,000,000	(millions)	10 x 10 x 10 x 10 x 10 x 10
10,000,000	(ten-millions)	10 x 10 x 10 x 10 x 10 x 10 x 10

We have a more simple way of showing this by using small numbers written up to the right of the tens to show how many times ten is multiplied by itself. We call these <u>exponents</u>. These small figures show the powers of ten. We would read 10^2 as "ten squared" or as "ten to the second power." This means 10 x 10 or 100. Expressed in another way $10^1 \times 10^1 = 10^2$. Here you will see that we can add exponents in order to multiply numbers that are expressed exponentially. This is not particularly time saving in the multiplication of small numbers, but it is very useful in dealing with large ones. It also forms the basis for some very necessary understandings in regard to place value and the base of a number system. These ideas are also used in developing the concept of logarithms.

It would be well at this point to consider a chart showing how the powers of ten can be used to indicate place value.

1	(ones)		10^0
10	(tens)	10 x 1	10^1
100	(hundreds)	10 x 10	10^2
1,000	(thousands)	10 x 10 x 10	10^3
10,000	(ten-thousands)	10 x 10 x 10 x 10	10^4
100,000	(hundred-thousands)	10 x 10 x 10 x 10 x 10	10^5
1,000,000	(millions)	10 x 10 x 10 x 10 x 10 x 10	10^6
10,000,000	(ten-millions)	10 x 10 x 10 x 10 x 10 x 10	10^7

We can see why ten to the zero (10^0) is equal to one if we do a problem using the powers of ten. We will start with an example to which we know the answer:

$$\begin{array}{r} 10 \\ \times\ 10 \\ \hline 100 \end{array} \qquad \text{or} \qquad \begin{array}{r} 10^1 \\ \times 10^1 \\ \hline 10^2 \end{array}$$

We have to get 10^2 for the answer since $10 \times 10 = 100$, and 100 is written with an exponent as 10^2. Now, if we do a division example, we will see why 10^0 is equal to one.

$$10\overline{)100}^{\,10} \qquad \text{or} \qquad 10^1\overline{)10^2}^{\,10^1}$$

This may also be expressed as a fraction:

$$\frac{10^2}{10^1} = 10^1$$

Since $100 \div 10 = 10$, we have to show the answer as ten. (Ten expressed with an exponent is 10^1; therefore, we can assume that we would subtract exponents to divide numbers; or $10^2 \div 10^1 = 10^{2-1}$ or 10^1.

Since $10 \div 10 = 1$, we can now show that $10^1 \div 10^1 = 10^0$, because we subtract exponents to divide numbers. In this case, we could show it as $10^1 \div 10^1 = 10^{1-1}$ or 10^0. Or, in other words, ones can be expressed as 10^0. From this, we might conclude that: any number (not a zero) whose exponent is zero (^{0}) is equal to one, or $N^0 = 1$.

We should now remember that our number system is based on a model group of 10. Each position in place value can be expressed as a power of ten. These powers of ten are written as exponents. Knowing these things we can: (1) write any number of quantities that we wish to; and, (2) we can set up a number system in any base other than ten that we may choose. It is necessary to understand this as a foundation to the mathematics that is basic to arithmetic.

Often in mathematics when we wish to generalize we let letters stand for numbers. If we let the letter B stand for <u>any base</u> that we may choose, we can make a chart that will show place value for any model group or base. This will help to understand just exactly how place value operates in our familiar base 10 system. We could compare this with base 10.

Base 10	10^6	10^5	10^4	10^3	10^2	10^1	10^0
Any Base	B^6	B^5	B^4	B^3	B^2	B^1	B^0

III. OTHER SYSTEMS AND BASES

Egyptian Number System

Babylonian Number System

Greek Number System

Roman Number System

Base 7

Suggestions for Teaching

THE EGYPTIAN NUMBER SYSTEM

Egypt is one of the oldest civilizations about which we have a record. From their great temples and tombs, which they constructed some four thousand years ago, we have been able to learn a great deal about their lives. Suprisingly enough, we know quite a bit about their mathematics. They seem to have been a clever folk, quick to invent mathematical ideas and methods that would help them in their daily lives. An arithmetic book has been found dating from about four thousand years ago.

The Egyptians and the people in the countries around this area appear to have used a kind of picture writing for their words and for the figures that they made. The Samarians, Babylonians, Arabians, and Hindus, later the Greeks and Romans, have all contributed to our mathematical system. The earliest of these used hieroglyphics, or pictures, to make their words and numerals. Man appears first to have done his figuring only by drawing crude pictures to represent numbers. For instance:

for one

for two sheep

It is easy to see how something like this could lead to a hieroglyphic writing and number system. There is some indication that men also wrote out words for each number. Instead of writing a symbol like 6 or 7, they would write out the word, six, and seven. It would be very difficult to do much figuring with this kind of number system. So, symbols probably were invented as a matter of convenience. One of the oldest of all these systems is the Egyptian. It is quite remarkable for its simplicity and usefulness. Their figures, like their written words, were hieroglyphics.

We have words that illustrate this idea. Water wheel, water shed, water snake, and the name Waterman might be written in hieroglyphics like this:

water shed	
water wheel	
water snake	
waterman	

The Egyptians just made marks for the number from 1 to 9. It is easier to read these if the marks are grouped, and this is how they look:

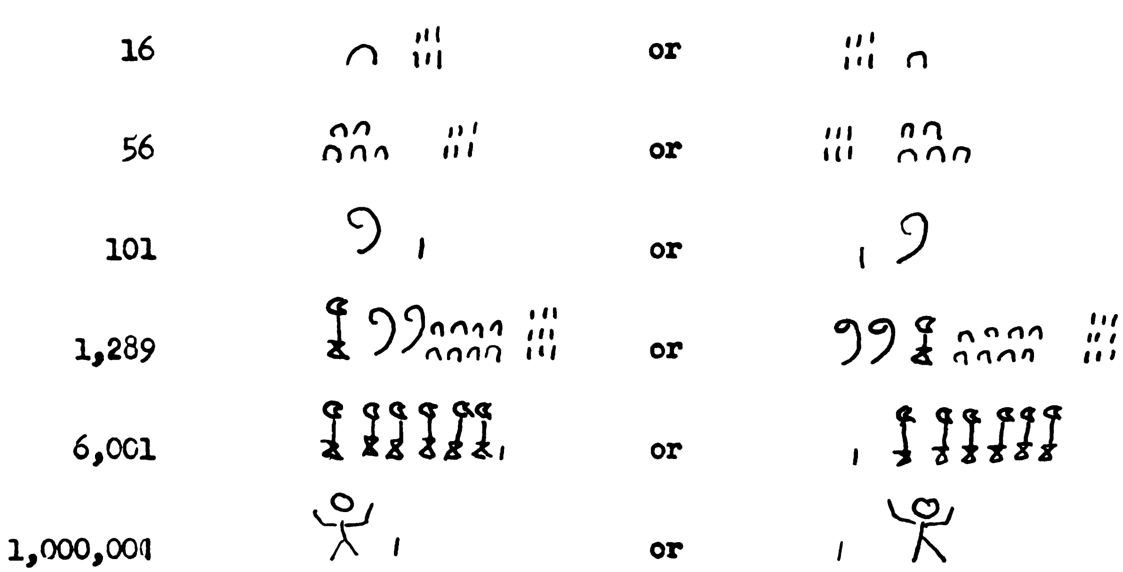
1	2	3	4	5	6	7	8	9
I	I I	I I I	I I I I	I I I I I	I I I I I I	I I I I I I I	I I I I I I I I	I I I I I I I I I

In addition to these, they had symbols for a sort of place value much like ours. It was based on 10.

10		heel bone
100		coil of rope
1,000		lotus flower
10,000		bent line
100,000		burbot
1,000,000		man in astonishment

We do not know why they picked these particular symbols. It is easy to imagine that a number like a million would be represented as a "man in astonishment". He probably was just overcome by such a large number. This reminds us of the very primitive number system of: "one, two, some, and a whole lot."

The Egyptians had no zero. They had no need of it in this system. The zero was to come many years later. If you write a few numbers, you can see that they would not miss a zero.



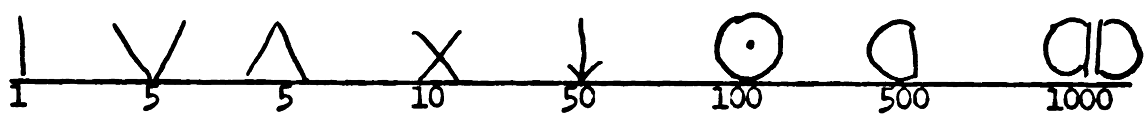
The Egyptians had a place value of sorts. Yet, it would make no difference to them if 128, , was written as or or , etc. We can see that they used the principle of addition in their system. It would make no difference if they added 10 + 6 to make 16, or 6 + 10 to make 16. This makes it rather easy to do problems in Egyptian hieroglyphics because all you have to do is to count the number of times a symbol appears and write it in tens. We will see how this system operated in the Roman number system later on.

ROMAN NUMBERS

In order to write Roman numbers, we use addition, subtraction, multiplication, and repetition. When these are understood, reading and writing these numerals becomes a simple process.

Roman numerals are written with the use of seven (and sometimes eight) symbols: I V X L C D M and sometimes $\overline{M}$. This is a very old number system. Over the centuries there have been changes in the symbols as we know them today. At the time Roman numerals were first used, people did not need very large numbers or the symbols to write them. As the culture of the people developed and their possessions increased, they had need of a better number system; so they changed the symbols that they used to fit their particular needs.

Very early Roman numerals were quite different from those of today. There were several ways to write the same number. Examples of such numerals have been found in old ruins.



EARLY ROMAN NUMERALS

There are many interesting books about the development of our number system. See:

Smith, David Eugene, and Jekuthiel Ginsburg, Numbers and Numerals. The National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W. Washington, D.C. (35 cents postpaid)

Smith, David Eugene, Number Stories of Long Ago. The National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D.C.

The National Council also has fine lists of books on mathematics in many areas.

THE USE OF ADDITION

V equals 5	X equals 10	L equals 50
C equals 100	D equals 500	M equals 1,000

A line over a numeral is sometimes used to multiply a number by 1,000; $\overline{M}$ would equal one thousand thousand or a million.

VI equals 6	MDCXX equals 1,000 plus 500 plus 100 plus 10 plus 10 (1,620)
VIII equals 8	LXV equals 50 plus 10 plus 5 (65)
XII equals 12	DCCC equals 500 plus 100 plus 100 plus 100 or (800)

With the exception of M (1,000), no symbol is repeated more than three times. When it becomes necessary to repeat a symbol four times, the next higher one is used and we subtract.

1	2	3	4	5	6	7	8	9	10	30	40
I	II	III	IV	V	VI	VII	VIII	IX	X	XXX	XL

50	80	90	100	200	300	400	500	1,000	1,000,000
L	LXXX	XC	C	CC	CCC	CD	D	M	$\overline{M}$

In order to write several thousands, the symbol M is usually repeated.

* * * * * * * * * * *

THE USE OF SUBTRACTION

Only one symbol is subtracted in writing Roman numerals. We would not write 80 as XXC because we would be subtracting more than one ten.

