ON A GENERALIZATION OF SUBNORMALITY IN INFINITE GROUPS

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ABSTRACT

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The concept of an f-subnormal subgroup is defined by R. E. Phillips in [1]. We say that the subgroup H is an f-subnormal subgroup of G (written $H \triangleleft f \triangleleft G$) if there exists a series

$$S : H = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = G$$

of finite length such that either $H_i \triangleleft H_{i+1}$ or $|H_{i+1}:H_i| < \infty$. We say that the series S is an f-series for H in G.. In this paper a number of questions concerning f-subnormal subgroups are investigated.

We study conditions under which the join of two f-subnormal subgroups is an f-subnormal subgroup. It is shown, for example, that whenever G is metabelian, finite-by-abelian, nilpotent-by-abelian, or FC-by-abelian, then the join of two finite f-subnormal subgroups of G is a finite f-subnormal subgroup. H. Wielandt showed that the join of two finite subnormal subgroups is a subnormal subgroup. In contrast to this result, we exhibit an abelian-by-finite

group which has two finite f-subnormal subgroups whose join is not an f-subnormal subgroup.

In the main result of Chapter IV we show that under certain restrictive conditions on the class $\mathfrak{X}$, the join of finitely many f-subnormal $\mathfrak{X}$ - subgroups is a $\mathfrak{X}$ - subgroup. This result implies that the join of finitely many f-subnormal solvable-by-finite-, $\check{\mathfrak{M}}_s$-, $\hat{\mathfrak{M}}_s$ - subgroups is a solvable-by-finite-, $\check{\mathfrak{M}}_s$ - , $\hat{\mathfrak{M}}_s$ - subgroup respectively, where $\check{\mathfrak{M}}_s$ $(\hat{\mathfrak{M}}_s)$ is the minimal (maximal) condition for subnormal subgroups.

In Chapter V f-subnormal $\check{\mathfrak{M}}_s$ - subgroups are studied. It is shown that if H and K are f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of the group G, then $|H : N_H(K)| < \infty$. This result is used to prove that if $\mathfrak{X}$ is a class closed under the taking of subgroups and homomorphic images, then the following are equivalent:

(1) If $G \in \mathfrak{X}$, the join of two finite f-subnormal subgroups of G is a finite f-subnormal subgroup.

(2) If $G \in \mathfrak{X}$, the join of two f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of G is an f-subnormal $\check{\mathfrak{M}}_s$ - subgroup.

A generalization of the concept of a nilgroup is investigated briefly in Chapter VI. We say that G is a β_f - group if for all $x \in G$, $\langle x \rangle \lhd f \lhd G$. A description of

β_f-groups in which the length of the shortest f-series is bounded by 1 or 2 for each element is given. It is shown that β_f-groups which satisfy $\check{\mathfrak{M}}_s$ are periodic groups in which $|G : Z(G)| < \infty$.

1. Phillips, Richard E., "Some generalizations of normal series in infinite groups." To appear in the J. Austral. Math. Soc.

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TABLE OF CONTENTS

Chapter I

Introduction and Notation

In this paper we will investigate a generalization of the concept of a subnormal subgroup. The definition is due to R. E. Phillips, [7].

<u>Definition</u> 1.1: We say that H is an f-subnormal subgroup of G $(H \triangleleft f \triangleleft G)$ if there exists a finite chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \dots \subseteq H_n = G$$

such that either $H_i \triangleleft H_{i+1}$ or $|H_{i+1} : H_i| < \infty$. We refer to such a series as an f-series for H in G.

H. Wielandt defined the concept of a subnormal subgroup in [17].

<u>Definition</u> 1.2: We say that H is a subnormal subgroup of G $(H \triangleleft\triangleleft G)$ if there exists a finite chain of subgroups

$$H = H_0 \triangleleft H_1 \triangleleft \dots H_{n-1} \triangleleft H_n = G .$$

<u>Notation</u> 1.3: If H is a subgroup of G, we write

$H \subseteq G$ and denote by $|G:H|$ the index of H in G. If H is a normal subgroup of G, we write $H \triangleleft G$. If N is a subset of G, N^G denotes the normal closure of N in G. When $N = \{x\}$, we write x^G for $\{x\}^G$. If $H \subseteq G$, then H^g denotes the conjugate of H by $g \in G$ and $\mathrm{Core}_G(H) = \bigcap_{g \in G} H^g$.

If H and K are subgroups of G, then $C_K(H)$ denotes the centralizer of H in K and $N_K(H)$ denotes the normalizer of H in K.

If $H \triangleleft\triangleleft G$, it can be shown that there exists a canonical normal series from H to G (see for example [8] or [10]). This canonical normal series from H to G is referred to as the standard series for H in G. It is well known that the standard series for H in G yields a normal series of shortest length from H to G.

Definition 1.4: If $H \triangleleft\triangleleft G$, the length of the standard series for H in G is called the index of subnormality for H in G and is denoted by $s(G,H)$.

H. Wielandt proved in [17] that whenever G has a composition series, the join of two subnormal subgroups of G is again subnormal in G. In contrast to this result,

D. Robinson has given in [10] examples of groups in which the join of two subnormal subgroups need not be a subnormal subgroup. Various conditions under which the join of two subnormal subgroups is a subnormal subgroup are discussed, for example, in [1], [8], [9], [10], [11], [12], [14], and [17].

In this paper we establish a number of sufficient conditions under which the join of two finite f-subnormal subgroups is an f-subnormal subgroup. In Theorem 3.18 we give an example of a group G which has two finite f-subnormal subgroups whose join is not an f-subnormal subgroup of G. One may compare this result with Theorem 7 of [17], in which H. Wielandt shows that the join of two finite subnormal subgroups is always a subnormal subgroup.

<u>Notation</u> 1.5: The unit element and the group of order one will both be denoted by 1.

<u>Definition</u> 1.6: A (group theoretic) class Γ is a collection of groups such that $1 \in \Gamma$ and whenever $G \in \Gamma$ and G_1 is isomorphic to G, then $G_1 \in \Gamma$.

<u>Notation</u> 1.7: (i) We will use the following notation for those classes of groups which will be referred to frequently.

$\mathfrak{A}$ - abelian groups

$\mathfrak{A}^{(n)}$ - solvable groups with derived length less than or equal to n

$\mathfrak{S}$ - solvable groups

$\mathfrak{F}$ - finite groups

$\mathfrak{N}$ ($\mathfrak{N}_c$) - nilpotent groups (of class $\leq c$)

$\mathfrak{F}_g$ - finitely generated groups

$\hat{\mathfrak{M}}, \hat{\mathfrak{M}}_n, \hat{\mathfrak{M}}_s$ - groups satisfying the maximal condition for subgroups, normal subgroups, subnormal subgroups

$\check{\mathfrak{M}}, \check{\mathfrak{M}}_n, \check{\mathfrak{M}}_s$ - groups satisfying the minimal condition for subgroups, normal subgroups, subnormal subgroups

(ii) If Γ_1 and Γ_2 are two classes of groups, then $\Gamma_1\Gamma_2$ is the class defined by

$$\Gamma_1\Gamma_2 = \{G \mid \exists N \lhd G,\ N \in \Gamma_1, \text{ and } G/N \in \Gamma_2\} .$$

(iii) If Γ is a class of groups, then

$$s\Gamma = \{H \mid \exists G \in \Gamma \text{ and } H \subseteq G\}$$
$$q\Gamma = \{H \mid \exists G \in \Gamma,\ N \lhd G, \text{ and } G/N = H\}$$
$$sn_0\Gamma = \{H \mid \exists G \in \Gamma,\ H \lhd\lhd G, \text{ and } |G:H| < \infty\}$$

(iv) If Γ_1 and Γ_2 are two classes of groups, then

$G \in \Gamma_1 \wedge \Gamma_2$ if $G \in \Gamma_1$ and $G \in \Gamma_2$;

$G \in \Gamma_1 \vee \Gamma_2$ if $G \in \Gamma_1$ or $G \in \Gamma_2$.

We define the property (*) in Definition 4.4. We say that the class $\mathfrak{X}$ has the property (*) if

(*) for any group G, the join of a finite number of subnormal $\mathfrak{X}$-subgroups of G is a $\mathfrak{X}$-subgroup.

It is known for example that the class $\mathfrak{S}$ has the property (*) [16 ; Theorem A].

We establish in Theorem 4.5 that if $\mathfrak{X} = sn_0\mathfrak{X}$ is a class of groups having the property (*), then the join of a finite number of f-subnormal $\mathfrak{X}\mathfrak{F}$-subgroups of a group G is a $\mathfrak{X}\mathfrak{F}$-subgroup of G.

In [11] and [12] J. E. Roseblade gives the following definition.

<u>Definition</u> 1.8: Let $\mathfrak{X}$ be any class of groups. We say that $\mathfrak{X}$ is a subnormal coalition class if, given that H and K are subnormal $\mathfrak{X}$-subgroups of G, the join $\langle H,K\rangle$ is a subnormal $\mathfrak{X}$ - subgroup of G.

It is clear that if $\mathfrak{X}$ is a subnormal coalition class, then $\mathfrak{X}$ has the property (*).

In [11] Roseblade showed that $\overset{\vee}{\mathfrak{M}}_s$ is a subnormal coalition class and in [12] he showed that $\overset{\wedge}{\mathfrak{M}}_s$ is a subnormal coalition class. Further results on subnormal coalition classes may be found in [8], [9], and [14].

In Chapter V we study f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of a group. We establish in Theorem 5.8 that if H and K are f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of G, then $|H : N_H(K)| < \infty$, where $N_H(K)$ denotes the normalizer of K in H. We showed in Corollary 4.12 that the join of finitely many f-subnormal $\check{\mathfrak{M}}_s$ - subgroups is a $\check{\mathfrak{M}}_s$ - subgroup. We use Theorem 5.8 to give an alternate proof of Corollary 4.12.

In Theorem 5.14 we establish that for certain classes of groups, the join of two finite f-subnormal subgroups is always an f-subnormal subgroup if and only if the join of two f-subnormal $\check{\mathfrak{M}}_s$ - subgroups is always an f-subnormal subgroup.

R. Baer defined in [1] the concept of a nilgroup. We generalize this concept in Definition 6.2. We say that G is a $\mathfrak{P}_f$-group if for all $x \in G$, $\langle x \rangle \triangleleft f \triangleleft G$. In Propositions 6.4 and 6.6 we examine $\mathfrak{P}_f$-groups for which there is a uniform bound on the length of the f - series for each element. In Proposition 6.15 we use the results of Chapter V to characterize $\mathfrak{P}_f$ - groups with $\check{\mathfrak{M}}_s$.

