

ELASTO-PLASTIC INCREMENTAL APPROACH
IN SLOPE STABILITY ANALYSIS

Thesis for the Degree of Ph. D.
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VISHARN POOPATH
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ABSTRACT

ELASTO-PLASTIC INCREMENTAL APPROACH IN SLOPE STABILITY ANALYSIS

By

Visharn Poopath

The formation of failure surfaces induced by incrementally increasing loads on soil slope embankments was studied through the use of the finite element method. The soil was assumed to behave as an ideal elastic, perfectly plastic and isotropic material. The effects of strain hardening and nonhomogeneity were considered.

Two embankment building processes, single-lift construction and incremental construction, were simulated by the superposition method. Loading was considered to result from an incrementally increasing soil unit weight for the single-lift construction, and from the application of surface forces to simulate a gradually increasing soil thickness for the incremental construction. The investigations included slopes completely within the elastic range, and slopes in the elasto-plastic range up to and including limiting equilibrium or failure conditions. Plastic deformations were assumed to be contained up until complete failure, and no analysis was considered beyond this point.

Very small values of modulus of elasticity were applied in the contained plastic regions during increments of load. Location and configuration of potential failure surfaces were determined by the slip-line method and compared with results of plasticity analyses by the method of Bishop (1954). Stability was also determined by an average factor of safety calculated from the factor of safety field in the elasto-plastic stages. These results are compared with the factor of safety calculated by the Bishop method.

It was found that for homogeneous soil slopes, the elasticity and elasto-plasticity analyses are in agreement with the Bishop analysis in determining the location and configuration of potential failure surfaces. This indicates that geometrical changes of the slope body due to applied loads, during the stages prior to failure are not large enough to cause appreciable error in the plasticity analysis at the final stage. The elasticity analysis provides an indication of the maximum height of slope at which first local yield occurs, the location of first yield, the area where tensile stresses develop, and the required strength characteristics of the soil to prevent slope failure.

For nonhomogeneous soil slopes, the elasto-plasticity analysis was found to be useful in studying failure behavior. The potential failure surfaces predicted from elasto-plastic stress fields differed considerably from potential failure surfaces predicted by Bishop's method for nonhomogeneous soil slopes.

Factors of safety calculated from elastic stress fields, elasto-plastic stress fields and by Bishop's method all were in good agreement.

ELASTO-PLASTIC INCREMENTAL APPROACH
IN SLOPE STABILITY ANALYSIS

By

Visharn Poopath

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DEDICATION

To the memory of my father.

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NOTATIONS

A, B	pore-pressure parameters
$\bar{B}$	a soil parameter
$b, dx, \Delta x$	width of slice
C_{ijkl}	fourth order tensor of elastic constants
$c, c_o, c', c_D, c_U, c_{CU}$	cohesion of soil; (prime(') denotes effective stress, D=drained, U=undrained, CU=consolidated-undrained)
C_w	compressibility of fluid
C_c	compressibility of soil skeleton
dN'	effective normal force acting on base of slice
dP_b	neutral normal force acting on base of slice
e	void ratio or volumetric strain
E	modulus of elasticity, normal force at interface of slices or internal energy
E'	effective normal force at interface of slices
E_{ii}	summation of principal strains
E_e	slope of stress-strain curve in elastic range
E_p	slope of stress-strain curve in plastic range
F	factor of safety or body force
G	specific gravity of soil skeleton
H	total height of slope
h, z	depth or height of slice

$[K]$	stiffness matrix
K	bulk modulus
$l, \Delta L$	length of slice base
n	porosity of soil
N, P	normal force acting on base of slice
p	hydrostatic stress or mean pressure
P_w	neutral normal force at interface of slices
Q	horizontal external force
q	pressure distribution along rupture surface
$\{r\}$	vector of nodal point displacements
$\{R\}$	vector of nodal point forces
R, R_o	radius of circular rupture surface
r_u	pore-pressure ratio
s	distance along rupture surface or length of slope surface
S	shear force acting on base of slice
$[S]$	stress transformation matrix
$T_{ij}, E_{ij}, \delta_{ij}, E'_{ij}$	second order tensor
u	pore-pressure or neutral stress
V	volume or velocity
$W, dW, \Delta W$	total weight of a slice or external work-done
X, T	shear force at interface of slices
$\{X\}$	a column matrix of body forces
x, y, z	rectangular coordinates

α	angle of inclination, angle of slip-line with major principal stress direction, angle at base of slice or angle of curvature
α_t	angle of line of thrust with horizon
β, i	angle of slip-line with minor principal stress direction (β) or angle of slope
γ, ω	unit weight of soil
γ'	unit weight of soil submerged in water
γ_w	unit weight of water
δ_{ij}	Kronecker delta
$\{\delta u\}$	a column matrix of incremental of deformations
$\{\delta \epsilon\}$	a column matrix of increment of unit deformation components
$\epsilon_x, \epsilon_y, \epsilon_z$	unit deformation in x, y and z directions
ϵ_e	unit deformation in elastic range
ϵ_p	unit deformation in plastic range
η, ξ	function of transformation
θ	angle of major principal stress direction with horizon
λ, μ	Lame's elastic constants
ν	Poisson's ratio
ρ	mass density of soil ($\frac{\gamma}{g}$)
$\sigma, \sigma_n, n, \sigma'$	normal stress
σ_1, σ'_1	major principal stress
σ_3, σ'_3	minor principal stress
σ_x, σ'_x	normal stress in x-direction
σ_y, σ'_y	normal stress in y-direction

prime (')
denotes
effective
stress

σ'_o	over burden pressure
σ'_c	preconsolidation pressure
$\{\sigma\}$	a column matrix of stress components
σ_o, τ_o	octahedral stress components
τ, t	shear stress
τ_{xy}	shear stress in x-y coordinates
τ_f, S	shear strength or shear stress at failure
$\phi, \phi', \phi_D, \phi_{CU}, \phi_{CD}$	angle of internal of friction (prime(') denotes effective stress, D=drained, CU=consolidated-undrained, CD=consolidated- drained)
$\Phi(\dots)$	function
Δ, d, δ	change; increment or decrement
Σ	summation

The Greek Alphabet

α	Alpha	ν	Nu
β	Beta	π	Pi
γ	Gamma	ρ	Rho
δ	Delta	σ	Sigma
ϵ	Epsilon	τ	Tau
ξ	Xi	ϕ	Phi
η	Eta	χ	Chi
θ	Theta	ψ	Psi
λ	Lambda	ω	Omega
μ	Mu		

CHAPTER I

INTRODUCTION

Civil engineers commonly solve problems in continuum mechanics by one of two assumptions concerning the material properties of the medium: the material is elastic (normally linearly elastic) or the material is rigid plastic. The theory of elasticity, the first assumption, is normally used to investigate the behavior of the material under loads which cause stresses below the yield stress. Elasticity solutions can provide a partial answer to the question, how strong is the body? in the form of a factor of safety or ratio of maximum safe load to the applied load. Plasticity theory, the second assumption, can be used to calculate the maximum safe load by limiting equilibrium methods.

For simple problems, such as uniaxial tension, elastic and plastic parts of deformation can be separated by a critical yield point. More complicated problems such that stress states at critical points do not reach the yield condition at the same time, fall into the elasto-plastic category. These problems can be reliably solved

by plasticity theory only if the material has very small elastic deformations in comparison with the plastic parts.

Currently the stability of slopes is most often considered in the realm of plasticity. A complete solution of stresses and strains in a mass of soil under arbitrary loading as the load is increased to failure is almost always neglected. Plastic limit theorems (Drucker, Prager and Greenburg, 1952) permit lower and upper bounds on the failure load to be found conveniently when the soil mass is idealized as perfectly plastic. Liam Finn (1967), Chen (1969), Lysmer (1970), and others, have extended the applications of the theory of plasticity to soil mechanics problems by introducing different forms of statically admissible stress fields for the lower bound solution and some forms of velocity fields as required for the application of the upper bound theorem. The maximum safe load at limit equilibrium (which is actually the upper-bound load) can also be predicted by the slip-line method (Sokolovski, 1965). To obtain the stability of slopes, equilibrium methods (Bishop, 1954; Morgenstern, 1965; Janbu, 1954) under an assumption of simultaneous failure along any complete surface are generally used to estimate an average factor of safety. The factor of safety is the ratio of resisting ability to the static stresses of the points along the assumed sliding surface.

Plasticity analyses do not simulate the actual behavior of slopes except at failure, when the factor of safety is equal to unity. For slopes having large factors of safety, equilibrium analyses may often provide a valid indication of stability, but they do not provide information concerning the deformations of slope, nor are they capable of indicating which zones around the slope are most critically stressed. These methods can be used to predict the location of the critical failure surface in slopes only when the soil is homogeneous throughout. In cases of nonhomogeneous soil the methods of prediction are no longer applicable. Some types of soils behave like rigid plastic materials (very small elastic strains), but most soils exhibit a certain range of elastic deformation that may cause a considerable amount of error in the plastic analyses. Elasticity theory is useful for design purposes in finding required soil strength and the locations of the most critical stresses. Elasto-plastic incremental theory is useful in obtaining the true mode of failure, and stress fields in the range between fully elastic, and complete failure can be investigated.

It is the purpose of this thesis to present a method of analysis for stress-strain behavior in the range between fully elastic and complete failure, by considering an incrementally increasing load. Location and configuration of the critical failure surface is then predicted by the

slip-line field obtained from the stress field at full failure. Behavior of elasto-plastic stress fields during increasing load can be observed. These results are compared with the classical equilibrium method of Bishop (1954).

Because of the difficulty in obtaining exact solutions of elasto-plasticity problems, especially for very complicated shapes and loading systems, numerical methods (finite element) are used to calculate the stress fields at different stages. The soil is assumed to have a piece-wise linear stress-strain relationship. Calculation in the elasto-plastic incremental range is done by superposition.

This study is limited to the investigation of the stability of slope embankments where loading is assumed to result from gradually increasing body forces or from gradually increasing surface loads simulating embankment construction. Loading is assumed to be performed within a short period of time so that the strength criterion can reasonably be represented by the undrained strength envelope (for saturated soils), but the failure surface is predicted by a graphical method of slip-line construction under the effective angle of internal friction concept.

CHAPTER II

REVIEW OF CURRENT METHODS OF STABILITY ANALYSIS

Many methods of stability analysis have been proposed to investigate the characteristics of failure surfaces and the factor of safety against sliding along these critical failure surfaces. Good agreement of results between natural failures and calculated failures has been obtained only for homogeneous and isotropic soils. Various types of trial failure surfaces have been investigated, but the latest acceptable method for homogeneous soil introduced by Bishop (1954) is analyzed by a cylindrical failure surface. However, it is apparent that because of the departures from homogeneity, the soil mass may slip along non-circular surfaces. In this case the most critical failure surface can be obtained later by trial and error using as a guide the circular failure surface from Bishop's method. For non-homogeneous soil the factor of safety can be obtained for individual slip surfaces (Morgenstern, 1965; Janbu, 1954). An advanced method to investigate the true failure surface in non-homogeneous soils has not been accomplished. Each individual case must be investigated separately, and very few general rules have been laid.

The same criterion of failure is applied in all of the methods mentioned above. The shearing strength as expressed by Coulomb's equation is used for the limit of equilibrium, and two-dimensional, plane strain is assumed. Most of the methods assume a constant cohesion and angle of shear resistance along the failure surface. The shear strength will vary only because of varying normal stress along the surface under consideration. Review of these methods is presented in this chapter.

2.1 The Culmann Method

Culmann (1866) assumed that rupture would occur along a plane surface. This assumption was the first rational one for the failure surface of a cut slope with a lateral retaining wall. For an ideal frictionless retaining wall, no shearing resistance develops along the interface of the soil and the wall (Figure 2-1a). The stress distribution within the soil mass will be the same as that in a semi-infinite soil mass under the influence of gravity (Figure 2-1b). The orientations of the principal stresses are self-parallel, and this condition is also true along the failure surfaces.

When the retaining wall is removed, the orientations of the principal stresses change around the cut area. Distortion of the failure surface will also occur. In practical work it is observed that the failure surface

distorts as shown on Figure 2-2. The radius of curvature is minimum at the toe and increases along the failure line to a maximum at the top surface.

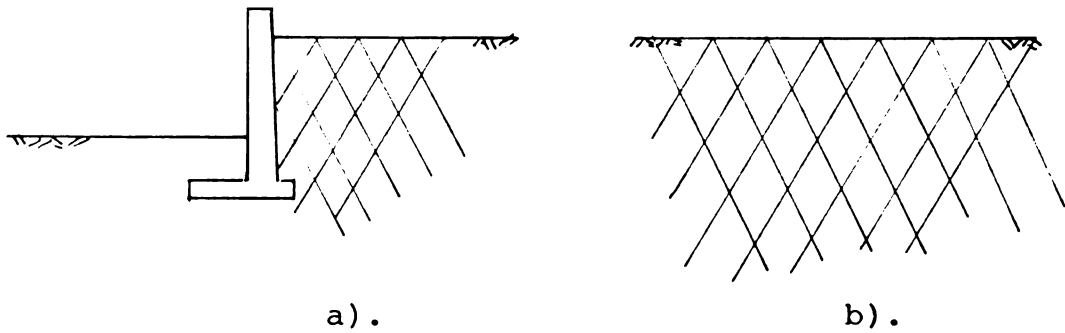


Figure 2-1.--Orientation of Slip-lines for a Slope with Lateral Restraint.

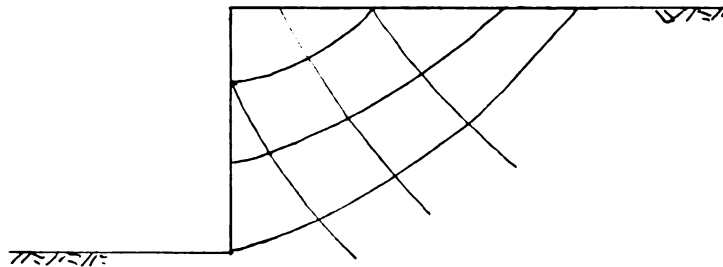


Figure 2-2.--Orientation of Slip-lines for a Slope Without Lateral Restraint.

2.2 Point to Point Stress Analysis Method

This method was investigated by two investigators: Brahtz (May and Brahtz, 1936), and Bennett (1951). The method

consists of investigating the state of stress at every point of interest in the slope vicinity. Brahtz developed a stress function in the form of Airy's stress function similar to those used in plane stress problems of the theory of elasticity. He introduced a comparative factor $K = 50/\gamma$, where the unit weight of saturated soil, γ , is in pounds per cubic foot. The condition of plasticity is given by

$$\tau = c + \sigma \tan \phi \quad (2-1)$$

where τ = available shear strength, c = cohesion of the soil, σ = normal stress at the shearing surface, and ϕ = angle of shearing resistance.

The safety factor is defined by the ratio of the available shearing strength to the shearing stress at each point of the potential zone of failure. This point to point stress analysis aims at finding the zone of failure, where the limiting equilibrium of the soil is reached or exceeded, characterized by lines of equal factor of safety against flow or failure. The factors of safety in this analysis are point functions and cannot be compared with the mean value of safety against total rupture along a continuous sliding surface.

In this analysis Brahtz assumed a horizontal soil layer (Figure 2-3a), so that the stresses at any depth z are given by

$$\sigma_z = \gamma z, \sigma_x = 0, \sigma_{xz} = 0 \quad (2-2)$$

where σ_z = vertical normal stress, σ_x = horizontal normal stress, and σ_{xz} = shearing stress on vertical and horizontal planes. Brahtz then removed the lateral soil to form a trapizoidal shaped dam (Figure 2-3b). He then attempted to correct the horizontal stress (σ_x) by assuming a stress function, which does not affect the basic vertical and shearing stresses, superimposed upon the basic stress system of equation (2-2). The resulting corrected system was thus obtained for the trapezoidal soil mass. His analysis was intended primarily for the purpose of dam design, and effects of pore pressure were also included for different situations of reservoir water.

Finally an empirical slope formula was developed as a guide in choosing the slope of a dam on both surfaces as follows:

$$\text{upstream slope, } S_1 = 275/(2c+h\gamma\tan\phi) \quad (2-3)$$

$$\text{downstream slope, } S_2 = (275h^2 - 1.7B(c+H\gamma\tan\phi)) / (H(2c+H\gamma\tan\phi)) \quad (2-4)$$

where h = depth of reservoir, H = total height of dam, B = width of crown, c = cohesion of soil, ϕ = angle of internal friction of soil, and γ = unit weight of saturated material.

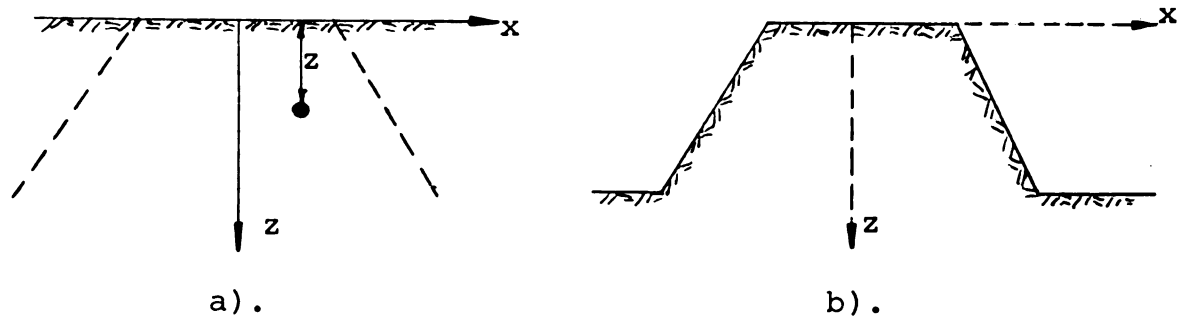


Figure 2-3.--Excavated Slope.

Bennett (1951) advocated point-to-point stress analysis without using stress functions but using the superposition method of the elastic stress distribution laws up to the limit of elastic equilibrium. He stated that the purpose of stability analysis is not to investigate the elastic deformations, but to explore the possibilities of a shear failure. The inelastic behavior of the soil will not have much effect on this exploration, but it may be unwise to ignore those features of stress analysis which are so particularly well adapted to study the borderline between elastic and plastic conditions. The study of probable magnitude and orientation of principal stresses, and direction of potential slip planes may be profitable.

He presented the result as a sketch of a system of orthogonal curves approximating the trajectories of the principal stresses (Figure 2-4). The potential surfaces of rupture are also drawn, making a constant angle with

the directions of principal stresses. He stated that a quantitative determination of the exact stresses within the soil slope is very difficult, and the only accurate current method was photoelasticity.

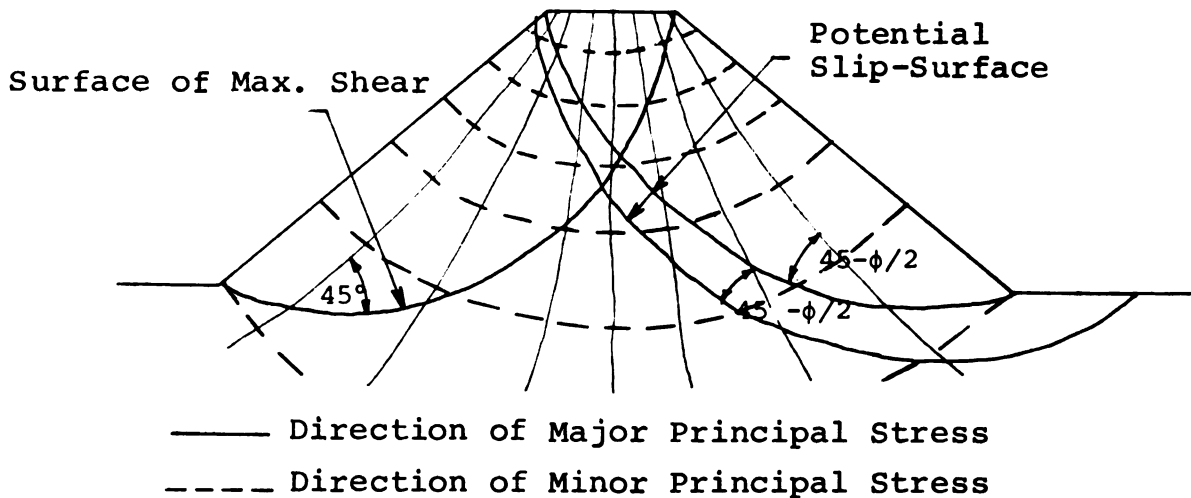


Figure 2-4.--Trajectories of Principal Stresses.

To investigate the stability of a slope, the points along a potential slip plane or within a zone of known weakness such as an unconsolidated clay layer in the foundation are examined. To account for the degree of consolidation the pore pressure is estimated from consolidation theory, and is subtracted from the total normal stress acting on any plane through the point. The shearing stress on the plane is assumed to be unaffected by the neutral stresses.

Bennett concluded in his paper that this method is a quick rough stress analysis at one or two points in

the critical zone, and it is therefore useful in preliminary layout of slopes. Finally, he recommended that the point-to-point stress analysis, as well as the sliding surface procedure, may be used in the analysis of any important embankment on a weak foundation and that the results be given equal weight in a final judgement of the safety factor.

