GENERALIZATION OF NORMAL SHOCK THEOREMS TO MAGNETOGASDYNAMICS WITH RADIATION

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RAM MOHAN SRIVASTAVA

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ABSTRACT

GENERALIZATION OF NORMAL SHOCK
THEOREMS TO MAGNETOGASDYNAMICS
WITH RADIATION

by Ram Mohan Srivastava

The present work is primarily concerned with the generalization of normal shock theorems by Courant and Friedrichs valid in isentropic, inviscid, non-heat conducting fluids to radiation-magnetogasdynamics. In the process, few generalized Rankine-Hugoniot relations and generalized Prandtl relation have been derived. Also, the generalized Hugoniot function has been defined, and the shape of the Hugoniot curve in the (p,v)-plane has been determined.

In the main part of this work one-dimensional, uniform, and steady state flow of an electrically conducting, fully ionized and compressible gas under a planar magnetic field perpendicular to the velocity vector has been assumed. Only first approximations for radiation parameters, for an optically thick medium, have been considered.

The shape of the Hugoniot curve, in the (p,v)-plane, has been found to be similar to the one in classical gasdynamics. The generalized Rankine-Hugoniot relations are in implicit form, hence, successive approximation technique has to be used to find the corresponding state behind the shock front. Theorems 1 and 3 refer to change in modified entropy across a shock front. Theorem 2 compares the pressure rise across a shock front to

that in reversible adiabatic change. Finally, Theorem 4 refers to the flow velocities in front and behind the shock wave.

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NOMENCLATURE

a_e	effective velocity of sound in radiation-magnetogasdynamics
a_r	Stefan-Boltzmann constant
B	magnetic flux density
c	velocity of light
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
D_r	Rosseland diffusion coefficient of radiation
e	specific internal energy = C_vT
e^*	total internal energy per unit mass $= C_vT + 3vp_r + vp_h$
E	electric field
E_r	total radiation energy per unit volume = a_rT^4
E_t	total internal energy per unit mass $= C_vT + \frac{1}{2}u^2 + E_r/\rho$
F_e	electromagnetic force
H	magnetic field
$\mathcal{H}$	Hugoniot function
i	electric current
j	mechanical equivalent of heat
J	electric current density
k	coefficient of heat conduction
m	constant = ρu
M	Mach number
M^*	critical Mach number

p	hydrostatic pressure
p_h	magnetic pressure $= \frac{1}{2} \mu_e H^2 = \frac{\theta^2}{2v^2}$
p_r	radiation pressure $= \frac{1}{3} E_r$
p_t	$= p + p_r$
p^*	total pressure $= p + p_r + p_h$
Pr	Prandtl number
Q	total heat flux
R	Gas Constant
Re	Reynolds number
R_p	radiation pressure number $= p_r/p$
R_h	magnetic pressure number $= p_h/p$
s^*	modified entropy
t	time
T	absolute temperature
u	flow velocity
v	specific volume
x	space coordinate
γ	ratio of specific heats $= C_p/C_v$
δ	shock wave thickness
ϵ	dielectric constant
ϵ_e	joule heat $= \vec{E}.\vec{J}$

θ	constant
	$= \sqrt{\mu_e} vH$
λ	mean free path
λ_r	Rosseland mean free path of radiation
μ	coefficient of viscosity
μ_e	magnetic permeability
ν_h	magnetic diffusivity
ρ	density (mass per unit volume)
ρ_e	excess electric charge
σ	electrical conductivity
$(\ )_1$	subscript 1 signifies the state in front of the shock wave
$(\ )_2$	subscript 2 signifies the state behind the shock wave

INTRODUCTION

In the magnetogasdynamics, the generalized Rankine-Hugoniot relations referring to a normal shock have been developed by several authors [4, 11, 13, 14]. In the radiation-magnetogasdynamics, the generalized Rankine-Hugoniot relation, as the ratio of velocities (i.e. $\frac{u_2}{u_1}$) only, has been developed by [15, 16]. The normal shock theorems, referring to some characteristic properties of normal shock in classical gasdynamics, have been developed by [2, 6, 28] and condensed by [3].

In the present work, we derive generalized Rankine-Hugoniot and Prandtl relations, not covered in [15, 16], in radiation-magnetogasdynamics. Further, we have generalized the normal shock theorems of [3] with their proofs, to radiation-magnetogasdynamics.

In Chapter I, some aspects of normal shocks have been discussed based upon the existing literature and a table of shock layer thickness have been compiled. In Chapter II the fundamental equations of radiation-magnetogasdynamics have been collected and reduced for the case of one-dimensional steady state flow. Chapter III contains the derivation of the generalized Rankine-Hugoniot and Prandtl relations. Further, modified first law of thermodynamics and modified Rankine-Hugoniot relations have been introduced in Section (3.4). In addition, several auxiliary inequalities have been derived in Section (3.5). In Chapter IV the generalized Hugoniot function has been defined, and the shape of the Hugoniot curve has been determined. The crucial part of the work, i.e., the generalizing the four normal shock theorems with their proofs, occupies Chapters V to VIII.

I. BRIEF REVIEW OF THE PRESENT STATUS OF THE SHOCK THEORY

1.1 Inviscid Isentropic Flow-Shock and Rankine-Hugoniot Relation (Continuum):

All the approaches in this chapter are phenomenological. The considerations referring to the shock phenomena should emphasize two kinds of assumptions which form the basis of any approach to the shock theory. The first group refers to the fundamental laws governing the flow, i.e., three conservation laws and equation of state. We use a unique nomenclature for these laws, namely fundamental dynamic laws (f.d.l.). The second group refers to the fundamental assumptions governing the structure of the shock (f.s.l.).

The Rankine-Hugoniot relation [3, 10, 20, 27] is derived with f.d.l. as standard laws of ideal, perfect gas. The f.s.l. assume that the shock is a step-wise transition, in all the variables, of zero thickness.

1.2 Viscous Flow Shocks in Continuous Media:

Shock in real gases exhibit very steep but continuous transition from the state 1 to the state 2. As the shock wave becomes very steep, viscous stresses and heat conduction effects become appreciable, no matter how small be the coefficient of viscosity and the coefficient of thermal conductivity, and so a particle of fluid is subject to diabatic effects. The effects of viscosity and heat conduction tend to wipe out discontinuities in velocity and temperature. Therefore these effects control the thickness of a shock wave. Below, we discuss briefly the fundamental assumptions in a few representative works on the subject.

In [21] a perfect gas, satisfying f.d.l. with two different uniform u, ρ, and T as initial and end boundary conditions, has been assumed. The f.s.l. assume special functions describing the velocity distribution inside the shock. The thickness of the shock is obtained from the entire formalism.

In [23] the "Shock-Thickness Reynolds Number" is derived for air. The f.d.l. and f.s.l. assumptions are the same as in [21]. The viscosity μ has been assumed to be proportional to T^n.

In [26] the f.d.l. assumptions are the same as in [21], the f.s.l. assume that the quotient μ/k remains approximately constant with temperature variation. No other constraints are introduced. This allows the author to calculate only the upper and lower bounds of the thickness but not the actual thickness of the shock itself.

In [9] the f.d.l. assumptions are the same as in [21], except the viscosity terms are retained in the momentum equation and neglected in the energy equation. The f.s.l. assume the inflection point (i.e. $d^2u/dx^2 = 0$, at $x = 0$) inside the shock wave and this is sufficient for the existence of the transition region. The results obtained are in close agreement with the exact solutions of [12, 19] for the structure of the shock wave and its thickness.

1.3 Kinetic Theory Treatment of the Viscous Flow Shocks:

In [1] a perfect gas, whose specific heat is independent of temperature, satisfies f.d.l. assumptions as that of [21]. The f.s.l. assumptions are the same as in [9]. The author concludes that the thickness of a moderately strong shock is of the order of mean free path and must be treated directly from the relevant

Boltzmann equation. Whereas very strong shocks have thickness less than the mean free path and even the Boltzmann equation cannot be used. Hence, the actual reference in [1] to the kinetic theory is a recommendation that the kinetic theory equations should be used in the shock theory.

In [25] it has been pointed out that if the increase of the coefficients of thermal conductivityand of viscosity with increasing temperature and pressure is taken into account, then the shock wave thickness for a perfect gas will never be less than the mean free path and hence the Boltzmann equation can be applied even for very strong shocks. The author takes the third approximation to the Boltzmann equation. The f.d.l. and f.s.l. assumptions are the same as in [1].

In [12] the conclusions of [25] has been modified for any gas whose μ and k has been assumed to be proportional to T^n, where n is a positive constant depending only on the gas in question and for $Pr = 3/4$. In this paper f.d.l. assumptions are the same as that in [25] and f.s.l. assumptions are the same as in [1].

In [8] the author tries to improve the results of [26], by taking the third approximation to the Boltzmann equation for f.d.l. and keeping f.s.l. assumptions the same as in [26]. But the author finds that the bounds of the shock wave thickness are not affected by the higher order Burnett terms.

1.4 Shocks in Magnetogasdynamics:

The governing relations are derived from the magnetogasdynamic equations describing the steady, one-dimensional flow of a viscous, heat conducting, electrically conducting, and compressible gas under a planar magnetic field perpendicular to the velocity vector (i.e. velocity and magnetic field vectors are in the same plane but perpendicular one to each other). If the magnetic field vector is parallel to the velocity vector, then it will not affect the gasdynamic equations.

In [13] a perfect gas, satisfying f.d.l. and Maxwell's equations with two different uniform u, ρ, T, and H as initial and boundary conditions, has been assumed. The f.s.l. assume that there is a point of inflection, in the transition region, for all the variables. As a special case the generalized Rankine-Hugoniot relations have been derived. Moreover, the structures of the shock wave of a finite thickness have been considered for a few special cases.

In [11] the f.d.l. and the f.s.l. assumptions are the same as in [13] along with Maxwell's equations. The author determines the shock profile and width of the transition region for a few special cases.

1.5 Shocks in Radiation-Magnetogasdynamics:

The governing relations are derived exactly in the same manner as that for shocks in Magnetogasdynamics, except that the radiation pressure is added to the gas pressure in the momentum and energy equations and the radiation field energy is added to the energy equation.

In [16] the generalized Rankine-Hugoniot, as well as sev-

eral limiting cases of Rankine-Hugoniot relations have been derived. The f.d.l. assumptions are the same as that of [13]. The f.s.l. assume that the shock is a step-wise transition, in all the variables, of zero thickness.

1.6 Fundamental Properties of the Shock Transition:

The literature reviewed in Sections (1.1) to (1.5) refers to a some sort of the phenomenological theory of shocks. Namely, it is assumed that the shock exists and the mathematical formalism helps to answer on such questions of how small or large is the thickness of the shock, etc. But there are many other important questions concerning the shock wave thoery which are of a more fundamental nature.

Let us only quote from [28] two such questions: (a) What are the conditions for the equation of state of a fluid under which shocks with their distinctive qualitative features may be produced; (b) the second question refers to the physical structure of the shock layer whose "infinitesimal" width is of the order of magnitude ϵ provided that heat conductivity and viscosity are small of the same order. Below, we present a brief review of the literature referring to this part of the shock theory.