IV equals 4	XL equals 40	CD equals 400
IX equals 9	XC equals 90	CM equals 900

As we write Roman numerals now, these are the only ones that we use subtraction for. Remember that we subtract to write the numbers with fours, (4, 40, 400) and the ones with nines (9, 90, 9000).

THE USE OF REPETITION

The only symbols that are repeated are:

I equal to 1	C equal to 100
X equal to 10	M equal to 1,000

Remember that with the exception of M, which stands for 1,000, these are only repeated one, two, or three times, never four.

II equals 2	CC equals 200
III equals 3	CCC equals 300
XX equals 20	MMMMMM equals 6,000

* * * * * * * * * * * *

THE USE OF MULTIPLICATION

In very early times, large numbers were not used a great deal. The Roman numerals that were written about the beginning of the Christian era were quite different than ours and very irregular. When it became necessary to express larger numbers, a bar, $\overline{}$, was drawn over the number to multiply it by a thousand. This was similar to the Greek M for expressing larger numbers.

$\overline{\text{XXX}}\text{XIV}$ equals 30 thousand plus 14, or 30,014

$\overline{\text{XX}}$ equals 20 thousand or, 20,000

$\overline{\text{MM}}\text{D}$ equals 2 million plus 500 thousand or 2,500,000. The bar over the two M's makes each of them one thousand thousand.

* * * * * * * * * * * *

TWO HANDY RULES

I. A smaller number after a larger number is added to the larger number.

II. A smaller number before a larger number is subtracted from the larger number.

BASE SEVEN

We do not give every separate number a separate name. Instead, we count in groups of numbers. This makes it possible to count to an infinite number with the use of only ten symbols. We call this group of numbers a model group or the base of the number system. This may be any size.

For the most part, we use a model group, or base, of ten. Many times we need to have a mathematical system based on some other model group, or base, such as 12, 7, 60, etc. We have used some of these model groups or bases for a long time without realizing it.

We can use a base of 12 in figuring linear measurements. If we think of these as a base of 12, it makes the addition and subtraction much more simple.

2 feet	8 inches	
+3 feet	6 inches	
5 feet	14 inches	
+1 foot	-12 inches	(removing 1 model group of 12 and "carrying" it)
6 feet	2 inches	

or

5 feet	2 inches	
-2 feet	6 inches	
4 feet	12+2 inches	("borrow", or regroup, 5 feet 2 inches into 4 feet 14 inches)
-2 feet	6 inches	
2 feet	8 inches	(subtraction can now be completed)

In figuring time, we really are using a base of 60.

4 hours	50 minutes	
+2 hours	20 minutes	(removing 1 model group of 60 minutes and "carrying" it as 1 hour)
6 hours	70 minutes	
+1 hour	-60 minutes	
7 hours	10 minutes	

The same operation may be performed for subtraction, We would regroup or "borrow" 1 model group of 60.

4 hours	20 minutes	
-1 hour	40 minutes	
3 hours	60+20 minutes	(regroup 4 hours 20 minutes to 3 hours 60+20 minutes)
-1 hour	40 minutes	
2 hours	40 minutes	

Science and industry find new uses for bases other than 10. It is sometimes desirable to organize a system of arithmetic to fit a particular situation. In order to be able to symbolize and to understand another number system using a different base or model group, it is often well to first learn to count in the system. It is extremely important to learn to count, and to understand counting, in any system because counting is the basis for the processes of multiplication, division, addition, and subtraction. A comparison with our familiar Base Ten will make this evident.

First, consider the number of symbols needed to write the system. The model group, or base, to be used dictates how many symbols will compose the number system. We have nine digits and a zero to write all our numbers in base 10: 1, 2, 3, 4, 5, 6, 7, 8, 9, and a zero, 0. We can do this by using place value. It would be almost impossible to use a different number of symbols with a base of 10 and place value as we use it. We give numbers two values to do this: (1) an amount which the number symbolizes (how many), and (2) an amount which it has because of its place.

If we count, we can see how this functions.

Count: "1, 2, 3, 4, 5, 6, 7, 8, 9 . . . ?????"

When we get to nine, we are forced to use place value simply because we have no more symbols to use. Our only alternative would be to invent more symbols. Therefore, we write one, zero (10) and call it ten. Ten is the name of this place. It is the size of the model group.

The 1 now, because of its place in the numeral, has a value of 1 ten, rather than just 1. Or, expressed in another way, it is equal to 1 x 10 plus 0 x 1. We would write this more conveniently as 1(10) + 0(1), with the understanding that the parenthesis indicates that ten is multiplied by 1. Were there a 2 in this place, as in the numeral 20, we would show this as 2(10) + 0(1). This indicates that the second place, or tens place, is now multiplied by 2 and that 0 ones are added. We simply call this number "twenty" and, because of learning and habit, it means two tens to us. It is unfortunate that we have this habit, because it tends to keep us from understanding what the number really is, and how the system actually functions.

Now, if instead of counting in groups of 10, we change the size of the group and count in groups of 7, we have changed the base, or model group, of the system. We now have, and need, only six digits and a zero. That is all that we can use with a base of 7. To understand counting we need to recall that the values of each place in place value are expressed as powers of the base. That is, we have ones, tens, hundreds (10 x 10), thousands (10 x 10 x 10) in Base Ten. Now we have: ones, sevens (7 x 1), forty-nines (7 x 7), three-hundred-forty-threes (7 x 7 x 7), etc. The only difference is that we are accustomed to the terms for place value in a base of 10 and have special names for them. In a base of seven, we can either make up new names for these places or use terms that indicate their value in base 10. Making up new names necessitates extra memory work, so it is easier simply to think of them in the terms we know.

Again, countin in the base 7 system will show how the system functions.

Count: "1, 2, 3, 4, 5, 6, . . . ????

We cannot use 7 because seven is our model group and is now written 1 group of sevens and no ones, or 10. In order to avoid mixing up this system with base 10, in which we call this place "ten", we will call this "one-oh". We might also call it 1 seven, but this would tend to be confusing. We have now changed the value of the second place to 7 instead of ten. We could show this by indicating the value for the places as we did before: 10 in base 7 is equal to 1(7) + 3(1); in base 10 it is equal to 1(10) + 0(1).

It helps to understand a number system if you count in the system and write out the value for each place. This is true of base ten also, and helps for real understanding of the system and how it functions. First, however, we need to be sure of the place values. A comparison with base ten will help to understand base 7. If need be, refer back to the section on place value to review how the value for each place is established by the powers of the base. A comparison of the two bases is of some help in understanding this.

ten-thousands (10,000 = 10 x 1,000)	thousands (1,000 = 10 x 100)	hundreds (100 = 10 x 10)	tens (10 = 10 x 1)	ones

Base Ten

three-hundred-forty-threes (343 = 7 x 49)	forty-nines 49 = 7 x 7)	sevens 7 = 7 x 1)	ones

Base Seven

The value for each place is established exactly the same in Base Ten as in Base Seven. If we think of the value for each place as a power of the base, it is much simpler. This means the number of times the base is multiplied by itself. 10^2 means 10 x 10; 7^2 would mean 7 x 7. Comparing this to base ten, we would have a chart like this:

BASE 10		BASE 7	
ones	10^0	ones	7^0
tens 10 x 1, or (ten ones)	10^1	sevens 7 x 1, or (seven ones)	7^1
hundreds 10 x 10, or (ten tens)	10^2	forty-nines 7 x 7, or (seven sevens)	7^2
thousands 10 x 10 x 10, or (ten hundreds)	10^3	three-hundred-forty-threes 7 x 7 x 7, or (seven forty-nines)	7^3
ten-thousands 10 x 10 x 10 x 10, or (ten thousands)	10^4	two-thousand-four-hundred-ones 7 x 7 x 7 x 7, or (seven three-hundred-forty-threes)	7^4
hundred-thousands 10 x 10 x 10 x 10 x 10, or (ten ten0thousands)	10^5	sixteen-thousand-eight-hundred-sevens 7 x 7 x 7 x 7 x 7, or (seven two-thousand-four-hundred-ones)	7^5

A COMPARISON OF PLACE VALUE IN BASE TEN AND SEVEN

10^6	10^5	10^4	10^3	10^2	10^1	10^0
1,000,000's	100,000's	10,000's	1,000's	100's	10's	1's
7^6	7^5	7^4	7^3	7^2	7^1	7^0
117,649's	16,807's	2,401's	343's	49's	7's	1's
	3	2	6	0	2	2

Imagine that we have the numeral 326,022. Each digit of this numeral has a value for the place in which it stands. The places are indicated in the chart above the numeral. For instance, hundred's place in a base of ten is only equal to forty-nine's place in a base of seven. This is reasonable when we think that the base is smaller. We would show the value of the above number in this way:

For base ten:

$$326{,}022 = 3(100{,}000) + 2(10{,}000) + 6(1{,}000) + 0(100) + 2(10) + 2(1)$$

We read this number "three hundred, twenty-six thousand, twenty-two". We do not have such names as thousand, etc., for the base of seven. That is we do not have a name that means to us: "thre-hundred-forty-threes", like "thousand" means ten tens. Therefore, we will use base ten number names for the places in Base Seven. This will help to understand the value for these places.

For base seven:

$$326{,}022 = 3(16{,}807) + 2(2{,}401) + 6(343) + 0(49) + 2(7) + 2(1)$$

We can read this number simply "three, two, six, zero, two, two," since we do not have place value names for base seven. We would write it with a little subscript indicating the base like this: $326{,}022_7$.

The value for this numeral in base seven is only 57,297 inBase ten, since the worth of each place is less in Base Seven. This is due to the fact that the model group is smaller.

Since counting may be considered as the basis of arithmetic, it would help to understand another model group if you write out a chart like the one following. Fill in the missing numbers indicated by . . ., and show the value for each place. Go as far as 100 at least in Base 10. This will make a counting chart that can be used later to construct addition and multiplication tables, with which it is possible to work problems in the new base.

BASE 10	BASE 7
1	1
2	2
3	3
4	4
5	5
6	6
7	10 1 seven + 0 ones
8	11 1 seven + 1 one
9	12 1 seven + 2 ones
10 1 ten + 0 ones	13 1 seven + 3 ones
.. ..	.. ..
14 1 ten + 4 ones	20 2 sevens + 0 ones
15 1 ten + 5 ones	21 2 sevens + 1 one
.. ..	.. ..
21 2 tens + 1 one	30 3 sevens + 0 ones
22 2 tens + 2 ones	31 3 sevens + 1 one
.. ..	.. ..
41 4 tens + 1 one	56 5 sevens + 6 ones
42 4 tens + 2 ones	60 6 sevens + 0 ones
43 4 tens + 3 ones	61 6 sevens + 1 one
.. ..	.. ..
48 4 tens + 8 ones	66 6 sevens + 6 ones
49 4 tens + 9 ones	100 1 forty-nine + 0 sevens + 0 ones
50 5 tens + 0 ones	101 1 forth-nine + 0 sevens + 1 one
.. ..	... .
101 1 hundred + 0 tens + 1 one	203 2 forty-nines + 0 sevens + 3 ones
342 3 hundreds + 4 tens + 2 ones	666 6 forty-nines + 6 sevens + 6 ones
343 3 hundreds + 4 tens + 3 ones	1000 1 three-hundred-forty-three + 0 forty-nines + 0 sevens + 0 ones
344 3 hundreds + 4 tens + 4 ones	
... .	....
2400 2 thousands + 4 hundreds + 0 tens + 0 ones	6666 6 three-hundred-forty-threes + 6 forty-nines + 6 sevens + 6 ones
2401 2 thousands + 4 hundreds + 0 thens + 1 one	10000 1 two-thousand-four-hundred-one + 0 three-hundred-forty-three + 0 forty-nines + 0 ones

Reading across on this chart it is possible to see what digits would represent the same number in each base. For example: 42 in Base Ten (4 tens + 2 ones) is written 60 in Base Seven (6 sevens and 0 ones); 43 in Base Ten is written 61 in Base Seven. (6 sevens would give you 42 in Base Ten plus one more would make 43.)

