THE EFFECTS OF UNITARY SYMMETRY UPON MESON PRODUCTION IN PROTON-ANTIPROTON ANNIHILATION

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ABSTRACT

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A theoretical investigation of meson production from proton-antiproton annihilation at total center-of-mass energies in the range 1.88 to 3.00 Bev is carried out. The Fermi Statistical Model is applied and is generalized, from an SU_2 invariant form, to include invariance under the transformations of the group SU_3 .

The calculations were done using the Control Data 3600 Computer. Phase space integrals were obtained with a relative error of less than 6% using a Monte Carlo technique. All combinations of mesons from the J=0-, 1-, and 2+ nonets, which are allowed by conservation laws, are included as possible final states. The calculation of the SU_3 statistical weights became increasingly complex with increasing particle multiplicity. It was therefore necessary to limit final state multiplicities to six or less particles under the assumption of SU_3 invariance. This placed an upper limit of approximately 2.5 Bev upon the energies which could be considered in this case. The SU_3 weights are tabulated. The SU_2 weights are much easier to calculate so that, in this case, final states of

up to ten particles and energies of up to 3.0 Bev are considered.

The results indicate that no substantial improvement is made by the generalization from SU_2 to SU_3 invariance or by the inclusion of the resonances comprising the vector and tensor nonets. Furthermore, it is still necessary to introduce unphysically large volume parameters into the model in order to obtain results consistent with experiment, a common difficulty in statistical model calculations. However, the theory adequately reproduces experimental results for pion distributions, kaon production rates and charged prong distributions at a number of energies.

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I. Introduction.

The proton-antiproton system and the myriad of decay modes to which it is strongly coupled, has been of considerable interest to experimentalists. With the advent of high-energy accelerators, a variety of experimental data at various energies has been obtained.[1-11] The theoretical analysis of the system and attempts to predict experimental results have been incomplete. There appears to be little hope of developing, in the near future, a quantitative theory of multiple particle production in strong interactions. Hence, one is forced to resort to phenomenological models in order to deal with systems with three or more particles in the final state.

One such model, which has succeeded in predicting some of the experimental data, is the Fermi Statistical Model.[12] The model assumes that in a high energy collision of two baryons in their center-of-mass system, all the available energy is released into a small volume surrounding the initial particles. The energy is then redistributed in a statistical way. That is, the relative probability for the formation of a particular final state can be calculated by assuming that all <u>allowed</u> final states have equal a priori probability of being formed. The single parameter in

the simple model is the volume Ω in which the interaction takes place. We would naturally expect this to be of the order of the cube of the range of the annihilation interaction.

The model, as proposed by Fermi, allows for refinements due to conservation laws in operation and hence, to consequences of symmetries which the system possesses. One of the first refinements, due to Fermi himself, was to build into the model the effects of isotopic spin invariance.[13] We have extended this by considering the effects of SU_3 invariance in applying the model to meson production in proton-antiproton annihilation.

The invariance of strong interactions under the transformations which are generated by the group SU_3 is currently being investigated. The main success of the theory has been in reproducing large segments of the mass spectrum of the hadrons, the strongly interacting particles. This spectacular success is only possible however in a "broken symmetry" scheme, that is, it is necessary to include in the strong interaction Hamiltonian a term which is not invariant under SU_3 transformations but transforms like one of the generators of SU_3. In a simple first-order perturbation calculation, this

term breaks the symmetry by splitting the degenerate SU_3 energy levels to obtain the experimentally measured masses.

When exact SU_3 invariance is applied to the dynamics of strong interactions, the theory meets only limited success.[14-17] An obvious explanation for this failure is that the broken-symmetry term in the reaction Hamiltonian cannot be ignored in dynamical calculations. A suggested way of by-passing this obvious difficulty is to consider interactions in which the energy available to the reactants is much greater than the mass differences accounted for by the broken-symmetry term.[18] Proton-antiproton annihilation into mesons is a reasonable candidate since, even when the annihilation occurs at rest, the energy available for meson production is large compared to the meson mass differences in an SU_3 multiplet. It is also possible that the effects of symmetry-breaking upon a many-particle system may be masked by the higher-order pure SU_3 effects.

The purpose of this calculation is then two-fold. First we wish to investigate the effect of SU_3 invariance upon a dynamical system of the type discussed above. The secondary aim is to carry out a

fairly complete calculation applying the Fermi Statistical Model to mesons produced in p-$\bar{p}$ annihilation emphasizing the effects of including recent experimental data on meson resonances. In this vein, the following calculation differs from those done previously in that all allowable combinations of 27 mesons, including the $J^P = 0^-$, 1^- and 2^+ nonets, have been included as possible final states. A Monte Carlo routine has been used to evaluate the phase space integrals. Final states of up to 6 particles have been included under the assumption of SU_3 invariance and states of up to 10 particles under the assumption of SU_2 invariance. All calculations have been done at total energies of 1.9, 2.3 and 2.5 BeV in the center-of-mass system, assuming invariance under both SU_2 and SU_3, and at 3.0 BeV for SU_2 invariance only.

II. The Statistical Model

A. Formulation of the Model.

The Statistical Model was proposed by Fermi in 1950 as an order-of-magnitude method of dealing with multiple particle production resulting from collisions between nucleons. The model attempted to give estimates of relative cross-sections for production of mesons and baryons from an initial two-nucleon state.[12] The model was quite crude in its original form, taking into account only energy conservation. Ordinary momentum was conserved only approximately by considering all particles to be either extremely relativistic (mesons) or non-relativistic (baryons) and then allowing the non-relativistic particles to carry off momentum. Subsequent to its introduction, the model was modified to include momentum conservation,[19] isotopic spin conservation[13] and final-state interactions.[20] In 1960, Hagedorn reformulated the model in a rigorous framework starting from S-matrix theory.[21] An important result of this formulation is Hagedorn's interpretation of the volume parameters. Under the assumption that the S-matrix dependence upon the isotopic spins, momenta and masses of the individual

particles may be neglected, the product of the volume parameters for a specified final state can be interpreted as the mean value of the square of the S-matrix element. This is a reasonable assumption since, in the statistical theory, the main dependence of the relative cross-section for the production of any state upon the individual masses and momenta is exhibited by the phase space integrals, and the dependence upon individual isotopic spins is exhibited by the weight factor. Hence, Hagedorn concludes, the product of the volume parameters should be dependent, at most, upon the total energy and the number of particles in the final state irrespective of their isotopic spins, masses or momenta. We shall refer to this conclusion later.

According to the modified statistical model, the unnormalized probability for the formation of a specific state consisting of n mesons from $p - \bar{p}$ annihilation at energy E, in the center-of-mass system, is:

$$S(m_1 m_2 \cdots m_n, E) = \left\{ \prod_{k=1}^{k_{max}} \Omega_k^{n_k} (2 s_k + 1)^{n_k} \right\}$$

$$\times \left(\frac{G_{SU_n}}{\prod_k n_k!} \right) \frac{\rho_n (m_1 m_2 \cdots m_n, E)}{(2\pi)^{3n-3}}$$

where:

m_i is the mass of the i^{th} particle.

n_k is the number of (identical) particles of the k^{th} kind so that $\sum_{k=1}^{k_{max}} n_k = n$.

s_k is the spin of the k^{th} kind of particle.

Ω_k is the volume parameter associated with the k^{th} kind of particle.

G_{SU_n} is a weight factor arising from the invariance of the interaction responsible for the production under the Unitary group SU_n.

f_n is the familiar phase space integral.

In this calculation we shall consider invariance under the groups SU_2 and SU_3. In the above formula, one must consider all the particles which fit into a single representation of SU_n as identical. We have assumed throughout that all particles which fit into a single irreducible representation of SU_2, i.e., all members of the same isotopic spin multiplet, are identical. In the case of SU_3 invariance we consider the above assignment to be a consequence of

symmetry breaking.*

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*If Su_3 invariance were strictly true, all particles in an irreducible representation of Su_3 would have to be considered identical. However, the members of a given "SU_3 multiplet" are actually split into its isotopic spin (SU_2) multiplets by the symmetry-breaking term.

B. Successes and failures of the model.

The refined statistical model has had moderate success in reproducing experimental data on distributions for the production of pions and charged prongs from annihilation of antiprotons.[22] Its most drastic failure has been that it predicts too large a rate of kaon production. This discrepancy, of course, can be minimized by introducing a volume parameter for kaons that is much smaller than those for other mesons. Also, in order to fit the experimental data, it has been necessary to introuce volume parameters of the order of the pion volume,* whereas general field theoretic arguments show the range of the annihilation potential to be of the order of the nucleon Compton wave-length.[23] Assuming that the volume parameter must be of the order of magnitude of the interaction volume, we would expect it to be of the order of $1/350\,\Omega_\pi$. As pointed out by Hagedorn, the requirement of unphysically large volume parameters may be due to the omission of possible final states, in particular, to those containing resonances.[21] Therefore, we have included as possible final states all combinations of the 27 mesons

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*The pion volume is the volume of a sphere whose radius is the pion Compton wave-length $\Omega_\pi = \frac{4}{3}\pi\left(\frac{\hbar}{m_\pi c}\right)^3$.

comprising the "so-called" psuedoscalar, vector and tensor nonets which are allowed by conservation of energy, charge, isotopic spin, baryon number and hypercharge. (The hypercharge Y is a quantity equivalent to the strangeness S and related to it and the baryon number B by $Y = B + S$. The isotopic spin projection quantum number is also conserved in strong interactions. Its conservation is guaranteed by the conservation of charge and hypercharge, the three quantities being related by $Q = T + 1/2\ Y$.)

A major source of difficulty with the model has been the accurate evaluation of the phase space integrals. Except for special cases, the integrals become so complex for n greater than three that approximations are unavoidable and the use of a high speed computor is necessitated.[24] In this calculation, the phase space integrals were evaluated to within approximately 6% relative error by a Monte Carlo technique developed by Cerulus and Hagedorn.[19] This technique, together with an error analysis, is briefly outlined in Appendix B.

III. The effects of Unitary Symmetry upon meson production in $p-\bar{p}$ annihilation.

A. Symmetry groups in physics.

It has always been the hope of physicists to be able to describe matter as a composite of a few fundamental constituents. Thus, discovery of new particles frequently leads to attempts to explain their existence as a part of a general scheme or spectroscopy. With the development of high energy accelerators has come the discovery of a large number of new "particles". Among the many attempts to classify these, the most successful and fruitful has been the model known as the "8-fold way". The basic assumption of this model is the invariance of a thus-far undiscovered super-strong interaction under the symmetry group SU_3 .

In the 1950's, experimental evidence for the charge independence of the two nucleon force led to the postulation of the conservation of isotopic spin in strong interactions.[25] In group theory language, this can be expressed as the invariance of the interaction under the special unitary group in two dimensions, SU_2 . SU_2 is the group of all unitary, unimodular (determinant = 1) 2 by 2 matrices. This is, in a sense, a generalization of the rotation group, having representations of all even dimensions (corresponding to half-integral values of

angular momentum) in addition to the odd dimensional representations (corresponding to integral values of angular momentum), which it has in common with the rotation group.

In applying the theory of group representations to physical systems,* symmetry with respect to a particular group is meant to imply that the Hamiltonian of the system remains invariant under all transformations of the form:

$$R(\theta_1, \theta_2 \cdots \theta_M) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{M} i \theta_k J_k \right)^n = e^{i \sum_{k=1}^{M} \theta_k J_k}$$

where:

J_k are operators which are called generators of the group. These may be chosen Hermitian in which case they are related in the usual way to physical observables. Of course, only those generators which mutually commute may be simultaneously diagonalized, the number of which is called the rank of the group.

θ_k are continuous parameters, the values of which specify the particular transformation.

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*In all that follows we limit ourselves to a consideration of irreducible representations of simple Lie groups.

An n-fold degenerate state of the system is represented by the basis states of an n-dimensional irreducible representation of the group, one basis state corresponding to one physical state.* There exists, for a group of rank ℓ, ℓ irreducible fundamental representations. These are fundamental in the sense that all irreducible representations of the group appear in the Kronecker products of the fundamental representations with themselves. An important aim of most models is to fit the basic constituents into the fundamental representations and then to form the structured physical systems from these according to the rules of group theory. For example, consider the isotopic spin group, SU_2, as applied to nuclear physics. The single fundamental representation of the rank 1 group SU_2 is a doublet ($I = 1/2$), the two basis states of which represent the neutron and the proton. Nuclei, of all possible Z and N, fit into the higher dimensional irreducible representations which are formed by taking Kronecker products of the fundamental representation with itself.

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*We note that, in considering the theory of symmetry groups in physics, there is first the group invariance properties and then the correspondence of the mathematical quantities with the physical system which we shall call the model.

B. Application of SU_3 symmetry to strong interactions.

In 1953, Gell-Mann and Nishijima, on the basis of the empirical evidence then available, postulated a new additive quantum number called the strangeness which is conserved in all strong interactions.[26] This additional quantum number could not be accounted for by SU_2 invariance alone. SU_2, being a rank 1 group, allows for only a single additive quantum number, this being identified with the i-spin projection I_z. Hence, there could either be a product symmetry at work or the strong interaction could be invariant under a higher rank group which contains SU_2 as a subgroup. A rank 2 group, having two mutually commuting generators, could account for both the additive quantum numbers S and I_z. An obvious candidate (among others) for this role is SU_3, the group of 3 by 3 unitary, unimodular matrices.

There were a number of different models suggested, each corresponding to a particular way of fitting the known particles into the representations of the group. The most famous of these was the $(N-\lambda)$ triplet model of Sakata.[27] In 1961, Gell-Mann and Ne'eman independently proposed the model which has come to be known as the "eightfold-way".[28,29] This model differed from that of Sakata in that no attempt was made to fit observed

particles into the fundamental representations of the group. Instead, it was recognized that a large number of the experimentally observed particles could be fit into certain higher dimensional irreducible representations of SU_3. The real strength of the model was that a simple modification of the SU_3 symmetry could yield known mass differences to a high degree of accuracy. It was later postulated that, consistent with the eightfold-way and the observed sprectrum of particles, three hypothetical particles called "quarks" and their antiparticles could be associated with the fundamental representations of SU_3.* The lower mass baryons are then formed by triplets of quarks according to the prescription $3 \otimes 3 \otimes 3 = 10 \oplus 8_1 \oplus 8_2 \oplus 1$ and mesons from a quark-antiquark pair according to $3 \otimes 3^* = 8 \oplus 1$. This adequately describes the known baryon decuplet, octet and singlet and the three meson nonets (see Tables 1, 2, 3). (However, a second baryon octet has not yet been detected). In addition, the search

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*SU_3, being a rank 2 group, has two fundamental representations. These are each of dimension 3 and are conjugate to each other.

Table 1. Properties of the psuedoscalar (J=0-) meson nonet.

Mesons	Average mass of the I-spin multiplet (Mev)	Isotopic spin	Hypercharge	Dimension of SU_3 representation
$\pi^{\pm}$	138	1	0	8
K^{+}	496	1/2	1	8
K^{0}	496	1/2	-1	8
η^{0}	549	0	0	8
η'^{0}	959	0	0	1

Table 2. Properties of the vector (J=1-) meson nonet.

Meson	Average mass of the I-spin multiplet (Mev)	Isotopic spin	Hypercharge	Dimension of SU_3 representation
$\rho^{\pm}$	765	1	0	8
K^{*+}	891	1/2	1	8
K^{*0}	891	1/2	-1	8
ϕ^{0}	1020	0	0	8†
ω^{0}	783	0	0	1†

Table 3. Properties of the tensor (J=2+) meson nonet.

Meson	Average mass of the I-spin multiplet (Mev)	Isotopic spin	Hypercharge	Dimension of SU_3 representation
$A2^{\pm}$	1324	1	0	8
K^{*+}	1405	1/2	1	8
K^{*0}	1405	1/2	-1	8
f^{0}	1253	0	0	8†
f'^{0}	1500	0	0	1†

†In order to fit the masses of the mesons in the vector and tensor nonets within the framework of the eightfold way, it is necessary to introduce mixing between the two isotopic spin singlets. Therefore, neither can be assigned to a definite irreducible representation. For simplicity, we assigned the particles as shown.

for quarks, which in this model must be of non-integral charge, has been fruitless.

In order to account for the mass splitting of the particles fitting into an irreducible representation of SU_3, (with pure SU_3 invariance, all particles fitting into a given irreducible representation have equal masses) it is necessary to introduce a symmetry-breaking term into the Hamiltonian. The term is chosen to transform like one of the generators of SU_3. This choice is in analogy to the breaking of SU_2 symmetry by the electromagnetic force where a term transforming like the isotopic spin projection operator I_z splits a degenerate set of particles of a given isotopic spin into their observed charge multiplets. With this choice, a perturbation calculation then yields the mass splittings of the mesons and baryons to reasonable accuracy. Up to the present, this has been the most impressive success of the "eightfold-way".

An obvious extension of this theory is to consider the effects of SU_3 invariance upon dynamical systems undergoing strong interactions. Using the simple rules for the expanding of a Kronecker product in terms of irreducible representations of the group, with a table of Clebsch-Gordan coefficients for SU_3, one can predict relationships between various cross-sections. In this crude

analysis, the symmetry-breaking term is always neglected. As pointed out by Levinson, Lipkin and Meshkov, this is reasonable only if the energy available to the system is much greater than the mass splittings of the particles involved in the interaction.[18] This analysis has shown, for the most part, poor agreement with experiment.[14-16]

C. The SU_2 weight factors.

It is our objective to extend the statistical theory to include the effects of invariance under the unitary groups SU_2 and SU_3. As pointed out in the previous chapter, invariance under these groups give rise to a multiplicative factor G_{SU_n} in the expression for the relative production probability for a given final state:

$$P(m_1 \cdots m_n) = \left\{ \prod_{k=1}^{k_{max}} \Omega_k^{n_k} (2S_k+1)^{n_k} \right\} \frac{G_{SU_n}\, \rho_n(m_1, m_2 \cdots m_n, E)}{(2\pi)^{3n-3} \prod_k n_k!}$$

where, in the n particle final state, there are n_k identical particles of the k^{th} kind with $n = \sum_{k=1}^{k_{max}} n_k$.

We now proceed to derive an expression for G_{SU_2} using an approach very close to that of Cerulus.[31]

Starting from S-matrix theory, with a definite initial state ψ_i, the transition probability to a set F of possible final states ψ_f is given by:

$$P = \sum_{f \in F} |\langle \psi_f | S | \psi_i \rangle|^2$$

$$= \sum_{\substack{\text{ISOTOPIC SPIN} \\ \text{VARIABLES}}} \sum_{\substack{\text{SPIN} \\ \text{VARIABLES}}} \int \cdots \int dp_1\, dp_2 \cdots dp_n\, |\langle \psi_f | S | \psi_i \rangle|^2$$

We assume that the wave functions are eigenfunctions of the momenta, masses, spins and isotopic spins of the individual particles (two nucleons in the initial state, n mesons in the final state). We represent all quantum numbers, except those associated with the isotopic spin, by a single parameter, α. In what follows, we shall represent the isotopic spin of a single particle by i and its projection by i_z.

For simplicity, we shall assume that the initial state is a "pure" isotopic spin state $\psi_i = |I_i\, I_{z_i};\alpha_i\rangle$. If it is an admixture of isotopic spin states, one merely has to average over its components as follows:

$$\psi_i = \sum_k a_k |I_k\, I_{z_k};\alpha_i\rangle$$

$$\text{where: } \sum_k |a_k|^2 = 1.$$

The final state is to be made up of n single particles each of which has a definite value of isotopic spin $|i_1 i_{z_1}\rangle |i_2 i_{z_2}\rangle \cdots \cdots |i_n i_{z_n}\rangle$. We transform from the set of product wave functions to those which are eigenfunctions of the total I-spin by a unitary transformation. We choose a scheme where the intermediate isotopic spins:

$$\vec{I}_2^{\,2} = (\vec{i}_1 + \vec{i}_2)^2$$

$$\vec{I}_3^{\,2} = (\vec{i}_1 + \vec{i}_2 + \vec{i}_3)^2$$

$$\vdots$$

$$\vec{I}_{n-1}^{\,2} = (\vec{i}_1 + \vec{i}_2 + \cdots + \vec{i}_{n-1})^2$$

are also diagonal. The transformation is:

$$|I_2, I_3 \cdots I_{n-1}\,;\, I, I_z\rangle = \sum_{i_{z_1}\, i_{z_2} \cdots\, i_{z_n}} C^{i_1\, i_2\, I_2}_{i_{z_1}\, i_{z_2}\, I_{z_2}} C^{I_2\, i_3\, I_3}_{I_{z_2}\, i_{z_3}\, I_{z_3}} \cdots$$

$$\cdots C^{I_{n-1}\, i_n\, I}_{I_{z_{n-1}}\, i_{z_n}\, I_z} \left(|i_1 i_{z_1}\rangle |i_2 i_{z_2}\rangle \cdots |i_n i_{z_n}\rangle\right)$$

where the C's are the familiar SU_2 Clebsch-Gordan coefficients. The above expression is generally written as:

$$|I_2, I_3 \cdots I_{n-1}\,;\, I, I_z\rangle = \sum_{i_{z_1}\, i_{z_2} \cdots\, i_{z_{n-1}}} \langle I_2 I_3 \cdots I_{n-1}\,;\, I, I_z \,|^{i_1\, i_2 \cdots\, i_n}_{i_{z_1}\, i_{z_2} \cdots\, i_{z_n}}\rangle$$

$$\times \left(|i_1 i_{z_1}\rangle |i_2 i_{z_2}\rangle \cdots\cdots |i_n i_{z_n}\rangle\right)$$

where the term in brackets is called the recoupling coefficient. We will make use of the fact that the recoupling coefficient is real since it is a product of Clebsch-Gordan coefficients. We also note that

$$\sum_{i_{z_1}\, i_{z_2} \cdots\, i_{z_n}} \langle I_2, I_3, \cdots I_{n-1}\,;\, I, I_z \,|^{i_1\, i_2 \cdots\, i_n}_{i_{z_1}\, i_{z_2} \cdots\, i_{z_n}}\rangle^2 = 1 .$$

We want to calculate the transition probability to a final state of n particles $|i_1 i_{z_1}\rangle |i_2 i_{z_2}\rangle \cdots |i_n i_{z_n}\rangle$, the admixture of which in the properly constructed final state $|I_2 I_3 \cdots I_{n-1}; I I_z\rangle$ is given by the square of the recoupling coefficient

$$\left\langle I_2, I_3, \cdots I_{n-1}; I\ I_z \middle| \begin{matrix} i_1 & i_2 & \cdots & i_n \\ i_{z_1} & i_{z_2} & & i_{z_n} \end{matrix} \right\rangle^2 .$$

Assuming the initial state is a pure isotopic spin state $\psi_i = |I_i I_{z_i}; \alpha_i\rangle$, our n particle state yields a contribution to the S-matrix element of:

$$\langle \psi_f | S | \psi_i \rangle = \left\langle I_2 I_3 \cdots I_{n-1}; I_f I_{z_f} \middle| \begin{matrix} i_1 & i_2 & \cdots & i_n \\ i_{z_1} & i_{z_2} & & i_{z_n} \end{matrix} \right\rangle$$

$$\times \langle I_2 I_3 \cdots I_{n-1}, I_f I_{z_f}; \alpha_f | S | I_i I_{z_i}; \alpha_i \rangle .$$

Conservation of isotopic spin demands that the S-matrix element be zero unless $I_f = I_i$ and $I_{z_f} = I_{z_i}$.

Therefore:

$$\langle \psi_f | S | \psi_i \rangle = \left\langle I_2 I_3 \cdots I_{n-1}; I_i I_{z_i} \middle| \begin{matrix} i_1 & i_2 & \cdots & i_n \\ i_{z_1} & i_{z_2} & & i_{z_n} \end{matrix} \right\rangle$$

$$\times \langle I_2 I_3 \cdots I_{n-1}, I_i I_{z_i}; \alpha_f | S | I_i I_{z_i}; \alpha_i \rangle .$$

Of course, the final state consisting of particles $|i_1 i_{z_1}\rangle |i_2 i_{z_2}\rangle \cdots |i_n i_{zn}\rangle$ contributing to a total isotopic spin (I_i, I_{zi}) may be realized through a number of sets of intermediate isotopic spins $(I_2, I_3, \ldots I_{n-1})$. We must therefore sum over these:

$$\langle \psi_f | S | \psi_i \rangle = \sum_{I_2, I_3 \cdots I_{n-1}} \langle I_2 I_3 \cdots I_{n-1}; I_i I_{zi} | \begin{matrix} i_1 & \cdots & i_n \\ i_{z_1} & \cdots & i_{zn} \end{matrix} \rangle$$

$$\times \langle I_2 I_3 \cdots I_{n-1}, I_i I_{zi}; \alpha_f | S | I_i I_{zi}; \alpha_i \rangle .$$

In order to obtain the transition probability we must take the absolute square of the term above. We now make a statistical assumption and neglect all interference terms by taking the absolute square within the summation.

$$|\langle \psi_f | S | \psi_i \rangle|^2 \approx \sum_{I_2 I_3 \cdots I_{n-1}} \langle I_2 I_3 \cdots I_{n-1}; I_i I_{zi} | \begin{matrix} i_1 & \cdots & i_n \\ i_{z_1} & \cdots & i_{zn} \end{matrix} \rangle^2$$

$$\times |\langle I_2 I_3 \cdots I_{n-1}, I_i I_{zi}; \alpha_f | S | I_i I_{zi}; \alpha_i \rangle|^2$$

The basis for this approximation is that if enough states are considered, there will tend to be an equal amount of positive and negative contributions

to the square of the S-matrix from the cross terms. This is essentially a random phase approximation.