In [2] the author derives several important theorems concerning the behavior of shock waves based on the three assumptions for the equation of state. In [28] the author derives some of the conclusions of [2] using more rigorous mathematical methods but made seven assumptions concerning the equation of state. In [3] the authors condensed the physical assumptions of [28] and rearranged those of [2]. The authors follow the method of [28] and prove four basic properties of the shock transition.

In [6] the author proves the existence and uniqueness of the shock layer for the general class of fluids considered in [28], for arbitrary end states satisfying the shock relations, with k and μ being arbitrary functions of the state.

In [22] the transient and steady state behaviors of normal shock waves are examined. The transient behavior is examined from dynamical considerations, while the stationary behavior of the shock is studied with the help of the Rayleigh line.

1.7 Table of Shock Layer Thickness:

Authors	Formulas	Remarks
Becker [1]	$\delta = \frac{4}{3}\mu \frac{\zeta_1+1}{\zeta_1} \cdot \sqrt{\frac{1}{\rho_1 p_1}} \cdot \sqrt{\frac{\zeta_1-1}{\pi\zeta_1+1}} \cdot \frac{\sqrt{\frac{\pi\zeta_1+1}{\pi+\zeta_1}}+1}{\sqrt{\frac{\pi\zeta_1+1}{\pi+\zeta_1}}-1}$ where: $\zeta_1 = \frac{\gamma+1}{\gamma-1}$, $\pi = \frac{p_2}{p_1}$	μ has been assumed constant.
Lieber, Romano and Lew [9]	$\delta = (\alpha-1)/(dV/d\xi)_{\xi=0}$ where: $\xi = x/\lambda_1$, $\alpha = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2}$, $V = u/u_1$, $(\frac{dV}{d\xi})_{\xi=0} = \frac{3}{4}\sqrt{\frac{\gamma\pi}{8}} \cdot \frac{M_1}{Z} \cdot [(V_1-1) - \frac{1}{\gamma M_1^2}(1 - \frac{1}{V_1^{\,n}})]$, $V_1\vert_{\xi=0} = (n/\gamma M_1^2)^{\frac{1}{n+1}}$, $Z = 0.36$, $n = -\{\ell n[1+\gamma M_1^2(1-\alpha)]/\ell n\,\alpha\}$	μ is retained only in the momentum equation and not in the energy equation.

Morduchow and Libby [12]	$\delta = \lambda_1 \delta' = -(u_1 - u_2)/(\frac{du}{dx})_{max}$, where: $\delta' = -(1-\alpha)/(\frac{dV}{d\xi})_{\xi=0}$, $\alpha = \frac{u_2}{u_1}$, $V = \frac{u}{u_1}$, $\xi = \frac{x}{\lambda_1}$, $(\frac{du}{d\xi})_{\xi=0} = -K\, M_1 (1 - \sqrt{\alpha})^2$, $K = 1.36$ for air .	μ has been assumed constant.
Pucket and Stewart [18]	$\frac{\rho^* u^* \delta}{\mu^*} = \frac{4}{3} \cdot \frac{u_1(u_1 - u_2)u^*}{u_1^2 + RT_1 - u_1(\gamma^*+1)RT^*}$.	Both μ and C_p vary with temperature.
Shapiro and Kline [21] (Perfect Gas)	$Re = \frac{\rho_1 u_1 \delta}{\mu^*} = \frac{D}{\gamma+1} \cdot \frac{M_1^*+1}{M_1^*-1} \cdot \left[1 \pm \sqrt{1 + \frac{8\gamma(\gamma+1)}{3\, Pr^*} \cdot \frac{1}{D^2} \cdot \frac{(M_1^*-1)^2}{M_1^*}}\right]$, where: $D \equiv \frac{4}{3} + \frac{2\gamma}{Pr^*} - \frac{\gamma+1}{2\, Pr^*} \cdot \frac{M_1^{*2}+1}{M_1^*}$.	$\pm$ sign depends upon the sign of D, so that Re is positive.
Shapiro and Kline [23]	$Re = \frac{\rho_1 a_1 \delta}{\mu_1} = \frac{D}{(\gamma+1)M_1^*} \cdot \frac{M_1^*+1}{M_1^*-1} \left[(\frac{\gamma+1}{2}) \cdot (1 - \frac{\gamma-1}{\gamma+1} M_1^*)\right]^{\frac{1}{2}-n}$. $\left[1 \pm \sqrt{1 + \frac{8\gamma(\gamma+1)}{3\, Pr^*} \cdot \frac{1}{D^2} \cdot \frac{(M_1^*-1)^2}{M_1^*}}\right]$ where: $D \equiv \frac{4}{3} + \frac{2\,\gamma}{Pr^*} - \frac{\gamma+1}{2\, Pr^*} \cdot \frac{M_1^{*2}+1}{M_1^*}$, $a_1 = \sqrt{\gamma RT_1}$. Note: $(\ )^*$ signifies state corresponding to the critical speed.	$\pm$ sign depends upon the sign of D, so that Re is positive; n is a positive constant in the formula $\mu \sim T^n$.

continued on next page

Taylor and Maccoll [5]	$\delta = \frac{D}{u_1-u_2} \cdot \ln \frac{u_1-u}{u-u_2}$ where: $D = [\frac{k\,j}{R}(1-\frac{1}{\gamma}) + \frac{4}{3}\cdot\frac{\mu}{\gamma-1}] \cdot \frac{2(\gamma-1)}{\rho_1(\gamma+1)}$ $\delta' = \frac{2\,D}{u_1-u_2} \ln 9 = \frac{4.4\,D}{u_1-u_2}$	δ' is the thickness where 8/10 of the change in velocity from u_1 to u_2 occurs.
Thomas [25]	$\delta = \frac{4}{3}\cdot(\frac{2}{\pi})^{\frac{1}{2}} \frac{\mu}{\rho} \left[\frac{(w_1-w_2)[(w_1+w_2)(w_1w_2-(w_1+w_2-1)w^2)]^{\frac{1}{2}}}{(w_1-w)(w-w_2)} \right]_{min}$ where: $w = \frac{ua}{b}$, $a = \rho u$, $b = au + p$	
von Mises [26]	$\frac{4}{3} \cdot \frac{\mu_1}{m} \cdot \frac{1-\alpha}{\gamma+1} \cdot I \le \delta \le \frac{4}{3} \cdot \frac{\mu_2}{m} \cdot \frac{1-\beta}{\gamma+1} \cdot I$ where: $\alpha = \frac{3\gamma}{4\,Pr} \cdot t'_2$, $\beta = \frac{3\gamma}{4\,Pr} \cdot t'_1$, if $Pr < \frac{3}{4}$ $\alpha = \frac{3\gamma}{4\,Pr} \cdot t'_1$, $\beta = \frac{3\gamma}{4\,Pr} \cdot t'_2$, if $Pr > \frac{3}{4}$ $I = 2 \ln [(\frac{1}{\epsilon} - 1)(\frac{u_1+u_2}{u_1-u_2})]$, $\epsilon < \frac{1}{2}$ and $t' = dt/du$, $\gamma = g\cdot\frac{C_p}{C_v}$	
v. Krzywoblocki [8]	The author tries to modify the results of [26] by taking into account the third approximation to the Boltzmann equation (Burnett's equation) but finds that the result of [26] are not affected by the higher order Burnett terms.	

II. FUNDAMENTAL ASPECTS OF SHOCKS IN RADIATION-MAGNETOHYDRODYNAMICS

2.1 Fundamental Systems of Equations:

The systems below refer to a one-dimensional non-steady flow [7, 14, 15].

(a) Hydrodynamic System:

Equation of State (perfect gas):

$$p = \rho RT. \tag{2.1.1}$$

Equation of Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0. \tag{2.1.2}$$

Equation of Motion:

$$\rho\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right) = -\frac{\partial p_t}{\partial x} + F_e + \frac{\partial}{\partial x}\left(\frac{4}{3} \mu \frac{\partial u}{\partial x}\right). \tag{2.1.3}$$

Equation of Energy:

$$\frac{\partial}{\partial t} (\rho E_t) + \frac{\partial}{\partial x} (\rho E_t u) = -\frac{\partial}{\partial x} (u p_t) + \frac{\partial}{\partial x}\left(\frac{4}{3} \mu u \frac{\partial u}{\partial x}\right) + \frac{\partial Q}{\partial x} + \epsilon_e. \tag{2.1.4}$$

(b) Electromagnetic System:

Maxwell's Equations:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}; \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2.1.5}$$

$$\nabla . \vec{B} = \nabla . \mu_e \vec{H} = 0; \quad \nabla . \epsilon \vec{E} = \rho_e. \tag{2.1.6}$$

Equation of Conservation of Electric Charge:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{J} = 0. \qquad (2.1.7)$$

Equation of Electric Current:

$$\vec{J} = \vec{i} + \rho_e \vec{u} = \sigma[\vec{E} + \mu_e(\vec{u} \times \vec{H})] + \rho_e \vec{u}. \qquad (2.1.8)$$

(c) Radiation System:

Equations of radiation pressure, energy, and flux:

$$p_r = \frac{1}{3} E_r, \; E_r = a_r T^4, \; Q_r = \frac{c}{4} \frac{E_r}{\rho} . \qquad (2.1.9)$$

The above equations are only first approximations of p_r, E_r, and Q_r respectively.

2.2 Reduction of the System of Equations:

The equations of Section (2.1) will be reduced for one-dimensional steady flow in Radiation-Magnetogasdynamics.

Equations of State and Continuity:

$$p = \rho RT, \; \rho u = \text{constant} = m. \qquad (2.2.1)$$

Equation of Motion: The term F_e in (2.1.3) is the x-component of the electromagnetic force, i.e.,

$$\vec{F}_e = \rho_e \vec{E} + \mu_e(\vec{J} \times \vec{H}). \qquad (2.2.2)$$

Assuming the plasma as a fully ionized gas in which the excess electric charge is zero (i.e. $\rho_e = 0$). Also in magnetogasdynamics the displacement current (i.e. $\epsilon\vec{E}$) is very small as compared to

curl of the magnetic field. Hence from $(2.1.5)_1$, we get:

$$\vec{J} = \nabla \times \vec{H}. \tag{2.2.3}$$

Therefore, after substituting the value of $\vec{J}$ from (2.2.3) and $\rho_e = 0$ into (2.2.2), we get:

$$\vec{F}_e = \mu_e[(\nabla \times \vec{H}) \times \vec{H}]. \tag{2.2.4}$$

Hence, x component of $\vec{F}_e = -\mu_e H \frac{\partial H}{\partial x}$.

