A STUDY OF THE STABILIZING AUTOMORPHISMS OF A FINITE GROUP

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ABSTRACT

A STUDY OF THE STABILIZING AUTOMORPHISMS OF A FINITE GROUP

By

Bruce Staal

One of the most important objects of investigation in finite group theory is the group of automorphisms of a finite group. If one can say something about the automorphisms of a group then conceivably he can say something about the group itself. Our interest here is to consider certain relationships between subgroups and series of subgroups of a group and subgroups and series of subgroups of its automorphism group.

Given a series $\bar{G}$ of subgroups of G; say $\bar{G}: G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$, we say that an automorphism α of G stabilizes $\bar{G}$ if $\alpha(x)x^{-1} \varepsilon G_{i+1}$ for all $x \varepsilon G_i$ for $i = 0, 1, \cdots, n - 1$. We obtain information about stabilizing automorphisms of a finite group building upon the work of Kaloujnine, P. Hall, and Baer. If A is a set of automorphisms of G then the multiplier group of A, M(G,A), is given by $M(G,A) = \langle \alpha(g)g^{-1}: g \varepsilon G, \alpha \varepsilon A \rangle$. It has been shown by P. Hall that the set of all stabilizing automorphisms of a series of a finite group G, denoted by Stab($\bar{G}$), and M(G,Stab($\bar{G}$)) are nilpotent groups.

After setting down more of the background definitions and theorems in Chapter I, we investigate in Chapter II the general properties of stabilizing automorphisms which are independent of the

particular series that is stabilized. It is shown that if α is a stabilizing automorphism then one can construct a standard series of shortest length which α stabilizes and this series is subnormal. This aids in obtaining the following results about the order of a stabilizing automorphism in terms of the order of the group.

1. Let G be a finite group and suppose $\alpha \in \mathrm{Aut}(G)$ stabilizes the series $\bar{G}: G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$. Then if G_1 is a π-group, α is a π-automorphism.
2. If α stabilizes a series of G and Fit(G), the Fitting subgroup of G, is a π-group then α is a π-automorphism.

Using these results it is shown that the inner automorphisms that stabilize some series of G are precisely those induced by elements of the Fitting subgroup.

Kaloujnine defined the following generalization of a nilpotent group. For $A \subseteq \mathrm{Aut}(G)$, let $M_o(G,A) = G$ and $M_{i+1}(G,A) = M(M_i(G,A),A)$. If there exists a natural number n such that $M_n(G,A) = (1)$ then G is said to be A-nilpotent. The usual idea of a nilpotent group occurs when A is the group of inner automorphisms of G. In Chapter III the stabilizing group, S(G), is defined to be the group generated by the stabilizing automorphisms of G. After examining some of the general properties of S(G) it is shown that S(G) is nilpotent if and only if G is S(G)-nilpotent. We then look at the conditions under which a group is Aut(G)-nilpotent and S(G)-nilpotent and conclude the chapter by considering the structure of S(G) for some of the well-known groups.

A subgroup H of G is a Hall subgroup if the order of H and the index of H in G are relatively prime. Baer has shown that if $\bar{H}: G = H_o \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = (1)$ is a series of Hall subgroups which are normal in G, then an automorphism α of G stabilizes $\bar{H}$ if and only if α is an inner automorphism induced by an element of $Z(\bar{H}) = \prod_{i=1}^{n} Z(H_i)$. If $G = H_1 \cdots H_n$ with $H_iH_j = H_jH_i$ and $(|H_i|, |H_j|) = 1, i \neq j, i,j = 1, \cdots, n$, the series $\bar{H}: G = H_1 \cdots H_n \geq H_1 \cdots H_{n-1} \geq \cdots H_1 \geq H_o = (1)$ can be formed. The series Baer considered have the above properties, but were also normal. We are able to get Baer's conclusion if $\bar{H}$ has a property which is an extension of the Sylow Tower property. It is also shown that Baer's conclusion holds for the inner automorphisms of G in the series $\bar{H}$ described above. The question of whether these inner automorphisms are the only stabilizing automorphisms for this type of series remains open.

If $\bar{G}: G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$ is a series of subgroups of G, an automorphism α of G is said to power stabilize $\bar{G}$ if for each $x \in G_i$ there is a natural number n and $y \in G_{i+1}$ such that $\alpha(x) = x^n y$. This generalizes the idea of a stabilizing automorphism and it may be noted that if $\bar{G}$ is a subnormal series, α is a power stabilizing automorphism of $\bar{G}$ if and only if α induces a power automorphism of $\frac{G_i}{G_{i+1}}$ for $i = 0, 1, \cdots, n - 1$. The following results are obtained in Chapter V in the direction of the results for stabilizing automorphisms.

1. If $\bar{G}$ is a subnormal series of subgroups of G then the set of power stabilizing automorphisms of $\bar{G}$ is supersolvable.
2. If $\bar{G}$ is a normal series of subgroups of G then the multiplier group for G determined by the power stabilizing automorphisms of $\bar{G}$ is supersolvable.

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INTRODUCTION

Most of our knowledge of abstract algebraic objects such as groups, rings, or vector spaces has been gained by studying either the substructures of the objects themselves or the homomorphic mappings into and out of the objects. The study of automorphisms of finite groups is an example of the latter of these two approaches. If one can say something about the automorphisms of a finite group then he can often say something about the group itself.

In this paper we consider certain relationships between subgroups and series of subgroups of a group, G, and the elements and certain subgroups of the automorphism group of G. A well-known example of this kind of result is

<u>Theorem</u>: If H is a normal subgroup of G and A is the set of automorphisms of G such that for each $\alpha \in A$ the following hold:

1) $\alpha(h) = h$ for all $h \in H$

2) $\alpha(g)g^{-1} \in H$ for all $g \in G$

then A is an abelian subgroup of the automorphism group.

Here a normal subgroup of a group can be thought of as singling out a certain abelian subgroup of the automorphism group.

If an automorphism, α, has the properties stated in the theorem with respect to a normal subgroup H, α is said to stabilize H in G. An automorphism of G is said to stabilize a subnormal series of G if it leaves the series invariant and induces the identity automorphism on the factor groups. An equivalent definition can be given which is applicable to any (possibly non-subnormal) series.

In Chapter I we will see that Kaloujnine, P. Hall and Baer have proved several results concerning stabilizing automorphisms and that as a result certain connections between the structure of the group and its automorphism group have been obtained. Here we set down the basic definitions and results which provide the background necessary for the remainder of the paper.

In Chapter II those properties of stabilizing automorphisms which are possessed irrespective of the nature of the series stabilized are examined. We look at how restrictive the condition that an automorphism stabilize some series is.

It is shown that a standard subnormal series of shortest length that is stabilized by a given stabilizing automorphism can be constructed. This aids in obtaining results about the order of a stabilizing automorphism in terms of the order of the group. Using these, a characterization of the inner automorphisms which are stabilizing automorphisms is found. These are precisely those inner automorphisms which are induced by elements of the Fitting subgroup of the group.

In Chapter III we consider the kind of substructure the stabilizing automorphisms form in the automorphism group. To aid in this we define the stability group of a group G, S(G), to be the group generated by the stabilizing automorphisms of G. In the case when S(G) is nilpotent its structure is shown to have a very interesting relation to the structure of G. The structure of S(G) is examined for many of the well-known groups and some relationships between the stabilizing group of G and those of its subgroups and factor groups are considered.

In [2] Baer proved that only certain inner automorphisms stabilize a series of normal Hall subgroups. In Chapter IV we look at the kinds of automorphisms which stabilize series with some but not all of the properties of a normal Hall series. Some extensions of Baer's work especially for inner automorphisms are obtained.

A stabilizing automorphism of a subnormal series induces the identity automorphism on factor groups. In Chapter V we consider automorphisms which induce power automorphisms on the factor groups of a subnormal series. It is shown that a group of such power stabilizing automorphisms for a subnormal series of a finite group is supersolvable. The multiplier group of a power stabilizing group of automorphisms for a normal series is also shown to be supersolvable.

The notation used is mostly standard and an index of notation is included at the end of the paper. All theorems, lemmas, corollaries, examples, and definitions are numbered consecutively in the order that they appear in each chapter with the first digit referring to the chapter number. The numbers in square brackets refer to the bibliography.

CHAPTER I

BACKGROUND DEFINITIONS AND THEOREMS

Our study in this paper is restricted to finite groups; accordingly, G will denote a finite group throughout. We let Aut(G) denote the group of automorphisms of G and Inn(G) the group of inner automorphisms of G.

If G is a group and H is a normal subgroup of $G(H \trianglelefteq G)$, then $\alpha \in \mathrm{Aut}(G)$ is said to stabilize H in G if $\alpha(h) = h$ for all $h \in H$ and $\alpha(g)g^{-1} \in H$ for all $g \in G$. It has long been known that the set of all such automorphisms for a given normal subgroup H form an abelian subgroup of Aut(G). An automorphism, α, with these properties is called a stabilizing automorphism of H in G and $\mathrm{Stab}(H,G) = \{ \alpha : \alpha$ stabilizes H in G $\}$ is called the stability group of H in G.

In [8] Kaloujnine generalized this notion as follows. Let $\bar{G}: G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$ be a series of subgroups of G. We will say that $\bar{G}$ is a <u>subnormal series</u> of G (denoted $\bar{G}: G = G_o \trianglerighteq G_1 \trianglerighteq \cdots G_n = (1)$) if $G_{i+1} \trianglelefteq G_i$ for $i = 0, 1, \cdots, n-1$. An element $\alpha \in \mathrm{Aut}(G)$ leaves $\bar{G}$ invariant if $\alpha(G_i) = G_i$, $i = 0, 1, \cdots, n$. If α leaves $\bar{G}$ invariant and $\bar{G}$ is a subnormal series then for each $i = 0, \cdots n-1$, $\alpha \in \mathrm{Aut}(G)$ induces an automorphism, $\bar{\alpha}_i$, on the quotient group $\frac{G_i}{G_{i+1}}$ defined by

$\bar{\alpha}_i(gG_{i+1}) = \alpha(g)G_{i+1}$ for $g \in G_i$. Now if α leaves the subnormal series $\bar{G}$ invariant and induces the identity automorphism on $\frac{G_i}{G_{i+1}}$ for $i = 0, \cdots, n-1$, we will say that α stabilizes $\bar{G}$ and in this case call α a stabilizing automorphism of $\bar{G}$.

Now α inducing the identity automorphism of $\frac{G_i}{G_{i+1}}$ is equivalent to $\bar{\alpha}_i(xG_{i+1}) = \alpha(x)G_{i+1} = xG_{i+1}$ for all $x \in G_i$. This is, in turn, equivalent to $\alpha(x)x^{-1} \in G_{i+1}$ for all $x \in G_i$. Using this equivalence it is possible to obtain a definition of stabilizing automorphism for any (possibly non-subnormal) series of G as follows.

<u>Definition 1.1</u>: Let G be a group, $\bar{G}$: $G = G_0 \geq G_1 \geq \cdots G_n = (1)$ a series of G, and $\alpha \in \text{Aut}(G)$. α is said to be a <u>stabilizing automorphism of $\bar{G}$</u> if α leaves $\bar{G}$ invariant and for all $x \in G_i$, $\alpha(x)x^{-1} \in G_{i+1}$, $i = 0, 1, \cdots, n-1$. Furthermore if there is a series of G which $\alpha \in \text{Aut}(G)$ stabilizes, then we call α a <u>stabilizing automorphism of G</u>.

The multiplier group is an important aid in the study of stabilizing automorphisms.

<u>Definition 1.2</u>: If $A \leq \text{Aut}(G)$ then $M(G,A) = \langle \alpha(g)g^{-1} : g \in G, \alpha \in A \rangle$ is the <u>multiplier group</u> of A in G.

The identity $h(\alpha(g)g^{-1})h^{-1} = \alpha(h)h^{-1}\,\alpha(hg)(hg)^{-1}$ shows that M(G,A) is a normal subgroup of G. If H is a normal subgroup of G such that each element of A induces the identity automorphism on $\frac{G}{H}$ then for $\alpha \in A$ $\bar{\alpha}(gH) = \alpha(g)H = gH$, so $\alpha(g)g^{-1} \in H$ for an arbitrary $g \in G$. Hence, $M(G,A) \leq H$. We summarize these results in

<u>Theorem 1.3</u>: If $A \leq \mathrm{Aut}(G)$, $M(G,A)$ is a normal subgroup of G and it is the smallest normal subgroup of G whose factor group remains fixed, elementwise, by A.

In view of Definition 1.2 the statement $\alpha(x)x^{-1} \in G_{i+1}$ for all $x \in G_i$ in definition 1.1 can be restated as $M(G_i,\alpha) \leq G_{i+1}$ (we write $M(G,\alpha)$ for $M(G,\{ \alpha \}))$.

The following are now defined recursively:

$M_0(G,A) = G$, $M_i(G,A) = M(M_{i-1}(G,A),A)$.

If $\bar{G}$ is a series of G, the set of all stabilizing automorphisms of $\bar{G}$ forms a group which is denoted by $\mathrm{Stab}(\bar{G})$, i.e.,

$\mathrm{Stab}(\bar{G}) = \{ \alpha \in \mathrm{Aut}(G): \alpha \text{ stabilizes } \bar{G} \}$.

In [8] Kaloujnine showed that if $M_i(G,\mathrm{Stab}(\bar{G}))$ are all normal subgroups of G, then $M(G,\mathrm{Stab}(\bar{G}))$ is nilpotent. In [5] P. Hall improved this with:

<u>Theorem 1.4</u> (P. Hall): Let $\bar{G}$ be a series of G, then $M(G,\mathrm{Stab}(\bar{G}))$ is nilpotent.

We now recall the definition of the class of nilpotent group. The <u>descending central series</u> of a group G:

$G = \gamma_0(G) \geq \gamma_1(G) \geq \cdots \geq \gamma_i(G) \geq \gamma_{i+1}(G) \geq \cdots$ is defined by $\gamma_0(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G),G]$ which is the subgroup generated by all the commutators $x^{-1}y^{-1}xy$ where $x \in \gamma_i(G)$ and $y \in G$. A finite group G is <u>nilpotent</u> if there is an integer n such that $\gamma_n(G) = (1)$ and the least such integer is called the <u>class</u> of the nilpotent group.

In [8] Kaloujnine proves:

Theorem 1.5 (Kaloujnine): Let $\bar{G}$ be a normal series of G of length m, then Stab($\bar{G}$) is nilpotent of class $\leq m - 1$.

An example is given to show that this is the best possible estimate of the class of this nilpotent group.

For the case in which $\bar{G}$ is not a normal series the best Kaloujnine could do was to show that Stab($\bar{G}$) was solvable of derived length $\leq m - 1$. Here we mean by the derived length of a finite solvable group the least integer n for which the term $G^{(n)}$ of the derived series is trivial; the derived series of G,

$G = G^{(o)} \geq G^{(1)} \geq \cdots \geq G^{(i)} \geq G^{(i+1)} \cdots$ is defined by $G^{(o)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Of course, a finite group is solvable if there is an integer n such that $G^{(n)} = (1)$.