In order to completely isolate all isotopic spin dependence into a single multiplicative factor, we shall assume that any isotopic spin dependence left in the S-matrix element is negligible

$$|\langle I_2 I_3 \cdots I_{n-1}, I_i I_{Zi}; \alpha_f | S | I_i I_{Zi}; \alpha_i \rangle|^2 \approx |\langle \alpha_f | S | \alpha_i \rangle|^2$$

so that we have:

$$|\langle \psi_f | S | \psi_i \rangle|^2 \approx \left\{ \sum_{I_2 I_3 \cdots I_{n-1}} \langle I_2 I_3 \cdots I_{n-1}; I_i I_{Zi} | {}^{i_1\ i_2\ \cdots\ i_n}_{i_{z_1}\ i_{z_2}\ \cdots\ i_{z_n}} \rangle^2 \right\} \times |\langle \alpha_f | S | \alpha_i \rangle|^2$$

$$= P^{I_i}_{i_{z_1} i_{z_2} \cdots i_{z_n}} |\langle \alpha_f | S | \alpha_i \rangle|^2,$$

where, following the notation of Cerulus:

$$P^{I_i}_{i_{z_1} i_{z_2} \cdots i_{z_n}} \equiv \sum_{I_2 I_3 \cdots I_{n-1}} \langle I_2 I_3 \cdots I_{n-1}; I_i I_{Zi} | {}^{i_1\ i_2\ \cdots\ i_n}_{i_{z_1}\ i_{z_2}\ \cdots\ i_{z_n}} \rangle^2 .$$

Finally, we must take into account the fact that the expression above assumes a definite order of the individual particles (as specified by their isotopic spin variables i, i_z) in the final state.

We want to find the transition probability to a final state consisting of n_1 particles of the first kind, n_2 of the second kind, etc. regardless of their order. We must therefore sum up the contributions to the S-matrix element due to final states which are distinct permutations of the state above. In doing this we make use of the fact that the value of $P^{I}_{i_{z_1}, i_{z_2} \cdots i_{z_n}}$ is independent of the order of the $\{ i_z \}$. (See Appendix A). Hence all the contributions are the same and we simply multiply the term above by the number of distinct permutations of the set $|i_1 i_{z_1}\rangle |i_2 i_{z_2}\rangle |i_3 i_{z_3}\rangle \cdots |i_n i_{z_n}\rangle$. This factor is:

$$\frac{\prod_{i=1}^{i_{max}} n_i!}{\prod_{k=1}^{k_{max}} n_k!} \quad \text{with} \quad n = \sum_{k=1}^{k_{max}} n_k = \sum_{i=1}^{i_{max}} n_i$$

where:

n_i is the number of particles with a given value of isotopic spin.

n_k is the number of identical particles of the k^{th} kind. Particles are considered identical if they have the same quantum numbers (i, i_z).

We now have the following expression for the square of the S-matrix element from an initial state of isotopic spin (I_i, I_{z_i}) to a final state consisting

of k_{max} sets of n_k identical particles of isotopic spin (i_k, i_{z_k}):

$$\sum_{\substack{\text{ISOTOPIC}\\ \text{SPIN}\\ \text{VARIABLES}}} |\langle \psi_f | S | \psi_i \rangle|^2 = \frac{\prod_i n_i!}{\prod_k n_k!} P^{I_i}_{i_{z_1} i_{z_2} \cdots i_{z_n}} |\langle \alpha_f | S | \alpha_i \rangle|^2 .$$

Substituting this into the expression for the transition probability gives:

$$P = \frac{\prod_i n_i!}{\prod_k n_k!} P^{I_i}_{i_{z_1} i_{z_2} \cdots i_{z_n}} \sum_{\substack{\text{SPIN}\\ \text{VARIABLES}}} \int \cdots \int d\vec{p}_1 \, d\vec{p}_2 \cdots d\vec{p}_n \, |\langle \alpha_f | S | \alpha_i \rangle|^2 .$$

We recall that the symbol α represents mass, momentum and spin coordinates. It can be shown rigorously (see the article by Hagedorn)[21] that the terms

$$\sum_{\substack{\text{SPIN}\\ \text{VARIABLES}}} \int \cdots \int d\vec{p}_1 \, d\vec{p}_2 \cdots d\vec{p}_n \, |\langle \alpha_f | S | \alpha_i \rangle|^2$$

give rise in the statistical model to the factors

$$\left\{ \prod_{k=1}^{k_{max}} \Omega_k^{n_k} (2S_k + 1)^{n_k} \right\} \frac{\rho_n (m_1, m_2, \ldots m_n, E)}{(2\pi)^{3n-3} \prod_k n_k!} .$$

A comparison of the expression for the transition probability given above to that given previously, allows us to identify the SU_2 weight factor for transition to

a final state consisting of individual particles $|i_1 i_{z1}\rangle |i_2 i_{z2}\rangle \cdots |i_n i_{zn}\rangle$ from an initial state of isotopic spin (I_i, I_{zi}) as:

$$G_{SU_2} = \frac{\prod_i n_i!}{\prod_k n_k!} P^{I_i}_{i_{z1}, i_{z2} \cdots i_{zn}}$$

where:

$$P^{I_i}_{i_{z1} i_{z2} \cdots i_{zn}} = \sum_{I_2 I_3 \cdots I_{n-1}} \langle I_2 I_3 \cdots I_{n-1}; I_i I_{zi} | {}^{i_1\ i_2\ \cdots\ i_n}_{i_{z1}\ i_{z2}\ \cdots\ i_{zn}} \rangle^2 .$$

For the case where the initial state is not a pure isotopic spin state but an admixture of the form:

$$\psi_i = \sum_j a_j |I_j I_{zj}\rangle$$

where:

$$\sum_j |a_j|^2 = 1$$

the SU_2 weight factor becomes

$$G_{SU_2} = \frac{\prod_i n_i!}{\prod_k n_k!} \sum_j |a_j|^2 P^{I_j}_{i_{z1}, i_{z2} \cdots i_{zn}} .$$

In the case of p-$\bar{p}$ annihilation, the initial state (consisting of a proton and antiproton) is an equal admixture of isotopic spins 1 and 0 so that:

$$G_{SU_2} = \frac{\prod_i n_i!}{\prod_k n_k!} \left\{ \frac{1}{2} P^{I=1}_{i_{z1}, i_{z2} \cdots i_{zn}} + \frac{1}{2} P^{I=0}_{i_{z1}, i_{z2} \cdots i_{zn}} \right\}$$

D. The SU_3 weight factors.

We now proceed to generalize the preceeding discussion to obtain the weight factor for the case of invariance under the group SU_3 . SU_3 differs from SU_2 in that it is not a simply reducible group. This gives rise to two basic changes in the Racah algebra for the group:

1. In general, an irreducible representation is not equal to its complex conjugate representation.
2. In a Kronecker product, a given irreducible representation may appear more than once.

We will represent the pair of eigenvalues of the Casimir operators of SU_3 by the single symbol* U and the basis states of the irreducible representation by $\phi_{I\, I_z\, Y}^{U\gamma}$ The symbol γ is necessary only if ϕ is a state belonging to a representation which appears more than once in a Kronecker product; I is the eigenvalue of the Casimir operator of the subgroup SU_2 (it is the isotopic spin); I_z and Y are the two additive quantum numbers for SU_3

---

*SU_3 is a rank two group and therefore has two Casimir operators. The eigenvalues of these, in a particular order, uniquely identify the irreducible representations.

and are respectively the eigenvalues of the operators representing the isotopic spin projection and the hypercharge.

We couple states in SU_3 as follows:

$$\phi^{U_\gamma}_{I I_z Y} = \sum_{\substack{i_{z_1} i_{z_2} \\ y_1 y_2}} \left(\begin{matrix} \mu_1 & \mu_2 \\ i_1 i_{z_1} y_1 & i_2 i_{z_2} y_2 \end{matrix} \middle| \begin{matrix} U_\gamma \\ I I_z Y \end{matrix} \right) \phi^{\mu_1}_{i_1 i_{z_1} y_1} \phi^{\mu_2}_{i_2 i_{z_2} y_2}$$

where: $I_z = i_{z_1} + i_{z_2}$

$Y = y_1 + y_2$.

where the term in parentheses is the SU_3 Clebsch-Gordan coefficient. We again start with the expression for the transition probability:

$$P = \sum_{\substack{SU_3 \\ \text{VARIABLES}}} \sum_{\substack{\text{SPIN} \\ \text{VARIABLES}}} \int \cdots \int d\vec{p}_1 \, d\vec{p}_2 \cdots d\vec{p}_n \, |\langle \psi_f | S | \psi_i \rangle|^2$$

and, as before, we shall attempt to explicitly to perform the summation (approximately) over the SU_3 variables.

We assume that the wave functions are eigenfunctions of the momenta, masses, spins and of the SU_3 variables of the individual particles filling the states. The SU_3 variables consist of the eigenfunctions of the SU_3 Casimir operator μ , of the SU_2 Casimir operator i

(the isotopic spin), of its projection i_z and of the hypercharge y . We again represent all other eigenvalues by the single parameter α .

We assume that the initial state is a pure SU_3 state of the form:

$$\psi_i = \phi^{U_i}_{I_i, I_{zi}, Y_i}.$$

We shall later generalize to the general case of an initial state which is an admixture of various SU_3 states. The final state is to contain n single particle states each of which is an SU_3 eigenstate:

$$| \phi^{u_1}_{i_1 i_{z_1} y_1} \rangle | \phi^{u_2}_{i_2 i_{z_2} y_2} \rangle \cdots | \phi^{u_n}_{i_n i_{z_n} y_n} \rangle.$$

We recall that the final state is to consist of mesons which fit into either the 8 or the 1 dimensional irreducible representation of SU_3 each with a given value of isotopic spin i , projection i_z and hypercharge y . (See Tables 1, 2, 3)

The final state is to be formed as an eigenstate of:

$$U_{f\gamma} = u_1 \otimes u_2 \otimes u_3 \otimes \cdots \otimes u_n$$

$$\vec{I}^2_f = (\vec{i}_1 + \vec{i}_2 + \vec{i}_3 + \ldots + \vec{i}_n)^2$$

which will be some linear combination of our product states above. We transform from the system of product states to those which are eigenfunctions of U_f and I_f by a unitary transformation. We choose a scheme in which:

$$
\begin{aligned}
U_{2\gamma} &= u_1 \otimes u_2 \\
\vec{I}_2^{\,2} &= (\vec{i}_1 + \vec{i}_2)^2 \\
U_{3\gamma} &= u_1 \otimes u_2 \otimes u_3 \\
\vec{I}_3^{\,2} &= (\vec{i}_1 + \vec{i}_2 + \vec{i}_3)^2 \\
&\vdots \\
U_{n-1\,\gamma} &= u_1 \otimes u_2 \otimes u_3 \cdots \otimes u_{n-1} \\
\vec{I}_{n-1}^{\,2} &= (\vec{i}_1 + \vec{i}_2 + \vec{i}_3 + \cdots + \vec{i}_{n-1})^2
\end{aligned}
$$

are also diagonal. The transformation is:

$$
\left| \begin{matrix} U_{2\gamma}, U_{3\gamma}, \cdots U_{n-1\,\gamma} ; U_{f\gamma} \\ I_2, I_3 \qquad\quad I_{n-1} ; I_f I_{zf} Y_f \end{matrix} \right\rangle = \sum \left(\begin{matrix} u_1 & u_2 \\ i_1 i_{z_1} y_1 & i_2 i_{z_2} y_2 \end{matrix} \middle| \begin{matrix} U_{2\gamma} \\ I_2 I_{z_2} Y_2 \end{matrix} \right)
$$

$$
\times \left(\begin{matrix} U_{2\gamma} & u_3 \\ I_2 I_{z_2} Y_2 & i_3 i_{z_3} y_3 \end{matrix} \middle| \begin{matrix} U_{3\gamma} \\ I_3 I_{z_3} Y_3 \end{matrix} \right) \cdots
$$

$$
\cdots \left(\begin{matrix} U_{n-1\,\gamma} & u_n \\ I_{n-1} I_{z_{n-1}} Y_{n-1} & i_n i_{z_n} y_n \end{matrix} \middle| \begin{matrix} U_{f\gamma} \\ I_f I_{zf} Y_f \end{matrix} \right) \left| \phi^{u_1}_{i_1 i_{z_1} y_1} \right\rangle \cdots \left| \phi^{u_n}_{i_n i_{z_n} y_n} \right\rangle .
$$

We again identify the product of Clebsch-Gordan

coefficients as the (real) recoupling coefficient and write this as

$$\left\langle \begin{matrix} U_{2\gamma}, & U_{3\gamma}, \dots & U_{n-1\,\gamma} \\ I_2, & I_3 & I_{n-1} \end{matrix} ; \begin{matrix} U_{f\gamma} \\ I_f \nu_f \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 & \dots & \mu_n \\ i_1\nu_1 & i_2\nu_2 & \dots & i_n\nu_n \end{matrix} \right\rangle$$

where we have introduced the standard notation

$$\nu \equiv (i_z, \gamma)$$

also

$$\sum_{\nu_1 \nu_2 \dots \nu_n} \left\langle \begin{matrix} U_{2\gamma} & U_{3\gamma} & \dots & U_{n-1\,\gamma} \\ I_2, & I_3, & \dots & I_{n-1} \end{matrix} ; \begin{matrix} U_{f\gamma} \\ I_f \nu_f \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 & \dots & \mu_n \\ i_1\nu_1 & i_2\nu_2 & \dots & i_n\nu_n \end{matrix} \right\rangle^2 = 1 .$$

We now identify the recoupling coefficient above as the admixture of our product state

$$\left| \phi^{\mu_1}_{i_1\nu_1} \right\rangle \left| \phi^{\mu_2}_{i_2\nu_2} \right\rangle \dots \left| \phi^{\mu_n}_{i_n\nu_n} \right\rangle$$

in the correctly constructed final state

$$\left| \begin{matrix} U_{2\gamma} & U_{3\gamma} & \dots & U_{n-1\,\gamma} \\ I_2, & I_3, & \dots & I_{n-1} \end{matrix} ; \begin{matrix} U_{f\gamma} \\ I_f \nu_f \end{matrix} \right\rangle .$$

Assuming the initial state to be of the form

$\psi_i = \phi^{U_i}_{I_i \nu_i} \equiv \left| {}^{U_i}_{I_i \nu_i} \right\rangle$, our n meson state has an S-matrix element equal to:

$$\left\langle \psi_f \middle| S \middle| \psi_i \right\rangle = \left\langle \begin{matrix} U_{2\gamma}, & U_{3\gamma} \dots & U_{n-1\,\gamma} \\ I_2, & I_3 & I_{n-1} \end{matrix} ; \begin{matrix} U_{f\gamma} \\ I_f \nu_f \end{matrix} \middle| \begin{matrix} \mu_1 & \dots & \mu_n \\ i_1\nu_1 & \dots & i_n\nu_n \end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix} U_{2\gamma} & U_{3\gamma} & \dots & U_{f\gamma} \\ I_2, & I_3 & \dots & I_f \nu_f \end{matrix} ; \alpha_f \middle| S \middle| \begin{matrix} U_i \\ I_i \nu_i \end{matrix} ; \alpha_i \right\rangle .$$

Invariance of the interaction under SU_3 demands that the transition probability be zero unless:

$$U_{f\gamma} = U_i$$

$$I_f = I_i$$

$$Y_f = Y_i$$ (This implies the two equalities $I_{zf} = I_{zi}$ and $Y_f = Y_i$)

(γ may take on more than one value since U_i may be contained γ times in the Kronecker product $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$)

Therefore:

$$\langle \psi_f | S | \psi_i \rangle = \left\langle \begin{matrix} U_{2\gamma}, & U_{3\gamma} \cdots & U_{n-1\gamma} \\ I_2, & I_3 & I_{n-1} \end{matrix} ; \begin{matrix} U_{i\gamma} \\ I_i Y_i \end{matrix} \middle| \begin{matrix} \mu_1 & \cdots & \mu_n \\ i_1 \gamma_1 & & i_n \gamma_n \end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix} U_{2\gamma}, & U_{3\gamma} \cdots & U_{n-1\gamma}, & U_{i\gamma} \\ I_2, & I_3 & I_{n-1}, & I_i Y_i \end{matrix} ; \alpha_f \middle| S \middle| \begin{matrix} U_i \\ I_i Y_i \end{matrix} ; \alpha_i \right\rangle .$$

Again, we must sum over intermediate states to obtain the transition probability to our n meson final state. However, in addition to summing over the intermediate SU_3 representations U_γ , we must sum over all intermediate isotopic spin states I which are contained in a given irreducible representation U_γ . We now have:

$$\langle \psi_f | S | \psi_i \rangle = \sum_{\substack{U_{2\gamma}, U_{3\gamma}, \cdots U_{n-1\gamma} \\ I_2, I_3, \cdots I_{n-1}}}{}' \left\langle \begin{matrix} U_{2\gamma}, \cdots & U_{n-1\gamma} \\ I_2 & I_{n-1} \end{matrix} ; \begin{matrix} U_{i\gamma} \\ I_i Y_i \end{matrix} \middle| \begin{matrix} \mu_1 & \cdots & \mu_n \\ i_1 \gamma_1 & & i_n \gamma_n \end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix} U_{2\gamma} & U_{3\gamma} \cdots & U_{n-1\gamma}, & U_{i\gamma} \\ I_2, & I_3 & I_{n-1}, & I_i Y_i \end{matrix} ; \alpha_f \middle| S \middle| \begin{matrix} U_i \\ I_i Y_i \end{matrix} ; \alpha_i \right\rangle$$

where the prime on the summation is meant to indicate that there is to be a sum over the γ associated with U_i if γ takes on more than one value.

We now take the absolute square of the S-matrix element and make the statistical assumption by neglecting the interference terms.

$$|\langle \psi_f | S | \psi_i \rangle|^2 \approx \sum_{\substack{U_{2\gamma} \\ I_2}}{}' \sum_{\substack{U_{3\gamma} \\ I_3}} \cdots \sum_{\substack{U_{n-1\gamma} \\ I_{n-1}}} \left\langle \begin{matrix} U_{2\gamma}, U_{3\gamma} \cdots U_{n-1\gamma} ; U_{i\gamma} \\ I_2, I_3 \cdots I_{n-1} ; I_i \nu_i \end{matrix} \middle| \begin{matrix} \mu_1, \mu_2 \cdots \mu_n \\ i_1\nu_1, i_2\nu_2 \cdots i_n\nu_n \end{matrix} \right\rangle^2$$

$$\times \left| \left\langle \begin{matrix} U_{2\gamma}, U_{3\gamma} \cdots U_{n-1\gamma}, U_{i\gamma} \\ I_2, I_3 \cdots I_{n-1}, I_i \nu_i \end{matrix} ; \alpha_f \middle| S \middle| \begin{matrix} U_i \\ I_i \nu_i \end{matrix} ; \alpha_i \right\rangle \right|^2 .$$

In order to isolate all SU_3 dependence in a single term, we assume any further SU_3 dependence in the S-matrix element is negligible:

$$\left| \left\langle \begin{matrix} U_{2\gamma} \cdots U_{n-1\gamma}, U_{i\gamma} \\ I_2 \cdots I_{n-1}, I_i \nu_i \end{matrix} ; \alpha_f \middle| S \middle| \begin{matrix} U_i \\ I_i \nu_i \end{matrix} ; \alpha_i \right\rangle \right|^2 \approx |\langle \alpha_f | S | \alpha_i \rangle|^2 .$$

So that we have

$$|\langle \psi_f | S | \psi_i \rangle|^2 \approx \left\{ \sum_{\substack{U_{2\gamma} \\ I_2}}{}' \cdots \sum_{\substack{U_{n-1\gamma} \\ I_{n-1}}} \left\langle \begin{matrix} U_{2\gamma} \cdots U_{n-1\gamma} ; U_{i\gamma} \\ I_2 \cdots I_{n-1} ; I_i \nu_i \end{matrix} \middle| \begin{matrix} \mu_1 \cdots \mu_n \\ i_1\nu_1 \cdots i_n\nu_n \end{matrix} \right\rangle^2 \right\} \times |\langle \alpha_f | S | \alpha_i \rangle|^2$$

$$\equiv P^{U_i, I_i}_{\nu_1 \nu_2 \nu_3 \cdots \nu_n} |\langle \alpha_f | S | \alpha_i \rangle|^2 .$$

where we have defined a P-factor for SU_3 in analogy to that for SU_2 .

Finally we must take into account the fact that a definite order has been assumed for the individual particles in the final state. Since we want the transition probability to a final state of n specified individual particles regardless of their order, we must sum the contributions due to states which are distinct permutations of the state above. It is shown in Appendix A that the sum over intermediate states of the recoupling coefficient is independent of the order of the constituents in the set $\left\{ \begin{matrix} \mu_i \\ i_i, \nu_i \end{matrix} \right\}$. Therefore all the contributions are the same and we must multiply the expression above by the number of distinct permutations of the set $\left| \begin{matrix} \mu_1 \\ i_1 \nu_1 \end{matrix} \right\rangle \left| \begin{matrix} \mu_2 \\ i_2 \nu_2 \end{matrix} \right\rangle \cdots \left| \begin{matrix} \mu_n \\ i_n \nu_n \end{matrix} \right\rangle$.

This factor is:

$$\frac{n!}{\prod_{k=1}^{k_{max}} n_k!}$$

where:

n is the number of particles in a given irreducible representation of SU_3 . For the weights we have calculated, n is the total number of particles in the final state.

n_k is the number of identical particles of the k^{th} kind.

In this case, particles are considered identical if they have the same quantum numbers (u, i, i_z, y).