2.3 Circular Arc Methods

The circular arc method was first proposed by K. E. Petterson (1915). The justification that circular arcs are close approximations of actual rupture surfaces comes from field investigations of a large number of actual slides. An infinite number of arcs may be drawn for a given slope. The critical arc, the one used in analysis, is defined as the one which is least stable. The method of circular arc analysis has been widely used and accepted as satisfactory. This method has undergone many refinements by many authors. A few of the most prominent methods will be presented.

2.3.1 ϕ -Circle Method

This method was developed by Professor Glennon Gilboy and Arthur Casagrande (Taylor, 1937). It was initially proposed in terms of a complete graphical solution of the slope problem. For any circle analyzed,

the method provides the required soil strength to obtain equilibrium of the free body diagram. They found that one graphical layout for any given slope and any given angle of internal friction, could represent various heights of slope merely by the use of different scales. Finally Taylor (1937) introduced a dimensionless parameter, $c/(FwH)$, called the "stability number," (c = actual unit cohesion for the soil, F = factor of safety, H = height of slope, and w = unit weight of the soil), that is used to describe the stability of slopes.

Figure 2-5 shows an arbitrary circular sliding surface, AB, through the toe of the slope. Possible circular arcs are of three configurations: circles passing through the toe of the slope, circles passing below the toe, and circles passing above the toe. Taylor developed only the first two cases. The problem is considered as two dimensional with unit thickness extended in the third dimension. Shearing resistance on both faces of the third dimension is neglected. It is assumed that the soil mass is in equilibrium and just begins to slide along the circular arc.

$$c_1 La = c_1 \bar{L} R \text{ or } a = \frac{R \bar{L}}{L} \quad (2-5)$$

where $\bar{L}$ is the length of arc AB. Thus the line of action of C may be found. P is the resultant force transmitted from grain to grain of the soil across the arc AB.

The intersection of W and C is known. Since the three forces, W, C and P, must be concurrent, P must also pass through this intersection. With W known in magnitude and direction and the direction of C also known, the force polygon may be constructed if a second point can be obtained on P to determine its direction. This is accomplished by the basic assumption of the ϕ -Circle Method.

P is made up of small elementary forces, such as p (Figure 2-5a) which, at failure, must act at an obliquity of ϕ to the arc AB. If the line of action of p is extended and a circle is drawn with center at O and tangent to the extended line, it will be seen that the p force for any other element of the arc AB will also be tangent to this circle. This circle is called the ϕ -circle. Any two such p forces will intersect just outside of the ϕ -circle so their resultant would pass just outside this circle, and by the same reasoning P must pass outside. The assumption introduced is that P is tangent to the ϕ -circle.

Mathematical derivation of the stability number is referenced to Taylor (1937), and the final results are presented.

Case a (Figure 2-5), circles pass through the toe:

$$\frac{c}{FwH} = \frac{\frac{1}{2} \csc^2 x (y \csc^2 y - \cot y) + \cot x - \cot i}{2 \cot x \cot v + 2} \quad (2-6)$$

(x, y, i and v are shown in Figure 2-5)

Case b (Figure 2-6), circles pass below the toe:

$$\frac{c}{FwH} = \frac{\frac{1}{2} \csc^2 x (y \csc^2 y - \cot y) + \cot x - \cot i - 2n}{2 \cot x \cot v + 2} \quad (2-7)$$

(x, y, i and n are shown on Figure 2-6. v can be obtained in the same fashion as in the first case)

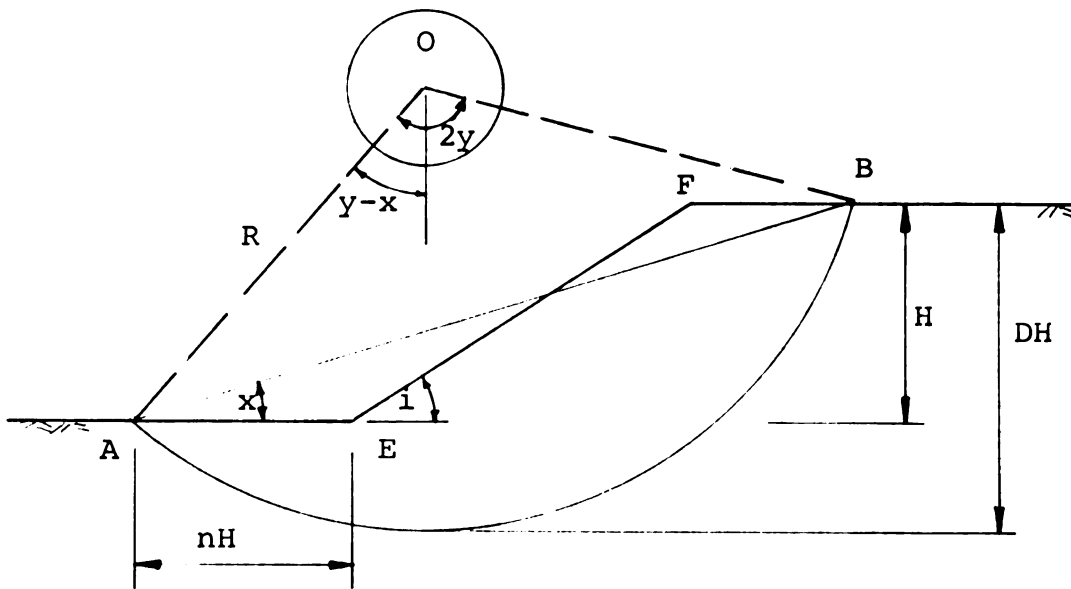


Figure 2-6.--Sketch Showing Elements of the ϕ -Circle Method for Circles Passing Below the Toe of Slope.

2.3.2 Jáký Method

Jáký (1936) proposed an analytical method of slope stability analysis based on the assumption of circular failure surfaces. Compatibility of strains is violated by assuming simultaneous failure along a circular failure surface, but equilibrium of stresses is still applied along the assumed failure surface.

Slopes of all kinds remain stable if they are not made steeper than a certain maximum angle, β . If a slope steeper than β is desired it must be supported by a retaining wall. The forces that keep a slope stable are the passive internal forces, friction and cohesion. Acting against the direction of sliding, these forces prevent movement on the potential surface of rupture. The relationship between the normal (n) and the tangential (t) stresses that act on the surface of rupture (Figure 2-7) may be expressed by the well known Coulomb formula

$$t = n \tan \phi + c \quad (2-8)$$

where ϕ is the angle of internal friction of the soil, and c is shearing resistance or cohesion.

Fr. Kötter (1903) was the first to derive a formula of the sliding surface for cohesionless materials. According to Kötter the following relationship exists between the pressure, q, and radius, $ds/d\alpha$, of the curved sliding surface

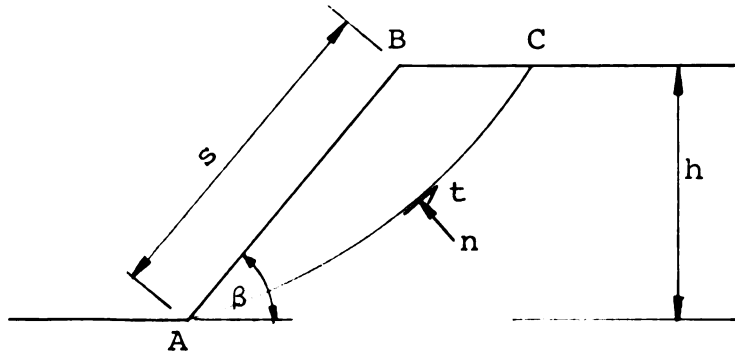


Figure 2-7.--Slope and Failure Configurations.

$$\frac{dq}{ds} - 2q \tan \phi = \frac{ds}{d\alpha} \sin(\alpha - \phi) \quad (2-9)$$

where α is the angle the sliding element makes with the horizontal, and ds is the length of the element (Figure 2-8).

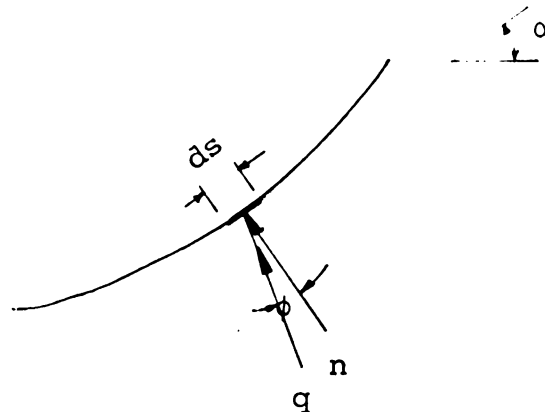


Figure 2-8.--Component of Forces on an Element of Sliding Surface.

If, in equation 2-9 we substitute

$$q = t/\sin\phi \quad \text{and}$$

$$\frac{dq}{d\alpha} = \frac{1}{\sin\phi} \frac{dt}{d\alpha} \quad \text{then}$$

$$\frac{dt}{d\alpha} - 2t \tan\phi = \frac{ds}{d\alpha} \gamma \sin\phi \sin(\alpha-\phi) \quad (2-10)$$

where t is the shearing stress distribution along the sliding surface.

As derived, this formula is valid only for cohesionless soils. In the following (Jáky, 1936) it will be shown that the Kötter formula is valid also for cohesive soils. Taking into consideration the most frequent practical case, when the terrain is horizontal, the equilibrium of differential forces on a small prismatical element may be given by the Cauchy equations (Figure 2-9).

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = \gamma \quad (2-11)$$

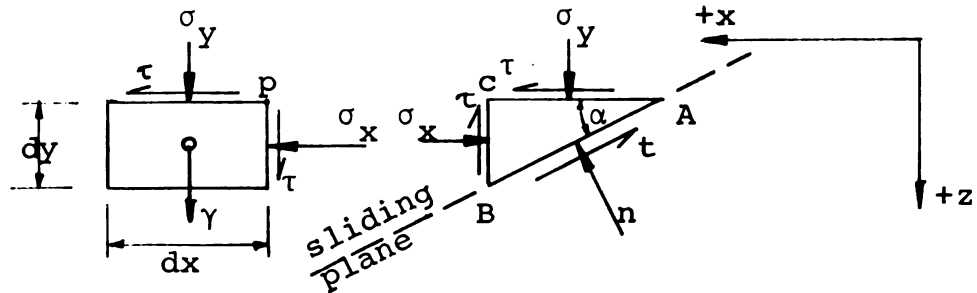


Figure 2-9.--Stress Components on an Elementary Prism; and Equilibrium of an Elementary Prism-force System at any Point on the Sliding Surface.

Equilibrium of forces on the sides of a small triangle that has one side on the sliding surface, to the forces on the other two sides, may be expressed with the formulae

$$t = \frac{\sigma_y - \sigma_x}{2} \sin 2\alpha + \tau \cos 2\alpha \quad (2-12)$$

$$n = \frac{\sigma_y + \sigma_x}{2} \cos 2\alpha - \tau \sin 2\alpha.$$

These expressions, if solved for σ_x and σ_y produce

$$\sigma_x = \tau \cot \alpha + n - t \cot \alpha \quad (2-13)$$

$$\sigma_y = \tau \tan \alpha + n + t \tan \alpha.$$

These may be substituted for σ_x and σ_y in the Cauchy equations

$$\frac{\partial \tau}{\partial x} \cot \alpha + \frac{\partial \tau}{\partial y} + \frac{\partial n}{\partial x} - \frac{\partial t}{\partial x} \cot \alpha = 0$$

$$\frac{\partial \tau}{\partial y} \tan \alpha + \frac{\partial \tau}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial t}{\partial x} \tan \alpha = \gamma$$

Because τ , the shearing stress, is a function merely of the x and y coordinates, the formula

$$\frac{\partial \tau}{\partial x} \cos \alpha + \frac{\partial \tau}{\partial y} \sin \alpha = \frac{\partial \tau}{\partial s}$$

gives the differential of τ by the ds element of arc.

Thus, the above equations may be changed to

$$\frac{\partial \tau}{\partial s} + \frac{\partial n}{\partial x} \sin \alpha - \frac{\partial t}{\partial x} \cos \alpha = 0 \quad (2-14)$$

$$\frac{\partial \tau}{\partial s} + \frac{\partial n}{\partial y} \cos \alpha + \frac{\partial t}{\partial y} \sin \alpha = \gamma \cos \alpha.$$

These equations remain valid regardless of the laws governing the condition at failure. The shear causing rupture in soils is taken as

$$t = n \tan \phi + c.$$

Therefore rupture will occur in such elements of the earth in which $(t - n \tan \phi)$ is maximum. This expression is a function of x, y , and α . Therefore its maximum will develop where the partial differential quotients by x, y , and α are zero. Thus, on the surface of rupture

$$\frac{\partial t}{\partial x} = \frac{\partial n}{\partial x} \tan \phi$$

$$\frac{\partial t}{\partial y} = \frac{\partial n}{\partial y} \tan \phi \quad (2-15)$$

$$\frac{\partial t}{\partial \alpha} = \frac{\partial n}{\partial \alpha} \tan \phi.$$

Equations (2-15) and (2-12) may be combined to yield

$$(\sigma_y - \sigma_x) \cos 2\alpha - 2\tau \sin 2\alpha = [-(\sigma_y - \sigma_x) \sin 2\alpha - 2\tau \cos 2\alpha] \tan \phi$$

or simply,

$$\tau = \frac{\sigma_y - \sigma_x}{2} \cot(2\alpha - \phi). \quad (2-16)$$

And, using equation (2-12),

$$\tau = t \frac{\cos(2\alpha - \phi)}{\cos\phi} . \quad (2-17)$$

This way we arrive at an expression for the shearing stress (τ) in terms of the sliding force (t). This expression is valid only for the surface of rupture.

A further step will determine the $\frac{\partial t}{\partial x}$ and $\frac{\partial t}{\partial y}$ partial differential quotients. Let us put $\frac{\partial n}{\partial x}$ and $\frac{\partial n}{\partial y}$ of equation (2-15) into equation (2-14). Then simple computation will result in the following:

$$\frac{\partial t}{\partial x} = \frac{\frac{d\tau}{ds} \sin\phi}{\sin(\alpha - \phi)} \frac{dt}{dy} \frac{\gamma \cos\alpha \sin\phi - \frac{d\tau}{ds} \sin\phi}{\cos(\alpha - \phi)} . \quad (2-18)$$

From equation (2-12) we obtain

$$\frac{\partial n}{\partial \alpha} = -2t$$

and from equation (2-8)

$$\frac{\partial t}{\partial \alpha} = \frac{\partial n}{\partial \alpha} \tan\phi .$$

Therefore

$$\frac{\partial t}{\partial \alpha} = -2t \tan\phi . \quad (2-19)$$

The total differential of the sliding stress, t , may be formed from the partial differential quotients, as that of any function with multiple variables

$$\frac{dt}{ds} = \frac{\partial t}{\partial x} \cos \alpha + \frac{\partial t}{\partial y} \sin \alpha + \frac{\partial t}{\partial \alpha} \frac{d\alpha}{ds}.$$

Substituting the respective values for the partial differential quotients, and arranging yields

$$\frac{dt}{ds} - 2t \tan \phi \frac{d\alpha}{ds} = \gamma \sin \phi \sin(\alpha - \phi),$$

and, multiplying by $\frac{ds}{d\alpha}$

$$\frac{dt}{d\alpha} - 2t \tan \phi = \gamma \frac{ds}{d\alpha} \sin \phi \sin(\alpha - \phi). \quad (2-20)$$

The result is wholly Fr. Kötter's formula for cohesionless material. In the above derivation, however, not only friction, but also cohesion has been taken into consideration. Thereby the validity of Kötter's formula is ascertained to be entirely general, and applicable to both cohesive and cohesionless materials.

There are, however, two unknowns in the formula, the sliding stress, t , and the shape of the surface of rupture. Of these two the curved surface of rupture, as pointed out before, is known from observations to be, for all practical purposes, a circular cylinder. This knowledge permits a simple integration of Kötter's differential equation, and the resulting formula will yield the unknown sliding stress, t .

Stresses at point C are fully determined by these conditions, because by combining equations (2-8), (2-12), and (2-17) the following equations, expressing the stress conditions in any point of the sliding surface, will be obtained

$$\begin{aligned}\sigma_y &= t \frac{1+\sin\phi \sin(2\alpha-\phi)}{\sin\phi \cos\phi} - c \cot\phi \\ \sigma_x &= t \frac{1-\sin\phi \sin(2\alpha-\phi)}{\sin\phi \cos\phi} - c \cot\phi \\ \tau &= t \frac{\cos(2\alpha-\phi)}{\cos\phi} .\end{aligned}\tag{2-23}$$

When $\tau = 0$, then $2\alpha - \phi = 90^\circ$. Therefore inclination of the sliding surface at point C is given by $\alpha_0 = 45 + \phi/2$. Because also $\sigma_y = 0$,

$$t_0 = c(1 - \sin\phi) .\tag{2-24}$$

Incorporating these with equation (2-21),

$$\begin{aligned}c(1 - \sin\phi) &= \frac{Ry \sin\phi}{1 + 4 \tan^2 \phi} [\cos(45 - \phi/2) \\ &\quad + 2 \tan\phi \sin(45 - \phi/2)] + K e^{2 \tan\phi (45 + \phi/2)} .\end{aligned}\tag{2-25}$$

Equation (2-25) permits determination of K. Its value, however, is unnecessary for our purposes. Another boundary stress-condition exists at point B

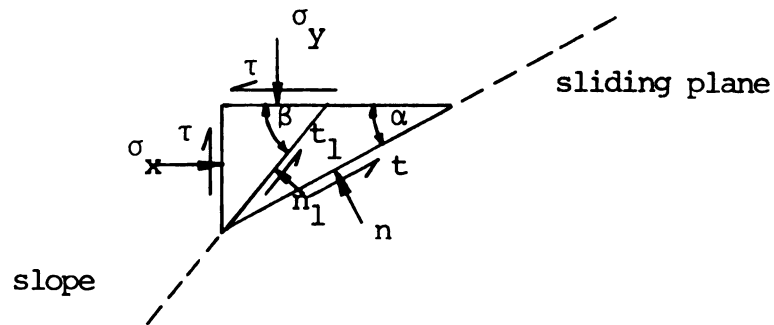


Figure 2-11.--Stress Components on any Plane Surface other than the Sliding Surface.

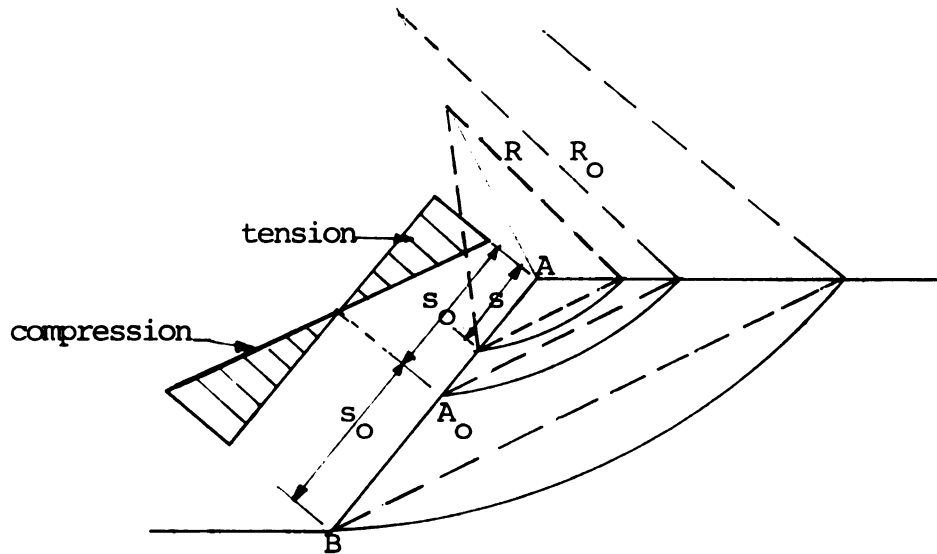


Figure 2-12.--Sketches Showing Tension and Compression Zones.

$$t_1 = 0$$

$$n_1 = 0.$$

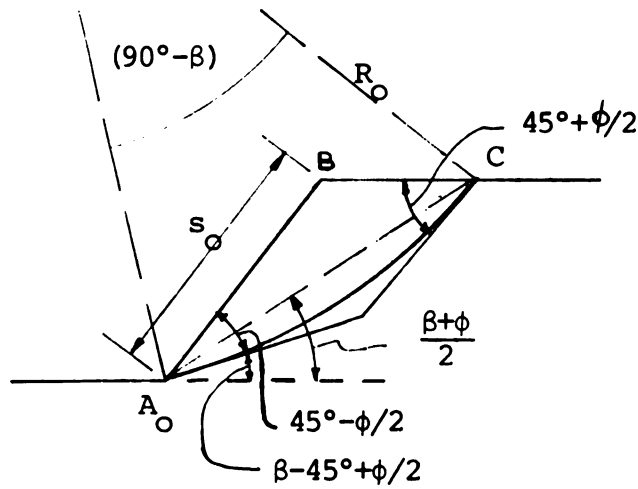


Figure 2-13.--Inclination of Tangent Lines at the Boundary Points.