<u>Notation</u> 1.9: If N and M are subsets of G, we denote by $\langle M, N \rangle$ the subgroup of G generated by N and M. We denote by $[M,N]$ the subgroup

$$\langle [m,n] \mid m \in M,\ n \in N \rangle,$$

where $[m,n]$ is the commutator of m and n.

Notation 1.10: We will often indicate by $H \overset{f}{\subseteq} K$ that H is a subgroup of finite index in K.

Notation 1.11: If G is a split extension of H by K, we write $G = H]K$.

Notation 1.12: We will denote by C_{p^∞} the group defined by the set of generators $X = \{x_1, x_2, \ldots\}$ and the set of relations $R = \{x_1^p = 1,\ x_{i+1}^p = x_i \mid 1 \leq i\}$.

Chapter II

f-Subnormal Subgroups

It is inherent in Definition 1.1 that an f-series for an f-subnormal subgroup H of a group G is of finite length. Hence, for every f-subnormal subgroup H of G, we can find an f-series of shortest length. We denote the length of such an f-series by $f(G,H)$.

Remark 2.1: If $H \triangleleft f \triangleleft G$, then $f(G,H)$ is an invariant of the pair (G,H). This seems to be the only meaningful invariant of (G,H).

Suppose that $f(G,H) = n$ and

$$S : H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

is an f-series for H in G. We can define functions g_1 and g_2 which have as their domain all triples (G,H,S), where $H \triangleleft f \triangleleft G$ and S is an f-series for H in G of length $f(G,H)$. Let

$$g_1(G,H,S) = \text{the number of non-normal "jumps" in } S$$

and

$$g_2(G,H,S) = \Sigma\left\{ |H_{i+1} : H_i| \;\middle|\; H_i \ntriangleleft H_{i+1} \text{ and } |H_{i+1} : H_i| < \infty \right\}.$$

Example 2.2 shows that neither g_1 nor g_2 is an invariant of (G,H) .

<u>Example</u> 2.2: Let

$$N_1 = \sum_{i=1}^{\infty} \langle x(1,i) \rangle \quad \text{and} \quad N_2 = \sum_{i=1}^{\infty} \langle x(2,i) \rangle ,$$

where the elements $x(i,j)$ are of order 2. Let N be the direct product of N_1 and N_2 ,

$$N = N_1 \oplus N_2 .$$

We define the automorphisms t_1 and t_2 of N by

$$x(1,i)^{t_1} = x(1,i) \qquad x(2,i)^{t_1} = x(1,i)+x(2,i)$$

$$x(1,i)^{t_2} = x(1,i)+x(2,i) \qquad x(2,i)^{t_2} = x(2,i)$$

Let $G = N\,] \langle t_1,t_2 \rangle$. It is readily verified that t_1 and t_2 are automorphisms of order 2 and that

$$\langle t_1,t_2 \rangle \simeq S_3 ,$$

the symmetric group on three letters. Consider the f-series

$$S^1 : \langle x(1,1) \rangle \triangleleft N \triangleleft G$$

and

$$S^2 : \langle x(1,1) \rangle \triangleleft (\langle x(1,1) \rangle \oplus \sum_{i=3}^{\infty} \langle x(1,i) \rangle) \oplus N_2 \overset{f}{\subseteq} G$$

for $\langle x(1,1) \rangle$ in G . Since $\langle x(1,1) \rangle$ is neither normal nor of finite index in G, $f(G, \langle x(1,1) \rangle) \geq 2$. The existence of the f-series S^1 implies that $f(G, \langle x(1,1) \rangle) = 2$. Hence S^1 and S^2 are f-series of length $f(G, \langle x(1,1) \rangle)$. We note that

$$g_1(G,H,S^1) = 0 \quad g_1(G,H,S^2) = 1$$
$$g_2(G,H,S^1) = 0 \quad g_2(G,H,S^2) = 12 .$$

It follows that neither g_1 nor g_2 is an invariant of the pair (G,H).

The following two lemmas are easy consequences of Definition 1.1.

Lemma 2.3: If $H \triangleleft f \triangleleft G$ and $K \subseteq G$, then $H \cap K \triangleleft f \triangleleft K$ and $f(K, H \cap K) \leq f(G,H)$.

Proof: If $f(G,H) = n$, then there exists an f-series

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

for H in G. Let us consider the series

$$\text{(a)} \quad H \cap K = B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n = K ,$$

where $B_i = H_i \cap K$ for $0 \leq i \leq n$. If $H_i \triangleleft H_{i+1}$, then $H_i \cap K \triangleleft H_{i+1} \cap K$. If $|H_{i+1} : H_i| < \infty$, then $|H_{i+1} \cap K : H_i \cap K| \leq |H_{i+1} : H_i| < \infty$. Thus the series (a) is an f-series for $H \cap K$ in K. Hence $H \cap K \triangleleft f \triangleleft K$ and $f(K, H \cap K) \leq n$ $\square$

Lemma 2.4: If $H \triangleleft f \triangleleft G$ and $K \triangleleft G$, then $HK/K \triangleleft f \triangleleft G/K$ and $f(G/K, HK/K) \leq f(G,H)$.

Proof: If $f(G,H) = n$, then there exists an f-series

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

for H in G . Let us consider the series

$$\text{(a)} \quad HK/K \subseteq H_1K/K \subseteq \ldots \subseteq H_nK/K = G/K .$$

If $H_i \triangleleft H_{i+1}$, then $H_iK/K \triangleleft H_{i+1}K/K$. If $|H_{i+1} : H_i| < \infty$, then $|H_{i+1}K/K : H_iK/K| \leq |H_{i+1} : H_i| < \infty$. Hence (a) is an f-series for HK/K in G/K . We conclude that $HK/K \triangleleft f \triangleleft G/K$ and $f(G/K, HK/K) \leq f(G,H) = n$. □

In what follows, frequent use will be made of Lemmas 2.3 and 2.4 without further reference.

Remark 2.5: A subgroup of finite index in a subnormal subgroup of G is an f-subnormal subgroup of G . Also, a subnormal subgroup of a subgroup which has finite index in G is an f-subnormal subgroup of G . We will show in Example 2.8 that not every f-subnormal subgroup of G need be of either of the above types.

We will need the notion of a wreath product in Lemma 2.7.

Definition 2.6: Let H and K be groups and let $B = \sum_{k \epsilon K} H_k$ be the direct sum of isomorphic copies of H indexed by the elements of K. Let $b \epsilon B$ with $b(k)$ as its k-th component. If $x \epsilon K$, we define b^x by the rule $b^x(k) = b(kx^{-1})$. The wreath product of H with K, $H \text{ wr } K$, is the split extension $B\,]\,K$.

For further information see [6].

Lemma 2.7: If A is a simple group and B is isomorphic to C_{p^∞}, then $A \text{ wr } B$ has no proper subgroups of finite index.

Proof: It will suffice to show that $A \text{ wr } B$ has no proper normal subgroups of finite index. Suppose $K \triangleleft A \text{ wr } B$ and $|A \text{ wr } B : K| < \infty$. Since B is divisible and $|B : B \cap K| < \infty$, $B \subseteq K$. It will now suffice to show that for some $b_0 \epsilon B$, $A_{b_0} \subseteq K$, for then $\langle A_{b_0}^B, B \rangle = A \text{ wr } B \subseteq K$ and $K = A \text{ wr } B$. Since $\sum_{b \epsilon B} A_b$ is an infinite group and

$$\left| \sum_{b \epsilon B} A_b : \left(\sum_{b \epsilon B} A_b \right) \cap K \right| < \infty ,$$

there exists $1 \neq x \epsilon \left(\sum_{b \epsilon B} A_b \right) \cap K$. Let $b_0 \epsilon B$ such that $x(b_0) \neq 0$. Let $y \epsilon A_{b_0}$ such that $[x(b_0), y] \neq 1$. Using the fact that

$$\left(\sum_{b \epsilon B} A_b \right) \cap K \triangleleft \sum_{b \epsilon B} A_b ,$$

we observe that

$$1 \neq [x(b_0),y] = x^{-1}x^{y} \in \left(\sum_{b\in B} A_b \right) \cap K$$

and

$$1 \neq x^{-1}x^{y} \in A_{b_0} \cap K .$$

Since $A_{b_0} \cap K \lhd A_{b_0}$ and A_{b_0} is simple, it follows that $A_{b_0} \subseteq K$. Hence $K = A \operatorname{wr} B$. □

<u>Example</u> 2.8: Let A_i be the alternating group on i letters and let C be an arbitrary infinite simple group. Let $G = C\wr_5 A_5$ be the split extension

$$\left(\sum_{i=1}^{5} C_i \right)] A_5 ,$$

where $C_i \simeq C$ and the elements of A_5 act as permutations on the coordinates of the elements of $\sum_{i=1}^{5} C_i$. Let $H = C\wr_4 A_4 = \left(\sum_{i=1}^{4} C_i \right)] A_4 \subset G$. Since $H^G \supseteq {A_4}^{A_5} = A_5$, $H^G \supseteq \langle {C_1}^{A_5}, A_5 \rangle = G$. Hence, H is not a subnormal subgroup of G. We note that H is an f-subnormal subgroup of G since

$$\text{(a)} \quad H = C\wr_4 A_4 \lhd \left(\sum_{i=1}^{5} C_i \right) \cdot A_4 \overset{f}{\subseteq} G$$

is an f-series for H in G. Since $|G:H|$ is not finite, $f(G,H) > 1$. It follows that $f(G,H) = 2$, since the series (a) shows that $f(G,H) \leq 2$.

Let $\overline{G} = G \operatorname{wr} B$, where $B \simeq C_{p^\infty}$ and p is a prime number. If we identify G with G_1, we observe that

$$H \lhd f \lhd \overline{G}$$

since

$$H \lhd f \lhd G$$

and

$$G \lhd \sum_{b \epsilon B} G_b \lhd \overline{G} \quad .$$

Since H is not a subnormal subgroup of G, H is not a subnormal subgroup of $\overline{G}$.

We will now show that $\overline{G}$ contains no proper subgroups K of finite index in $\overline{G}$. If we denote by $A_5(b)$ the subgroup of G_b corresponding to the subgroup A_5 of G, then the subgroup $\left(\sum_{b \epsilon B} A_5(b) \right) \cdot B$ of $\overline{G}$ is isomorphic to A_5 wr B . It follows from Lemma 2.7 that $\left(\sum_{b \epsilon B} A_5(b) \right) \cdot B$ has no subgroups of finite index. Hence $\left(\sum_{b \epsilon B} A_5(b) \right) \cdot B \subseteq K$. If we denote by $C_1(b)$ the subgroup of G_b corresponding to the subgroup C_1 of G, then the subgroup $\left(\sum_{b \epsilon B} C_1(b) \right) \cdot B$ of $\overline{G}$ is isomorphic to C_1 wr B . Again, it follows from Lemma 2.7 that $\left(\sum_{b \epsilon B} C_1(b) \right) \cdot B \subseteq K$. It now follows that

$$< \sum_{b \epsilon B} A_5(b), \sum_{b \epsilon B} C_1(b) > \; = \sum_{b \epsilon B} G_b \subseteq K$$

and

$$\overline{G} = \left(\sum_{b \epsilon B} G_b \right) \cdot B = K \; .$$

We conclude that $\overline{G}$ contains no subgroup K such that $H \lhd\lhd K$ and $|\overline{G} : K| < \infty$.