We therefore have the following expression for the square of the S-matrix element (summed over SU_3 variables) from an initial state specified by the quantum numbers (U_i, I_i, I_{zi}, Y_i) to a final state consisting of k_{max} sets of n_k identical particles with quantum numbers (u_k, i_k, i_{zk}, y_k):

$$\sum_{\substack{SU_3 \\ \text{VARIABLES}}} |\langle \psi_f | S | \psi_i \rangle|^2 = \frac{n!}{\prod_k n_k!} P^{U_i, I_i}_{\nu_1 \nu_2 \nu_3 \cdots \nu_n} \times |\langle \alpha_f | S | \alpha_i \rangle|^2 .$$

Substituting this into the expression for the transition probability gives:

$$P = \frac{n!}{\prod_k n_k!} \left\{ \sum'_{\substack{U_{2\gamma} \cdots U_{n-1\gamma} \\ I_2 \quad\quad I_{n-1}}} \left\langle \begin{matrix} U_{2\gamma} \cdots U_{n-1\gamma}, U_{i\gamma} \\ I_2 \quad\quad I_{n-1}, I_i \nu_i \end{matrix} \middle| \begin{matrix} u_1 \cdots u_n \\ i_1 \nu_1 \cdots i_n \nu_n \end{matrix} \right\rangle^2 \right\}$$

$$\times \sum_{\substack{\text{SPIN} \\ \text{VARIABLES}}} \int \cdots \int d\vec{p}_1 \, d\vec{p}_2 \cdots d\vec{p}_n \, |\langle \alpha_f | S | \alpha_i \rangle|^2 .$$

As in the previous section a comparison with the general expression for the transition probability allows us to identify the SU_3 weight factor G_{SU_3}. The following

weight factor is for a transition to a final state consisting of individual particles $|\phi^{\mu_1}_{i_1\nu_1}\rangle|\phi^{\mu_2}_{i_2\nu_2}\rangle\cdots|\phi^{\mu_n}_{i_n\nu_n}\rangle$ from an initial state $\phi^{U_i}_{I_i\nu_i}$.

$$G_{SU_3} = \frac{n!}{\prod_k n_k!} \; P^{U_i, I_i}_{\nu_1 \nu_2 \cdots \nu_n}$$

It only remains to generalize this expression for the case where the initial state is a linear combination of "pure" SU_3 states. We construct our initial state by coupling the states representing the proton and the antiproton using SU_3 Clebsch-Gordan coefficients. We recall that the proton and the antiproton, respectively, fit into the $(I=1/2, I_z=1/2, Y=1)$ and the $(I=1/2, I_z=-1/2, Y=-1)$ slots of 8-dimensional irreducible representations of SU_3*. They therefore couple to $I=0$ or 1 with $I_z=Y=0$. Our initial state is:

$$\psi_i = \sum_{k=1}^{6} a_k(I=0)\,\phi^{U_k}_{000} + b_k(I=1)\,\phi^{U_k}_{100}$$

where a_k and b_k are SU_3 Clebsch-Gordon coefficients satisfying the relation $\sum_{k=1}^{6} |a_k|^2 + |b_k|^2 = 1$. The index k runs over the 6 irreducible representations in the Kronecker product $8 \otimes 8 = 27 \oplus 10 \oplus 10^* \oplus 8_1 \oplus 8_2 \oplus 1$. Substituting the Clebsch-Gordan coefficients into the

---

*The 8-dimensional representation is equal to its complex conjugate, i.e. $8 = 8^*$

expression above yields:

$$\psi_i = \sum_{U_k=1}^{6} \left(\begin{matrix} 8 & 8 \\ 1/2\ 1/2\ 1 & 1/2\ -1/2\ -1 \end{matrix} \middle| \begin{matrix} U_k \\ 0\ 0\ 0 \end{matrix} \right) \phi^{U_k}_{000} + \left(\begin{matrix} 8 & 8 \\ 1/2\ 1/2\ 1 & 1/2\ -1/2\ -1 \end{matrix} \middle| \begin{matrix} U_k \\ 1\ 0\ 0 \end{matrix} \right) \phi^{U_k}_{100} .$$

For simplicity we may combine the two terms and write:

$$\psi_i = \sum_{U_k} \sum_{I_k} \left(\begin{matrix} 8 & 8 \\ 1/2\ 1/2\ 1 & 1/2\ -1/2\ -1 \end{matrix} \middle| \begin{matrix} U_k \\ I_k\ 0\ 0 \end{matrix} \right) \phi^{U_k}_{I_k 00}$$

$$= \sum_{U_k} \sum_{I_k} a^{U_k}_{I_k} \phi^{U_k}_{I_k 00} .$$

While U_k runs over the set $\{27, 10, 10^*, 8_1, 8_2, 1\}$, I_k takes on each of the values 0 and 1. Using this initial state, the weight factor becomes:

$$G_{SU_3} = \frac{n!}{\prod_k n_k!} \sum_{U_k I_k} \sum_{\substack{U_{2\gamma} \ldots U_{n-1\gamma} \\ I_2 \quad\ I_{n-1}}} \sum_{\gamma_k} |a^{U_k}_{I_k}|^2$$

$$\times \left\langle \begin{matrix} U_{2\gamma} & U_{3\gamma} \ldots & U_{n-1\gamma} \\ I_2 & I_3 \ldots & I_{n-1} \end{matrix} ; \begin{matrix} U_k \gamma_k \\ I_k \nu_k \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 & \ldots & \mu_n \\ i_1\nu_1 & i_2\nu_2 & \ldots & i_n\nu_n \end{matrix} \right\rangle^2$$

where the prime has been dropped. Also we must explicitly sum over both of the 8 dimensional representations in the set $\{U_k\}$. We express the final result in terms of the SU_3 P-factor:

$$G_{SU_3} = \frac{n!}{\prod_k n_k!} \sum_{\substack{U_k \\ I_k}} \sum_{\substack{U_{2\gamma}, \ldots U_{n-1\gamma} \\ I_2 \quad\ I_{n-1}}} |a^{U_k}_{I_k}|^2 \, P^{U_k I_k}_{\nu_1 \nu_2 \ldots \nu_n}$$

where we recall:

$$P^{U_k\, I_k}_{\nu_1 \nu_2 \cdots \nu_n} = \sum_{\substack{U_{2\gamma}\, U_{3\gamma} \cdots U_{n-1\,\gamma} \\ I_2,\ I_3 \cdots I_{n-1}}}{}' \left\langle \begin{matrix} U_{2\gamma} & U_{3\gamma} \cdots U_{n-1\,\gamma} \\ I_2, & I_3 \cdots I_{n-1} \end{matrix} ; \begin{matrix} U_k \\ I_k \end{matrix} \gamma \nu_k \middle| \begin{matrix} \mu_1 & \cdots \mu_n \\ i_1 \nu_1, & \cdots i_n \nu_n \end{matrix} \right\rangle^2 .$$

We also define the product of the P-factor and the permutation factor as:

$$P^{*\,U,I}_{\nu_1 \nu_2 \cdots \nu_n} \equiv \frac{n!}{\prod_k n_k!}\, P^{U,I}_{\nu_1 \nu_2 \cdots \nu_n} .$$

The SU_3 weight factors and terms $P^{*\,U,I}_{\nu_1 \nu_2 \cdots \nu_n}$ used in the calculation are listed in table 4 for given values of $(\nu_1 \nu_2 \nu_3 \cdots \nu_n)$. A check on these is obtained by noting that the sum $\sum_{\nu_1 \nu_2 \cdots \nu_n} P^{*\,U,I}_{\nu_1 \nu_2 \cdots \nu_n}$ is equal to the number of times that the irreducible representation $U_{(I)}$ appears in the Kronecker product $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$. The sums agree with these values to within 1%. The discrepancy is annoying. It may be due to a misprint in the table of SU_3 Clebsch-Gordan coefficients* or, less likely, to computer round-off error.

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* The table of SU_3 Clebsch-Gordan coefficients that we have used is:

Kuriyan, Lurie' and Macfarlane, Jour. of Math. Phys. 6 722 (1965).

Table 4. SU_3 weight factors for states of mesons from the 8-dimensional representation.

SU labels (i,i_z,y) of the individual particles	This state filled with U=0- mesons	P-factors for SU_3 $P^{*U,I}_{\nu_1,\nu_2,\ldots,\nu_n}$ I=0 U below 27	8	1	I=1 U below 27	10	10^*	8	Total SU_3 weight*
(1 0 0)(1 0 0)	$\pi^0\pi^0$	.0084	.2000	.1250	0	0	0	0	.0763
(1 0 0)(0 0 0)	$\pi^0\eta^0$	0	0	0	.6000	.5000	.5000	.4000	.2367
(0 0 0)(0 0 0)	$\eta^0\eta^0$	.6750	.2000	.1250	0	0	0	0	.1263
(1 1 0)(1-1 0)	$\pi^+\pi^-$	.0167	.4000	.2500	0	.1667	.1667	.6667	.3358
(½ ½ 1)(½-½-1)	K^+K^-	.1500	.6000	.2500	.2000	.1667	.1667	.4667	.3792
(½-½ 1)(½ ½-1)	$K^0\bar{K}^0$	.1500	.6000	.2500	.2000	.1667	.1667	.4667	.3792
Totals		1.00	2.00	1.00	1.00	1.00	1.00	2.00	
Number of times U appears in $8\otimes 8$		1	2	1	1	1	1	2	
(1 0 0)(1 0 0)(1 0 0)	$\pi^0\pi^0\pi^0$	0	0	0	.1095	.0916	.0916	.2967	.0954
(1 0 0)(1 0 0)(0 0 0)	$\pi^0\pi^0\eta^0$	.4107	.5000	.0750	0	0	0	0	.1902
(1 0 0)(0 0 0)(0 0 0)	$\pi^0\eta^0\eta^0$	0	0	0	.6429	.2500	.2500	.5000	.2226
(0 0 0)(0 0 0)(0 0 0)	$\eta^0\eta^0\eta^0$	.2893	.3000	.0250	0	0	0	0	.1148
(1 1 0)(1-1 0)(1 0 0)	$\pi^+\pi^-\pi^0$	.1500	.6000	.2500	.4429	.3642	.3642	1.188	.6046
(1 1 0)(1-1 0)(0 0 0)	$\pi^+\pi^-\eta^0$	.8214	1.000	.1500	.9000	.5000	.5000	.6000	.6937
(½ ½ 1)(½-½-1)(1 0 0)	$K^+K^-\pi^0$	.2821	.5333	.1750	.6238	.5216	.4475	1.091	.6008
(½ ½ 1)(½-½-1)(0 0 0)	$K^+K^-\eta^0$	1.318	1.200	.2250	.7286	.4167	.4167	.5333	.7537
(½-½ 1)(½ ½-1)(1 0 0)	$K^0\bar{K}^0\pi^0$	.2821	.5333	.1750	.6238	.5216	.4475	1.091	.6008
(½-½ 1)(½ ½-1)(0 0 0)	$K^0\bar{K}^0\eta^0$	1.318	1.200	.2250	.7286	.4167	.4167	.5333	.7537
(½-½-1)(½-½ 1)(1 1 0)	$K^-K^0\pi^+$	.5643	1.067	.3500	.6190	.4506	.5247	1.086	.8027
(½ ½ 1)(½ ½-1)(1-1 0)	$K^+\bar{K}^0\pi^-$	.5643	1.067	.3500	.6190	.4506	.5247	1.086	.8027
Totals		6.00	8.00	2.00	6.04	3.98	3.98	8.01	
Number of times U appears in $8\otimes 8\otimes 8$		6	8	2	6	4	4	8	

*Equal to the P-factors $P^{*U,I}_{\nu_1,\nu_2,\nu_3,\ldots,\nu_n}$ averaged over the initial state components.

Table 4. (Continued)

SU labels (i, i_z, y) of the individual particles	This state filled with J=0- mesons	P-factors for SU_3 $P^{n;U,I}_{\nu_1,\nu_2,\ldots,\nu_n}$ I=0 U below			I=1 U below				Total SU_3 weight
		27	8	1	27	10	10^*	8	
(1 0 0)(1 0 0)(1 0 0)(1 0 0)	$\pi^0\pi^0\pi^0\pi^0$	.0537	.1057	.0371	0	0	0	0	.0404
(1 0 0)(1 0 0)(1 0 0)(0 0 0)	$\pi^0\pi^0\pi^0\eta^0$	0	0	0	.5872	.3477	.3477	.4228	.2153
(1 0 0)(1 0 0)(0 0 0)(0 0 0)	$\pi^0\pi^0\eta^0\eta^0$	.8696	.5143	.1250	0	0	0	0	.2351
(1 0 0)(0 0 0)(0 0 0)(0 0 0)	$\pi^0\eta^0\eta^0\eta^0$	0	0	0	.6348	.3571	.3571	.3429	.2030
(0 0 0)(0 0 0)(0 0 0)(0 0 0)	$\eta^0\eta^0\eta^0\eta^0$	.3616	.1429	.0375	0	0	0	0	.0747
(1 1 0)(1-1 0)(1 0 0)(1 0 0)	$\pi^+\pi^-\pi^0\pi^0$	.4928	.8451	.2969	.7500	.5714	.5714	1.333	.8042
(1 1 0)(1-1 0)(1 0 0)(0 0 0)	$\pi^+\pi^-\pi^0\eta^0$	1.757	1.600	.3000	2.395	1.365	1.365	1.702	1.513
(1 1 0)(1-1 0)(0 0 0)(0 0 0)	$\pi^+\pi^-\eta^0\eta^0$	1.739	1.029	.2500	1.093	.5714	.5714	.8000	.8611
(1 1 0)(1 1 0)(1-1 0)(1-1 0)	$\pi^+\pi^+\pi^-\pi^-$	.3214	.6349	.2222	.4554	.3909	.3909	.8889	.5604
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ -1)$(1 0 0)(1 0 0)	$K^+K^-\pi^0\pi^0$	.6607	.9365	.2778	.8063	.5309	.5305	1.016	.7714
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ -1)$(1 0 0)(0 0 0)	$K^+K^-\pi^0\eta^0$	1.707	1.352	.2667	2.654	1.467	1.425	1.870	1.510
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ -1)$(0 0 0)(0 0 0)	$K^+K^-\eta^0\eta^0$	2.346	1.457	.3000	1.063	.5238	.5238	.6762	1.002
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ -1)$(1 1 0)(1-1 0)	$K^+K^-\pi^+\pi^-$	1.989	2.806	.8222	2.342	1.661	1.552	3.069	2.312
$(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$(1 0 0)(1 0 0)	$K^0\bar{K}^0\pi^0\pi^0$	.6607	.9365	.2778	.8063	.5309	.5305	1.016	.7714
$(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$(1 0 0)(0 0 0)	$K^0\bar{K}^0\pi^0\eta^0$	1.707	1.352	.2667	2.654	1.467	1.425	1.870	1.510
$(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$(0 0 0)(0 0 0)	$K^0\bar{K}^0\eta^0\eta^0$	2.346	1.457	.3000	1.063	.5238	.5238	.6762	1.002
$(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$(1 1 0)(1-1 0)	$K^0\bar{K}^0\pi^+\pi^-$	1.989	2.806	.8222	2.342	1.661	1.552	3.069	2.312
$(\frac{1}{2}\ -\frac{1}{2}\ -1)(\frac{1}{2}\ -\frac{1}{2}\ 1)$(1 1 0)(1 0 0)	$K^-K^0\pi^+\pi^0$	1.336	1.867	.5333	2.070	1.379	1.479	2.725	1.808
$(\frac{1}{2}\ -\frac{1}{2}\ -1)(\frac{1}{2}\ -\frac{1}{2}\ 1)$(1 1 0)(0 0 0)	$K^-K^0\pi^+\eta^0$	3.414	2.705	.5333	2.655	1.422	1.464	1.870	2.076
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$(1-1 0)(1 0 0)	$K^+\bar{K}^0\pi^-\pi^0$	1.336	1.867	.5333	2.070	1.379	1.479	2.725	1.808
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$(1-1 0)(0 0 0)	$K^+\bar{K}^0\pi^-\eta^0$	3.414	2.705	.5333	2.655	1.422	1.464	1.870	2.076
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ -1)(\frac{1}{2}\ -\frac{1}{2}\ -1)$	$K^+K^+K^-K^-$	.7143	.8254	.2222	.6577	.4321	.3679	.6843	.6211
$(\frac{1}{2}\ \frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ -1)(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)$	$K^+K^-K^0\bar{K}^0$	3.029	3.225	.8222	2.595	1.549	1.657	2.695	2.453
$(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ -\frac{1}{2}\ 1)(\frac{1}{2}\ \frac{1}{2}\ -1)(\frac{1}{2}\ \frac{1}{2}\ -1)$	$K^0K^0\bar{K}^0\bar{K}^0$	.7143	.8254	.2222	.6577	.4321	.3679	.6843	.6211
Totals		32.96	32.00	8.00	33.01	19.98	19.94	32.00	
No. of times U appears in 8●8●8●8		33	32	8	33	20	20	32	

Table 4. (Continued)

SU labels (i,i_z,y) of the individual particles	This state filled with J=0- mesons	P-factors for SU_3 $P^{*\,U,I}_{\nu_1,\nu_2,\ldots,\nu_8}$ I=0 U below 27	8	1	I=1 U below 27	10	10*	8	Total SU_3 weight
(100) (100) (100) (100) (100)	$\pi^\circ\pi^\circ\pi^\circ\pi^\circ\pi^\circ$	0	0	0	.0939	.0624	.0624	.1250	.0490
(100) (100) (100) (100) (000)	$\pi^\circ\pi^\circ\pi^\circ\pi^\circ\eta^\circ$	.4222	.3448	.0661	0	0	0	0	.1434
(100) (100) (100) (000) (000)	$\pi^\circ\pi^\circ\pi^\circ\eta^\circ\eta^\circ$	0	0	0	1.067	.5155	.5155	.6896	.3535
(100) (100) (000) (000) (000)	$\pi^\circ\pi^\circ\eta^\circ\eta^\circ\eta^\circ$	1.101	.6696	.1071	0	0	0	0	.2969
(100) (000) (000) (000) (000)	$\pi^\circ\eta^\circ\eta^\circ\eta^\circ\eta^\circ$	0	0	0	.6362	.2679	.2679	.3348	.1864
(000) (000) (000) (000) (000)	$\eta^\circ\eta^\circ\eta^\circ\eta^\circ\eta^\circ$	.2411	.1384	.0179	0	0	0	0	.0618
(110) (1-10) (100) (100) (100)	$\pi^+\pi^-\pi^\circ\pi^\circ\pi^\circ$	.6607	.9365	.2778	1.127	.7489	.7489	1.501	.9529
(110) (1-10) (100) (100) (000)	$\pi^+\pi^-\pi^\circ\pi^\circ\eta^\circ$	3.377	2.760	.5282	3.991	2.188	2.188	2.810	2.566
(110) (1-10) (100) (000) (000)	$\pi^+\pi^-\pi^\circ\eta^\circ\eta^\circ$	3.911	2.429	.5000	4.272	2.060	2.060	2.760	2.499
(110) (1-10) (000) (000) (000)	$\pi^+\pi^-\eta^\circ\eta^\circ\eta^\circ$	2.202	1.339	.2143	1.527	.7292	.7292	.8095	1.057
(110) (110) (1-10) (1-10) (100)	$\pi^+\pi^+\pi^-\pi^-\pi^\circ$	1.321	1.873	.5556	2.055	1.380	1.380	2.746	1.807
(110) (110) (1-10) (1-10) (000)	$\pi^+\pi^+\pi^-\pi^-\eta^\circ$	2.536	2.068	.3968	2.689	1.452	1.452	1.873	1.808
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ (100) (100) (100)	$K^+K^-\pi^\circ\pi^\circ\pi^\circ$	.7256	.8121	.2117	1.279	.8396	.7662	1.384	.9092
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (100) (100) (100)	$K^\circ\bar{K}^\circ\pi^\circ\pi^\circ\pi^\circ$	.7256	.8121	.2117	1.279	.8396	.7662	1.384	.9092
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ (100) (100) (000)	$K^+K^-\pi^\circ\pi^\circ\eta^\circ$	4.063	2.986	.5853	3.601	1.900	1.900	2.436	2.519
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (100) (100) (000)	$K^\circ\bar{K}^\circ\pi^\circ\pi^\circ\eta^\circ$	4.063	2.968	.5853	3.601	1.900	1.900	2.436	2.519
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ (100) (000) (000)	$K^+K^-\pi^\circ\eta^\circ\eta^\circ$	3.652	2.196	.4226	4.901	2.438	2.372	2.971	2.570
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (100) (000) (000)	$K^\circ\bar{K}^\circ\pi^\circ\eta^\circ\eta^\circ$	3.652	2.196	.4226	4.901	2.438	2.372	2.971	2.570
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ (000) (000) (000)	$K^+K^-\eta^\circ\eta^\circ\eta^\circ$	3.188	1.768	.3036	1.377	.6548	.6548	.7321	1.225
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (000) (000) (000)	$K^\circ\bar{K}^\circ\eta^\circ\eta^\circ\eta^\circ$	3.188	1.768	.3036	1.377	.6548	.6548	.7321	1.225
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ (110) (1-10) (100)	$K^+K^-\pi^+\pi^-\pi^\circ$	6.545	7.344	1.930	9.027	5.678	5.465	9.633	7.022
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (110) (1-10) (100)	$K^\circ\bar{K}^\circ\pi^+\pi^-\pi^\circ$	6.545	7.344	1.930	9.027	5.678	5.465	9.633	7.022
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ (110) (1-10) (000)	$K^+K^-\pi^+\pi^-\eta^\circ$	12.15	8.907	1.754	10.82	5.751	5.648	7.323	7.543
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (110) (1-10) (000)	$K^\circ\bar{K}^\circ\pi^+\pi^-\eta^\circ$	12.15	8.907	1.754	10.82	5.751	5.648	7.323	7.543
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ (110) (100) (100)	$K^-K^\circ\pi^+\pi^\circ\pi^\circ$	2.901	3.250	.8452	3.825	2.353	2.471	4.124	3.045
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (1-10) (100) (100)	$K^+\bar{K}^\circ\pi^-\pi^\circ\eta^\circ$	2.903	3.248	.8470	3.833	2.351	2.466	4.126	3.046
$(\frac{1}{2}\,\text{-}\frac{1}{2}\,\text{-}1)$ $(\frac{1}{2}\,\text{-}\frac{1}{2}\,1)$ (110) (100) (000)	$K^-K^\circ\pi^+\pi^\circ\eta^\circ$	8.045	5.869	1.167	9.596	5.018	5.090	6.503	5.829
$(\frac{1}{2}\,\frac{1}{2}\,1)$ $(\frac{1}{2}\,\frac{1}{2}\,\text{-}1)$ (1-10) (100) (000)	$K^+\bar{K}^\circ\pi^-\pi^\circ\eta^\circ$	8.045	5.869	1.167	9.596	5.018	5.090	6.503	5.829

Table 4. (Continued)

SU labels (i, i_z, y) of the individual particles	This state filled with J=0- mesons	P-factors for SU_3 $P^{*\,U,I}_{\nu_1,\nu_2,\ldots}$ I=0 U below 27	8	1	I=1 U below 27	10	10^*	8	Total SU_3 weight
$(\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (110)\ (000)\ (000)$	$K^-K^0\pi^+\eta^0\eta^0$	7.304	4.393	.8452	4.896	2.373	2.439	2.967	3.554
$(\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (1\bar{1}0)\ (000)\ (000)$	$K^+\bar{K}^0\pi^-\eta^0\eta^0$	7.304	4.393	.8452	4.896	2.373	2.439	2.976	3.554
$(\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (110)\ (110)\ (1\bar{1}0)$	$K^-K^0\pi^+\pi^+\pi^-$	4.353	4.873	1.270	5.100	3.124	3.298	5.506	4.277
$(\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (1\bar{1}0)\ (1\bar{1}0)\ (110)$	$K^+\bar{K}^0\pi^-\pi^-\pi^+$	4.353	4.873	1.270	5.100	3.124	3.298	5.506	4.277
$(\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (100)$	$K^+K^+K^-K^-\pi^0$	1.902	1.777	.4277	2.469	1.535	1.345	2.255	1.742
$(\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (100)$	$K^0K^0\bar{K}^0\bar{K}^0\pi^0$	1.902	1.777	.4277	2.469	1.535	1.345	2.255	1.742
$(1\ 1\ 1)\ (1\ 1\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (000)$	$K^+K^+K^-K^-\eta^0$	3.857	2.651	.5159	2.746	1.450	1.370	1.777	2.073
$(1\ -1\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (1\ 1\ -1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (000)$	$K^0K^0\bar{K}^0\bar{K}^0\eta^0$	3.857	2.651	.5159	2.746	1.450	1.370	1.777	2.073
$(1\ 1\ 1)\ (1\ -1\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (100)$	$K^+K^-K^0\bar{K}^0\pi^0$	7.493	7.013	1.684	9.816	5.721	5.720	8.930	6.895
$(1\ 1\ 1)\ (1\ -1\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (000)$	$K^+K^-K^0\bar{K}^0\eta^0$	15.43	10.37	2.016	10.92	5.534	5.692	7.054	8.193
$(\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (110)$	$K^-K^0K^0\bar{K}^0\pi^+$	5.581	5.242	1.261	5.484	3.088	3.316	5.019	4.402
$(\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (1\bar{1}0)$	$K^+K^+K^-\bar{K}^0\pi^-$	5.581	5.242	1.261	5.484	3.088	3.316	5.019	4.402
$(\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ -1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (110)$	$K^+K^-K^-K^0\pi^+$	5.696	5.338	1.288	5.580	3.323	3.132	5.040	4.461
$(\frac{1}{2}\ \frac{1}{2}\ 1)\ (\frac{1}{2}\ -\frac{1}{2}\ 1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (\frac{1}{2}\ \frac{1}{2}\ -1)\ (1\bar{1}0)$	$K^+K^0\bar{K}^0\bar{K}^0\pi^-$	5.738	5.314	1.270	5.581	3.278	3.169	5.010	4.448
Totals		178.7	144.7	32.0	179.6	99.66	99.39	145.0	
No. times U appears in 8 8 8 8 8		180	145	32	180	100	100	145	

Table 4. (Continued)