Now (2.1.3) after some simplification is reduced to:

$$mdu + dp_t + \mu_e H\, dH - d(\frac{4}{3} \mu \frac{du}{dx}) = 0. \tag{2.2.5}$$

After integrating (2.2.5), we get:

$$mu + p_t + \frac{\mu_e}{2} H^2 - \frac{4}{3} \mu \frac{du}{dx} = \text{constant} = mc_1. \tag{2.2.6}$$

Equation of the Magnetic Field:

For steady state case $(2.1.5)_2$ is reduced to:

$$\nabla \times \vec{E} = 0. \tag{2.2.7}$$

When excess electric charge is approximately zero (i.e. $\rho_e = 0$), then (2.1.8) is reduced to:

$$\vec{J} = \sigma[\vec{E} + \mu_e(\vec{u} \times \vec{H})]. \tag{2.2.8}$$

Substituting the value of $\vec{E}$ from (2.2.8) into (2.2.7), we get:

$$\nabla \times [\frac{\vec{J}}{\sigma} - \mu_e(\vec{u} \times \vec{H})] = 0. \tag{2.2.9}$$

Now substituting the value of $\vec{J}$ from (2.2.3) into (2.2.9), we get after some simplication the equation of the magnetic field, for one-dimensional steady state case as:

$$d(uH) - d(\nu_h \frac{dH}{dx}) = 0, \tag{2.2.10}$$

where: $\nu_h = \frac{1}{\mu_e \sigma}$, is called magnetic diffusivity.

After integrating (2.2.10), we get:

$$uH - \nu_h \frac{dH}{dx} = \text{constant} = F. \tag{2.2.11}$$

Equation of Energy:

For one-dimensional steady state case (2.1.4) can be simplified as:

$$\frac{d}{dx}(\rho E_t u) = - \frac{d}{dx}(up_t) + \frac{d}{dx}(\frac{4}{3}\mu u \frac{du}{dx}) + \frac{dQ}{dx} + \epsilon_e. \tag{2.2.12}$$

where:
$$\epsilon_e = \vec{E} \cdot \vec{J} = - \mu_e \frac{dH}{dx}[uH - \nu_h \frac{dH}{dx}],$$

$$Q = k\frac{dT}{dx} + D_r \frac{d}{dx}(E_r),$$

$$D_r = \frac{\lambda_r c}{3} = \frac{c}{3K_r \rho},$$

$$\lambda_r = \lambda_{ro}\left(\frac{T_o}{T}\right)^{n_1} \cdot \left(\frac{\rho_o}{\rho}\right)^{n_2},$$

where subscript o indicates conditions at some reference point, exponent n_2 is a positive number while n_1 may be positive at low temperature range and negative at high temperature range. These exponents are determined experimentally.

Now substituting the above expressions in (2.2.12) and after some simplification we get:

$$md\left(C_v T + \frac{u^2}{2} + \frac{E_r}{\rho}\right) + d(up_t) - \frac{4}{3} d\left(\mu\ u \frac{du}{dx}\right)$$

$$- d\left(k \frac{dT}{dx}\right) - d\left(D_r \frac{dE_r}{dx}\right) + F\ \mu_e dH = 0. \qquad (2.2.13)$$

After integrating (2.2.13), we get:

$$m\left(C_v T + \frac{u^2}{2} + \frac{E_r}{\rho}\right) + up_t - \frac{4}{3} \mu\ u \frac{du}{dx} - k \frac{dT}{dx} - D_r \frac{dE_r}{dx} + F\ \mu_e H$$

$$= \text{constant} = mc_2. \qquad (2.2.14)$$

III. FUNDAMENTAL EQUATIONS FOR NORMAL SHOCK

3.1 Generalized Rankine-Hugoniot Relations for an Optically Thick Medium:

The fundamental equations governing the flow field, with no variation in x-direction and separated by a shock wave, are obtained from (2.2.1), (2.2.6), (2.2.11), and (2.2.14). These equations are:

$$p = \rho RT \; ; \; \rho u = \text{constant} = m, \tag{3.1.1}$$

$$mu + p + p_r + p_h = \text{constant} = mc_1, \tag{3.1.2}$$

$$m\left(C_v T + \frac{u^2}{2} + \frac{E_r}{\rho} + \mu_e \frac{H^2}{2\rho}\right) + u(p + p_r + p_h) = \text{constant} = mc_2, \tag{3.1.3}$$

$$uH = \text{constant} = F, \text{ or } vH = \text{constant} = \frac{\theta}{\sqrt{\mu_e}} . \tag{3.1.4}$$

The generalized Rankine-Hugoniot relation [15, 16], as the ratio of velocities, is:

$$\frac{u_2}{u_1} = \frac{1}{2}\left[\frac{\gamma_e - 1}{\gamma_e + 1} + \frac{2\gamma_e(P_e + h_1^2)}{\gamma_e + 1}\right] + \frac{1}{2}\left[\left\{\frac{\gamma_e - 1}{\gamma_e + 1} + \frac{2\gamma_e(P_e + h_1^2)}{\gamma_e + 1}\right\}^2 + 8 \cdot \frac{2-\gamma_e}{\gamma_e + 1} h_1^2\right]^{\frac{1}{2}} . \tag{3.1.5}$$

where: $h_1 = H_1/\left(2mu_1/\mu_e\right)^{\frac{1}{2}}$, $P = \frac{p_1}{\rho_1 u_1^2} = \frac{1}{\gamma M_1^2}$, and the so-called effective values of P and of the ratio of specific heats, as defined in [15], for radiation-magnetogasdynamics are:

$$P_e = (R_{p1} + 1) f(R_{p2}) P, \tag{3.1.6}$$

$$\gamma_e = \frac{4(\gamma-1)R_{p2} + \gamma}{3(\gamma-1)R_{p2} + 1} , \tag{3.1.7}$$

$$f(R_p) = [\frac{u}{u_1} - \frac{(R_p+1)(8R_{p1}+r^2+1)}{(R_{p1}+1)(8R_p+r^2+1)}]/(\frac{u}{u_1} - 1), \qquad (3.1.8)$$

$$R_p = p_r/p = \text{radiation pressure number}, \qquad (3.1.9)$$

$$r^2 = (\gamma+1)/(\gamma-1).$$

The generalized Rankine-Hugoniot relation, as the ratio of pressures, derived from (3.1.1) and (3.1.2), is:

$$\frac{p_2}{p_1} = \frac{1 + R_{p1} + R_{h1} + \gamma M_1^2 (1 - u_2/u_1)}{1 + R_{p2} + R_{h2}}, \qquad (3.1.10)$$

where: $R_h = (\mu_e \frac{H^2}{2})/p = p_h/p$ = magnetic pressure.number.

The generalized Rankine-Hugoniot relation, as the ratio of temperatures, derived from (3.1.1) and (3.1.10), is:

$$\frac{T_2}{T_1} = \frac{u_2}{u_1}\left[\frac{1 + R_{p1} + R_{h2} + \gamma M_1{}^2(1 - u_2/u_1)}{1 + R_{p2} + R_{h2}}\right]. \qquad (3.1.11)$$

The generalized Rankine-Hugoniot relations for a viscous, heat-conducting gas without radiation and magnetic field (17), actually equivalent to isentropic flow, are:

$$\frac{u_2}{u_1} = \frac{1}{\gamma+1} (\gamma-1 + \frac{2}{M_1^2}), \qquad (3.1.12)$$

$$\frac{p_2}{p_1} = \frac{1}{\gamma+1} (2\gamma\, M_1^2 - \gamma+1), \qquad (3.1.13)$$

$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2} (M_1^2 - 1)[\gamma + \frac{1}{M_1^2}]. \qquad (3.1.14)$$

Using (3.1.12), the velocity ratio (3.1.5) can also be written as:

$$\frac{u_2}{u_1} = \frac{\alpha_1}{\gamma+1} \left(\gamma - 1 + \frac{2}{M_1^2}\right), \qquad (3.1.15)$$

where: α_1 = velocity correction factor:

$$\alpha_1 = \frac{\dfrac{\gamma_e - 1}{\gamma_e + 1} + \dfrac{2\gamma_e (P_e + h_1^2)}{\gamma_e + 1} + \left[\left\{\dfrac{\gamma_e - 1}{\gamma_e + 1} + \dfrac{2\gamma_e (P_e + h_1^2)}{\gamma_e + 1}\right\}^2 + 8 \, \dfrac{2 - \gamma_e}{\gamma_e + 1} \, h_1^2\right]^{\frac{1}{2}}}{\dfrac{2}{\gamma+1} \left(\gamma - 1 + \dfrac{2}{M_1^2}\right)} \qquad (3.1.16)$$

Since from $(3.1.1)_2$, $\frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$; hence from (3.1.15) we also get:

$$\frac{\rho_1}{\rho_2} = \frac{\alpha_1}{\gamma+1} \left(\gamma - 1 + \frac{2}{M_1^2}\right). \qquad (3.1.17)$$

Using (3.1.13), the pressure ratio (3.1.10) can also be written as:

$$\frac{p_2}{p_1} = \frac{\alpha_2}{\gamma+1} \left(2\gamma M_1^2 - \gamma + 1\right), \qquad (3.1.18)$$

where: α_2 = pressure correction factor:

$$\alpha_2 = \frac{1 + R_{p1} + R_{h1} + \gamma M_1^2 (1 - u_2/u_1)}{1 + R_{p2} + R_{h2}} \cdot \left(\frac{\gamma + 1}{2\gamma \, M_1^2 - \gamma + 1}\right). \qquad (3.1.19)$$

Using (3.1.14), the temperature ratio (3.1.11) can also be written as:

$$\frac{T_2}{T_1} = \alpha_3 \left\{1 + \frac{2(\gamma-1)}{(\gamma+1)^2} \cdot (M_1^2 - 1) \left[\gamma + \frac{1}{M_1^2}\right]\right\}. \qquad (3.1.20)$$

where: α_3 = temperature correction factor:

$$\alpha_3 = \frac{u_2}{u_1} \left[\frac{1 + R_{p1} + R_{h1} + \gamma M_1^2 (1 - u_2/u_1)}{1 + R_{p2} + R_{h2}}\right] \cdot \frac{1}{1 + \dfrac{2(\gamma-1)}{(\gamma+1)^2} \cdot (M_1^2 - 1) \cdot \left[\gamma + \dfrac{1}{M_1^2}\right]} \cdot \qquad (3.1.21)$$

The correction factors (i.e. α_1, α_2, and α_3) are equal to unity for viscous, heat-conducting gas without radiation and magnetic field, and for isentropic flow.

3.2 Generalized Prandtl Relation:

The generalized Prandtl relation, for two uniform states separated by a shock, is derived from $(3.1.1)_2$ and (3.1.2). The result is:

$$u_1 u_2 = \frac{p_1 - p_2}{\rho_1 - \rho_2} + \frac{p_{r1} - p_{r2}}{\rho_1 - \rho_2} + \frac{p_{h1} - p_{h2}}{\rho_1 - \rho_2} \tag{3.2.1}$$

The Prandtl relation for viscous, heat-conducting gas without radiation and magnetic field is [3]:

$$u_1 u_2 = \frac{p_1 - p_2}{\rho_1 - \rho_2} . \tag{3.2.2}$$

Using (3.2.2), the generalized Prandtl relation (3.2.1) can also be written as:

$$u_1 u_2 = \alpha_4 \left[\frac{p_1 - p_2}{\rho_1 - \rho_2}\right], \tag{3.2.3}$$

where: α_4 = correction factor for Prandtl relation:

$$\alpha_4 = (1 + \frac{p_{r1} - p_{r2}}{p_1 - p_2} + \frac{p_{h1} - p_{h2}}{p_1 - p_2}). \tag{3.2.4}$$

The correction factor, α_4, is equal to unity for viscous, heat-conducting gas without radiation and magnetic field, and for isentropic flow.