The tables on the next page are constructed to show multiplication and addition in Base Seven. It would help to make a table of your own and then to compare it with these. Refer to the counting chart that you made previously.

For example, in constructing the chart, think: 6 x 6 = 36 in Base Ten. Refer to the counting chart, 36 is written 51 in Base Seven, so write 51 in the space for the answer to 6 x 6. Do this for all the spaces. Remember that since you are working in a Base of Seven, you will be able to use only 6 digits on your chart. The seventh digit would become 10 in Base Seven, or 1 group of 7 ones plus 0 ones.

It is difficult to start thinking in another base when you first start working in it; although, if we thought as carefully in Base Ten as we are forced to think in Base Seven, we would probably understand arithmetic much better. For beginning, think the answer to problems in Base Ten and convert to Base Seven. Do this for a while, and you will probably find yourself thinking in Base Seven.

I. $4_7 \times 5_7 = 26_7$

Think: 4 x 5 = 20 in Base Ten. (Four and five are written then same in Base Ten and Base Seven.) But twenty in Base Ten is written 20, in Base Seven 26. This is because two tens are the same as two sevens plus six ones. You are working in groups of sevens instead of tens. Refer to your counting chart for the answers.

II. $4_7 \times 6_7 = 33_7$

Think: 4 x 6 = 24 in Base Ten, but this is written 33 in Base Seven. Write 33 in the space for the answer to 4 x 6. (This gives you 3 sevens plus 3 ones which equals 24 in Base Ten.)

III. $4_7 \times 10_7 = 40_7$

Think: This is the same as 4 x 7 in Base Ten, but seven in Base Seven is written 10 (or 1 group of seven ones). The problem is done exactly as in

Base Ten, but the 1's now stand for groups of sevens instead of tens. The answer 40_7 is equal to 4 sevens or 28 in Base 10.

I. 10 x 10 = 100 in Base Ten, and $10_7 \times 10_7 = 100_7$ in Base Seven, BUT the ones in the second example stand for groups of sevens and forty-nines instead of tens and hundreds as in the first one.

ADDITION TABLE IN BASE SEVEN

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

MULTIPLICATION TABLE IN BASE SEVEN

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

It is unfortunate that it is often possible to learn to do arithmetic problems by rote with little or no understanding of the processes involved. Doing some simple problems in another number system helps to understand how a system really functions, since it is impossible to do them without this knowledge. The following problems are worked in both Base Ten and Base Seven so that you can see hos a 7 would be carried as a model group in the same way that a 10 is. Learning to manipulate a model group of 7 will make plain how the base of a system functions.

	BASE TEN		BASE SEVEN	
1.	4	$4(1)$	4_7	$4(1)$
	$+2$	$2(1)$	$+2_7$	$2(1)$
	6	$6(1)$	6_7	$6(1)$

Since there is no digit with a value over 7, the problem is the same as in Base Ten.

2.	4	$4(1)$	4_7	$4(1)$
	$+3$	$3(1)$	$+3_7$	$3(1)$
	7	$7(1)$	10_7	$1(7) + 0(1)$

Now we are forced to "carry" in the Base Seven problem because 7 is the model group. The two answers have the same value, but they are in different bases.

3.	3	$3(1)$	3_7	$3(1)$
	$+8$	$8(1)$	$+11_7$	$1(7) + 1(1)$
	11	$1(10) + 1(1)$	14_7	$1(7) + 4(1)$

In the problem $3 + 8 = 11$, we are forced to "carry" in the Base Ten situation, but not in Base Seven, we do not have a group of 7 to carry. The actual value of the two answers is, however, the same.

4.	23	$2(10) + 3(1)$	32_7	$3(7) + 2(1)$
	$+23$	$2(10) + 3(1)$	$+32_7$	$3(7) + 2(1)$
	46	$4(10) + 6(1)$	64_7	$6(7) + 4(1)$

The process of addition is the same, except that the model group in one is ten and the other is seven.

5.	19	$1(10) + 9(1)$	25_7	$2(7) + 5(1)$
	$+24$	$2(10) + 4(1)$	$+33_7$	$3(7) + 3(1)$
	43	$4(10) + 3(1)$	61_7	$6(7) + 1(1)$

In the case of the problem: $19 + 24 = 43$ in Base ten, we have to "carry" 1 ten. Then the problem is converted into Base Seven, we still have to "carry" but in this case, we are working with the new base of seven, so this is our model group. The answers have different notation, but the same value.

HOW TO WORK A PROBLEM IN BASE SEVEN

ADDITION

BASE TEN		BASE SEVEN	
34	3(10)+ 4(1)	46_7	4(7) + 6(1)
+25	2(10) + 5(1)	$+34_7$	3(7) + 4(1)
59	5(10) + 9(1)	113_7	1(49) + 1(7) + 3(1)

Think: 6 + 4 = 10. Ten in Base Seven is written on a counting chart as 13. (Call this "one three" so that you do not mix it up with Base Ten "thirteen"). Put down 3 and "carry" 1 <u>seven</u>, 4 +3 = 7. This however, really is 4 sevens plus 3 sevens are 7 sevens, plus the one that you are "carrying" makes 8 sevens. Eight on a counting chart is written 11 (call this "one one" so as not to mix it up with Base Ten "eleven".) Put down <u>11</u>. This gives you the answer, using place value of: 1(49) + 1(7) + 3(1). It is the same process you would use in base ten, but the model group is 7.

342	3(100) + 4(10) + 2(1)	666_7	6(49) + 6(7) + 6(1)
+ 66	6(10) + 6(1)	$+123_7$	1(49) + 2(7) + 3(1)
408	4(100) + 0(10) + 8(1)	1122_7	1(343) + 1(49) + 2(7) + 2(1)

THINK: 6 + 3 = 9, but 9 in Base Seven is written <u>one two</u> (12). Put down 2 and "carry" 1 <u>seven</u>. 6 + 2 = 8. Eight is written <u>one one</u> (11), add the 1 seven that you are "carrying". This makes <u>one two</u> (12). Put down 2 and "carry" 1. In this case, it is 1 forty-nine. 6 + 1 = 7. Seven in Base Seven is written <u>one oh</u> (10). Add the one forty-nine that you are "carrying". This makes <u>one one</u> (11). The answer is $1{,}122_7$ or 1(343) + 1(40) + 2(7) + 2(1) which would equal 408.

ADDITION IN BASE SEVEN

These are a series of problems of increasing difficulty in Base Seven. They could be mimeographed and used with students either for enrichment or understanding. They might be presented in such a way that each student could go as far as possible in the series. The problems have been worked in Base Ten first and the answers indicated. This should make for more meaning than working in Base Seven alone with no way of comparing with Base Ten. It is not necessary to indicate the base with a subscript when it is understood, therefore, these are omitted. Encourage students to look for the pattern of counting.

	BASE TEN		BASE SEVEN	
1.	$\begin{array}{r}5\\ \underline{5}\\ 10\end{array}$	1(10)+ 0(1)	$\begin{array}{r}5\\ \underline{5}\\ 13\end{array}$	1(7) + 3(1)
2.	$\begin{array}{r}6\\ \underline{6}\\ 12\end{array}$	1(10) + 2(1)	$\begin{array}{r}6\\ \underline{6}\\ 15\end{array}$	1(7) + 5(1)
3.	$\begin{array}{r}7\\ \underline{7}\\ 14\end{array}$	1(10) + 4(1)	$\begin{array}{r}10\\ \underline{10}\\ 20\end{array}$	2(7) + 0(1)
4.	$\begin{array}{r}8\\ \underline{8}\\ 16\end{array}$	1(10) + 6(1)	$\begin{array}{r}11\\ \underline{11}\\ 22\end{array}$	2(7) + 2(1)
5.	$\begin{array}{r}9\\ \underline{9}\\ 18\end{array}$	1(10) + 8(1)	$\begin{array}{r}12\\ \underline{12}\\ 24\end{array}$	2(7) + 4(1)
6.	$\begin{array}{r}10\\ \underline{10}\\ 20\end{array}$	2(10) + 0(1)	$\begin{array}{r}13\\ \underline{13}\\ 26\end{array}$	2(7) + 6(1)
7.	$\begin{array}{r}11\\ \underline{11}\\ 22\end{array}$	2(10) + 2(1)	$\begin{array}{r}14\\ \underline{14}\\ 31\end{array}$	3(7) + 1(1)
8.	$\begin{array}{r}12\\ \underline{12}\\ 24\end{array}$	2(10) + 4(1)	$\begin{array}{r}15\\ \underline{15}\\ 33\end{array}$	3(7) + 3(1)
9.	$\begin{array}{r}13\\ \underline{13}\\ 26\end{array}$	2(10) + 6(1)	$\begin{array}{r}16\\ \underline{16}\\ 35\end{array}$	3(7) + 5(1)
10.	$\begin{array}{r}14\\ \underline{14}\\ 28\end{array}$	2(10) + 8(1)	$\begin{array}{r}20\\ \underline{20}\\ 40\end{array}$	4(7) + 0(1)
11.	$\begin{array}{r}35\\ \underline{35}\\ 70\end{array}$	7(10) + 0(1)	$\begin{array}{r}50\\ \underline{50}\\ 130\end{array}$	1(49) + 3(7) + 0(1)

HOW TO WORK MULTIPLICATION PROBLEMS IN BASE SEVEN

BASE TEN	BASE SEVEN
1. $\begin{array}{r} 8 \\ \underline{\times 3} \\ 24 \end{array}$ 2(10 + 4(1) = 24	$\begin{array}{r} 11 \\ \underline{\times 3} \\ 33 \end{array}$ 3(7) + 3(1) = 24

THINK: To convert 8 to Base Seven, eight equals 1 seven plus 1 one. Write one one (11), read it "one one". There is no regrouping necessary because the ones do not combine to an amount over the base of 7.

2. $\begin{array}{r} 15 \\ \underline{\times 4} \\ 60 \end{array}$ 6(10) + 0(1) = 60	$\begin{array}{r} 21 \\ \underline{\times 4} \\ 114 \end{array}$ 1(49) + 1(7) + 4(1) = 60

THINK: Fifteen in Base Seven is equal to 2 sevens plus 1 one. Write 21. Read it "two one". To multiply, think: 4 x 1 = 4. Four in Base Seven is simply written 4 because it is not over the model group of 7. Write down 4. 4 x 2 = 8. Eight in Base Seven is equal to 1 seven plus 1 one. Write 11. Read it "one one". The answer is the same amount as in Base Ten. The notation is different.