The t_1 and n_1 stresses that act on a plane inclined at β (Figure 2-11) may be calculated by substituting t_1 , n_1 , and β in equation (2-12) for t , n , and α . Combining equation (2-12) with equation (2-23) yields

$$t_1 = t \frac{\cos(2\beta - 2\alpha + \phi)}{\cos\phi}$$

$$n_1 = t \frac{1 - \sin\phi \sin(2\beta - 2\alpha + \phi)}{\sin\phi \cos\phi} - c \cot\phi. \quad (2-26)$$

At point B, $t_1 = 0$. Therefore $\alpha_1 = \beta - 45 + \phi/2$. Also $n_1 = 0$, and therefore $t_1 = c(1 + \sin\phi)$. These values of α_1 and t_1 put into equation (2-21) give

$$c(1+\sin\phi) = \frac{R\gamma\sin\phi}{1+4\tan^2\phi} [\cos(\beta-45-\phi/2) + 2\tan\phi\sin(\beta-45-\phi/2)] + Ke^{2\tan\phi(\beta-45+\phi/2)} \quad (2-27)$$

Now, by eliminating the constant K from equation (2-25) and (2-27), the radius of the sliding cylinder, R, may be computed

$$R = \frac{c}{\gamma} \frac{1+\sin\phi}{\sin\phi} (1+4\tan^2\phi) \frac{1-\tan^2(45-\phi/2) e^{2\tan\phi(\beta-90^\circ)}}{[\cos(\beta-45-\phi/2) + 2\tan\phi\sin(\beta-45-\phi/2)] - [\cos(45-\phi/2) + 2\tan\phi\sin(45-\phi/2)]}$$

For simplifying this formula let us introduce the following notations:

$$\tan \lambda = 2\tan\phi \quad (2-28)$$

$$\omega = 45^\circ + \phi/2 + \lambda.$$

Angles λ and ω are functions of ϕ only. Therefore, in a particular case they are constants. Thus

$$R_o = \frac{c}{\gamma} \frac{1+\sin\phi}{\sin\phi\cos\lambda} \frac{1-\tan^2(45-\phi/2) e^{\tan\lambda(\beta-90^\circ)}}{\cos(\beta-\omega) - \sin\omega e^{\tan\lambda(\beta-90^\circ)}} \quad (2-29)$$

This R_o is the radius of a cylinder that intercepts the slope at point A_o (Figure 2-12). At A_o there are neither normal nor tangential stresses, but there is the neutral axis of the bent slope. The bending causes tension in the

upper half, A_OA , and compression in the lower half, A_OB . When the algebraic sum of the normal forces equals zero, the slope is in balance. Its length, AB , with a linear distribution of stresses, is equal to $2s_O$ (Figure 2-12). It is of interest to note that the plane of rupture solution predicts a slope subjected to bending moment.

Inclination of the tangent to the failure surface through point A_O is given by $\alpha_1 = \beta - 45 + \phi/2$ (Figure 2-13). Inclination of the tangent through point C is given by $\alpha_O = 45 + \phi/2$. Thus A_OC is inclined at an angle $\frac{\beta+\phi}{2}$. In words, the angle of the chordplane of the cylinder, halves the angle $(\beta - \phi)$ between the slope and the angle of repose. The central angle of the arc with R_O radius is given by $(90^\circ - \beta)$. From Figure 2-13

$$\overline{AC} = 2R_O \sin(45^\circ - \beta/2), \quad \text{and}$$

$$s_O = \frac{\overline{AC} \sin(\frac{\beta+\phi}{2})}{\sin\beta}, \quad \text{therefore}$$

$$s_O = 2R_O \frac{\sin(45^\circ - \beta/2) \sin(\frac{\beta+\phi}{2})}{\sin\beta}. \quad (2-30)$$

The sliding cylinders intercept the slope invariably at a constant angle $\alpha_1 = \beta - 45 + \phi/2$. The tangential stress, t_1 , is everywhere zero, and from equation (2-26)

we obtain $2\beta - 2\alpha_1 - \phi = 90^\circ$. Therefore, R is a linear function of s (equation 2-30). In other words an increase in length of the slope will be followed by a proportional increase in the R radius of the sliding cylinder.

Linear relationships exist between t and R ($\alpha = \text{constant}$), between n_1 and t (equation 2-26), and between R and s (equation 2-30). Therefore n_1 is proportion to the length s of the slope, giving a triangular stress-distribution (Figure 2-12). The maximum length of the slope is

$$s = 2s_0 = \frac{4R_0 \sin(45^\circ - \beta/2) \sin(\frac{\beta + \phi}{2})}{\sin \beta}$$

which together with R_0 's formula gives

$$s = \frac{4c}{\gamma} \frac{1 + \sin \phi}{\sin \phi \cos \lambda} \frac{\sin(45^\circ - \beta/2) \sin(\frac{\beta + \phi}{2})}{\sin \beta} \quad (2.31)$$

$$\frac{1 - \tan^2(45^\circ - \phi/2) e^{\tan \lambda (\beta - 90^\circ)}}{\cos(\beta - \omega) - \sin \omega e^{\tan \lambda (\beta - 90^\circ)}} \cdot$$

This is the polar equation of the limit curve for cohesive soils.

Taylor (1937) commented that Jáký's method leads to results which are somewhat on the unsafe side.

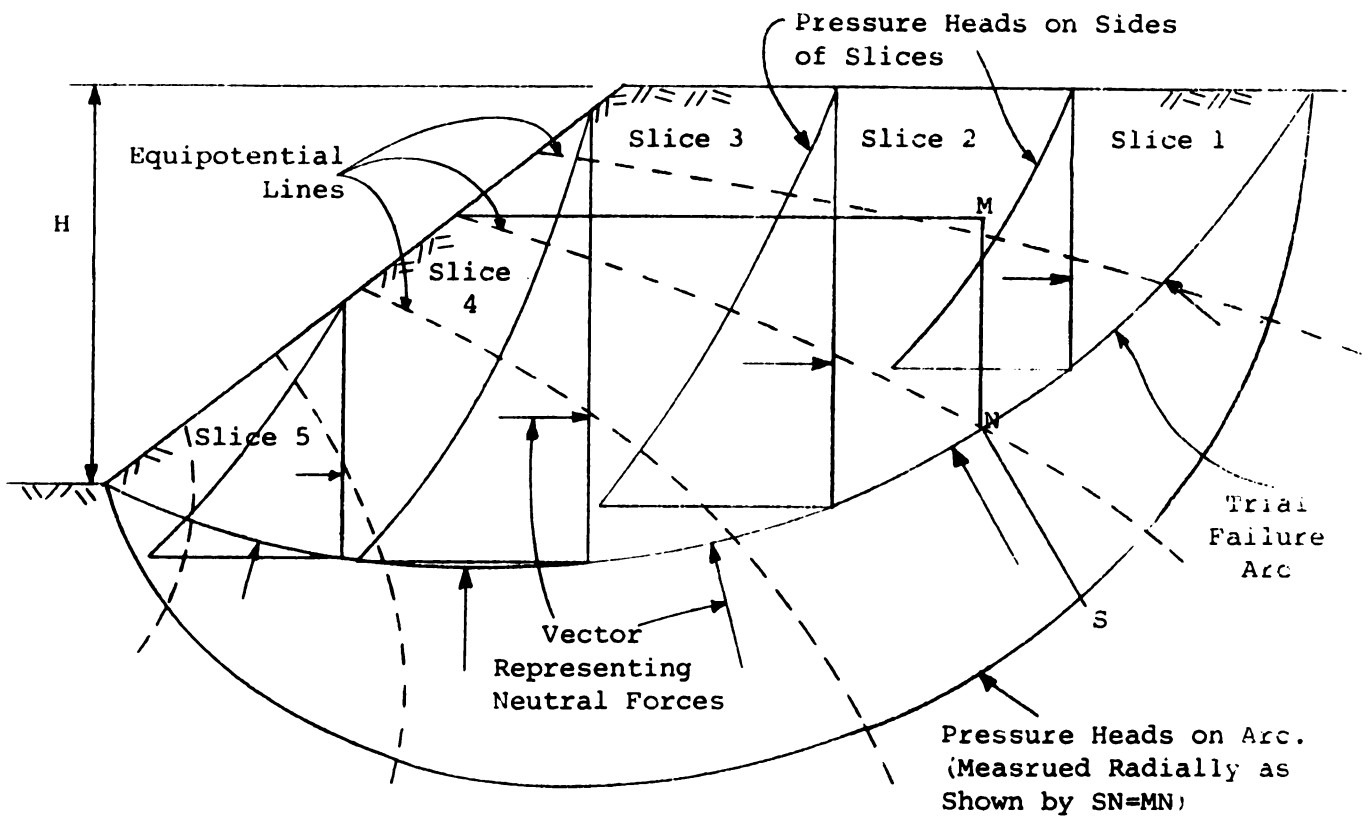
2.3.3 The Method of Slices

The method of slices was developed by W. Fellenius (1936). It is based on the static analysis of the mass above any trial failure arc, with this mass considered to be made up of vertical slices.

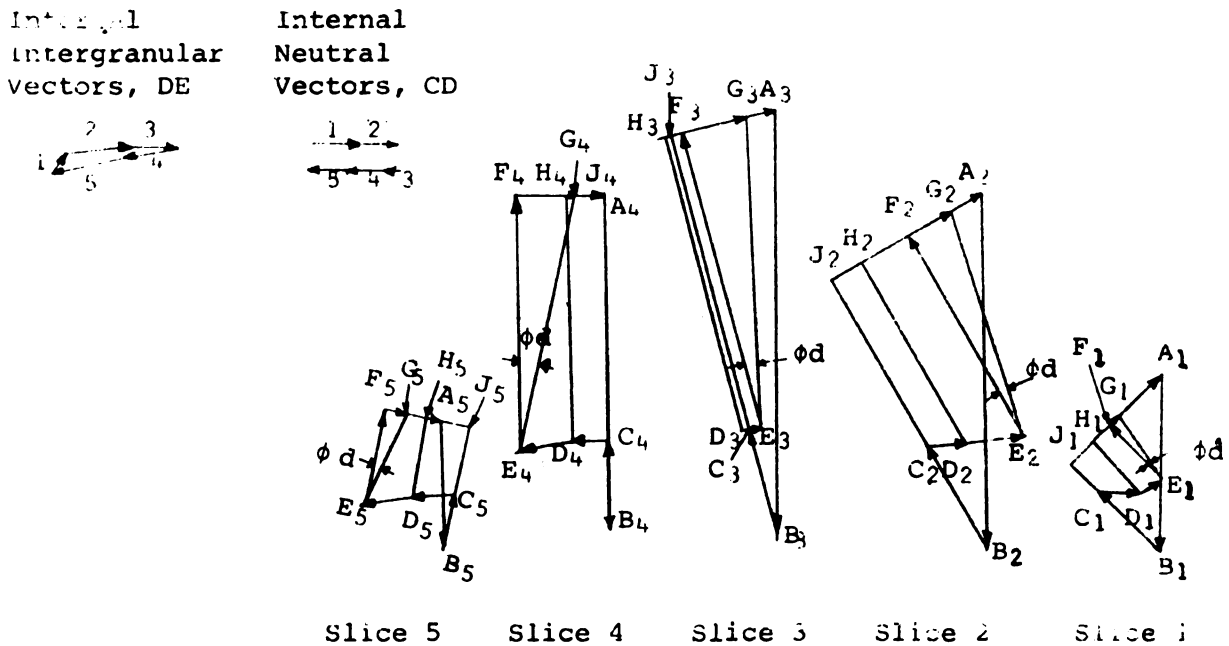
Before the method and its assumption can be truly understood a clear picture is required of all forces entering the analysis, and these forces will be explained in some detail. Seepage forces are included in the following explanation, although these were not included in the original form of the slices method.

Let it be assumed that the circular failure arc shown on the section of Figure 2-14a is an arbitrarily chosen trial arc. This section is assumed to be in homogeneous soil through which a steady state of seepage is occurring as represented by the equipotential lines, which are shown by dashed curves. This section has been arbitrarily divided into five vertical slices of equal width.

From the equipotential lines the neutral pressure may be determined at any point on the section. For example, at point N, in Figure 2-14a, the pressure head is MN. Point S is determined by setting the distance NS equal to MN. A number of points obtained in the same manner as S give the curved line through S which is a pressure head diagram. On the sides of the slices pressure head diagrams are also shown.



(a) Cross Section Showing Slices and Neutral Forces.



(b) Vector Polygons.

Figure 2-14.--Stability Analysis by the Slices Method.

The actuating force may be taken as either the combination of total weights and boundary neutral forces or the combination of submerged weights and seepage forces. The former will be used. Thus the forces acting on any slice are its total weight, the neutral forces acting on its sides and on its base, and the resultant intergranular forces on its sides and on its base. Granular forces on the base consist of the cohesion, the normal component of intergranular force P_N , and the frictional resistance that is developed by P_N and that is equal to $P_N \tan \phi_D$.

In Figure 2-14a, vector polygons are given which show all forces for all slices. For purposes of explanation any slice may be used, since notations are the same for each slice except for subscript. The total weight of any slice is equal to the volume multiplied by the total unit weight, and is represented by vector AB. The neutral force across the base is represented by vector BC, which acts normal to the arc. The combined neutral force on the two sides of the slice is represented by vector CD, acting horizontally. For the steady seepage case, with the flow net known, vectors BC and CD are completely defined, and from the pressure head diagrams, shown in (a), these vectors may be determined.

Vector DE represents the combination of intergranular force on the two sides of the slice. The magnitude of the force represented by DE is dependent on strains and stress-strain characteristics of the soil; it is therefore statically indeterminate. The vectors DE in the figure are merely arbitrarily chosen values that are reasonable; to obtain a solution some assumption relative to these forces must be made.

Reverting to considerations of the entire mass, it may be noted that the total weight of the mass must be transmitted across the rupture arc. The sum of all boundary forces, including both neutral and intergranular components, must equal the total weight, regardless of the distribution among the several slices. The assumption mentioned at the end of the previous paragraph is, therefore, an assumption relative to distribution only and thus it is called a distributional assumption. Fortunately the various possible distributions usually lead only to minor differences in the stability conditions.

The remaining forces are the normal and tangential components of the intergranular forces across the arc. These forces are represented by EF and FA, and are commonly designated by P_N and P_S . Once the assumption giving DE is chosen and point E is thus known, EF and FA can be determined. The requirement for equilibrium is that the

shearing force, P_s , be supplied by friction and cohesion. The figures show that P_s is supplied by the sum of the frictional force FG , which is equal to $P_N \tan \phi_D$, and the cohesional force C_D , represented in the figure by GA . This may be written

$$P_s = P_N \tan \phi_D + C_D$$

With respect to the mass as a whole the requirement for stability is expressed in terms of the equilibrium of moments about the center of the circle. In the case under consideration boundary neutral pressures exist only on the circular arc and introduce no moment. Thus the actuating moment is that of the total weight. If weight, AB , is separated into shearing and normal components AJ and JB , designated respectively by W_s and W_N , the normal component has no moment, and the actuating moment may be written $R\sum W_s$, in which R is the radius and the moment is clockwise.

For a correct consideration of the equilibrium of individual slices, forces CD and DE , which act on the sides of slices, must be included. However, relative to the mass as a whole they form closed polygons, as shown in the upper left on Figure 2-14b, and the summation of their moments must be zero. Force EF also has no moment about the center of the circle. Thus the only other forces introducing moment are FG and GA . Forces FG give

a moment which may be expressed as $R\Sigma(FG)$, or $R\tan\phi_D\Sigma P_N$. Forces GA give the moment $R\Sigma(GA)$, or $Rc_D L_a$, in which L_a is the length of the arc. These moments are counter-clockwise, and together they form the resisting moment.

Equating the actuating and resisting moments gives the expression of moment equilibrium,

$$R\Sigma W_s = R\tan\phi_D\Sigma P_N + Rc_D L_a \quad (2-32)$$

All terms in this expression are moments, and therefore they are vector quantities. If the equation is divided by R it becomes

$$\Sigma W_s = \tan\phi_D\Sigma P_N + c_D L_a \quad (2-33)$$

The terms of this equation have the dimension of forces, but since they are tangential summations they are not vector quantities in the ordinary sense.

In equation (2-33) the only term which depends on the distributional assumption, and thus the only term which offers any difficulty in the analysis is ΣP_N . If the resultants of lateral intergranular pressures DE are correct, as shown on Figure 2-14 b, the vectors EF, representing P_N , are correct.

Fellenius proposed the use of the assumption that the lateral forces are equal on the two sides of each slice. This is not simply an assumption relative to the

distribution, since in addition it involves other factors which cannot be fully explained without an unduly long discussion. A small amount of study shows that it gives conditions in individual slices which are entirely incorrect. For example, it indicates that slices 1 and 2 have much less resisting force than is required for equilibrium, and that slices 4 and 5 have much more resisting force than needed. However, these inconsistencies counteract each other to a large degree, and for the mass as a whole this assumption, which is the characteristic assumption of the slices method, gives fairly reasonable results. In analyses of the type shown on Figure 2-14, where the effects of seepage are included, there are two possible interpretations of the Fellenius assumption. These interpretations are considered separately in the two following paragraphs.

In the first interpretation it is assumed that the lateral intergranular forces are in balance on the sides of each slice. According to this assumption forces DE all become equal to zero, and ΣP_N becomes $\Sigma(DH)$.

In the second interpretation it is assumed that the lateral combined forces are in balance on the sides of each slice. On this basis combinations of forces CD and DE all become zero and ΣP_N becomes $\Sigma(CJ)$.

The use of the second interpretation does not require the determination of the neutral pressures on the sides of the slices, and thus it saves a considerable amount of labor. Since it gives conservative results, it is probably the more satisfactory procedure for general use.

2.3.4 Bishop's Method

In 1954 a modified method in the stability analysis of slopes was proposed by Bishop. Consideration was given to the effect of pore pressure distribution and variation of shear parameters on the position and shape of the rupture surface. Failure is assumed to occur simultaneously along the surface. The surface is also assumed to be a circular arc. The problem is assumed to be one of plane strain.

Under these conditions a quantitative estimate of the factor of safety can be obtained by examining the conditions of equilibrium when incipient failure is postulated, and comparing the strength necessary to maintain limiting equilibrium with the available strength of the soil. The factor of safety (F) is thus defined as the ratio of the available shear strength of the soil to that required to maintain equilibrium. The shear strength mobilized is, therefore, equal to s , where

$$s = \frac{1}{F} \{ c' + (\sigma_n - u) \tan \phi' \} \quad (2-34)$$

and c' denotes cohesion, ϕ' denotes the angle of shearing resistance, σ_n denotes the total normal stress, and u denotes pore-pressure.

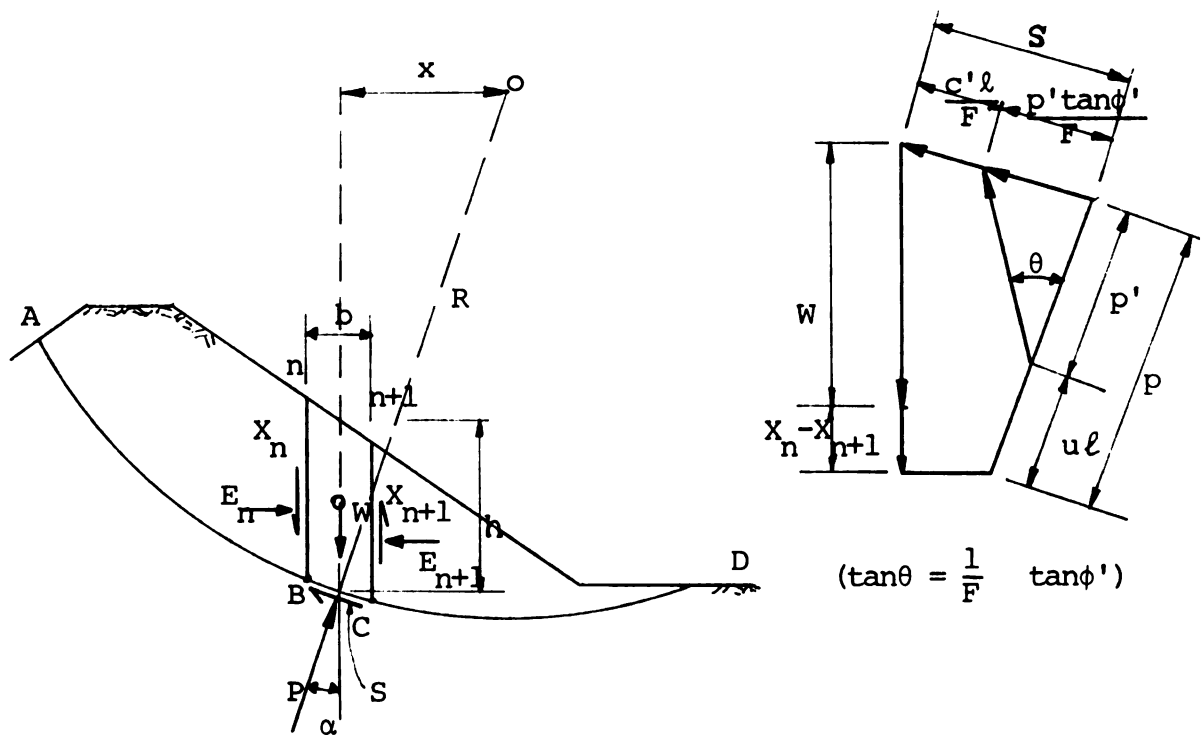
In order to examine the equilibrium of the mass of soil above the slip surface it follows from equation (2-34) that it is necessary to know the value of the normal stress at each point on this surface, as well as the magnitude of the pore-pressure. It is possible to estimate the value of the normal stress by the following method developed by Fellenius (1927, 1936), in which the conditions for the statical equilibrium of the slice of the soil lying vertically above each element of the sliding surface are fully satisfied.