3.3 Shock relation in terms of change in internal energy:

The change in internal energy, between two uniform states separated by a shock, is derived from $(3.1.1)_2$, (3.1.2), and (3.1.3). The result is, with $e = c_v T$:

$$e_2-e_1 = (v_1-v_2).\frac{p_1+p_2+p_{r1}+p_{r2}+p_{h1}+p_{h2}}{2} + 3(v_1 p_{r1}-v_2\ p_{r2}) + (v_1 p_{h1}-v_2 p_{h2}). \tag{3.3.1}$$

The change in internal energy for viscous, heat-conducting gas without radiation and magnetic field is [3]:

$$e_2 - e_1 = (v_1 - v_2).\frac{p_1 + p_2}{2} \tag{3.3.2}$$

Using (3.3.2), equation (3.3.1) can also be written as:

$$e_2 - e_1 = \alpha_5 (v_1 - v_2).\frac{p_1 + p_2}{2} \tag{3.3.3}$$

where: α_5 = correction factor for internal energy:

$$\alpha_5 = 1 + \frac{v_1(7p_{r1}+p_{r2}+3p_{h1}+p_{h2}) - v_2(p_{r1}+7p_{r2}+p_{h1}+3p_{h2})}{(v_1 - v_2).(p_1 + p_2)}. \tag{3.3.4}$$

The correction factor, α_5, is equal to unity for viscous, heat-conducting gas without radiation and magnetic field, and for isentropic flow.

3.4 Modified Equations:

The first law of thermodynamics, for the case of radiation-magnetogasdynamics, is modified as [4, 24]:

$$dQ = de^* + p^* dv, \tag{3.4.1}$$

where: $dQ = T\, dS^*.$

e^* = total internal energy per unit mass,

$= c_v T + 3v\, p_r + v p_h,$

p^* = total pressure,

$= p + p_r + p_h.$

S^* = modified entropy.

The generalized Rankine-Hugoniot relation (3.3.1) can be modified as:

$$e_2^* - e_1^* = (v_1 - v_2) \cdot \frac{p_1^* + p_2^*}{2} \; . \qquad (3.4.2)$$

Let us introduce the Hugoniot function:

$$\mathcal{H} = e^* - e_1^* + (v - v_1) \cdot \frac{p_1^* + p^*}{2} \; . \qquad (3.4.3)$$

For $p = p_2$, $v = v_2$, (3.4.3) reduces to the modified Rankine-Hugoniot relation, with $\mathcal{H} = 0$, across the normal shock as shown in (3.4.2).

3.5 Auxiliary Inequalities:

Lemma 1: The value of γ_e is everywhere, greater than or equal to $\frac{4}{3}$ and less than or equal to γ.

Proof: The proof is divided into two parts.

1. (i) $\frac{4}{3} \le \gamma_e$ everywhere.

From (3.1.7) we have:

$$\gamma_e = \frac{4(\gamma-1)R_{p2} + \gamma}{3(\gamma-1)R_{p2} + 1} \, , \text{ or } \gamma_e - \frac{4}{3} = \frac{3\gamma-4}{3\{3(\gamma-1)R_{p2} + 1\}} \; . \qquad (3.5.1)$$

According to [15], we have for all gases:

$$3\gamma - 4 \geq 0 \quad \text{and} \quad \{3(\gamma-1)R_{p2} + 1\} > 0, \; R_{p2} \geq 0. \tag{3.5.2}$$

The case $3\gamma - 4 = 0$ when T approaches infinity. Therefore from $(3.5.1)_2$ and (3.5.2), we get the following inequality:

$$\gamma_e - \frac{4}{3} \geq 0, \text{ or } \frac{4}{3} \leq \gamma_e, \text{ Q.E.D.} \tag{3.5.3}$$

1. (ii) $\gamma_e \leq \gamma$, everywhere.

From $(3.5.1)_1$ we can write:

$$\gamma - \gamma_e = \frac{(\gamma-1)(3\gamma-4)R_{p2}}{3(\gamma-1)R_{p2} + 1} . \tag{3.5.4}$$

Since $(\gamma-1) > 0$ everywhere, hence, from (3.5.2) and (3.5.4) we get:

$$\gamma - \gamma_e \geq 0, \text{ or } \gamma_e \leq \gamma, \text{ Q.E.D.} \tag{3.5.5}$$

Now combining the two inequalities, $(3.5.3)_2$ and $(3.5.5)_2$, we get:

$$\frac{4}{3} \leq \gamma_e \leq \gamma, \text{ Q.E.D.} \tag{3.5.6}$$

<u>Lemma 2</u>: $\frac{1}{7} \leq \frac{\gamma_e - 1}{\gamma_e + 1} \leq \frac{\gamma-1}{\gamma+1}$, everywhere.

<u>Proof</u>: The proof is divided into two parts.

2. (i) $\frac{1}{7} \leq \frac{\gamma_e - 1}{\gamma_e + 1}$, everywhere.

From $(3.5.3)_2$, we have:

$$\frac{4}{3} \leq \gamma_e, \text{ or } \frac{4}{3}\gamma_e + \frac{4}{3} \leq \frac{4}{3}\gamma_e + \gamma_e, \text{ or } \frac{4}{3}(\gamma_e+1) \leq \frac{7}{3}\gamma_e,$$

or $\frac{4}{7}(\gamma_e+1) \le \gamma_e$, or $\frac{4}{7}(\gamma_e+1) - 1 \le \gamma_e - 1$, or $\frac{4}{7} - \frac{1}{\gamma_e+1} \le \frac{\gamma_e-1}{\gamma_e+1}$,

$$\text{or} \quad \frac{1}{7} + \left(\frac{3}{7} - \frac{1}{\gamma_e+1}\right) \le \frac{\gamma_e-1}{\gamma_e+1} . \tag{3.5.7}$$

From $(3.5.3)_1$, we have:

$$\gamma_e - \frac{4}{3} \ge 0, \text{ or } (\gamma_e+1) - \left(\frac{4}{3} + 1\right) \ge 0,$$

$$\text{or} \quad (\gamma_e+1) - \frac{7}{3} \ge 0, \text{ or } \frac{3}{7} - \frac{1}{\gamma_e+1} \ge 0. \tag{3.5.8}$$

Therefore, from (3.5.7) and $(3.5.8)_2$, we get:

$$\frac{1}{7} \le \frac{\gamma_e-1}{\gamma_e+1}, \text{ Q.E.D.} \tag{3.5.9}$$

2. (ii) $\frac{\gamma_e-1}{\gamma_e+1} \le \frac{\gamma-1}{\gamma+1}$, everywhere.

From $(3.5.5)_2$ we have:

$$\gamma_e \le \gamma, \text{ or } \gamma_e + 1 \le \gamma + 1,$$

$$\text{or} \quad \frac{1}{\gamma+1} \le \frac{1}{\gamma_e+1}, \text{ or } 0 \le \left[\frac{1}{\gamma_e+1} - \frac{1}{\gamma+1}\right]. \tag{3.5.10}$$

Again from $(3.5.5)_2$, we have:

$$\gamma_e \le \gamma, \text{ or } \gamma_e + \gamma_e\gamma \le \gamma + \gamma_e\gamma,$$

$$\text{or} \quad \gamma_e(\gamma+1) \le \gamma(\gamma_e+1), \text{ or } \frac{\gamma_e}{\gamma_e+1} \le \frac{\gamma}{\gamma+1},$$

or $\frac{\gamma_e}{\gamma_e+1} - \frac{1}{\gamma+1} \le \frac{\gamma}{\gamma+1} - \frac{1}{\gamma+1}$, or $\frac{\gamma_e}{\gamma_e+1} - \frac{1}{\gamma_e+1} + \left[\frac{1}{\gamma_e+1} - \frac{1}{\gamma+1}\right] \le \frac{\gamma-1}{\gamma+1}$,

$$\text{or} \quad \frac{\gamma_e-1}{\gamma_e+1} + \left[\frac{1}{\gamma_e+1} - \frac{1}{\gamma+1}\right] \le \frac{\gamma-1}{\gamma+1}. \tag{3.5.11}$$

Therefore, from $(3.5.10)_2$ and (3.5.11) we get:

$$\frac{\gamma_e-1}{\gamma_e+1} \le \frac{\gamma-1}{\gamma+1}, \quad \text{Q.E.D.} \tag{3.5.12}$$

Now combining the two inequalities (3.5.9) and (3.5.12), we get:

$$\frac{1}{7} \le \frac{\gamma_e-1}{\gamma_e+1} \le \frac{\gamma-1}{\gamma+1}, \quad \text{Q.E.D.} \tag{3.5.13}$$

Lemma 3: The ratio of the specific volumes, $\frac{v_1}{v_2}$, is less than 7 everywhere.

Proof: From (3.1.5) we get:

$$\frac{u_2}{u_1} > \frac{\gamma_e-1}{\gamma_e+1}, \tag{3.5.14}$$

since all other terms in (3.1.5) are greater than zero. Then from continuity equation, $\frac{u_2}{u_1} = \frac{v_2}{v_1}$, and (3.5.14) we get:

$$\frac{v_2}{v_1} > \frac{\gamma_e-1}{\gamma_e+1}, \text{ or } \frac{v_2}{v_1} - \frac{1}{7} > \frac{\gamma_e-1}{\gamma_e+1} - \frac{1}{7}. \tag{3.5.15}$$

From (3.5.13), $\frac{\gamma_e-1}{\gamma_e+1} - \frac{1}{7} \ge 0$, hence from (3.5.15) we get:

$$\frac{v_2}{v_1} - \frac{1}{7} > 0, \text{ or } \quad 7 - \frac{v_1}{v_2} > 0. \quad \text{Q.E.D.} \tag{3.5.16}$$

Lemma 4: $d^2p^*|_1 > 0$ at the point of state 1.

Proof: By definition, $p^* = p + p_r + p_h$,

$$\text{or} \qquad p^* = p + \frac{a_r}{3R^4}(pv)^4 + \frac{\theta^2}{2v^2}, \tag{3.5.17}$$

Thus, $p^* = g(v,p)$, where $p = p(v,S^*)$.