3. $\begin{array}{r} 53 \\ \underline{\times 24} \\ 212 \\ \underline{106} \\ 1272 \end{array}$ 1(1000) + 2(100) + 7(10) + 2(1) = 1272	$\begin{array}{r} 104 \\ \underline{\times 33} \\ 315 \\ \underline{315} \\ 3465 \end{array}$ 3(343) + 4(49) + 6(7) + 5(1) = 1272

THINK: Base Ten 53 is written 104 in Base Seven because 53 has 1 forty-nine + 0 sevens + 4 ones. To multiply, think: 3 x 4 = 12 in Base Ten. Twelve in Base Seven is written 15. Put down 5 and carry 1 seven. 3 x 0 = 0. Add the 1 seven. Write 1. 3 x 1 = 3. Put down 3. Do the same for the next line, and add.

4. $\begin{array}{r} 79 \\ \underline{\times 7} \\ 553 \end{array}$ 5(100) + 5(10) + 3(1) = 553	$\begin{array}{r} 142 \\ \underline{\times 10} \\ 1420 \end{array}$ 1(343) + 4(49) + 2(7) + 0(1)= 553

THINK: In the base seven problem the model group is 7, so one zero stands for 1 seven. The process is the same as base ten, but the notation is different. The answer will have the same value.

5. $\begin{array}{r} 17 \\ \underline{\times 3} \\ 51 \end{array}$ 5(10) + 1(1) = 51	$\begin{array}{r} 23 \\ \underline{\times 3} \\ 102 \end{array}$ 1(49) + 0(7) + 2(1) = 51

THINK: 3 x 3 = 9. (Written in Base Seven as 12.) Write 2, carry 1 seven. 3 x 2 = 6. (Written in Base Seven as 6.) Add the 1 seven making 7 sevens. In Base seven this is written 10. Put down 10. Place value operates here the same as in Base Ten.

MULTIPLICATION BASE SEVEN

It might be best to do problems in Base Ten first and then convert them to Base Seven, and check as indicated.

	BASE TEN	BASE SEVEN	
1.	49 3 147	100 3 300	3(49) + 0(7) + 0(1) = 147
2.	24 2 48	33 2 66	6(7) + 6(1) = 48
3.	14 4 56	20 4 110	1(49) + 1(7) + 0(1) = 56
4.	21 6 126	30 6 240	2(49) + 4(7) + 0(1) = 126
5.	35 7 245	50 10 500	5(49) + 0(7) + 0(1) = 245
6.	42 8 336	60 11 60 60 660	6(49) + 6(7) + 0(1) = 336
7.	349 8 2793	1006 11 1006 1006 11066	1(2401) + 1(343) + 0(49) + 6(7) + 6(1) = 2792
8.	1072 12 2144 1072 12864	3061 15 21425 3061 52335	5(2401) + 2(343) + 3(40) = 3(7) + 5(1) = 12864
9.	364 14 1456 364 5096	1030 20 20600	2(2401) + 0(343) + 6(49) + 0(7) + 0(1) = 5096
10.	98 4 392	200 4 1100	1(343) + 1(49) + 0(7) + 0(1) = 392

SUBTRACTION PROBLEMS IN BASE SEVEN

Subtraction in Base Seven is exactly the same as in Base Ten, except that we would "borrow", or regroup, sevens instead of tens. If we think about some proglems, this is clear.

1.
$$\begin{array}{r} 20 \\ -6 \\ \hline 14 \end{array} \qquad \text{or} \qquad \begin{array}{rrl} & 1 \text{ ten} & 0+10 \text{ ones} \\ & \cancel{2} \text{ tens} + & \cancel{0} \text{ ones} \\ - & & 6 \text{ ones} \\ \hline & 1 \text{ ten} + & 4 \text{ ones} \end{array}$$

It is a mistake if we do not think of the 1 ten that we "borrow" as being <u>added</u> to the ones in the one's place. We are inclined to write the one <u>beside</u> the zero, in this case, and with little thought simply call it ten.

2.
$$\begin{array}{r} 22 \\ -6 \\ \hline 16 \end{array} \qquad \text{or} \qquad \begin{array}{rrl} & 1 \text{ ten} & 2+10 \text{ ones} \\ & \cancel{2} \text{ tens} + & \cancel{2} \text{ ones} \\ - & & 6 \text{ ones} \\ \hline & 1 \text{ ten} + & 6 \text{ ones} \end{array}$$

The ten ones that we "borrow" are <u>added</u> to the 2 ones that are already in one's place and this makes 12. If <u>we do</u> a problem in Base Seven, we can see this more clearly. We <u>must</u> remember that we are regrouping in terms of the base or model group.

3. **BASE TEN** — **BASE SEVEN**

$$\begin{array}{r} 22 \\ -6 \\ \hline 16 \end{array} \qquad \begin{array}{r} 31 \\ -6 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{rrl} & 2 \text{ sevens} & 1+7 \text{ ones (or 8 ones)} \\ & \cancel{3} \text{ sevens} + & \cancel{1} \text{ ones} \\ & & 6 \text{ ones} \\ \hline & 2 \text{ sevens} + & 2 \text{ ones} = 16 \end{array}$$

It is easy to see that it is NECESSARY TO ADD the model group (7) to the ones that are already in the one's place and not to simply write the 1 beside the 1 in the one's place and call it eleven. We may do this, if we have been able to think in the system that 11-6=2. But, until one thinks in the system, or had memorized the facts, it is almost impossible to do this. Thinking of the model group as being added when it is "borrowed" maked it plain that 11 in Base Seven is composed of 1(7) + 1(1) or is equal to 8 in Base Ten.

Students can gain an understanding of subtraction by making up problems, converting them to Base Seven, and then working them in both systems. The problems always should be proved according to the above pattern until the students are very sure of themselves in the new system.

Long division in Base Seven is easily done as multiple subtraction; therefore, the subtraction is not developed further here.

LONG DIVISION IN BASE SEVEN

It is possible to do division in Base Seven either as multiple subtraction or in the form, 2/‾8‾, sometimes called the "divided-by" form. For beginners, it is probably easier to do it as multiple subtration. Following are some exampled. Note that this method carries on the ideas presented under subtraction.

BASE TEN	BASE SEVEN	
1. 7 r.3	10 r.3	
7/52	10/103	1 forty-nine + 0 sevens + 3 ones
49	-10 1	- 1 seven + 0 ones
3	63	6 sevens + 3 ones
	-10 1	- 1 seven + 0 ones
	53	5 sevens + 3 ones
	-10 1	- 1 seven + 0 ones
	43	4 sevens + 3 ones
	-10 1	- 1 seven + 0 ones
	33	3 sevens + 3 ones
	-10 1	- 1 seven + 0 ones
	23	2 sevens + 3 ones
	-10 1	- 1 seven + 0 ones
	13	1 seven + 3 ones
	-10 1	- 1 seven + 0 ones
	3 10 1(7)+0(1)	3 ones

Proof: 7	10 1(7)+0(1)
x7	x10 1(7)+0(1)
49	100 1(49)+0(7)+0(1)
+3	+3
52	103 1(49)+0(7)+3(1) = 52 in base ten.

2. 12 r.2	15 r.2
8/98	11/200
	-66 6
	101
	-66 6
	2 15 1(7) + 5(1) = 12

THINK: Take 6 eights away from 200. You have to decompose 200 to 6 sevens + 7 ones. Complete the subtraction thinking in groups of 7's.

Proof: 12	15
x8	x11
96	15
+2	15
98	165
	+ 2
	200

THINK: 1 x 5 = 5, 1 x 1 = 1, and repeat. Add the remainder. Remember 5 + 2 = 10, or one group of seven. Carry 1 seven. This makes 6 sevens plus the one you are carrying. Put down the zero and carry again. You now have 1 + 1 = 2. Your answer is 2(49) + 0(7) + 0(1).

DIVISION PROBLEMS IN BASE SEVEN

BASE TEN		BASE SEVEN	
1. 11 r. 3		14 r.3	
9/ 102		12/ 204	
-63	7	-120	10
39		54	
-36	+4	-51	+4
3	11	3	14 1(7) + 4(1) = 11
Proof: 11		14	
x 9		x12	
99		31	
+3		14	
102		201	
		+ 3	
		204	2(49) + 0(7) + 4(1) = 102

Remember as you work in base seven, you must either convert each number to base seven as you go along, or else <u>think</u> in the base.

29 r.3		41 r.3	
2. 7/ 206		10/ 413	
-49	7	-100	10
157		313	
-49	7	-100	10
108		213	
-49	7	-100	10
59		113	
-49	7	-100	10
10		13	
- 7	1	- 10	1
3		3	41 4(7) + 1(1) = 29
Proof: 29		41	
x7		x10	
203		410	
+ 3		+ 3	
206		413	4(49) + 1(7) + 3(1) = 206

In this base seven problem, 10 stands for 1 group of 7 ones instead of ten as in base ten. Then 10 x 10 = 100, however, 100 in base seven is equal to 1 forty-nine instead of one hundred.

40 r. 10	55 r.13	
4. 12/490	15/1300	
48	114	
10	130	
	114	
	13	
Proof: 40	55	
x12	x15	
80	404	
40	55	
480	1254	
10	13	
490	1300	1(343)+3(49)+0(7)=0(1)=490

This problem is done with the "dividedby" method. It is more difficult because you have to think in the base.

PLACE VALUE CHART OF OTHER BASES

BASE	B^6	B^5	B^4	B^3	B^2	B^1	B^0
10	1,000,000's	100,000's	10,000's	1,000's	100's	10's	1's
7	117,649's	16,807's	2,401's	343's	49's	7's	1's
2	64's	32's	16's	8's	4's	2's	1's
3	729's	243's	81's	27's	9's	3's	1's
5	15,625's	3,125's	625's	125's	25's	5's	1's

By using place value, we can make a number system in any base. This helps us to understand the structure of our own Base Ten, because we are forced to think of it as it functions as a base or model group, not as something we have learned by rote. This also helps to give us an insight into pure mathematics.

We have given no bases with a model group larger than ten, because this would mkke it necessary to invent extra symbols. However, a base, such as base twelve would function in the same way as any that used place value and a model group.

To convert to any base, follow this procedure using the appropriate place values. To convert 469 in base ten to base seven:

$$\begin{array}{rl} 469 & \\ \underline{-343} & \text{1 three-hundred-forty-three} \\ 126 & \\ \underline{-98} & \text{2 forty-nines} \\ 28 & \\ \underline{-28} & \text{4 sevens} \\ 0 & \text{0 ones} \end{array}$$

THINK: Take the largest place value out of 469 that you can. This is 343. You will have 1 three-hundred-forty-three. Then take out all you can of the next largest place value. This is 2 forty-nines. The next digit in the base seven number will be 2. Take out 4 sevesn. You have used up all the 469 but you must hold the one's place with a zero, since you have no ones in this numeral. If you did not put a zero on the end to hold one's place, this would be only a three place number and the values would be wrong.

After you have subtracted as indicated above, it is possible to simply read down the column. This will be the new numeral in the new base. In this case, 1,240: 1(343) + 2(49) + 4(7) + 0(1) = 469.

SUGGESTIONS FOR TEACHING

1. Teach counting in the base first. Have students write out the values for the place value according to the pattern below. Do this for as far as the base squared at least. In Base Seven, this would be 49 in Base Ten.