Consider the equilibrium of the mass of soil (of unit thickness) bounded by the circular arc ABCD, of radius R and center at O (Figure 2-15). In the case where no external forces act on the surface of the slope, equilibrium must exist between the weight of the soil above ABCD and the resultant of the total forces acting on ABCD.

The following definitions are given:

- E_n , and E_{n+1} denote the resultants of the total horizontal forces on the sections n and $n+1$ respectively.
- X_n , and X_{n+1} denote the vertical shear forces.
- W denotes the total weight of the slice of soil.

P	denotes the total normal force acting on its base.
S	denotes the shear force acting on its base.
h	denotes the height of the element.
b	denotes the breadth of the element.
l	denotes the length BC.
α	denotes the angle between BC and the horizontal.
x	denotes the horizontal distance of the slice from the center of rotation.



a) Force Components on a Sliced Element

b) Force Polygon.

Figure 2-15.--Stability Analysis by the Bishop Method.

The total normal stress is σ_n where

$$\sigma_n = \frac{P}{\ell} \quad (2-35)$$

Hence from equation (2-34) the magnitude of the shear strength mobilized to satisfy the conditions of limiting equilibrium is s , where

$$s = \frac{1}{F} \left\{ c' + \left(\frac{P}{\ell} - u \right) \tan \phi' \right\}. \quad (2-36)$$

The shear force, S , acting on the base of the slice is equal to $s\ell$, and thus, equating the moment about O of the weight of soil within ABCD with the moment of the external forces acting on the sliding surface, we obtain

$$\sum W x = \sum SR = \sum s \ell R \quad (2-37)$$

It follows, therefore, from equation (2-36) that

$$F = \frac{R}{\sum W x} \sum [c' \ell + (P - u \ell) \tan \phi']. \quad (2-38)$$

From the equilibrium of the soil in the slice above BC, we obtain P , by resolving forces in a direction normal to the slip surface

$$P = (W + X_n - X_{n+1}) \cos \alpha - (E_n - E_{n+1}) \sin \alpha. \quad (2-39)$$

The expression of F thus becomes

$$F = \frac{R}{\sum W x} \sum [c' \ell + \tan \phi' (W \cos \alpha - u \ell) \quad (2-40)$$

$$+ \tan \phi' \{ (X_n - X_{n+1}) \cos \alpha - (E_n - E_{n+1}) \sin \alpha \}].$$

Since there are no external forces on the face of the slope, it follows that

$$\Sigma (X_n - X_{n+1}) = 0 \quad (2-41a.)$$

and

$$\Sigma (E_n - E_{n+1}) = 0 \quad (2-41b.)$$

However, except in the case where ϕ' is constant along the slip surface and α is also constant (i.e., in a plane slip surface), the terms in equation (2-40) containing X_n and E_n do not disappear. A simplified form of analysis, suggested by Krey (1926), Terzaghi (1929) and also presented by May (1936) as a graphical method, implies that the sum of those terms,

$$\Sigma \tan \phi' \{ (X_n - X_{n+1}) \cos \alpha - (E_n - E_{n+1}) \sin \alpha \},$$

may be neglected without serious loss in accuracy.

Putting $x = R \sin \alpha$, the simplified form may be written as

$$F = \frac{1}{\Sigma W \sin \alpha} \Sigma [c' \ell + \tan \phi' (W \cos \alpha - u \ell)]. \quad (2-42)$$

The pore-pressures are often expressed as a function of the total weight of the column of soil above the point considered, i.e.

$$u = \bar{B} \left(\frac{W}{b} \right) \quad (2-43)$$

where $\bar{B}$ is a soil parameter based either on field data or laboratory tests. In the case, putting $l = b \sec \alpha$, the expression for factor of safety can be further simplified to

$$F = \frac{1}{\sum W \sin \alpha} \sum [c' l + \tan \phi' W (\cos \alpha - \bar{B} \sec \alpha)]. \quad (2-44)$$

This expression permits the rapid and direct computation of the value of F which is necessary if sufficient trial circles are to be used to locate the most critical surface. However, the values of F are, in general, found to be conservative, and may lead to uneconomical design (Bishop, 1954). This is especially marked where conditions permit deep slip circles around which the variation in α is large. To derive a method of analysis which largely avoids this error, it is convenient to return to equation (2-38). If we denote the effective normal force, $(P - ul)$, by P' (Figure 2-15b) and resolve the force on the slice vertically, then we obtain, on rearranging

$$P' = \frac{W + X_n - X_{n+1} - l \left(u \cos \alpha + \frac{c'}{F} \sin \alpha \right)}{\cos \alpha + \frac{\tan \phi' \sin \alpha}{F}} .$$

Substituting in equation (2-38) and putting $l = b \sin \alpha$ and $x = R \sin \alpha$.

$$F = \frac{1}{\sum W \sin \alpha} \sum [\{c' b + \tan \phi' (W(1-\bar{B}) + (X_n - X_{n+1}))\} \frac{\sec \alpha}{1 + \tan \phi' \tan \alpha}]. \quad (2-45)$$

The values of $(X_n - X_{n+1})$ used in this expression are found by successive approximation, and must satisfy the conditions given in equation (2-41). In addition, the position of the lines of thrust between the slices should be reasonable, and no unbalanced moment should be implied in any slice. The factor of safety against sliding on the vertical sections should also be satisfactory, though since the slip surface assumed is only an approximation to the actual one, overstress may be implied in the adjacent soil.

The condition $\sum (X_n - X_{n+1}) = 0$ in equation (2-41a) can also be satisfied directly by selecting appropriate values of X_n . The corresponding sum, $\sum (E_n - E_{n+1})$ can be readily computed in terms of the expression used in equation (2-45).

Resolving the forces on a slice tangentially we obtain the expression

$$(W + X_n - X_{n-1}) \sin \alpha + (E_n - E_{n+1}) \cos \alpha = S$$

or

$$(2-46)$$

$$(E_n - E_{n+1}) = S \sec \alpha - (W + X_n - X_{n+1}) \tan \alpha.$$

Now, if equation (2-45) is written

$$F = \frac{1}{\sum W \sin \alpha} \sum [m] , \quad (2-47)$$

then

$$S = \frac{m}{F}, \quad (2-48)$$

and hence

$$\sum (E_n - E_{n+1}) = \sum \left[\frac{m}{F} \sec \alpha - (W + X_n - X_{n+1}) \tan \alpha \right]. \quad (2-49)$$

The X-values must, therefore, also satisfy the condition that

$$\sum \left[\frac{m}{F} \sec \alpha - (W + X_n - X_{n+1}) \tan \alpha \right] = 0. \quad (2-50)$$

In the practical point of view it is of interest to note that although there are a number of different distributions of $(X_n - X_{n+1})$ which satisfy equation (2-50), the corresponding variations in the value of F are found to be insignificant (less than 1% in a typical case). It should also be noted that since the error in the simplified method given in equation (2-42) varies with the central angle of the arc, it will lead to a different location for the critical circle than that given by the more rigorous method.

Bishop also developed a more general case for partially submerged slopes.

Slip surfaces predicted by Bishop's method correlate well with observed field data for homogeneous soil.

2.4 Composite Surface of Sliding

In many instances the geometrical or geological conditions of the problem are such that the surface of sliding may not be even approximately circular. The following methods have been proposed in calculating the factor of safety against sliding for noncircular rupture surfaces.

2.4.1 Janbu's Method

The method was proposed by Janbu (1954). A sliding mass with a noncircular surface of sliding is shown on Figure 2-16.

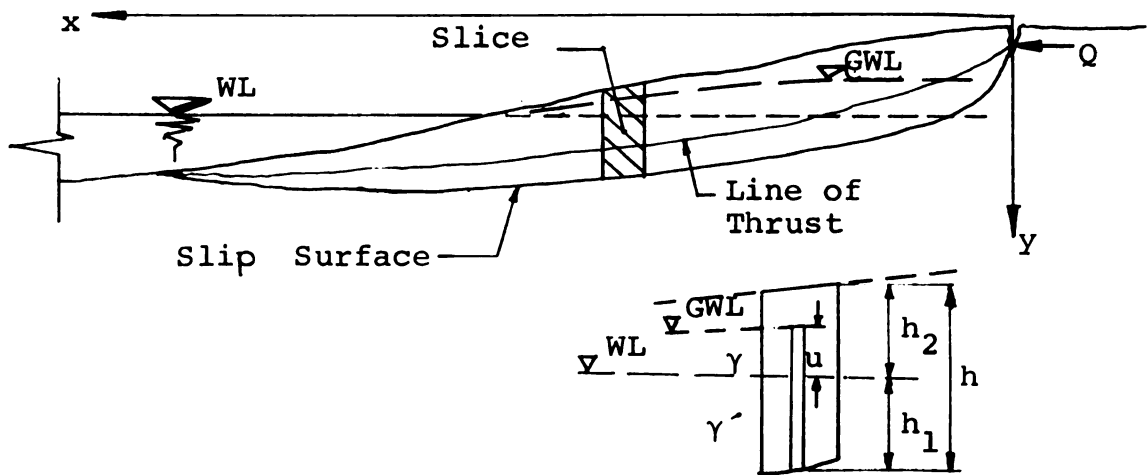
The soil mass above the slip surface is divided into an appropriate number of slices. Equilibrium in each slice must be maintained as follows:

Moment equilibrium about point M (Figure 2-17), yields

$$(T + \frac{1}{2}\Delta T)\Delta x + (E + \frac{1}{2}\Delta E)\Delta Y_O = 0 \quad (2-51)$$

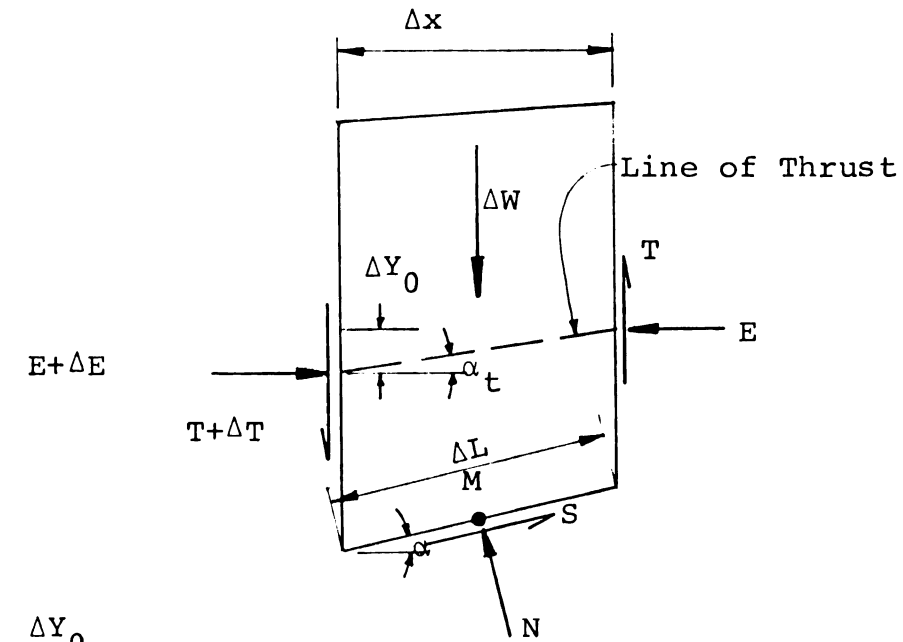
For a vertical section (Figure 2-18) $\Delta E = \Delta T = 0$.

Hence,



Notations: $p = \frac{\Delta W}{\Delta x} = \gamma h$ when above water level, or
 $p = \gamma' h_1 + \gamma h_2$ when partially submerged
 u = pore pressure, or excess pore pressure above WL.
 $t = \frac{\Delta T}{\Delta x}$ derivative of T with respect to x .
 Q = horizontal external force at top of slip surface.

Figure 2-16.--Definition Sketch for Janbu's Method.



$$\frac{\Delta Y_0}{\Delta x} = \tan \alpha_t$$

$$L = \frac{\Delta x}{\cos \alpha} ; S = s_e \Delta L$$

Figure 2-17.--Enlarged Slice.

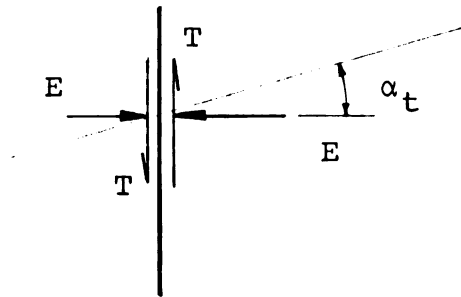


Figure 2-18.--Forces on a Slice Boundary.

$$T = -\tan\alpha_t(E) \quad (2-52)$$

Equilibrium of each slice in y and x-directions
(Figure 2-16) yields

$$\Delta W + \Delta T = S \sin\alpha + N \cos\alpha \quad (2-53)$$

and

$$\Delta E = -S \cos\alpha + N \sin\alpha \quad (2-54)$$

Equilibrium of the whole mass in y and x-directions
(Figure 2-16) yields

$$\Sigma(S \sin\alpha + N \cos\alpha) = \Sigma\Delta W, \text{ i.e. } \Sigma\Delta T = 0 \quad (2-55)$$

and

$$\Sigma(-S \cos\alpha + N \sin\alpha) = -Q, \text{ i.e. } \Sigma\Delta E + Q = 0 \quad (2-56)$$

Combining (2-53) $\sin\alpha$ - (2-54) $\cos\alpha$, yields

$$(\Delta W + \Delta T)\sin\alpha - \Delta E \cos\alpha = S. \quad (2-57)$$

Introducing (2-57) into criterion (2-56), yields

$$Q + \Sigma(\Delta W + \Delta T)\tan\alpha = \Sigma S/\cos\alpha = \Sigma s_e \Delta x / \cos^2\alpha. \quad (2-58)$$

Coulomb's equation for shear strength, is given by

$$s_e = c_e + \left(\frac{N}{\Delta L} - u\right)\tan\phi_e \quad (2-59)$$

in which, according to the definition of the safety factor, F ,

$$c_e = \frac{c'}{F} ; \quad \tan \phi_e = \frac{\tan \phi'}{F} \quad (2-60)$$

where c' and $\tan \phi'$ denote the available shear strength characteristics.

Introducing for N from (2-53) into (2-59) and solving for S_e ,

$$S_e = \frac{c_e + (p+t-u) \tan \phi_e}{1 + \tan \alpha \tan \phi_e} \quad (2-61)$$

When combining (2-60), (2-61) and (2-58) and solving for F , one finds

$$F = \frac{\frac{\sum [c' + (p+t-u) \tan \phi'] \Delta x}{n_\alpha}}{Q + \sum (p+t) \tan \alpha \Delta x} = \frac{\sum A}{Q + \sum B} \quad (2-62)$$

where, $n_\alpha = \cos^2 \alpha (1 + \tan \alpha \tan \phi' / F)$.

From the moment equilibrium condition (Figure 2-16) one gets,

$$T = -\tan \alpha_t E. \quad (2-63)$$

Since $E_x = \sum_0^x E$, equation (2-62) and equation (2-57) yield,

$$T_x = -\tan \alpha_t \sum_0^x (B - \frac{A}{F}) \quad (2-64)$$

From which T and ΔT are obtained for all slices. The implicit equations (2-62) and (2-64) are solved by successive approximations.

2.4.2 Morgenstern's Method

The method proposed by Morgenstern et al. (1965), is used to calculate the factor of safety of non-circular slip surfaces. The method can be used to consider a wide variety of surfaces and any configuration composed of soils with differing shear strength properties. It is also able to make an allowance for complex pore-pressure distributions and hence treat stability problems in terms of effective stress. The main theoretical justification for the method is that all equilibrium conditions are satisfied. (The methods formerly presented in this paper do not satisfy all the equilibrium conditions). It is also an application of the method of slices. The solution of the equations is currently obtained using a digital computer.

We investigate the equilibrium of the potential sliding mass shown on Figure (2-19a). In this Figure: (1) the equation of the assumed slip surface is $y = y(x)$; (2) the equation of the surface of the slope, which is taken as known, is $y = z(x)$; (3) the equation of the position of action of the effective horizontal thrust between slices is unknown and given by $y = y'_t(x)$; and (4) the line of thrust of the internal water pressure is $y = h(x)$.

The forces acting on an infinitesimal slice of width dx of the potential sliding mass are shown on Figure (2-19b). In this Figure:

E' denotes the lateral thrust on the side of the slice in terms of effective stresses.

X denotes the vertical shear force on the side of the slice.

dW denotes the total weight of the slice.

P_w denotes the resultant water pressure acting on the side of the slice.

dP_b denotes the water pressure on the base of the slice.

dN' denotes the effective normal pressure on the base of the slice.

dS denotes the shear force acting along the base of the slice.

α denotes the inclination of the base of the slice with respect to the horizontal.

The condition that there be no rotation of the slice is satisfied if the sum of the moments about the center of the base of the slice is equal to zero. By taking moments about the mid-point of the base of the slice we find that

$$E' \left[(y - y'_t) - \left(-\frac{dy}{2}\right) \right] + P_w \left[(y - h) - \left(-\frac{dy}{2}\right) \right] - (E' + dE') \left[y + dy - y'_t - dy'_t + \left(-\frac{dy}{2}\right) \right] \\ - X \frac{dx}{2} - (X + dX) \frac{dx}{2} - (P_w + dP_w) \left[(y + dy) - (h + dh) - \frac{dy}{2} \right] - dP_b g = 0 \quad (2-65)$$

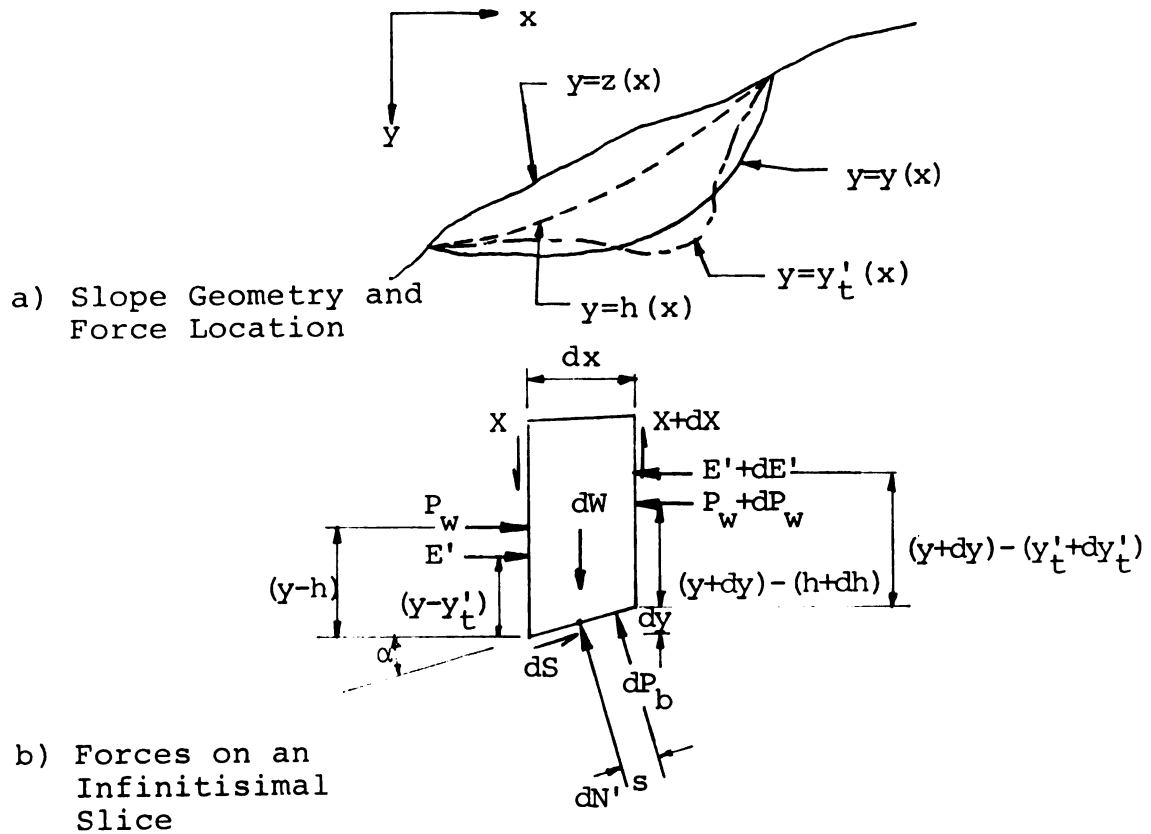


Figure 2-19.--Potential Sliding Mass Used for Derivation of Morgenstern's Equations.