SU labels (i,i_z,y) of the individual particles	This state filled with J=0- mesons	P-factors for SU_3 $P^{*\,U,I}_{\nu_1,\nu_2,\ldots,\nu_k}$ I=0 U below 27	8	1	I=1 U below 27	10	10*	8	Total SU_3 weight
(100)(100)(100)(100)(100)(100)	$\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0$	.0519	.0583	.0156	0	0	0	0	.0233
(100)(100)(100)(100)(100)(000)	$\pi^0\pi^0\pi^0\pi^0\pi^0\eta^0$	0	0	0	.5173	.2751	.2751	.3500	.1792
(100)(100)(100)(100)(000)(000)	$\pi^0\pi^0\pi^0\pi^0\eta^0\eta^0$	1.012	.6585	.1293	0	0	0	0	.2963
(100)(100)(100)(000)(000)(000)	$\pi^0\pi^0\pi^0\eta^0\eta^0\eta^0$	0	0	0	1.655	.7977	.7977	.8779	.5033
(100)(100)(000)(000)(000)(000)	$\pi^0\pi^0\eta^0\eta^0\eta^0\eta^0$	1.423	.6920	.1256	0	0	0	0	.3300
(100)(000)(000)(000)(000)(000)	$\pi^0\eta^0\eta^0\eta^0\eta^0\eta^0$	0	0	0	.5824	.2790	.2790	.2768	.1693
(000)(000)(000)(000)(000)(000)	$\eta^0\eta^0\eta^0\eta^0\eta^0\eta^0$	.2131	.0938	.0173	0	0	0	0	.0463
(110)(1-10)(100)(100)(100)(100)	$\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0$	.9341	1.050	.2813	1.385	.8590	.8590	1.500	1.052
(110)(1-10)(100)(100)(100)(000)	$\pi^+\pi^-\pi^0\pi^0\pi^0\eta^0$	4.875	3.583	.7024	6.206	3.302	3.302	4.199	3.679
(110)(1-10)(100)(100)(000)(000)	$\pi^+\pi^-\pi^0\pi^0\eta^0\eta^0$	8.816	5.267	1.035	8.840	4.326	4.326	5.375	5.230
(110)(1-10)(100)(000)(000)(000)	$\pi^+\pi^-\pi^0\eta^0\eta^0\eta^0$	6.375	3.536	.6071	6.615	3.193	3.193	3.511	3.628
(110)(1-10)(000)(000)(000)(000)	$\pi^+\pi^-\eta^0\eta^0\eta^0\eta^0$	2.847	1.384	.2511	1.728	.7924	.7924	.8839	1.171
(110)(110)(1-10)(1-10)(100)(100)	$\pi^+\pi^+\pi^-\pi^-\pi^0\pi^0$	3.430	3.857	1.025	4.892	3.120	3.120	5.393	3.810
(110)(110)(1-10)(1-10)(100)(000)	$\pi^+\pi^+\pi^-\pi^-\pi^0\eta^0$	9.752	7.167	1.405	11.38	6.045	6.045	7.703	7.000
(110)(110)(1-10)(1-10)(000)(000)	$\pi^+\pi^+\pi^-\pi^-\eta^0\eta^0$	6.608	3.952	.7753	5.851	2.893	2.893	3.583	3.682
(110)(110)(110)(1-10)(1-10)(1-10)	$\pi^+\pi^+\pi^+\pi^-\pi^-\pi^-$	1.038	1.167	.3125	1.352	.8673	.8673	1.500	1.097
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{-1}{2}-1)$ (100)(100)(100)(100)	$K^+K^-\pi^0\pi^0\pi^0\pi^0$	1.128	1.073	.2625	1.332	.7708	.7653	1.208	.9825
$(\frac{1}{2}\frac{-1}{2}1)(\frac{1}{2}\frac{1}{2}-1)$ (100)(100)(100)(100)	$K^0\bar{K}^0\pi^0\pi^0\pi^0\pi^0$	1.128	1.073	.2625	1.332	.7708	.7653	1.208	.9825
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{-1}{2}-1)$ (100)(100)(100)(000)	$K^+K^-\pi^0\pi^0\pi^0\eta^0$	4.572	3.116	.6091	6.578	3.441	3.334	4.270	3.572
$(\frac{1}{2}\frac{-1}{2}1)(\frac{1}{2}\frac{1}{2}-1)$ (100)(100)(100)(000)	$K^0\bar{K}^0\pi^0\pi^0\pi^0\eta^0$	4.572	3.116	.6091	6.578	3.441	3.334	4.270	3.572
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{-1}{2}-1)$ (100)(100)(000)(000)	$K^+K^-\pi^0\pi^0\eta^0\eta^0$	10.30	5.985	1.120	8.165	3.939	3.939	4.702	5.278
$(\frac{1}{2}\frac{-1}{2}1)(\frac{1}{2}\frac{1}{2}-1)$ (100)(100)(000)(000)	$K^0\bar{K}^0\pi^0\pi^0\eta^0\eta^0$	10.30	5.985	1.120	8.165	3.939	3.939	4.702	5.278
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{-1}{2}-1)$ (100)(000)(000)(000)	$K^+K^-\pi^0\eta^0\eta^0\eta^0$	5.998	3.185	.5491	7.656	3.612	3.559	3.984	3.767
$(\frac{1}{2}\frac{-1}{2}1)(\frac{1}{2}\frac{1}{2}-1)$ (100)(000)(000)(000)	$K^0\bar{K}^0\pi^0\eta^0\eta^0\eta^0$	5.998	3.185	.5491	7.656	3.612	3.559	3.984	3.767
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{-1}{2}-1)$ (000)(000)(000)(000)	$K^+K^-\eta^0\eta^0\eta^0\eta^0$	3.925	1.942	.3315	1.607	.7329	.7329	.7961	1.387
$(\frac{1}{2}\frac{-1}{2}1)(\frac{1}{2}\frac{1}{2}-1)$ (000)(000)(000)(000)	$K^0\bar{K}^0\eta^0\eta^0\eta^0\eta^0$	3.925	1.942	.3315	1.607	.7329	.7329	.7961	1.387
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{-1}{2}-1)$ (110)(1-10)(100)(100)	$K^+K^-\pi^+\pi^-\pi^0\pi^0$	15.72	14.98	3.636	20.60	12.29	11.75	18.93	14.61
$(\frac{1}{2}\frac{-1}{2}1)(\frac{1}{2}\frac{1}{2}-1)$ (110)(1-10)(100)(100)	$K^0\bar{K}^0\pi^+\pi^-\pi^0\pi^0$	15.72	14.98	3.636	20.60	12.29	11.75	18.93	14.61

Table 4. (Continued)

SU labels (i,i_z,y) of the individual particles	This state filled with J=0- mesons	P-factors for SU_3 $P^{\alpha}_{\mu_1\mu_2\ldots\mu_n}$ I=0 U below 27	8	1	I=1 U below 27	10	10*	8	Total SU_3 weight
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)(100)(000)$	$K^+K^-\pi^+\pi^-\pi^0\eta^0$	41.43	28.25	5.508	46.06	23.91	23.48	29.85	27.79
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)(100)(000)$	$K^0\bar{K}^0\pi^+\pi^-\pi^0\eta^0$	41.43	28.25	5.508	46.06	23.91	23.48	29.85	27.79
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)(000)(000)$	$K^+K^-\pi^+\pi^-\eta^0\eta^0$	30.76	17.89	3.340	24.42	11.91	11.74	14.12	15.80
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)(000)(000)$	$K^0\bar{K}^0\pi^+\pi^-\eta^0\eta^0$	30.76	17.89	3.340	24.42	11.91	11.74	14.12	15.88
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(110)(1\text{-}10)(1\text{-}10)$	$K^+K^-\pi^+\pi^+\pi^-\pi^-$	11.23	10.70	2.598	13.15	7.900	7.527	12.14	9.811
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(110)(1\text{-}10)(1\text{-}10)$	$K^0\bar{K}^0\pi^+\pi^+\pi^-\pi^-$	11.23	10.70	2.598	13.15	7.900	7.527	12.14	9.811
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(100)(100)(100)$	$K^-K^0\pi^+\pi^0\pi^0\pi^0$	4.468	4.262	1.024	6.292	3.646	3.755	5.820	4.346
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(1\text{-}10)(100)(100)(100)$	$K^+\bar{K}^0\pi^-\pi^0\pi^0\pi^0$	4.468	4.262	1.024	6.292	3.646	3.755	5.820	4.346
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(100)(100)(000)$	$K^-K^0\pi^+\pi^0\pi^0\eta^0$	18.28	12.45	2.438	19.68	10.06	10.20	12.78	12.05
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(1\text{-}10)(100)(100)(000)$	$K^+\bar{K}^0\pi^-\pi^0\pi^0\eta^0$	18.28	12.45	2.438	19.68	10.06	10.20	12.78	12.05
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(100)(000)(000)$	$K^-\bar{K}^0\pi^+\pi^0\eta^0\eta^0$	20.32	11.85	2.201	21.65	10.42	10.56	12.53	12.19
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(1\text{-}10)(100)(000)(000)$	$K^+\bar{K}^0\pi^-\pi^0\eta^0\eta^0$	20.32	11.85	2.201	21.65	10.42	10.56	12.53	12.19
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(000)(000)(000)$	$K^-K^0\pi^+\eta^0\eta^0\eta^0$	12.00	6.369	1.098	7.657	3.556	3.609	3.982	5.240
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(1\text{-}10)(000)(000)(000)$	$K^+\bar{K}^0\pi^-\eta^0\eta^0\eta^0$	12.00	6.369	1.098	7.657	3.556	3.609	3.982	5.240
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(110)(1\text{-}10)(100)$	$K^-K^0\pi^+\pi^+\pi^-\pi^0$	17.87	17.05	4.095	23.04	13.35	13.78	21.35	16.51
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)(1\text{-}10)(100)$	$K^+\bar{K}^0\pi^+\pi^-\pi^-\pi^0$	17.87	17.05	4.095	23.04	13.35	13.78	21.35	16.51
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(110)(1\text{-}10)(000)$	$K^-K^0\pi^+\pi^+\pi^-\eta^0$	27.43	18.69	3.655	26.23	13.40	13.61	17.03	16.97
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)(1\text{-}10)(000)$	$K^+\bar{K}^0\pi^+\pi^-\pi^-\eta^0$	27.43	18.69	3.655	26.23	13.40	13.61	17.03	16.97
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(100)(100)$	$K^+K^+K^-K^-\pi^0\pi^0$	4.484	3.786	.8677	5.247	3.036	2.750	4.240	3.578
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(100)(100)$	$K^0K^0\bar{K}^0\bar{K}^0\pi^0\pi^0$	4.484	3.785	.8677	5.247	3.036	2.750	4.240	3.578
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(100)(000)$	$K^+K^+K^-K^-\pi^0\eta^0$	10.93	6.982	1.333	12.08	6.225	5.924	7.479	7.046
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(100)(000)$	$K^0K^0\bar{K}^0\bar{K}^0\pi^0\eta^0$	10.93	6.982	1.333	12.08	6.225	5.924	7.479	7.046
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(000)(000)$	$K^+K^+K^-K^-\eta^0\eta^0$	9.671	5.452	.9940	6.294	3.058	2.938	3.490	4.429
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(000)(000)$	$K^0K^0\bar{K}^0\bar{K}^0\eta^0\eta^0$	9.671	5.452	.9948	6.294	3.058	2.938	3.490	4.429
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(100)(100)$	$K^+K^-K^0\bar{K}^0\pi^0\pi^0$	17.84	14.69	3.318	20.74	11.42	11.53	16.80	14.07
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(100)(000)$	$K^+K^-K^0\bar{K}^0\pi^0\eta^0$	43.00	27.51	5.260	47.94	24.08	24.15	29.68	27.87
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(000)(000)$	$K^+K^-K^0\bar{K}^0\eta^0\eta^0$	38.60	21.52	3.887	25.08	11.84	12.05	13.88	17.58
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)$	$K^+K^+K^-K^-\pi^+\pi^-$	13.44	11.31	2.584	13.93	8.150	7.279	11.33	10.05
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)$	$K^0K^0\bar{K}^0\bar{K}^0\pi^+\pi^-$	13.44	11.31	2.584	13.93	8.150	7.279	11.33	10.05
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(\frac{1}{2}\frac{1}{2}{-1})(110)(1\text{-}10)$	$K^+K^-K^0\bar{K}^0\pi^+\pi^-$	53.09	44.02	9.977	55.01	30.43	30.59	44.81	39.48

Table 4. (Continued)

SU labels (i,i_z,y) of the individual particles	This state filled with J=0- mesons	P-factors for SU_3 I=0 U below 27	8	1	$P^{U,I}_{\nu_1,\nu_2,\ldots,\nu_n}$ I=1 U below 27	10	10*	8	Total SU_3 weight
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(110)(100)$	$K^-K^0K^0\bar{K}^0\pi^+\pi^0$	20.50	16.79	3.765	24.46	13.34	13.77	19.64	16.33
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}\frac{1}{2}-1)(1\text{-}10)(100)$	$K^+K^+K^-\bar{K}^0\bar{\pi}^-\pi^0$	20.50	16.79	3.765	24.46	13.34	13.77	19.64	16.33
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(110)(000)$	$K^-K^0K^0\bar{K}^0\pi^+\eta^0$	32.07	20.55	3.931	26.82	13.30	13.73	16.63	17.88
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}\frac{1}{2}-1)(1\text{-}10)(000)$	$K^+K^+K^-\bar{K}^0\bar{\pi}^-\eta^0$	32.07	20.55	3.931	26.82	13.30	13.73	16.63	17.88
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(110)(100)$	$K^+K^-K^-K^0\pi^+\pi^0$	20.47	16.79	3.765	24.40	13.78	13.18	19.64	16.31
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(\frac{1}{2}\frac{1}{2}-1)(1\text{-}10)(100)$	$K^+K^0\bar{K}^0\bar{K}^0\bar{\pi}^-\pi^0$	20.47	16.79	3.765	24.40	13.78	13.18	19.64	16.31
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(110)(000)$	$K^+K^-K^-K^0\pi^+\eta^0$	32.78	20.97	4.004	27.13	13.77	13.50	16.77	18.15
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(\frac{1}{2}\frac{1}{2}-1)(1\text{-}10)(000)$	$K^+K^0\bar{K}^0\bar{K}^0\bar{\pi}^-\eta^0$	32.78	20.97	4.004	27.13	13.77	13.50	16.77	18.15
$(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(110)(110)$	$K^-K^-K^0K^0\pi^+\pi^+$	6.838	5.571	1.244	6.959	3.731	3.956	5.608	4.985
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(\frac{1}{2}\frac{1}{2}-1)(1\text{-}10)(1\text{-}10)$	$K^+K^+\bar{K}^0\bar{K}^0\bar{\pi}^-\bar{\pi}^-$	6.838	5.571	1.244	6.959	3.731	3.956	5.608	4.985
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}-1)$	$K^+K^+K^+K^-K^-K^-$	1.795	1.417	.3125	1.635	.9843	.7920	1.221	1.192
$(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}-1)$	$K^0K^0K^0\bar{K}^0\bar{K}^0\bar{K}^0$	1.795	1.417	.3125	1.635	.9843	.7920	1.221	1.192
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(\frac{1}{2}\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}-1)$	$K^+K^+K^-K^-K^0\bar{K}^0$	16.42	12.48	2.693	14.63	8.081	7.468	10.83	10.60
$(\frac{1}{2}\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}-\frac{1}{2}1)(\frac{1}{2}\frac{1}{2}-1)(\frac{1}{2}-\frac{1}{2}-1)$	$K^+K^-K^0K^0\bar{K}^0\bar{K}^0$	16.42	12.48	2.693	14.63	8.081	7.468	10.83	10.60
Totals		989.7	700.5	145.0	995.2	525.2	520.4	701.1	
No. times U appears in 8⊗8⊗8⊗8⊗8⊗8		999	702	145	999	525	525	702	

IV. Calculations.

The relative production rates for all allowed states consisting of mesons (from those listed in tables 1, 2 and 3) produced in p-$\bar{p}$ annihilation at a total center-of-mass energy E, were computed from:

$$S(m_1, m_2 \cdots m_n, E) = \left\{ \prod_{k=1}^{k_{max}} \Omega^k (2S_k+1)^{n_k} \right\} \frac{G_{SU_n} \rho_n(m_1 \cdots m_n, E)}{(2\pi)^{3n-3} \prod_k n_k!}.$$

The total rate (the sum of the individual relative rates) was normalized to unity.

The weight factors G_{SU_n} were computed according to the technique set up in chapter III. The SU_3 weights are listed in table 5. The SU_2 weights have not been listed; a table listing some of these is included in the paper by Cerulus.[31] Because of the complexity of the numerical calculation, the SU_3 weights were obtained only for states containing up to a maximum of six particles; the SU_2 weights, which are more tractable, were obtained for states of up to ten particles. It was therefore not possible to carry out calculations, under the assumption of SU_3 invariance, at total energies exceeding approximately 2.5 Bev (in the center-of-mass). At center-of-mass energies above 2.5 Bev, the rate of production for final states of seven or more particles begins to become

appreciable. Reference to table 6 shows that the phase space integrals for states consisting of all pions are indeed negligible for $n \geq 7$ at 2.5 Bev but are not negligible at 3 Bev. From table 6 one also sees that it is reasonable to assume that rates of states consisting of more than 10 particles are negligible at 3 Bev. We have therefore limited ourselves to total center-of-mass energies less than or equal to 2.5 Bev involving final states of up to 6 particles (with resonances included) assuming SU_3 invariance and energies less than or equal to 3 Bev for final states of up to 10 particles assuming SU_2 invariance. The calculations presented in figures 1 to 16 have been carried out at total center-of-mass energies of 1.88, 2.29 and 2.50 Bev for SU_3 invariance and at energies of 1.88, 2.29, 2.50 and 3.00 Bev for SU_2 invariance. The latter correspond to laboratory antiproton kinetic energies of 0, 0.92, 1.45 and 2.92 Bev respectively.

The phase space factors ρ_n were calculated with a relative error of less than or equal to 6% using a Monte Carlo technique outlined in Appendix B, (the method of estimating the relative error is also described there). Resonant particles had to be specially treated as they are not observed directly but decay rapidly into pions and kaons. These were treated by first calculating the production rate for the state containing the resonant particles and then "allowing it to decay". The rate for

Table 5. Important decay modes and branching ratios used in the calculation.

Particle	Mass (Mev)	Important decay modes Partial modes*	Fraction (%)
η	549	$(\gamma\gamma)$	38.7
		$3\pi^0$ or $\pi(2\gamma)$	30.8
		$\pi^+\pi^-\pi^0$	25.0
		$\pi^+\pi^-(\gamma)$	5.5
η'	959	$\eta\, 2\pi$	76.0
		$\pi^+\pi^-(\gamma)$	24.0
ρ	765	2π	100
K^*	891	$K\pi$	100
ϕ	1020	$K^0\bar{K}^0$	38.0
		K^+K^-	30.0
		$\pi\rho$ or 3π	32.0
ω	783	$\pi^+\pi^-\pi^0$	100
A2	1324	$\pi\rho$	100
K^*	1405	$K\pi$	50.0
		$K^*\pi$	50.0
f	1253	2π	100
f'	1500	K^+K^-	30.0
		$K^0\bar{K}^0$	30.0
		$K\,K^*(891)$	40.0

*Branching ratios into charge states, if not explicitly shown, were obtained from isotopic spin invariance.

Table 6. Values of phase space integrals for states of 2 to 10 pions.

The integrals have been calculated from:

$$\rho = \frac{1}{(2\pi)^{3N-3}} \int \cdots \int d\vec{p}_1\, d\vec{p}_2 \cdots d\vec{p}_N\, \delta\left(\sum_{i=1}^{N} \vec{p}_i\right) \delta\left(E - \sum_{i=1}^{N} (p_i^2 + m_i^2)^{1/2}\right)$$

Those obtained from the Monte Carlo routine have errors attached.

Total energy in CM. (Bev)	Kinetic energy of $\vec{p}$ in lab (Bev)	N	Phase space integral ρ (in pion mass units with $\hbar = c = 1$)
1.88 (annihilation at rest)	0	2	1.16
		3	$1.98 \pm .08$
		4	$.681 \pm .027$
		5	$.0721 \pm .0029$
		6	$.00266 \pm .00016$
		7	$(3.36 \pm .20) \times 10^{-5}$
		8	$(1.87 \pm .11) \times 10^{-8}$
		9	$(2.54 \pm .15) \times 10^{-10}$
		10	$(7.94 \pm .48) \times 10^{-14}$
2.50	1.45	2	2.06
		3	$8.85 \pm .35$
		4	$7.84 \pm .31$
		5	$2.43 \pm .15$
		6	$.314 \pm .019$
		7	$.0172 \pm .0010$
		8	$(5.77 \pm .35) \times 10^{-4}$
		9	$(6.97 \pm .42) \times 10^{-6}$
		10	$(4.42 \pm .27) \times 10^{-8}$
3.00	2.92	2	2.98
		3	$22.7 \pm .9$
		4	34.9 ± 1.4
		5	19.0 ± 0.8
		6	$5.16 \pm .31$
		7	$.665 \pm .040$
		8	$.050 \pm .003$
		9	$.00148 \pm .00009$
		10	$(3.00 \pm .18) \times 10^{-5}$

this was then divided among the various final states according to the individual branching ratios. The branching ratios for the necessary decay modes are listed in table 5.

Two "models" were used in the calculations, each corresponding to a particular way of specifying the volume parameters. It was found that if the volume parameters are chosen to be the same for all particles and equal to the pion volume, the model predicts that the percentage of annihilations at rest in which a K-pair is produced (kaons from annihilation are always produced in pairs due to conservation of strangeness) is approximately 40%. Experiment shows it to be of the order of 4%. Furthermore, the theoretical value could not be substantially improved by varying the volume parameter (see table 7). Another obvious approach, consistent with SU_3 invariance, is to specify that all mesons in a given nonet have the same volume parameter. This leads to even more abundant kaon production (as shown in table 8). It was therefore deemed necessary to introduce something into the volume parameters which would suppress kaon production. One way to do this is to allow the volume parameter for a particular particle to vary as the reciprocal of the mass of the particle (in pion units) to some power. Hence, in this model, which was originally introduced by Kalbfleisch,[32] the

Table 7. Results for the statistical model with a single volume parameter.

Total CM energy (Bev)	Quantity	Exp. value*	$\Omega = \Omega_\pi$ SU_2	$\Omega = \Omega_\pi$ SU_3	$\Omega = .1\,\Omega_\pi$ SU_2	$\Omega = .1\,\Omega_\pi$ SU_3	$\Omega = .01\,\Omega_\pi$ SU_2	$\Omega = .01\,\Omega_\pi$ SU_3	$\Omega = .001\,\Omega_\pi$ SU_2	$\Omega = .001\,\Omega_\pi$ SU_3
1.88	Average no. of pions produced with a kaons ($\bar{n}_\pi$)	4.88±.24[a]	4.27	4.40	3.94	4.34	3.87	4.33	3.87	4.33
1.88	Ave. no. of pions produced with 2 kaons ($\bar{n}_{\pi\ 2K}$)	--	1.52	1.68	1.39	1.66	1.38	1.67	1.38	1.67
1.88	% of annih. yielding K-pair (%K)	4.0±1.0[a]	32.6	42.5	43.5	34.8	45.6	33.3	45.9	33.2
2.29	$\bar{n}_\pi$	5.2±.2[b]	5.01	5.09	4.55	4.83	4.41	4.79	4.40	4.79
2.29	$\bar{n}_{\pi\ 2K}$	2.4±.1[b]	2.25	2.36	1.77	2.01	1.63	1.93	1.61	1.92
2.29	%K	10.3±1.1[b]	36.4	57.2	42.3	40.5	44.1	35.5	44.3	34.9
2.50	$\bar{n}_\pi$	5.4±.35[b]	5.40	5.45	4.82	5.03	4.56	4.90	4.51	4.89
2.50	$\bar{n}_{\pi\ 2K}$	2.6±.2[b]	2.66	2.74	2.17	2.28	1.98	2.13	1.95	2.11
2.50	%K	13±3[b]	40.3	64.1	49.0	48.6	53.0	42.6	53.6	41.8

*The experimental values listed at 2.50 Bev are actually for a total center-of-mass energy equal to 2.43 Bev.

Experimental values from: [a] Agnew et al, Phys. Rev. 118, 1371 (1960)

[b] G. R. Lynch, Rev. of Mod. Phys. 33, 395 (1961)

Table 8. Results for the statistical model with a separate volume parameter for each meson nonet.

The volume parameters are proportional to $1/\bar{m}^3$, where $\bar{m}$ is approximately equal to the average mass of the particles in the nonet. All calculations have been done under the assumption of SU_3 invariance.

Total CM energy (Bev)	Quantity	Experimental* value	$\Omega_{0-}=1$, $\Omega_{1-}=.125$, $\Omega_{2+}=.037$	$\Omega_{0-}=.1$, $\Omega_{1-}=.0125$, $\Omega_{2+}=.0037$	$\Omega_{0-}=.01$, $\Omega_{1-}=.00125$, $\Omega_{2+}=.00037$	$\Omega_{0-}=.001$, $\Omega_{1-}=.000125$, $\Omega_{2+}=.000037$
1.88	Ave. number of pions produced without kaons ($\bar{n}_\pi$)	$4.88 \pm .24$[a]	4.02	3.93	3.94	3.94
1.88	Ave. number of pions produced with 2 kaons ($\bar{n}_{\pi\,2K}$)	--	1.40	.848	.600	.561
1.88	Percent. of annihilations producing kaons ($\%K$)	4.0 ± 1.0[a]	59.8	37.6	29.2	28.1
2.29	$\bar{n}_\pi$	$5.2 \pm .2$[b]	4.58	4.16	4.12	4.12
2.29	$\bar{n}_{\pi\,2K}$	$2.4 \pm .1$[b]	2.07	1.40	1.10	1.04
2.29	$\%K$	10.3 ± 1.1[b]	76.2	52.3	34.9	32.0
2.50	$\bar{n}_\pi$	$5.4 \pm .35$[b]	4.91	4.32	4.20	4.19
2.50	$\bar{n}_{\pi\,2K}$	$2.6 \pm .2$[b]	2.42	1.63	1.25	1.15
2.50	$\%K$	13 ± 3[b]	80.3	58.5	37.3	32.9

*The experimental values listed at 2.50 Bev are actually for a total center-of-mass energy equal to 2.43 Bev.

Experimental values from: [a]Agnew et al, Phys. Rev. 118, 1371 (1960)

[b]G. R. Lynch, Rev. of Mod. Phys. 33, 395 (1961)

volume parameter is given by:

$$\Omega_i = \Omega_o \left(\frac{1}{m_i}\right)^B$$

where Ω_o is measured in units of the pion volume. The other model which we have used is one in which all the volume parameters are the same except the one which is associated with kaons and kaon resonances. The production of kaons may then be reduced by decreasing the kaon volume parameter.

The results of the calculations are shown in tables 9 and 10 and in figures 1 to 16. In all cases, the parameters were chosen to give the "best fit" to the experimental data.*

The results at rest and at a total center-of-mass energy of 2.5 Bev are shown in tables 9 and 10. The agreement appears to be quite good, with the values of the volume parameters under the assumptions of SU_3 invariance and of SU_2 invariance being roughly the same. A minor difference is that the value of the parameter B in the Kalbfleisch model is about 1 for invariance under SU_2 and

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*In doing this, use was made of a search program written by Professor Peter Signell which varies the parameters to minimize the value of χ^2 for the experimental data to be fit.

about 2 for invariance under SU_3, resulting in smaller volume parameters for all particles (except the pions) under the assumption of an SU_3 invariant matrix element. The only general conclusion we draw from these results is that the volume parameters decrease with energy and that there appears to be no marked difference between the two assumptions (SU_2 or SU_3 invariance) as far as the present overall experimental data are concerned.

Also shown in tables 9 and 10 is the fraction of all annihilations resulting in the production of states containing certain resonances. These are, in most cases, consistent with experimental values.

Figures 1 to 14 show various quantities plotted as a function of energy. Since the results shown in tables 9 and 10 seem to indicate that the volume parameters decrease with energy, it was decided to modify the models above by including a multiplicative factor of $(2M_p/E_{c.M.})$ in each volume parameter. This is the Lorentz contraction term introduced in the original theory by Fermi. It was found that this modification slightly improved the fit.