In [5] P. Hall was also able to improve this result with

Theorem 1.6 (P. Hall): Let $\bar{G}$ be a series of G, then Stab($\bar{G}$) is nilpotent of class $\leq \frac{m(m-1)}{2}$, where $\bar{G}$ is of length m.

Hall also gives an example to show that this is the best possible estimate of the nilpotence class, i.e., there is a group with a non-normal series in which Kaloujnine's estimate of $m - 1$ for the nilpotence class of the stabilizing group is too small and $\frac{m(m-1)}{2}$ is needed.

Thus it is apparent that the study of stabilizing automorphisms gives a connection between some of the nilpotent normal subgroups of G (M(G,Stab($\bar{G}$)) and some of the nilpotent subgroups of Aut(G) (Stab($\bar{G}$)).

Definition 1.7: If π is a set of primes then a natural number n is called a π-number if its prime factorization contains only primes from π. A group G is a π-group if its order is a π-number and $\alpha \in Aut(G)$ is a π-automorphism if its order is a π-number.

The following result is known and it is a consequence of the stronger Corollary 2.16 in this paper.

Theorem 1.8: If $\alpha \in Aut(G)$ stabilizes a series of G and G is a π-group, then α is a π-automorphism.

A subgroup $H < G$ is called a Hall subgroup of G provided $(|H|, [G:H]) = 1$ where $[G:H]$ is the index of H in G and $|H|$ is the order of H. In [2] Baer proves the following concerning the automorphisms that stabilize a normal series of Hall subgroups of G. We use the notation Z(G) for the center of G and π_x denotes the inner automorphism of G induced by $x \in G$, i.e., $\pi_x(g) = x^{-1}gx$ for $g \in G$.

Theorem 1.9 (Baer): Let $\bar{H}: G = H_o \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = (1)$ be a normal series of Hall subgroups of G. Then $\alpha \in Aut(G)$ stabilizes $\bar{H}$ if and only if $\alpha = \pi_x$ where $x \in Z(\bar{H}) = \prod_{i=1}^{n} Z(H_i)$.

Using these basic results as our starting point we shall obtain more information about the stabilizing automorphisms of a group.

CHAPTER II

THE STABILITY AUTOMORPHISMS OF A FINITE GROUP

In this chapter we investigate the general properties of stabilizing automorphisms which are independent of a particular stabilizing series in order to discover what characterizes these automorphisms in the whole group of automorphisms.

Suppose $A \subseteq \mathrm{Stab}(\bar{G})$ where $\bar{G}$: $G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$ is a series of G. Now $M(G,A) \subseteq \prod_{\alpha\varepsilon A} M(G,\alpha) \subseteq G_1$. If we assume $M_i(G,A) \subseteq G_i$, then $M_{i+1}(G,A) = M(M_i(G,A),A) \subseteq M(G_i,A) \subseteq \prod_{\alpha\varepsilon A} M(G_i,\alpha) \subseteq G_{i+1}$. Thus it follows by induction that $M_k(G,A) \subseteq G_k$ $k = 1, \cdots, n$. In particular $M_n(G,A) \subseteq G_n = (1)$ so $M_n(G,A) = (1)$. Let m be the smallest natural number such that $M_m(G,A) = (1)$.

Let $x \; \varepsilon \; M(G,A)$, $\alpha \; \varepsilon \; A$ and consider $\alpha(x)x^{-1} = y \; \varepsilon \; M(G,A)$ then $\alpha(x) = y\,x \; \varepsilon \; M(G,A)$ so for all $\alpha \; \varepsilon \; A$, α is an automorphism of $M(G,A)$. Further, if z is a generator of $M_2(G,A)$, that is, $z = \alpha(x)x^{-1}$ for some $\alpha \; \varepsilon \; A$ and $x \; \varepsilon \; M(G,A)$ then both $\alpha(x)$ and x^{-1} are in $M(G,A)$ so $z \; \varepsilon \; M(G,A)$. Hence $M_2(G,A) \subseteq M(G,A)$.

Proceeding inductively, we assume α induces an automorphism of $M_k(G,A)$ for all $\alpha \; \varepsilon \; A$ and $M_{k+1}(G,A) \subseteq M_k(G,A)$. If $x \; \varepsilon \; M_{k+1}(G,A)$, then $x \; \varepsilon \; M_k(G,A)$ so that $\alpha(x)x^{-1} \; \varepsilon \; M_{k+1}(G,A)$ for all $\alpha \; \varepsilon \; A$ and hence $\alpha(x) \; \varepsilon \; M_{k+1}(G,A)$ for all $\alpha \; \varepsilon \; A$.

If z is a generator of $M_{k+2}(G,A)$, that is, $z = \alpha(x)x^{-1}$ for some $\alpha \in A$ and $x \in M_{k+1}(G,A)$ then both $\alpha(x)$ and x^{-1} are in $M_{k+1}(G,A)$ so $z \in M_{k+1}(G,A)$ and we have that $M_{k+2}(G,A) \subseteq M_{k+1}(G,A)$.

A series $\bar{M}$ can be constructed as follows:

$\bar{M}$: $G = M_0(G,A) \trianglerighteq M_1 = M_1(G,A) \trianglerighteq \cdots \trianglerighteq M_m = M_m(G,A) = (1)$

This series is subnormal since $M(G,A) \trianglelefteq G$ and hence $M_{k+1}(G,A) \trianglelefteq M_k(G,A)$.

The above argument then shows that A stabilizes $\bar{M}$ and $M_k(G,A)$ is the smallest possible subgroup which can appear as the k^{th} subgroup in a series which A stabilizes. This is summarized in:

Theorem 2.1: Let $A \subseteq \mathrm{Stab}(\bar{G})$ for a series

$\bar{G}$: $G = G_0 \geq G_1 \geq \cdots \geq G_n = (1)$ of the group G. Then there is a least natural number m such that $M_m(G,A) = (1)$ and the series

$\bar{M}$: $G = M_0 \trianglerighteq M_1 = M_1(G,A) \trianglerighteq \cdots \trianglerighteq M_m = M_m(G,A) = (1)$

is a subnormal series stabilized by A. Furthermore, $\bar{M}$ is a series of least length stabilized by A.

From [1] we get the following:

Theorem 2.2: If $A \subseteq \mathrm{Aut}(G)$, then $M(G,A) = M(G,\langle A\rangle)$.

Proof:

If $\alpha,\beta \in A$, then $\alpha\beta(g)g^{-1} = [\ \alpha(\beta(g))(\beta(g))^{-1}\ ]\ (\beta(g)g^{-1}) \in M(G,A)$.

Therefore $M(G,\langle A\rangle) \subseteq M(G,A)$. Since $A \leq \langle A\rangle$, $M(G,A) \subseteq M(G,\langle A\rangle)$ so $M(G,A) = M(G,\langle A\rangle)$.

So we see that for many subsets of $\mathrm{Stab}(\bar{G})$ in Theorem 2.1 we may get the same series $\bar{M}$.

It is natural to ask whether $\bar{M}$ might indeed always be a normal series. The following theorem of P. Hall in [3] is used to show that this is not the case.

Theorem 2.3 (P. Hall): There exists a nilpotent group G of class 2 with a series of subgroups of length 3 whose stability group is of class 3.

Let B be the stability group of class 3 in the theorem. Form the series

$$\bar{M}: M_o = G \trianglerighteq M_1 = M_1(G,B) \trianglerighteq M_2 = M_2(G,B) \trianglerighteq M_3 = M_3(G,B) = (1)$$

Now M_2 cannot be a normal subgroup of G for if it were then $\bar{M}$ would be a normal series and by Theorem 1.3 B would have to be nilpotent of class ≤ 2.

An important special case of Theorem 2.1 is that in which A consists of a single stabilizing automorphism. We state this in

Corollary 2.4: Let $\alpha \in Aut(G)$ be a stabilizing automorphism of G. Then there is a least natural number m such that $M_m(G,\alpha) = (1)$ and the series

$$\bar{M}: M_o = G \trianglerighteq M_1 = M_1(G,\alpha) \trianglerighteq \cdots \trianglerighteq M_m = M_m(G,\alpha) = (1)$$

is a subnormal series of shortest length which is stabilized by α.

In what follows we will find a better estimate for the order of a stabilizing automorphism than that of Theorem 1.8 and characterize the stabilizing automorphisms of some well-known groups.

In the proof of the following theorem we will use the semi-direct product in a special case. We will give a restricted definition of such a product and the basic results we will use. A more general discussion of semi-direct products can be found in [10] pages 212-217.

Let G be a group and A be an abelian subgroup of Aut(G). We form the semi-direct product $K = G\cdot[A]$ in the following manner. K is the set of all pairs (g,α) where $g \in G$, $\alpha \in A$ with binary operation $(g,\alpha)(h,\beta) = (g\alpha(h),\alpha\beta)$. Now K is a group and K has the property that if we form the inner automorphism of K induced by $(1,\alpha)$ (we will denote it by π_α rather than $\pi_{(1,\alpha)}$) then this inner automorphism acts on elements of G in K (i.e., $(g,1)$) in the same way as α acts on G as the following shows:

$$\begin{aligned}\pi_\alpha(g,1) &= (1,\alpha)(g,1)(1,\alpha)^{-1}\\ &= (\alpha(g),\alpha)(1,\alpha^{-1})\\ &= (\alpha(g),1)\end{aligned}$$

The following theorem characterizes the stabilizing automorphisms of a p-group in terms of their orders:

<u>Theorem 2.5</u>: If G is a p-group, then α is a stabilizing automorphism of G if and only if α is a p-automorphism.

<u>Proof</u>:

If α is a stabilizing automorphism the fact that α is a p-automorphism follows from Theorem 1.8.

The converse is proved by showing that $M(G,\alpha) \subsetneq G$ when G is a p-group and α is a p-automorphism. Then since $M(G,\alpha)$ is a p-group and α is a p-automorphism of $M(G,\alpha)$ there must be a natural number m such that $M_m(G,\alpha) = (\ 1\ )$. The series $\bar{M}$ is then stabilized by α proving the result.

Case 1: G is an abelian p-group

$M(G,\alpha) = \langle \alpha(g)g^{-1} : g \in G \rangle = \{ \alpha(g)g^{-1} : g \in G \}$

since if $x,y \in G$ $\alpha(x)x^{-1}\alpha(y)y^{-1} = \alpha(xy)(xy)^{-1}$ and $(\alpha(x)x^{-1})^{-1} = \alpha(x^{-1})(x^{-1})^{-1}$. But since α is a p-automorphism of a p-group it is not fixed point free, thus $M(G,\alpha) = \{ \alpha(g)g^{-1} : g \in G \} \subsetneq G$.

Case 2: G is a non-abelian p-group

Form the semi-direct product $G\cdot[\langle\alpha\rangle] = K$. Since α is a p-automorphism, $\langle\alpha\rangle$ is a p-group so K is a p-group. Also since G is non-abelian so is K. Now the inner automorphism π_α of K acts on the elements of G as a subgroup of K the same as α acts on the elements of G as a group.

Thus $M(G\cdot[\langle 1\rangle],\pi_\alpha) \simeq M(G,\alpha)$

Now $M(G\cdot[\langle 1\rangle],\pi_\alpha) \subseteq K'$ and $K' \subseteq G\cdot[\langle 1\rangle]$.

If $K' = G\cdot[\langle 1\rangle]$ then $\frac{K}{G\cdot[\langle 1\rangle]} = \frac{K}{K'}$ but since $\langle\alpha\rangle$ is cyclic and

$\langle\alpha\rangle \simeq \frac{K}{G\cdot[\langle 1\rangle]}$ it follows that $\frac{K}{K'}$ is cyclic. This is impossible since K is non-abelian. So $K' \neq G\cdot[\langle 1\rangle]$ and $K' \subsetneq G\cdot[\langle 1\rangle]$. Hence $M(G\cdot\langle 1\rangle,\pi_\alpha) \subseteq K' \subsetneq G\cdot[\langle 1\rangle]$ from which it follows that $M(G,\alpha) \subsetneq G$.

Theorem 2.6: If G is nilpotent and the factorization of G into the direct product of its Sylow subgroups is given by $G = P_1 \times \cdots \times P_n$ where P_i is a p_i-group, then $\alpha \in \mathrm{Aut}(G)$ stabilizes a series of G if and only if $\alpha|P_i$ is a p_i-automorphism $i = 1, \cdots, n$.

Proof:

Since $\alpha|_{P_i}$ is an automorphism of P_i, α stabilizes a series of G if and only if $\alpha|_{P_i}$ stabilizes a series of P_i for $i = 1, \cdots, n$. By Theorem 2.5 this is the case if and only if $\alpha|_{P_i}$ is a p_i-automorphism $i = 1, \cdots, n$.

Corollary 2.7: If G is nilpotent and $g \in G$ then π_g is a stabilizing automorphism of G.

Proof:

Referring to the factorization of G in Theorem 2.6 we write $g \in G$ as $g = g_1 g_2 \cdots g_n$, $g_i \in P_i$. Since in a nilpotent group the Sylow subgroups centralize each other $\pi_{g|_{P_i}} = \pi_{g_1 \cdots g_n|_{P_i}} = \pi_{g_i|_{P_i}}$ which is a p_i-automorphism and the result follows from the theorem.

Thus the stabilizing automorphisms of a nilpotent group can be characterized in terms of the orders of the induced automorphisms on the Sylow subgroups.

Example 2.8: This is an example to show that if G is not nilpotent the conclusion of Corollary 2.7 may not hold.

Let $G = S_3$, the symmetric group on three letters. Then

$Aut(S_3) \cong S_3 \cong Inn(S_3)$.

Any stabilizing automorphism of S_3 must stabilize

$S_3 \trianglerighteq A_3 \trianglerighteq (1)$ since this is the only non-trivial subnormal series of S_3. But only the inner automorphisms induced by elements of A_3 stabilize this series. Thus the remaining inner automorphisms are not stabilizing automorphisms.

It may be noted that this example also shows that the inner automorphisms need not be contained in the group generated by the stabilizing automorphisms.

Example 2.9: This is an example to show that, in general, the set of stabilizing automorphisms is not a group.

Let $G = V$ the Klein 4-group. Only the automorphisms of order 2 stabilize a series of V. But $\mathrm{Aut}(V) \cong S_3$ and the elements of order 2 generate S_3.

A characterization is now given of the stabilizing automorphisms of any finite group using the characterization of the stabilizing automorphisms of a nilpotent group given in Theorem 2.6.

Theorem 2.10: Let G be a finite group. $\alpha \in \mathrm{Aut}(G)$ is a stabilizing automorphism of G if and only if $M(G,\alpha)$ is nilpotent and $\alpha|_{P_i}$ is a p_i-automorphism $i = 1, \cdots, k$. Where $P_1 \times \cdots \times P_k = M(G,\alpha)$ is the factorization of the nilpotent group $M(G,\alpha)$ into the direct product of its Sylow subgroups.