After simplifying and proceeding to the limit as $dx \rightarrow 0$, it can be readily shown that

$$X = \frac{d}{dx}(E' y_t') - y \frac{dE'}{dx} + \frac{d}{dx}(P_w h) - y \frac{dP_w}{dx} . \quad (2-66)$$

For equilibrium in the N direction, we find

$$dN' + dP_b = dW \cos \alpha - dX \cos \alpha - dE' \sin \alpha - dP_w \sin \alpha . \quad (2-67)$$

From equilibrium in the S direction, we find

$$dS = dE' \cos\alpha + dP_w \cos\alpha - dX \sin\alpha + dW \sin\alpha. \quad (2-68)$$

The Mohr-Coulomb failure criterion in terms of effective stresses may be expressed as

$$dS = \frac{1}{F} [c' dx \sec\alpha + (dN') \tan\phi'] \quad (2-69)$$

where c' is the cohesion intercept, ϕ' is the angle of shearing resistance, and F denotes the factor of safety.

It should be noted that equation (2-69) also constitutes a definition of the factor of safety. The factor of safety with respect to shear strength has been adopted here. It is that value by which the shear strength parameters must be reduced in order to bring the potential sliding mass into a state of limiting equilibrium. It is clear that the factor safety with respect to moment ratios cannot be utilized in non-circular analyses where the shape of the sliding surface is arbitrary.

Eliminating dS from equations (2-68) and (2-69) we obtain

$$\frac{1}{F} [c' dx \sec\alpha + (dN') \tan\phi'] = dE' \cos\alpha + dP_w \cos\alpha - dX \sin\alpha + dW \sin\alpha. \quad (2-70)$$

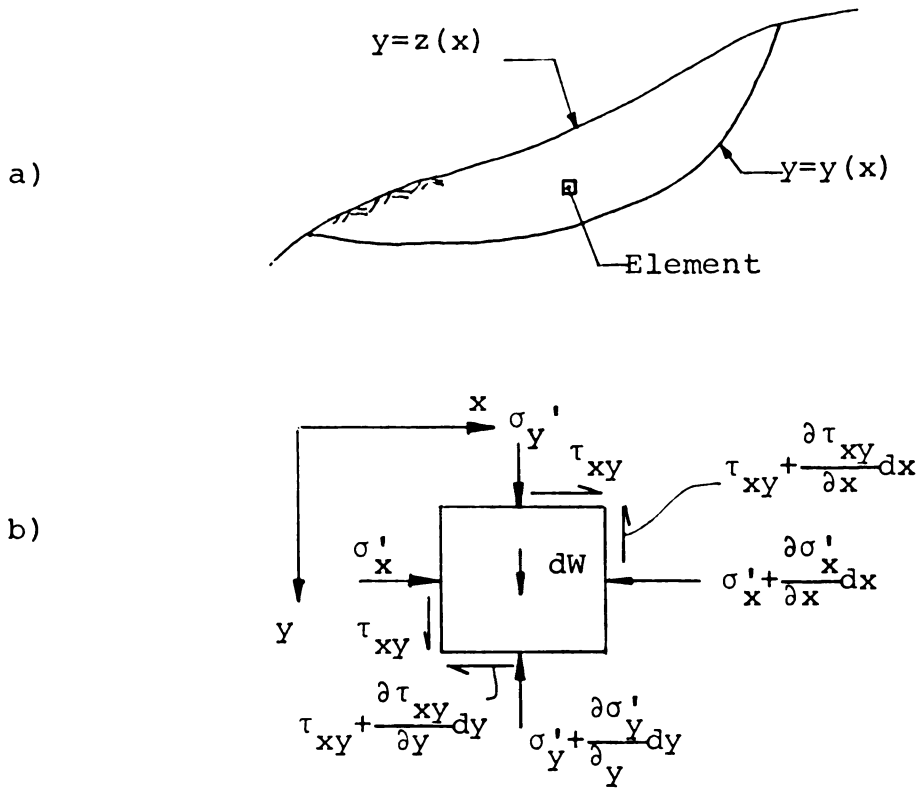


Figure 2-20.--An Element at an Interface Between Two Slices.

Eliminating dN' from equations (2-67) and (2-70), and dividing by $dx \cos \alpha$ yields

$$\begin{aligned} \frac{c'}{F} \sec^2 \alpha + \frac{\tan \phi'}{F} \left[\frac{dW}{dx} - \frac{dX}{dx} - \frac{dE'}{dx} \tan \alpha - \frac{dP_w}{dx} \tan \alpha - \frac{dP_b}{dx} \sec \alpha \right] \\ = \frac{dE'}{dx} + \frac{dP_w}{dx} - \frac{dX}{dx} \tan \alpha + \frac{dW}{dx} \tan \alpha. \end{aligned} \quad (2-71)$$

In the specified co-ordinate system, $\tan \alpha = -\frac{dy}{dx}$, and equation (2-71) becomes

$$\begin{aligned} & \frac{c'}{F} [1 + (\frac{dy}{dx})^2] + \frac{\tan \phi'}{F} \left\{ \frac{dW}{dx} - \frac{dX}{dx} + \frac{dE'}{dx} \frac{dy}{dx} + \frac{dP_w}{dx} \frac{dy}{dx} - r_u \frac{dW}{dx} [1 + (\frac{dy}{dx})^2] \right\} \\ & = \frac{dE'}{dx} + \frac{dP_w}{dx} + \frac{dX}{dx} \frac{dy}{dx} - \frac{dW}{dx} \frac{dy}{dx}. \end{aligned} \quad (2-72)$$

where

$$dP_b = r_u dW \sec \alpha, \quad (2-73)$$

and r_u is the pore-pressure ratio defined by Bishop and Morgenstern (1960).

Therefore, the two governing differential equations are

$$(1) \quad X = \frac{d}{dx} (E' Y_t') - Y \frac{dE'}{dx} + \frac{d}{dx} (P_w h) - Y \frac{dP_w}{dx}$$

and

$$\begin{aligned} (2) \quad & \frac{dE'}{dx} \left[1 - \frac{\tan \phi'}{F} \frac{dy}{dx} \right] + \frac{dX}{dx} \left[\frac{\tan \phi'}{F} + \frac{dy}{dx} \right] = \frac{c'}{F} [1 + (\frac{dy}{dx})^2] \\ & + \frac{dP_w}{dx} \left[\frac{\tan \phi'}{F} \frac{dy}{dx} - 1 \right] + \frac{dW}{dx} \left\{ \frac{\tan \phi'}{F} + \frac{dy}{dx} - r_u [1 + (\frac{dy}{dx})^2] \frac{\tan \phi'}{F} \right\} \end{aligned} \quad (2-74)$$

If y is specified as some function of x , we have, in general, a statically indeterminate problem involving the unknown functions E' , X and Y_t' , and the two governing differential equations. The indeterminacy arises from our lack of knowledge of the stresses existing in the soil mass. If the stresses could be determined, the displacements could be predicted using a representative stress-strain

relationship. This would obviate the need of doing limit equilibrium analyses. It is our inability to obtain an adequate stress analysis that justifies the application of limit equilibrium methods. At the same time, not knowing the stresses makes it necessary to invoke an assumption in order to render the problem statically determinate.

There are three classes of assumptions that can be made:

1. The distribution of normal pressure along the sliding surface can be assumed. The friction circle analysis is an example of a method that adopts this type of assumption. We have chosen to eliminate the normal pressure from the equilibrium equations and thereby place the burden of the indeterminacy on the internal forces.

2. An assumption may be made regarding the position of the line of thrust. For example if

$$y - y_t = a(y - z) \quad (2-75)$$

the moment equilibrium equation, neglecting the pore-pressure terms, becomes

$$X = E \frac{dy}{dx} - a \frac{d}{dx}(E(y - z)). \quad (2-76)$$

Equations (2-76) and (2-74) now define a statically determinate problem where the magnitude of a and F must be found by satisfying the appropriate boundary conditions. An

attempt (Morgenstern et al., 1965) has been made to obtain solutions using equations (2-74) and (2-76) but ill-conditioned functions arise in the solution of the differential equations and it was not found possible to obtain a satisfactory numerical procedure.

3. Assumptions may be made regarding the relation between the E' and X forces. If we isolate an element at the interface between two slices, as shown on Figure (2-20a), the effective stresses acting on this element will be as given on Figure (2-20b). For a specified geometry and slip surface the internal forces are determined by

$$E' = \int_z^y \sigma'_x(y) dy \quad (2-77)$$

and

$$X = \int_z^y \tau_{xy}(y) dy \quad (2-78)$$

We may therefore assume that

$$X = \lambda f(x) E'. \quad (2-79)$$

If $f(x)$ is specified the problem is statically determinate and λ and F may be found from a solution of the differential equations that satisfies the appropriate boundary conditions. The function $f(x)$ can take any prescribed form in principle. However, the behavior of soil imposes certain limitations so that only a certain range of functions will be reasonable in practice. Estimates of the function can

be obtained from elasticity theory. More reliable field observations of internal stresses in soil slopes may also be useful in estimating the distribution of the internal forces.

To simplify the equations it has been found convenient to define $\lambda f(x)$ by using the total horizontal stress E instead of the effective stress E' . Thus we define

$$E = E' + P_w \quad (2-80)$$

and the point of application y_t of the total stress by

$$E y_t = E' y_{t'} + P_w h \quad (2-81)$$

Then instead of (2-79) we assume

$$X = \lambda f(x) E. \quad (2-82)$$

In order to investigate the stability of a soil mass with any slope and properties it has been assumed that the potential sliding body may be divided into a number of finite slices by vertical lines with co-ordinates $x_0, x_1, \dots, x_n$. This division is carried out so that within each slice the portion of the slip surface is linear. Also within each slice, the interface between different soil types and pore pressure zones are linear and the function f defined by equation (2-82) depends linearly on x . Hence within each slice we have

$$y = Ax+B \quad (2-83)$$

$$\frac{dW}{dx} = px+q \quad (2-84)$$

$$\text{and} \quad f = kx+m. \quad (2-85)$$

Equations (2-83) and (2-84) allow a section with any arbitrary shape to be approximated in the analysis. Equation (2-85) assumes that the ratio of the internal forces changes linearly over a segment of the sliding body. This assumption is not unduly restrictive because k and m may be chosen to vary from segment to segment and any continuous distribution of internal forces can be approximated in this way.

Using equations (2-80) to (2-85), equation (2-66) becomes

$$X = \frac{d}{dx}(Ey_t) - y \frac{dE}{dx} \quad (2-86)$$

and equation (2-74) becomes

$$(Kx+L) \frac{dE}{dx} + KE = Nx+P \quad (2-87)$$

where

$$K = \lambda k \left(\frac{\tan \phi'}{F} + A \right) \quad (2-87a)$$

$$L = m\left(\frac{\tan\phi'}{F} + A\right) + 1 - A \frac{\tan\phi'}{F} \quad (2-87b)$$

$$N = p\left[\frac{\tan\phi'}{F} + A - r_u(1+A^2) \frac{\tan\phi'}{F}\right] \quad (2-87c)$$

and
$$P = \frac{c'}{F}(1+A^2) + q\left[\frac{\tan\phi'}{F} + A - r_u(1+A^2) \frac{\tan\phi'}{F}\right]. \quad (2-87d)$$

Equation (2-87) can be integrated across each slice in turn starting with the value $E = 0$ at the beginning of the slip surface. If, for each slice, x is measured from the beginning of that slice, then the solution which satisfies

$$E = E_i \text{ when } x = 0 \quad (2-88)$$

is

$$E = \frac{1}{L+Kx} \left[E_i L + \frac{Nx^2}{2} + P_x \right]. \quad (2-89)$$

The value of E at the end of the slice is determined and this gives the starting value of E for the next slice unless the end of the slip surface has been reached. The boundary condition to be satisfied at the end of the slip surface is

$$E = E_n \text{ when } x = x_n \quad (2-90)$$

where E_n is usually zero.

Satisfying equation (2-89) and its boundary conditions is alone insufficient to insure complete equilibrium

since equation (2-86) must also be satisfied. This can be done by determining y_t from the values of E found from equations (2-89) and (2-82), provided they satisfy the following necessary condition. By integrating equation (2-86) we have

$$M = E(y_t - y) = \int_{x_0}^x (X - E \frac{dy}{dx}) dx. \quad (2-91)$$

Since in general $M = 0$ when $x = x_n$, then

$$M_n = \int_{x_0}^{x_n} (X - E \frac{dy}{dx}) dx = 0. \quad (2-92)$$

If equation (2-92) is satisfied, values of y_t can be found from equation (2-91) thereby insuring that each slice is in moment equilibrium.

Hence in order to find values of λ and F such that all the equations of equilibrium are satisfied, we start with guessed values of λ and F and then integrate across all the slices to obtain the values of E_n and M_n which in general will not both be zero. Then, by a systematic iterative method of modifying λ and F , values are finally obtained for which E_n and M_n are zero.

There is one complication, however, which will be described here. In some cases there is more than one pair of values of λ and F which satisfy the equations for a given surface and one must decide which is the physically

meaningful solution. The rule which has been adopted is that the values of λ and F should make the values of $(L + Kx)$ positive for the complete range of x of the slip surface. Use of this rule appears to produce unique values of λ and F (Morgenstern et al., 1965). From equation (2-89) it can be seen that if $(L + Kx)$ is zero for any value of x , E becomes infinite at that point. This is not physically possible. Although the values of $(L + Kx)$ are discontinuous across any of the boundaries of the slices $x_1, x_2, \dots, x_n$, one should not expect $(L + Kx)$ to change sign across these boundaries. If we consider a very thin slice near each x_i , in which some soil properties are very quickly varying but continuous, then if $(L + Kx)$ changes sign across a discontinuity, it should be zero at some point in the corresponding thin hypothetical region. This indicates that $(L + Kx)$ should be of the same sign for all x , and from experience it appears that it should be positive due to the dominance of the contribution of unity to L in equation (2-87b).

Some special assumptions relating the X and E forces are of interest. If it is assumed that

$$X = \frac{E dy}{dx} \quad (2-93)$$

it can be shown that

$$Y_t = Y \quad (2-94)$$

and the factor of safety can be found from a modified form of equation (2-90) alone. The assumption stated by equation (2-93) is similar to that adopted in the conventional circular analysis, Bishop (1954). Although this assumption simplifies the numerical calculations it is not acceptable because the implied internal forces are likely to be physically inadmissible. There is also no reason for the relationship between the internal forces to depend solely upon the inclination of an arbitrarily chosen slip surface.

2.5 The Method of Characteristics

The method of characteristics was a development of the ideas suggested by Rankine (1857). Kötter (1903) considered differential equations of equilibrium, and the condition of limiting equilibrium at each point from a set of equations of limiting equilibrium and then transformed them to curvilinear coordinates. Further development of this theory was done by Prandtl (1920), Von Karman (1927) and Caquot (1934). However, due to the absence of a general method, all these investigations found only a limited application in practice.

In 1939 a general method was derived by Sokolovski. The general idea of the method is to derive an exact theory of limiting equilibrium. To do this the material is assumed to be perfectly plastic (the strain state is at or beyond yield point) throughout the continuous region to which

this theory is applied. The limiting stress is assumed to obey Coulomb's criterion, and no strain hardening occurs beyond the limiting stresses. Within the region of interest, the compatability of strain is violated, but equilibrium conditions are still satisfied.

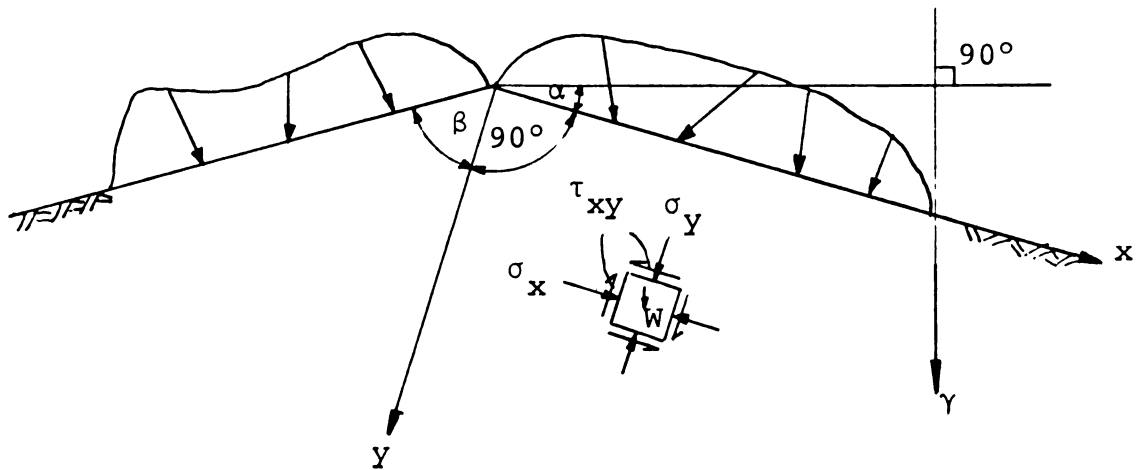


Figure 2-21.--Coordinates and Loading Systems.

In the mathematical analysis, plain strain is assumed. Referring to Figure 2-21, the z-direction is perpendicular to the paper, and the stress components $\tau_{yz} = \tau_{zx} = 0$. The remaining components σ_x , σ_y , σ_z and τ_{xy} are independent of the coordinate z. For convenience we take a compressive stress to be positive and a tensile stress to be negative.

From the equilibrium condition (anywhere within the soil mass)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \gamma \sin \alpha \quad (2-95a)$$

and

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \gamma \cos \alpha \quad . \quad (2-95b)$$

With the Coulomb failure criterion (in terms of σ_x , σ_y and τ_{xy})

$$\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 = \frac{\sin^2 \phi}{4} (\sigma_x + \sigma_y + 2c \cot \phi)^2, \quad (2-96)$$

and the given boundary conditions, the problem can be considered as statically determinate.

Sokolovski (1960) developed a procedure called "the method of characteristics and finite difference" to solve the problem as follows.

The first consideration was to assume the existence of slip lines when limiting equilibrium is reached. At the limiting equilibrium the two-dimensional problem concerns three stress components, σ_x , σ_y and τ_{xy} , as indicated in equations (2-95). The three stress components can be expressed in terms of two functions σ and θ , shown in Figure 2-22, where σ is the distance from the point of

intersection of the Mohr-Coulomb envelope to the center of the respective Mohr circles, and θ is the angle between the direction of the major principal stress and the x-axis. That is

$$\sigma_x = \sigma(1 + \sin\phi \cos 2\theta) - \psi \quad (2-97a)$$

$$\sigma_y = \sigma(1 - \sin\phi \cos 2\theta) - \psi \quad (2-97b)$$

$$\tau_{xy} = \sigma \sin\phi \sin 2\theta. \quad (2-97c)$$

Equations (2-97) were obtained from the state of stress at yield condition (Coulomb's circle). Substituting equations (2-97) into the equilibrium condition (2-95) one obtains the basic set of equations

$$(1 + \sin\phi \cos 2\theta) \frac{\partial \sigma}{\partial x} + \sin\phi \sin 2\theta \frac{\partial \sigma}{\partial y} - 2\sigma \sin\phi (\sin 2\theta \frac{\partial \sigma}{\partial x} - \cos 2\theta \frac{\partial \theta}{\partial y}) = \gamma \sin \alpha \quad (2-98a)$$

$$\sin\phi \sin 2\theta \frac{\partial \sigma}{\partial x} + (1 - \sin\phi \cos 2\theta) \frac{\partial \sigma}{\partial y} + 2\sigma \sin\phi (\cos 2\theta \frac{\partial \theta}{\partial x} + \sin 2\theta \frac{\partial \theta}{\partial y}) = \gamma \cos \alpha \quad (2-98b)$$

Multiplying the first of these expressions by $\sin(\theta \pm \mu)$ and the second by $-\cos(\theta \pm \mu)$, where $\mu = \frac{\pi}{4} - \frac{\phi}{2}$, after some elaboration we get the two expressions

$$\left[\frac{\partial \sigma}{\partial x} - 2\sigma \tan\phi \frac{\partial \theta}{\partial x} - \gamma \frac{\sin(\alpha - \phi)}{\cos\phi} \right] \cos(\theta - \mu) + \left[\frac{\partial \sigma}{\partial y} - 2\sigma \tan\phi \frac{\partial \theta}{\partial y} - \gamma \frac{\cos(\alpha - \phi)}{\cos\phi} \right] \sin(\theta - \mu) = 0 \quad (2-99a)$$