Suppose that L is a subgroup of $\overline{G}$ such that $L \lhd\lhd \overline{G}$ and $|L : H| < \infty$. Then $L \cap G_1 = L \cap G \lhd\lhd G$ and $|L \cap G : H| < \infty$. Hence $H \overset{f}{\subseteq} L \cap G \lhd\lhd G$. We showed that $H^G = G$. It follows that $(L \cap G)^G = G$ and $L \cap G = G$. Since $|G : H|$ is not finite, we conclude that the subgroup L cannot exist.

It will be of interest to know when an f-subnormal subgroup is a subnormal subgroup.

<u>Definition</u> 2.9: Let Γ be a class of groups. We say that G belongs to the class $L\Gamma$ if every finitely generated subgroup of G is contained in a Γ-subgroup of G.

<u>Lemma</u> 2.10: If G is an $L\mathfrak{N}$- group, then $H \lhd f \lhd G$ if and only if $H \lhd\lhd G$.

<u>Proof</u>: If $H \lhd f \lhd G$, let

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

be an f-series for H in G. Suppose that

$$|H_{i+1} : H_i| < \infty ,$$

then

$$L = H_{i+1}/\mathrm{Core}_{H_{i+1}}(H_i)$$

is a finite $L\mathfrak{N}$-group since the class $L\mathfrak{N}$ is closed under homomorphic images [4;p. 222]. Hence $L \in \mathfrak{N}$. It is well known that in a nilpotent group every subgroup is subnormal [4;p. 225]. Thus

$$H_i/\mathrm{Core}_{H_{i+1}}(H_i) \lhd\lhd H_{i+1}/\mathrm{Core}_{H_{i+1}}(H_i)$$

But then $H_i \lhd\lhd H_{i+1}$ and $H \lhd\lhd G$. □

The following lemma shows that in a certain sense f-subnormal subgroups are not too far removed from being subnormal subgroups.

<u>Lemma</u> 2.11: If $H \lhd f \lhd G$, then there exists $\overline{H} \lhd\lhd G$ such that $\overline{H} \subseteq H$, $|H : \overline{H}| < \infty$, and $s(G,\overline{H}) \leq f(G,H)$.

<u>Proof</u>: We will prove the lemma by induction on $f(G,H)$.

Suppose $f(G,H) = 1$. If $H \lhd G$, there is nothing to prove. If $|G : H| < \infty$, we may choose $\overline{H} = \mathrm{Core}_G(H)$.

Let us now assume that the assertion is true for $n-1$ and that $f(G,H) = n$. Let

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

be an f-series for H in G. If $H_{n-1} \lhd G$, then $f(H_{n-1},H) = n-1$ and the induction hypothesis yields the existence of $\overline{H} \lhd\lhd H_{n-1}$ such that $H \supseteq \overline{H}$, $|H : \overline{H}| < \infty$, and $s(H_{n-1}, \overline{H}) \leq n-1$. But then $\overline{H} \lhd\lhd G$ and $s(G, \overline{H}) \leq n$.

If $|G : H_{n-1}| < \infty$, we let $L = \mathrm{Core}_G(H_{n-1})$. Now let us consider the series

(a) $H \cap L \subsetneq H_1 \cap L \subsetneq \ldots \subsetneq H_{n-1} \cap L = L$.

If $H_i \lhd H_{i+1}$, then $H_i \cap L \lhd H_{i+1} \cap L$ and if $|H_{i+1} : H_i| < \infty$, $|H_{i+1} \cap L : H_i \cap L| < \infty$. Thus, (a) is an f-series for $H \cap L$ in L. Since $f(L, H \cap L) < n$, the induction hypothesis yields the existence of $\overline{H} \lhd\lhd L$ with the property that $|H \cap L : \overline{H}| < \infty$ and $s(L, \overline{H}) \leq n-1$. Now, $|H : H \cap L| < \infty$ and $L \lhd G$ imply the desired conclusion. □

A useful consequence of Lemma 2.11 is the following

<u>Corollary</u> 2.12: If H is an infinite f-subnormal subgroup of G which has no proper subgroups of finite index, then $H \lhd\lhd G$. □

<u>Definition</u> 2.13: If Σ is a class of groups, we say that Σ is a radical class if $s\Sigma = \Sigma$ and every group G has a normal Σ-subgroup, $\Sigma(G)$, containing all the normal Σ-subgroups of G. We refer to $\Sigma(G)$ as the Σ-radical of G.

<u>Lemma</u> 2.14 [7; Theorem 3.1]: If Σ is a radical class such that $\Sigma = \Sigma\mathfrak{F}$ and H is an f-subnormal subgroup

of G, then $\Sigma(H)$ is an f-subnormal subgroup of $\Sigma(G)$. □

It readily follows from Definition 2.13 that $(L\mathfrak{F})\mathfrak{F} = L\mathfrak{F}$ is a radical class. Hence, we have as a consequence of Lemma 2.14

Lemma 2.15 [7; Corollary 3.1]: If H and K are locally finite f-subnormal subgroups of G, then $\langle H,K\rangle$ is locally finite. In particular, the join of finitely many finite f-subnormal subgroups is finite. □

Next we present a few comments concerning the intersection of f-subnormal subgroups.

Proposition 2.16: If H and K are f-subnormal subgroups of G , then $H\cap K \lhd f \lhd G$ and $f(G,H\cap K) \leq f(G,H) + f(G,K)$. □

Proposition 2.16 follows readily from Lemma 2.3.

Corollary 2.17: The intersection of a finite number of f-subnormal subgroups of G is an f-subnormal subgroup of G . □

The intersection of an arbitrary set of f-subnormal subgroups of G need not be an f-subnormal subgroup of G . This is shown in the following example.

Example 2.18: Let G be the infinite dihedral group

$$G = \langle a,b \mid bab = a^{-1}, b^2 = 1 \rangle \quad .$$

Let $H_i = \langle a^{2^i}, b\rangle$, then $H_i \lhd\lhd G$, $s(G,H_i) = i$, and $\bigcap_{i=1}^{\infty} H_i = \langle b\rangle$.

Suppose $\langle b\rangle$ is an f-subnormal subgroup of G, then there exists an f-series

$$\langle b\rangle = A_0 \subset A_1 \subset \ldots \subset A_n = G \ .$$

Since $A_1 \supsetneq A_0$, there is an element $a^m b^i$, $m \neq 0$ such that $a^m b^i \in A_1$. Hence $a^m \in A_1$. Let n_0 be the least positive integer such that $a^{n_0} \in A_1$. Then $A_1 = \langle a^{n_0}, b \mid ba^{n_0}b = a^{-n_0}, b^2\rangle$. Since A_1 is an infinite group, $\langle b\rangle \lhd A_1$. But this cannot occur since $b^{a^{n_0}} = ba^{2n_0} \notin \langle b\rangle$. Hence $\langle b\rangle$ is not an f-subnormal subgroup of G .

Chapter III

The Join of f-Subnormal Subgroups

It is well known that the join of two subnormal subgroups need not be a subnormal subgroup. It can be shown, using the example given by D. Robinson in [8], that there exists an $\mathfrak{A}\mathfrak{N}_2$ - group in which the join of two subnormal subgroups need not be an f-subnormal subgroup. In this chapter we study conditions which imply that the join of two f-subnormal subgroups is an f-subnormal subgroup.

In the main result of this chapter, Theorem 3.18, we give an example of a group in which the join of two finite f-subnormal subgroups is not an f-subnormal subgroup. One may wish to compare this result with Theorem 7 of [17] in which H. Wielandt shows that the join of two finite subnormal subgroups of a group is always a subnormal subgroup.

The following lemma is an immediate consequence of Lemma 2.4.

<u>Lemma</u> 3.1: If $H \triangleleft f \triangleleft G$ and $K \triangleleft G$, then $\langle H,K\rangle \triangleleft f \triangleleft G$ and $f(G,HK) \le f(G,H)$. □

<u>Lemma</u> 3.2: If H and K are f-subnormal subgroups

of G such that K has an f-series

$$\text{(a)} \quad K = K_0 \subset K_1 \subset \ldots \subset K_n = G$$

and $K_i^H = K_i$ for $i = 0,1,2,\ldots n$, then $J = \langle H,K \rangle \lhd f \lhd G$ and $f(G,J) \leq f(G,H) \cdot f(G,K)$.

Proof: Let K_i and K_{i+1} be members of the f-series (a) for K in G . If $K_i \lhd K_{i+1}$, then $K_i \lhd HK_{i+1}$. By Lemma 2.3, $H \lhd f \lhd HK_{i+1}$. An application of Lemma 3.1 shows that $HK_i \lhd f \lhd HK_{i+1}$ and $f(HK_{i+1},HK_i) \leq f(G,H)$. If $|K_{i+1} : K_i| < \infty$, then $|HK_{i+1} : HK_i| < \infty$. Thus, in any case, $HK_i \lhd f \lhd HK_{i+1}$ and $f(HK_{i+1},HK_i) \leq f(G,H)$. We conclude that $HK \lhd f \lhd G$ and $f(G,HK) \leq f(G,K) \cdot f(G,H)$. □

Lemma 3.3: If $H \lhd\lhd G$, $K \lhd f \lhd G$, and $H^K = H$, then $\langle H,K \rangle \lhd f \lhd G$ and $f(G,\langle H,K \rangle) \leq s(G,H) \cdot f(G,K)$.

Proof: Let

$$H = H_n \lhd H_{n-1} \lhd \ldots \lhd H_1 \lhd H_0 = G$$

be the standard series for H in G . We will show by induction on i that $H_i^K = H_i$. We note that $H_0^K = G^K = G$. Suppose $H_{i-1}^K = H_{i-1}$, then $H_i = H^{H_{i-1}}$ and hence $H_i^K = (H^{H_{i-1}})^K = (H^K)^{H_{i-1}} = H^{H_{i-1}} = H_i$. An application of Lemma 3.2 yields the desired conclusion. □

Proposition 3.4: Let H and K be f-subnormal subgroups of G such that $H^K = H$ and $|K| < \infty$, then $\langle H,K\rangle \lhd f \lhd G$ and $f(G,\langle H,K\rangle) \leq f(G,H)\cdot f(G,K)$.