Proof:

If α stabilizes a series of G then it follows from Theorem 1.4 that $M(G,\alpha)$ is nilpotent; by Corollary 2.4 α stabilizes

$\bar{M}$: $M_0 = G \trianglerighteq M_1 = M(G,\alpha) \trianglerighteq \cdots \trianglerighteq M_n(G,\alpha) = (1)$.

Thus α stabilizes a series of $M(G,\alpha)$ and by Theorem 2.6 $\alpha|_{P_i}$ is a p_i-automorphism $i = 1, \cdots, k$.

Conversely, since $M(G,\alpha)$ is nilpotent and α is an automorphism of $M(G,\alpha)$, according to Theorem 2.6 α stabilizes a series of $M(G,\alpha)$. Also α induces the identity automorphism on $\frac{G}{M(G,\alpha)}$, so α stabilizes a series of G.

Note that Example 2.8 shows that the product of two stabilizing automorphisms need not be a stabilizing automorphism. The following shows when such a product is a stabilizing automorphism.

Lemma 2.11: Let G be a finite group. If $\alpha,\beta \in Aut(G)$ are stabilizing automorphisms of G, then $M(G,\alpha\beta)$ is nilpotent.

Proof:

By Theorem 2.10 $M(G,\alpha)$ and $M(G,\beta)$ are nilpotent and it follows from Theorem 1.3 that $M(G,\alpha)$ and $M(G,\beta)$ are normal subgroups of G.

Let $g \in G$. The identity

$$\alpha\beta(g)g^{-1} = \alpha(\beta(g)g^{-1})(\beta(g)g^{-1})^{-1}(\beta(g)g^{-1})(\alpha(g)g^{-1})$$

shows that $M(G,\alpha\beta) \subseteq M(G,\alpha)\ M(G,\beta)$ so $M(G,\alpha\beta)$ is nilpotent.

Theorem 2.12: Let G be a finite group. If $\alpha,\beta \in Aut(G)$ are stabilizing automorphisms of G then $\alpha\beta$ stabilizes a series of G if and only if $\alpha\beta|_{P_i}$ is a p_i-automorphism $i = 1, \cdots, k$ where

$M(G,\alpha\beta) = P_1 x \cdots x P_k$, P_i the p_i-Sylow subgroup of the nilpotent group $M(G,\alpha\beta)$.

Proof:

The Theorem follows from Theorem 2.10 and Lemma 2.11.

The Klein 4-group shows that even for an abelian p-group one cannot get that α,β stabilize imply $\alpha\beta$ stabilizes for all $\alpha,\beta \in Aut(G)$. This indicates that restrictions must be placed on the stabilizing automorphisms themselves rather than on the group structure to get results in this direction.

Theorem 2.13: If $\alpha,\beta \in Aut(G)$ are stabilizing automorphisms of G and $\langle\alpha,\beta\rangle$ is nilpotent, then $\alpha\beta$ is a stabilizing automorphism of G.

Proof:

Since by Theorem 1.3 and Theorem 2.4 $M(G,\alpha)$ and $M(G,\beta)$ are normal nilpotent subgroups of G, $M(G,\alpha)\ M(G,\beta)$ is a normal nilpotent subgroup of G.

Let $M(G,\alpha)\ M(G,\beta) = P_1 x \cdot\cdot\cdot x P_n$ be the factorization of $M(G,\alpha)\ M(G,\beta)$ into the direct product of its Sylow subgroups.

α is an automorphism of $M(G,\alpha)\ M(G,\beta)$ since if $x \in M(G,\alpha)\ M(G,\beta)$, $\alpha(x)x^{-1} \in M(G,\alpha)$ so $\alpha(x) \in M(G,\alpha)\ M(G,\beta)$.

Similarly β is an automorphism of $M(G,\alpha)\ M(G,\beta)$.

Since α and β stabilize the nilpotent group $M(G,\alpha)\ M(G,\beta)$, $\alpha|_{P_i}$ and $\beta|_{P_i}$ are p_i-automorphisms for $i = 1, \cdot\cdot\cdot, n$.

Let $\langle\alpha,\beta\rangle = Q_1 x \cdot\cdot\cdot x Q_m$ be the factorization of $\langle\alpha,\beta\rangle$ into the direct product of its Sylow subgroups. Then write $\alpha = (\alpha_1, \cdot\cdot\cdot, \alpha_m)$ and $\beta = (\beta_1, \cdot\cdot\cdot, \beta_m)$ where $\alpha_i, \beta_i \in Q_i$, $i = 1, \cdot\cdot\cdot, m$.

Since $\alpha|_{P_i}$ and $\beta|_{P_i}$ are p_i-automorphisms there exists j such that $\alpha|_{P_i} = \alpha_j$ and $\beta|_{P_i} = \beta_j$. So $\alpha\beta|_{P_i} = \alpha|_{P_i}\ \beta|_{P_i} = \alpha_j\beta_j$ and $\alpha_j\beta_j \in Q_j$ which is a p_i-group. Thus $\alpha\beta|_{P_i}$ is a p_i-automorphism for $i = 1, \cdot\cdot\cdot, n$.

Since $M(G,\alpha\beta) \subseteq M(G,\alpha)\ M(G,\beta)$ the factorization of the nilpotent group $M(G,\alpha\beta)$ into the direct product of its Sylow subgroups can be written $M(G,\alpha\beta) = P'_1 x \cdot\cdot\cdot x P'_n$ where $P'_i \subseteq P_i$ for $i = 1, \cdot\cdot\cdot, n$. Then $\alpha\beta|_{P'_i}$ is a p_i-automorphism for $i = 1, \cdot\cdot\cdot, n$. Hence, by Theorem 2.12 $\alpha\beta$ is a stabilizing automorphism of G.

Corollary 2.14: If $\alpha, \beta \in Aut(G)$ are stabilizing automorphisms of G and $\alpha\beta = \beta\alpha$ then $\alpha\beta$ is a stabilizing automorphism of G.

Proof:

Since $\alpha\beta = \beta\alpha$, $\langle\alpha, \beta\rangle$ is abelian and hence nilpotent.

We now examine the orders of the stabilizing automorphisms and obtain results which extend Theorem 1.8. This will make it possible to obtain more information about automorphisms of a finite group which are stabilizing automorphisms.

Theorem 2.15: Let G be a finite group and $\alpha \in Aut(G)$. Suppose further that k is the least natural number such that $M_k(G,\alpha) = M_{k+1}(G,\alpha)$ then α is a π-automorphism where $\pi = \pi' \cup \pi''$ with $\alpha|_{M_k(G,\alpha)}$ a π'-automorphism and $M(G,\alpha)$ a π''-group.

Proof:

Suppose $\bar{M}$ is given by

$$\bar{M}: M_o = G \supsetneq M_1 = M_1(G,\alpha) \supsetneq \cdots \supsetneq M_k = M_k(G,\alpha) = M_{k+1}(G,\alpha)$$

k being the least natural number such that $M_k(G,A) = M_{k+1}(G,A)$.

We proceed by induction on the length of $\bar{M}$. Suppose first that $M_1(G,A) = M_o(G,A)$. Then the theorem says that α is a π-automorphism where $\alpha|_{M_o(G,\alpha) = G}$ is a π-automorphism which certainly is the case.

Suppose the theorem is true when $\bar{M}$ is of length k - 1. Consider α such that $\bar{M}: M_o = G \supsetneq M_1(G,\alpha) \supsetneq \cdots \supsetneq M_k(G,\alpha) = M_{k+1}(G,\alpha)$ is of length k.

Now α is an automorphism of $M_1(G,\alpha)$ and the $\bar{M}$ series for $\alpha|_{M_1(G,\alpha)}$ is of length k - 1.

By induction

$\alpha|_{M_1(G,\alpha)}$ is a σ-automorphism where $\sigma = \pi' \cup \sigma''$ with

$\alpha|_{M_k(G,\alpha)}$ a π'-automorphism and $M_2(G,\alpha)$ a σ''-group.

If $g \in G$, $\alpha(g)g^{-1} = x \in M_1(G,\alpha)$ so $\alpha(g) = xg$ and

$\alpha^2(g) = \alpha(\alpha(g)) = \alpha(xg) = \alpha(x)\alpha(g) = \alpha(x)xg$. Then in general

$\alpha^n(g) = \alpha^{n-1}(k) \cdots \alpha(x)xg$.

Let m be the least natural number such that $\alpha^m(x) = x$. Then since $x \in M_1(G,\alpha)$ $\alpha(x) = \alpha|_{M_1(G,\alpha)}(x)$ and, since $\alpha|_{M_1(G,\alpha)}$ is a σ-automorphism it follows that m is a σ-number.

Suppose $M_1(G,\alpha)$ is a π''-group, then $\sigma'' \subseteq \pi''$.

Now $\alpha^{m+1}(g) = x\,\alpha^{m-1}(x) \cdots \alpha(x)xg$ and

$\alpha^{2m}(g) = (\alpha^{m-1}(x) \cdots \alpha(x)x)^2 g$.

So that, in general, $\alpha^{tm}(g) = (\alpha^{m-1}(x) \cdots \alpha(x)x)^t g$. Hence $\alpha^{tm}(g) = g$ if and only if $(\alpha^{m-1}(x) \cdots \alpha(x)x)^t = 1$. If $s = |\alpha^{m-1}(x) \cdots \alpha(x)x|$ which is a π''-number then sm is a $\sigma \cup \pi'' = \pi' \cup \sigma'' \cup \pi'' = \pi' \cup \pi''$ number since $\sigma'' \subseteq \pi''$.

Set $\pi = \pi' \cup \pi''$.

Now let q be the π-number with the highest exponents appearing on the primes in the factorizations of the numbers ms obtained as described above as g ranges over all of G.

Then if $g \in G$, m and s can be found as above and there is a natural number a such that $q = ams$ and $\alpha^q(g) = (\alpha^{ms})^a(g) = g$.

Since this is true for all $g \in G$, $|\alpha| \,\big|\, q$. But q is a π-number and thus α is a π-automorphism.

This theorem has two corollaries which give information about the order of a stabilizing automorphism.

<u>Corollary 2.16</u>: Let G be a finite group and suppose $\alpha \in Aut(G)$ stabilizes the series

$\bar{G}: G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$. Then if G_1 is a π-group, α is a π-automorphism.

<u>Proof</u>:

Since α is a stabilizing automorphism of G there is a natural number n such that $M_n(G,\alpha) = (1)$. By Theorem 2.15 α is a π'-automorphism where $M(G,\alpha)$ is a π'-group. Now $M(G,\alpha) \subseteq G_1$ so $\pi' \subseteq \pi$ and hence α is a π-automorphism.

It may be noted that when $\bar{G}$ is the particular series $\bar{M}$ the corollary says that if α is a stabilizing automorphism and $M(G,\alpha) = M_1(G,\alpha)$ is a π-group then α is a π-automorphism.

The corollary which follows gives information about the order of a stabilizing automorphism, α, which is independent of the particular α chosen.

<u>Corollary 2.17</u>: If α stabilizes a series of G and Fit(G), the Fitting subgroup of G, is a π-group then α is a π-automorphism.

<u>Proof</u>:

α stabilizes $\bar{M}$: $G = M_o \trianglerighteq M_1(G,\alpha) \trianglerighteq \cdots \trianglerighteq M_n(G,\alpha) = (1)$.

By Theorems 1.3 and 1.4 $M_1(G,\alpha)$ is nilpotent and $M_1(G,\alpha) \trianglelefteq G$ thus $M_1(G,\alpha) \subseteq Fit(G)$.

Then α stabilizes $G \trianglerighteq Fit(G) \trianglerighteq M_1(G,\alpha) \trianglerighteq \cdots \trianglerighteq M_n(G,\alpha) = (1)$. So by Theorem 2.15 if Fit(G) is a π-group α is a π-automorphism.

<u>Example 2.18</u>: Let $G = S_3$, the symmetric group on three letters.

Now $Fit(S_3) = A_3$. Corollary 2.17 then says that any stabilizing automorphism of S_3 is a 3-automorphism showing that none of the 2-automorphisms of S_3 can stabilize a series of S_3.

We shall now determine exactly which inner automorphisms of a finite group stabilize a series of G.

Let $[x,y] = x^{-1}y^{-1}xy$ denote the commutator of the elements x and y of G. We define inductively the following series of commutators in G. $[x,y]_o = y$, $[x,y]_n = [x,[x,y]_{n-1}]$.

We call an element x of G an <u>Engel element</u> of G if for each $y \in G$ there exists a natural number n(y) such that $[x,y]_{n(y)} = 1$.

Let Fit(G) denote the Fitting subgroup of G and define inductively $Fit_o(G) = (1)$, $Fit_n(G)$ is the subgroup of G corresponding to $Fit\left(\frac{G}{Fit_{n-1}(G)}\right)$. It's known that all elements of Fit(G) are Engel elements and that if G is a finite solvable group, then Fit(G) is exactly the set of all Engel elements of G. Also $Fit_n(G)$ is a normal subgroup of G and solvable if G is any finite group.

<u>Theorem 2.19</u>: Let G be a finite group. If for $g \in G$, π_g, the inner automorphism induced by g, stabilizes a series of G, then g is an Engel element of G.

<u>Proof</u>:

Suppose $g \in G$ such that π_g stabilizes a series of G. We must show that for each $x \in G$ there exists a natural number n(x) such that $[g,x]_{n(x)} = 1$.

Since π_g is a stabilizing automorphism there exists a natural number n such that $M_n(G,\pi_g) = (1)$. Now $[g,x] = g^{-1}x^{-1}gx = \pi_g(x^{-1})x \in M_1(G,\pi_g)$. Proceeding by induction suppose $[g,x]_j \in M_j(G,\pi_g)$ then

$$[\,g,x\,]_{j+1} = [\,g,[\,g,x\,]_j\,]$$
$$= g^{-1}[\,g,x\,]_j^{-1}g[\,g,x\,]_j$$
$$= \pi_g([\,g,x\,]^{-1}_j)[\,g,x\,]_j$$

thus $[\,g,x\,]_{j+1} \in M_{j+1}(G,\pi_g)$

therefore $[\,g,x\,]_n \in M_n(G,\pi_g) = (\,1\,)$.

Since this is true for all $x \in G$ we conclude that g is an Engel element of G.

Proposition 2.20: If for $g \in G$, $M(G,\pi_g)$ is nilpotent then $g \in Fit_n(G)$ where n is the least integer such that $Fit_n(G) = Fit_{n+1}(G)$.

Proof:

If n is the least natural number such that $Fit_n(G) = Fit_{n+1}(G)$, then for $x \in G$, $\pi_g(x^{-1})x \in Fit(G)$ since $M(G,\pi_g)$ is nilpotent and normal.

But $Fit(G) \subseteq Fit_n(G)$, so $\pi_g(x^{-1})x \in Fit_n(G)$, that is $g^{-1}x^{-1}gx \in Fit_n(G)$ thus $gFit_n(G) \in Z(\frac{G}{Fit_n(G)})$.