To sum up, one can vary the volume parameters to get very good agreement with experiment at a single energy and fairly good agreement at a number of energies. In general one has equal success assuming invariance under

SU_2 or SU_3. However, the values of the volume parameters necessary to fit the data are, in all cases, at least an order of magnitude greater than the "physical" value of approximately .001 Ω_π consistent with the size of the range of the annihilation potential.

Table 9. Results for the best fit to the experimental data for annihilation at rest.

Symmetry group Model Values of parameters		SU_3 Kalbfleisch $\Omega_8 = 6.46$ $B = 1.69$	SU_2 Kalbfleisch $\Omega_0 = 5.56$ $B = .96$	SU_3 Small K $\Omega = 7.63$ $\Omega_K = 1.24$	SU_2 parameter $\Omega = 5.67$ $\Omega_K = 2.23$
Quantity	Exp. value	Theoretical values			
Average pion multiplicity	$4.88 \pm .24$[a]	4.67	4.70	4.93	4.83
Percent of annih. producing K-pair	4.0 ± 1.0[a]	4.6	4.6	4.6	4.9
Fraction of annih. producing prongs (no kaons):					
0 prongs	$.032 \pm .005$[b]	.029	.024	.025	.019
2 prongs	$.426 \pm .011$[b]	.432	.437	.439	.441
4 prongs	$.458 \pm .010$[b]	.466	.467	.462	.468
6 prongs	$.038 \pm .020$[b]	.028	.028	.040	.030
Production of resonances:					
Fraction resulting in ρ production	$\gtrsim .250$[b]	.083	.238	.422	.505
Fraction resulting in ω production	$\gtrsim .045$[b]	.015	.037	.098	.096
Fraction resulting in η production	$\gtrsim .014$[b]	.086	.070	.300	.148

Experimental values from:

[a] Agnew, Phys. Rev. 118, 1371 (1960)

[b] C. Baltay, Phys. Rev. 145, 1103 (1966)

Table 10. Results for the best fit to the experimental data at a total center-of-mass energy of 2.5 Bev.

Symmetry group Model Values of parameters		SU_3 Kalbfleisch $\Omega_0 = 1.59$ $B = 1.90$	SU_2 Kalbfleisch $\Omega_0 = 1.78$ $B = 1.01$	SU_3 Small K parameter $\Omega = .509$ $\Omega_K = .105$	SU_2 Small K parameter $\Omega = .479$ $\Omega_K = .148$
Quantity	Exp. value*	Theoretical values			
Ave. pion multplcty without kaons	$5.4 \pm .35$[a]	4.65	5.10	5.33	5.19
Ave. pion multplcty with K-pair	$2.6 \pm .20$[a]	2.59	2.59	2.73	2.63
Percent of annih. producing K-pair	13 ± 3[a]	13.8	13.7	19.3	14.5
Fraction of annih. producing pions with K-pair:					
0 pions	$.01 \pm .01$[b]	.0016	.0061	.0153	.0160
1 pion	$.05 \pm .03$	.068	.081	.075	.093
2 pions	$.41 \pm .17$	.387	.365	.262	.313
3 pions	$.45 \pm .17$	.429	.421	.491	.426
4 pions	$.09 \pm .09$	.107	.116	.128	.131
5 pions	$.01 \pm .01$	.006	.010	.023	.016
Fraction of annih. producing (pion) prongs with K-pair:					
0 prongs	$.06 \pm .02$[b]	.059	.076	.060	.061
2 prongs	$.62 \pm .23$	.533	.516	.388	.429
4 prongs	$.32 \pm .21$	.395	.395	.536	.499
6 prongs	$.005 \pm .005$	.001	.013	.015	.012
Production of resonances:					
Fraction of annih. resulting in ρ production		.182	.460	.619	.739
Fraction of annih. resulting in ω production		.032	.081	.333	.225
Fraction of annih. resulting in η production		.110	.105	.281	.200

*All experimental values are for a total CM energy of 2.43 Bev. The calculation was done at 2.50 Bev.

Experimental values from:

[a] G. R. Lynch, Rev. of Mod. Phys. 33, 395 (1961)

[b] G. R. Kalbfleisch, UCRL-9597 (1961)

Reference to figures 1 to 16.

The experimental values appearing on the graphs are from the following sources:

All experimental values for pion multiplicities and charged prongs accompanied by a kaon pair are from:

G. R. Kalbfleisch, UCRL-9597 (1961)

Other values used, listed by energy (kinetic energy of the antiproton in the laboratory), are:

0 Bev	C. Baltay, P. Franzini, G. Lutjens, J.C. Severiens, D. Tycko, and D. Zanello, Phys. Rev. 145, 1103 (1966)
0.47 Bev	S. Goldhaber, G. Goldhaber, W. Powell, and R. Silberberg, Phys. Rev. 121, 1525 (1961)
0.92 Bev 1.26 Bev	G. R. Lynch, Rev. of Mod. Phys. 33, 395 (1961)
2.86 Bev	T. Ferbel, A. Firestone, J. Sandweiss, H. Taft, M. Gailloud, T. Morris, W. Willis, A. Bachman, P. Baumel, and R. Lea, Phys. Rev. 143, 1096 (1966)
2.99 Bev	C. Baltay, J. Lach, J. Sandweiss, H. Taft, N. Yeh, D. Stonehill, and R. Stump, Phys. Rev. 142, 932 (1966)

Figure 1. Average number of pions accompanying 0 and 2 kaons under the assumption of SU_3 invariance.

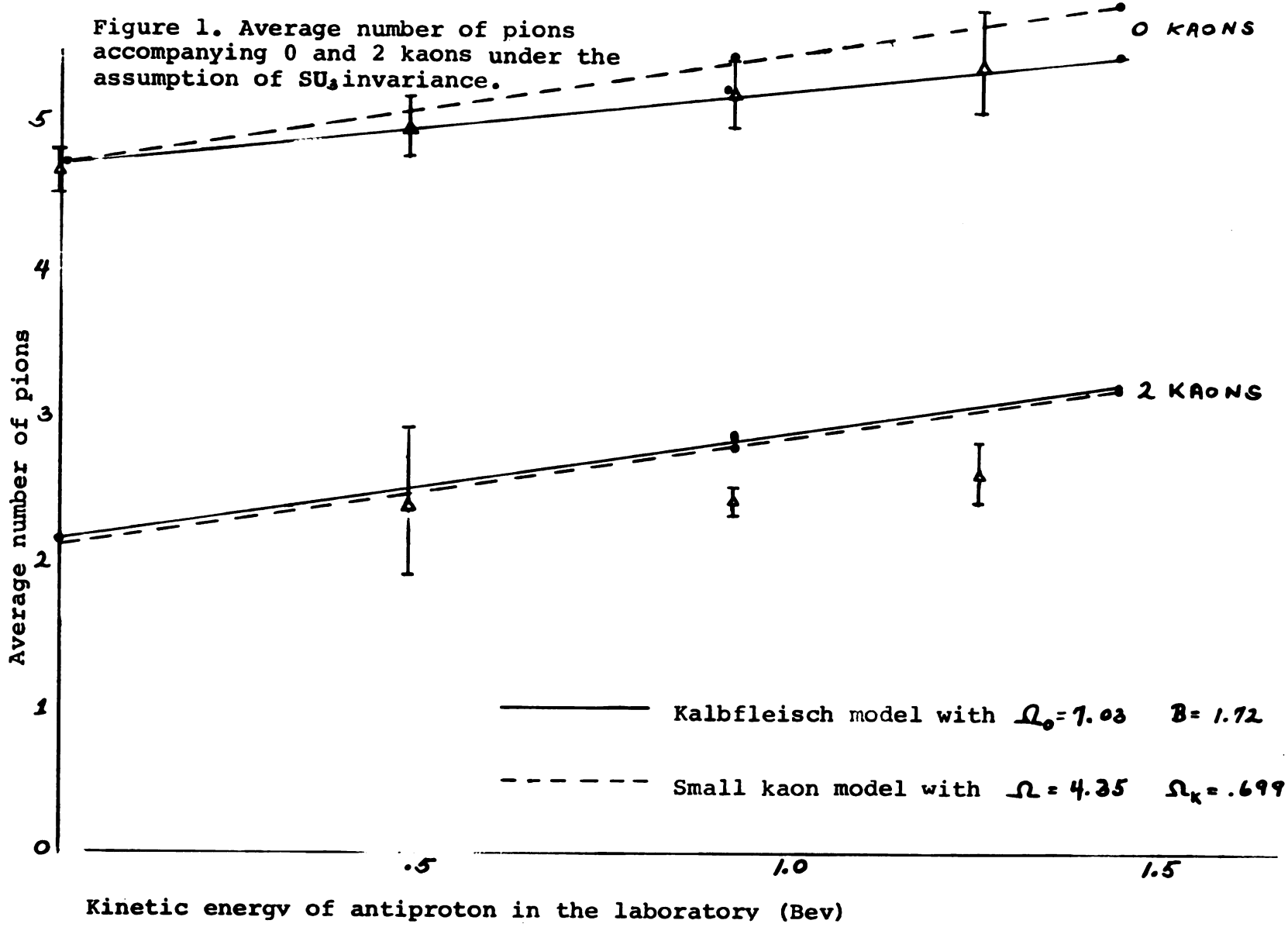
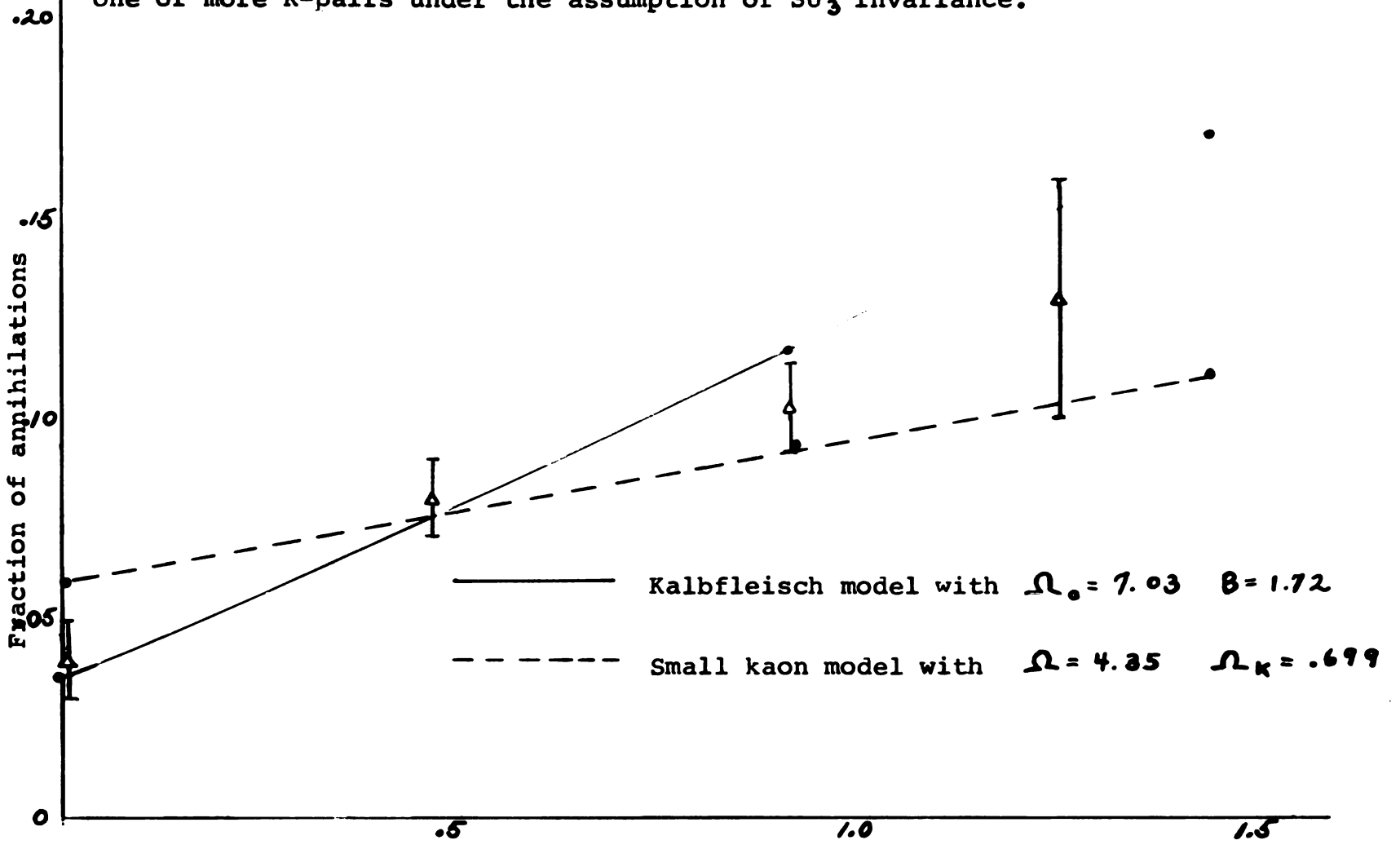
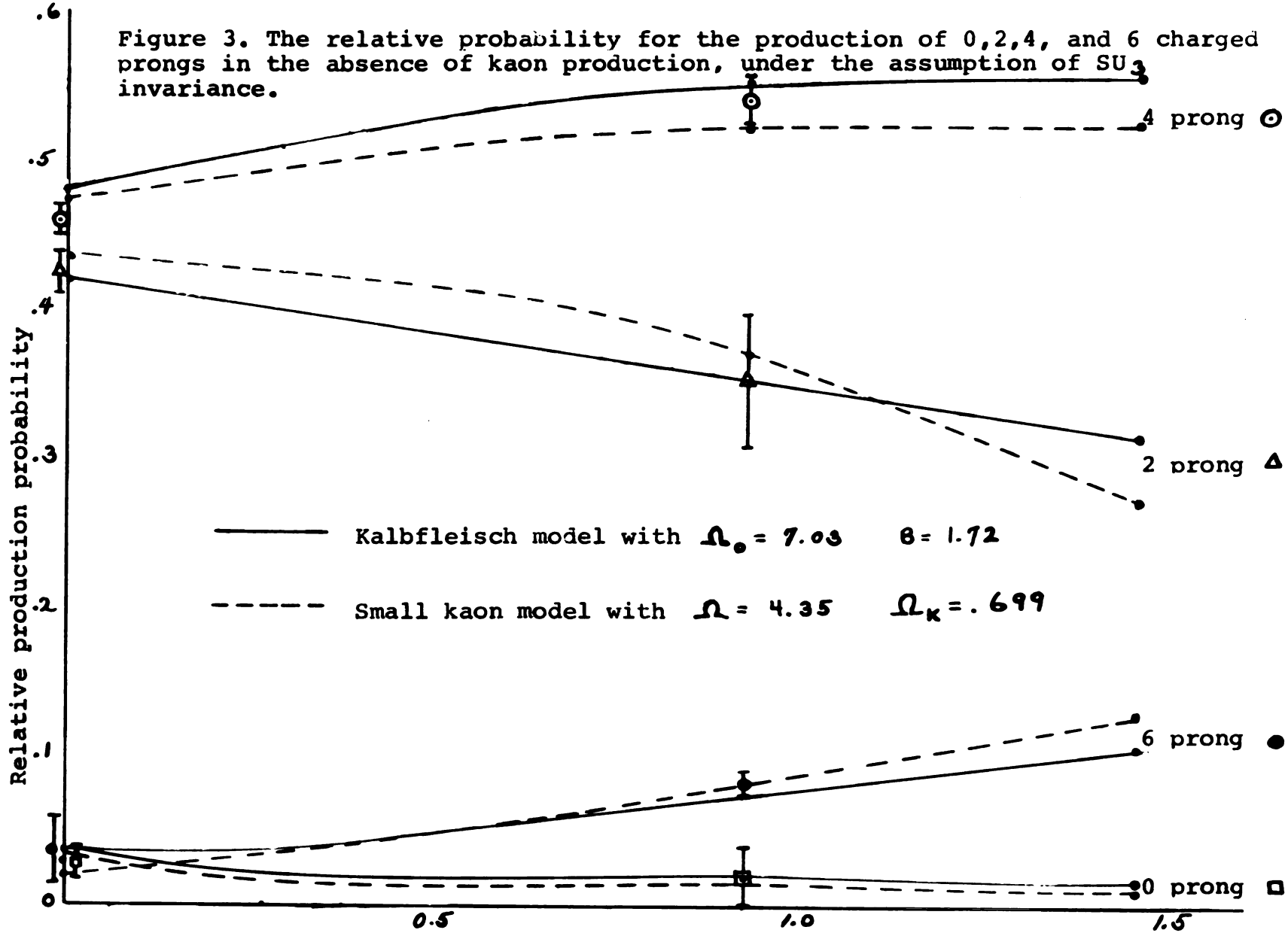


Figure 2. Fraction of all annihilations producing states having one or more K-pairs under the assumption of SU_3 invariance.



Kinetic energy of the antiproton in the laboratory (Bev)

Figure 3. The relative probability for the production of 0,2,4, and 6 charged prongs in the absence of kaon production, under the assumption of SU_3 invariance.



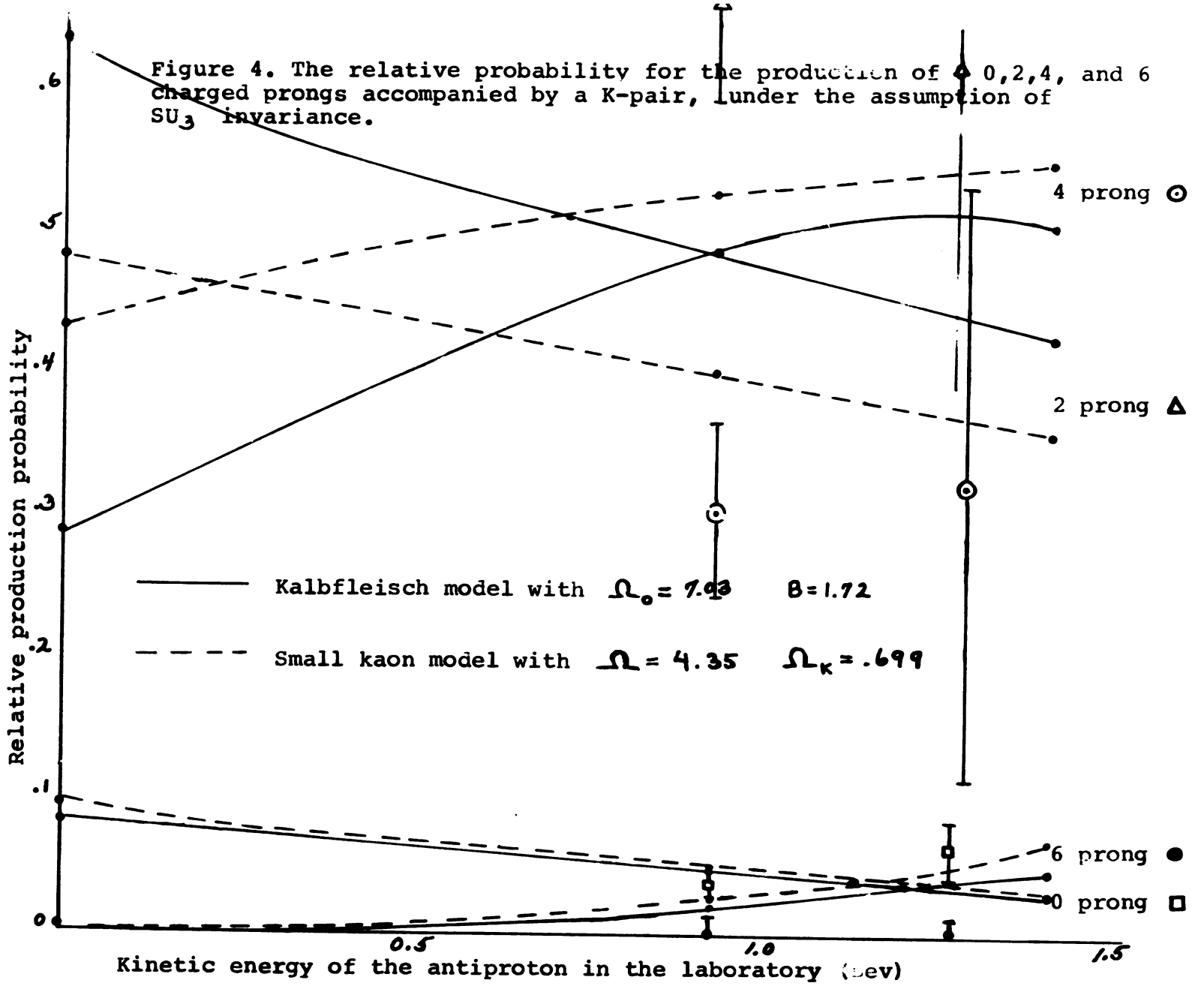
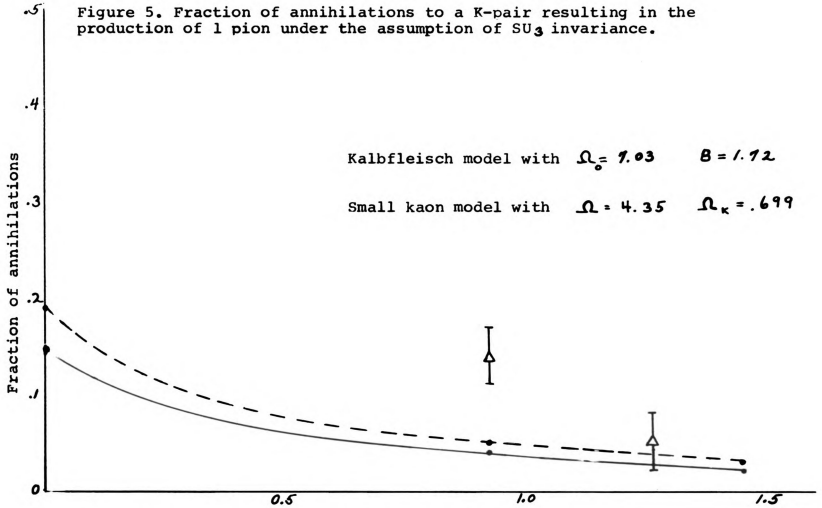


Figure 4. The relative probability for the production of 0,2,4, and 6 charged prongs accompanied by a K-pair, under the assumption of SU_3 invariance.

Figure 5. Fraction of annihilations to a K-pair resulting in the production of 1 pion under the assumption of SU_3 invariance.



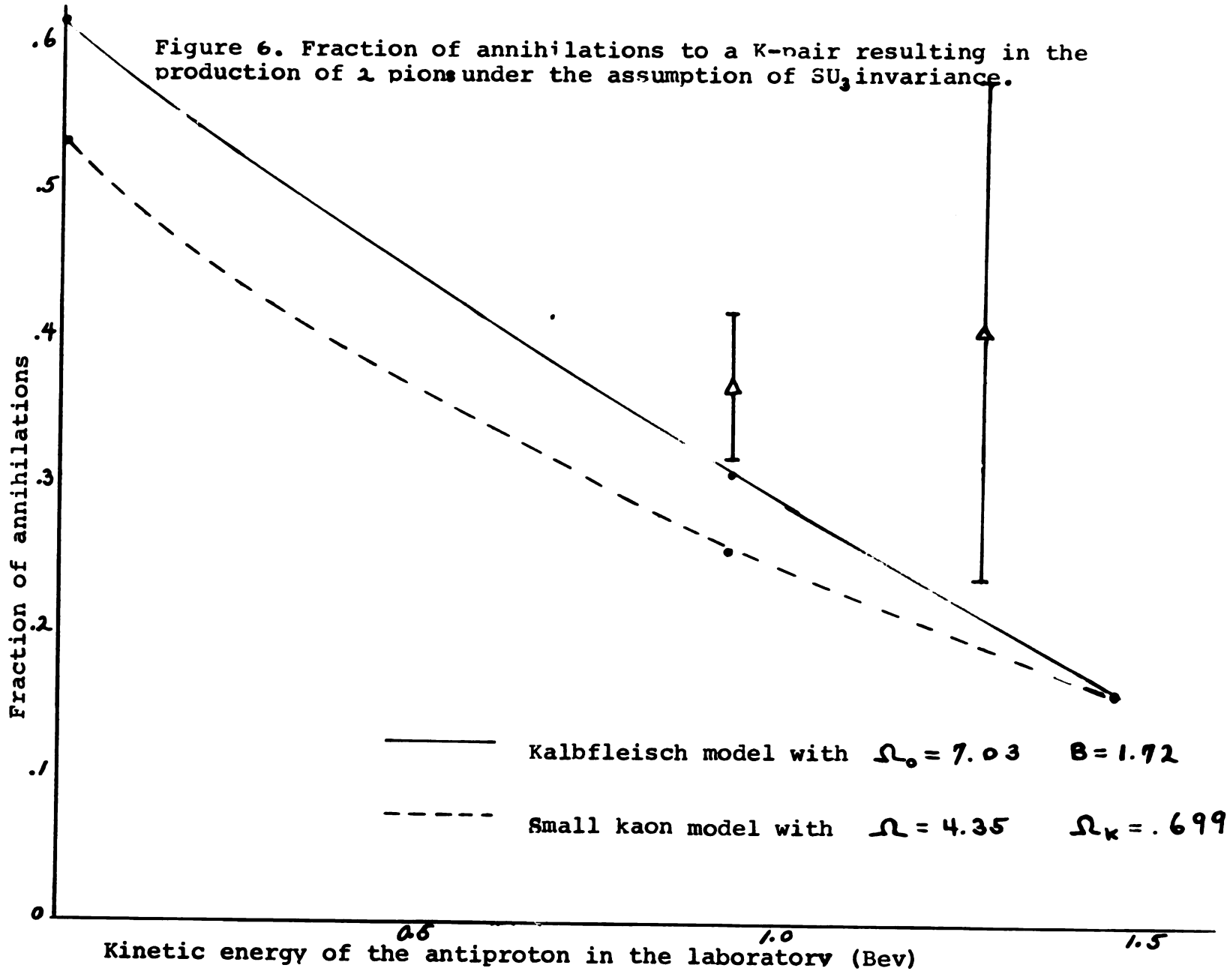


Figure 6. Fraction of annihilations to a K-pair resulting in the production of 2 pions under the assumption of SU_3 invariance.
Fraction of annihilations
.6
.5
.4
.3
.2
.1
0
0.6
1.0
1.5
Kinetic energy of the antiproton in the laboratory (Bev)
Kalbfleisch model with $\Omega_0 = 7.03$ $B = 1.72$
Small kaon model with $\Omega = 4.35$ $\Omega_K = .699$

Figure 7. Fraction of annihilations to a K-pair resulting in the production of 3 pions under the assumption of SU_3 invariance.