Proof: Let

$$\text{(i)} \quad H = H_0 \subset H_1 \subset H_2 \subset \dots \subset H_n = G$$

be an f-series for H in G. Let

$$B_i = \bigcap_{k \in K} H_i^{\,k} .$$

Consider the series

$$\text{(ii)} \quad H = B_0 \subseteq B_1 \subseteq B_2 \dots \subseteq B_n = G .$$

If $H_i \lhd H_{i+1}$, then $\bigcap_{k\in K} H_i^{\,k} \lhd \bigcap_{k\in K} H_{i+1}^{\,k}$. If $|H_{i+1} : H_i| < \infty$, then $|\bigcap_{k\in K} H_{i+1}^{\,k} : \bigcap_{k\in K} H_i^{\,k}| \leq |H_{i+1} : H_i|^{|K|} < \infty$. Hence, (ii) is an f-series for H in G. Since $B_i^{\,K} = B_i$ for $0 \leq i \leq n$, Lemma 3.2 implies that $\langle H,K\rangle \lhd f \lhd G$ and $f(G,\langle H,K\rangle) \leq f(G,H)\cdot f(G,K)$. □

As a consequence of Proposition 3.4 we have

Corollary 3.5: If G is any group and G^1 is torsion free, then the join of two finite f-subnormal subgroups of G is a finite f-subnormal subgroup of G .

Proof: Let G be a group such that G^1 is torsion free. If H and K are two finite f-subnormal subgroups of G, then $\langle H,K\rangle \in \mathfrak{F}$ by Lemma 2.15. Hence $[H,K] = 1$ and $H^K = H$. It now follows from Proposition 3.4 that $\langle H,K\rangle \triangleleft f \triangleleft G$. $\square$

Theorem 3.6: If H and K are two finite f-subnormal subgroups of G, then the following are equivalent:

(i) $\langle H,K\rangle \triangleleft f \triangleleft G$

(ii) $H^K \triangleleft f \triangleleft G$

(iii) $[H,K] \triangleleft f \triangleleft G$.

Proof: Since H and K are finite f-subnormal subgroups of G, it follows from Lemma 2.15 that $\langle H,K\rangle \in \mathfrak{F}$ and consequently H^K and $[H,K]$ are finite subgroups of G.

Suppose that $\langle H,K\rangle \triangleleft f \triangleleft G$. Since H^K and $[H,K]$ are normal subgroups of $\langle H,K\rangle$, H^K and $[H,K]$ are f-subnormal subgroups of G. Thus, (i) implies (ii) and (i) implies (iii).

On the other hand, if $H^K \triangleleft f \triangleleft G$, it follows from Proposition 3.4 that $\langle H^K,K\rangle = \langle H,K\rangle$ is an f-subnormal subgroup of G. Thus (ii) implies (i).

If $[H,K] \triangleleft f \triangleleft G$, it follows from Proposition 3.4 that $\langle [H,K],H\rangle = H^K \triangleleft f \triangleleft G$. Hence, (iii) implies (ii). $\square$

Corollary 3.7: Let $\mathfrak{X}$ be a class of groups with the property

(A) If $G \in \mathfrak{X}$, then every finite subgroup of G is f-subnormal in G.

If $G \in \mathfrak{X}\mathfrak{A}$, then the join of two finite f-subnormal subgroups of G is an f-subnormal subgroup of G.

Proof: If $G \in \mathfrak{X}\mathfrak{A}$, G has a normal subgroup N such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{A}$. Let H and K be two finite f-subnormal subgroups of G. Since $G/N \in \mathfrak{A}$, $[H,K] \subseteq N$. It follows from Lemma 2.15 that $[H,K] \in \mathfrak{F}$. Hence $[H,K] \lhd f \lhd N \lhd G$ and $[H,K] \lhd f \lhd G$. It now follows from Theorem 3.6 that $\langle H,K \rangle \lhd f \lhd G$. $\square$

Definition 3.8:

(i) We say that G is an FC-group $(G \in FC)$ if for every element $x \in G$, $|G : C_G(x)| < \infty$.

(ii) For any group G we define the FC-center of G, $FC(G)$, by

$$FC(G) = \{x \mid |G : C_G(x)| < \infty\}.$$

It is readily verified that $FC(G)$ is a characteristic subgroup of G.

Lemma 3.9: Every finite subgroup of an FC-group is an f-subnormal subgroup.

<u>Proof</u>: If $G \in FC$ and H is a finite subgroup of G, then $|G : C_G(h)| < \infty$ for every element $h \in H$. Since H is finite,

$$|G : C_G(H)| = |G : \bigcap_{h \in H} C_G(h)| \leq \prod_{h \in H} |G : C_G(h)| < \infty .$$

and

$$H \triangleleft H \cdot C_G(H) \overset{f}{\subseteq} G$$

is an f-series for H in G. Hence, H is an f-subnormal subgroup of G. □

<u>Remark</u> 3.10: The classes $\mathfrak{A}, \mathfrak{Z}, \mathfrak{N}$ and FC satisfy the property (A) of Corollary 3.7. It follows from Lemma 2.15 and Corollary 3.7 that whenever $G \in \mathfrak{A}^{(2)}, \mathfrak{Z}\mathfrak{A}, \mathfrak{N}\mathfrak{A}$, or $(FC)\mathfrak{A}$, the join of two finite f-subnormal subgroups of G is a finite f-subnormal subgroup of G.

<u>Proposition</u> 3.11: Let H and K be f-subnormal subgroups of G. If $|G : N_G(H)| < \infty$, then $\langle H, K\rangle \triangleleft f \triangleleft G$ and $f(G, \langle H,K \rangle) \leq 2 \cdot f(G,K)$.

<u>Proof</u>: Since $|G : N_G(H)| < \infty$, $|K : N_K(H)| < \infty$.

Let $\overline{H} = H^K$. Then there exist elements $k_1, k_2, \ldots, k_n \in K$ such that $\overline{H} = \langle H^{k_1}, H^{k_2}, \ldots, H^{k_n} \rangle$. Since $|G : \bigcap_{i=1}^{n} N_G(H^{k_i})| \leq |G{:}N_G(H)|^n < \infty$, $|G : N_G(\overline{H})| < \infty$. Hence,

$$\overline{H} \triangleleft N_G(\overline{H}) \overset{f}{\subseteq} G$$

is an f-series for $\overline{H}$ in G. Since $\overline{H}^K = \overline{H}$ and $N_G(\overline{H})^K = N_G(\overline{H})$, Lemma 3.2 implies that

$$\langle H,K\rangle = \langle \overline{H},K\rangle \lhd f \lhd G$$

and

$$f(G,\langle H,K\rangle) \leq 2 \cdot f(G,K) \ . \qquad \square$$

<u>Corollary</u> 3.12: If H and K are two f-subnormal subgroups of the FC-group G and $H \in \mathfrak{F}$, then $\langle H,K\rangle \lhd f \lhd G$.

<u>Proof</u>: If H is a finite subgroup of the FC-group G, $|G : N_G(H)| < \infty$. Proposition 3.11 shows that $\langle H,K\rangle \lhd f \lhd G$. $\square$

<u>Definition</u> 3.13: If we denote the center $Z(G)$ of G by $Z_1(G)$, then for the ordinal number α we define $Z_{\alpha+1}(G)$ by

$$Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G)) \ .$$

If β is a limit ordinal, we define $Z_\beta(G)$ by

$$Z_\beta(G) = \bigcup_{\alpha<\beta} Z_\alpha(G) \ .$$

We define the hypercenter of $G, Z_\infty(G)$, by

$$Z_\infty(G) = \bigcup\{Z_\alpha(G) \mid \alpha \text{ an ordinal number}\} \ .$$

We say that G is hypercentral if $Z_\infty(G) = G$.

The following concept will be useful in the proof of Proposition 3.15.

<u>Definition</u> 3.14: If

$$S : H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

is an f-series for H in G, we define $n(G,H,S)$ to be the number of factors in the series S which are not finite.

<u>Proposition</u> 3.15: If G is a split extension of N by H such that $N \in \mathfrak{A}$ and $H \in \mathfrak{F}$, then $H \triangleleft f \triangleleft G$ if and only if $Z_\infty(G) = Z_n(G)$ for some integer n and $|G : Z_\infty(G)| < \infty$.

<u>Proof</u>: Let

$$S : H = H_0 \subset H_1 \subset \ldots \subset H_m = G$$

be an f-series for H in G. If $n(G,H,S) = k$, we write the f-series S

$$\text{(i)} \quad H = H_{\beta_0} \subseteq H_{\alpha_1} \triangleleft H_{\beta_1} \subseteq H_{\alpha_2} \triangleleft H_{\beta_2} \subseteq \ldots \subseteq H_{\alpha_k} \triangleleft H_{\beta_k} \subseteq H_{\alpha_{k+1}} = G,$$

where $H_{\beta_i}/H_{\alpha_i} \notin \mathfrak{F}, 1 \leq i \leq k$, and $|H_{\alpha_{i+1}} : H_{\beta_i}| < \infty, 0 \leq i \leq k$.

We proceed to prove the proposition by induction on $n(G,H,S)$. If $n(G,H,S) = 0$, $G \in \mathfrak{F}$ and the assertion is trivially true.

We now state our induction hypothesis: If $\overline{G}$ is a split extension of $\overline{N}$ by $\overline{H}$ such that $\overline{N} \in \mathfrak{A}$ and $\overline{H} \in \mathfrak{Z}$, then for some integer n $Z_\infty(\overline{G}) = Z_n(\overline{G})$ and $|\overline{G} : Z_\infty(\overline{G})| < \infty$ whenever $\overline{H}$ has an f-series $\overline{S}$ such that $n(\overline{G},\overline{H},\overline{S}) < k$.

Now, suppose that G is a split extension of N by H such that $N \in \mathfrak{A}$, $H \in \mathfrak{Z}$, and H has an f-series S as given in (i) with $n(G,H,S) = k$. Since $H_{\alpha_1} \in \mathfrak{Z}$ and $H_{\alpha_1} \triangleleft H_{\beta_1}$, $|H_{\beta_1} : C_{H_{\beta_1}}(H_{\alpha_1})| < \infty$. Since $|G : N| < \infty$, it follows that

$$|H_{\beta_1} : N \cap C_{H_{\beta_1}}(H_{\alpha_1})| < \infty .$$

But then $|H_{\beta_1} : H_{\beta_1} \cap Z(G)| < \infty$, since $N \cap C_{H_{\beta_1}}(H_{\alpha_1}) \subseteq Z(G)$.