But $Z(\frac{G}{Fit_n(G)}) \subseteq Fit(\frac{G}{Fit_n(G)}) = (\,1\,)$ therefore $g \in Fit_n(G)$.

The following theorem gives a characterization of the inner automorphisms which stabilize a series of G:

Theorem 2.21: If $g \in G$ then π_g is a stabilizing automorphism of G if and only if $g \in Fit(G)$.

Proof:

If $g \in Fit(G)$, since $Fit(G) \trianglelefteq G$, $M(G,\pi_g) \subseteq Fit(G)$. By Corollary 2.7, since $Fit(G)$ is nilpotent, π_g stabilizes a series of $Fit(G)$. Therefore π_g stabilizes a series of G.

If π_g stabilizes a series of G, we proceed by first considering solvable groups.

Case 1: G is solvable.

Suppose π_g stabilizes a series of G. Then by Theorem 2.19, g is an Engel element and since G is solvable, Fit(G) is exactly the set of Engel elements of G. Thus $g \in \text{Fit}(G)$.

Case 2: G is any finite group.

Let n be the least natural number such that $\text{Fit}_n(G) = \text{Fit}_{n+1}(G)$. Since π_g stabilizes a series of G, $M(G,\pi_g)$ is nilpotent. Then by Proposition 2.20 $g \in \text{Fit}_n(G)$.

Now $\text{Fit}_n(G)$ is solvable and π_g is an automorphism of $\text{Fit}_n(G)$, so since π_g stabilizes a series of G it must stabilize a series of $\text{Fit}_n(G)$.

It then follows from Case 1 that $g \in \text{Fit}(\text{Fit}_n(G))$. Now $\text{Fit}(\text{Fit}_n(G))$ is a characteristic subgroup of $\text{Fit}_n(G)$ which in turn is normal in G.

Therefore $\text{Fit}(\text{Fit}_n(G)) \trianglelefteq G$ and nilpotent so that $\text{Fit}(\text{Fit}_n(G)) \subseteq \text{Fit}(G)$ and $g \in \text{Fit}(G)$.

CHAPTER III

THE STABILIZING GROUP OF A FINITE GROUP

Example 2.9 shows that, in general, the set of all stabilizing automorphisms of a group is not itself a group. In this chapter the properties of the subgroup of Aut(G) generated by the stabilizing automorphisms will be investigated. By studying this group more information is obtained about stabilizing automorphisms and their relationship with the automorphism group.

Definition 3.1: The stabilizing group of the group G is given by

$\langle \alpha \in Aut(G) : \alpha$ is a stabilizing automorphism of $G \rangle$.

This group will be denoted by S(G).

Theorem 3.2: S(G) is a normal subgroup of Aut(G).

Proof:

If α is a generator of S(G), that is, a stabilizing automorphism of G, then there is a natural number n such that $M_n(G,\alpha) = (1)$. To prove that $S(G) \trianglelefteq Aut(G)$ it suffices to show that for each $\beta \in Aut(G)$ there is a natural number m such that $M_m(G,\alpha^\beta) = (1)$, since then α^β is a stabilizing automorphism and therefore α^β is in S(G).

We use induction to show that

$$M_i(G,\alpha^\beta) \subseteq \beta^{-1}(M_i(G,\alpha))$$

Let $g \in G$, then there is an $h \in G$ such that

$$g = \beta^{-1}(h) \text{ and } \alpha^{\beta}(g)g^{-1} = \beta^{-1}\alpha\beta(\beta^{-1}(h))(\beta^{-1}(h))^{-1}$$

$$= \beta^{-1}\alpha(h)\beta^{-1}(h^{-1})$$

$$= \beta^{-1}(\alpha(h)h^{-1})$$

thus $M(G,\alpha^{\beta}) = M_1(G,\alpha^{\beta}) \subseteq \beta^{-1}(M_1(G,\alpha)$.

Assume $M_{i-1}(G,\alpha^{\beta}) \subseteq \beta^{-1}(M_{i-1}(G,\alpha))$.

Let x be a generator of $M_i(G,\alpha^{\beta})$, then there exists $y \in M_{i-1}(G,\alpha^{\beta})$ such that $x = \alpha^{\beta}(y)y^{-1}$.

By the induction hypothesis there exists a $z \in M_{i-1}(G,\alpha)$ such that $y = \beta^{-1}(z)$.

Then $x = \alpha^{\beta}(y)y^{-1}$

$$= \beta^{-1}\alpha\beta(\beta^{-1}(z)(\beta^{-1}(z))^{-1}$$

$$= \beta^{-1}\alpha(z)\beta^{-1}(z^{-1})$$

$$= \beta^{-1}(\alpha(z)z^{-1})$$

Therefore $M_i(G,\alpha^{\beta}) \subseteq \beta^{-1}(M_i(G,\alpha))$ and $M_n(G,\alpha^{\beta}) \subseteq \beta^{-1}(M_n(G,\alpha)) = (1)$ thus $M_n(G,\alpha^{\beta}) = (1)$ completing the proof.

From [1] we have the following:

<u>Theorem 3.3</u>: If A is a normal subgroup of $\mathrm{Aut}(G)$ then $M(G,A)$ is a characteristic subgroup of G.

<u>Proof</u>:

Let $g \in G$, $\alpha \in A$, $\beta \in \mathrm{Aut}(G)$.

Then $\beta(\alpha(g)g^{-1}) = \beta\ \alpha\ \beta^{-1}(\beta(g))(\beta(g))^{-1}$.

Since $A \trianglelefteq Aut(G)$, $\beta \alpha \beta^{-1} \varepsilon A$.

Therefore $\beta(\alpha(g)g^{-1}) \varepsilon M(G,A)$ and $M(G,A)$ is a characteristic subgroup of G.

Corollary 3.4: $M(G,S(G))$ is a characteristic subgroup of G.

Proof:

This follows from Theorem 3.2 and Theorem 3.3.

The following theorem follows directly from Corollary 2.7.

Theorem 3.5: If G is a finite nilpotent group then $Inn(G) \leq S(G)$ where $Inn(G)$ is the group of inner automorphisms of G.

Theorem 3.6: If G is a finite group, then $M(G,S(G))$ is nilpotent.

Proof:

Let $\alpha_1, \cdots, \alpha_n$ be the stabilizing automorphisms of G, then $S(G) = < \alpha_1, \cdots, \alpha_n >$.

Set $E = \{ \alpha_1, \cdots, \alpha_n \}$.

By Theorem 2.2 $M(G,E) = M(G,<E>) = M(G,S(G))$.

Now $M(G,E) = < \alpha_i(g)g^{-1}: g \varepsilon G, \alpha_i \varepsilon E >$.

So $M(G,E) = \prod_{i=1}^{n} M(G,\alpha_i)$

Since α is a stabilizing automorphism by Theorem 1.4 $M(G,\alpha_i)$ is nilpotent $i = 1, \cdots, n$.

Also $M(G,\alpha_i) \trianglelefteq G$ $i = 1, \cdots, n$ by Theorem 1.3.

Therefore $\prod_{i=1}^{n} M(G,\alpha_i) = M(G,E) = M(G,S(G))$ is nilpotent.

Corollary 3.7: If $S(G)$ contains a fixed point free automorphism, then G is nilpotent.

Proof:

If S(G) contains a fixed point free automorphism, then $G = M(G,S(G))$. Therefore by Theorem 3.6 G is nilpotent.

The following example gives a simple application of Theorem 3.6.

Example 3.8: Let $G = S_3$, the symmetric group on 3 letters.

$Aut(S_3) \simeq S_3$ and if $\alpha \in Aut(S_3)$ has order 2 then $M(S_3,\alpha) = S_3$. Theorem 3.6 then gives that $\alpha \notin S(S_3)$.

A generalization of nilpotent groups due to Kaloujnine in [8] will now be described.

Definition 3.9: Let $A \leq Aut(G)$. If there exists a natural number n such that $M_n(G,A) = (1)$, then we will say that G is A-nilpotent. Further if n is the least natural number such that $M_n(G,A) = (1)$, then G will be said to be A-nilpotent of class n.

It may be noted that in this generalization the usual idea of a nilpotent group occurs when $A = Inn(G)$.

Proposition 3.10: Let G be a finite group with $H \leq Fit(G)$. Then G is A-nilpotent where $A = \{ \pi_h : h \in H \}$.

Proof:

Since $H \leq Fit(G) \trianglelefteq G$ we have $M(G,A) = [G,H] \subseteq [G,Fit(G)] \subseteq Fit(G) = \gamma_1(Fit(G))$. Proceeding by induction suppose $M_j(G,A) \subseteq \gamma_j(Fit(G))$.

Then $M_{j+1}(G,A) = M(M_j(G,A),A)$

$$= [M_j(G,A),H]$$

$$\subseteq [\gamma_j(\mathrm{Fit}(G)),H]$$

$$\subseteq [\gamma_j(\mathrm{Fit}(G)),\mathrm{Fit}(G)]$$

$$= \gamma_{j+1}(\mathrm{Fit}(G))$$

Thus $M_k(G,A) \subseteq \gamma_k(\mathrm{Fit}(G))$.

Since Fit(G) is nilpotent there exists a natural number n such that $\gamma_n(\mathrm{Fit}(G)) = (\ 1\ )$.

Therefore $M_n(G,A) = (\ 1\ )$ and G is A-nilpotent.

The following theorem establishes that if S(G) is nilpotent, S(G) must stabilize some series of G. Thus if S(G) is nilpotent all of its elements are stabilizing automorphisms.

In view of Definition 3.9 it may be noted that the following are equivalent properties of $A \leq \mathrm{Aut}(G)$.

1. A stabilizes a series of G.

2. G is A-nilpotent.

Theorem 3.11: If G is a finite group then S(G) is nilpotent if and only if G is S(G)-nilpotent.

Proof:

If G is S(G)-nilpotent, then S(G) stabilizes a series of G. Thus by Theorem 1.4 S(G) is nilpotent.

Conversely, suppose S(G) is nilpotent. Form the semi-direct product $H = G\cdot[S(G)]$. Let $A = \{\ \pi_\alpha \in \mathrm{Aut}(H)\colon \alpha \in S(G)\ \}$.

We now show that there exists a natural number n such that $M_n(H,A) \subseteq G\cdot[1] \simeq G$.

If we write elements of H in the form (g,α) where $g \in G$ and $\alpha \in S(G)$ then multiplication in H of the second coordinates is independent of the first coordinates and is the multiplication of S(G).

Now $M(H,A) = [\ H,(\ 1\ )\cdot S(G)\ ]$, so the second coordinate of $M(H,A)$ is contained in $\gamma_2(S(G))$.

Proceeding by induction we assume that the second coordinate of $M_k(H,A)$ is contained in $\gamma_{k+1}(S(G))$.

$$\text{Then } M_{k+1}(H,A) = M(M_k(H,A),A)$$

$$= [\ M_k(H,A),(\ 1\ )\cdot S(G)\ ]$$

The second coordinate of $M_k(H,A)$ is contained in $\gamma_{k+1}(S(G))$ so the second coordinate of $M_{k+1}(H,A)$ is contained in $[\ \gamma_{k+1}(S(G)),S(G)\ ]$ which is $\gamma_{k+2}(S(G))$.

Therefore by induction the second coordinate of $M_m(H,A)$ is contained in $\gamma_{m+1}(S(G))$ for $m = 1,2, \cdots$.

Then since S(G) is nilpotent there is an n such that $\gamma_{n+1}(S(G)) = 1$ and hence $M_n(H,A) \subseteq G\cdot[1]$.

Let α be a generator of S(G), that is, α is a stabilizing automorphism of G. From the work above there is a natural number n such that $M_n(H,\pi_\alpha) \subseteq G\cdot[1]$. Also since α is a stabilizing automorphism of G there is a natural number m such that $M_m(G,\alpha) = (\ 1\ )$.

Now $M_{n+j}(H,\pi_\alpha) \subseteq M(G,\alpha)\cdot[1]$ so that $M_{m+n}(H,\pi_\alpha) = (\ 1\ )$. Therefore π_α is an inner stabilizing automorphism of H.

By Theorem 2.21 $\alpha \in \mathrm{Fit}(H)$ and then since α is any generator of S(G) we can conclude that $(\ 1\ )\cdot[S(G)] \subseteq \mathrm{Fit}(H)$.

It follows from Proposition 3.10 that H is A-nilpotent. A then stabilizes $\bar{M}$: $H = M_0(H,A) \geq M_1(H,A) \geq \cdots \geq M_n(H,A) = (1)$.

It will now be shown that $M_i(G,S(G))$ is contained in the first coordinate of $M_i(H,A)$.

Let $\alpha \in S(G)$, $x \in G$, so that $\alpha(x)x^{-1}$ is a generator of $M(G,S(G))$. Now $\pi_\alpha(x,1)(x,1)^{-1} \in M(H,A)$ and $\pi_\alpha(x,1)(x,1)^{-1} = \alpha(x,1)\alpha^{-1}(x,1)^{-1} = (\alpha(x)x^{-1},1)$ so $M(G,S(G))$ is contained in the first coordinate of $M(H,A)$.

Proceeding by induction we assume that $M_{k-1}(G,S(G))$ is contained in the first coordinate of $M_{k-1}(H,A)$.

If $\alpha \in S(G)$ and $x \in M_{k-1}(G,S(G))$ the induction hypothesis gives that $(x,1) \in M_{k-1}(H,A)$ so $\pi_\alpha(x,1)(x,1)^{-1} = (\alpha(x)x^{-1},1)$ is in $M_k(H,A)$. Therefore $M_k(G,S(G))$ is contained in the first coordinate of $M_k(H,A)$. From the work above we know that there exists a natural number n such that $M_n(H,A) = 1$.

Then it must be that $M_n(G,S(G)) = (1)$ since it is contained in the first coordinate of $M_n(H,A)$. Thus G is S(G)-nilpotent.

The relationship between the nilpotence class of S(G) and the S(G)-class of G will now be investigated when S(G) is nilpotent. It will be shown that the S(G)-class of G imposes a bound on the nilpotence class of S(G) but that no bound is imposed on the S(G)-class of G by the nilpotence class of S(G).

Theorem 3.12: If $S(G)$ is nilpotent and m is the $S(G)$-class of G then the nilpotence class of $S(G)$ is $\leq \frac{m(m-1)}{2}$.

Proof:

Let $\bar{M}$ be the series

$$\bar{M}: G = M_0 \gneq M_1(G,S(G)) \gneq \cdots \gneq M_m(G,S(G)) = (1).$$

It is shown in the proof of Theorem 3.11 that $M_m(G,S(G)) = (1)$.

Now $\text{Stab}(\bar{M}) = S(G)$ so by Theorem 1.6 the nilpotence class of $S(G)$ is $\leq \frac{m(m-1)}{2}$.