Therefore, $p^* = g(v,p(v,S^*))$,

$$\text{or} \qquad p^* = G(v,S^*), \tag{3.5.18}$$

where, v and S^* are assumed to be independent agruments, so that:

$$dp^* = G_{,v}dv + G_{,S*}dS^* \equiv p^*_{,v}dv + p^*_{,S*}dS^*, \tag{3.5.19}$$

$$\text{and} \qquad d^2p^* = (p^*_{,vv}dv + p^*_{,S*v}dS^*)dv + \frac{\partial}{\partial S^*}(dp^*)dS^*. \tag{3.5.20}$$

At the point of state 1, we have from (5.2.4), (see below), $dS^*_1 = 0$, thus:

$$d^2p^*_1 = \{p^*_{,vv}(dv)^2\}|_1. \tag{3.5.21}$$

Differentiating (3.5.17) twice with respect to v, we obtain:

$$p^{*}_{,vv}\Big|_1 = \{\frac{3\theta^2}{v^4} + \frac{4a_r}{R^4}(pv)^2(p + vp_{,v})^2\}\Big|_1$$

$$+ \{\frac{4}{3}\frac{a_r}{R^4}(pv)^3.(2p_{,v} + vp_{,vv})\}\Big|_1 + \{p_{,vv}\}\Big|_1. \qquad (3.5.22)$$

$$\text{Let} \quad p^{*}_{,vv}\Big|_1 = \{I\} + \{II\} + \{III\}. \qquad (3.5.23)$$

(i) $\{I\} > 0$ everywhere, since each term of $\{I\}$ is positive everywhere.

(ii) From (4.2.5) and (4.2.10), we get: respectively,

$$p_{,v}\Big|_1 = -\frac{\frac{\gamma p_1}{\gamma-1} + 16\, p_{r1}}{v_1[\frac{1}{\gamma-1} + 12.\frac{p_{r1}}{p_1}]},$$

$$p_{,vv}\Big|_1 = -\frac{\frac{64\, p_{r1}}{v_1} + (r^2 + \frac{124\, p_{r1}}{p_1})p_{,v}\Big|_1 + \frac{36\, p_{r1}}{p_1^2}v_1(p_{,v}\Big|_1)^2}{v_1[\frac{1}{\gamma-1} + \frac{12\, p_{r1}}{p_1}]},$$

$$\text{or} \quad p_{,vv}\Big|_1 = \frac{\gamma.\frac{152-24\gamma}{(\gamma-1)^2}.p_{r1} + 16.\frac{112+33\gamma}{(\gamma-1)}.\frac{p_{r1}^2}{p_1} + \frac{\gamma(\gamma+1)p_1}{(\gamma-1)^3} + 12288.\frac{p_{r1}^3}{p_1^2}}{v_1^2[\frac{1}{\gamma-1} + \frac{12\, p_{r1}}{p_1}]^3} \qquad (3.5.24)$$

Therefore,

$$(2p_{,v}+vp_{,vv})\big|_1 = \frac{-2\{(\frac{\gamma p_1}{\gamma-1} + 16\, p_{r1})(\frac{1}{\gamma-1} + \frac{12\, p_{r1}}{p_1})^2\} + \gamma.\frac{152-24\gamma}{(\gamma-1)^2}.p_{r1} + 16.\frac{112+33\gamma}{(\gamma-1)}.\frac{p_{r1}^2}{p_1} + \frac{\gamma(\gamma+1)p_1}{(\gamma-1)^3} + 12288.\frac{p_{r1}^3}{p_1^2}}{v_1[\frac{1}{\gamma-1} + 12.\frac{p_{r1}}{p_1}]^3}$$

or

$$(2p_{,v}+vp_{,vv})\big|_1 = \frac{\frac{\gamma}{(\gamma-1)^2}.p_1 + 8.\frac{3\gamma(3-\gamma)+4(\gamma-1)}{(\gamma-1)^2}.p_{r1}+16.\frac{64+15\gamma}{(\gamma-1)}.\frac{p_{r1}^2}{p_1} + 7680.\frac{p_{r1}^3}{p_1^2}}{v_1[\frac{1}{\gamma-1} + 12.\frac{p_{r1}}{p_1}]^3} \tag{3.5.25}$$

From (3.5.25) for $1 < \gamma < 2$, we get:

$$(2p_{,v} + vp_{,vv})\big|_1 > 0,$$

since all other functions on the right hand side of (3.5.25) are greater than zero everywhere.

Therefore, $\{II\} > 0$ at the point of state 1, since all other quantities in $\{II\}$ is positive everywhere.

(iii) From (3.5.24), $\{III\} > 0$ at the point of state 1 for $1 < \gamma < 2$.

Hence, from (3.5.23), $p^*_{,vv}|_1 > 0$ and then from (3.5.21), we get:

$$d^2 p_1^* > 0, \quad \text{Q.E.D.} \tag{3.5.26}$$

3.6 Table of Stepwise Shock Relations

$$\frac{u_2}{u_1} = \frac{\alpha_1}{\gamma+1}\left(\gamma-1+\frac{2}{M_1^2}\right), \quad \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}.$$

$$\frac{p_2}{p_1} = \frac{\alpha_2}{\gamma+1}\left(2\gamma M_1^2 - \gamma+1\right).$$

$$\frac{T_2}{T_1} = \alpha_3\left\{1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\cdot(M_1^2-1).\left[\gamma+\frac{1}{M_1^2}\right]\right\}.$$

$$u_1u_2 = \alpha_4\left(\frac{p_1-p_2}{\rho_1-\rho_2}\right).$$

$$e_2-e_1 = \alpha_5\,(v_1-v_2)\,\frac{p_1+p_2}{2}.$$

where:

$$\alpha_1 = \left[\frac{\gamma_e-1}{\gamma_e+1} + \frac{2\gamma_e(P_e+h_1^2)}{\gamma_e+1}\right] + \left\{\left[\frac{\gamma_e-1}{\gamma_e+1} + \frac{2\gamma_e(P_e+h_1^2)}{\gamma_e+1}\right]^2 + 8\,\frac{2-\gamma_e}{\gamma_e+1}\,h_1^2\right\}^{\frac{1}{2}} \Big/ \left[\frac{2}{\gamma+1}\left(\gamma-1+\frac{2}{M_1^2}\right)\right],$$

$$\alpha_2 = 1 + R_{p1} + R_{h1} + \gamma M_1^2\left(1-\frac{u_2}{u_1}\right)\left[\frac{\gamma+1}{2\gamma M_1^2-\gamma+1}\right]\Big/(1+R_{p2}+R_{h2}),$$

$$\alpha_3 = \frac{u_2}{u_1}\left[\frac{1+R_{p1}+R_{h1}+\gamma M_1^2\left(1-\frac{u_2}{u_1}\right)}{1+R_{p2}+R_{h2}}\right]\Big/\left\{1+\frac{2(\gamma-1)}{(\gamma+1)^2}(M_1^2-1)\left[\gamma+\frac{1}{M_1^2}\right]\right\},$$

$$\alpha_4 = 1 + \frac{p_{r1}-p_{r2}}{p_1-p_2} + \frac{p_{h1}-p_{h2}}{p_1-p_2},$$

$$\alpha_5 = 1 + \frac{v_1(7p_{r1}+p_{r2}+3p_{h1}+p_{h2}) - v_2(p_{r1}+7p_{r2}+p_{h1}+3p_{h2})}{(v_1-v_2)\cdot(p_1+p_2)}.$$

IV. GENERALIZED HUGONIOT FUNCTION

4.1 Generalized Hugoniot Function:

Transferring all quantities from the right hand side to the left hand side in the generalized Rankine-Hugoniot relation (3.3.1) and abandoning the subscript 2 in this equation, we obtain a function defined as the generalized Hugoniot function. This function is, with $e = \frac{pv}{\gamma - 1}$:

$$\mathcal{H} = \frac{pv}{\gamma-1} - \frac{p_1 v_1}{\gamma-1} + \frac{1}{2}\,(v - v_1)\;.\;\{p + p_1 + \frac{a_r}{3R^4}\,[(pv)^4 + (p_1 v_1)^4] + \frac{\theta^2}{2}\,[\frac{1}{v^2} + \frac{1}{v_1^2}]\} + \frac{a_r}{R^4}\,\{p^4 v^5 - p_1^4 v_1^5\} + \frac{\theta^2}{2}\,\{\frac{1}{v} - \frac{1}{v_1}\}. \qquad (4.1.1)$$

The right side of (4.1.1) is a function of p and v only, the variables at the state 1 being fixed. Therefore the Hugoniot function, $\mathcal{H} = \mathcal{H}(v,p)$. We get the Hugoniot relation across a normal shock by substituting $\mathcal{H}(v,p) = 0$ in (4.1.1), and $p \equiv p_2$, and $v \equiv v_2$.

4.2 Hugoniot Curve:

The graph of the Hugoniot relation, $\mathcal{H}(p,v) = 0$, in the (p,v)-plane is called the Hugoniot curve. The general shape of this curve can be determined if the signs of first and second derivatives of p with respect to v are known. With $d\mathcal{H} = 0$ along this curve, we have:

$$d\mathcal{H} = \mathcal{H}_{,v}\,dv + \mathcal{H}_{,p}\,dp = 0, \qquad (4.2.1)$$

thus,
$$\frac{dp}{dv} = - \frac{\mathcal{H}_{,v}}{\mathcal{H}_{,p}} . \qquad (4.2.2)$$

From (4.1.1), we have:

$$\mathcal{H}_{,v} = \{\frac{1}{2} pr^2 + \frac{1}{2}(p_1 + p_{r1})\} + \{\frac{1}{2} p_r(35 - 4 \frac{v_1}{v})\} + \{\frac{1}{2}(p_{h1}-p_h)-p_h(1- \frac{v_1}{v})\}, \qquad (4.2.3)$$

$$\mathcal{H}_{,p} = \frac{1}{2} v \{r^2 + 28 R_p - \frac{v_1}{v} (1 + 4 R_p)\}. \qquad (4.2.4)$$

Therefore:

$$\frac{dp}{dv} = - \frac{\{pr^2 + (p_1+p_{r1})\} + \{p_r(35-4 \frac{v_1}{v})\} + \{(p_{h1}-p_h)-2p_h(1- \frac{v_1}{v})\}}{v r^2 + 28 R_p - \frac{v_1}{v} (1 + 4 R_p)\}} . \qquad (4.2.5)$$

Equation (4.2.5) gives the slope at a point of the Hugoniot curve in (p,v)-plane.

<u>Theorem 4.2</u>: The derivative $\frac{dp}{dv}$ is everywhere less than zero along the entire Hugoniot curve.

<u>Proof</u>: The proof is divided into two parts.

1. (i) $\mathcal{H}_{,v} > 0$ along the entire Hugoniot curve.

From (4.2.3), let:

$$\mathcal{H}_{,v} = \{I\} + \{II\} + \{III\}. \qquad (4.2.6)$$

(i.a) $\{I\} > 0$ everywhere, since $r^2 > 0$ for $\gamma > 1$ and all the functions in $\{I\}$ are positive everywhere.

(i.b) $\{II\} = \frac{1}{2} p_r[35 - \frac{4v_1}{v}] = \frac{1}{2} p_r[(35 - 28) + 4(7 - \frac{v_1}{v})]$,

or $\{II\} = \frac{1}{2} p_r[7 + 4(7 - \frac{v_1}{v})]$.

Therefore, $\{II\} > 0$ everywhere from (3.5.16).