Let us now consider the group $G/Z(G)$. $G/Z(G)$ is a split extension of $NZ(G)/Z(G)$ by $HZ(G)/Z(G)$ such that $NZ(G)/Z(G) \in \mathfrak{A}$ and $HZ(G)/Z(G) \in \mathfrak{Z}$. The f-series

$$\overline{S}: HZ(G)/Z(G) = H_{\beta_0}Z(G)/Z(G) \subseteq H_{\alpha_1}Z(G)/Z(G) \triangleleft H_{\beta_1}Z(G)/Z(G) \ldots$$
$$\ldots H_{\alpha_k}Z(G)/Z(G) \triangleleft H_{\beta_k}Z(G)/Z(G) \subseteq H_{\alpha_{k+1}}Z(G)/Z(G) = G/Z(G)$$

satisfies the inequality $n(G/Z(G), HZ(G)/Z(G), \overline{S}) < k$. An application of the induction hypothesis yields the existence of an integer n such that $Z_\infty(G/Z(G)) = Z_n(G/Z(G))$ and $|G/Z(G) : Z_\infty(G/Z(G))| < \infty$. But then $Z_\infty(G) = Z_{n+1}(G)$ and $|G : Z_\infty(G)| < \infty$.

The converse is a special case of Proposition 3.16. □

Proposition 3.16: If G is a group such that $|G : Z_n(G)| < \infty$ for some integer n, then every subgroup of G is an f-subnormal subgroup.

Proof: Let H be a subgroup of G, then

$$H \lhd Z_1(G)H \lhd \ldots \lhd Z_n(G)H \overset{f}{\subseteq} G$$

is an f-series for H in G since $|G : Z_n(H)| < \infty$ and $H^{Z_i(G)} = \langle H, [H, Z_i] \rangle \subseteq \langle H, Z_{i-1} \rangle$, $1 \le i \le n$. □

Corollary 3.17: If $G = N] H$ such that $N \in \mathfrak{A}$ and $H \in \mathfrak{F}$, then $H \lhd f \lhd G$ implies that every subgroup K of G is subnormal in a subgroup having finite index in G. Consequently, every subgroup of G is an f-subnormal subgroup of G.

Proof: If $H \lhd f \lhd G$, then $|G : Z_k(G)| < \infty$ for some integer k. For any subgroup K of G,

$$K \lhd KZ_1(G) \lhd \ldots \lhd KZ_k \overset{f}{\subseteq} G$$

is an f-series for K in G. The corollary follows. □

Theorem 3.18: There exists a group $G \in \mathfrak{A}^{(3)} \wedge \mathfrak{A}\mathfrak{F}$ such that the join of two finite f-subnormal subgroups H and K of G is not f-subnormal in G.

Proof: We let G be the group of Example 2.2. G is a split extension of $N = N_1 \oplus N_2$ by $< t_1, t_2 >$, where $N \in \mathfrak{A}$ and $<t_1, t_2> \simeq S_3$, the symmetric group on three letters. Hence, $G \in \mathfrak{A}^{(3)} \wedge \mathfrak{A}\mathfrak{F}$. We recall that t_i centralizes N_i and N/N_i, $i = 1,2$. The subgroups $< t_1 >$ and $< t_2 >$ are f-subnormal subgroups of G since

$$< t_i > \triangleleft N_i < t_i > \triangleleft N < t_i > \overset{f}{\subseteq} G$$

is an f-series for $< t_i >$ in G, $i = 1,2$.

We now verify that $Z(G) = 1$. Suppose $n \cdot t \in Z(G)$, where $n \in N$ and $t \in < t_1, t_2 >$. If $t \neq 1$, there exists $\bar{t} \in < t_1, t_2 >$ such that $t^{\bar{t}} \neq t$. But then $(nt)^{\bar{t}} = nt$ and $n^{-1} n^{\bar{t}} = t(t^{\bar{t}})^{-1}$. Since $N \cap < t_1, t_2 > = 1$, $t^{\bar{t}} = t$, contradicting the fact that $t^{\bar{t}} \neq t$. Hence $t = 1$. Since $nt = n \cdot 1 = n \in N$, $n = n_1 + n_2$, where $n_1 \in N_1$ and $n_2 \in N_2$. If $n_1 \neq 0$, then $n^{t_2} \neq n$. If $n_2 \neq 0$, then $n^{t_1} \neq n$. Hence $n = 1$. We conclude that $Z(G) = 1$. Now, Proposition 3.15 implies that $< t_1, t_2 >$ cannot be f-subnormal in G. □

Remark 3.19: If $\mathfrak{X}$ is the class $\mathfrak{A}^{(3)}, \mathfrak{A}\mathfrak{F}, (FC)\mathfrak{F}$, or $\mathfrak{A}(FC)$ and $G \in \mathfrak{X}$, then the join of two finite f-subnormal subgroups of G need not be f-subnormal in G.

Chapter IV

The Join of f-Subnormal $\mathfrak{X}\mathfrak{F}$ - Subgroups

In order to prove the main result of this section, we will need the following technical lemma.

Lemma 4.1: Suppose $H \lhd G$, $|G : H| = n < \infty$, and $K \subseteq H$. If $A = \{a(1), a(2), \ldots, a(n)\}$ is a right transversal for H in G such that $1 \in A$ and $G = \langle K, A \rangle$, then there exists a finite subset L of H such that

$$H = \langle K^a, L \mid a^{-1} \in A \rangle .$$

Proof: Let $a, b \in A$ and $k \in K$. Since $K \subseteq H \lhd G$, $akb^{-1} \in H$ if and only if $a = b$. For $a(i)$, $a(j) \in A$, $1 \le i, j \le n$, we define $a(i,j) \in A$ uniquely by the equation $a(i)a(j)a(i,j)^{-1} \in H$.

Let $\overline{H}$ be defined by

$$\overline{H} = \langle K^a, a(i)a(j)a(i,j)^{-1} \mid a^{-1} \in A, 1 \le i, j \le n \rangle .$$

Since $H \lhd G, \overline{H} \subseteq H$. If $g \in H$, then

$$g = g_1^{\epsilon_1} g_2^{\epsilon_2} \ldots g_m^{\epsilon_m}$$

for some elements $g_i \in K \cup A$ and $\epsilon_i = \pm 1, 1 \le i \le m$. Set $a(i_0) = 1$.

There exists a unique element $a(i_1) \in A$ such that $a(i_0)g_1^{\epsilon_1} \in Ha(i_1)$. Hence $a(i_0)g_1^{\epsilon_1}a(i_1)^{-1} \in H$. It is easily verified in fact that $(a(i_0)g_1^{\epsilon_1}a(i_1)^{-1})^{\epsilon_1}$ is a (displayed) generator of $\overline{H}$. Suppose that for all j, $1 \leq j < \ell \leq m$, we have chosen $a(i_j) \in A$ such that $a(i_{j-1})g_j^{\epsilon_j}a(i_j)^{-1} \in \overline{H}$. We then choose $a(i_\ell) \in A$ as the unique element satisfying the equation $a(i_{\ell-1})g_\ell^{\epsilon_\ell} \in Ha(i_\ell)$. Then $(a(i_{\ell-1})g_\ell^{\epsilon_\ell}a(i_\ell)^{-1})^{\epsilon_\ell}$ is a (displayed) generator of $\overline{H}$ and $a(i_{\ell-1})g_\ell^{\epsilon_\ell}a(i_\ell)^{-1} \in \overline{H}$. But then

$$g = (a(i_0)g_1^{\epsilon_1}a(i_1)^{-1})(a(i_1)g_2^{\epsilon_2}a(i_2)^{-1}) \ldots$$
$$(a(i_{m-1})g_m^{\epsilon_m}a(i_m)^{-1})a(i_m),$$

where $a(i_{\ell-1})g_\ell^{\epsilon_\ell}a(i_\ell)^{-1} \in \overline{H}$, $1 \leq \ell \leq m$. Since $\overline{H} \subseteq H$, it follows that $a(i_m) = 1$. Consequently, $g \in \overline{H}$ and $H = \overline{H}$. The lemma now follows if we set

$$L = \{a(i)a(j)a(i,j)^{-1} \mid 1 \leq i,j \leq n\}. \qquad \square$$

<u>Definition</u> 4.2: If $\mathfrak{X}$ is a class of groups, we define $\theta_{\mathfrak{X}}(G)$ by

$$\theta_{\mathfrak{X}}(G) = \langle H \mid H \lhd\lhd G \text{ and } H \in \mathfrak{X} \rangle .$$

We note that $\theta_{\mathfrak{N}}(G)$ is the Baer radical of G (see for example [8; p. 101]).

<u>Remark</u> 4.3: It is clear from Definition 4.2 that for any class $\mathfrak{X}$

(i) $\Theta_{\mathfrak{X}}(G)$ is a characteristic subgroup of G and

(ii) whenever T is a finite subset of $\Theta_{\mathfrak{X}}(G)$, there exist a finite number of subnormal $\mathfrak{X}$-subgroups $H_1, H_2, \ldots .H_n$ of G such that

$< T > \subseteq < H_1, H_2, \ldots, H_n >$.

<u>Definition</u> 4.4: We say that the class $\mathfrak{X}$ has the property (*) if

(*) for any group G, the join of a finite number of subnormal $\mathfrak{X}$ - sugroups of G is a $\mathfrak{X}$ - subgroup of G .

We recall that, for an arbitrary class $\mathfrak{X}$, the class $sn_0\mathfrak{X}$ is defined by

$sn_0\mathfrak{X} = \{H | H \lhd\lhd G \in \mathfrak{X}$ and $|G : H| < \infty\}$.

<u>Theorem</u> 4.5: If $\mathfrak{X} = sn_0\mathfrak{X}$ is a class of groups which has the property (*), then the join of a finite number of f-subnormal $\mathfrak{X}\mathfrak{F}$ - subgroups of a group G is a $\mathfrak{X}\mathfrak{F}$-subgroup of G .