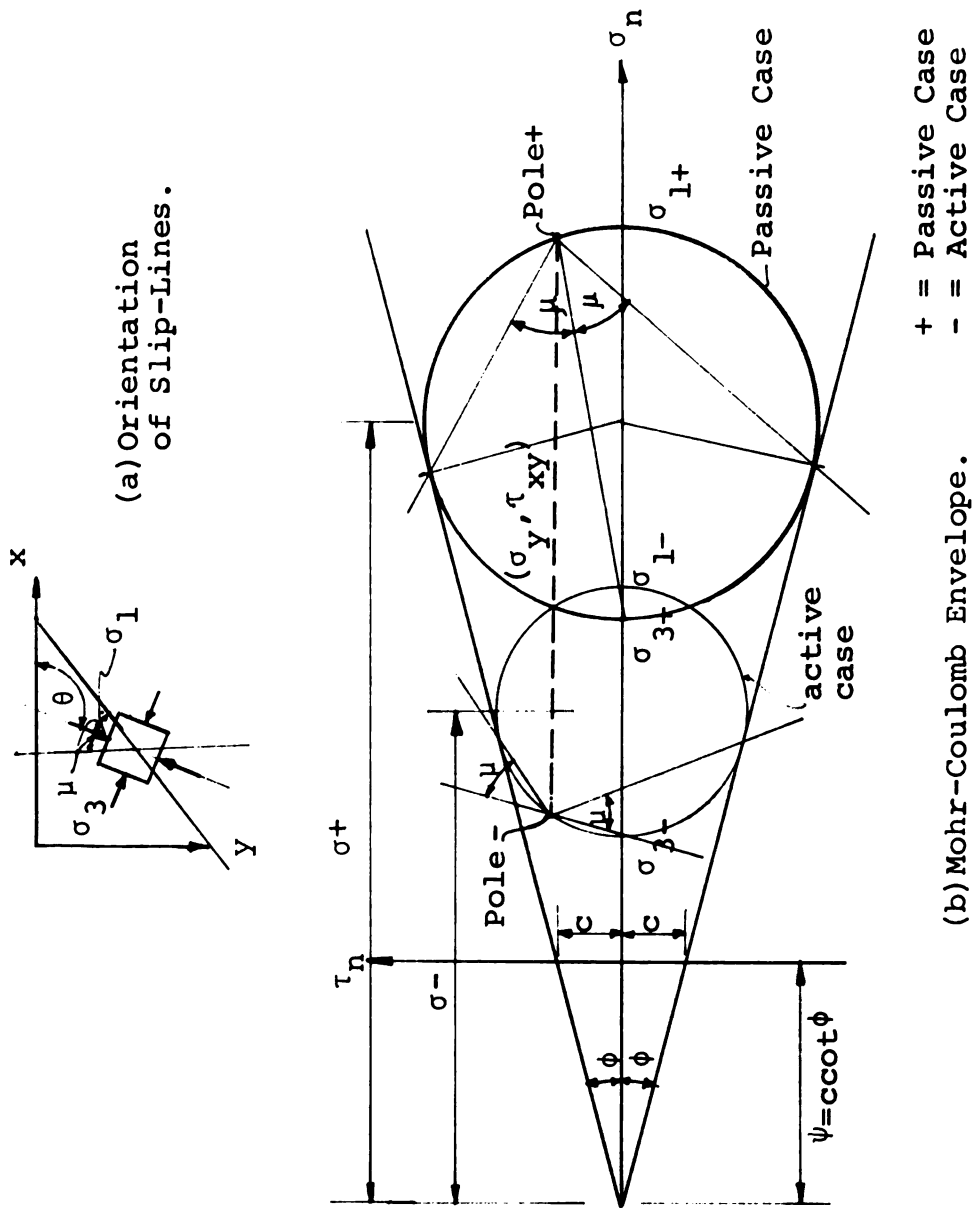


Figure 2-22.--Stress States at Limiting Equilibrium.

and

$$\left[\frac{\partial \sigma}{\partial x} + 2\sigma \tan \phi \frac{\partial \theta}{\partial x} - \gamma \frac{\sin(\alpha + \phi)}{\cos \phi} \right] \cos(\theta + \mu) + \left[\frac{\partial \sigma}{\partial y} + 2\sigma \tan \phi \frac{\partial \theta}{\partial y} - \gamma \frac{\cos(\alpha + \phi)}{\cos \phi} \right] \sin(\theta + \mu) = 0. \quad (2-99b)$$

By introducing

$$\chi = \frac{1}{2} \cot \phi \ln \frac{\sigma}{c} \quad (2-100)$$

and recognizing that

$$\frac{\partial \sigma}{\partial x} = 2\sigma \tan \phi \frac{\partial \chi}{\partial x}; \quad \frac{\partial \sigma}{\partial y} = 2\sigma \tan \phi \frac{\partial \chi}{\partial y},$$

upon substitution in equations (2-99), we find

$$\frac{\partial \xi}{\partial x} + \tan(\theta + \mu) \frac{\partial \xi}{\partial y} = a = \frac{\gamma \cos(\theta + \alpha - \mu)}{2\sigma \sin \phi \cos(\theta + \mu)} \quad (2-101a)$$

and,

$$\frac{\partial \eta}{\partial x} + \tan(\theta - \mu) \frac{\partial \eta}{\partial y} = b = - \frac{\gamma \cos(\theta + \alpha + \mu)}{2\sigma \sin \phi \cos(\theta - \mu)} \quad (2-101b)$$

where $\xi = \chi + \theta$

and $\eta = \chi - \theta$.

Equations (2-101) are "Sokolovski's basic equations."

Noting the relationships

$$\sigma = c \exp[(\xi + \eta) \tan \phi] \quad (2-102a)$$

and

$$\theta = \frac{\xi - \eta}{2} \quad (2-102b)$$

we see that the problem has been transformed to the determination of the functions ξ and η , ($\xi \stackrel{?}{=} \xi(x,y)$, $\eta \stackrel{?}{=} \eta(x,y)$). By considering equations (2-101) and the expressions for the total differentials,

$$\begin{aligned} d\xi &= \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \\ d\eta &= \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy, \end{aligned} \quad (2-103)$$

and using Cramer's rule, we have

$$\frac{\partial \xi}{\partial x} = \frac{a \, dy - \tan(\theta + \mu) \, d\xi}{dy - \tan(\theta + \mu) \, dx} \quad (a)$$

$$\frac{\partial \xi}{\partial y} = \frac{d\xi - a \, dx}{dy - \tan(\theta + \mu) \, dx} \quad (b)$$

(2-104)

$$\frac{\partial \eta}{\partial x} = \frac{b \cdot dy - \tan(\theta - \mu) \, d\eta}{dy - \tan(\theta - \mu) \, dx} \quad (c)$$

$$\frac{\partial \eta}{\partial y} = \frac{d\eta - b \, dx}{dy - \tan(\theta - \mu) \, dx} \quad (d)$$

Equations (2-104) can be used to obtain solutions of slip lines within the soil mass and solution of lines of discontinuity.

To obtain the solution of slip lines within the soil mass we set the numerators and the denominators of

Equation (2-104) equal to zero simultaneously and obtain two families of characteristic curves (η and ξ).

For the η -characteristics

$$dy = dx \tan(\theta - \mu) \quad (a)$$

and (2-105)

$$d\sigma - 2\sigma \tan\phi \, d\theta = \frac{\gamma}{\cos\phi} [\sin(\alpha - \phi) dx + \cos(\alpha - \phi) dy]. \quad (b)$$

For the ξ -characteristics,

$$dy = dx \tan(\theta + \mu) \quad (a)$$

and (2-106)

$$d\sigma + 2\sigma \tan\phi \, d\theta = \frac{\gamma}{\cos\phi} [\sin(\alpha + \phi) dx + \cos(\alpha + \phi) dy]. \quad (b)$$

To obtain the solution of the lines of discontinuity, we set only the denominators of Equation (2-104) equal to zero (the four partial derivatives $\frac{\partial \xi}{\partial x}$, $\frac{\partial \xi}{\partial y}$, $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$ will become discontinuous and in turn the stresses are also discontinuous) and obtain the locus of the lines of discontinuity

$$\frac{dy}{dx} = \tan(\theta - \mu); \text{ and } \frac{dy}{dx} = \tan(\theta + \mu). \quad (2-107)$$

The lines of discontinuity are actually envelopes of the slip lines obtained in the first part (equations (2-105) and (2-106)). Since along the line of discontinuity

equilibrium is violated, the restricted region in which the equations of equilibrium and strength relation can yield meaningful answers is the area within the envelope only.

To obtain solutions of the differential equations, equations (2-105) and (2-106), Sokolovski integrated the expressions by a finite-difference procedure. Since it is a long and tedious procedure, explanation is referred to Harr (1966, pp. 252-324). Harr also presents solutions by this method of some problems in bearing capacity and slope stability. The final solution obtained from the analytical method is in the form of the maximum load that the structure can sustain.

2.6 Numerical Method

A numerical procedure has been developed (Bell, 1969) to approximate the location of the critical rupture surface and to assess the importance of its shape in slope-stability calculations. The analytical procedure is an extension of the slip analysis developed by Bishop (neglecting the unbalance of vertical shear forces on an individual slice). A system of equations is set up with points on any assumed sliding surface as unknown variables. By the use of an optimization process, the depths can be obtained which yield the smallest factor of safety (F). The method can be used to analyze homogeneous soil slopes only.

Bell (1969) has shown that the modified factor of safety from the Bishop approximate method of slices for a statically loaded finite homogeneous slope under a homogeneous pore-pressure distribution, Figure 2-23, may be expressed by

$$F = F^* \frac{\left[\sum \frac{N^* \sec \alpha_i + (1-r_u) \frac{h_i}{h_o} \sec \alpha_i}{1 + \frac{\tan \alpha_i}{F^*}} \right]}{\sum \frac{h_i}{h_o} \sin \alpha_i} \quad (2-108)$$

where $F^* = \frac{F}{\tan \phi'}$ (2-109)

$$N^* = \frac{c'}{\gamma h_o \tan \phi'} \quad (2-110)$$

$$r_u = \frac{u}{\gamma h} \quad (2-111)$$

where u = the homogeneous pore-pressure acting on the potential sliding surface, γ = the total unit weight of the soil, and the strength parameters c' and ϕ' are the effective cohesion and effective friction angle respectively. The geometric variables α_i and h_i are defined by Figure 2-23, and h_o is any convenient reference dimension (usually taken to be the total height of the slope). For a particular slope geometry, the dependent dimensionless variable, F^* , is a function of only two independent dimensionless variables, the modified stability number N^* and the homogeneous pore-pressure ratio r_u .

$$F = \Phi(d_1, d_2, d_3 \dots d_{n-1}, x_n, x_o) \quad (2-112)$$

in which n = number of slices of equal width, d_i = depth of the right side of the i^{th} slice, x_o = horizontal coordinate of the toe of potential slip surface, and x_n = horizontal coordinate of the head of potential slip surface. The relationship between the primary space variables and those of equation (2-108) may be approximated for relatively narrow slice widths by

$$h_i = \frac{(d_{i-1} + d_i)}{2} \quad (2-113)$$

$$b = \frac{(x_n - x_o)}{n} \quad (2-114)$$

$$z_i = y_i - d_i \quad (2-115)$$

$$\alpha_i = \frac{\tan^{-1}(z_i - z_{i-1})}{b} \quad (2-116)$$

in which the slope geometry is specified by $y = F(x)$ and Figure 2-23.

The solution of the problem, in general terms, now involves the minimization of the factor of safety with respect to the primary space variables listed by equation (2-112). The adoption of modified dimensionless parameters such as those of equation (2-108) through (2-111) simplifies both the calculations and the presentation of

results. The following expression is adopted effectively in computation

$$F^* = \phi(\bar{d}_1, \bar{d}_2, \bar{d}_3 \dots \bar{d}_{n-1}, \bar{x}_n, \bar{x}_0) \quad (2-117)$$

in which each dimensionless space variable listed is defined as the ratio of the real distance to the reference height, h_0 , associated with the slope. In mathematical terms, the solution is stated explicitly by

$$\frac{\partial F^*}{\partial \bar{d}_i} = 0 \quad (i = 1, 2 \dots \dots, n-1) \quad (2-118)$$

$$\frac{\partial F^*}{\partial \bar{x}_n} = 0 \quad (2-119)$$

and
$$\frac{\partial F^*}{\partial \bar{x}_0} = 0 \quad (2-120)$$

Three restrictions are ultimately placed on the form of the potential slip surface. The first is already contained in equations (2-112), (2-117), and (2-118), where vertical cracks at the head or toe of the potential slip surface are not permitted. The second restriction is that the potential slip surface must be planar or concave upward. The third restriction prevents the sliding surface from penetrating a rigid stratum below.

The first restriction could well be invalid at the head of the surface when a cohesive slope is near, or at, yield and this is deserving of further investigation. The second restriction is based on a lack of field evidence to the contrary, and for constant slice width, it is numerically required that

$$z_{i-1} - 2z_i + z_{i+1} \geq 0 \quad (2-121)$$

or the dimensionless equivalent statement that

$$\bar{z}_{i-1} - 2\bar{z}_i + \bar{z}_{i+1} \geq 0. \quad (2-122)$$

An additional restraint was considered but not invoked which involved the angle of inclination of the potential slip surface at soil-air or soil-water interfaces. Detailed laboratory or field studies on the shape of failure surfaces would aid in clarification of this point, at least for the yield condition.

Part of the results (Bell, 1969) obtained by the numerical method is presented on Figure 2-24 which shows shapes of critical potential slip surfaces in comparison with circular slip surfaces. Stability analysis by this method was found to be in agreement with the method introduced by Bishop and Morgenstern (1960).

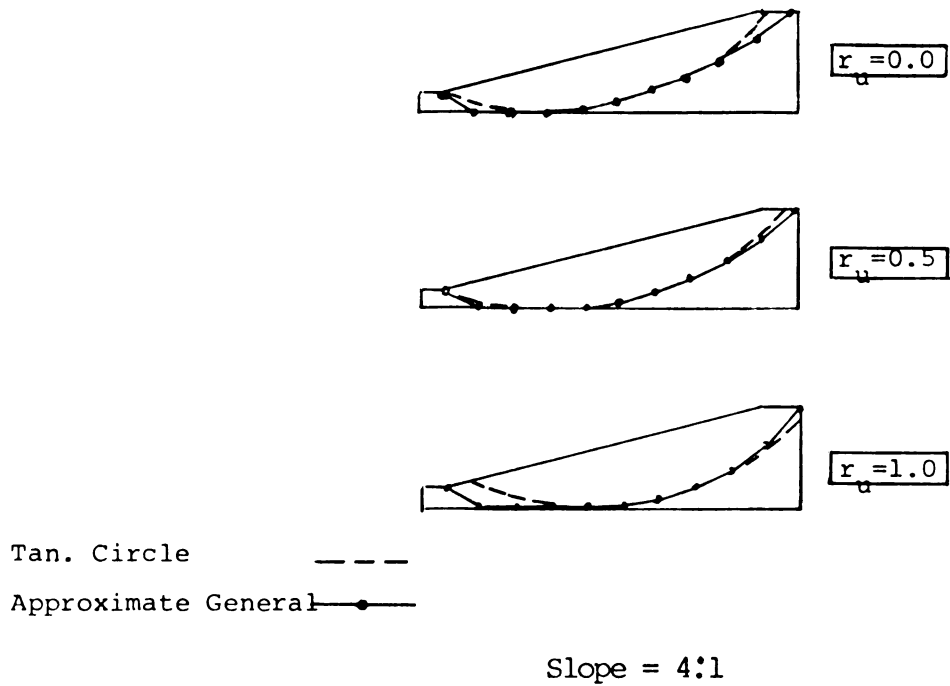


Figure 2-24.--Shape of Critical Potential Slip Surfaces with Variable Pore-Pressure, (After Bell, 1969).

CHAPTER III

CONCEPTS IN ELASTICITY AND PLASTICITY, AND BEHAVIOR OF SOILS UNDER LOAD

3.1 Rheological Properties of Materials

The general behavior of materials under stress according to a certain assumption can be used to approximate soil-deformation characteristics. Investigations of material behavior follow two main lines of inquiry: the study of the effect of the assumption of a property or quality, or the examination of the effect of a quantitative assumption. Two stress-deformation relationships must be considered in two different categories: one that connects the volume change with the hydrostatic stress, and the other which relates shearing or distortional deformation to the shearing stress. Furthermore, the variation of the behavior in time and with temperature may also enter in one or both relationships.

There are two kinds of ideal materials: elastic solids (Figure 3-1a) and viscous fluids (Figure 3-1b). (If the behavior of the material under an applied stress is dependent only on the applied stress and not on the previous history of stress or deformation in time, the substance is usually referred to as "ideal".)

3.1.1 Elastic Solids

A certain level of hydrostatic or shearing stress applied to elastic solids causes an immediate deflection or deformation of the body; later, when the stress is removed, the body returns to its original shape and size, and there is a one to one relationship between the state of stress and the state of strain, for a given temperature. Heat transfer is considered insignificant, and all the input work is assumed converted to internal energy in the form of recoverable stored elastic strain energy, which can be recovered as work when the body is unloaded. In general, however, the major part of the work input into a deforming material is not recoverably stored, but dissipated by the deformation process, causing an increase in the body's temperature and eventually being conducted away as heat. The relationship of the deformation of the material to the applied stress can be of any manner, linear or non-linear. If the relationship is linear or the strain is directly proportional to the applied stress, the material is referred to as a Hookean Elastic Solid. The classical elasticity equations, often called the Generalized Hooke's Law, are nine equations expressing the stress components as linear homogeneous functions of the nine strain components

$$T_{ij} = C_{ijkl} E_{kl} \quad (3-1)$$

where T_{ij} is the stress tensor, E_{kl} is the strain tensor, and C_{ijkl} is the tensor of elastic constants.

Due to the symmetry of T_{ij} and E_{kl} , for elastically isotropic material, the eighty-one independent elastic constants can be expressed in terms of just two elastic moduli

$$T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij} \quad (3-2)$$

where λ and μ are the Lamé's elastic constants, and δ_{ij} is the Kronecker delta. Contraction yields

$$T_{ii} = (3\lambda + 2\mu) E_{ii}, \text{ or } E_{kk} = T_{kk} / (3\lambda + 2\mu) \quad (3-3)$$

Substitution of (3-3) into (3-2) yields

$$E_{ij} = - \frac{\lambda \delta_{ij}}{2(3\lambda + 2\mu)} T_{kk} + \frac{1}{2\mu} T_{ij} \quad (3-4)$$

The two Lamé's elastic constants, λ and μ , are related to the more familiar shear modulus, G , Young's modulus, E , and Poisson's ratio, ν , as follows:

$$\mu = G = \frac{E}{2(1+\nu)} \text{ and } \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (3-5)$$

Equation (3-4) can also be written in the form

$$E_{ij} = - \frac{\nu}{E} T_{kk} \delta_{ij} + \frac{1+\nu}{E} T_{ij} \quad (3-6)$$

Equation (3-6) can be represented by two relationships as follows:

$$T'_{ij} = 2GE'_{ij} \quad (\text{distortion part}) \quad (3-7)$$

and $P = -Ke$ (hydrostatic part)

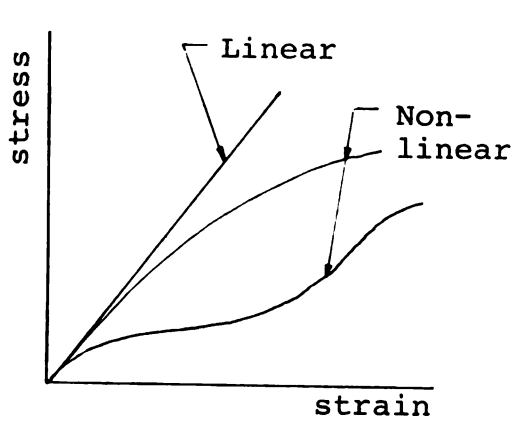
where $T'_{ij} = T_{ij} - \frac{1}{3}T_{kk}\delta_{ij}$ = the deviatoric stress,

$E'_{ij} = E_{ij} - \frac{1}{3}E_{kk}\delta_{ij}$ = the deviatoric strain,

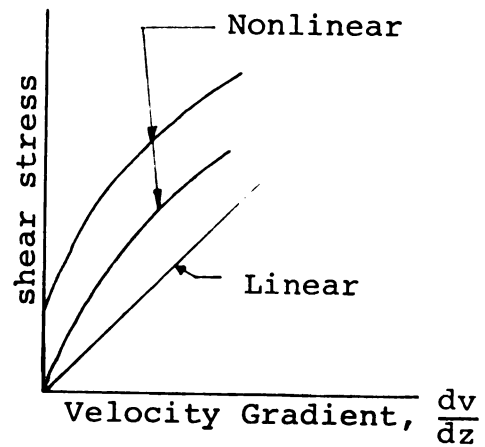
$P = -\frac{T_{kk}}{3}$ = the mean pressure,

$e = E_{kk}$ = the volumetric strain,

$K = \frac{E}{3(1-2\nu)}$ = the bulk modulus.



a) Ideally Elastic Materials.



b) Ideally Viscous Fluid.

Figure 3-1.--Stress-deformation Relationships of Ideal Materials.

3.1.2 Viscous Fluids

The qualitative postulate for a viscous fluid is that the rate of shearing deformation of the material depends on the applied stress. If a direct proportionality between the deformation velocity gradient normal to the shearing surfaces and the applied shear stress is assumed as a quantitative description of the quality of the substance, the fluid is said to be Newtonian, with viscosity being the proportionality constant. However, a further relationship between hydrostatic stress and volumetric strain must be specified for a fluid. Quantitatively the material may or may not be compressible.

3.1.3 Plastic Materials

The two substances considered in sections 3.1.1 and 3.1.2 have the qualities of ideal elasticity and fluidity for which various quantitative stress-deformation relationships can be suggested. If, however, a material experiences finite deformations up to a certain level of stress only, after which it deforms continuously, it is said to be "plastic" and to flow plastically above the yield stress. If it is postulated that no deformation takes place up to the limiting stress, and when the stress is reached, the material deforms continuously without stress increase, the material is referred to as "ideally plastic."