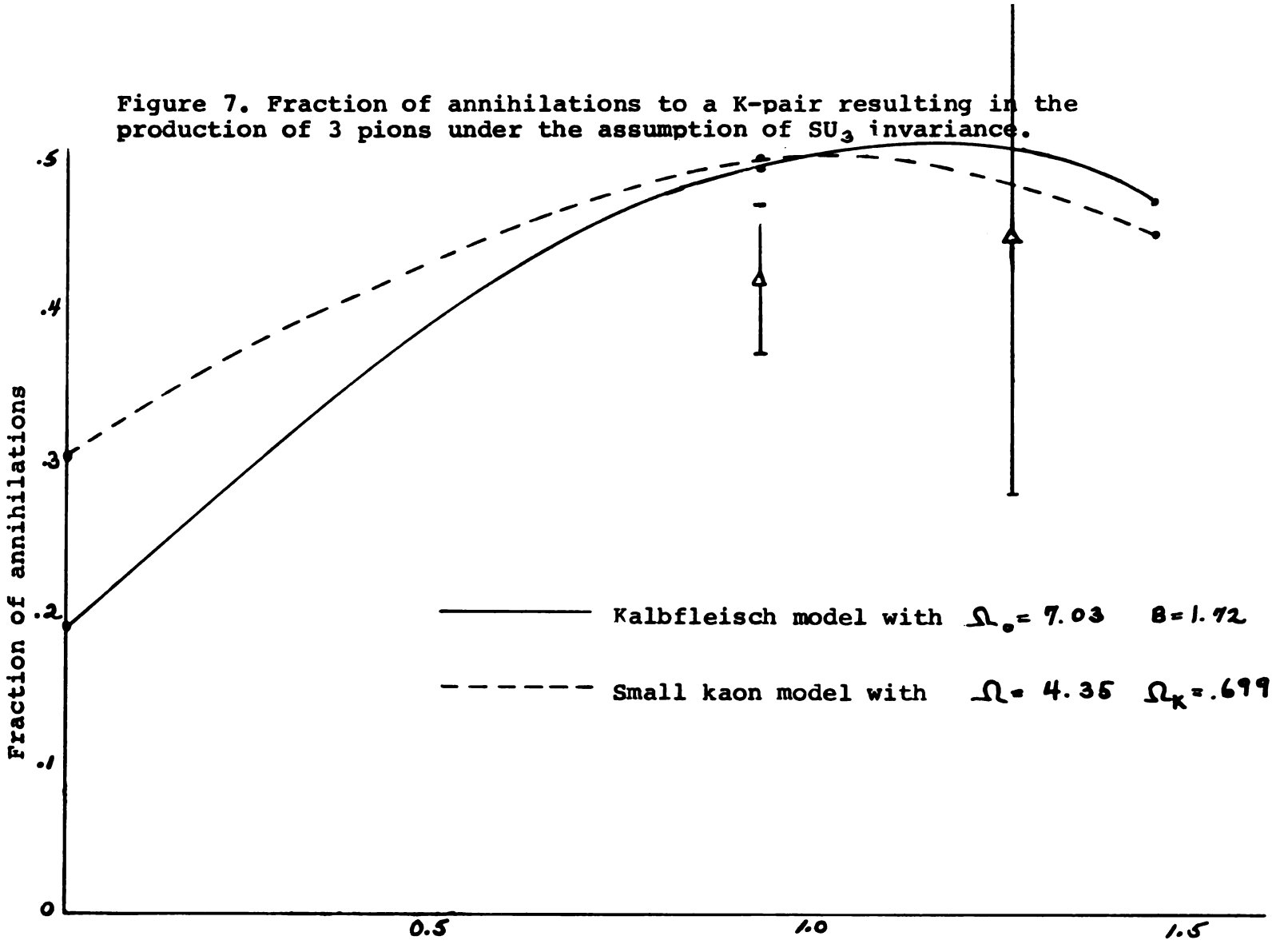
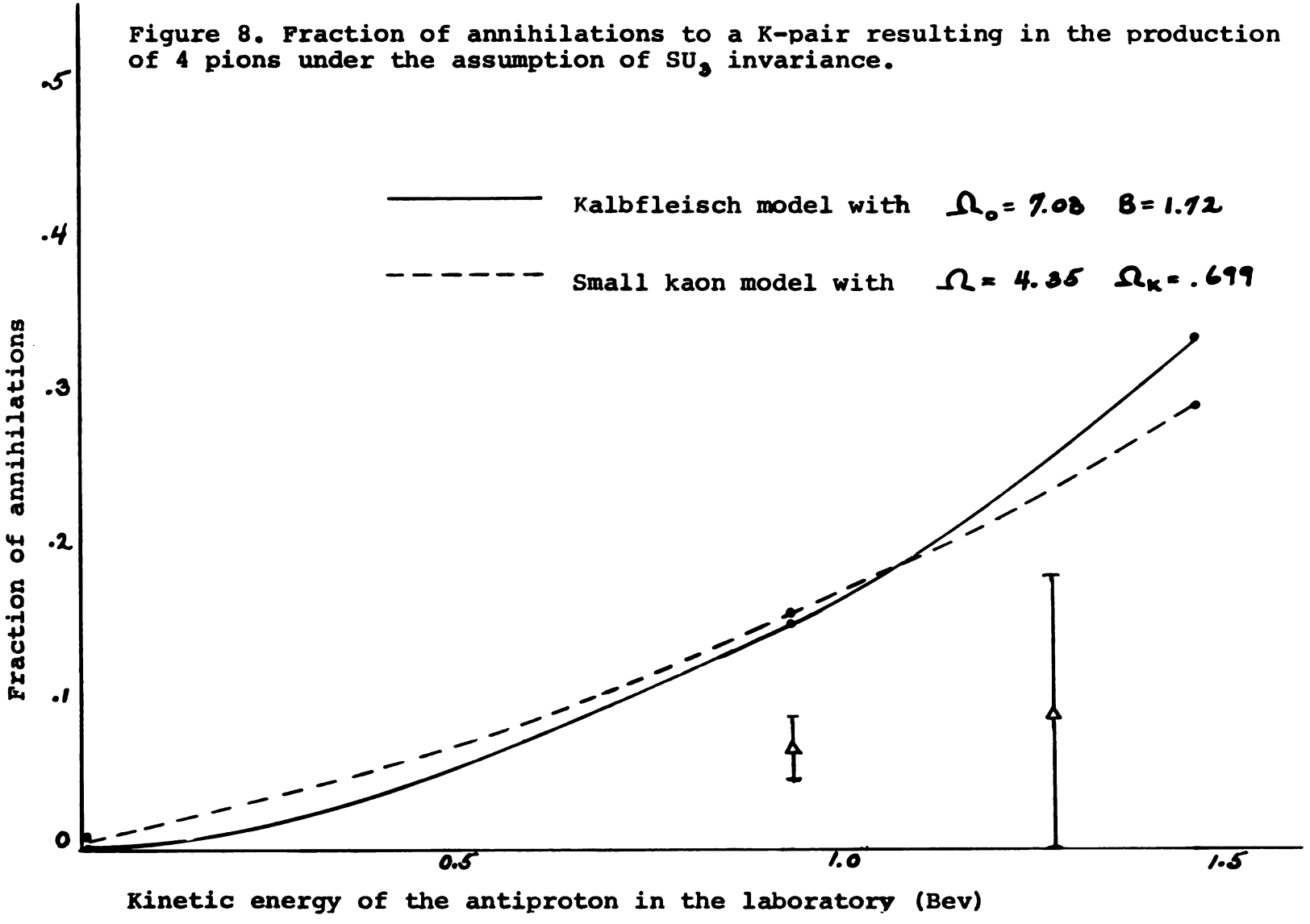


Figure 8. Fraction of annihilations to a K-pair resulting in the production of 4 pions under the assumption of SU_3 invariance.



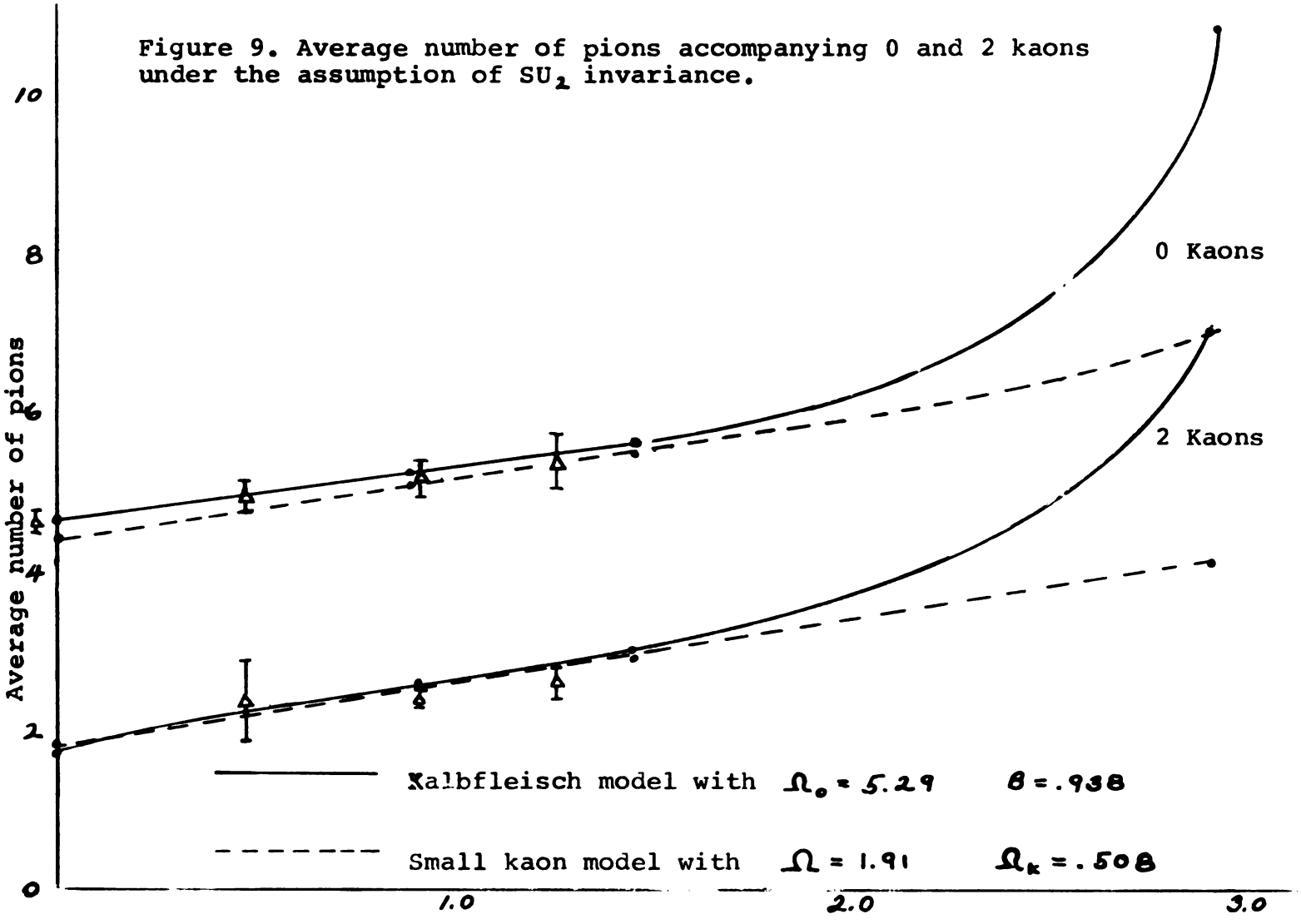


Figure 9. Average number of pions accompanying 0 and 2 kaons under the assumption of SU_2 invariance.
Average number of pions
10
8
6
4
2
0
0 Kaons
2 Kaons
Kalbfleisch model with $\Omega_0 = 5.29$ $\beta = .938$
Small kaon model with $\Omega = 1.91$ $\Omega_k = .508$
1.0
2.0
3.0
Kinetic energy of the antiproton in the laboratory (Bev)

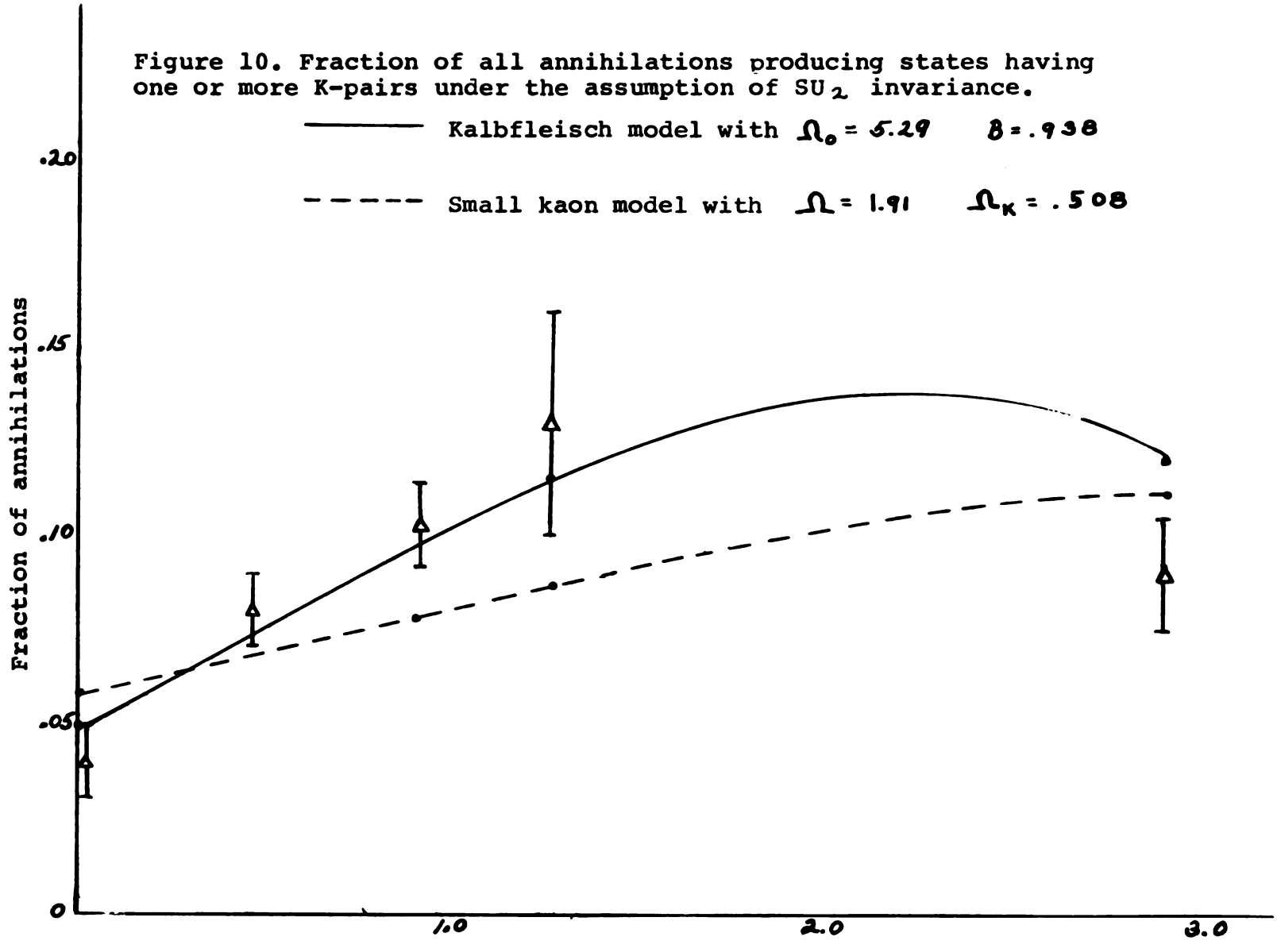


Figure 10. Fraction of all annihilations producing states having one or more K-pairs under the assumption of SU_2 invariance.

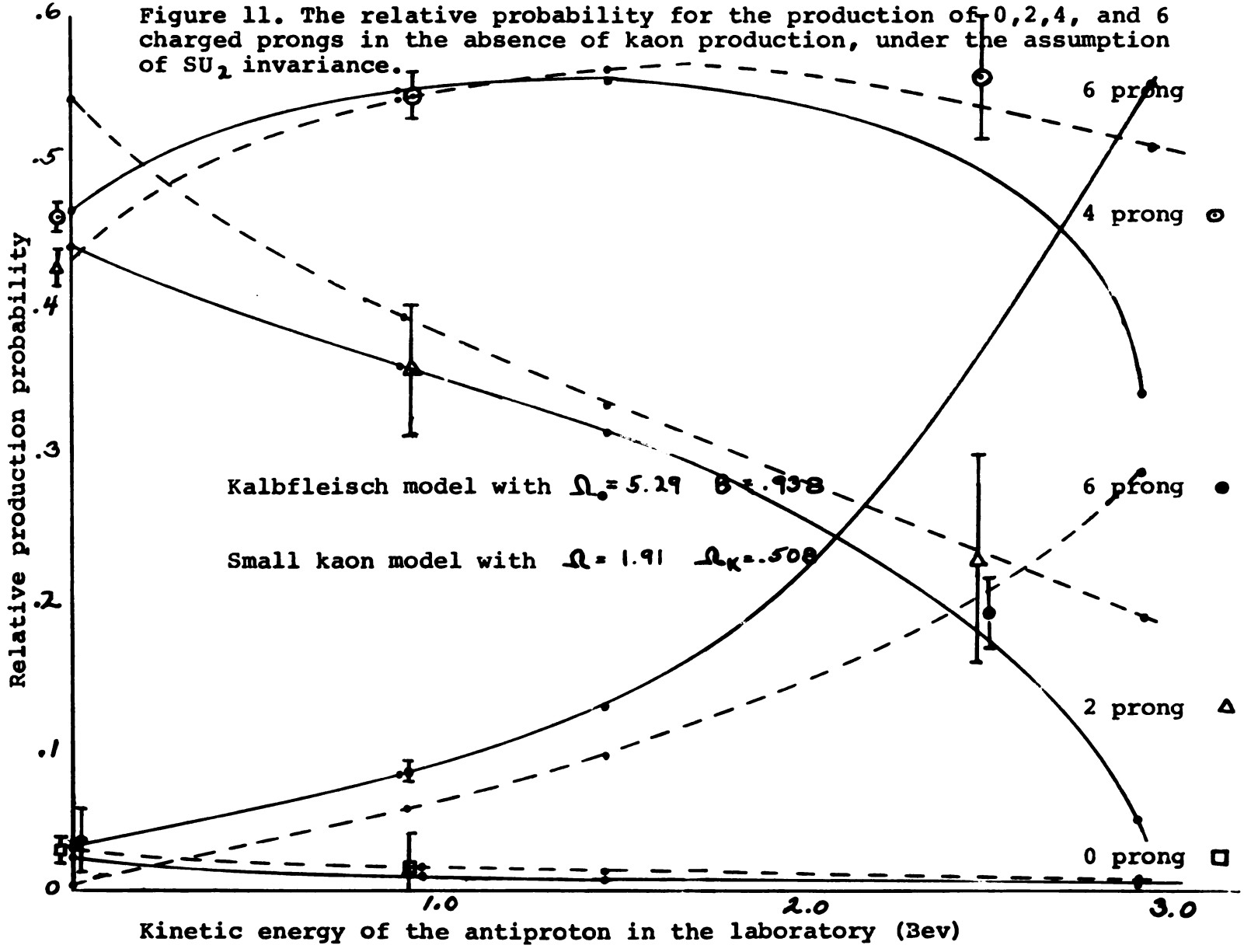


Figure 11. The relative probability for the production of 0,2,4, and 6 charged prongs in the absence of kaon production, under the assumption of SU_2 invariance.

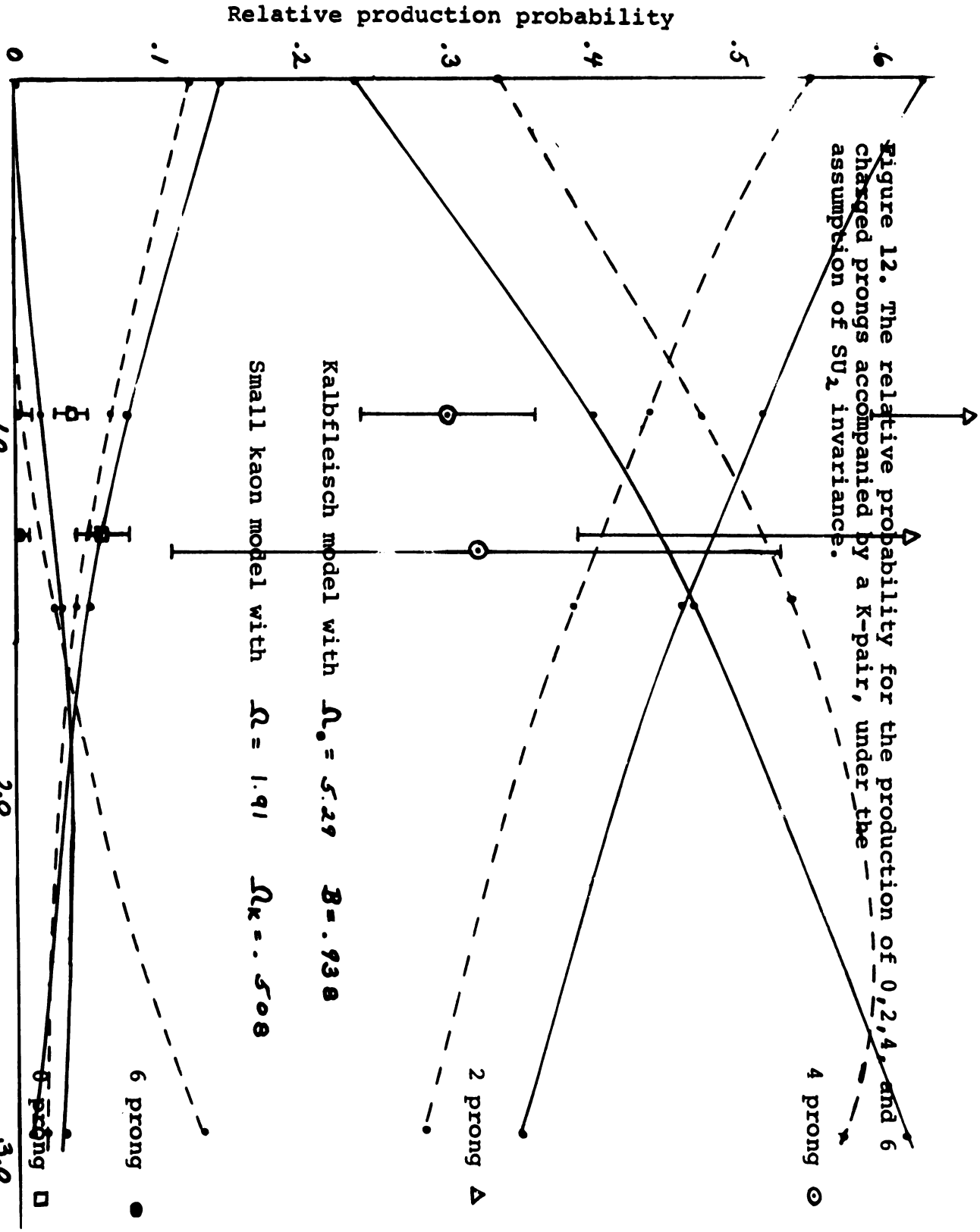
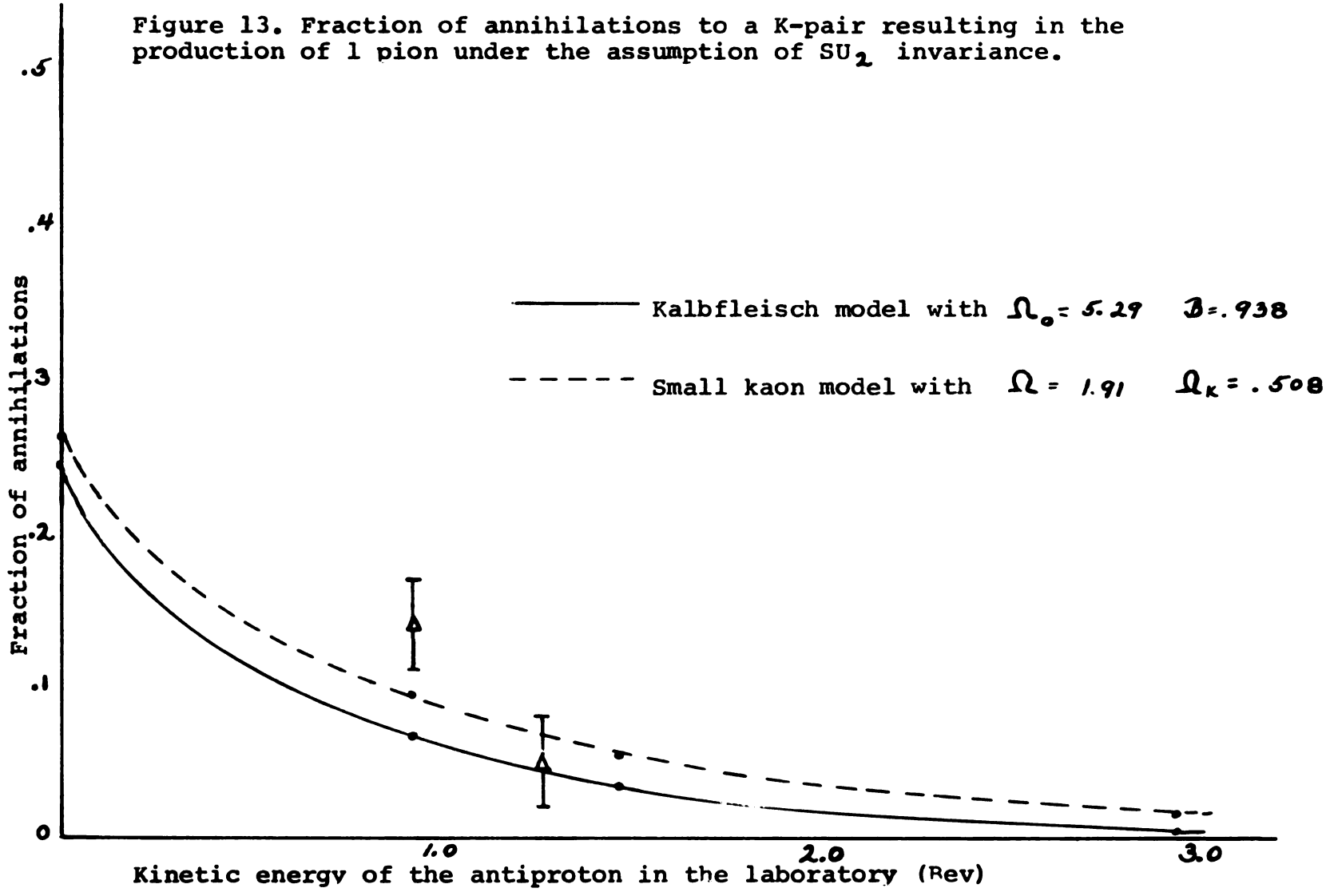


Figure 12. The relative probability for the production of 0, 2, 4, and 6 charged prongs accompanied by a K-pair, under the assumption of SU_2 invariance.