<u>Proof</u>: Let $H_1, H_2, \ldots, H_n$ be f-subnormal $\mathfrak{X}\mathfrak{F}$-subgroups of G . We may assume, without loss of generality, that $G = < H_1, H_2, \ldots, H_n >$. Since $H_i \in \mathfrak{X}\mathfrak{F}$, there exists $K_i \lhd H_i$

such that $|H_i : K_i| < \infty$ and $K_i \in \mathfrak{X}$ for $1 \le i \le n$. We note that $K_i, 1 \le i \le n$, is an f-subnormal $\mathfrak{X}$-subgroup of G. It follows from Lemma 2.11 that for each subgroup K_i there exists a subnormal subgroup F_i of G such that $F_i \subseteq K_i$ and $|K_i : F_i| < \infty$. Since $\mathfrak{X} = sn_0\mathfrak{X}$, $F_i \in \mathfrak{X}, 1 \le i \le n$. Since $|H_i : F_i| < \infty$ and $F_i \subseteq \Theta_{\mathfrak{X}}(G)$, it follows that $H_i\Theta_{\mathfrak{X}}(G)/\Theta_{\mathfrak{X}}(G)$ is a finite f-subnormal subgroup of $G/\Theta_{\mathfrak{X}}(G)$. An application of Lemma 2.15 shows that

$$G/\Theta_{\mathfrak{X}}(G) = \langle H_1, \ldots, H_n \rangle \Theta_{\mathfrak{X}}(G)/\Theta_{\mathfrak{X}}(G) \in \mathfrak{F} .$$

Let T and T_i be right transversals for $\Theta_{\mathfrak{X}}(G)$ in G and F_i in $H_i, 1 \le i \le n$, respectively such that $1 \in T$. Since $G = \langle H_1, H_2, \ldots, H_n \rangle$,

$$G = \langle F_1, F_2, \ldots, F_n, T_1, T_2, \ldots, T_n \rangle .$$

Each of the subsets $T_i, 1 \le i \le n$, is finite, hence there exists a finite subset T_0 of $\Theta_{\mathfrak{X}}(G)$ such that

$$G = \langle F_1, F_2, \ldots, F_n, T_0, T \rangle .$$

It now follows from the finiteness of T_0 and the definition of $\Theta_{\mathfrak{X}}(G)$ that there exist subnormal $\mathfrak{X}$-subgroups $L_1, L_2, \ldots, L_\ell$ of G such that

$$\langle T_0 \rangle \subseteq \langle L_1, L_2, \ldots, L_\ell \rangle .$$

Hence,

$$G = \langle F_1, F_2, \ldots, F_n, L_1, \ldots, L_\ell, T \rangle .$$

If we let

$$K = \langle F_1, \ldots, F_n, L_1, \ldots, L_\ell \rangle,$$

an application of Lemma 4.1 yields the existence of a finite subset L of $\Theta_{\mathfrak{X}}(G)$ such that

$$\Theta_{\mathfrak{X}}(G) = \langle F_1{}^t, \ldots, F_n{}^t, L_1{}^t, \ldots, L_\ell{}^t, L \mid t^{-1} \in T \rangle .$$

Again, it follows from the definition of $\Theta_{\mathfrak{X}}(G)$ and the finiteness of L that there exist subnormal $\mathfrak{X}$ - subgroups $M_1, M_2, \ldots, M_m$ of G such that

$$\langle L \rangle \subseteq \langle M_1, M_2, \ldots, M_m \rangle .$$

Hence,

$$\Theta_{\mathfrak{X}}(G) = \langle F_1{}^t, \ldots, F_n{}^t, L_1{}^t, \ldots, L_\ell{}^t, M_1, \ldots, M_m \mid t^{-1} \in T \rangle .$$

Since $\mathfrak{X}$ has the property (*), it now follows that $\Theta_{\mathfrak{X}}(G) \in \mathfrak{X}$. Consequently,

$$G = \langle H_1, H_2, \ldots, H_n \rangle \in \mathfrak{X}\mathfrak{J} . \qquad \square$$

In the remainder of this section we will point out a number of interesting consequences of Theorem 4.5.

Definition 4.6: If $\mathfrak{X}$ is a class of groups, we define the class $n_0\mathfrak{X}$ by

$$n_0\mathfrak{X} = \{G \mid G \text{ is generated by finitely many normal } \mathfrak{X}\text{-subgroups of } G\}.$$

Lemma 4.7 [12; Theorem 1]: Every class $\mathfrak{X} = n_0\mathfrak{X}$ contained in $\hat{\mathfrak{M}}_s$ is a subnormal coalition class. $\square$

Corollary 4.8: The join of finitely many f-subnormal $\hat{\mathfrak{M}}_s$-subgroups of G is a $\hat{\mathfrak{M}}_s$-subgroup of G. $\square$

Since $\mathfrak{N} \wedge \mathfrak{Z}_g = n_0(\mathfrak{N} \wedge \mathfrak{Z}_g) = sn_0(\mathfrak{N} \wedge \mathfrak{Z}_g)$ is a subclass of $\hat{\mathfrak{M}}_s$, we have as an immediate consequence of Lemma 4.7 and Theorem 4.5

Corollary 4.9: The join of finitely many f-subnormal $\mathfrak{N}\mathfrak{Z} \wedge \mathfrak{Z}_g$-subgroups of a group G is a $\mathfrak{N}\mathfrak{Z}$-subgroup of G. $\square$

Remark 4.10: Corollary 4.9 may also be deduced from Corollary 3.2 of [7].

Lemma 4.11 [11; Theorem 1]: Every subclass $\mathfrak{X} = n_0\mathfrak{X}$ of $\check{\mathfrak{M}}_s$ is a subnormal coalition class. $\square$

Corollary 4.12: The join of finitely many f-subnormal $\check{\mathfrak{M}}_s$-subgroups of G is a $\check{\mathfrak{M}}_s$-subgroup of G. $\square$

Remark 4.13: We will give another proof of Corollary 4.12 in Theorem 5.9.

The following theorem is proved by S. E. Stonehewer in [16].

Lemma 4.14 [16; Theorem A]: In any group G, the join of finitely many subnormal $\mathfrak{S}$ - subgroups of G is an $\mathfrak{S}$ - subgroup of G. $\square$

Theorem 4.5 and Lemma 4.14 yield

Corollary 4.15: In any group G, the join of finitely many f-subnormal $\mathfrak{S}\mathfrak{F}$ - subgroups is an $\mathfrak{S}\mathfrak{F}$ - subgroup. $\square$

Chapter V

f-Subnormal $\check{\mathfrak{M}}_s$-Subgroups

In this chapter we examine the f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of a group. In Theorem 5.8 we show that whenever H and K are f-subnormal $\check{\mathfrak{M}}_s$ - subgroups, $|H : N_H(K)| < \infty$. We use this result to obtain an alternate proof of Corollary 4.12, which we give as Theorem 5.9. In Theorem 5.14, we show that for certain classes of groups the join of two finite f-subnormal subgroups is always an f-subnormal subgroup if and only if the join of two f-subnormal $\check{\mathfrak{M}}_s$ - subgroups is always an f-subnormal subgroup.

<u>Lemma</u> 5.1: If $G \in \check{\mathfrak{M}}_s$ and $H \triangleleft f \triangleleft G$, then $H \in \check{\mathfrak{M}}_s$.

<u>Proof</u>: It is clear that $K \in \check{\mathfrak{M}}_s$ whenever $K \triangleleft G$ and $G \in \check{\mathfrak{M}}_s$.

Let $H \subseteq G, G \in \check{\mathfrak{M}}_s$ and $|G : H| < \infty$. Then, $|G : \mathrm{Core}_G(H)| < \infty$ and $\mathrm{Core}_G(H) \in \check{\mathfrak{M}}_s$. Since $H/\mathrm{Core}_G(H) \in \mathfrak{F}$, $H/\mathrm{Core}_G(H) \in \check{\mathfrak{M}}_s$. Hence $H \in \check{\mathfrak{M}}_s\check{\mathfrak{M}}_s = \check{\mathfrak{M}}_s$.

Now, by induction on $f(G,H)$, it follows that every f-subnormal subgroup H of G belongs to $\check{\mathfrak{M}}_s$. □

<u>Lemma</u> 5.2: If $G \in \check{\mathfrak{M}}_S$, then G satisfies the minimum condition for subgroups of finite index. □

<u>Lemma</u> 5.3: If $H_1 \subseteq H_2 \subseteq G \in \check{\mathfrak{M}}_S$ and if

(i) $|G : N_G(H_2)| < \infty$

and

(ii) $|H_2 : H_1| < \infty$,

then $|G : N_G(H_1)| < \infty$.

<u>Proof</u>: If H_1 and H_2 are as indicated, then $H_1 \overset{f}{\subseteq} H_2 \triangleleft N_G(H_2) \overset{f}{\subseteq} G$. Hence H_1 and H_2 are f-subnormal subgroups of G and it follows from Lemma 5.1 that H_1 and H_2 are $\check{\mathfrak{M}}_S$ - subgroups of G . Let F be the minimum subgroup of finite index in H_2 . F is a characteristic subgroup of H_2 and $F \subseteq H_1$. Since $H_2 \triangleleft K = N_G(H_2)$, $F \triangleleft K$. We consider

$$H_1/F \subseteq H_2/F \triangleleft K/F .$$

Since $H_2/F \in \mathfrak{J}$ and $H_2/F \triangleleft K/F$, $|K/F : N_{K/F}(H_1/F)| < \infty$. Hence, $|K : N_K(H_1)| < \infty$ and $|G : N_K(H_1)| < \infty$. It follows that $|G : N_G(H_1)| < \infty$. □

<u>Lemma</u> 5.4 [11; Theorem 3]: If $H \triangleleft \triangleleft G$ and $G \in \check{\mathfrak{M}}_S$ then $|G : N_G(H)| < \infty$. □

<u>Theorem</u> 5.5: If $G \in \check{\mathfrak{M}}_S$ and $H \triangleleft f \triangleleft G$, then $|G : N_G(H)| < \infty$.

<u>Proof</u>: We prove the assertion by induction on $f(G,H)$.

If $f(G,H) = 1$, then either $H \lhd G$ or $|G:H| < \infty$. In either case the assertion follows trivially.

Suppose that $f(G,H) = n$ and $H = H_n \subset H_{n-1} \subset \ldots \subset H_1 = G$ is an f-series for H in G. By the induction hypothesis $|G : N_G(H_{n-1})| < \infty$. By Lemma 5.1, $K = N_G(H_{n-1}) \in \check{\mathfrak{M}}_s$ and we have

$$H \subseteq H_{n-1} \lhd K \in \check{\mathfrak{M}}_s .$$

If $H \lhd H_{n-1}$, then by Lemma 5.4 $|K : N_K(H)| < \infty$. If $|H_{n-1} : H| < \infty$, then by Lemma 5.3 $|K : N_K(H)| < \infty$. Hence $|G : N_K(H)| < \infty$, from which we conclude that $|G : N_G(H)| < \infty$. □

<u>Corollary</u> 5.6: Let $G \in \check{\mathfrak{M}}_s$ and suppose H and K are f-subnormal subgroups of G. Then $J = \langle H,K \rangle$ is an f-subnormal $\check{\mathfrak{M}}_s$ - subgroup of G.

<u>Proof</u>: Once we have shown that $J \lhd f \lhd G$, we can conclude, using Lemma 5.1, that $J \in \check{\mathfrak{M}}_s$. Since $H \lhd f \lhd G$ and $G \in \check{\mathfrak{M}}_s$, $|G : N_G(H)| < \infty$. Proposition 3.11 shows that $J = \langle H,K \rangle \lhd f \lhd G$. □

<u>Corollary</u> 5.7: If $G \in \check{\mathfrak{M}}_s$ and $H_\lambda \lhd f \lhd G$ for $\lambda \in \Lambda$, then

(i) $\bigcap_{\lambda \in \Lambda} H_\lambda \lhd f \lhd G$

and

(ii) $\langle H_\lambda \mid \lambda \in \Lambda \rangle \lhd f \lhd G$.