(i. c) $\{III\} = \{\frac{\theta^2}{4v_1^2} - \frac{\theta^2}{4v^2} - \frac{\theta^2}{2v^2} (1 - \frac{v_1}{v})\}$,

$$\text{or} \quad \{III\} = \frac{\theta^2 (v_1 - v)^2 (2v_1 + v)}{4\, v^3\, v_1^2} .$$

Therefore, $\{III\} > 0$ everywhere, since v and v_1 are always positive. Thus, from (4.2.6), we get:

$$\mathcal{N}_{,v} > 0,$$

along the entire Hugoniot curve. Q.E.D.

2. (ii) $\mathcal{N}_{,p} > 0$ along the entire Hugoniot curve.

From (4.2.4), let:

$$\mathcal{N}_{,p} = \frac{1}{2} v[\{(1+4R_p)(\frac{\gamma_e+1}{\gamma_e-1} - \frac{v_1}{v})\} + \{\frac{\gamma+1}{\gamma-1} + 28\ R_p - \frac{\gamma_e+1}{\gamma_e-1}(1+4R_p)\}],$$

let
$$\mathcal{N}_{,p} = \frac{1}{2} v[\{I\} + \{II\}]. \qquad (4.2.7)$$

(ii.a) $\{I\} > 0$ everywhere from $(3.5.15)_1$.

(ii.b) From (3.1.7), $\frac{\gamma_e+1}{\gamma_e-1} = \frac{7(\gamma-1)R_p + \gamma+1}{(\gamma-1)(R_p+1)}$. (4.2.8)

From (4.2.8), we have:

$$\{II\} = \frac{6R_p(3\gamma-4)}{(\gamma-1)(R_p+1)} .$$

Therefore, $\{II\} \geq 0$ everywhere, since $\gamma \geq \frac{4}{3}$ from (3.5.6). Thus, from (4.2.7) we get: $\mathcal{N}_p > 0$ along the entire Hugoniot curve. Q.E.D.

Since $\mathcal{N}_{,v} > 0$ and $\mathcal{N}_{,p} > 0$ everywhere along the Hugoniot curve, hence, from (4.2.2) $\frac{dp}{dv} < 0$ along the entire Hugoniot curve. Q.E.D.

With $d^2\mathcal{N} = 0$ and $dp = \frac{dp}{dv} dv$ along the Hugoniot curve, we can write from (4.2.1) after differentiation:

$$d^2\mathcal{H} = [\mathcal{H}_{,vv} + 2\mathcal{H}_{,pv}(\frac{dp}{dv}) + \mathcal{H}_{pp}(\frac{dp}{dv})^2 + \mathcal{H}_{,p}(\frac{d^2p}{dv^2})](dv)^2 = 0. \qquad (4.2.9)$$

Since $dv \neq 0$, therefore:

$$\mathcal{H}_{,vv} + 2\mathcal{H}_{,pv}(\frac{dp}{dv}) + \mathcal{H}_{,pp}(\frac{dp}{dv})^2 + \mathcal{H}_{,p}(\frac{d^2p}{dv^2}) = 0,$$

$$\text{or} \quad \frac{d^2p}{dv^2} = -\left[\frac{\mathcal{H}_{,vv} + 2\mathcal{H}_{,pv}(\frac{dp}{dv}) + \mathcal{H}_{,pp}(\frac{dp}{dv})^2}{\mathcal{H}_{,p}}\right], \qquad (4.2.10)$$

Substituting the value of $\frac{dp}{dv}$ from (4.2.2), we get:

$$\frac{d^2p}{dv^2} = \frac{2\mathcal{H}_{,pv}\mathcal{H}_{,p}\mathcal{H}_{,v} - \mathcal{H}_{,vv}(\mathcal{H}_{,p})^2 - \mathcal{H}_{,pp}(\mathcal{H}_{,v})^2}{(\mathcal{H}_{,p})^3}. \qquad (4.2.11)$$

Differentiating (4.2.3) with respect to v, we get:

$$\mathcal{H}_{,vv} = \frac{2p_r}{v^2}(35v - 3v_1) + \frac{3p_h}{v^2}(v - v_1). \qquad (4.2.12)$$

Differentiating (4.2.4) with respect to v and p, we get:

$$\mathcal{H}_{,pv} = \frac{r^2}{2} + \frac{2p_r}{pv}(35v - 4v_1). \qquad (4.2.13)$$

$$\mathcal{H}_{,pp} = \frac{6p_r}{p^2}(7v - v_1). \qquad (4.2.14)$$

Equation (4.2.11) gives the second derivative of p with respect to v at a point of the Hugoniot curve in (p,v)-plane. The second derivative is positive everywhere along the Hugoniot curve. To verify this statement the positive real roots, of (4.1.1) with $\mathcal{H}(p,v) = 0$, are determined by programming this Hugoniot relation on CDC 3600 computer [see Appendix A].Then corresponding to these roots the values of $\frac{d^2p}{dv^2}$, are calculated by programming (4.2.11) on CDC 3600 computer which gives $\frac{d^2p}{dv^2}$ always positive for each and every root of the Hugoniot relation. Hence, it can be inferred that $\frac{d^2p}{dv^2}$ is positive everywhere along the Hugoniot curve.

Since $\frac{dp}{dv} < 0$ and $\frac{d^2p}{dv^2} > 0$ along the entire Hugoniot curve, hence, the shape of the Hugoniot curve is convex downwards in (p,v)-plane.

V. THEOREM 1

5.1 The Present Formulation of Theorem 1:

Courant and Friedrichs [3] have proven a series of theorems referring to the mathematical formulation of the description of a normal shock in an isentropic flow. We quote below the first of these theorems (denoted by letters C.F.):

Theorem 1 [C.F.]: "The increase of entropy across a [normal] shock front is of the third order in the shock strength."

Here the shock strength refers to any of the differences $p_2 - p_1$, $|v_2 - v_1|$, or $|u_2 - u_1|$.

5.2 Theorem 1 in Radiation-Magnetogasdynamics:

We generalized the theorem and proof by [C.F.] to the case of radiation-magnetogasdynamics.

Theorem 1: "The increase of modified entropy across a normal shock front in radiation-magnetogasdynamics is of the third order in the shock strength."

Proof: The proof is a straightforward one. Along the Hugoniot curve $d\mathcal{H} = 0$, (4.2.1), hence from (3.4.3), we get by differentiation:

$$2d\mathcal{H} = 2de^{*} + (p^{*} + p_1^{*})dv + (v - v_1)dp^{*} = 0. \qquad (5.2.1)$$

From the modified first law of thermodynamics (3.4.1), we get:

$$de^{*} = TdS^{*} - p^{*}dv. \qquad (5.2.2)$$

Substituting the value of de^{*} from (5.2.2) into (5.2.1), we get:

$$2TdS^* + (p_1^* - p^*)dv + (v - v_1)dp^* = 0. \qquad (5.2.3)$$

At the point of state 1, we have:

$$p^* = p_1^*, \; v = v_1,$$

and thus from (5.2.3) with $T_1 \neq 0$, we get:

$$dS_1^* = 0. \qquad (5.2.4)$$

Differentiating (5.2.3) again along the Hugoniot curve and considering v as the independent variable, we get:

$$2d(TdS^*) + (v - v_1)d^2p^* = 0. \qquad (5.2.5)$$

Therefore at the point of state 1, we have from (5.2.5):

$$d(TdS^*)|_1 = 0, \text{ or } (Td^2S^* + dTdS^*)|_1 = 0,$$

But $dS_1^* = 0$ from (5.2.4), hence:

$$d^2S_1^* = 0. \qquad (5.2.6)$$

Differentiating (5.2.5) again along the Hugoniot curve and considering v as the independent variable, we get:

$$2d^2(TdS^*) + dv\, d^2p^* + (v - v_1)d^3p^* = 0. \qquad (5.2.7)$$

Therefore at the point of state 1, we have from (5.2.7):

$$2d^2(TdS^*)|_1 = -(dv\, d^2p^*)|_1,$$

$$\text{or} \quad 2(Td^3S^* + d^2TdS^* + 2dT\, d^2S^*)|_1 = -(dv\, d^2p^*)|_1,$$

But $dS_1^* = 0$ and $d^2S_1^* = 0$ from (5.2.4) and (5.2.6) respectively, hence:

$$d(Td^3S^*)\big|_1 = -(dv\ d^2p^*)\big|_1. \tag{5.2.8}$$

Therefore from (5.2.8), we get for $T_1 > 0$:

$$d^3S_1^* > 0, \tag{5.2.9}$$

when $dv_1 < 0$, since $d^2p_1^* > 0$ from (3.5.26). Hence from (5.2.9) the increase of the modified entropy is exactly of the third order in the shock strength. Q.E.D.

VI. THEOREM 2

6.1 The Present Formulation of Theorem 2:

[C.F.] have proven the following theorem 2 in an isentropic flow:

Theorem 2 [C.F.]: "The pressure rise across a [normal] shock front agrees with the pressure rise in the adiabatic [reversible] change up to terms of the second order in the shock strength." Also, [C.F.] had shown that geometrically the Hugoniot curve and the adiabatic [reversible] curve, passing through the point of state 1, have a contact of second order at this point. It is assumed here that the initial state and one quantity (say specific volume) in the final state, i.e. at the end of the shock process, are the same for both, the isentropic flow and Hugoniot curve (process). This assumption implies that the both curves, isentropic and Hugoniot, pass through the point 1, but do not meet at the final state.

6.2 Theorem 2 in Radiation-Magnetogasdynamics:

We generalize the theorem and proof by [C.F.] to the case of radiation-magnetogasdynamics.

Theorem 2: "The pressure rise across a normal shock front in radiation-magnetogasdynamics is not equal to the pressure rise in the reversible adiabatic change."

Geometrically, the Hugoniot curve and the reversible adiabatic curve, passing through the point of state 1, intersect each other at this point. Additionally, the slope of the adiabatic curve at 1 is steeper than the slope of the Hugoniot curve at the same point.

<u>Proof</u>: The proof is divided into two parts.

1. The Hugoniot curve intersects the adiabatic curve at the point of state 1.

We obtain the slope of the Hugoniot curve at the point of state 1, from (4.2.5) after substituting $p = p_1$, $v = v_1$, $p_r = p_{r1}$, and $p_h = p_{h1}$:

$$\frac{dp}{dv}\Big|_{H1} = \frac{p_1(r^2+1) + 32\, p_{r1}}{v_1\{(r^2-1) + 24\, R_{p1}\}} ,$$

$$\text{or}\quad \frac{dp}{dv}\Big|_{H1} = -\frac{\gamma p_1^2 + 16(\gamma-1)p_1 p_{r1}}{v_1\{p_1 + 12(\gamma-1)p_{r1}\}} . \tag{6.2.1}$$

The equation of the reversible adiabatic curve, passing through the point of state 1, is:

$$pv^{\gamma} = p_1 v_1^{\gamma}. \tag{6.2.2}$$

We obtain the slope of the adiabatic curve at the point of state 1 by differentiating (6.2.2) with respect to v and then substituting $p = p_1$ and $v = v_1$. The result is:

$$\frac{dp}{dv}\Big|_{A1} = -\frac{\gamma p_1}{v_1} . \tag{6.2.3}$$

Since the slope of the Hugoniot curve (6.2.1) is not equal to the slope of the adiabatic curve (6.2.3) and the point of state 1 is

common in both curves, hence the Hugoniot curve intersects the adiabatic curve at the point of state 1. Thus the pressure rise across a normal shock front, in radiation-magnetogasdynamics, is not equal to the pressure rise in the reversible adiabatic change at all. Q.E.D.