BASE TEN		BASE SEVEN	
1	1(1)	1	1(1)
2	2(1)	2	2(1)
3	3(1)	3	3(1)
4	4(1)	4	4(1)
5	5(1)	5	5(1)
6	6(1)	6	6(1)
7	7(1)	10	1(7) + 0(1)
8	8(1)	11	1(7) + 1(1)
9	9(1)	12	1(7) + 2(1)
10	1(10) + 0(1)	13	1(7) + 3(1)
11	1(10) + 1(1)	14	1(7) + 4(1)
12	1(10) + 2(1) . . . etc.	15	1(7) + 5(1) . . . etc.

2. After a counting chart has been made, use it to make addition and multiplication tables in the base. This helps the student to begin to THINK in the base.

3. Use these tables to work simple problems. Probably the problems that students make up themselves would be best to use. Work them in Base Ten first, and then convert, compare, and prove the problems back to base ten as suggested.

4. Have children produce a chart or booklet explaining what they understand about another number system other than Base Ten.

5. Give oral demonstrations of place value. Take place value pockets and change the base from ten to something else, and show how the values for the places would have to change.

6. Let slow children make simple counting charts in another base.

7. Write a story or develop the idea that a people with twelve fingers might have a number system based on twelve. This would require the invention of two additional symbols.

IV. ARITHMETIC PROCESSES

Addition

Subtraction

Multiplication

Division

Review and Reinforcement Suggestions

ADDITION

The process of addition for the most part is usually well established prior to the grades for which this material is written; however, some attention will be given to it here. Unfortunately, some students have learned their addition facts by drill and have little or no real understanding of what they are actually doing.

In addition, we are combining groups of irregular counting. Actually, addition is a matter of regrouping in whatever base you may be working. in our system of Base Ten, we group and regroup numbers based in this system. For example: 5 + 7 = 12. We are saying a subgroup of 5 and a subgroup of 7 are regrouped to form a subgroup of 1 ten and 2 ones. In addition, we are regrouping numbers based on ten. In another base, such as Base Seven, we would be regrouping numbers based on 7. While it is not advisable to present addition to primary children in such terms as above, it is possible to help them to understand that they are dealing with groups of numbers rather than mere facts.

36	3 tens	6 ones
54	5 tens	4 ones
90	8 tens	10 ones
	1 ten	-10 ones
	9 tens	0 ones

Think: In our number system, we have no digit or symbol higher than 9. So we say 6+4 gives us 1 ten and 0 ones. We "carry" 1 group of ones to the ten's column, or place, and this gives us 9 groups of tens plus no ones. It is easy to see that 8 tens + 10 ones would give us the same face amount; however, we could not write the numeral as 810 for this would be read "eight-hundred-ten". So, we are forced to combine the 8 tens and the 10 ones to read 9 tens or ninety.

Addition is still counting, but it is much easier and faster to count by groups rather than by ones. Instead of the student just saying 8 + ' =15, he should understand and think of this as 1 group of tens and 1 group of 5 ones. Writing out these problems as we have indicated helps this process.

	1 group of hundreds			
11	1 group of tens	OR	367	
367			294	
294			11	(7 + 4=11)
661			150	(60 + 90 = 150)
			500	(300 + 200 = 500)
			661	

Here, we are regrouping, or "carrying" ones, tens, and hundreds. THINK, 11 ones regroup to 1 ten and 1 one. 16 tens regroup to 1 hundred and 6 tens.

The second illustration shows how the problem could be done by simply adding the value for each place. This would Avoid "carrying".

SUBTRACTION

Subtraction can be taught as the inverse process of addition. Instead of combining subgroups as we do in addition, we subtract subgroups. For example: 7 - 5 = 2. We are saying: "A group of 7 less a subgroup of 5 equals a subgroup of 2". We can apply this to any combination except where the digit of the subtrahend is greater than the corresponding digit of the minuend. Then we must use decomposition, or "borrowing", before we can subtract.

		(4 tens) (3+10 ones)
53	or	$\cancel{5}$ tens + $\cancel{3}$ ones
-36		-3 tens + 6 ones
17		1 tne + 7 ones

THINK: We cannot subtract a group of 6 from a group of 3, so we regroup by decomposing (rearranging) the 5 tens into 4 tens and 10 ones. Now we regroup the ones by adding the subgroup of 3 ones, to the ten ones we "borrowed" to make 13 ones from which to subtract the 6 ones of the subtrahend. The student should understand that this is what he is doing when he "borrows" from the next place and puts a <u>1</u> beside the digit from which he will subtract. He is <u>increasing</u> the minuend digit by the amount of the model group of the system.

		(2+10 hundreds)	(1+10 tens)	
	7 thousands	$\cancel{2}$ hundred	$\cancel{1}$ ten	3+10 ones
8323	$\cancel{8}$ thousands +	$\cancel{3}$ hundreds +	$\cancel{2}$ tens +	$\cancel{3}$ ones
-2756	-2 thousands +	7 hundreds +	5 tens +	6 ones
5567	5 thousands +	5 hundreds +	6 tens +	7 ones

THINK: We decompose the 2 tens to read 1 ten and 10 ones . . . regrouped, this gives us 1 ten <u>plus</u> 3 ones or 13 ones. The 3 hundred decomposes to 2 hundred and 10 tens ... regrouped to give us 10 tens <u>plus</u> 1 ten, or 11 tens. The 8 thousand regroups to read 7 thousand and 10 hundreds, which regroups to give us 10 hundreds plus 2 hundreds or 12 hundreds.

This process might be explained by showing that the rearrangement will still equal the original amount:

$$7{,}000 + 1{,}200 + 110 + 13 = 8{,}323$$

Subtraction can also be done by the <u>additive method</u>. This is considered by some to be a better one since it employes the use of the addition facts with no new combinations to learn. However, the "take away" method is most commonly used. Essentially it is only a different way of <u>thinking</u> of the process.

<u>TAKE AWAY</u>			<u>ADDITIVE</u>		
538	We think:	8-6=?	538	We think:	6+?=8
-426		3-2=?	-426		2+?=3
112		5-4=?	112		4+?=5

MULTIPLICATION

An approach to multiplication through a thorough understanding of place value opens many interesting and worthwhile activities that can be used to motivate students and increase understanding. Following are given some different forms of working multiplication problems. Students, investigating these, may find it worthwhile to invent others.

```
1.  12    1 ten + 2 ones           12
    x3          x 3 ones           x3
    --   ---------------           --
    36   3 tens+ 6 ones            30  (3 x 10)
                                   +6  (3 x  2)
                                   --
                                   36
```

```
2.  14    1 ten   +  4 ones
    x3            x  3 ones
    --   -----------------
    42    3 tens  + 12 ones
         +1 ten   - 10 ones   (remove 1 ten and "carry" it)
         -----------------
          4 tens  +  2 ones
```

```
3.  14                    14                  14
    x3                    x3                  14
    --                    --                  14
    12   (3 x 4)          30   (3 x 10)       --
   +30   (3 x 10)         12   (3 x  4)       12   (4+4+4)
    --                    --                  30   (10+10+10)
    42                    42                  --
                                              42
```

```
4.  243                   243                    243
    x24                   x24                    x24
    ---                   ----                   ----
     12   (4 x 3)         4860   (20 x 243)      4860   (20 x 243)
    160   (4 x 40)         972   ( 4 x 243)        12   (4 x 3)
    800   (4 x 200)       ----                    160   (4 x 40)
     60   (20 x 3)        5832                    800   (4 x 200)
    800   (20 x 40)                              ----
   4000   (20 x 200)                             5832
   ----
   5832
```

Such different ways as these can make multiplication have a "new"look and furnish an excellent means of checking problems. They may not be used very successfully if a student does not have a good idea of place value. These methods should not be presented to students as something that they are required to memorize. The should be the result and natural outcome of understandins acquired in studying place value and the function of the number system.

When students are given freedom to invent methods of doing some process they should be required, however, to prove their invention is sound and will result in the correct answer.

DIVISION

Division, or dividing numbers, can be thought of as subtracting in a special way. If we take the problem:

$$3\overline{)12}$$

We may want to know how many threes are in 12, how much 1/3 of 12 would be, or how many times we can take 3 away from 12.

Putting the problem down like this: $3\overline{)12}$ is just a convenient way of arranging it so that we can find out what we want to know. The well-known "box" has absolutely nothing to do with this.

We could also put the problem down like this and find the answer:

$$\begin{array}{rll} 12 & & \\ \underline{-3} & 1 & \text{(three taken away)} \\ 9 & & \\ \underline{-3} & 1 & \text{(another three taken away)} \\ 6 & & \\ \underline{-3} & 1 & \text{(another three taken away)} \\ 3 & & \\ \underline{-3} & \underline{1} & \text{(the last three taken away)} \\ 0 & 4 & \end{array}$$

We have taken away 4 threes, or 4 x 3 = 12. It took 4 threes to use up all of the 12. We could find out how many 3 inch pieces there were in twelve inches in the same way, simply by taking a 12 inch strip of paper and cutting it up, then count the pieces.

$$\begin{array}{rll} 54 & \text{inches} & \text{(the length of the snake in inches)} \\ \underline{-12} & 1 & \text{(take away one foot of 12 inches)} \\ 42 & & \\ \underline{-12} & 1 & \text{(take away another foot of 12 inches)} \\ 30 & & \\ \underline{-12} & 1 & \text{(take away another foot of 12 inches)} \\ 18 & & \\ \underline{-12} & \underline{1} & \text{(take away the last whole foot)} \\ 6 & 4 & \text{(there are 6 inches left)} \end{array}$$

The snake was 4 twelve inches, or 4 feet, long. There were six inches left over.

A better way to do this problem would be like this:

$$\begin{array}{rll} 54 & \text{inches} & \\ \underline{-24} & 2 & \text{(take away 2 twelves)} \\ 30 & & \\ \underline{-24} & \underline{2} & \text{(take away 2 more twelves)} \\ 6 & 4 & \end{array}$$

The snake was 4 feet 6 inches long. The children have divided 54 by 12: or we could say 54 ÷ 12 = 4, remainder 6; or 12/54, etc. It does not make any difference how we put it down to work it, if we are careful, we will get the right answer.