Example 3.13: Let $G = C_{p^n}$, the cyclic group of order p^n where p is an odd prime and $n \geq 2$. Then $\text{Aut}(C_{p^n}) \cong C_{(p-1)p^{n-1}}$ and $\text{Aut}(C_{p^n})$ has $C_{p^{n-1}}$ as a normal p-Sylow subgroup. Thus $S(C_{p^n}) \cong C_{p^{n-1}}$ and $S(C_{p^n})$ is nilpotent with nilpotence class 1.

We show that there exists $\alpha \in S(C_{p^n})$ such that $M_{n-1}(C_{p^n},\alpha) \neq 1$ which is sufficient to show that the $S(C_{p^n})$-class of C_{p^n} is at least n.

Let a be a generator of C_{p^n}. Define $\alpha \in \text{Aut}(C_{p^n})$ by $\alpha(a) = a^{p+1}$. Then α is an automorphism since a^{p+1} is also a generator of C_{p^n}.

Now for the natural number m, $\alpha(a^m)a^{-m} = a^{(p+1)m}a^{-m} = a^{pm}$.

So $M_1(C_{p^n},\alpha)$ contains a^{pm} for all m.

Thus $C_{p^{n-1}} \subseteq M_1(C_{p^n},\alpha)$ and $C_{p^{n-1}} \cong M_1(C_{p^n},\alpha)$.

Proceeding by induction we assume $M_k(C_{p^n},\alpha) \cong C_{p^{n-k}} = \langle a^{p^k}\rangle$.

Let $x \in C_{p^{n-k}}$, then $x = a^{mp^k}$ for some natural number m

then $\alpha(a^{mp^k})a^{-mp^k} = a^{mp^k(p+1)}a^{-mp^k}$

$= a^{mp^{k+1}}$

Thus $M_{k+1}(C_{p^n},\alpha)$ contains $a^{mp^{k+1}}$ for all m.

Therefore $M_{k+1}(C_{p^n},\alpha) = \langle a^{p^{k+1}} \rangle \simeq C_{p^{n-(k+1)}}$.

In particular for k = n-1 it follows that

$M_{n-1}(C_{p^n},\alpha) \simeq C_{p^{n-(n-1)}} = C_p \neq (1)$.

Example 3.14: This is an example to show that it may happen that $M(G,\alpha) \subseteq Fit(G)$ for all $\alpha \in S(G)$ and yet S(G) does not consist entirely of stabilizing automorphisms.

Let V be the Klein 4-group, then $Aut(V) = S_3$ and $Fit(V) = V$. Then $M(V,\alpha) \subseteq Fit(V)$ for all $\alpha \in Aut(V)$, while $S(V) = S_3$ and is not all stabilizing automorphisms.

The problem of when a group is Aut(G)-nilpotent will now be considered.

Lemma 3.15: Let P be a p-group, then P is Aut(P)-nilpotent if and only if Aut(P) is a p-group.

Proof:

Suppose Aut(P) is a p-group. By Theorem 2.5 all automorphisms of P are stabilizing automorphisms so S(P) = Aut(P). Then S(P) is a p-group and thus is nilpotent so by Theorem 3.11 P is Aut(P)-nilpotent.

Conversely, if Aut(P) contains a p'-automorphism it cannot stabilize a series of P by Theorem 2.5.

Lemma 3.15 may be restated as follows: a p-group P is Aut(P)-nilpotent if and only if it admits no p'-automorphisms.

Example 3.16: Let C_{2^n} be the cyclic group of order 2^n. Now

$\mathrm{Aut}(C_{2^n}) \simeq C_2 \times C_{2^{n-1}}$, so the lemma says C_{2^n} is $\mathrm{Aut}(C_{2^n})$-nilpotent.

Example 3.17: Let D be the dihedral group of order 8. Now

$\mathrm{Aut}(D) \simeq D$, so again the lemma says that D is Aut(D)-nilpotent.

Theorem 3.18: A finite group G is Aut(G)-nilpotent if and only if G is nilpotent and its Sylow subgroups are p-groups that admit no p'-automorphisms.

Proof:

Suppose G is Aut(G)-nilpotent. G must be nilpotent since otherwise there exists a $g \in G$ such that $g \notin \mathrm{Fit}(G)$ and by Theorem 2.21 π_g could not be a stabilizing automorphism.

Conversely, if G is nilpotent, let $G = P_1 \times \cdots \times P_n$ be the factorization of G into the direct product of its Sylow subgroups.

Then $\mathrm{Aut}(G) = \mathrm{Aut}(P_1) \times \cdots \times \mathrm{Aut}(P_n)$.

But G is then Aut(G)-nilpotent if and only if P_i is $\mathrm{Aut}(P_i)$-nilpotent for $i = 1, \cdots, n$.

Lemma 3.15 then give the desired conclusion.

We now consider when the group G is S(G)-nilpotent.

Theorem 3.19: Let P be a finite p-group. P is S(P) nilpotent if and only if Aut(P) has a normal p-Sylow subgroup.

Proof:

Suppose Aut(P) has a normal p-Sylow subgroup S_p. Then $S(P) = S_p$ since S(P) is generated by all the p-elements of Aut(P). Therefore S(P) is nilpotent and by Theorem 3.11, P is S(P)-nilpotent.

Conversely, if P is S(P)-nilpotent, then S(P) must be a p-group. But S(P) includes all of the p-elements of Aut(P) and therefore it must be the unique p-Sylow subgroup of Aut(P).

<u>Example 3.20</u>: This is an example of a non-abelian p-group P $(p \nmid 2)$ which is S(P)-nilpotent.

Let P be the non-abelian p-group of order p^3 for $p \nmid 2$ with generators a and b satisfying the following relations:

$a^{p^2} = 1$, $b^p = 1$, and $bab^{-1} = a^{1+p}$. Then according to [11] (page 151), $|Aut(P)| = p^3(p-1)$.

Thus Aut(P) has a normal p-Sylow subgroup and by Theorem 3.19, P is S(P)-nilpotent.

We may think of Theorem 3.19 as saying that the problem of finding S(P)-nilpotent p-groups is the same as the problem of finding p-groups in which Aut(P) has a normal p-Sylow subgroup. This latter problem has been solved in [6] in which it is shown that Aut(P) has a normal p-Sylow subgroup if and only if the invariants of P are distinct. Thus for P an abelian p-group we can state Theorem 3.19 as follows:

<u>Theorem 3.21</u>: If P is a finite abelian p-group then P is S(P)-nilpotent if and only if the invariants of P are distinct.

It is clear that this can be extended to any abelian group as follows:

<u>Theorem 3.22</u>: If G is an abelian group, then G is S(G)-nilpotent if and only if the invariants of S_p are distinct for each p-Sylow subgroup, S_p, of G.

Some examples of $S(G)$ for various groups will now be examined. The next theorem gives a large class of examples of $S(G)$ and will give an example of a group G for which $S(G)$ is not solvable.

Theorem 3.23: Let P be an elementary abelian p-group of order p^n, then

$S(P) \simeq SL(n,p)$.

Proof:

According to Theorem 2.5, $S(P)$ is generated by the set of p-automorphisms of P.

Now $Aut(P) \simeq GL(n,p)$ and $\left|\frac{GL(n,p)}{SL(n,p)}\right| = p-1$.

Thus since $SL(n,p) \trianglelefteq GL(n,p)$, $SL(n,p)$ contains all of the p-elements of $GL(n,p)$.

So $S(P) \subseteq SL(n,p)$.

On the other hand $SL(n,p)$ is generated by the transvections and the transvections of $GL(n,p)$ all have order p. So $SL(n,p) \subseteq S(P)$.

Hence $SL(n,p) = S(P)$.

Corollary 3.24: If P is an elementary abelian 2-group then $S(P) = Aut(P)$.

Proof:

$$\left|\frac{GL(n,2)}{SL(n,2)}\right| = 2-1 = 1.$$

Therefore $Aut(P) = GL(n,2) = SL(n,2) = S(P)$.

Example 3.25: Let G be the elementary abelian 2-group of order 2^3. Then $S(G) \simeq SL(3,2) \simeq A_5$. So for this group $S(G)$ is not solvable.

We now find $S(S_n)$ where S_n is the symmetric group on n letters.

For $n \geq 5$, $S_n \trianglerighteq A_n \trianglerighteq (1)$ is the only sub-normal series of S_n and if $x \in S_n$, π_x is not the identity automorphism on A_n for any $x \in S_n$ so $S(S_n) = (1)$.

For $n = 2$, $\mathrm{Aut}(S_2) = (1)$ so $S(S_2) = (1)$.

For $n = 3$, only the automorphisms induced by elements of A_3 stabilize the series $S_3 \trianglerighteq A_3 \trianglerighteq (1)$ so $S(S_3) = A_3$.

For $n = 4$, there are several possibilities for sub-normal series S_4 using A_4, V, the Klein 4-group, and its subgroups. Checking these it is found that $S(S_4) = V$.

The automorphisms induced on subgroups and factor groups by stabilizing automorphisms are now considered.

Theorem 3.26: Let $N \trianglelefteq G$, $\alpha \in \mathrm{Aut}(G)$ and $\bar{\alpha}$ the automorphism of $\frac{G}{N}$ induced by α, then $M_n(\frac{G}{N},\bar{\alpha}) = \frac{M_n(G,\alpha)N}{N}$.

Proof:

We proceed by induction. Certainly,

$$M_o(\frac{G}{N},\bar{\alpha}) = \frac{G}{N} = \frac{GN}{N} = \frac{M_o(G,\alpha)N}{N}.$$

Suppose $M_k(\frac{G}{N},\bar{\alpha}) = \frac{M_k(G,\alpha)N}{N}$.

Then $M_{k+1}(\frac{G}{N},\bar{\alpha}) = \langle\, \bar{\alpha}(\bar{x})(\bar{x})^{-1} \colon \bar{x} \in M_k(\frac{G}{N},\bar{\alpha}) \,\rangle$ and

$\frac{M_{k+1}(G,\alpha)N}{N} = \langle\, \alpha(x)x^{-1}N \colon x \in M_k(G,\alpha) \,\rangle$ if $\alpha(x)x^{-1}N$ for $x \in M_k(G,\alpha)$ is a generator of $\frac{M_{k+1}(G,\alpha)N}{N}$, then, by the induction hypothesis, $\bar{x} \in M_k(\frac{G}{N},\bar{\alpha})$.

Then there is a $y \in M_i(G,\alpha)$ such that

$\bar{x} = \bar{y} \quad \alpha(\bar{x})(\bar{x})^{-1} = \alpha(\bar{y})(\bar{y})^{-1}$

$$= (y)y^{-1}N \in \frac{M_{k+1}(G,\alpha)N}{N}.$$

Therefore $M_{k+1}(\frac{G}{N},\bar{\alpha}) = \frac{M_{k+1}(G,\alpha)N}{N}$.

Corollary 3.27: If $\alpha \in S(G)$ and $N \trianglelefteq G$ then $\bar{\alpha} \in S(\frac{G}{N})$.

Proof:

Since $\alpha \to \bar{\alpha}$ is a homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(\frac{G}{N})$ it suffices to show that if α is a generator of $S(G)$, then $\bar{\alpha} \in S(\frac{G}{N})$. So let α be a stabilizing automorphism of G. Then there is an n such that $M_n(G,\alpha) = (1)$ and by Theorem 3.26 $M_n(\frac{G}{N},\bar{\alpha}) = \frac{M_n(G,\alpha)N}{N} = \frac{(1)N}{N} = (\bar{1})$.

Therefore $\bar{\alpha} \in S(\frac{G}{N})$.

If we consider the homomorphism Φ: $\mathrm{Aut}(G) \to \mathrm{Aut}(\frac{G}{N})$ defined by $\Phi(\alpha) = \bar{\alpha}$, the corollary shows that $\Phi|_{S(G)}: S(G) \to S(\frac{G}{N})$.

Now $\mathrm{Ker}\Phi = \langle \alpha \in \mathrm{Aut}(G): M(G,\alpha) \subseteq N \rangle$

so $\mathrm{Ker}\Phi|_{S(G)} = \langle \alpha \in S(G): M(G,\alpha) \subseteq N \rangle$.

Thus $S(\frac{G}{N})$ contains a subgroup isomorphic with $\frac{S(G)}{\mathrm{Ker}\Phi|_{S(G)}}$.

Theorem 3.28: If $H \leq G$, $\alpha \in \mathrm{Aut}(G)$ with $\alpha|_H$ an automorphism of H, then $\alpha \in S(G)$ implies $\alpha|_H \in S(H)$.

Proof:

$M_i(H,\alpha|_H) \subseteq M_i(G,\alpha)$.

Since $\alpha \in S(G)$, there is an n such that $M_n(G,\alpha) = (1)$, thus $M_n(H,\alpha|_H) = (1)$ and $\alpha|_H \in S(H)$.

Suppose H is a characteristic subgroup of G, then Φ: $Aut(G) \to Aut(H)$ defined by $\Phi(\alpha) = \alpha|_H$ is an homomorphism. In particular, the theorem says that $\Phi|_{S(G)}$ is an homomorphism of $S(G) \to S(H)$. So S(H) contains a subgroup isomorphic with $\frac{S(G)}{Ker\Phi|_{S(G)}}$ where $Ker\Phi|_{S(G)} = \langle \alpha \in S(G): \alpha|_H = 1 \rangle$.

<u>Definition 3.29</u>: A group is <u>characteristically simple</u> if it contains no proper characteristic subgroups.

<u>Theorem 3.30</u>: If G is a characteristically simple finite group then either $S(G) = (1)$ or G is an abelian p-group.

<u>Proof</u>:

By Corollary 3.4 M(G,S(G)) is a characteristic subgroup of G. So if G is characteristically simple $M(G,S(G)) = (1)$ or G.

If $M(G,S(G)) = (1)$, $S(G) = (1)$.

If $M(G,S(G)) = G$, by Theorem 3.6 G is nilpotent.

Since the p-Sylow subgroups of a nilpotent group are characteristic, G must be a p-group, but then Z(G) is non-trivial and characteristic, so G is abelian.

From [10] we have that a finite abelian p-group is characteristically simple if and only if P is not cyclic and of order p^n $n > 1$.

<u>Theorem 3.21</u>: If $\frac{H}{K}$ is a characteristic factor of G, then $S(\frac{H}{K}) = (1)$ or $S(\frac{H}{K}) \simeq SL(n,p)$.

Proof:

Since $\frac{H}{K}$ is a characteristic factor, it is characteristically simple and the direct product of isomorphic simple groups. By Theorem 3.30 $S(\frac{H}{K}) = (1)$ or $\frac{H}{K}$ is an abelian p-group. But the only abelian p-groups that are direct products of isomorphic simple groups are elementary abelian p-groups. Then by Theorem 3.23 $S(\frac{H}{K}) \simeq SL(n,p)$ where $|\frac{H}{K}| = p^n$.