The constitutive equations for plasticity are not well established. Malvern (1969) states that several different idealizations have been proposed, all failing to account completely for the phenomena observed, but some nevertheless are quite useful. Most of the applications made have used the perfectly plastic theories.

Most applications of plasticity theory fall into one of two categories, as follows:

(a) Contained plastic deformations. When large deformations are prevented by surrounding elastic material or by elastic redundant members of a statically indeterminate structure, the plastic deformation is said to be contained. Collapse will not occur even though some parts of the body are loaded beyond the elastic limit. For such contained deformation it is reasonable to neglect work-hardening, and to use as the deformation variable the small-strain tensor, whose increments for small strain are negligibly different from the increments of natural strain.

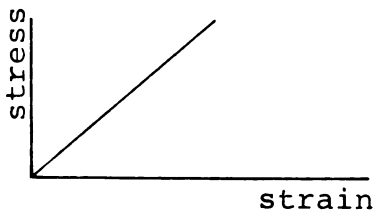
(b) Uncontained plastic flow. In metal-forming processes (e.g., drawing, extrusion, rolling) the problem is to produce a desired large deformation by cold-working. The cold-working involves hardening, which may be desired in order to have a stronger finished product, but which interferes with the process itself not only by necessitating greater working forces, but also because nonuniform

hardening may produce residual stresses. It may be reasonable to neglect the elastic-strain increments in comparison to the plastic-strain increments at large strains and to take as the deformation variables either the rate-of-deformation tensor or the natural-strain increment.

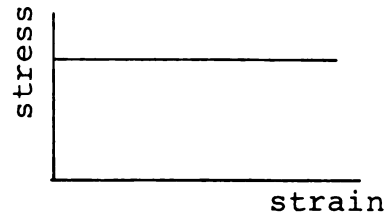
The two categories above represent two opposite extremes of possible problems. One of these two extremes is satisfactorily applied to most practically important problems of metals. Problems intermediate between these two extremes seldom occur in metals but may occur in non-metals, for example, in soil mechanics or in polymers. Stresses associated with deformations of such materials are also much more rate-dependent than is the case with metals, where the usual theories of plasticity neglect rate-dependence altogether.

3.2 Idealized Stress-strain Curves

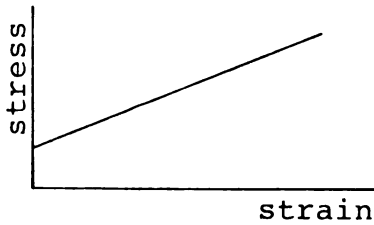
Idealized stress-strain curves frequently used for applications in mathematical calculations are illustrated in Figure 3-2. It has already been seen that cases (b) and (d) can be used to characterize contained plastic deformations, while (b) and (c) can represent uncontained plastic flow (where elastic strains are neglected). According to Malvern (1969), perfectly plastic material has proved to be quite useful in limit analysis of contained deformation.



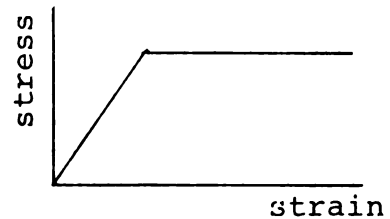
a) Perfectly Elastic, Brittle.



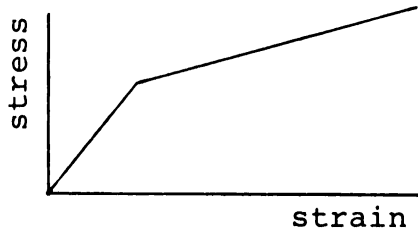
b) Rigid, Perfectly Plastic.



c) Rigid, Linear Strain Hardening



d) Elastic, Perfectly Plastic.



e) Elastic, Linear Strain Hardening.

Figure 3-2.--Idealized Stress-Strain Curves.

Limit analysis, without analyzing the detailed contained deformation up to limit load, seeks to find the limit load at which uncontained plastic flow would occur. If the geometry change of structure below the limit load is negligible, the same limit load is predicted by rigid, plastic theory as by elastic-plastic theory.

3.3 Mohr's Theory of Failure

Mohr's theory of failure assumes that the critical shear stress is not necessarily equal to the maximum shear stress but depends also on the normal stress acting on the shear plane. The limiting stress states are determined by the function $\tau_f = f(\sigma')$, where σ' = effective normal stress on the shear plane (see Figure 3-3).

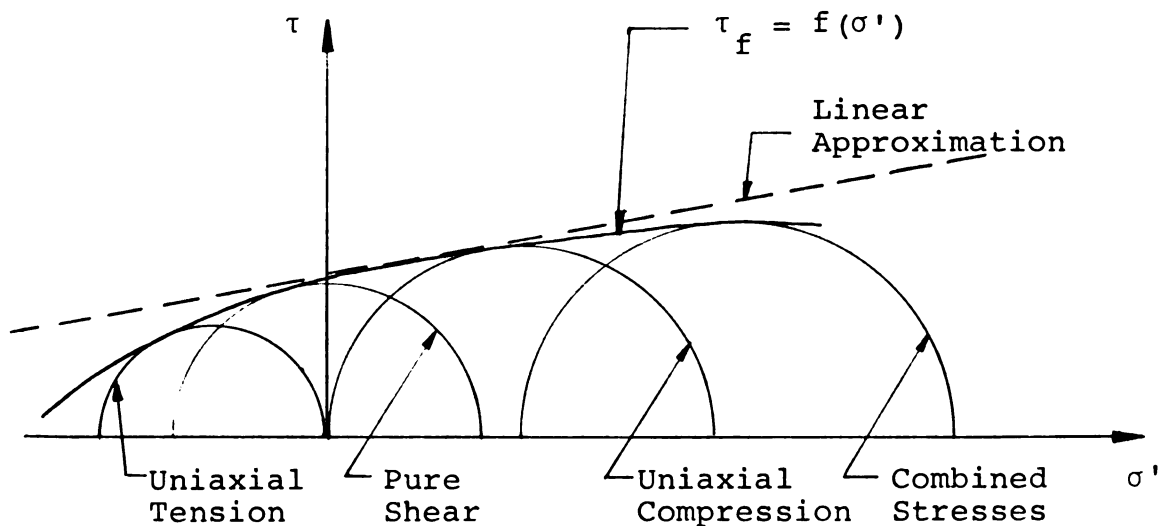


Figure 3-3.--Failure Envelope of Mohr's Stress Circles.

The curve representing this function in the system of rectangular coordinates, (σ', τ) , is the envelope of all Mohr circles representing the limiting stress states.

The failure envelope of normally consolidated cohesive soils is a good approximation to the straight line

$$\tau_f = c'_o + \sigma' \cdot \tan \phi' \quad (3-8)$$

where τ_f = shear strength, σ' = effective normal stress, and the definitions of c'_o and ϕ' are shown in Figure 3-4. This straight line approximation of Mohr's failure envelope expresses Coulomb's failure criterion.

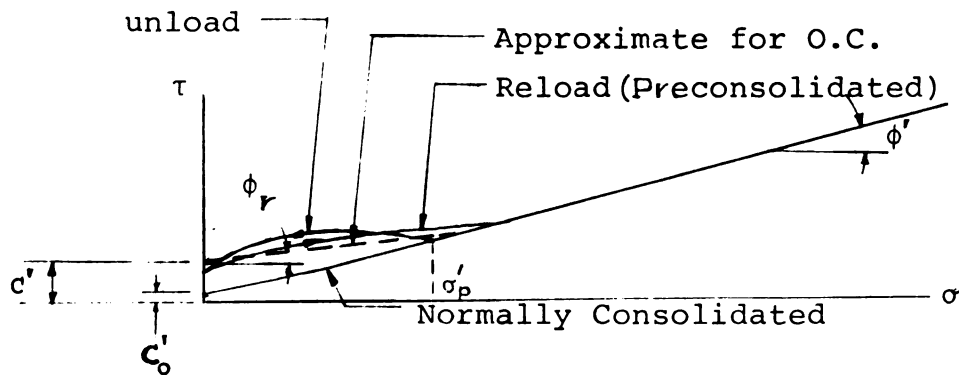


Figure 3-4.--Common Forms of Stress Envelopes for Normally Consolidated and Preconsolidated Soils.

The envelopes of the limiting stress states for unloading or reloading of preconsolidated cohesive soils are usually curved lines forming a hysteresis loop as shown on Figure 3-4. The preconsolidation load is denoted by σ'_p in this figure. If, in the first approximation, this hysteresis is replaced by a straight line, Coulomb's failure criterion can also be applied to over consolidated clays.

$$\tau_f = c' + \sigma' \tan \phi_r. \quad (3-9)$$

For normally consolidated cohesive soils, c'_0 is small in value and can usually be neglected. From equation (3-8), we have

$$\tau_f = \sigma' \cdot \tan \phi'. \quad (3-10)$$

For plane-strain problems the failure condition is assumed to be independent of the intermediate stress. Using the notation in Figure 3-5, Coulomb's failure criterion can be expressed as follows:

$$\frac{\sigma'_1 - \sigma'_3}{2} = (c' \cot \phi' + \frac{\sigma'_1 + \sigma'_3}{2}) \sin \phi', \quad (3-11)$$

$$\text{or} \quad \sigma'_1 - \sigma'_3 = 2c' \cos \phi' + (\sigma'_1 + \sigma'_3) \sin \phi'. \quad (3-12)$$

The influence of the mean principal stress is partly taken into account when using the failure law

$$\tau_{\text{oct}} = f(\sigma'_{\text{oct}}). \quad (3-13)$$

Coulomb's failure theory has been experimentally proven to represent the characteristics of soils, and will be used as the failure criterion in this paper (even though the effect of the intermediate principal stress as assumed is not always true).

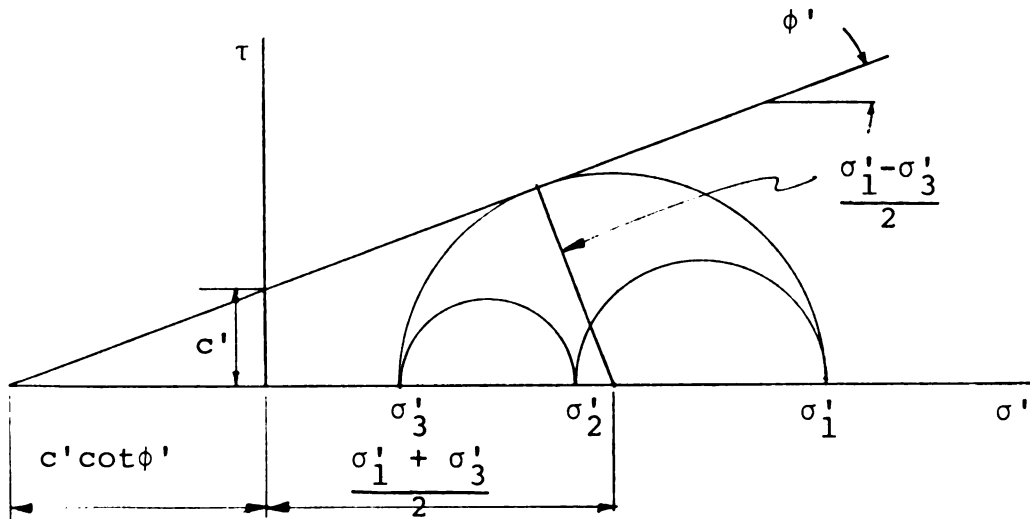


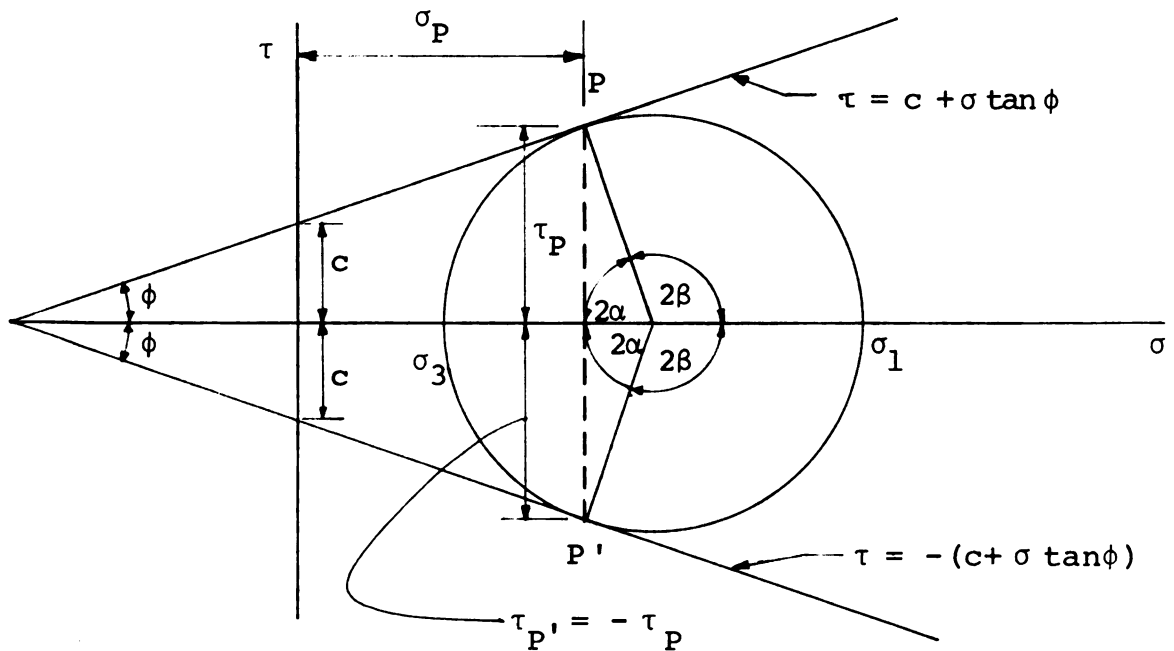
Figure 3-5.--Mohr's Circles for Plane-Strain Conditions.

3.4 Slip Lines

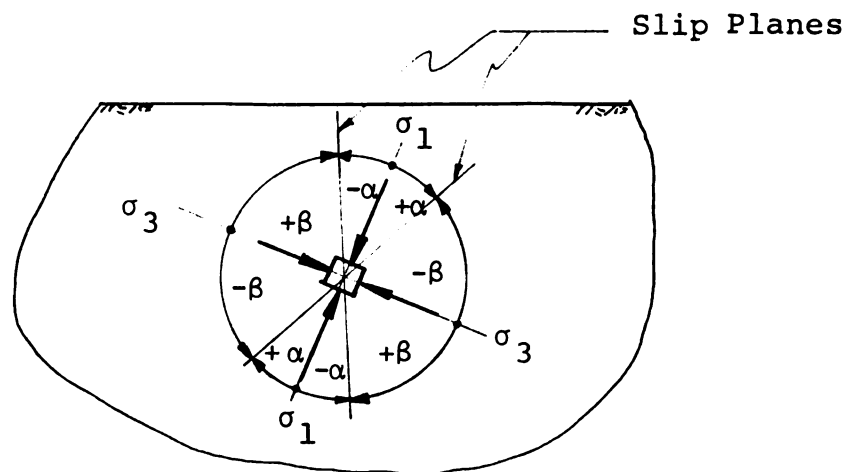
The stress state at any point in a continuous body can be represented graphically by Mohr's stress circle on the $\sigma - \tau$ plane (Figure 3-6a). According to Coulomb's criterion, the strength envelope is represented by two straight lines on the $\sigma - \tau$ plane. The equations for this envelope are

$$\tau = \pm (c + \sigma \tan \phi). \quad (3-14)$$

As the stress state reaches the yield point, Mohr's circle will touch the envelope at two points, P and P'. The stress conditions at points P and P' will represent stress states on the slip planes (or surfaces) at any



a) Mohr's Circle Representing Stress Condition at Yield.



b) Orientation of Slip Planes with Respect to the Principal Stresses.

Figure 3-6.--Graphical Representation of Stress at a Point.

point in the medium. The two surfaces are symmetrically oriented relative to the directions of the principal stresses (Figure 3-6b). The rupture surfaces are mutually oriented at the acute angle of $(\frac{\pi}{2} - \phi)$, or at the obtuse angle $(\frac{\pi}{2} + \phi)$. With respect to the major principal stress, the rupture surface makes an angle.

$$\alpha = \pm (\frac{\pi}{4} - \phi / 2), \quad (3-15)$$

and relative to the minor principal stress, an angle

$$\beta = \pm (\frac{\pi}{4} + \phi / 2). \quad (3-16)$$

If a plastic region forms within the medium and the stress field is known within the region, a field of slip surfaces can be graphically plotted within the zone.

3.5 Behavior of Soils Under Load

Soil behavior under load depends heavily on the type of soil. Two major categories, cohesionless and cohesive, are conventionally classified for soils, with different strength characteristics being obtained for each of these soil types.

3.5.1 Behavior of Cohesionless Soils

Cohesionless soils are normally composed of bulky grains ranging in shape from angular to well rounded. The behavior of granular soils under normal and shear stresses is presented briefly in this section.

When loads are applied, see Figure 3-7, the soil particles deform more or less elastically in the first stage. Secondly, there is local crushing at the most highly stressed points of contacts. Thirdly, both the elastic distortion and the crushing cause slight translation and rotation of the grains, increasing the size of some of the voids and decreasing others.

Both the previous confining stress and the stress level at the beginning of a stress increment influence the deformation resulting from the stress increment. The higher the degree of confinement, the greater the previous crushing and local adjustments, and therefore the less additional strain produced by additional increments of shear stress.

If the shear stress is increased further, two additional responses are evident. First, the particles tend to roll across one another (Figure 3-7). The resistance depends on their angle of contact and is proportional to the confining stress σ . The total resistance to rolling is the statistical sum of the behavior of all the particles. The second mechanism is sliding of one grain across the other. The resistance to sliding is essentially friction, which is proportional to the confining stress. A third mechanism involves the interference and interlocking of the corners of the more angular, irregular particles.

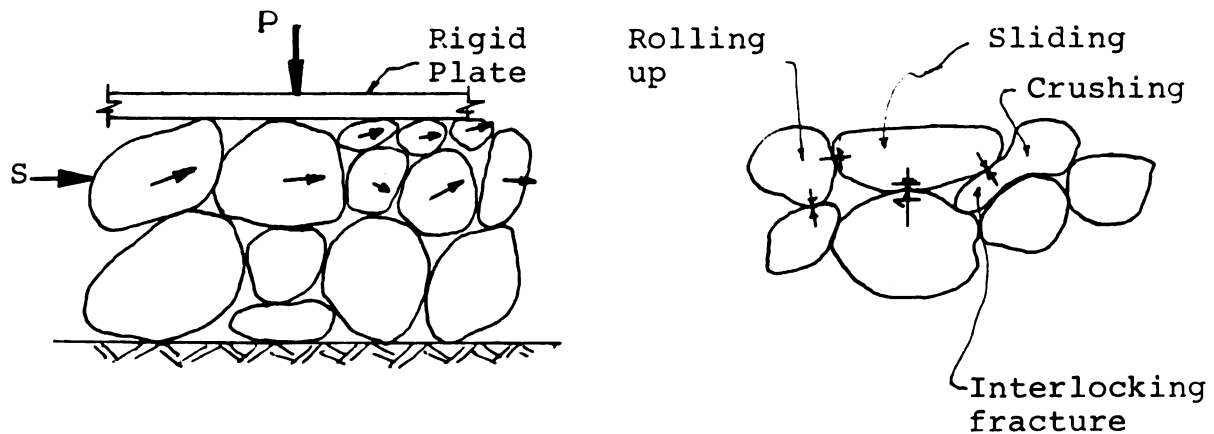


Figure 3-7.--Mechanism of Deformation and Shear in a Mass of Bulky Grains.

If the shear stress becomes sufficiently large, the effect of the distortion, crushing, shifting, rolling and sliding of the grains will be continuous movement and distortion of the mass, or "shear failure."

A typical stress-strain curves for initially loose and initially dense material subjected to an increasing shear stress with a constant confining stress, σ_3 , are shown on Figure 3-8. Both curves exhibit strains that are approximately proportional to stress at low stress levels, suggesting a relatively large component of elastic distortion. If the stress is reduced, the unloading stress-strain curve is nearly the same as the loading curve. However, not all the strain is recovered upon unloading, indicating some particle re-orientation and point crushing, with consequent loss of energy. The

hysteresis loss, the area of the stress-strain loop, represents this energy lost. Loose soils with larger voids and fewer points of contact exhibit greater strains and less recovery of strain upon unloading than dense soils. At higher stresses the strains are proportionally greater, indicating greater crushing and repositioning.