Figure 13. Fraction of annihilations to a K-pair resulting in the production of 1 pion under the assumption of SU_2 invariance.



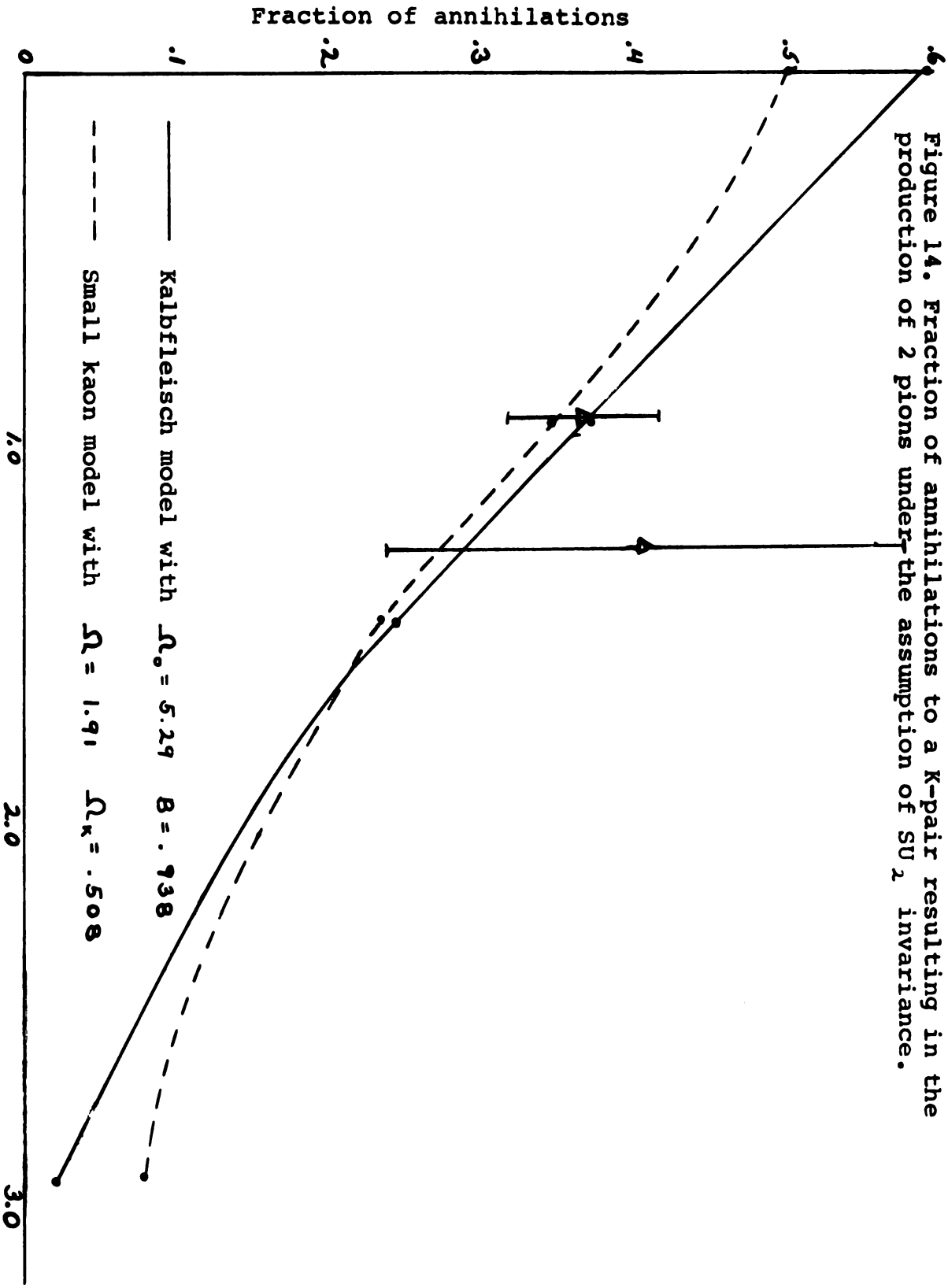


Figure 14. Fraction of annihilations to a K-pair resulting in the production of 2 pions under the assumption of SU_2 invariance.

Figure 15. Fraction of annihilations to a K-pair resulting in the production of 3 pions under the assumption of SU_2 invariance.

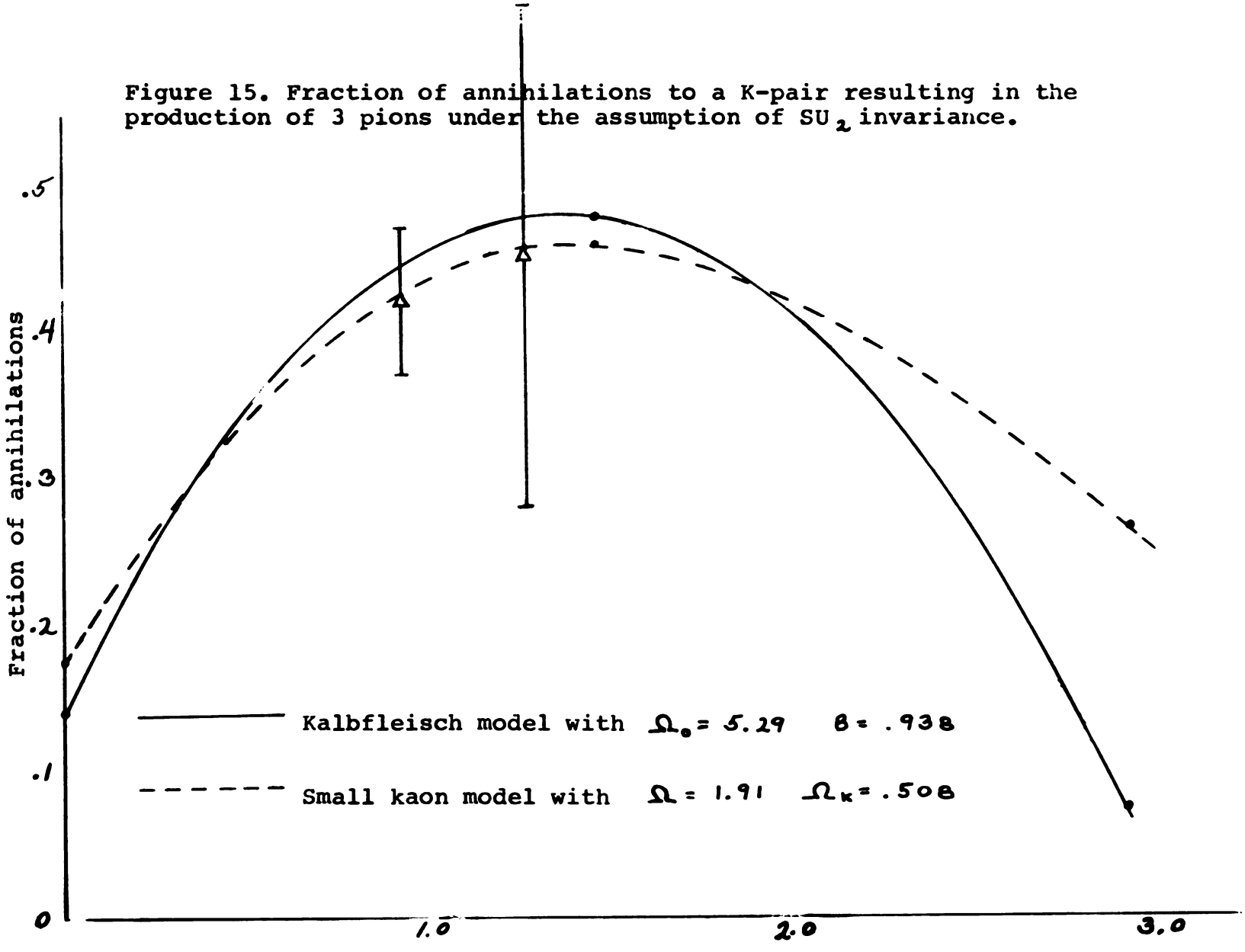
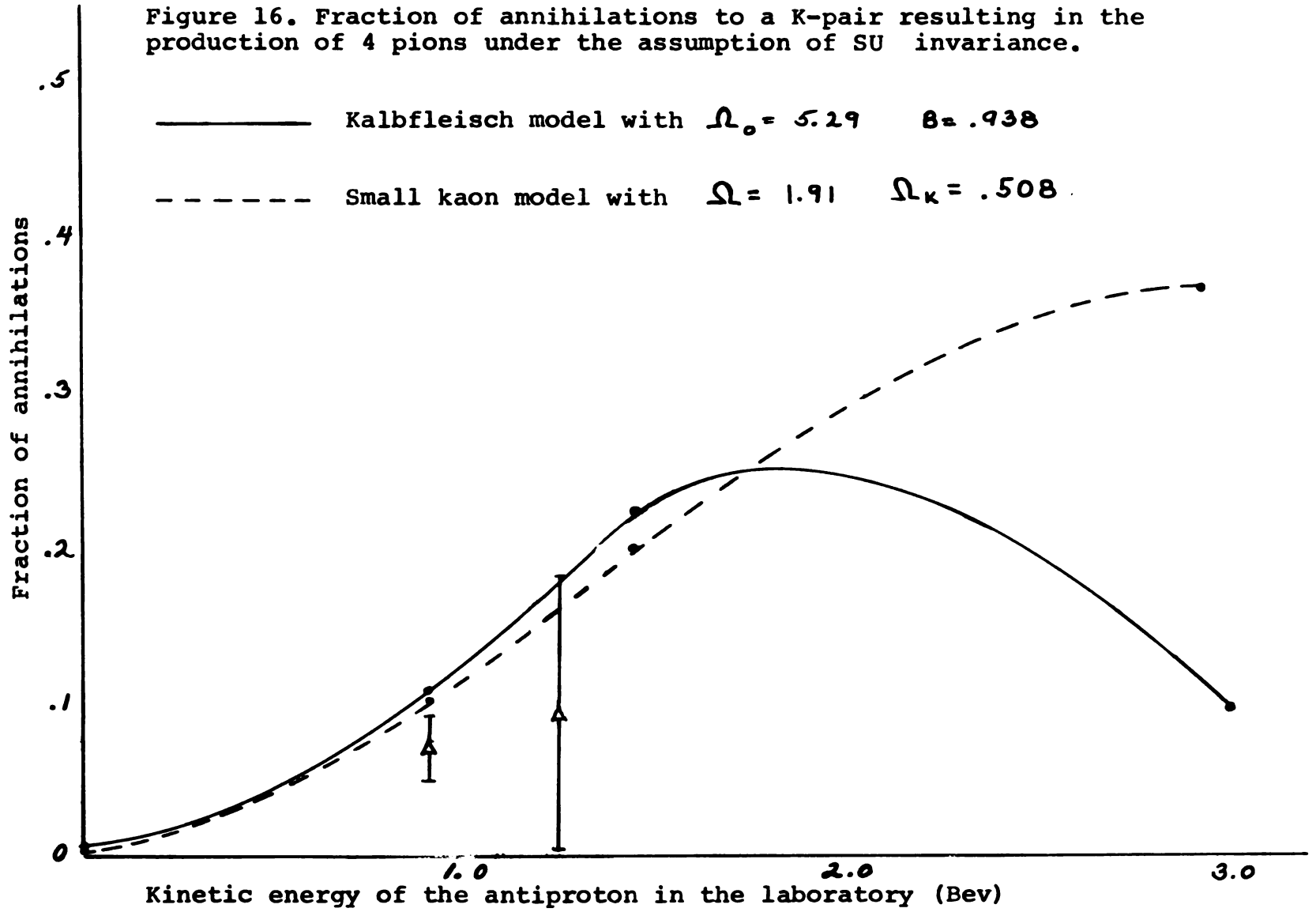


Figure 16. Fraction of annihilations to a K-pair resulting in the production of 4 pions under the assumption of SU invariance.



V. Conclusions.

We have attempted to reproduce experimental results for the production of kaons, pions and charged prongs from $p-\bar{p}$ annihilation using a simple statistical model with the modifications mentioned below. Previous statistical calculations were plagued with over copious production of kaons and required unphysically large volume parameters in order to obtain results consistent with experiment. It was hoped that the inclusion of a large number of resonances (which have been observed to appear quite frequently among the annihilation products) would alleviate these difficulties. Including the resonances seemed reasonably promising since, to get the same final answer, the volume parameters required can always be reduced by including more final states which are involved in the statistical process. The second modification consisted of building into the model the assumption of invariance under SU_3. However, the results presented show that the improvement, based mainly upon whether the experimental values can be fit with "small" volume parameters, is slight. The fit to the experimental data is fairly good; however, this is achieved only after the introduction of modifications to suppress kaon production and still involves rather large volume parameters.

It is difficult to isolate the effects of SU_3 invariance upon the model. As might be expected, the differences between the results obtained under the assumption of SU_3 invariance and those obtained under SU_2 invariance show up predominately at low (2 and 3) particle multiplicities and are averaged out in a statistical calculation where multiplicities of up to six or more particles are considered. One can see from the results that no substantial difference is incurred in this calculation by the generalization from SU_2 to SU_3 invariance.

Finally, it is worthwhile to mention a suggestion which may help to alleviate the difficulties mentioned above. If one uses a single parameter statistical model (with the volume parameter of the order of $0.1\,\Omega_\pi$) to calculate the fraction of annihilations at rest producing a kaon pair, one obtains a result which is approximately an order of magnitude larger than the experimental value. Furthermore, when the volume parameter is reduced to a value of the order of the "physical" range ($.01\,\Omega_\pi$ to $.001\,\Omega_\pi$) the kaon production is not changed significantly while the average pion multiplicity falls well below the experimental value. Maglic' has suggested a model in which kaon pairs are still produced in abundance but, at low relative

energy for example, immediately annihilate into pions.[33] This effect would not only reduce the kaon production, but would also substantially raise the average pion multiplicity so that reasonable results might be expected with small volume parameters. This line of thought can be investigated by including in the model a $K^- \bar{K}$ final state interaction. The preliminary calculations made by Maglic' indicate that a reasonable modification along these lines give results which go in the direction of further improving the model.

APPENDIX A:

I. Proof that $P^{I}_{i_{z_1}\cdots i_{z_n}}$ is independent of the order of $i_{z_1}, i_{z_2} \cdots i_{z_n}$.

We wish to interchange the i^{th} and $(i+1)^{th}$ particles going from a scheme in which:

$$\vec{I}_{i-1} + \vec{i}_i = \vec{I}_i$$
$$\vec{I}_i + \vec{i}_{i+1} = \vec{I}_{i+1}$$

to one in which:

$$\vec{I}_{i-1} + \vec{i}_{i+1} = \vec{I}$$
$$\vec{I} + \vec{i}_i = \vec{I}_{i+1}.$$

The basis states transform via the $6j$ symbol:

$$|I_{i+1}, M_{i+1}\rangle_{I_i} = \sum_{I} (2I_i+1)^{1/2} (2I+1)^{1/2} \times \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix} |I_{i+1}, M_{i+1}\rangle_{I}$$

where $|I, M\rangle_{I'}$ implies the final state (I, M) is formed via the intermediate value I'. We then introduce this change of basis into the recoupling coefficient in the P-factor $P^{I}_{i_{z_1} i_{z_2} \cdots i_{z_n}}$.

In what follows we write the intermediate state in parentheses; $i, i_2\ (I_2)$ implies $\vec{i} + \vec{i}_2 = \vec{I}_2$.

$$\rho^{I_f}_{i_{z_1} i_{z_2} \cdots i_{z_n}} = \sum_{I_2 I_3 \cdots I_{n-1}} \left| \left\langle I_2, I_3 \cdots I_{n-1}, I_f \, I_{z_f} \middle| \begin{matrix} i_1 & i_2 & \cdots & i_n \\ i_{z_1} & i_{z_2} & \cdots & i_{z_n} \end{matrix} \right\rangle \right|^2$$

$$= \sum_{I_2 I_3 \cdots I_{n-1}} \Bigg[\Bigg| \sum_{I} (2I_i+1)^{1/2} \, (2I+1)^{1/2}$$

$$\times \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix} \left\langle i_1 i_2(I_2), I_2 i_3(I_3) \cdots I_{i-1} \, i_{i+1}(I), I \, i_i(I_{i+1}) \cdots \right.$$

$$\left. \cdots I_{n-1} \, i_n(I_f) \middle| \begin{matrix} i_1 & \cdots & i_{i+1} & , & i_i & , & \cdots & i_n \\ i_{z_1} & \cdots & i_{z_{i+1}} & , & i_{z_i} & , & \cdots & i_{z_n} \end{matrix} \right\rangle \Bigg|^2 \Bigg].$$

We may write the above square as a sum over I and I' and isolate the sum over I_i:

$$= \sum_{I_2 I_3 \cdots I_{i-1}, I_{i+1}, \cdots I_{n-1}} \sum_{I, I'} (2I+1)^{1/2} \, (2I'+1)^{1/2}$$

$$\times \left\langle i_1 i_2(I_2) \cdots I_{i-1} \, i_{i+1}(I), I \, i_i(I_{i+1}) \cdots \middle| \begin{matrix} i_1 & \cdots & i_n \\ i_{z_1} & \cdots & i_{z_n} \end{matrix} \right\rangle \left\langle i_1 i_2(I_2) \cdots \right.$$

$$\left. \cdots I_{i-1} \, i_{i+1}(I'), I' i_i(I_{i+1}) \cdots I_{n-1} \, i_n(I_f) \middle| \begin{matrix} i_1 & \cdots & i_{i+1} & , & i_i & \cdots & i_n \\ i_{z_1} & \cdots & i_{z_{i+1}} & , & i_{z_i} & \cdots & i_{z_n} \end{matrix} \right\rangle$$

$$\times \sum_{I_i} (2I_i+1) \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix} \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I' \end{Bmatrix}$$

From the unitarity condition on the 6j symbols we have:

$$\sum_{I_i} (2I_i+1) \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix} \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I' \end{Bmatrix} = \frac{1}{2I+1} \delta_{I,I'}.$$

After performing the sum over I' by using the δ, we have:

$$\rho^{I_f}_{i_{z_1} i_{z_2} \cdots i_{z_n}} = \sum_{I_2 \cdots I_{i-1}, I_{i+1} \cdots I_{n-1}} \sum_{I} \left\langle i_1 i_2(I_2) \cdots I_{i-1} \, i_{i+1}(I), \cdots \right.$$

$$\left. \cdots I \, i_i(I_{i+1}) \cdots I_{n-1} \, i_n(I_f) \middle| \begin{matrix} i_1 & \cdots & i_{i+1} & , & i_i & , & \cdots & i_n \\ i_{z_1} & \cdots & i_{z_{i+1}} & , & i_{z_i} & , & \cdots & i_{z_n} \end{matrix} \right\rangle^2.$$

If we relabel I as I_i, it is clear that the right hand side is equal to ρ^{I_f} with i^{th} and $(i+1)^{th}$ states interchanged. Therefore:

$$\rho^{I_f}_{i_{z_1} i_{z_2} \cdots i_{z_i}, i_{z_{i+1}} \cdots i_{z_n}} = \rho^{I_f}_{i_{z_1} i_{z_2} \cdots i_{z_{i+1}}, i_{z_i} \cdots i_{z_n}}.$$

II. Proof that $\rho^{U,I}_{\nu_1 \nu_2 \cdots \nu_n}$ is independent of the order of $\nu_1 \nu_2 \cdots \nu_n$.

We wish to prove that the sum over intermediate states of the recoupling coefficient for SU_3 is independent of the order of the individual states.

We wish to go from a basis in which:

$$U_{\gamma_{i-1}} \otimes \mu_i = U_{\gamma_i} \qquad \vec{I}_{i-1} + \vec{i}_i = \vec{I}_i$$

$$U_{\gamma_i} \otimes \mu_{i+1} = U_{\gamma_{i+1}} \qquad \vec{I}_i + \vec{i}_{i+1} = I_{i+1}$$

to one in which:

$$U_{\gamma_{i-1}} \otimes \mu_{i+1} = U_{\gamma} \qquad \vec{I}_{i-1} + \vec{i}_{i+1} = \vec{I}$$

$$U_{\gamma} \otimes \mu_i = U_{\gamma_{i+1}} \qquad \vec{I} + \vec{i}_i = \vec{I}_{i+1}.$$

In this case, the transformation will have both SU_3 and SU_2 6j symbols. The notation used for the SU_3 transformation is according to Sharpe and Derome.[34] We again write the intermediate states in parentheses. The SU_3 and SU_2 parts transform respectively as:

$$|U_{i-1}\,\mu_i\,(U_{\gamma_i}),\,U_i\,\mu_{i+1}\,(U_{\gamma_{i+1}})\rangle = \sum_U [U]^{1/2}\,[U_i]^{1/2}$$

$$\times \begin{Bmatrix} U_{i-1} & \mu_i & U_i^* \\ U^*_{i+1} & \mu_{i+1} & U^* \end{Bmatrix}_{\gamma_{i+1}\quad\gamma_i}^{\gamma_2\,\gamma_3} |U_{i-1}\,\mu_{i+1}(U_{\gamma_3}),\,U_{\mu_i}(U_{i+1\,\gamma_3})\rangle$$

$$|I_{i-1}\,i_i(I_i),\,I_i\,i_{i+1}(I_{i+1})\rangle = \sum_I [I_i]^{1/2}\,[I]^{1/2}$$

$$\times \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix} |I_{i-1}\,i_{i+1}(I),\,I\,i_i\,(I_{i+1})\rangle$$

where summation is implied over repeated γ indices and $[I]$ represents the dimension of the irreducible representation labeled by I.