Proof: Since $G \in \check{\mathfrak{M}}_S$, $\bigcap_{\lambda\in\Lambda} N_G(H_\lambda)$ has finite index in G . Hence, since

$$\bigcap_{\lambda\in\Lambda} N_G(H_\lambda) \subseteq N_G\left(\bigcap_{\lambda\in\Lambda} H_\lambda\right)$$

and

$$\bigcap_{\lambda\in\Lambda} N_G(H_\lambda) \subseteq N_G(\langle H_\lambda \mid \lambda \in \Lambda \rangle) ,$$

the normalizer in G of both $\bigcap_{\lambda\in\Lambda} H_\lambda$ and $\langle H_\lambda \mid \lambda \in \Lambda \rangle$ has finite index in G . Hence,

$$\bigcap_{\lambda\in\Lambda} H_\lambda \lhd N_G\left(\bigcap_{\lambda\in\Lambda} H_\lambda\right) \overset{f}{\subseteq} G$$

and

$$\langle H_\lambda \mid \lambda \in \Lambda \rangle \lhd N_G(\langle H_\lambda \mid \lambda \in \Lambda \rangle) \overset{f}{\subseteq} G$$

are f-series for $\bigcap_{\lambda\in\Lambda} H_\lambda$ and $\langle H_\lambda \mid \lambda \in \Lambda \rangle$ respectively. The corollary now follows. $\square$

Theorem 5.8: If H and K are f-subnormal $\check{\mathfrak{M}}_S$-subgroups of G , then $|H : N_H(K)| < \infty$.

Proof: Suppose that the theorem is false. Then there exists a group G with f-subnormal $\check{\mathfrak{M}}_S$-subgroups H_0 and K_0 such that $|H_0 : N_{H_0}(K_0)| \nless \infty$. Since $H_0 \in \check{\mathfrak{M}}_S$, there exists a subnormal subgroup H of H_0 minimal with respect to the existence of an f-subnormal $\check{\mathfrak{M}}_s$ - subgroup K_1 of G such that $|H : N_H(K_1)| \not< \infty$. Since $K_1 \in \check{\mathfrak{M}}_s$, there exists a subnormal subgroup K of K_1 such that K is minimal with respect to $|H : N_H(K)| \not< \infty$. It follows from Lemma 5.1 that H and K are f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of G .

Let F be the minimum subgroup of finite index in H . Since $H \in \check{\mathfrak{M}}_s$, F exists and $|H : F| < \infty$. Suppose that $F \underset{+}{\subset} H$, then $|F : N_F(K)| < \infty$ by our choice of H and K . But $N_F(K) = F \cap N_H(K)$. Hence $|H : N_H(K)| < \infty$, contradicting our choice of H and K . Hence $H = F$. Since H has no proper subgroups of finite index, H normalizes every proper subnormal subgroup of K by our choice of H and K . In particular, H normalizes Q , where Q is the product of all the proper normal subgroups of K . Thus

$$Q \triangleleft J = \langle H, K \rangle .$$

Let $U = HQ/Q, V = K/Q$, and $W = J/Q$. Then U and V are f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of W . Also, U has no proper subgroups of finite index and V is simple. An application of Corollary 2.12 shows that $U \triangleleft\triangleleft W$. If V is an infinite simple group, then V would have no subgroups of finite index, and consequently would also be subnormal in W . In this case, we can apply Lemma 4.11 to conclude that $W \in \check{\mathfrak{M}}_s$.

If V is a finite simple group, then U^V is the join of a finite number of conjugates of U and, applying Lemma 4.11, we conclude that $U^V \in \check{\mathfrak{M}}_s$ and $U^V \lhd\lhd W$. In fact, $U^V \lhd W$ since $W = \langle U, V \rangle$. Hence $W/U^V \simeq V/U^V \cap V \in \check{\mathfrak{M}}_s = q\check{\mathfrak{M}}_s$. We conclude that $W \in \check{\mathfrak{M}}_s \check{\mathfrak{M}}_s = \check{\mathfrak{M}}_s$.

In either case, $W \in \check{\mathfrak{M}}_s$. But $W = J/Q$ and $Q \in \check{\mathfrak{M}}_s$ by Lemma 5.1. Hence, $J \in \check{\mathfrak{M}}_s \check{\mathfrak{M}}_s = \check{\mathfrak{M}}_s$. By Theorem 5.5, $|J : N_J(K)| < \infty$. We conclude that $|H : N_J(K) \cap H| = |H : N_H(K)| < \infty$. □

We now use the results of this section to give another proof of Corollary 4.12.

<u>Theorem</u> 5.9: If $H_1, H_2, \ldots, H_n$ are f-subnormal $\check{\mathfrak{M}}_s$-subgroups of G then $J = \langle H_1, H_2, \ldots, H_n \rangle \in \check{\mathfrak{M}}_s$.

<u>Proof</u>: Let $F_i, 1 \le i \le n$, be the minimum subgroup of finite index in H_i. It follows from Lemma 5.2 that F_i exists, $F_i \in \check{\mathfrak{M}}_s$, and $|H_i : F_i| < \infty$. By Corollary 2.12, $F_i \lhd\lhd G$.

Let $F = \langle F_i \mid 1 \le i \le n \rangle$. Suppose F has a proper subgroup L of finite index, then $|F_i : F_i \cap L| < \infty$ and $L \supseteq F_i, 1 \le i \le n$. Hence $L = F$. Thus, F has no proper subgroups of finite index. It follows from Lemma 4.11 that F is a subnormal $\check{\mathfrak{M}}_s$-subgroup of G. Hence, since F has no proper subgroups of finite index, F normalizes every f-subnormal $\check{\mathfrak{M}}_s$-subgroup of G. The subgroup F has finite index in the group $\langle H_i, F\rangle = H_iF$, hence F is the minimum subgroup of finite index in H_iF, $1 \le i \le n$. In particular, $F^{H_i} = F$ for $1 \le i \le n$ and $F \triangleleft J$.

Now, consider the groups

$$\overline{J} = J/F, \overline{H}_i = H_iF/F, 1 \le i \le n .$$

The groups $\overline{H}_i$ are finite f-subnormal subgroups of $\overline{J}$. Hence, it follows from Lemma 2.15 that $\overline{J} \in \mathfrak{J}$. Since $F \in \check{\mathfrak{M}}_s$, $J \in \check{\mathfrak{M}}_s \mathfrak{J} = \check{\mathfrak{M}}_s$ □

We obtain the following corollary from the proof of Theorem 5.9.

Corollary 5.10: Suppose $H_1, H_2, \ldots, H_n$ are f-subnormal $\check{\mathfrak{M}}_s$-subgroups of G. If F_i and F denote the minimum subgroup of finite index in H_i and $\langle H_i \mid 1 \le i \le n\rangle$ respectively, then

$$F = \langle F_i \mid 1 \le i \le n\rangle \triangleleft\triangleleft G$$

and

$$\langle H_1, H_2, \ldots, H_n\rangle / F \in \mathfrak{J}$$ □

Corollary 5.11: Let $H \triangleleft\triangleleft G$ and $K \triangleleft f \triangleleft G$ be such that H and K are $\check{\mathfrak{M}}_s$-subgroups of G. Then $\langle H, K\rangle \triangleleft f \triangleleft G$.

and $\langle H,K\rangle \in \check{\mathfrak{M}}_s$.

<u>Proof</u>: It follows from Theorem 5.8 that $|K:N_K(H)| < \infty$. Thus H^K is the join of a finite number of conjugates of H . Since $\check{\mathfrak{M}}_s$ is a subnormal coalition class (Lemma 4.11), $H^K \lhd\lhd G$ and $H^K \in \check{\mathfrak{M}}_s$. Lemma 3.3 now implies that $\langle H^K,K\rangle = \langle H,K\rangle \lhd f \lhd G$. It follows from Theorem 5.9 that $\langle H,K\rangle \in \check{\mathfrak{M}}_s$. □

<u>Remark</u> 5.12: It follows from Theorem 3.18 and the observation that $\mathfrak{F}$ is contained in $\check{\mathfrak{M}}_s$ that the join of two f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of a group G need not be f-subnormal in G .

<u>Lemma</u> 5.13 [9; Lemma 4.3]: If F is a subnormal $\check{\mathfrak{M}}_s$ - subgroup of G such that F has no proper subgroups of finite index, then $s(G,F) \leq 2$.

<u>Proof</u>: If $x \in G$, it follows from Theorem 5.8 that $F^x \subseteq N_G(F)$. Hence $F^G \subseteq N_G(F)$. We conclude that

$$F \lhd F^G \lhd G .$$ □

<u>Theorem</u> 5.14: If $\mathfrak{X} = q\mathfrak{X} = s\mathfrak{X}$ is a class of groups, then the following are equivalent:

(1) If $G \in \mathfrak{X}$, the join of two finite f-subnormal subgroups of G is f-subnormal in G .

(2) If $G \in \mathfrak{X}$, the join of two f-subnormal $\check{\mathfrak{M}}_s$ - subgroups is f-subnormal in G .

Proof: Since $\mathfrak{F} \subseteq \check{\mathfrak{M}}_s$, it is clear that property (2) implies property (1).

Suppose that H and K are f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of the $\mathfrak{X}$ - group G . If we let $J = \langle H,K \rangle$, then $J \in \check{\mathfrak{M}}_s$ by Theorem 5.9. Let F, F_H, and F_K be the minimum subgroups of finite index in J,H, and K respectively. By Corollary 5.10, $F = F_H \cdot F_K$. Lemma 5.13 shows that $F \triangleleft F^G \triangleleft G$. Hence $F \triangleleft F^G J$. We now observe that HF/F and KF/F are finite f-subnormal subgroups of $F^G J/F$ and $F^G J/F \in \mathfrak{X} = s\mathfrak{X} = q\mathfrak{X}$. Hence (1) implies that

$$\langle HF/F, KF/F \rangle = J/F \triangleleft f \triangleleft F^G J/F .$$

Thus $J \triangleleft f \triangleleft F^G J$. Also HF^G/F^G and KF^G/F^G are finite f-subnormal subgroups of G/F^G, where $G/F^G \in q\mathfrak{X} = \mathfrak{X}$. Hence, it follows from (1) that $\langle HF^G/F^G, KF^G/F^G \rangle = JF^G/F^G \triangleleft f \triangleleft G/F^G$. Thus, $F^G J \triangleleft f \triangleleft G$. The statements $J \triangleleft f \triangleleft F^G J$ and $JF^G \triangleleft f \triangleleft G$ imply that $J \triangleleft f \triangleleft G$. □

Corollary 5.15: Let $\mathfrak{X} = s\mathfrak{X} = q\mathfrak{X}$ be a class of groups such that every finite subgroup of a $\mathfrak{X}$ - group G is f-subnormal in G . If $G \in \mathfrak{X}\mathfrak{A}$, then the join of two f-subnormal $\check{\mathfrak{M}}_s$ - subgroups of G is an f-subnormal $\check{\mathfrak{M}}_s$ - subgroup of G .