It is difficult to do very complicated long division problems with this kind of a subtractive process unless students have a very good background in place value. After this has been built up, division done with any of the processes can become meaningful. Following are some problems of increasing difficulty. These should demonstrate the process.

$$
\begin{array}{r r}
15\overline{)65} & \\
\underline{-15} & 1 \\
50 & \\
\underline{-15} & 1 \\
35 & \\
\underline{-15} & 1 \\
20 & \\
\underline{-15} & \underline{1} \\
5 & 4
\end{array}
\qquad \text{or} \qquad
\begin{array}{r}
1+1+1+1=4 \text{ r.}5 \\
15\overline{)65} \\
\underline{-15} \\
50 \\
\underline{-15} \\
35 \\
\underline{-15} \\
20 \\
\underline{-15} \\
5
\end{array}
$$

A slow student might do the problem this way. One with more talent might speed the process up in the following manner.

$$
\begin{array}{r r}
15\overline{)65} & \\
\underline{-30} & 2 \\
35 & \\
\underline{-30} & \underline{2} \\
5 & 4
\end{array}
$$

Some students would see that 4 x 15 = 60 and know the answer. With this method, however, there is offered to students of many abilities an understandable way of doing the division process.

$$
\begin{array}{r r}
6\overline{)90} & \\
\underline{-60} & 10 \\
30 & \\
\underline{-30} & \underline{5} \\
 & 15
\end{array}
\qquad\qquad
\begin{array}{r r}
46\overline{)489} & \\
\underline{460} & \underline{10} \\
29 & 10
\end{array}
$$

Stop before doing a problem and think how many times the divisor should be taken out ... 1 time, 10 times, 100 times, etc. Do not try to take out so many times the divisor the multiplication becomes involved. For instance, it would be foolish to take out 15 times the divisor. This would make the problem more difficult and invite error.

```
15/650
   150    10  (start with 10 times the divisor)
   500
   300    20  (it is easy to see 20 times the divisor now)
   200
   150    10  (it would be foolish to use 13 here)
    50
    45     3  (this is easier to multiply)
     5
```

```
       100+200+100+30+5+4=439 r. 4   (the answer may be put in any
15/6589                                convenient and meaningful way)
   1500   100
   5089
   3000   200
   2089
   1500   100
    589
    450    30
    139
     75     5
     64
     60     4
      4   439
```

THINK: There are 439 fifteens in 650. It is possible to find out the answer simply by taking them out until you have used up all the 650. It would be foolish to take them out 1 at a time. Take them out in groups.

The form in which the division problem is placed often can obscure the meaning. Such an approach as multiple subtraction will tend to: (1) increase understanding; (2) make the division process easier; and (3) relate back to the place value and subtraction concepts that the child already has mastered.

It should be understood that some problems are done more conveniently one way, some another. It would be well to help children understand many ways and then use whatever way best suits the situation. When a child can do this, he is using mathematics intelligently and not just repeating a rote, a meaningless process.

IV. FRACTIONS

History and Development

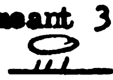
Understandings Needed

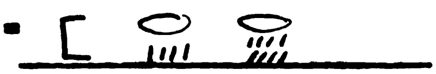
Methods

HISTORY AND DEVELOPMENT

People think of the amounts of things in two ways: as whole things, and as parts of things. It would appear that for some time during history, man did not think much about parts of things. Whatever accounts he kept and dealings that he had, were done, for the most part, in whole amounts. A way to "put down" or express fractions may have been difficult to invent. Ways to add, subtract, multiply and divide fractions were probably very confusing. We have records of very early fractions that were quite different from ours.

The word <u>fraction</u> comes from a Roman, or Latin, word, <u>frangere</u> which means to <u>break</u>. We have words like fracture that have much the same meaning. At first, people seemed to just express parts of things in any convenient way. Then, they tried to write fractions in such a way that all the numerators were 1. The Egyptians did this in the following manner.

Since ||| meant 3, they invented a symbol that means 1 part of three . So  stood for 1/3. They had a special symbol for 2/3, ; and for 1/2 . Other than these two, all fractions were unit fractions, or fractions with numerators of 1. This resulted in some unusual ways of figuring.

$$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} =$$ 

$$\frac{4}{15} = \frac{1}{5} + \frac{1}{15} =$$ 

$$\frac{8}{12} = \frac{1}{2} + \frac{1}{6} =$$ 

There was no set way of expressing fractions. We believe that some merchants "in the know" who had invented one used it to their own advantage. It is interesting to try to do this. You must start with the largest fraction that you man use, subtract this from the one that you want to express, use the next largest, etc.

We do much the same thing with money. So, you see, we are not so advanced after all. Suppose that you buy something for 26¢. You give the clerk $1.00. You will get 74¢ change. The clerk will say: "26¢, 27¢, 28¢, 29¢, 30¢, 40¢, 50¢, $1.00. She has given you:

$$\frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{10} + \frac{1}{10} + \frac{1}{2}$$

Your change is equal to 74/100 or a dollar. Therefore:

$$\frac{74}{100} = \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{10} + \frac{1}{10} + \frac{1}{2}$$

UNDERSTANDINGS NEEDED

Fractions may be confusing to children because we tell them that they mean so many things. We say:

$\frac{1}{2}$ — 1 number you have / 2 name of parts

or $\frac{1}{2}$ shows 1 part of two equal parts

or $\frac{1}{2}$ shows the relationship between 1 and 2

or $\frac{1}{2}$ means 1 divided by 2

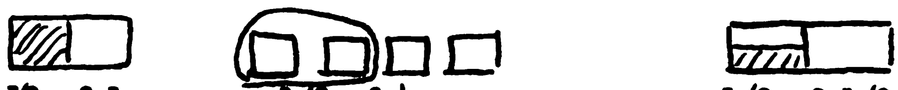
By the time that a child has been told all of this, it is probable that he does not know what to think and just gives up and memorizes everything that he needs to know about fractions in order to "pass". It would seem well at this point to make clear some of the basic understandings that a child needs, and to show some of the ways that these may be demonstrated.

Few of the things that we deal with in our daily lives come in wholes. We are always using parts of things. This is also true for children. Fractions can be put in the tangible context of a child's life, using the things that he knows. These can be used to help him understand the reality and usefulness of fractions. What he sees and can touch, he will remember better. There are many things that can be used in this way to help with fractions.

When fractions are approached through their historical development, we see that the manner of writing them is just one of convenience. Fractions can be just a logical, convenient way of figuring with parts of things. This brings us to the first understanding.

1. Fractions are Parts of Things

This can be shown graphically or with things drawn from children's environment.



1/2 of 1 1/2 of 4 1/2 of 1/2

This also can be demonstrated. Use liquid measures and show the following facts. Demonstrate them, list them, make graphs of them, and then draw conclusions.

FACT	GRAPHIC
2 cups equal 1 pint	

Conclusion: 1 cup = 1/2 pt.

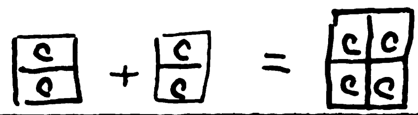
1/2 cup = 1/4 pt.

..... etc.

cup + cup = pint

FACT	GRAPHIC
2 cups equal 1 pint	

2 pints equal 1 quart



pint + pint = quart

Conclusion: 1 c = 1/4 qt.

2 c = 2/4 qt. (or 1/2 qt.)

3 c = 3/4 qt.

or 3 c = $\frac{1\text{-}1/2}{2}$ of a qt.

These can be carried as far as the talents of students permit. Slow children may not go very far. More gifted ones will be able to draw quite advanced conclusions. Introducing fractions in this manner, at any grade level, puts the symbolization and the concrete illustration on the child's level of thinking. The next step would be to do simple problems, and then more difficult ones with denominate numbers. Below are some illustrations.

$$\begin{array}{rr} 1\text{ pt.} & 2\text{ c.} \\ +1\text{ pt.} & 1\text{ c.} \\ \hline 2\text{ pt.} & 3\text{ c.} \\ +1\text{ pt.} & -2\text{ c.} \\ \hline 3\text{ pt.} & 1\text{ c.} \end{array}$$

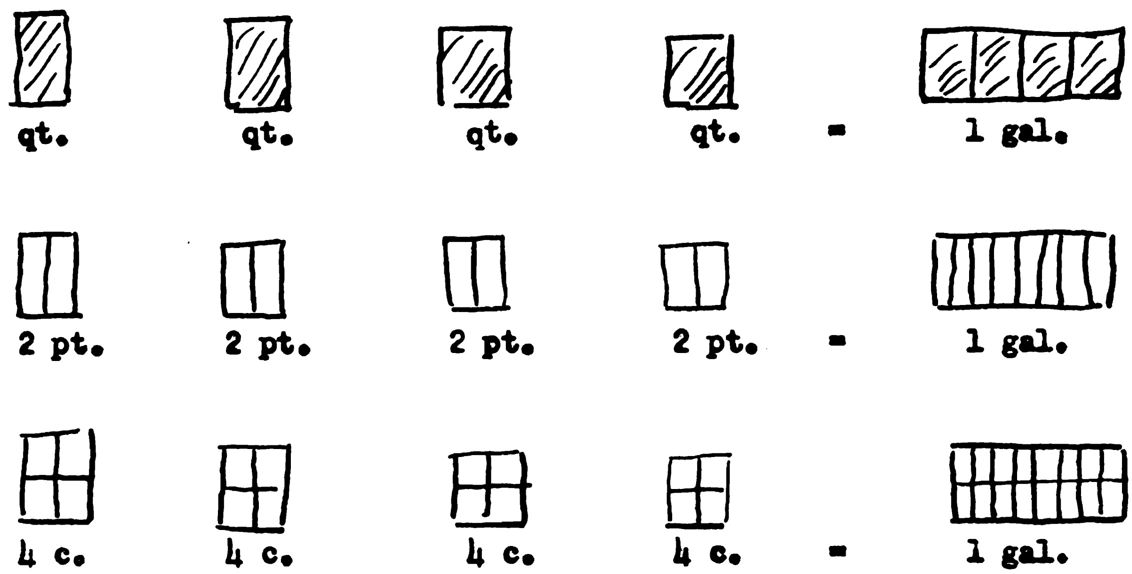
$$\begin{array}{l} 1/2\text{ qt.} \\ 1/2\text{ qt.} \\ \hline 2/2\text{ qt.} = 1\text{ qt.} \end{array}$$

$$\begin{array}{l} 1\ 1/2\text{ gal.} \\ 1\ 1/2\text{ gal.} \\ \hline 2\ 2/2\text{ gal.} = 3\text{ gal.} \end{array}$$

After children have demonstrated these facts, drawn conclusions from the facts, made graphics of the facts, and worked problems regarding them, they may see reasons for writing fractions as we do, the logic behind reducing fractions, and ways of combining them.

It is possible to help children make up very interesting displays or graphics developing any of the facts that they use. They could demonstrate that:

4 quarts = 1 gallon
8 pints = 1 gallon
16 cups = 1 gallon



Conclusion from the above: 1 pint = 1/8 gallon
1 cup =1/16 gallon
2 pints = 2/8 gallon = 1/4 gallon, etc.

We can also use a number line to show the sequence of fractions. It is possible to count in fractions. This helps to put them in a logical arrangement for children. Children could make any kind of a number line for demonstrating fractions that was at their own particular level of thinking. Below are suggestions.

0 1/2 2/2 3/2 4/2 5/2 ...

0 1/2 2/2=1 3/2=1 1/2 4/2=2 5/2=2 1/2 ...

0 1/2 1 1 1/2 2 2 1/2

... 5/2 4/2 3/2 2/2 1/2 0

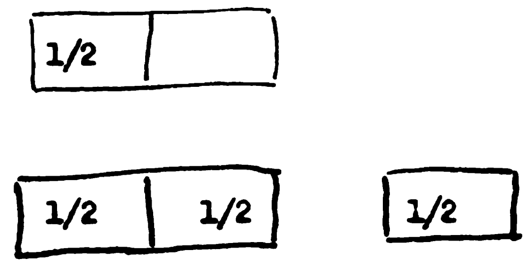
2/1/2 2 1 1/2 1 1/2 0

Any logical sequence makes a good number line for fractions. Counting forwards, backwards, and by reducing helps children to see how fractions relate. This is also excellent to help them see reduction of fractions.