The P-factor for SU_3 invariance is:

$$P^{U_f\,I_f}_{\nu_1\nu_2\cdots\nu_n} = \sum_{\substack{U_{2\gamma},\,\ldots\,U_{n-1\,\gamma} \\ I_2 \qquad\quad I_{n-1}}}{}' \left\langle \begin{matrix} U_{2\gamma}, & \ldots U_{n-1\,\gamma}, & U_{f\,\gamma} \\ I_2 & I_{n-1} & I_f\,\nu_f \end{matrix} \middle| \begin{matrix} \mu_1 & \ldots & \mu_n \\ i_1\nu_1 & & i_n\nu_n \end{matrix} \right\rangle^2 .$$

We suppress the indices on the γ's wherever possible; the γ's above are, in general, all different. The $'$ indicates a summation over the γ associated with U_f. We introduce the 6j symbols into the equation above:

$$= \sum_{\substack{U_{2\gamma}\ldots U_{n-1\gamma} \\ I_2 \ldots I_{n-1}}}{}' \Bigg[\Bigg| \sum_{\substack{U \\ I}} [U^{1/2}]\,[U_i]^{1/2}\,[I]^{1/2}\,[I_i]^{1/2} \times \begin{Bmatrix} U_{i-1} & \mathcal{U}_i & U_i^* \\ U^*_{i+1} & \mathcal{U}_{i+1} & U^* \end{Bmatrix}_{\gamma_{i+1}\ \gamma_i}^{\gamma\gamma_3} \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix}$$

$$\times \left\langle \begin{matrix} \mathcal{U}_1 & \mathcal{U}_2(U_{2\gamma}) & \ldots & U_{i-1\gamma} & \mathcal{U}_{i+1}(U_{\gamma_3}), U_{\gamma_3}\,\mathcal{U}_i(U_{i+1\gamma}) & \ldots \\ i_1 & i_2\ (I_2) & & I_{i-1} & i_{i+1}\ (I) \quad I\ \ i_i\ (I_{i+1}) & \end{matrix} \right.$$

$$\left. \begin{matrix} \ldots & U_{n-1\gamma} & \mathcal{U}_n(U_{i\gamma}) \\ & I_{n-1} & i_n\ (I_i) \end{matrix} \Big| \begin{matrix} \mathcal{U}_1 & \mathcal{U}_2 & \ldots & \mathcal{U}_n \\ i_1\nu_1 & i_2\nu_2 & \ldots & i_n\nu_n \end{matrix} \right\rangle \Bigg|^2 \Bigg]$$

and write the square as a double sum:

$$= \sum_{\substack{U_{2\gamma}\ U_{3\gamma}\ldots U_{n-1\gamma} \\ I_2\ I_3 \quad\ldots\quad I_{n-1}}}{}' \sum_{\substack{U\ U' \\ I\ I'}} [U]^{1/2}\,[U']^{1/2}\,[U_i]\,[I]^{1/2}\,[I']^{1/2}\,[I_i]$$

$$\begin{Bmatrix} U_{i-1} & \mathcal{U}_i & U_i^* \\ U^*_{i+1} & \mathcal{U}_{i+1} & U^* \end{Bmatrix}^{*}_{\gamma_{i-1}\ \gamma_i}{}^{\gamma\gamma_3} \begin{Bmatrix} U_{i-1} & \mathcal{U}_i & U_i^* \\ U^*_{i+1} & \mathcal{U}_{i+1} & U'^* \end{Bmatrix}_{\gamma_{i+1}\ \gamma_i}^{\gamma'\gamma_3'} \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I \end{Bmatrix} \begin{Bmatrix} I_{i-1} & i_i & I_i \\ I_{i+1} & i_{i+1} & I' \end{Bmatrix}$$

$$\times \left\langle \begin{matrix} \ldots & U_{i-1\gamma} & \mathcal{U}_{i+1}(U_{\gamma_3}) & \ldots \\ & I_{i-1} & i_{i+1}\ (I) & \end{matrix} \Big| \begin{matrix} \mathcal{U}_1 & \ldots & \mathcal{U}_n \\ i_1\nu_1 & \ldots & i_n\nu_n \end{matrix} \right\rangle \left\langle \begin{matrix} \ldots & U_{i-1\gamma} & \mathcal{U}_{i+1}(U'_{\gamma'_3}) & \ldots \\ & I_{i-1} & i_{i+1}\ (I') & \end{matrix} \Big| \begin{matrix} \mathcal{U}_1 & \ldots & \mathcal{U}_n \\ i_1\nu_1 & \ldots & i_n\nu_n \end{matrix} \right\rangle$$

(where there is no sum over the γ_{i+1}'s and the γ_i's).

First consider the SU_2 part. The sum over I_i is done first, yielding a term $1/[I]\ \delta_{I,I'}$ from the unitarity of the 6j symbols. This gives us, isolating the sum over U_i:

$$= \sum_{\substack{U_{2\gamma}\ldots U_{i-1\gamma}\,U_{i+1\gamma}\ldots U_{n-1\gamma} \\ I_2 \quad\ldots\quad I_{i-1}\,I_{i+1}\quad\ldots\quad I_{n-1}}}{}' \sum_{\substack{U\ U' \\ I}} [U]^{1/2}\,[U']^{1/2} \left\langle \begin{matrix} \ldots & (U_{\gamma_3}) & \ldots \\ & (I) & \end{matrix} \Big| \ldots \begin{matrix} \mathcal{U}_n \\ i_n\nu_n \end{matrix} \right\rangle$$

$$\left\langle \begin{matrix} \ldots & (U'_{\gamma'_3}) & \ldots \\ & (I') & \end{matrix} \Big| \ldots \begin{matrix} \mathcal{U}_n \\ i_n\nu_n \end{matrix} \right\rangle \Bigg[\sum_{U_i} [U_i] \begin{Bmatrix} U_{i-1} & \mathcal{U}_i & U_i^* \\ U^*_{i+1} & \mathcal{U}_{i+1} & U^* \end{Bmatrix}^{*}_{\gamma_{i+1}\ \gamma_i}{}^{\gamma\gamma_3}$$

$$\times \begin{Bmatrix} U_{i-1} & \mathcal{U}_i & U_i^* \\ U^*_{i+1} & \mathcal{U}_{i+1} & U'^* \end{Bmatrix}_{\gamma_{i+1}\ \gamma_i}^{\gamma'\gamma_3'} \Bigg]$$

From the unitarity of the SU_3 6j symbols, we can show

(see below)* that the sum over U_i yields: $\frac{1}{U}\delta_{U,U'}\delta_{\gamma}^{\gamma'}\delta_{\gamma_3}^{\gamma_3'}$.

Hence, we have:

$$\rho^{U_f I_f}_{\gamma_1 \gamma_2 \cdots \gamma_n} = \sum'_{\substack{U_2 \gamma \cdots U_{i-1}\gamma, U_{i+1}\gamma \cdots U_{n-1}\gamma \\ I_2 \quad\quad I_{i-1}, I_{i+1} \quad\quad I_{n-1}}} \sum_{U} \sum_{I} \sum_{\gamma_3} \left\langle {}^{1,\ 1,2(\frac{1}{2})\cdots}_{i,\ i_2(I_2)} \right.$$

$$\cdots {}^{U_{i-1}\gamma\ \ u_{i+1}(U_{\gamma_3})\ U_{\gamma_3}\ u_i(U_{\gamma_{i+1}})\ \cdots\ U_{n-1}\gamma\ u_n(U_{f\gamma})}_{I_{i-1}\ \ i_{i+1}\ (I)'\ I\ \ i_i\ (I_{i+1})'\ \cdots\ I_{n-1}\ \ i_n\ (I_f)} \left| {}^{u_1 \cdots u_{i+1}\ \ u_i \cdots u_n}_{i_1 \gamma_1 \cdots i_{i+1}\gamma_{i+1}\ \ i_i \gamma_i \cdots i_n \gamma_n} \right\rangle^2 .$$

Relabeling $U \to U_i \quad I \to I_i \quad \gamma_3 \to \gamma_i$,

we see that the right hand side is the SU_3 P-factor with the i^{th} and the $(i+1)^{th}$ states interchanged.

---

*The sum over U_i is almost identical to the unitarity condition given by Sharpe and Derome. One can easily rewrite the summation above in the standard form by first lowering the γ indices using the A matrices and then switching the order of the indices using the λ's . This gives:

$$\sum_{U_i} [U_i] \left\{ {}^{U_{i-1}\ \ u_i\ \ U_i^*}_{U_{i+1}^*\ u_{i+1}\ U^*} \right\}^*_{\gamma_{i+1}\ \gamma_i}{}^{\gamma\gamma_3} \left\{ {}^{U_{i-1}\ \ u_i\ \ U_i^*}_{U_{i+1}^*\ u_{i+1}\ U'^*} \right\}_{\gamma_{i+1}\ \gamma_i}{}^{\gamma'\gamma_3'}$$

$$\propto \sum_{U_i} [U_i] \left\{ {}^{U_{i-1}^*\ \ u_{i+1}\ \ U}_{U_{i+1}\ \ u_i\ \ U_i} \right\}^*_{\gamma\gamma_{i+1}\gamma_i\gamma_3} \left\{ {}^{U_{i-1}^*\ \ u_{i+1}\ \ U'}_{U_{i+1}\ \ u_i\ \ U_i} \right\}_{\gamma'\gamma_{i+1}\gamma_i\gamma_3'}$$

$$= \frac{1}{[U]} \delta_{U,U'}\ \delta_{\gamma}^{\gamma'}\ \delta_{\gamma_3}^{\gamma_3'}$$

by unitarity of the 6j symbols.

APPENDIX B: Outline of a Monte Carlo method of evaluating phase space integrals.*

All calculations which follow are done in dimensionless units.

The phase space integral for n particles with total energy E and total momentum P is:

$$\rho_n(E,P) = \frac{1}{(2\pi\hbar)^{3n-3}} \int\cdots\int dp_1\, dp_2 \cdots dp_n\, \delta\Big(P - \sum_{i=1}^{n} p_i\Big)\, \delta\Big(E - \sum_i e_i\Big)$$

where p_i, e_i are respectively the individual particle momenta and energies. In what follows, we drop the $1/(2\pi\hbar)^{3n-3}$ and write the integral as $\rho_n^*(E,P)$.

The directional part is isolated and related to the well-known "random walk" problem.

$$d\vec{p}_i = p_i^2\, dp_i\, d\epsilon_i \qquad d\epsilon_i = \sin\theta_i\, d\theta_i\, d\phi_i$$

$$\rho_n^*(E,P) = \int\cdots\int p_1^2 p_2^2 \cdots p_n^2\, \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\Big)\, dp_1 \cdots dp_n$$

$$\times \int\cdots\int \delta\Big(\vec{P} - \sum_{i=1}^{n} p_i \hat{\epsilon}_i\Big)\, d\epsilon_1\, d\epsilon_2 \cdots d\epsilon_n$$

We separate the last integral as follows:

$$\omega_n(P; p_1 p_2 \cdots p_n) = \frac{1}{4\pi^n} \int\cdots\int \delta\Big(\vec{P} - \sum_{i=1}^{n} p_i \hat{\epsilon}_i\Big)\, d\epsilon_1 \cdots d\epsilon_n .$$

---

*What follows is an outline of the paper: F. Cerulus and R. Hagedorn: Nuovo Cimento Supplement 9, 646 (1951). This article is referred to throughout the discussion as I.

The function ω_n is the probability that, for fixed magnitudes of the momenta $p_1, p_2 \cdots p_n$ but random directions $\hat{\epsilon}_1, \hat{\epsilon}_2 \cdots \hat{\epsilon}_n$, the total momentum $\sum_{i=1}^{n} p_i \hat{\epsilon}_i$ lies in the neighborhood dP of P. For given arguments $(P; p_1 \cdots p_n)$, ω_n may be calculated using the theory of random walk. We note that ω_n is a function only of the magnitude of the momenta. We have now reduced the phase space integral to:

$$\rho_n^*(E,P) = (4\pi)^n \int_0^\infty \cdots \int_0^\infty p_1^2 p_2^2 \cdots p_n^2 \; \delta\left(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\right) \times \omega_n(P; p_1 p_2 \cdots p_n) \, dp_1 \, dp_2 \cdots dp_n .$$

In what follows, we restrict ourselves to the C.M. system so that $\vec{P} = 0$. Let us change the variables of integration from momenta to energies by:

$$p_i^2 = e_i^2 - m_i^2$$

Our integral becomes:

$$\rho_n^*(E,0) = (4\pi)^n \int_{m_1}^\infty \cdots \int_{m_n}^\infty e_1 \sqrt{e_1^2 - m_1^2} \; e_2 \sqrt{e_2^2 - m_2^2} \cdots \cdots e_n \sqrt{e_n^2 - m_n^2} \; v_n(P=0; e_1 e_2 \cdots e_n) \, \delta\left(E - \sum_{i=1}^{n} e_i\right) de_1 \cdots de_n$$

where:

$$v_n(P=0; e_1 e_2 \cdots e_n) \equiv \omega_n(P=0; p_1 = \sqrt{e_1^2 + m_1^2} \cdots p_n) .$$

Let: $\mu_i(e) = e\sqrt{e^2 - m_i^2} \quad e \geq m_i$

$$\rho_n^*(E,0) = (4\pi)^n \int_{m_1}^\infty \cdots \int_{m_n}^\infty \mu_1(e_1) \, \mu_2(e_2) \cdots \mu_n(e_n) \times v_n(0; e_1 \cdots e_n) \, \delta\left(E - \sum_i e_i\right) de_1 \, de_2 \cdots de_n .$$

A transformation to kinetic energies is made:

$$t_i = \epsilon_i - m_i .$$

Let: $\Phi(t_1 t_2 \cdots t_n) = \mu_1(t_1+m_1)\,\mu_2(t_2+m_2)\cdots\mu_n(t_n+m_n)$

$$\therefore\quad \rho_n^*(E,0) = (4\pi)^n \int_0^\infty \cdots \int_0^\infty \Phi(t_1 \cdots t_n)\,\delta(T - \Sigma t_i)\; dt_1 \cdots dt_n$$

We now order the kinetic energies:

$$\begin{aligned} T_1 &= t_1 \\ T_2 &= t_1 + t_2 \\ &\vdots \\ T_n &= t_1 + t_2 + \cdots + t_n \end{aligned}$$

The inverse transformation is:

$$\begin{aligned} t_1 &= T_1 \\ t_2 &= T_2 - T_1 \\ &\vdots \\ t_n &= T_n - T_{n-1} \end{aligned}$$

$$\therefore\quad \rho_n^* = (4\pi)^n \left\{ \int_0^\infty dT_1 \int_{T_1}^\infty dT_2 \cdots \int_{T_{n-1}}^\infty dT_n \right.$$

$$\left. \times \Phi(T_1, T_2 - T_1 \cdots T_n - T_{n-1})\,\delta(T - T_n) \right\}$$

where, it is clear that

$$0 \le T_1 \le T_2 \le \cdots \le T_{n-1} \le T_n .$$

We carry out the integration over dT_n making use of the δ function. The above relation then applies that the upper limits on all the integrals may be changed to T.

$$\rho_n^* = (4\pi)^n \int_0^T dT_1 \int_{T_1}^T dT_2 \cdots \int_{T_{n-2}}^T dT_{n-1}\; \Phi(T_1, T_2 - T_1, \cdots T - T_{n-1})$$

and we have: $0 \leq T_1 \leq T_2 \leq \cdots \leq T_{n-1} \leq T$.

We are now ready to apply simple Monte Carlo theory to the integral above. For integrals of the form:

$$I = \int_{a_1}^{b_1} dT_1 \int_{a_2}^{b_2} dT_2 \cdots \int_{a_n}^{b_n} dT_n \; f(T_1 T_2 \cdots T_n)\, \rho(T_1 T_2 \cdots T_n)$$

where $\rho(T_1 T_2 \cdots T_n)$ is an n dimensional density function defined, in the usual way, by:

1) $\rho(T_1 \cdots T_n) \geq 0$ over the allowed range of $(T_1 T_2 \cdots T_n)$

2) $\int_{a_1}^{b_1} dT_1 \int_{a_2}^{b_2} dT_2 \cdots \int_{a_n}^{b_n} dT_n \; \rho(T_1 \cdots T_n) = 1$

and $f(T_1, T_2, \ldots T_n)$ is a weight function.[35]

It can be shown that,

$$I = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f\left(T_1^{(i)}, T_2^{(i)}, \ldots T_n^{(i)}\right)$$

where the set $(T_1 \cdots T_n)$ is chosen randomly but weighted according to the distribution function $\rho(T_1 T_2 \cdots T_n)$.

We may rewrite our integral as:

$$\rho_n^*(E, 0) = (4\pi)^n \frac{T^{n-1}}{(n-1)!} \left[\int_0^T dT_1 \int_{T_1}^T dT_2 \cdots \int_{T_{n-2}}^T dT_{n-1} \right.$$

$$\left. \times \; \mathbb{P}(T_1, T_2 - T_1, \ldots T - T_{n-1}) \; \frac{(n-1)!}{T^{n-1}} \right]$$

we make the correspondence:

$$\rho(T_1 T_2 \cdots T_{n-1}) = \text{Constant} = \frac{(n-1)!}{T^{n-1}}$$

This represents a uniform distribution in $(n-1)$ dimensions where the intervals for each variable are chosen as:

$[0, T]$ for T_1

$[T_1, T]$ for T_2

$\vdots$

$[T_{n-2}, T]$ for T_{n-1}

It is clear that the variables satisfy the relation:

$$0 \leq T_1 \leq T_2 \leq \cdots \leq T_{n-1} \leq T.$$

We must show that ρ ,integrated over the limits of our integral, yields unity.

$$\int\cdots\int \rho\,(T_1 \cdots T_{n-1})\,dT_1 \cdots dT_{n-1} = \int_0^T dT_1 \int_{T_1}^T dT_2 \cdots \int_{T_{n-2}}^T dT_{n-1} \quad \frac{(n-1)!}{T^{n-1}}$$

If we allow all the lower limits to be replaced by 0 , we introduce a factor $\frac{1}{(n-1)!}$ into the integral above. (It is clear that a set chosen randomly has a probability $\frac{1}{(n-1)!}$ of being in increasing order). Therefore, the integral over the density function is equal to:

$$\int\cdots\int \rho\,dT_1 \cdots dT_{n-1} = \int_0^T dT_1 \int_0^T dT_2 \cdots \int_0^T dT_{n-1}\,\frac{1}{T^{n-1}} = 1.$$

Hence, simple Monte Carlo calculations give

$$\rho_n^*(E,0) = (4\pi)^n \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \Phi\left(T_1^{(i)},\ T_2^{(i)} - T_1^{(i)},\ \cdots T - T_{n-1}^{(i)}\right)$$

where the set $(T_1 \cdots T_{n-1})$ is generated by choosing $n-1$ random numbers uniformly from the interval $[0,T]$ and relabeling them so that:

$$T_1^{(i)} \leq T_2^{(i)} \leq \cdots \leq T_{n-1}^{(i)}$$

In order to actually calculate the phase space integrals, it is convenient to express all quantities in terms of the individual energies. We have:

$$\Phi\left(T_1, T_2 - T_1, \cdots T - T_{n-1}\right) = e_1\sqrt{e_1^2 - m_1^2}\cdots e_n\sqrt{e_n^2 - m_n^2}\ \upsilon_n(0; e_1 \cdots e_n).$$

We recall that the function υ_n is evaluated using the

theory of the random walk. Refering to I, the result is:

$$v_n = -\frac{1}{(n-3)!\,\pi\, 2^{n+1}\sqrt{e_1^2-m_1^2}\cdots\sqrt{e_n^2-m_n^2}} \times \left\{ \sum_{\substack{\sigma_2\sigma_3\cdots\sigma_n=\pm 1 \\ \sigma_1=1}} \sigma_1\sigma_2\cdots\sigma_n \left[S_q\left(\sum_{i=1}^{n}\sigma_i\sqrt{e_i^2-m_i^2}\right)\right]\left(\sum_{i=1}^{n}\sigma_i\sqrt{e_i^2-m_i^2}\right)^{n-3}\right\}$$

where: all σ_i (except $\sigma_1 = +1$) take on the values 1 and -1.

$$S_q(x) = \begin{cases} +1 & \text{for } x>0 \\ -1 & \text{for } x\le 0. \end{cases}$$

Hence, we get:

$$f_n^*(E,0) = \frac{(E-M)^{n-1}(2\pi)^{n-1}}{(n-1)!\,(n-3)!} \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\psi_n\left(e_1^{(i)}e_2^{(i)}\cdots e_n^{(i)}\right)$$

where

$$\psi_n(e_1e_2\cdots e_n) = -e_1e_2\cdots e_n\left\{\sum_{\sigma_2\sigma_3\cdots\sigma_n}\sigma_1\sigma_2\cdots\sigma_n\, S_q\left(\sum_i\sigma_i\sqrt{e_i^2-m_i^2}\right) \times\left[\sum_i\sigma_i\sqrt{e_i^2-m_i^2}\right]^{n-3}\right\}$$

The method of calculation is:

1) Generate a set of kinetic energy variables $\{T\}$ by obtaining $n-1$ random numbers in the interval 0 to T.

2) From these calculate the individual energies $e_i = T_i - T_{i-1} + m_i \quad i = 1, n \quad (T_0 = 0,\ T_n = T)$.

3) Knowing the energies and masses, the function $\psi_n(e_1e_2\cdots e_n)$ is calculated.

4) Steps 1-3 are done N times, and the phase space integral is obtained by multiplying the average

value of these by the factor $\frac{(E-M)^{n-1}(2\pi)^{n-1}}{(n-1)!\,(n-3)!}$.

We now proceed with a brief outline of the error analysis.

It is clear that, as one increases the value of N, the value calculated for the integral approaches the true value. The error analysis has been done in reference I and is quite lengthy. The method used for estimating the error is briefly outlined below.

Before starting the calculation, one chooses an integer N_0 which is much greater than one but small enough so that N_0 contributions are not expected to give a good approximation to the integral. We then do the calculation, checking the convergence after successive sets of N_0 terms.

Let us consider a calculation done in λ steps of N_0 contributions each, so that $N = \lambda N_0$. We define the partial sums as follows:

$$S_\mu = \sum_{i=(\mu-1)N_0+1}^{\mu N_0} \psi(e_1^{(i)} e_2^{(i)} \cdots e_n^{(i)})$$

where $\mu = 1, 2, \cdots \lambda$.

We then have:

$$\rho_\lambda = \frac{A}{\lambda} \sum_{\mu=1}^{\lambda} S_\mu \qquad \rho_{EXACT} = \lim_{\lambda \to \infty} \rho_\lambda$$

where: $$A = \frac{(E-M)^{n-1}(2\pi)^{n-1}}{(n-1)!\,(n-3)!\,N_0}.$$

We want to approximate the integral by some value ρ_λ calculated with $\lambda >> 1$ so that $N = \lambda N_0$ contributions yields a good approximation to the true value ρ .

At this point a problem is encountered that is a consequence of the Monte Carlo method. We want to approximate the exact phase space integral by the average of λ partial sums ρ_λ and hope to obtain some information about the error in ρ_λ . The straight-forward approach to this is to calculate ρ_λ many times independently and from these values calculate an RMS error. Yet we calculate ρ_λ but one time and do not wish to do so again; indeed, if we did so we would use the value 2λ as a new λ' and get a more accurate approximation $\rho_{\lambda'}$. Hence, we work toward obtaining an estimate of the (probable) error in without having to calculate it more than once for a given value of λ .

We assume that for fixed, large enough λ , the approximate values ρ_λ are normally distributed about the exact value ρ . The error analysis yields the result that the standard deviation $(\delta\rho)_\lambda$ divided by the mean value ρ is:

$$\frac{(\delta\rho)_\lambda}{\rho} = \frac{\overline{(\rho_\lambda - \rho)^2}}{\rho} \approx \left\{ \frac{1}{\lambda} \left(\frac{1}{\bar{S}^{(\lambda)}} \right)^2 \frac{\sum_{\mu=2}^{\lambda} \left(\bar{S}^{(\mu)} - S_\mu \right)^2}{\sum_{\mu=2}^{\lambda} \frac{\mu-1}{\mu}} \right\}^{1/2}$$

where $\bar{S}^{(k)} = \frac{1}{k} \sum_{\mu=1}^{k} S_\mu .$

We note that the expression above is easily calculated, along with ρ_λ , for successive values of λ . It is also expected that, as λ increases, the standard deviation $(\delta\rho)_\lambda$ and therefore $\frac{(\delta\rho)_\lambda}{\rho}$ will decrease.

We now make use of the following property of the Gaussian distribution. If ρ_λ is normally distributed about ρ with a standard deviation $(\delta\rho)_\lambda$ then, for an arbitrary value ρ_λ , the probability that:

$$|\rho - \rho_\lambda| \leq (\delta\rho)_\lambda \text{ is } \approx .68$$

$$|\rho - \rho_\lambda| \leq 2(\delta\rho)_\lambda \text{ is } \approx .95$$

$$|\rho - \rho_\lambda| \leq 3(\delta\rho)_\lambda \text{ is } \approx .9995$$

We can now arrive at a solution to the problem above. The particular value we calculate for ρ_λ is assumed to be one of a set of values $\{\rho_\lambda\}$ which are normally distributed about the true value ρ . Using the formula above, we calculate the approximate $(\delta\rho)_\lambda / \rho$ and use this to obtain limits on the relative error in our calculated value ρ_λ . Let us define:

$$z_\lambda \equiv \left| \frac{(\delta\rho)_\lambda}{\rho} \right|$$

then:

$$\frac{|\rho_\lambda - \rho|}{\rho} \leq Z_\lambda \quad \text{with} \approx 68\% \text{ certainty}$$

$$\frac{|\rho_\lambda - \rho|}{\rho} \leq 2Z_\lambda \quad \text{with} \approx 95\% \text{ certainty}$$

$$\frac{|\rho_\lambda - \rho|}{\rho} \leq 3Z_\lambda \quad \text{with} \approx 99.95\% \text{ certainty}$$

In our calculation, we chose $N_0 = 10$ and varied λ in increments of 1 starting with $\lambda = 5$ until the value of Z_λ became $\leq .06$. Hence, with $N = \lambda N_0$ terms, we obtained phase space integrals with a relative error less than 6% with 68% certainty or less than 12% with 95% certainty.

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