2. The slope of the reversible adiabatic curve is steeper than the slope of the Hugoniot curve at the point of state 1. i.e.,

$$\frac{dp}{dv}\Big|_{H1} - \frac{dp}{dv}\Big|_{A1} > 0. \tag{6.2.4}$$

From (6.2.1) and (6.2.3), we get:

$$\frac{dp}{dv}\Big|_{H1} - \frac{dp}{dv}\Big|_{A1} = - \frac{\gamma p_1^2 + 16(\gamma-1)p_1 p_{r1}}{v_1\{p_1 + 12(\gamma-1)p_{r1}\}} + \frac{\gamma p_1}{v_1},$$

$$\text{or} \quad \frac{dp}{dv}\Big|_{H1} - \frac{dp}{dv}\Big|_{A1} = \frac{4(\gamma-1)(3\gamma-4)p_1 p_{r1}}{v_1\{p_1 + 12(\gamma-1)p_{r1}\}} = \frac{N}{D}. \tag{6.2.5}$$

2. (i) $D > 0$ for $\gamma > 1$.

2. (ii) $N > 0$, since $\gamma > \frac{4}{3}$ from (3.5.2). Thus, from (6.2.5), we get:

$$\frac{dp}{dv}\Big|_{H1} - \frac{dp}{dv}\Big|_{A1} > 0. \quad \text{Q.E.D.} \tag{6.2.6}$$

VII. THEOREM 3

7.1 The Present Formulation of Theorem 3:

[C.F.] have proven the following theorem 3 in an isentropic flow:

Theorem 3 [C.F.]: "Along the whole Hugoniot curve the entropy increases with decreasing specific volume."

7.2 Theorem 3 in Radiation-Magnetogasdynamics:

We generalize the theorem and proof by [C.F.] to the case of radiation-magnetogasdynamics.

Theorem 3: "Along the whole hugoniot curve the entropy, S^*, increases with decreasing specific volume in radiation-magnetogasdynamics."

Proof: The generalized equation of state can be written as:

$$p^* = p^*(S^*,v). \tag{7.2.1}$$

Equation (7.2.1) implies that, $S^* = S^*(p^*,v)$. Therefore:

$$S^* = S^*(p^*(S^*,v),v). \tag{7.2.2}$$

Assuming S^* and v as independent arguments and differentiating (7.2.1) and (7.2.2), we obtain:

$$dp^* = p^*_{,v}dv + p^*_{,S^*}dS^*, \tag{7.2.3}$$

,

$$dS^* = S^*_{,p^*}dp^* + S^*_{,v}dv. \tag{7.2.4}$$

Substituting the value of dp^* from (7.2.3) into (7.2.4), we get:

$$dS^* = S^*_{,p*}p^*_{,S*}dS^* + (S^*_{,p*}p^*_{,v} + S^*_{,v})dv. \qquad (7.2.5)$$

Since S^* and v are assumed to be independent arguments, therefore from (7.2.5) we get:

$$S^*_{,p*}p^*_{,S*} = 1, \qquad (7.2.6)$$

$$S^*_{,p*}p^*_{,v} + S^*_{,v} = 0. \qquad (7.2.7)$$

From (3.4.1) and (3.4.3) with $\mathcal{K} = 0$, we obtain:

$$TdS^* = \frac{1}{2}\, d\, \{(v_1 - v).(p_1^* + p^*)\} + p^* dv,$$

$$\text{or} \quad dS^* = \frac{v_1 - v}{2T}\, dp^* + \frac{p^* - p_1^*}{2T}\, dv. \qquad (7.2.8)$$

Since $v_1 > v$ and $p^* > p_1^*$, which can be obtained from (3.1.10), everywhere along the Hugoniot curve; thus along this curve, we get by comparing (7.2.4) and (7.2.8):

$$S^*_{,p*} > 0; \; S^*_{,v} > 0. \qquad (7.2.9)$$

Now from (7.2.6) and $(7.2.9)_1$, we get:

$$p^*_{,S*} > 0. \qquad (7.2.10)$$

From (7.2.7), using (7.2.9), we get:

$$p^*_{,v} = -\frac{S^*_{,v}}{S^*_{,p*}} < 0. \qquad (7.2.11)$$

From (3.5.22), we have:

$$p^*_{,vv} = \frac{3\theta^2}{v^4} + \frac{4a_r}{R^4}(pv)^2[(p + vp_{,v})^2 + \frac{pv}{3}(2p_{,v} + vp_{,vv})] + p_{,vv}. \qquad (7.2.12)$$

Equation (7.2.12) is put on the CDC 3600 computer [see Appendix A] and the values of $p^*_{,vv}$ are evaluated at the number of points along the Hugoniot curve. The result gives $p^*_{,vv}$ always positive for each and every calculated points of the Hugoniot curve, thus:

$$p^*_{,vv} > 0. \tag{7.2.13}$$

Differentiating (7.2.11) with respect to v, we get:

$$dp^*_{,v} = -\left[\frac{S^*_{,vv}S^*_{,p^*}-S^*_{,v}S^*_{,p^*v}}{S^{*\,2}_{,p^*}} + \left\{\frac{S^*_{,p^*v}S^*_{,p^*}-S^*_{,v}S^*_{,p^*p^*}}{S^{*\,2}_{,p^*}}\right\}p^*_{,v}\right]. \tag{7.2.14}$$

Substituting the value of $p^*_{,v}$ from (7.2.11) into (7.2.14), we obtain:

$$p^*_{,vv} = -\left[\frac{S^*_{,vv}S^{*\,2}_{,p^*}-2S^*_{,p^*v}S^*_{,p^*}S^*_{,v}+S^*_{,p^*p^*}S^{*2}_{,v}}{S^{*\,3}_{,p^*}}\right]. \tag{7.2.15}$$

But $p^*_{,vv} > 0$ and $S^*_{,p^*} > 0$ from (7.2.13) and $(7.2.9)_1$ respectively, thus from (7.2.15), we get:

$$S^*_{,vv}S^{*\,2}_{,p^*} - 2S^*_{,p^*v}S^*_{,p^*}S^*_{,v} + S^*_{,p^*p^*}S^{*2}_{,v} < 0. \tag{7.2.16}$$

The increasing character of the entropy, S^*, along the Hugoniot curve, is proven below by showing that $dS^* > 0$, along this curve except at the point of state 1 where $dS^*|_1 = 0$.

The Hugoniot curve in (p^*,v)-plane, $\mathcal{H}(p^*,v) = 0$, is also convex downwards from (7.2.11) and (7.2.13).

Let a ray $\mathcal{R}$, in the (p^*, v)-plane, is represented in the parametric form as:

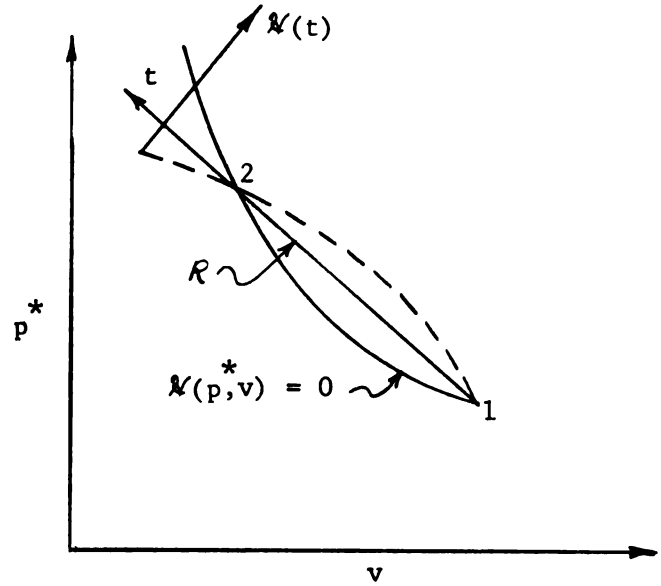

Fig. 1.

$$p^* = p_1^* + at; \; v = v_1 + bt, \tag{7.2.17}$$

where: $a = p_2^* - p_1^*$, $b = v_2 - v_1$. Hence,

$$dp^* = adt; \; dv = bdt. \tag{7.2.18}$$

Therefore, from (7.2.8), we get:

$$2T\, dS^* + (p_1^* - p^*)dv + (v - v_1)dp^* = 2d\mathcal{N}. \tag{7.2.19}$$

Substituting the value of p^*, v, dp^*, and dv, from (7.2.17) and (7.2.18), into (7.2.19), we get:

$$T\, dS^* = d\mathcal{N}, \text{ along the ray } \mathcal{R}. \tag{7.2.20}$$

Considering both S^* and $\mathcal{N}$ as functions of t along $\mathcal{R}$, therefore if either $S^*(t)$ or $\mathcal{N}(t)$ is stationary (i.e. their particular characteristic properties, like the maximum value of $S^*(t)$ or $\mathcal{N}(t)$ do not change their location, are invariant) then other is also stationary which is seen from (7.2.20). Since the Hugoniot curve, $\mathcal{H}(p^*,v) = 0$, is convex at 1, hence ray $\mathcal{R}$ cannot coincide with this Hugoniot curve. Since $\mathcal{H}(t)_1 = \mathcal{H}(t)_2 = 0$ (both are lying on the Hugoniot curve defined as $\mathcal{H} = 0$), implies that $\mathcal{H}(t)$ possesses at least one extremum in between. Therefore at the point of extremum the entropy, $S^*(t)$, is likewise stationary, or:

$$dS^* = S^*_{,p*}dp^* + S^*_{,v}dv. \qquad (7.2.21)$$

Now substituting the value of dp^* and dv from (7.2.18) into (7.2.21), and after dividing the resulting equation by dt we get:

$$\frac{dS^*}{dt}\bigg|_{extr.} = [S^*_{,p*}a + S^*_{,v}b]\big|_{extr.} = 0. \qquad (7.2.22)$$

Therefore, $S^*_{,v}/S^*_{,p*} = -a/b$. Next:

$$d\left(\frac{dS^*}{dt}\right) = \frac{\partial}{\partial p^*}\left(\frac{dS^*}{dt}\right)dp^* + \frac{\partial}{\partial v}\left(\frac{dS^*}{dt}\right)dv,$$

or

$$\frac{d^2S^*}{dt^2} = S^*_{,p*p*}a^2 + 2S^*_{,p*v}ab + S^*_{,vv}b^2,$$

$$= \frac{b^2}{S^{*\,2}_{,p*}}\left(S^*_{,p*p*}S^{*2}_{,v} - 2S^*_{,p*v}S^*_{,p*}S^*_{,v} + S^*_{,vv}S^{*\,2}_{,p*}\right). \qquad (7.2.23)$$

But the expression, inside the parenthesis, on the right hand side of (7.2.23) is always less than zero from (7.2.16) and $\frac{b^2}{S^{*2}_{,p*}} > 0$, therefore,

$$\frac{d^2S^*}{dt^2} < 0. \qquad (7.2.24)$$

Thus S^* and consequently $\mathcal{V}$ possess one and only one single stationary point on $\mathcal{R}$ between the point of states 1 and 2.