CHAPTER IV

AUTOMORPHISMS WHICH STABILIZE HALL-TYPE SERIES

It was noted in Chapter I that Baer has shown in [2] that if $\bar{H}: G = H_o \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = (1)$ is a series of normal Hall subgroups of G, then $\alpha \in Aut(G)$ stabilizes $\bar{H}$ if and only if $\alpha = \pi_x$ where $x \in Z(\bar{H}) = \prod_{i=1}^{n} Z(H_i)$. In this Chapter we seek to generalize this result.

We will examine the stabilizing automorphisms of series of the form $\bar{H}: G = H_1 \cdots H_n \geq H_1 \cdots H_{n-1} \geq \cdots \geq H_1 \geq H_o = (1)$ where $H_iH_j = H_jH_i$ and $(|H_i|, |H_j|) = 1 \quad i \neq j \quad i,j = 1, \cdots, n$. We may note that the series which Baer examined do enjoy these properties but were also normal.

We now describe how an interesting special case of this generalization may be obtained from Hall's work on solvable groups.

<u>Definition 4.1</u>: A collection of Sylow subgroups $\{ P_1, \cdots, P_n \}$ of G is called Sylow basis for G if the collection is pairwise permutable ($P_iP_j = P_jP_i$, for $i \neq j \quad i,j = 1, \cdots, n$) and $P_1 \cdots P_n = G$.

Hall has shown that G is a solvable group if and only if it has a Sylow basis. Hall showed further that any set of permutable Sylow subgroups of a solvable group G is included in some Sylow basis of G and two Sylow bases for a given solvable group are conjugate.

Given a Sylow basis $\{ P_1, \cdots, P_n \}$ for a solvable group, series of the following type can be constructed

$\bar{P}: G = P_1 \cdots P_n \geq P_1 \cdots P_{n-1} \geq \cdots \geq P_1 \geq P_o = (1)$. These series have the properties of the series we will consider in this chapter.

We recall the following definitions to be used in what follows:

Definition 4.2: A finite group G is said to be a Sylow Tower Group if there is a normal series

$\bar{G}: G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (1)$ with the property that for $i = 1, \cdots, n, \frac{G_i}{G_{i+1}}$ is isomorphic with a Sylow subgroup of G.

All Sylow Tower Groups are solvable and the class of all such Groups properly contains the supersolvable groups.

Definition 4.3: For $H \leq G$, we define the core of H in G, written $\text{Core}_G H$, as $\text{Core}_G H = \bigcap_{x \in G} H^x$.

This says that $\text{Core}_G H$ is the largest subgroup of H which is normal in G.

We now prove some results in the direction of those of Baer. The following theorem will be used to obtain Baer's result for Sylow Tower Groups.

<u>Theorem</u> <u>4.4</u>: Suppose $G = H_1 \cdots H_n$, $(|H_i|,|H_j|) = 1 \quad i \neq j$ and

$H_iH_j = H_jH_i$, $i,j = 1, \cdots, n$. Furthermore suppose there

exists an ordering of the H_i such that

$\bar{H}_i$: $G = H_{i_1} \cdots H_{i_n} \trianglerighteq H_{i_1} \cdots H_{i_{n-1}} \trianglerighteq \cdots \trianglerighteq H_{i_1} \trianglerighteq (1)$

is a normal Hall series of G. Then if $\alpha \in \mathrm{Aut}(G)$ stabilizes

$\bar{H}$: $G = H_1 \cdots H_n \geq H_1 \cdots H_{n-1} \geq \cdots \geq H_1 \geq (1)$,

$\alpha = \pi_x$ where $x \in Z(\bar{H}) = \prod_{i=1}^{n} Z(H_i)$.

<u>Proof</u>:

We will show that α stabilizes $\bar{H}_i$ and use Baer's result in

Theorem 1.9.

Since normal Hall subgroups are characteristic

$\alpha(H_{i_1} \cdots H_{i_j}) = H_{i_1} \cdots H_{i_j} \quad j = 1, \cdots, n.$

Let $x_{i_1} \cdots x_{i_j} \in H_{i_1} \cdots H_{i_j}$ then

$$\alpha(x_{i_1} \cdots x_{i_j})(x_{i_1} \cdots x_{i_j})^{-1} =$$

$$\alpha(x_{i_1} \cdots x_{i_{j-1}})\alpha(x_{i_j})(x_{i_1} \cdots x_{i_j})^{-1}$$

from above $\alpha(x_{i_1} \cdots x_{i_{j-1}}) \in H_{i_1} \cdots H_{i_{j-1}}$.

Since α stabilizes $\bar{H}$ $\alpha(x_{i_j})x_{i_j}^{-1} \in H_1 \cdots H_{i_j-1}$.

Also $\alpha(x_{i_j})x_{i_j}^{-1} \in H_{i_1} \cdots H_{i_j}$.

Then $\alpha(x_{i_j})x_{i_j}^{-1} \varepsilon (H_1 \cdots H_{i_j-1} \cap H_{i_1} \cdots H_{i_j}) \subseteq H_{i_1} \cdots H_{i_{j-1}}$

thus

$\alpha(x_{i_1} \cdots x_{i_j})(x_{i_1} \cdots x_{i_j})^{-1} \varepsilon H_{i_1} \cdots H_{i_{j-1}}$

for $j = 2, \cdots, n$.

α also stabilizes $H_1 \cdots H_{i_1} > \cdots > H_1 > (1)$ so

$M(H_1 \cdots H_{i_1}, \alpha) \subseteq H_1 \cdots H_{i_1-1}$ and

$M(H_{i_1}, \alpha) \subseteq M(H_1 \cdots H_{i_1}, \alpha) \subseteq H_1 \cdots H_{i_1-1}$.

Since H_{i_1} is characteristic $\alpha(H_{i_1}) = H_{i_1}$ and

$M(H_{i_1}, \alpha) \subseteq H_{i_1}$.

Therefore $M(H_{i_1}, \alpha) \subseteq H_{i_1} \cap H_1 \cdots H_{i_1-1} = (1)$ so that

α is the identity on H_{i_1}.

Hence α stabilizes $\bar{H}_i$ and by Theorem 1.9 $\alpha = \pi_x$

$x \varepsilon Z(\bar{H}_i) = \prod_{j=1}^{n} Z(H_{i_j}) = \prod_{i=1}^{n} Z(H_i) = Z(\bar{H})$.

<u>Corollary 4.5</u>: Let G be a finite Sylow Tower Group and $P_1, \cdots, P_n$ be a Sylow basis for G. If α stabilizes

$\bar{P}: G = P_1 \cdots P_n \geq P_1 \cdots P_{n-1} \geq \cdots \geq P_1 \geq (1)$, then

$\alpha = \pi_x$ for some $x \varepsilon Z(\bar{P}) = \prod_{i=1}^{n} Z(P_i)$.

<u>Proof</u>:

A Sylow Tower Group has the property that its Sylow bases satisfy the hypotheses of Theorem 4.4.

<u>Proposition 4.6</u>: Let G be a finite group and $H < G$, $\alpha \in Aut(G)$. Then if $M(G,\alpha) \subseteq H$, $\alpha|_H$ is an automorphism of H.

<u>Proof</u>:

Let $h \in H$. $\alpha(h)h^{-1} \in M(G,\alpha) \subseteq H$ so $\alpha(h) \in H$.

The following result shows that a non-trivial central automorphism cannot stabilize series of the type now being considered.

<u>Theorem 4.7</u>: Suppose $G = H_1 \cdots H_n$, $(|H_i|,|H_j|) = 1$ $i \neq j$, and $H_iH_j = H_jH_i$ $i,j = 1, \cdots, n$. If $\alpha \in Aut(G)$ is a central automorphism and α stabilizes

$\bar{H}$: $G = H_1 \cdots H_n > \cdots > H_1 > 1$, then $\alpha = (1)$.

<u>Proof</u>:

$Z(G) \trianglelefteq G$ so for $i = 1, \cdots, n$ $Z(G)H_i$ is a group. By Proposition 4.6 α is an automorphism of $Z(G)H_i$.

Now $([Z(G)H_i:H_i], |H_i|) = 1$ so H_i is complemented in $Z(G)H_i$.

Thus there exists $z \in Z(G)$ such that $\alpha(H_i) = H_i^z = H_i$. Therefore $M(H_i,\alpha) \subseteq H_i$.

But since α stabilizes $\bar{H}$

$M(H_i,\alpha) \subseteq M(H_1 \cdots H_i,\alpha) \subseteq H_1 \cdots H_{i-1}$ so

$M(H_i,\alpha) \subseteq H_1 \cdots H_{i-1} \cap H_i = (1)$ thus $\alpha = 1$ on H_i $i = 1, \cdots, n$.

The following characterizes those inner automorphisms that stabilize $\bar{H}$.

Theorem **4.8**: Suppose $G = H_1 \cdots H_n$, $(|H_i|, |H_j|) = 1$ $i \neq j$,

$H_iH_j = H_jH_i$ $i,j = 1, \cdots, n$ and

$\bar{H}$: $G = H_1 \cdots H_n \geq H_1 \cdots H_{n-1} \geq \cdots \geq H_1 \geq (1)$.

Then for $x \in G$, π_x stabilizes $\bar{H}$ if and only if

$x = x_1 \cdots x_n$, $x_i \in H_i$ with $x_j \in Z(H_1 \cdots H_j)$ and

$x_1 \cdots x_k \in \mathrm{Core}_{H_1 \cdots H_{k+1}}(H_1 \cdots H_k)$ $j = 1, \cdots, n$

$k = 1, \cdots, n-1$. (*)

Proof:

The proof is by induction on the length of $\bar{H}$. For $n = 1$, π_x stabilizes $1 < H_1$ if and only if $x \in Z(H_1)$, that is, $\pi_x = 1$.

Let $x = x_1 \cdots x_n$ satisfy (*) then $\pi_x = \pi_{x_1 \cdots x_{n-1}}$ since $x_n \in Z(H_1 \cdots H_n) = Z(G)$.

So $\pi_{x_1 \cdots x_{n-1}} = \pi_x$ is an automorphism of $H_1 \cdots H_{n-1}$.

By the induction hypothesis, since

$x_i \in Z(H_1 \cdots H_i)$, $i = 1, \cdots, n-1$.

$x_1 \cdots x_j \in \mathrm{Core}_{H_1 \cdots H_{j+1}}(H_1 \cdots H_j)$ $j = 1, 2, \cdots, n-2$ and π_x stabilizes $H_1 \cdots H_{n-1} \geq \cdots \geq H_1 \geq (1)$.

It remains to show that $M(G, \pi_x) \subseteq H_1 \cdots H_{n-1}$.

Let $g \in G$, $\pi_x(g)g^{-1} = x^{-1}g\, x\, g^{-1}$

$$= (x_1 \cdots x_n)^{-1}\, g\, (x_1 \cdots x_n)g^{-1}$$

$$= (x_1 \cdots x_{n-1})^{-1}\, g\, (x_1 \cdots x_{n-1})g^{-1}$$

since $x_n \in Z(G)$.

But $g(x_1 \cdots x_{n-1})g^{-1} \in H_1 \cdots H_{n-1}$ since

$x_1 \cdots x_{n-1} \in \mathrm{Core}_{H_1 \cdots H_n}(H_1 \cdots H_{n-1})$.

So $\pi_x(g)g^{-1} \in H_1 \cdots H_{n-1}$ and $M(G,\pi_x) \subseteq H_1 \cdots H_{n-1}$.

Conversely, suppose π_x stabilizes $\bar{H}$. Then π_x also stabilizes

$H_1 \cdots H_{n-1} > \cdots > H_1 > (1)$.

So on $H_1 \cdots H_{n-1}$ $\pi_x = \pi_{y_1 \cdots y_{n-1}}$ where

$y_i \in Z(H_1 \cdots H_i)$ $i = 1, \ldots, n-1$ and

$y_1 \cdots y_j \in \mathrm{Core}_{H_1 \cdots H_{j+1}}(H_1 \cdots H_j)$ $j = 1, \ldots, n-1$ by the induction

hypothesis.

Hence $\pi_{x(y_1 \cdots y_{n-1})^{-1}} = 1_{H_1 \cdots H_{n-1}} =$

$$\pi_{(y_1 \cdots y_{n-1})^{-1}x} \text{ on } H_1 \cdots H_{n-1}$$

then $\pi_{(y_1 \cdots y_{n-1})^{-1}x}(y_1 \cdots y_{n-1}) =$

$$x^{-1}(y_1 \cdots y_{n-1})(y_1 \cdots y_{n-1})(y_1 \cdots y_{n-1})^{-1}x$$

$$= y_1 \cdots y_{n-1}.$$

So $(y_1 \cdots y_{n-1})x = x(y_1 \cdots y_{n-1})$. (**)

Write $x = x_1 \cdots x_n$, $x_i \in H_i$

then $\pi_{x(y_1 \cdots y_{n-1})^{-1}}(x_1 \cdots x_{n-1}) = x_1 \cdots x_{n-1}$.

So

$$(y_1 \cdots y_{n-1})(x_1 \cdots x_n)^{-1}(x_1 \cdots x_{n-1})(x_1 \cdots x_n)$$

$$(y_1 \cdots y_{n-1})^{-1} = x_1 \cdots x_{n-1}$$

and $(y_1 \cdots y_{n-1})x_n^{-1}(x_1 \cdots x_n)(y_1 \cdots y_n)^{-1} = x_1 \cdots x_{n-1}$.

Hence

$$(x_1 \cdots x_n)(y_1 \cdots y_{n-1})^{-1} = x_n(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1}).$$

By (**) we get

$$[(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})]x_n =$$

$$x_n[(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})]. \qquad (***)$$

Therefore

$$\pi_{[(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})]x_n} =$$

$$\pi_{x_n[(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})]}.$$

Now if $H_1 \cdots H_{n-1}$ is a π-group

$\pi_{(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})}$ is a π-automorphism.

But since $(|H_n|, |H_1 \cdots H_{n-1}|) = 1$, H_n is a π'-group so that

π_{x_n} is a π'-automorphism and $(|\pi_{x_n}|, |\pi_{(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})}|) = 1$.

Now $|\pi_{x_n}| \,\Big|\, |\pi_{(y_1 \cdots y_{n-1})^{-1}(x_1 \cdots x_{n-1})x_n}|$ on $H_1 \cdots H_{n-1}$.

By Corollary 2.16 since $\pi_{(y_1 \cdots y_n)^{-1}(x_1 \cdots x_n)}$ stabilizes

$H_1 \cdots H_{n-1} \geq H_1 \cdots H_{n-2} \geq \cdots \geq H_1 \geq (1)$

$|\pi_{(y_1 \cdots y_n)^{-1}(x_1 \cdots x_n)}|$ has the same primes as

$|H_1 \cdots H_{n-2}|$ and so is a π-automorphism on $H_1 \cdots H_{n-1}$. Since

π_{x_n} is a π'-automorphism of $H_1 \cdots H_{n-1}$ it must then be the identity on

$H_1 \cdots H_{n-1}$, that is, $x \in C_G(H_1 \cdots H_{n-1})$.