Proof: The corollary is a consequence of Corollary 3.7, Theorem 5.14, and Theorem 5.9. $\square$

Remark 5.16: In the statement of Corollary 5.15, we may take for the class $\mathfrak{X}$ the classes $\mathfrak{Z}, \mathfrak{A}, \mathfrak{N}$, or FC .

Chapter VI

β_f - Groups

The following definition is due to R. Baer [1].

Definition 6.1: We say that G is a nilgroup if for all $x \in G$, $\langle x \rangle \lhd\lhd G$.

We generalize the concept of a nilgroup in the following definition.

Definition 6.2: We say that G is a β_f - group if for all $x \in G$, $\langle x \rangle \lhd f \lhd G$.

It is clear that every nilgroup is a β_f - group. On the other hand, since the class $\mathfrak{F}$ is contained in β_f and not every finite group is a nilgroup, the class β_f properly contains the class of nilgroups.

Definition 6.3: A non-abelian group in which every subgroup is normal is a Hamiltonian group.

The structure of Hamiltonian groups is well known. A description may be found in 9.7.4 of [15].

It is clear that every Hamiltonian group is a β_f - group. We have the following partial converse.

<u>Proposition</u> 6.4: If G is a non-abelian β_f-group such that for all $x \in G$, $f(G,\langle x\rangle) = 1$, then G is a finitely generated FC-group or a Hamiltonian group.

<u>Proof</u>: If $x \in G$, either $\langle x\rangle \lhd G$ or $|G : \langle x\rangle| < \infty$. In either case, $|G : C_G(x)| < \infty$. Hence G is an FC-group.

If there is an element $x \in G$ such that $|G : \langle x\rangle| < \infty$, then G is a finitely generated FC-group and $|G : Z(G)| < \infty$.

On the other hand, if for all $x \in G$, $\langle x\rangle \lhd G$, then every subgroup of G is a normal subgroup. In this case, by Definition 6.3, G is a Hamiltonian group. □

<u>Lemma</u> 6.5 [15; 7.1.7]: If G is a finitely generated group having a subgroup K such that $|G : K| < \infty$, then there exists $N \operatorname{char} G$ such that $N \subset K$ and $|G : N| < \infty$. □

<u>Proposition</u> 6.6: Let G be a β_f-group such that for all $x \in G$, $f(G,\langle x\rangle) \leq 2$. Then, $G/FC(G)$ is a nilgroup in which each cyclic subgroup $\langle \overline{x}\rangle$ has subnormal defect $s(G/FC(G),\langle \overline{x}\rangle) \leq 2$.

<u>Proof</u>: Let $x \in G$ such that $\langle x\rangle$ is not subnormal in G with subnormal defect $s(G,\langle x\rangle) \leq 2$. Then, there exists a subgroup K of G (K may be equal to G) such that either

(a) $\langle x\rangle \overset{f}{\subseteq} K \lhd G$

or

(b) $\langle x \rangle \lhd K \overset{f}{\subseteq} G$.

Suppose $\langle x \rangle \overset{f}{\subseteq} K \lhd G$. Then $K \in \hat{\mathfrak{M}}\hat{\mathfrak{M}} = \hat{\mathfrak{M}} \subseteq \mathfrak{F}_g$. By Lemma 6.5, there exists $N \subseteq \langle x \rangle$ such that $N \operatorname{char} K$ and $|K : N| < \infty$. Hence $N \lhd G$. Let us consider

$$\langle x \rangle / N \overset{f}{\subseteq} K/N \lhd G/N .$$

Since $K/N \in \mathfrak{F}$ and $K/N \lhd G/N$, $|G/N : N_{G/N}(\langle x \rangle/N)| < \infty$. We conclude that $|G : N_G(\langle x \rangle)| < \infty$ and hence $|G : C_G(x)| < \infty$.

Suppose that $\langle x \rangle \lhd K \overset{f}{\subseteq} G$. Then $|G : N_G(\langle x \rangle)| < \infty$. Since $N_G(\langle x \rangle)/C_G(\langle x \rangle) \in \mathfrak{F}$, we obtain that $|G : C_G(\langle x \rangle)| < \infty$.

Let $N_1 = \{x \in G \mid \langle x \rangle$ has an f-series (a) or (b)$\}$. By the above remarks, it is clear that the set N_1 is contained in $FC(G)$, the FC-center of G . Since every element of $FC(G)$ has an f-series (b), $FC(G) \subseteq N_1$. We conclude that $N_1 = FC(G)$.

If $x \notin FC(G) = N_1$, we must have $\langle x \rangle \lhd x^G \lhd G$ by the definition of N_1 . Hence, in the group $G/FC(G)$ every cyclic subgroup $\langle \bar{x} \rangle$ is subnormal with subnormal defect $s(G/FC(G), \langle \bar{x} \rangle) \le 2$ and $G/FC(G)$ is a nilgroup. $\square$

We need the following commutator notation. We write $[x, 1y]$ for $[x,y]$ and $[x,(n+1)y]$ for $[[x,ny],y]$. We write $\Gamma_1(G)$ for G and $\Gamma_{n+1}(G)$ for $[\Gamma_n(G),G]$.

<u>Definition</u> 6.7: We say that G satisfies the nth Engel condition if $[x,ny] = 1$ for all $x,y \in G$.

Groups satisfying the third-Engel condition are investigated by H. Heineken in [2].

<u>Theorem</u> 6.8 [2; Hauptsatz 1 and 2]: If G is a group satisfying the third-Engel condition, then

(i) G is locally nilpotent and

(ii) $\Gamma_5(G)$ is contained in the direct product of the Sylow 2 - and Sylow 5 - subgroups of G . □

As an immediate consequence of Proposition 6.6 and Theorem 6.8 we have

<u>Corollary</u> 6.9: Let $G \in \beta_f$ such that for all $x \in G$, $f(G,\langle x\rangle) \leq 2$. If $G/FC(G)$ has no elements of order two or five, then $G/FC(G) \in \mathfrak{N}_4$. □

<u>Definition</u> 6.10: We say that G is locally normal if every finite subset of G is contained in a finite normal subgroup of G .

<u>Lemma</u> 6.11 (Dietzmann's lemma) [4; p. 154]: If M is a finite normal periodic subset of a group G, then $\langle M\rangle$ is a finite normal subgroup of G . □

<u>Lemma</u> 6.12 [15; 15.1.16]: If G is an FC-group

then $G/Z(G)$ is a periodic FC-group. □

Lemma 6.13 [5; Theorem 3.2]: Let G be a locally nilpotent group satisfying $\hat{\mathfrak{M}}_n$. Then G is a finitely generated nilpotent group and hence satisfies $\hat{\mathfrak{M}}$. □

We have the following corollary to Proposition 6.6.

Corollary 6.14: Let G be a β_f-group such that for all $x \in G, f(G,\langle x\rangle) \leq 2$. If $G \in \hat{\mathfrak{M}}_n$, then $G \in \hat{\mathfrak{M}}$.

Proof: Let $\bar{Z} = Z(FC(G))$. Since $\bar{Z}$ char $FC(G)$ char G, $\bar{Z} \lhd G$. Thus, if $x \in \bar{Z}$, $x^G \subseteq \bar{Z}$. Since $G \in \hat{\mathfrak{M}}_n$, there exist elements $x_1, x_2, \ldots, x_m$ of $\bar{Z}$ such that

$$\bar{Z} = x_1^G x_2^G \ldots x_m^G .$$

Since $\bar{Z}$ is abelian and $|G : C(x_i)| < \infty$ for $i = 1,2,\ldots,m$, we conclude that $\bar{Z}$ is the direct sum of a finite number of cyclic groups. Hence $\bar{Z} \in \hat{\mathfrak{M}}$.

We now consider the subgroup $FC(G)/\bar{Z}$ of $G/\bar{Z}$. Since $FC(G)/\bar{Z}$ is periodic by Lemma 6.12, it follows from Lemma 6.11 that $FC(G)/\bar{Z}$ is finite. It was shown in Proposition 6.6 that $G/FC(G)$ is locally nilpotent. Since $G/FC(G) \in \hat{\mathfrak{M}}_n$, Lemma 6.13 implies that $G/FC(G) \in \hat{\mathfrak{M}}$. Hence $G \in \hat{\mathfrak{M}}\hat{\mathfrak{M}}\hat{\mathfrak{M}} = \hat{\mathfrak{M}}$. □

Using the results of Chapter V, we have the following proposition.

<u>Proposition</u> 6.15: If G is a β_f - group satisfying $\check{\mathfrak{M}}_s$, then G is a periodic FC-group with $|G : Z(G)| < \infty$.

<u>Proof</u>: Let $x \in G$. Since $G \in \beta_f, \langle x \rangle \triangleleft f \triangleleft G$. Now, Lemma 5.1 implies that $\langle x \rangle \in \check{\mathfrak{M}}_s$. Hence G is a periodic group. It follows from Theorem 5.5 that $|G : N_G(\langle x \rangle)| < \infty$. Hence $|G : C_G(x)| < \infty$ for all $x \in G$ and $G \in FC$. If F is the minimum subgroup of finite index in G, $F \subset C_G(x)$ for all $x \in G$ and $|G : Z(G)| < \infty$. □

As consequences of Proposition 3.16 and Proposition 6.15, we have the following corollaries.

<u>Corollary</u> 6.16: If G is a β_f - group with $\check{\mathfrak{M}}_s$, then every subgroup of G is f-subnormal in G. □

<u>Corollary</u> 6.17: If G is a $\check{\mathfrak{M}}_s$ - group, then every subgroup of G is f-subnormal in G if and only if $|G : Z(G)| < \infty$.

<u>Proof</u>: If every subgroup of G is f-subnormal in G, then $G \in \beta_f \wedge \check{\mathfrak{M}}_s$. It follows from Proposition 6.15 that $|G : Z(G)| < \infty$.

The converse is a special case of Proposition 3.16.□

<u>Remark</u> 6.18: According to Proposition 3.16, if

there exists a positive integer n such that $|G : Z_n(G)| < \infty$, then every subgroup of G is f-subnormal in G. We leave as an open question whether the converse of Proposition 3.16 is true.

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