A number line is also a fine help in teaching children that there are proper and improper fractions, while numbers and mixed numbers. This can be shown with things drawn from the child's environment and with graphics also. Below are some illustrations.

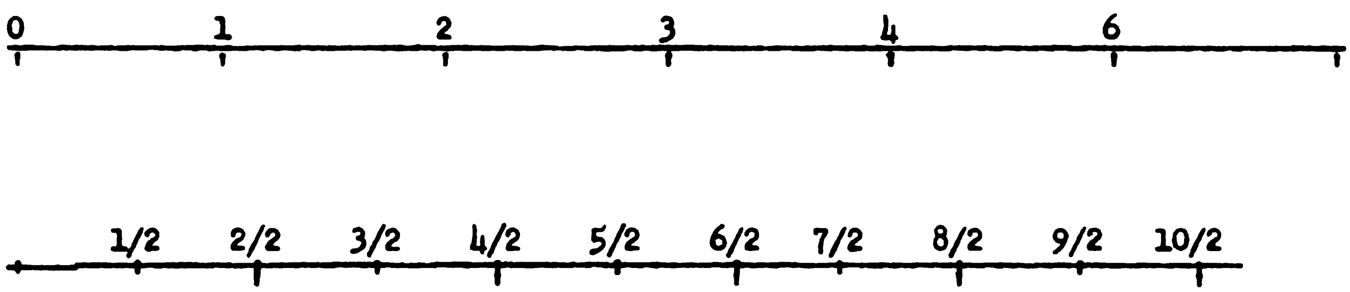
proper

improper or
mixed number



1 + 1/2 = 3/2

To show that all fractions can be changed to higher terms use measures and demonstrate or use a number line like the one below. These dan be made of any convenient material. Oak tag is good and can be kept in desks. Wrapping paper makes a good one to put up for the whole class to use. This also makes a fine project for a group of students to do.



2. A Fraction is a Ratio

This is difficult for some children to understand. Perhpas a problem will help to demonstrate this idea. Again, this is putting the fraction concept into a concrete situation drawn from the child's experience.

John has 5 marbles, 2 are blue. What part are blue?

Solution: Let each marble represent 1/5 of the total. Therefore, 2 marbles, or 2/5 are blue.

THINK: 1 marble = 1/5
+1 marble = 1/5
2 marbles = 2/5

3. Fractions are Composed of Other Numbers

Just like whole numbers, fractions are composed of other numbers. The number 6 is made of: 1+1+1+1+1+1 and 2+2+2 and 3+3, etc. 4/5 is made of 2/5 + 2/5, or 1/5 + 1/5 + 1/5 + 1/5, etc.

Also, 4/5 has the same value as any of these combinations. This is a fine way to help children see ideas that relate to lowest common denominators and to reducing. Opportunities to express fractions in as many different ways as one can are good. Another method that might lead to understanding would be to do problems in fractions as the Egyptians did. Below are some illustrations. Remember that it makes it easier to start with the largest fraction that you can.

Directions:

1. Express the following as unit fractions.

   6/8 = 1/2 + 1/8 + 1/8

   4/9 = 1/3 + 1/9

2. Express the following as unit fractions starting with the first fraction given.

   1 1/2 = 1/16 + _______

   3/10 = 1/5 + _______

Many such activities can be made up by students and teachers to give meaning to lowest common denominators and reducing.

4. A Fraction is a Division Example

This is further developed in the section on division of fractions. It should be emphasized that division can be expressed in any convenient way, such as:

$3\overline{)4}$ $\qquad$ $4 \div 3 = 4/3$

4. Fractions can be Added, Subtracted, Multiplied and Divided

This needs to be made plain with concrete materials and with symbols. It needs to be illustrated that unless this were true fractions would be of no use to the culture. It would be interesting for children to see if they could get along without fractions for a period of time, and then give a report on how they made out.

5. Fractions can be Estimated

Just like whole numbers, it is possible to estimate fractions. Children need practice in this. They should have opportunities to practice, such as doing demonstrations, making charts and comparisons, etc.

Such expressions as: $6/3 = ? > 1$

$3/5 = ? > 1/10$

$1/2 = ? < 6/8$, etc.

should be encouraged and explored. These can be suited to each child's talent. Children should be encouraged to demonstrate their reasons and the method they use to arrive at an answer.

METHODS WITH FRACTIONS

While it is fairly easy to visualize and to demonstrate addition and subtraction of fractions, this is more difficult to do with multiplication and division. Sometimes the method for these processes is emphasised so much that the meaning is obscured. Concentrating on the method until it is memorised may make it possible to "do the problems" but it helps very little with understandings and lasting learning.

Division and multiplication of fractions are closely related. They will be presented together here as a way of "thinking about fractions" rather than a process. There are several "methods" popular for teaching these. If the understandings behind these are mastered, the method as such, should develop of itself as a convenient way of doing the manipulation. This section will be divided into: (1) Understandings Basic to Division of Fractions, and (2) The Methods of Doing Division of Fractions. These will be accompanied with suggestions for classroom activities.

1. Basic Understandings

1. Children need to understand the relationship between multiplication and division. They should know that:

   $3\overline{)12}$ means also: $\underline{\ ?\ } \times 3 = 12$

   When a child does the first problem, or ones like it, he should also understand the second one and the way the two relate. Then

   $3/4 \div 1 = 3/4$, and $\underline{\ ?\ } \times 1 = 3/4$

   would be the next step in understanding the process with fractions. This helps to make answers more meaningful.

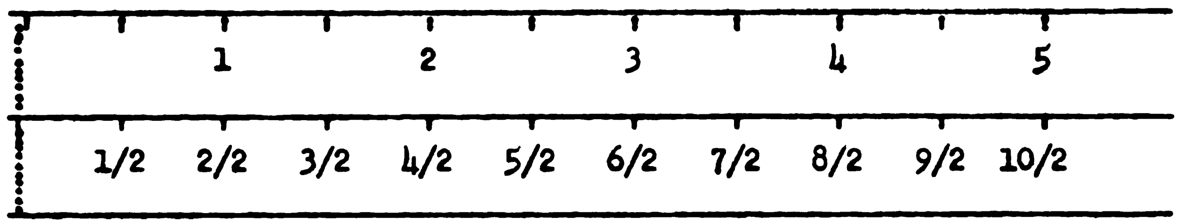
2. Children need to be able to change whole numbers and mixed numbers to improper fractions. If they have acquired the basic understanding that fractions are parts of things, this can be demonstrated with tangible items and then carried to symbols. They should be led to see that:

$$1\ 1/2 \times 4 = \quad ?$$

$$1\ 1/2 + 1\ 1/2 + 1\ 1/2 = \underline{\qquad}$$

or with a number line or ruler:

------ 1 1/2------|------1 1/2-------|-------1 1/2------|



or they can see that the problem could be done just like multiplication of whole numbers.

$$\begin{array}{rl} 3 & \\ \underline{1\ 1/2} & \\ 3 & (1 \times 3) \\ \underline{1\ 1/2} & (1/2 \text{ of } 3) \\ 4\ 1/2 & (\text{total of } (1x3) + (1/2 \times 3) \end{array}$$

or $1\ 1/2 = 3/2$, so we have, finally:

$$3/2 \times 3/1 = 9/2 = 4\ 1/2$$

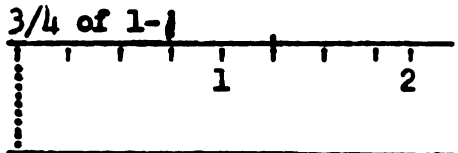
Children need <u>many</u> experiences with each problem. They need to see it in the abstract and the concrete. The need to be able to find several ways of doing the same problem. Time might better be spent doing the same problem many ways and understanding it, than by doing many problems without understanding.

Children should be encouraged to invent reasonable, logical, and convenient ways of doing problems like this. They may make reports or do demonstrations, or make charts and graphs to explain their thinking.

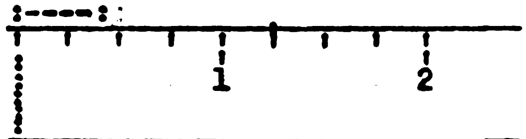
3. How to multiply common fractions and mixed numbers may be approached in much the same manner as the above. Helping children to see the problem in many meaningful ways is better than dwelling on one <u>method</u> as such.

3/4 x 1 1/2 = 3/4 of 1 plus 3/4 of 1/2 would be easy to see if inches were used.

3/4 of 1-|



3/4 of 1/2



Children can count the eighths and see that they will have 9/8 or 1 1/2. They can see that 3/4 = 6/8. Half of 3/4 or 6/8 is then 3/8. So, they will have 6/8 + 3/8 or 9/8 for the answer. Children need to be encouraged to think problems through logically.

To help in understanding that multiplication by fractions results in a smaller number than the multiplicand, introduce the expression taken so many times. Use this in the place of multiplied by, or times. Help children to understand that:

1 1/2 x 3/4 = means 3/4 taken one and one-half (1 1/2) times.

This may be shown in a sequence or number line relationship.

1 x 3/4 =	means 3/4 taken 1 time
2 x 3/4 =	means 3/4 taken 2 times
3 x 3/4 =	means 3/4 taken 3 times, etc.

Now the half way points may be shown:

1 x 3/4 =	means 3/4 taken 1 time
1 1/2 x 3/4=	means 3/4 taken 1 and 1/2 times
2 x 3/4 =	means 3/4 taken 2 times
2 1/2 x 3/4 =	means 3/4 taken 2 and 1/2 times, etc.

The problem may be done just as they have done multiplication with whole numbers in the past.

$$\begin{array}{rl} 3/4 & \\ \times\ 1\ 1/2 & \\ \hline 3/4 & (1 \times 3/4) \\ 3/8 & (1/2 \times 3/4) \\ \hline 3/4+3/8 & \text{(total of the two partial products)} \\ \text{or} \quad 9/8 & \text{(combining with a common denominator)} \end{array}$$

Such investigations as these, lead to understandings and promote insights that are often missed in rote situations. They are meaningful because they are based on things that the child already knows.

4. Children need to understand division as a process in order to understand division and multiplication of fractions.

$$\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$$ means

there are 4 threes in 12
there are 4 groups of three in 12
3 can be subtracted from 12 four times
4 x 3 = 12

In the same way:

6/8 ÷ 3/8 = 2 means there are 2 three-eighths in 6/8
there are 2 groups of 3/8's in 6/8
or

$$6/8 = \begin{matrix}(1/8) & (1/8) & (1/8) \\ (1/8) & (1/8) & (1/8)\end{matrix}$$

3/8 can be subtracted twice from 6/8
2 x 3/8 = 6/8

Finding relationships and ways to express them such as these helps children to see that the answer <u>2</u> is not wrong or foolish, although it may first seem that way to them.

5. The fact that fractions can be written in different forms, all having the same value should be plain to children. This comes easily from the basic understanding that "fractions are parts of things". As illustrated on page 131, 1 cup is equal to 1/16 of a gallon. Taking this as a starting point, children can make true statements, such as:

1 cup = 1/16 gallon
1 pint= 2 cups

therefore, 2 cups = 2/16 of a gallon

1 pint equals 1/8 of a gallon
2 cups equal 1 pint

therefore,
1 pint equals 2/16 of a gallon

Children should be encouraged to make their own observations, record them and draw conclusions. The lesson in the appendix on the fractional or combo ruler helps children to see these relationships.