Now $\pi_x(x_1 \cdots x_{n-1}) = (x_1 \cdots x_{n-1})$ so

$x_n(x_1 \cdots x_{n-1}) = (x_1 \cdots x_{n-1})x_n$ and

$\pi_{x_n}\pi_{x_1 \cdots x_{n-1}} = \pi_{x_1 \cdots x_{n-1}}\pi_{x_n}$ so that $|\pi_{x_n}| \,\Big|\, |\pi_x|$.

Again by Corollary 2.16 since π_x stabilizes

$H_1 \cdots H_n \geq H_1 \cdots H_{n-2} \geq \cdots \geq H_1 \geq (1)$, π_x is a π-automorphism of G. So since π_{x_n} is a π'-automorphism it must be that

$\pi_{x_n} = 1$ on $H_1 \cdots H_n = G$.

Therefore $\pi_{x_1 \cdots x_n} = \pi_{x_1 \cdots x_{n-1}}$

and $\pi_{x_1 \cdots x_{n-1}}$ stabilizes $\bar{H}$ and by the induction hypothesis

$x_i \in Z(H_1 \cdots H_i)$, $i = 1, \cdots, n-1$.

$x_1 \cdots x_j \in \mathrm{Core}_{H_1 \cdots H_{j+1}}(H_1 \cdots H_j)$ $j = 1, \cdots, n-2$.

We already have that $x_n \in Z(H_1 \cdots H_n)$.

So we need only show that

$x_1 \cdots x_{n-1} \in \mathrm{Core}_{H_1 \cdots H_n}(H_1 \cdots H_{n-1})$.

Let $g \in H_1 \cdots H_n = G$

then $\pi_{x_1 \cdots x_{n-1}}(g^{-1})g \in H_1 \cdots H_{n-1}$

that is, $(x_1 \cdots x_{n-1})^{-1}g^{-1}(x_1 \cdots x_{n-1})g \in H_1 \cdots H_{n-1}$

and $g^{-1}(x_1 \cdots x_{n-1})g \in H_1 \cdots H_{n-1}$.

This says that all the conjugates of $x_1 \cdots x_{n-1}$ are in $H_1 \cdots H_{n-1}$ so that $x_1 \cdots x_{n-1} \in \mathrm{Core}_{H_1 \cdots H_n}(H_1 \cdots H_{n-1})$.

Whether these are the only automorphisms that stabilize this type of series is an open question.

CHAPTER V

POWER STABILIZING AUTOMORPHISMS

Let $\bar{G}$: $G = G_o \geq G_1 \cdots \geq G_n = (1)$ be a series of subgroups of G.

Definition 5.1: $\alpha \in Aut(G)$ is a power stabilizing automorphism of $\bar{G}$ if for each $x \in G_i$ there is a natural number n and $y \in G_{i+1}$ such that $\alpha(x) = x^n y$.

In case G is a subnormal series, Definition 5.1 can be restated as: $\alpha \in Aut(G)$ is a power stabilizing automorphism of $\bar{G}$ if α induces a power automorphism of $\frac{G_i}{G_{i+1}}$ for $i = 0, 1, \cdots, n-1$.

Theorem 5.2: If α and β are power stabilizing automorphisms of $\bar{G}$, then so is $\alpha\beta$.

Proof:

Let $x \in G_i$ $\alpha\beta(x) = \alpha(x^n y)$, $y \in G_{i+1}$

$$= \alpha(x^n)\,\alpha(y)$$

$$= (x^n)^m y'\alpha(y),\ y' \in G_{i+1}$$

$$= x^{mn}(y'\alpha(y))$$

Now $y'\alpha(y) \in G_{i+1}$ so $\alpha\beta$ is a power stabilizing automorphism of $\bar{G}$.

Theorem 5.2 makes possible the following:

Definition 5.3: The power stabilizing group of $\bar{G}$, denoted by PStab($\bar{G}$), is PStab($\bar{G}$) = { α:α is a power stabilizing automorphism of $\bar{G}$ }.

We proceed to work out the basic properties of $PStab(\bar{G})$.

Let $PAut(G)$ denote the set of power automorphisms of G which is a normal subgroup of $Aut(G)$. The properties of power automorphisms can be found in [4].

<u>Theorem</u> <u>5.4</u>: If $\bar{G}$ is a series of characteristic subgroups of G, then

$PStab(\bar{G}) \trianglelefteq Aut(G)$.

<u>Proof</u>:

Let $\alpha \in PStab(\bar{G})$, $\theta \in Aut(G)$

Then α induces $\bar{\alpha}_i$ a power automorphism of $\frac{G_i}{G_{i+1}}$. Let $\bar{\theta}_i$ be the automorphism of $\frac{G_i}{G_{i+1}}$ induced by θ (since G_i and G_{i+1} are characteristic subgroups of G).

Then $\overline{\theta^{-1}\alpha\theta}_i = \bar{\theta}_i^{-1}\bar{\alpha}_i\bar{\theta}_i$ is a power automorphism of $\frac{G_i}{G_{i+1}}$ since $PAut(\frac{G_i}{G_{i+1}}) \trianglelefteq Aut(\frac{G_i}{G_{i+1}})$.

So $\theta^{-1}\alpha\theta$ is a power stabilizing automorphism of $\bar{G}$.

<u>Example</u> <u>5.5</u>: Let $G = S_3$ and $\bar{G}$: $S_3 \trianglerighteq A_3 \trianglerighteq (1)$. Then every automorphism of S_3 is a power stabilizing automorphism of $\bar{G}$, that is, $PStab(\bar{G}) \cong S_3$. Thus $PStab(\bar{G})$ need not be nilpotent and Theorem 1.6 does not extend for $PStab(\bar{G})$.

<u>Theorem</u> <u>5.6</u>: If $\alpha \in Aut(G)$ is a power stabilizing automorphism of $\bar{G}$: $G = G_o \geq G_1 \geq \cdot \cdot \cdot \geq G_n = (1)$ and $\bar{\bar{G}}$ is a refinement of $\bar{G}$, then α is a power stabilizing automorphism of $\bar{\bar{G}}$.

Proof:

Let $H \leq K$ be two successive groups in $\bar{\bar{G}}$. Then since $\bar{\bar{G}}$ is a refinement of $\bar{G}$ there is an i such that $G_{i+1} \leq H \leq K \leq G_i$. Let $k \in K$. Since $k \in G_i$, $\alpha(k) = k^n y$, $y \in G_{i+1}$. So $\alpha(K) = K$ and since $G_{i+1} \subsetneq H$, $y \in H$.

Therefore α is a power stabilizing automorphism of $\bar{\bar{G}}$.

Theorem 5.7: Let $\bar{G}$: $G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$ be any series of G. If $\bar{\bar{G}}$: $G = G_o \geq G_{i_1} \geq \cdots \geq G_{i_k} = (1)$ is a series of G with G_{i_j} some of the groups in $\bar{G}$, then $\mathrm{Stab}(\bar{\bar{G}}) \trianglelefteq \mathrm{PStab}(\bar{G})$.

Proof:

Since $\bar{G}$ is a refinement of $\bar{\bar{G}}$, Theorem 5.6 gives that $\mathrm{PStab}(\bar{\bar{G}}) \subsetneq \mathrm{PStab}(\bar{G})$.

Let $\theta \in \mathrm{PStab}(\bar{G})$, $\alpha \in \mathrm{Stab}(\bar{\bar{G}})$. It suffices to show that $\theta^{-1}\alpha\theta(x) = xk$, $k \in G_{i_{j+1}}$ for all $x \in G_{i_j}$.

$$\text{Let } x \in G_{i_j} \quad \theta^{-1}\alpha\theta(x) = \theta^{-1}\alpha(x^m y) \quad y \in G_{i_{j+1}}$$
$$= \theta^{-1}(x^m yz) \quad z \in G_{i_{j+1}}$$
$$= \theta^{-1}(x^m y)\theta^{-1}(z)$$
$$= x\theta^{-1}(z) \quad \theta^{-1}(z) \in G_{i_{j+1}}$$

Therefore $\mathrm{Stab}(\bar{\bar{G}}) \trianglelefteq \mathrm{PStab}(\bar{G})$.

Corollary 5.8: If $\bar{G}$: $G = G_o \geq G_1 \geq \cdots \geq G_n = (1)$ is any series of G, then $\mathrm{Stab}(\bar{G}) \trianglelefteq \mathrm{PStab}(\bar{G})$.

Proof:

This is Theorem 5.7 with $\bar{\bar{G}} = \bar{G}$.

We now describe briefly how Baer's work on supersolvable immersion in [3] may be used to prove that PStab($\bar{G}$) is supersolvable if G is supersolvable.

Definition 5.9 (Baer): A normal subgroup H of G is supersolvably immersed in G if for every homomorphism θ of G with $H^\theta \neq (1)$ there is a cyclic normal subgroup $A \neq (1)$ of G^θ such that $A \leq H^\theta$.

Theorem 5.10: If $H \trianglelefteq G$ and H has a series

$\bar{H}$: $H = H_o \geq H_1 \geq \cdots \geq H_n = (1)$ with $\frac{H_i}{H_{i+1}}$ cyclic of prime order $i = 0, \cdots, n-1$ and $H_i \trianglelefteq G$ $i = 0,1, \cdots, n$ then H is supersolvably immersed in G.

Proof:

Let θ be a homomorphism of G with $H^\theta \neq (1)$. Then there is a largest i such that $H_i^\theta \neq (1)$. Now H_i^θ is cyclic and $H_i^\theta \trianglelefteq G^\theta$, so H is supersolvably immersed in G.

The following Theorem is from [3]:

Theorem 5.11 (Baer): If H is supersolvably immersed in G, then the elements of G induce in H a supersolvable group of automorphisms.

Let G be supersolvable and

$\bar{G}$: $G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (1)$ be a chief series of G. Then $\frac{G_i}{G_{i+1}}$ is cyclic of prime order. Form the semi-direct product G·[PStab($\bar{G}$)], then using Theorem 5.10 we get that G is supersolvably immersed in

$G \cdot [PStab(\bar{G})]$. It then follows from Theorem 5.11 that the elements of $G \cdot [PStab(\bar{G})]$ induce in G a supersolvable group of automorphisms. Since this group contains a subgroup isomorphic with $PStab(\bar{G})$, $PStab(\bar{G})$ is supersolvable. This conclusion is stated as

<u>Theorem 5.12</u>: Let G be a supersolvable group and $\bar{G}$ a chief series of G, then $PStab(\bar{G})$ is supersolvable.

<u>Corollary 5.13</u>: If G is supersolvable and $\bar{G}$ is any series of G, then $PStab(\bar{G})$ is supersolvable.

<u>Proof</u>:

Refine $\bar{G}$ to $\bar{\bar{G}}$ a chief series of G.

By Theorem 5.6, $PStab(\bar{G}) \subseteq PStab(\bar{\bar{G}})$ and by Theorem 5.12 $PStab(\bar{\bar{G}})$ is supersolvable.

Therefore $PStab(\bar{G})$ is supersolvable.

We will now show that if G is solvable and $\bar{G}$ is any subnormal series of G, then $PStab(\bar{G})$ is supersolvable. Because of the length of the argument, it is convenient to split up the proof. We proceed by proving the result for composition series (Lemma 5.15 and Theorem 5.16) and then appeal to Theorem 5.6 to obtain the final conclusion (Corollary 5.17).

<u>Lemma 5.14</u>: If G is a non-abelian simple group, the $PAut(G) = (1)$.

<u>Proof</u>:

Let $\alpha \in PAut(G)$. Then since power automorphisms are central, if $g \in G$, $\alpha(g)g^{-1} \in Z(G) = (1)$ because G is non-abelian and simple. Therefore α is the identity automorphism.

<u>Lemma 5.15</u>: Let $\bar{G}$: $G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (1)$ be a composition series for G. Then $\frac{PStab(\bar{G})}{Stab(\bar{G})}$ is supersolvable.

Proof:

Define a series of subgroups of $PStab(\bar{G})$ as follows:

$C_n = PStab(\bar{G})$

$C_i = \{ \alpha \in C_{i+1}: \alpha \text{ induces the identity on } \frac{G_i}{G_{i+1}} \} \quad i = 0,1, \cdots, n-1.$

Then $C_o = Stab(\bar{G})$. We now show that $C_i \trianglelefteq PStab(\bar{G}) \quad i = 0, \cdots, n.$

Let $\alpha \in C_{n-1}$, $\beta \in PStab(\bar{G})$, $x \in G_{n-1}$.

Then $\beta^{-1}\alpha\beta(x) = \beta^{-1}\alpha(x^n)$, β is a power automorphism of G_{n-1}

$= \beta^{-1}(x^n)$, α fixes $x^n \in G_{n-1}$

$= x$

thus $C_{n-1} \trianglelefteq PStab(\bar{G})$.

Proceeding by induction suppose that $C_{i+1} \trianglelefteq PStab(\bar{G})$.

Let $\alpha \in C_i$, $\beta \in PStab(\bar{G})$, $x \in G_i$

$\beta^{-1}\alpha\beta(x) = \beta^{-1}\alpha(x^n y)$ $y \in G_{i+1}$, β is a power automorphism of $\frac{G_i}{G_{i+1}}$

$= \beta^{-1}(x^n y y')$ $y' \in G_{i+1}$, α fixes $\frac{G_i}{G_{i+1}}$ element-wise

$= x\beta^{-1}(y')$ $\beta(G_{i+1}) = G_{i+1}$ implies $\beta^{-1}(y') \in G_{i+1}$.

So $\beta^{-1}\alpha\beta$ induces the identity automorphism on $\frac{G_i}{G_{i+1}}$. Since we are assuming $C_{i+1} \trianglelefteq PStab(\bar{G})$, $\beta^{-1}\alpha\beta \in C_{i+1}$ for $\alpha \in C_{i+1}$. It then follows that $C_i \trianglelefteq PStab(\bar{G})$.

We now show that $\frac{C_i}{C_{i+1}}$ is isomorphic with a group of power automorphisms of $\frac{G_{i-1}}{G_i}$.

Define $\Psi(\alpha C_{i-1})$ to be the automorphism of $\frac{G_{i-1}}{G_i}$ induced by α for $\alpha \in C_i$.

If $\alpha C_{i-1} = \beta C_{i-1}$, then $\alpha\beta^{-1} \in C_{i-1}$ so $\alpha\beta^{-1}$ induces the identity automorphism on $\frac{G_{i-1}}{G_i}$.

Therefore α and β induce the same automorphism of $\frac{G_{i-1}}{G_i}$. Ψ is then a well-defined mapping and a homomorphism of $\frac{C_i}{C_{i+1}}$ into $\frac{G_{i-1}}{G_i}$.