From the fact that S^* has just one maxima between the states 1 and 2, we infer the inequalities:

$$\frac{dS^*}{dt}\Big|_1 > 0, \tag{7.2.25}$$

$$\frac{dS^*}{dt}\Big|_2 < 0. \tag{7.2.26}$$

The inequality (7.2.26) excludes the possibility of the magnitude of S^* being stationary along the Hugoniot curve at the point of state 2, otherwise ray $\mathcal{R}$ would be tangent to the Hugoniot curve at such a point. Thus, $d\mathcal{V} = 0$ at this point would therefore imply $\frac{dS^*(t)}{dt}\Big|_2 = 0$, in contradiction to (7.2.26). Thus, it has been proven that the entropy, S^*, increases along the Hugoniot curve with decreasing specific volume in radiation-magnetogasdynamics. Q.E.D.

VIII. THEOREM 4

8.1 The Present Formulation of Theorem 4:

[C.F.] have proven the following theorem 4 in an isentropic flow:

Theorem 4 [C.F.]: "The flow velocity relative to the shock front is supersonic at the front side, subsonic at the back side."

8.2 Theorem 4 in Radiation-Magnetogasdynamics:

We generalize the theorem and proof by [C.F.] to the case of radiation-magnetogasdynamics.

Theorem 4: "The flow velocity relative to the shock front is greater than the effective speed of sound at the front side, and is less than the effective speed of sound at the back side."

Proof: It can be demonstrated [see Appendix B] that infinitesimal pressure disturbance propagates with the effective speed of sound, a_e, in radiation-magnetogasdynamics:

$$p^*_{,\rho} = a_e^2; \text{ thus } p^*_{,v} = -\rho^2 a_e^2$$

and from (7.2.11), we get:

$$\frac{S^*_{,v}}{S^*_{,p*}} = -p^*_{,v} = \rho^2 a_e^2. \tag{8.2.1}$$

From (7.2.18), we get:

$$\frac{dp^*}{dt} = p^*_2 - p^*_1, \ \frac{dv}{dt} = v_2 - v_1, \tag{8.2.2}$$

and from (7.2.4), we obtain after dividing this equation by dt:

$$\frac{dS^*}{dt} = S^*_{,p^*} \frac{dp^*}{dt} + S^*_{,v} \frac{dv}{dt} . \tag{8.2.3}$$

Hence at the point of state 1, we get from (8.2.3) using (7.2.25) and $(7.2.9)_1$:

$$\frac{dp^*}{dt} + \left.\frac{S^*_{,v}}{S^*_{,p^*}}\right|_1 \frac{dv}{dt} > 0. \tag{8.2.4}$$

After eliminating $\frac{dp^*}{dt}$ and $\frac{dv}{dt}$ from (8.2.4) using (8.2.2), we get:

$$(p_2^* - p_1^*) + \left.\frac{S^*_{,v}}{S^*_{,p^*}}\right|_1 (v_2 - v_1) > 0. \tag{8.2.5}$$

Substituting the value of $\frac{S^*_{,v}}{S^*_{,p^*}}$ from (8.2.1) into (8.2.5), we get:

$$(p_2^* - p_1^*) + \rho_1^2 a_{e1}^2 (v_2 - v_1) > 0. \tag{8.2.6}$$

Similarly we get, at the point of state 2 from (7.2.26):

$$(p_2^* - p_1^*) + \rho_2^2 a_{e2}^2 (v_2 - v_1) < 0. \tag{8.2.7}$$

Since $v_2 < v_1$ in a normal shock, then $v_1 - v_2 > 0$; therefore from (8.2.6), we get:

$$\frac{p_2^* - p_1^*}{v_1 - v_2} > \rho_1^2 a_{e1}^2, \tag{8.2.8}$$

and from (8.2.7), we get:

$$\frac{p_2^* - p_1^*}{v_1 - v_2} < \rho_2^2 a_{e2}^2. \tag{8.2.9}$$

Using $(3.1.1)_2$, we get from (3.2.1):

$$m^2 = \frac{p_1^* - p_2^*}{v_2 - v_1} = \frac{p_2^* - p_1^*}{v_1 - v_2} , \tag{8.2.10}$$

where: $m = \rho_1 u_1 = \rho_2 u_2$.

Thus, from (8.2.10) and (8.2.8), we get:

$$u_1^2 > a_{e1}^2, \tag{8.2.11}$$

Also from (8.2.10) and (8.2.9), we get:

$$u_2^2 < a_{e2}^2. \tag{8.2.12}$$

Hence, the flow velocity relative to the shock front is greater than the effective speed of sound at the point of state 1 from (8.2.11), and is less than the effective speed of sound at the point of state 2 from (8.2.12). Q.E.D.

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APPENDIX A

Fortran Program

The following fortran program for H-CURVE, help us to verify the statements that $\frac{d^2p}{dv^2}$ and $p^*_{,vv}$ are both greater than zero along the entire Hugoniot curve. In this program SUBROUTINE POLYRT, supplied by the MSU Computer Center, has been used to find the roots of (4.1.1), with $\mathcal{H}(p,v) = 0$. The variables used in the program are:

AR = a_r, GMA = γ, R = Gas Constant, V1 = v_1, V = v, P1 = p_1, THETA = θ, PR1 = p_{r1}, PH1 = p_{h1}, P = p, PR = p_r, PH = p_h, HV = $\mathcal{H}_{,v}$, HP = $H_{,p}$, HVV = $\mathcal{H}_{,vv}$, HPV = $\mathcal{H}_{,pv}$, HPP = $\mathcal{H}_{,pp}$, DPV = dp/dv, DSPV = d^2p/dv^2, PSTVV = $p^*_{,vv}$.

```
      PROGRAM  H CURVE
      DIMENSION P(100),V(100),FR(100),FI(100),RR(100),RI(100),DSPV(100),
     1PSTVV(100)
      READ 100, AR,GMA,R,V1,P1,THETA,DELTA2
  100 FORMAT (8E10.2)
      RSQ=(GMA+1)/(GMA-1)  $  NORDER=4
      PR1=AR*(P1*V1)**4/(3*R**4)  $  PH1=THETA**2/(2*V1**2)
      DO 50 I=1,12
      V(I)=V1*(15-I)/14.
      FR(1)=P1*(V(I)-RSQ*V1)/2+PR1*(V(I)-7*V1)/2+THETA**2/2*((1/V(I)**2+
     11/V1**2)*(V(I)-V1)/2+(1/V(I)-1/V1))
      FR(2)=(RSQ*V(I)-V1)/2  $  FR(3)=FR(4)=0
      FR(5)=AR/R**4*((V(I)-V1)/6+V(I))*V(I)**4
      FI(1)=FI(2)=FI(3)=FI(4)=FI(5)=0
      CALL POLYRT (FR,FI,NORDER,RR,RI,DELTA2)
      P(I)=RR(1)
      PRINT 225, I, V(I), (RR(J),RI(J), J=1,4)
  225 FORMAT (5X*I = *I2,5X*V = *E16.8,10X*RR*14X*RI*//(33X,2E20.8))
      PR=AR*(P(I)*V(I))**4/(3*R**4)  $  PH=THETA**2/(2*V(I)**2)
      HV=P(I)*RSQ/2+(P1+PR1)/2+PR*(35-4*V1/V(I))/2+(PH1-PH)/2-PH*(1-V1/
     1V(I))
      HP=V(I)/2*(RSQ+28*PR/P(I)-V1/V(I)*(1+4*PR/P(I)))
      HVV=2*PR/V(I)**2*(35*V(I)-3*V1)+3*PH/V(I)**2*(V(I)-V1)
      HPV=RSQ/2+2*PR/(P(I)*V(I))*(35*V(I)-4*V1)
      HPP=6*PR/P(I)**2*(7*V(I)-V1)
      DPV=-HV/HP
      DSPV(I)=-(HVV+2*HPV*DPV+HPP*DPV**2)/HP
      PSTVV(I)=3*THETA**2/V(I)**4+4*AR*(P(I)*V(I))**2/R**4*(P(I)+V(I)*
     1DPV)**2+4/3*AR/R**4*(P(I)*V(I))**3*(2*DPV+V(I)*DSPV(I))+DSPV(I)
   50 CONTINUE
      PRINT 250
  250 FORMAT (//4X*I*14X*V*20X*P*14X*DSPV*17X*PSTVV*//)
      PRINT 300,(I,V(I),P(I),DSPV(I),PSTVV(I),I=1,12)
  300 FORMAT (I5,4E20.8)
      END
```

APPENDIX B

Effective Speed of Sound in Radiation-Magnetogasdynamics

Considering the speed of sound, a_e, in radiation-magnetogasdynamics, it is derived by means of the small perturbation theory [27], in which the second and higher order terms of small quantities are negligible in comparison to the first-order terms. For the simplicity sake we restrict our presentation to one dimension and time. Consider a small perturbation of the state of rest, caused by an initial disturbance: to each (x,t) there will correspond small values of u, $p - p_0$, etc. Hence, the equation of continuity (2.1.2) is reduced to:

$$\rho_0 \frac{\partial u}{\partial x} = - \frac{\partial \rho}{\partial t}, \tag{B-1}$$

and the equation of motion (2.1.3), using (2.2.4), is reduced to:

$$\rho_0 \frac{\partial u}{\partial t} = - \frac{\partial p^*}{\partial x}, \text{ or } \rho_0 \frac{\partial u}{\partial t} = - \frac{\partial p^*}{\partial \rho}\Big|_0 \frac{\partial \rho}{\partial x} . \tag{B-2}$$

Now differentiating (B-1) with respect to t and $(B-2)_2$ with respect to x, we get: respectively,

$$\rho_0 \frac{\partial^2 u}{\partial t \partial x} = - \frac{\partial^2 \rho}{\partial t^2} , \tag{B-3}$$

$$\rho_0 \frac{\partial^2 u}{\partial t \partial x} = - \frac{\partial p^*}{\partial \rho}\Big|_0 \frac{\partial^2 \rho}{\partial x^2} . \tag{B-4}$$

Eliminating the value of $\rho_0 \frac{\partial^2 u}{\partial t \partial x}$, using (B-3), from (B-4), we get:

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial p^*}{\partial \rho}\bigg|_0 \frac{\partial^2 \rho}{\partial x^2} . \qquad \text{(B-5)}$$

The one dimensional wave equation for ρ, with the small disturbance propagation velocity, a_e, can be written as:

$$\frac{\partial^2 \rho}{\partial t^2} = a_e^2 \frac{\partial^2 \rho}{\partial x^2} . \qquad \text{(B-6)}$$

Now comparing (B-5) and (B-6), we obtain the effective speed of sound, a_e, in radiation-magnetogasdynamics as:

$$a_e^2 = \frac{\partial p^*}{\partial \rho}\bigg|_0 . \qquad \text{(B-7)}$$