Making tables of equal fractions helps children to see this. Start with something that they know.

1/2 = 2/4, 4/8, 8/16, etc.

1/3 = 2/6, 4/12, etc.

Encourage the children to see that they can do this with tangible objects and with symbols. Then encourage them to <u>draw the conclusion</u> that they are multiplying the fraction, top and <u>bottom, by the same</u> number. After they have discovered this, help them see that the reason this multiplication does not change the value of the fractions is that n/n is always equal to one.

$\frac{2}{2} \times \frac{1}{2} = \frac{2}{4}$ $\frac{2}{2}$ equals 1, therefore $\frac{2}{4} = \frac{1}{2}$ because the <u>fraction has been multiplied by the equivalent of 1.</u> This leads <u>directly to the discovery of lowest common denominators.</u>

6. Children need to have help in finding a lowest common denominator and help in a way to put it down so that it makes sense to them and can be read. Many children in the fifth and sixth grades have difficulty with fractions because their small muscle development has not reached a point where they can write them. If children have been taught that n/n = 1, they might be encouraged to write a problem involving lowest common denominators like the following:

$$\frac{2}{2} \times \frac{1}{2} = \frac{2}{4}$$

$$\frac{1}{1} \times \frac{3}{4} = \frac{3}{4}$$

$$\frac{5}{4} = 1\ 1/4$$

It should be emphasised that in every case they are multiplying by the equivalent of 1.

7. In line with the above, they should see how to change a fraction to higher terms. They should see this graphically, tangibly, and then with symbols.

$$\frac{3}{3} \times \frac{2}{5} = \frac{6}{15}$$

THINK; 2/5 is equal to how many fifteenths? If I multiply the denominator by 3, that will give me fifteenths. I can multiply the numerator then by 3 and I will have 6 fifteenths. This is all right to do because 3/3 is equal to 1. Multiplying a number by 1 does not change its value.

8. In order to shorten difficult multiplication processes, children should know how to cancel. However, they should know that cancelling is the same as dividing the fraction, or fractions, by the equivalent of 1 or n/n.

$$\frac{2}{3} \times \frac{3}{4} \times \frac{1}{2} =$$

$$\frac{2}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{3}}}{4} \times \frac{1}{2} = \quad \text{(dividing by } \tfrac{3}{3}\text{)}$$

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{4}}} \times \frac{1}{\cancel{2}} = \quad \text{(dividing by } \tfrac{2}{2}\text{)}$$

UNDERSTANDINGS BASIC TO DIVISION OF FRACTIONS

1. Dividing by a number less than one will yield an answer greater than ONE. This should be thoroughly understood or the quotient in a division example is quite meaningless. Following are some suggestions as to how it may be approached.

   Show division in a sequence of examples.

$$
\begin{aligned}
&3 \div 3 = 1 \\
&3 \div 2 = 1\ 1/2 \\
&3 \div 1 = 3 \\
&3 \div 1/2 = 6 \\
&3 \div 1/4 = 12, \text{ etc.}
\end{aligned}
$$

   Another way to show this would be to indicate the number of halves or fourths in three. This can be done with symbols or a number line.

$$
3 \div \frac{1}{2} = \qquad
\begin{array}{r}
1 = \frac{2}{2} \\[1em]
1 = \frac{2}{2} \\[1em]
1 = \frac{2}{2} \\
\hline
3 = \frac{6}{2}
\end{array}
$$

   If a youngster knows "division as a process" as listed in the basic requirements, he can now see that $3 \div 1/2$ can mean how many halves in in three, etc.

$$
3 \div \frac{1}{2} = \quad
\begin{array}{ccccccc}
\hline
0 & & 1 & & 2 & & 3 \\[1em]
0 & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} \\
\hline
\end{array}
$$

   If the problem is one of the type: $3 \div 2/4 =$, we have much the same situation, except that the fourths are taken two at a time. If the youngsters have been introduced to this phase, they can see that now the question asks how many groups of two-fourths are there in three? This can be shown graphically and symbolically also:

$3 \div \frac{2}{4} =$

$$1 = \frac{4}{4} = \left(\frac{2}{4} + \frac{2}{4}\right)$$

$$1 = \frac{4}{4} = \left(\frac{2}{4} + \frac{2}{4}\right)$$

$$1 = \frac{4}{4} = \left(\frac{2}{4} + \frac{2}{4}\right)$$

$$3 = \frac{2}{4} + \frac{2}{4} + \frac{2}{4} + \frac{2}{4} + \frac{2}{4} + \frac{2}{4} = \text{(6 two-fourths)}$$

or there are 6 two-fourths in 3. The question is one of putting a label on the answer. It might be well for a time to have children label their answers for fraction problems in division.

$$3 \div 2/4 = 6 \text{ two-fourths}$$

Logically, now students can be lead to see that we have to find out <u>how many fourths there are in 3 and then take them 2 at a time.</u> This <u>means we have to multiply 3 by 4 and divide by 2, or INVERT and</u> MULTIPLY.

$$3 \div \frac{2}{4} =$$

$$3 \times \frac{4}{2} = \frac{12}{2}$$ the number of fourths in 3 = 6(2/4's) taken two at a time.

The same thing can be done with a number line and the problem shown graphically.

$3 \div \frac{2}{4} =$

:-2/4-:-2/4-:-2/4-:-2/4-:-2/4-:-2/4-:
0 1 2 3

This is a logical approach to division of fractions. We simply have a convenient way of putting down the process. Children should have many experiences in "thinking about" what the process means. They should be encouraged to "find out for themselves and to experiment".

2. Another <u>method</u> for doing division of fractions is called the <u>common denominator method</u>. Children frequently discover this for themselves. Simply change the fractions to a lowest common denominator and divide as with whole numbers.

$$3 \div \frac{2}{4} =$$

$$\frac{12}{4} \div \frac{2}{4} = 6$$

or

$$2 \text{ fourths} \overline{\big)\, 12 \text{ fourths}}^{\;6 \text{ times}}$$

This is a reasonable and understandable method for children that has an explanation lying within their previous experiences with fractions.

3. A method that might be called the reciprocal method is possible to teach if children have mastered the idea that $n/n = 1$; and that, for this reason, you can multiply both terms of a fraction by the same number without changing its value.

The difficulty in understanding this method is sometimes the difficulty of understanding the term reciprocal. Reciprocal refers to TWO numbers. Or TWO NUMBERS WHICH ARE MULTIPLIED TOGETHER SO THAT A PRODUCT OF 1 IS OBTAINED ARE RECIPROCALS.

$3 \times ? = 1 \quad 3 \times 1/3 = 1$ (3 and 1/3 are reciprocals)

$1/2 \times ? = 1 \quad 1/2 \times 2/1 = 1$, etc.(1/2 and 2/1 are reciprocals)

We can state our division example as a fraction (complex fraction) and show this in a manner that will still be within the context of the previous experiences the children have had with fractions.

$$3 \div \frac{1}{4} =$$

$$\frac{3}{\frac{1}{4}}$$

If we multiply both terms by the same number, we are multiplying by 1 and this will not change the value of the fraction.

If we can make the denominator of this complex fraction 1, we will have the answer.

Therefore, multiply both terms by the reciprocal of the denominator and the denominator will then be 1.

$$\frac{\frac{4}{1} \times \frac{3}{1}}{\frac{4}{1} \times \frac{1}{4}} = \frac{\frac{4}{1} \times \frac{3}{1}}{1} = \frac{4}{1} \times \frac{3}{1} = \frac{12}{1} = 12$$

We have shown this in a round-about method. Now we can conclude, though, that the reciprocal of the divisor will be the divisor inverted, this will make the divisor 1 if we multiply both terms of the fraction by this reciprocal. So, all we have to do is to invert the divisor in the first place and multiply. This method gives a reason for inversion. Sometimes the reason is too difficult for all but a few youngsters. The important thing to remember is to give them a great many experiences, and many ways of thinking logically about fraction division. Without doubt, students and teachers can think of many additional ways of demonstrating the process. Time should be taken to investigate these. It will be well spent in developing interest in mathematics and real understanding.

V. DECIMAL FRACTIONS

Decimal and Common Fractions

An Extension of Our Number System

Changing Common Fractions to Decimal Fractions

Because decimal fractions all have denominators with powers of ten, we can leave the denominators off and write them as whole numbers if we can find a way to do it.

In order to do this, we put a • after the whole number and call all the numbers after the decimal point decimal fractions. The first place after the decimal point is tenths' place, the next, hundredths', etc., just as the places for the whole numbers.

WE CALL THIS WAY OF WRITING DECIMAL FRACTIONS AS AN EXTENSION OF OUR NUMBER SYSTEM. WE HAVE JUST EXTENDED THE SYSTEM IN ORDER TO WRITE THE DECIMAL FRACTIONS.

millions	10 x 100,000 = 1,000,000
hundred-thousands	10 x 10,000 = 100,000
ten-thousands	10 x 1,000 = 10,000
thousands	10 x 100 = 1,000
hundreds	10 x 10 = 100
tens	10 x 1 = 10
ones	
•	DECIMAL POINT
tenths	1/10 of 1 =
hundredths	1/10 of 1/10 =
thousandths	1/10 of 1/100 =
ten-thousandths	1/10 of 1/1000=

Reading decimal fractions is comparatively simple if we refer to the chart. The next problem is how to change all fractions to decimal fractions. To approach this logically and within the context of what the children know about fractions and lowest common denominators, it better than presenting a rule to be memorized.

Some common fractions can be changed to decimal fractions with little or no margin of error. Some examples will demonstrate this.

$$\frac{1}{2} = \frac{?}{10} \qquad \frac{1}{2} = \frac{?}{100}$$

$$\frac{5 \times 1}{5 \times 2} = \frac{5}{10} \qquad \frac{50 \times 1}{50 \times 2} = \frac{50}{100}$$

However:

$$\frac{1}{4} = \frac{?}{10} \qquad \frac{1}{4} = \frac{?}{100}$$

$$\frac{2 \times 1}{2 \times 4} = \frac{2}{10} \qquad \frac{25 \times 1}{25 \times 4} = \frac{25}{100}$$

In the first example, we have a choice of multiplying 4 by 2 1/2 in order to make the decimal come out even. This is foolish, since the purpose of decimals is to make them easy to use. So, we carry the fraction one more place and it comes out even. Our error would be less if it came out 1/2 of a hundredth than if it came out 1/2 of a tenth.

WE ARE MULTIPLYING BOTH TERMS OF THESE FRACTIONS BY THE EQUIVALENT OF 1, JUST AS WE DID WITH LOWEST COMMON DENOMINATORS BEFORE.

In order to find out what to multiply the common fraction denominator by, THINK: "What do I multiply 4 by to get ten?" Or, "What do I multiply 4 by to get 100?", etc. A problem like the following, is more difficult. Students may actually need to perform the division in order to find out what number to use as a multiplier.

$$\frac{3}{16} = \frac{?}{100} \qquad 16\overline{)100}^{\,6} \qquad \frac{6}{6} \times \frac{3}{16} = \frac{18}{100}$$

It may be necessary for children to divide 100 by 16 in this case before they can complete the equation. 18/100 is as close as we can come with whole numbers.

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