Suppose αC_{i-1} and βC_{i-1} are mapped to the same automorphism of $\frac{G_{i-1}}{G_i}$. Then if $x \in G_{i-1}$ and $\bar{\alpha}$ and $\bar{\beta}$ are the induced automorphisms of α and β respectively on $\frac{G_{i-1}}{G_i}$

$\bar{\alpha}(xG_i) = \bar{\beta}(xG_i) = \alpha(x)G_i = \beta(x)G_i$

So $\alpha\beta^{-1}(x) \in G_i$, that is, $\overline{\alpha\beta^{-1}}$ is the identity automorphism of $\frac{G_{i-1}}{G_i}$. Thus $\alpha\beta^{-1} \in C_{i-1}$ and $\alpha C_i = \beta C_i$, so that Ψ is one to one.

It may be noted that since $\alpha \in PStab(\bar{G})$. $\Psi(\alpha C_{i-1})$ is a power automorphism of $\frac{G_{i-1}}{G_i}$.

Then we have that $\frac{C_i}{C_{i+1}}$ is isomorphic with a group of power automorphisms of $\frac{G_{i-1}}{G_i}$.

Since $\bar{G}$ is a composition series of G, $\frac{G_{i-1}}{G_i}$ is either cyclic of prime order or non-abelian simple.

<u>Case</u> <u>1</u>: $\frac{G_{i-1}}{G_i}$ is cyclic of prime order.

Then $\text{Aut}(\frac{G_{i-1}}{G_i})$ is cyclic, so $\frac{C_i}{C_{i-1}}$ is cyclic.

<u>Case</u> <u>2</u>: $\frac{G_{i-1}}{G_i}$ is non-abelian simple.

By Lemma 5.14 $\text{PAut}(\frac{G_{i-1}}{G_i}) = (\ 1\ )$. Then since $\frac{C_i}{C_{i-1}}$ is isomorphic with a subgroup of $\text{Aut}(\frac{G_{i-1}}{G_i})$, $\frac{C_i}{C_{i-1}} = 1$.

Thus there is a series of normal subgroups of $\text{PStab}(\bar{G})$ from $\text{PStab}(\bar{G})$ to $\text{Stab}(\bar{G})$ with cyclic factor groups. So $\frac{\text{PStab}(\bar{G})}{\text{Stab}(\bar{G})}$ is supersolvable.

<u>Theorem</u> <u>5.16</u>: Let G be solvable and

$\bar{G}$: $G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (\ 1\ )$ be a composition series for G. Then $\text{PStab}(\bar{G})$ is supersolvable.

<u>Proof</u>:

Since Lemma 5.15 gives a normal series from $\text{PStab}(\bar{G})$ to $\text{Stab}(\bar{G})$ with cyclic factor groups, it remains to find a series with cyclic factor groups for $\text{Stab}(\bar{G})$ with series elements normal in $\text{PStab}(\bar{G})$.

Define $S_i = \text{Stab}(G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_i \trianglerighteq 1)$
$i = 0,1, \cdots, n-1$.

Then $S_o = (\ 1\ )$ and $S_{n-1} = \text{Stab}(\bar{G})$

Since S_i is the stability group of a subseries of $\bar{G}$ by Theorem 5.7 $S_i \trianglelefteq \text{PStab}(\bar{G})$ $i = 0,1, \cdots, n-1$.

Form the series

$\mathrm{Stab}(\bar{G}) = S_{n-1} \trianglerighteq S_{n-2} \trianglerighteq \cdots \trianglerighteq S_o = (1).$

We will show that between S_{k-1} and S_k there is a series

$S_{k-1} = T_{k-1} \trianglelefteq T_k \trianglelefteq \cdots \trianglelefteq T_n = S_k$ with $T_i \trianglelefteq \mathrm{PStab}(\bar{G})$ $i = k, \cdots, n$

and $\frac{T_i}{T_{i-1}}$ cyclic $i = k+1, \cdots, n$. This will complete the proof of the theorem.

Let $U_i = \mathrm{Stab}(G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_{k-1} \geq G_i \geq 1)$

$i = k, \cdots, n$ and then define $T_{n+k-i} = U_i \cap S_k$.

Now $U_k = S_k$ so $T_n = T_{n+k-k} = U_k \cap S_k = S_k \cap S_k = S_k$ also

$U_n = S_{k-1}$ so $T_k = T_{n+k-n} = U_n \cap S_k = S_{k-1} \cap S_k = S_{k-1}$.

By Theorem 5.7 $U_i \trianglelefteq \mathrm{PStab}(\bar{G})$ $i = k, \cdots, n$.

It may be noted that if $x \in G_{k-1}$ and $\alpha(x)x^{-1} \in G_{i+1}$ then $\alpha(x)x^{-1} \in G_i$ since $G_i \subsetneq G_{i+1}$.

Any element of T_{n+k-j} induces the identity automorphism on G_k $j = k, \cdots, n$ since $T_{n+k-j} = U_k \cap S_k$ and elements of S_k induce the identity automorphism on G_k. Thus $T_{n+k-i} \subseteq T_{n+k-(i+1)}$ $i = k, \cdots, n-1$.

So we have a series

$S_{k-1} = T_k \trianglelefteq T_{k-1} \trianglelefteq \cdots \trianglelefteq T_n = S_k$.

Let $x_j \in G_j$ be such that $x_j G_{j+1}$ is a generator of the cyclic group $\frac{G_j}{G_{j+1}}$ ($\frac{G_j}{G_{j+1}}$ is cyclic because G is solvable and $\bar{G}$ is a composition series).

Then $\langle x_{n-1} \rangle = G_{n-1}$ and $G_j = \langle G_{j+1}, x_j \rangle$ $j = 0, 1, \cdots, n-1$. For $j = k, \cdots, n$ $\alpha(x_j) = x_j$ since $\alpha \in S_k$.

Since $\alpha \in U_i$ $\alpha(x_{k-1}) = x_{k-1} g_i$, some $g_i \in G_i$ so write

$g_i = x_i^n g_{i+1}$, some $g_{i+1} \in G_{i+1}$, n an integer then

$\alpha(x_{k-1}) = x_{k-1} x_i^n g_{i+1}$.

Choose α so that $|n|$ is minimal.

Let $\beta \in T_i$. Then as with α above

$\beta(x_{k-1}) = x_{k-1} x_i^m g'_{i+1}$, m an integer, $g'_{i+1} \in G_{i+1}$.

If $r = (m,n)$, there exist integers s,t such that $r = mt + ns$.

Then

$$\begin{aligned}
\alpha^s\beta^t(x_{k-1}) &= \alpha^s\beta^{t-1}(x_{k-1} x_i^m g'_{i+1}) \\
&= \alpha^s\beta^{t-2}(x_{k-1} x_i^m g'_{i+1} x_i^m g'_{i+1}) \\
&= \alpha^s\beta^{t-2}(x_{k-1} x_i^{2m} g''_{i+1}),\ g''_{i+1} \in G_{i+1} \\
&= \alpha^s(x_{k-1} x_i^{tm} k),\ k \in G_{i+1} \\
&= x_{k-1} x_i^{tm+sn} h,\ h \in G_{i+1} \\
&= x_{k-1} x_i^r h
\end{aligned}$$

By the minimality of $|n|$ and since $|r| < |n|$ it must be that $|r| = |n|$. Then $n|m$ and there is an integer a such that $m = na$.

Since $\beta(x_{k-1}) = x_{k-1} x_i^m g'_{i+1}$

$$\begin{aligned}
x_{k-1} = \beta^{-1}\beta(x_{k-1}) &= \beta^{-1}(x_{k-1})\ \beta^{-1}(x_i^m g'_{i+1}) \\
&= \beta^{-1}(x_{k-1})\ x_i^m g'_{i+1},\ \text{since } \beta \text{ is the identity of } G_i
\end{aligned}$$

so $\beta^{-1}(x_{k-1}) = x_{k-1}(x_i^{m}\, g'_{i+1})^{-1}$

$$= x_{k-1}\, g_{i+1}^{-1}\, x_i^{-m}$$

$$= x_{k-1}\, x_i^{-m}\, k,\ k \in G_{i+1}$$

then $\alpha^{a}\beta^{-1}(x_{k-1}) = \alpha^{a}(x_{k-1}\, x_i^{-m}\, k)$

$$= x_{k-1}\, x_i^{m}\, x_i^{-m}\, h,\ h \in G_{i+1}$$

$$= x_{k-1}\, h$$

Therefore $\alpha^{a}\beta^{-1} \in T_{i-1}$ so $\beta\, T_{i-1} = (\alpha\, T_{i-1})^{a}$ and $\dfrac{T_i}{T_{i-1}}$ is cyclic, completing the proof.

Corollary 5.17: Let G be solvable and $\bar{G}$ be a series of subnormal subgroups of G. Then PStab($\bar{G}$) is supersolvable.

Proof:

Refine $\bar{G}$ to $\bar{\bar{G}}$ a composition series for G. By Theorem 5.16, PStab($\bar{\bar{G}}$) is supersolvable. Then by Theorem 5.6 PStab($\bar{G}$) $\subsetneq$ PStab($\bar{\bar{G}}$). Thus PStab($\bar{G}$) is supersolvable.

Subsequent to the proof of Corollary 5.17 Schmid proved the following Theorem in [9] which makes it possible to generalize Corollary 5.17 for any finite group.

Theorem 5.18 (Schmid): An automorphism group of a finite group G which normalizes (fixes) a composition series of G and induces a supersolvable group of automorphisms on the non-abelian composition factors is supersolvable.

The following proof was suggested by Professor Deskins:

Theorem 5.19: Let $\bar{G}$: $G_o = G \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (1)$ be a composition series for G, then PStab($\bar{G}$) is supersolvable.

Proof:

Let $\frac{G_i}{G_{i+1}}$ be a non-abelian factor of $\bar{G}$. Since $\bar{G}$ is a composition series for G, $\frac{G_i}{G_{i+1}}$ is simple so $Z(\frac{G_i}{G_{i+1}}) = (1)$.

Now $\alpha \in$ PStab($\bar{G}$) induces a power automorphism on $\frac{G_i}{G_{i+1}}$ and Cooper has shown in [4] that a power automorphism is central. But $Z(\frac{G_i}{G_{i+1}}) = (1)$ so α induces the trivial automorphism on $\frac{G_i}{G_{i+1}}$. Therefore by Theorem 5.18 PStab($\bar{G}$) is supersolvable.

Corollary 5.20: If $\bar{G}$ is a subnormal series of G then PStab($\bar{G}$) is supersolvable.

Proof:

Refine $\bar{G}$ to $\bar{\bar{G}}$ a composition series for G. Then by Theorem 5.19 PStab($\bar{\bar{G}}$) is supersolvable and by Theorem 5.6 PStab($\bar{G}$) $\subseteq$ PStab($\bar{\bar{G}}$) so PStab($\bar{G}$) is supersolvable.

If $\bar{G}$ is a normal series we can determine the structure of M(G,A) where A $\subseteq$ PStab($\bar{G}$).

Theorem 5.21: If $\bar{G}$: $G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (1)$ is a normal series of G and $\alpha \in$ PStab($\bar{G}$), then the inner automorphisms induced by $\alpha(g)g^{-1}$ are power stabilizing automorphisms for all $g \in G$.

Proof:

Let $g \in G$ and $\frac{K}{G_{i+1}}$ be a subgroup of $\frac{G_i}{G_{i+1}}$.

Now $\left(\frac{K}{G_{i+1}}\right)^{gG_{i+1}} = \frac{K^g}{G_{i+1}}$ is also a subgroup of $\frac{G_i}{G_{i+1}}$ since $G_k, G_{i+1} \trianglelefteq G$.

Now α induces $\bar{\alpha}_i$ a power automorphism of $\frac{G_i}{G_{i+1}}$ so

$$\left(\left(\frac{K}{G_{i+1}}\right)^{gG_{i+1}}\right)^{\bar{\alpha}_i} = \frac{K^{g^\alpha}}{G_{i+1}} = \frac{K^g}{G_{i+1}}.$$

Therefore $\frac{K^{g^\alpha g^{-1}}}{G_{i+1}} = \frac{K}{G_{i+1}}$.

Hence $\pi_{\alpha(g)g^{-1}}$ induces a power automorphism of $\frac{G_i}{G_{i+1}}$ and $\pi_{\alpha(g)g^{-1}}$ is then a power stabilizing automorphism of $\bar{G}$.

Theorem 5.22: Let $\bar{G}$: $G = G_o \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = (1)$ be a normal series for G. If $A \subseteq \mathrm{PStab}(\bar{G})$, then $M(G,A)$ is supersolvable.

Proof:

By Theorem 5.21 if $B = \langle \pi_{\alpha(g)g^{-1}} : \alpha \in A,\ g \in G \rangle$, $B \subseteq \mathrm{PStab}(\bar{G})$. Then B is supersolvable by Theorem 5.19.

Now $B \simeq \frac{M(G,A)}{M(G,A) \cap Z(G)}$.

So $\frac{M(G,A)}{M(G,A) \cap Z(G)}$ is supersolvable.

From $M(G,A) \cap Z(G) \subseteq Z(G)$ it follows that $M(G,A)$ is supersolvable.

INDEX OF NOTATION

I. Relations:

$\subseteq$	Is a subset of
$\subsetneq$	Is a proper subset of
$\leq$	Is a subgroup of or is less than or equal to
$\lneq$	Is a proper subgroup of
$\trianglelefteq$	Is a normal subgroup of
$\in$	Is an element of

II. Operations:

G^α or $\alpha(G)$	Image of the set G under the mapping α
g^α or $\alpha(g)$	Image of the element g under the mapping α
$\frac{G}{H}$	Factor group
X	Direct product of groups
[G:H]	Index of H in G
$\langle\ \rangle$	Subgroup generated by
{ }	Set whose members are
$\|G\|$	The number of elements in G
$\|g\|$	The order of the element g
(m,n)	The greatest common divisor of the integers m and n
$\alpha\|_H$	Restriction of the mapping α to the set H

III. Groups and Sets:

Aut(G)	The automorphism group of G
Z(G)	The center of G
Fit(G)	The Fitting subgroup of G
M(G,A)	Subgroup generated by $\alpha(g)g^{-1}$ where $g \in G$, $\alpha \in A \subseteq \mathrm{Aut}(G)$
$M_{i+1}(G,A)$	$M(M_i(G,A),A)$ where $M_o(G,A) = G$
$\bar{G}$	Chain of subgroups of the group G; $\bar{G}$: $G = G_o \geq G_1 \geq \cdot \cdot \cdot \geq G_n = 1$
C_n	Cyclic group of order n
S_n	The symmetric group on n letters
A_n	The alternating group on n letters
GL(n,p)	General linear group of nonsingular nxn matrices over a field of order p
SL(n,p)	Special linear group of nxn unimodular matrices over a field of order p
Stab($\bar{G}$)	The group of stabilizing automorphisms of $\bar{G}$
S(G)	The group generated by the stabilizing automorphisms of G
PAut(G)	The group of power automorphisms of G
PStab($\bar{G}$)	The group of power stabilizing automorphisms of $\bar{G}$
P_i	A p_i-Sylow subgroup of G
G·[A]	The semi-direct product of G with A

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