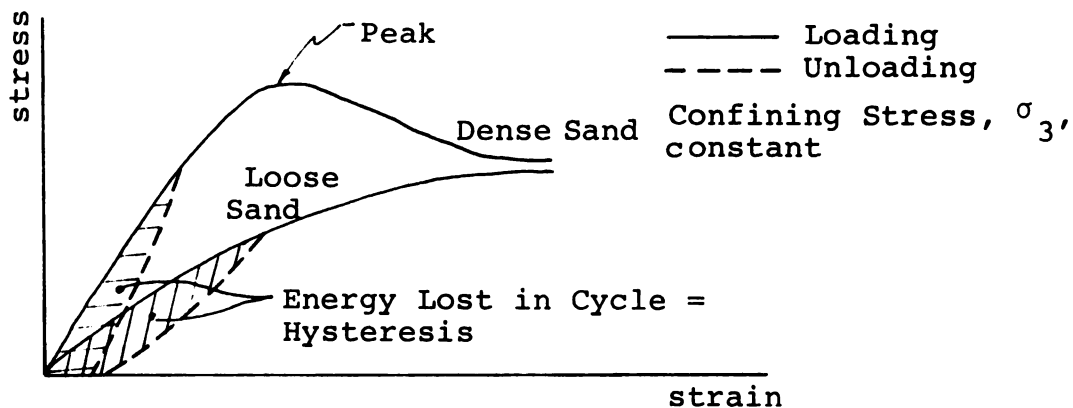


Figure 3-8.--Stress-Strain in Cohesionless Soil.

The results of tests on dry cohesionless soils show that the shear stress at failure, termed the shear strength, S , is nearly proportional to the normal effective stress on the failure surface, σ' . The Mohr's envelope is approximately a straight line through the origin which makes an angle of ϕ' with the σ -axis (see Figure 3-9). The equation for soil strength is given by

$$S = \sigma' \tan \phi'. \quad (3-17)$$

The angle ϕ' is termed the angle of internal friction.

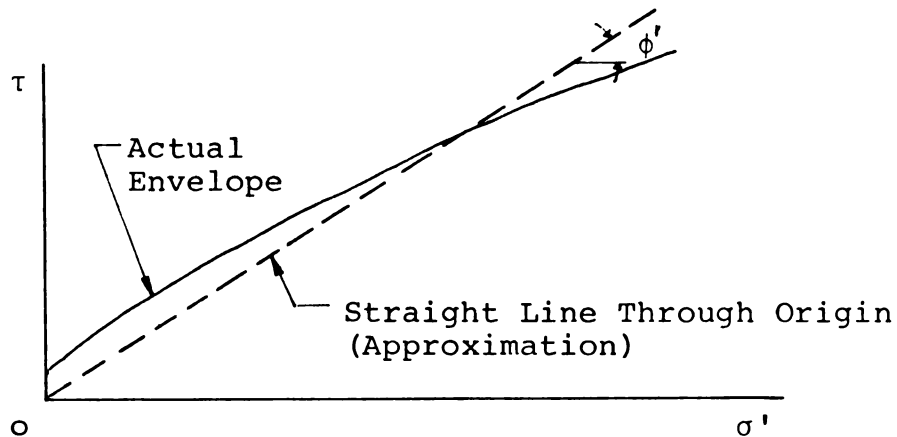


Figure 3-9.--Mohr's Envelope for Cohesionless Soil.

In general the major factors in failure are rolling and sliding. The sliding resistance of the grains is determined by the effective stress, the coefficient of friction between the minerals, the surface roughness, and the angle of contact between the grains. These in turn depend on the grain shape and the soil structure as reflected in the relative density. The resistance to rolling depends on the particle shape, the gradation (particle sizes), and the relative density. As a result the angle of internal friction is greater than the angle of friction between the minerals, and it varies with grain shape, gradation, and relative density. The Mohr's envelope is not always perfectly straight nor does it always pass through the origin because the resistance to rolling is present even with no confinement. At very high stresses the envelope may be curved concave downward due to fracture of some of the grains.

3.5.2 Behavior of Cohesive Soils

In cohesive soils, the shearing process is more complex. Like cohesionless soil, clay is made up of discrete particles which must slide or rotate for shear to take place. However, there are a number of significant differences:

(a) The soil is relatively compressible, therefore when a load is applied to saturated clay, it is initially supported by neutral stresses and is not immediately transmitted to the soil structure.

(b) The permeability of the clay is so low that the neutral stresses produced by the load are dissipated very slowly. Therefore it may be months or even decades before the soil structure feels the full stress increases.

(c) There are significant forces developed between the particles of clay by their mutual attraction and repulsion.

Because of the slow changes in the neutral stress and the corresponding slow changes in effective stress, the strength of clays is defined in terms of neutral stress dissipation. Three basic conditions are defined:

- (1) Drained or slow shear (consolidated-drained).
- (2) Consolidated-undrained.
- (3) Undrained (or unconsolidated-undrained) or quick shear.

In drained shear there is no neutral stress change and any increase in total stress produces a corresponding increase in effective stress. The soil consolidates, reducing the void ratio and water content. As a result of consolidation the particle spacings are reduced and inter-particle bonds are increased in proportion to the confining stress. The shear strength, therefore, increases in proportion to the effective confining stress, and Mohr's envelope is a straight line through the origin (Figure 3-10). The angle, ϕ_D , is called the angle of shear resistance of apparent internal friction (typical values lie between 5° to 30°). When a clay has been preconsolidated to a

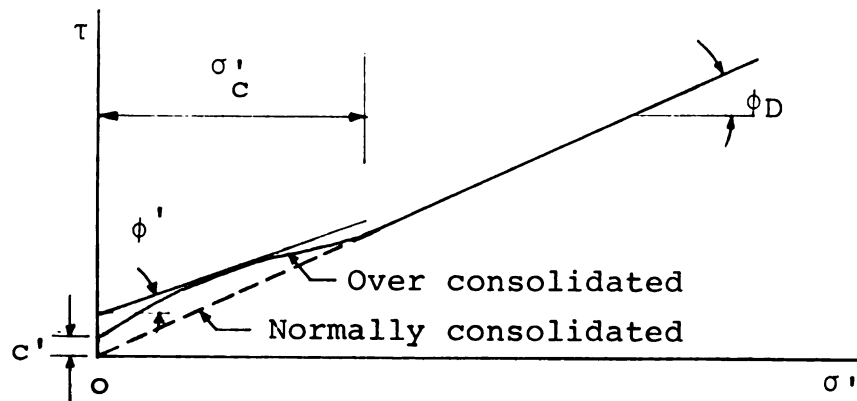


Figure 3-10.--Mohr's Envelope for Saturated-Clay in Drained Shear.

stress of $\bar{\sigma}_c$ and then unloaded, the particles do not return to their original spacing and previously higher void ratio. As a result the interparticle attractive force is not reduced and the strength at stresses less than the preconsolidation load is no longer proportional to the effective confining pressure but is some what higher (Figure 3-10). Strength under confining pressures above the preconsolidation load is given by

$$S = \sigma' \tan \phi_D. \quad (3-18)$$

Strength under confining pressures below the preconsolidation load is given by

$$S = c' + \sigma' \tan \phi'$$

where c' = intercept on the τ axis, and ϕ' = is as shown on Figure 3-10.

The drained shear condition represents the strength of soil developed by a long-term stress change. However, it can be used for any problem involving shear in saturated clays if effective stresses are used. Many slope failures are due to reduction of soil strength taking place over a long time.

In consolidated-undrained shear, the soil consolidates completely under some confining stress, with a corresponding reduction in void ratio and water content.

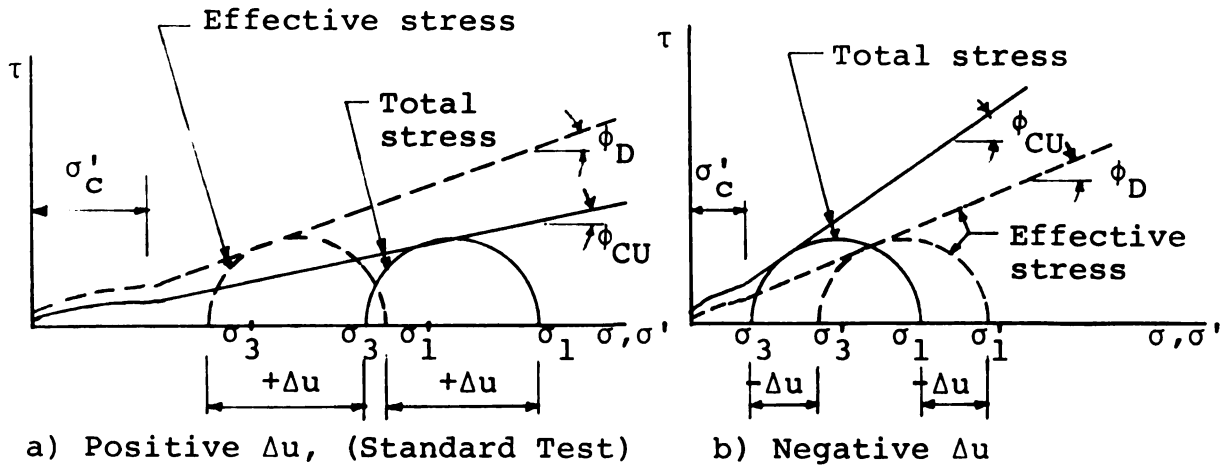


Figure 3-11.--Mohr's Envelope for Saturated Clay in Consolidated-Undrained Shear.

A plot of the effective stresses will give the Mohr's envelope for drained shear (Figure 3-11). A different envelope will be produced, if total stresses are plotted. The total stress circles are displaced horizontally to the right or left by an amount equal to the neutral stress ($\pm u$). The neutral stress developing during the undrained shear process depends on two factors: sign of stress changes and over-consolidation ratio of the soil sample. Normally consolidated clays develop positive pore pressure under increasing total stress and vice versa, while heavily over-consolidated clays give opposite results. Envelopes of the total stresses are presented in solid lines (Figure 3-11). The straight line portion represents results of normally consolidated clays. The curved portion represents

results of over-consolidated clays. The straight line portion of the apparent Mohr's envelopes of total stresses will pass through the origin and will have an apparent angle of shear resistance, ϕ_{CU} , which is about half of ϕ_D for the standard triaxial test (Bishop et al., 1957).

The effective stress envelope is always the same under any stress condition, while the apparent Mohr's envelope changes with changes of stress condition. The apparent angle of shear resistance, ϕ_{CU} , is less than ϕ_D under the condition which the total stresses are greater than the effective stresses (Figure 3-11a), and vice versa (Figure 3-11b).

This test is frequently employed in the analysis of embankment foundations, where the clay soil is first fully consolidated by its own weight and later subjected to a sudden increase in stress by the embankment load.

In undrained shear, both the confining and shear stresses are applied so rapidly that virtually no consolidation takes place. The soil void ratio and water content remain essentially unchanged, and neutral stress supports all of the added loads. The soil in its initial state supported an overburden pressure σ'_0 (or a pre-consolidation load σ'_c), under which it consolidated to

establish its void ratio, water content, and interparticle spacing. The soil strength resulting from this initial effective stress can be obtained by using the Mohr's envelope of drained shear shown on Figure 3-12. An increased confining pressure, $\Delta\sigma_3$, and axial load $\Delta\sigma_1$, is carried almost entirely by neutral stress, and the void ratio, interparticle spacing, and resulting undrained soil strength remain essentially unchanged.

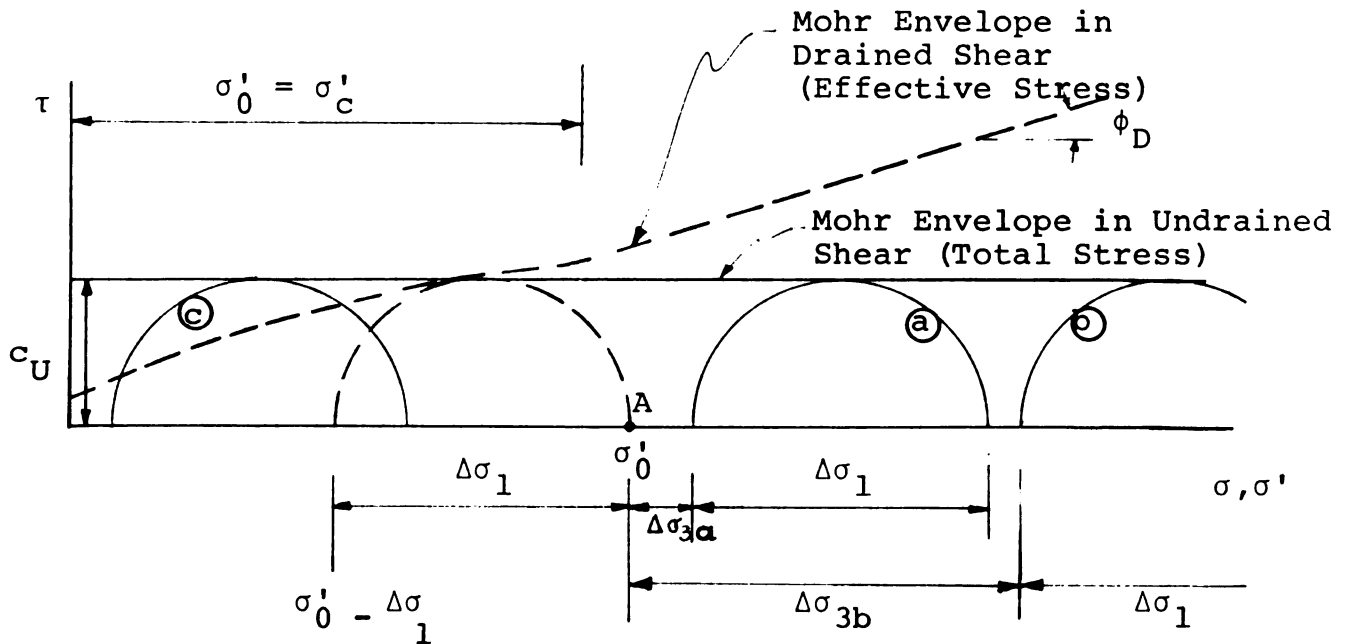


Figure 3-12.--Mohr's Envelope for Saturated Clay in Undrained Conditions.

The stress conditions during loading are tabulated in Table 3-1.

TABLE 3-1.--Tabulation of Stresses for Undrained Loading.*

Loading	Total Stress	Neutral Stress	Effective Stress
Overburden (initial condition)	$\sigma_1 = \sigma'_0$	$u = 0$	$\sigma'_1 = \sigma'_0$
	$\sigma_3 = \sigma'^{**}_0$	$u = 0$	$\sigma'_3 = \sigma'_0$
Adding confinement, $\Delta\sigma_3$	$\sigma_1 = \sigma'_0 + \Delta\sigma_3$	$u = \Delta\sigma_3$	$\sigma'_1 = \sigma'_0$
	$\sigma_3 = \sigma'_0 + \Delta\sigma_3$	$u = \Delta\sigma_3$	$\sigma'_3 = \sigma'_0$
Adding axial load, $\Delta\sigma_1$	$\sigma_1 = \sigma'_0 + \Delta\sigma_3 + \Delta\sigma_1$	$u = \Delta\sigma_1 + \Delta\sigma_3$	$\sigma'_1 = \sigma'_0$
	$\sigma_3 = \sigma'_0 + \Delta\sigma_3$	$u = \Delta\sigma_1 + \Delta\sigma_3$	$\sigma'_3 = \sigma'_0 - \Delta\sigma_1$

*The pore pressure, parameter, A, has been assumed equal to 1

**In many cases the minor principal stress from the overburden load will be less than the major principal stress (i.e., $K_0 < 1$) but this does not alter the neutral stress effects described in the table.

The effective minor principal stress is independent of the added confining stress, and therefore the effective major principal stress at failure and the strength depend only on the original overburden stress, σ'_0 , and the effective (drained shear) envelope. A plot of the total stress, the solid lines on Figure 3-12, shows a series of Mohr's circles. All have the same diameter, and the resulting envelope of total stresses is a horizontal straight line. The intercept of the envelope on the τ -axis is approximately equal to the shear strength (or cohesion) of the soil in its original condition, consolidated by the overburden stress, σ'_0 . The apparent angle of friction, ϕ_U , is zero. However, the angle of the failure plane α is determined by ϕ_D , and is not 45° as might be assumed with $\phi_U = 0$. Only soils for which the difference between ϕ_D and the true angle of internal friction (Leonards, 1962) is negligible are considered.

The undrained strength depends on the original overburden stress, σ'_0 (or σ'_c , consolidation stress), and on the drained Mohr's envelope. In a compressible soil such as an unsaturated clay, the overburden stress is related to the void ratio by the stress-void ratio curve. As a result the undrained strength of a saturated clay increases with decreasing void ratio and also decreasing water content.

Since most construction proceeds rapidly compared with the rate of clay consolidation, undrained strength is used in most problems of design. Even where the construction is so slow that some strength increase will develop, the undrained strength is frequently used because it is the minimum strength and therefore conservative. Caution must be exercised in using the undrained shear strength in the analysis of the problems where the final stress is less than the original overburden load (or consolidation stress), such as the design of excavated slopes or the study of land slides.

Undrained shear involves little or no volume change, but only distortion of the mass (Poisson's ratio is nearly 0.5). The shape of the stress-strain curve (Figure 3-13) depends largely on the interparticle bonding imposed by preconsolidation and by the structure. For undisturbed clays, the initial portion of the curve is straight, probably reflecting the distortion of the bonds. The curve flattens as increasing numbers of bonds are broken.

If a sample of undisturbed saturated clay is completely remolded without changing in water content, and then tested, it will be found that the undrained strength has been reduced. The difference between the undisturbed and remolded soil is greatest in sensitive

clays. Because of their flocculent structure and edge to face bonding, they are relatively rigid and have much higher initial values of Young's Modulus for a given water content than an insensitive clay with a more oriented structure. The loss

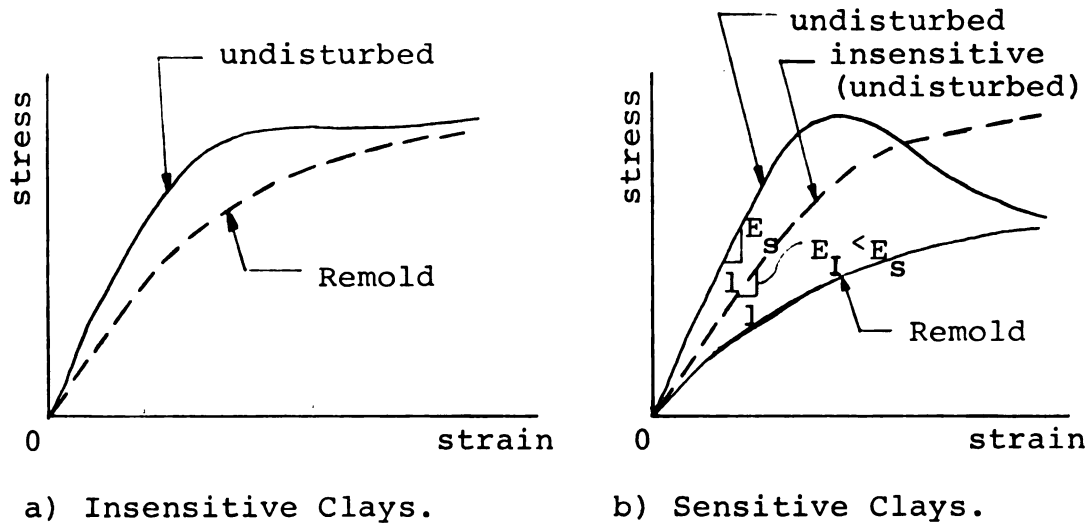


Figure 3-13.--Stress-strain Curves of Clays in Undrained Shear.

of strength in remolded clays is caused by a break down in the soil structure and a loss of the interparticle attractive forces and bonds. In clays originally with a dispersed structure, the loss is small, while in clays with a highly flocculent structure or soils with a well-developed skeletal structure, the loss in strength can be large.

The sensitive clay reaches a peak strength similar to that of dense sand, and then becomes weaker with

increasing strain (Figure 3-14b). The strength remaining after large strain (after failure) is the "residual strength" and is approximately equal to the remolded strength (Figure 3-14a).

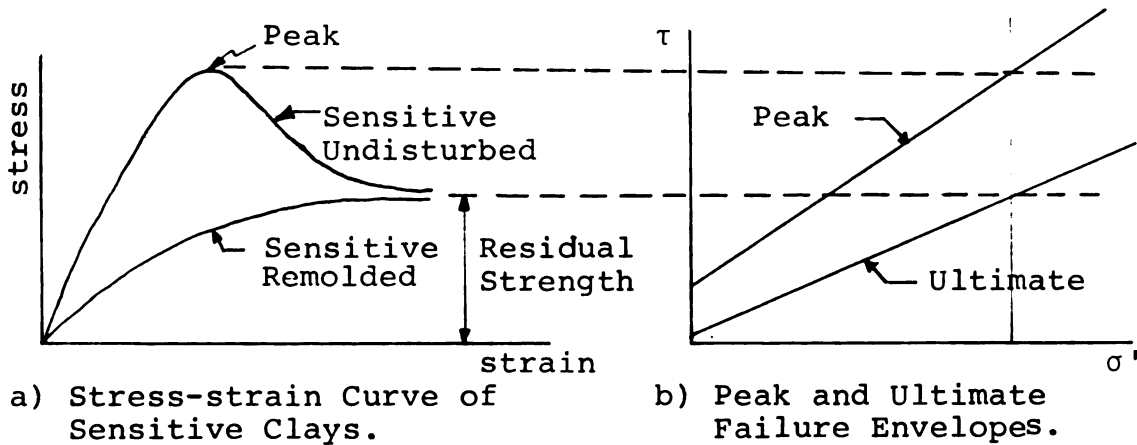


Figure 3-14.--Peak and Residual Strengths of Sensitive Clays.

The reduction of strength can occur in slope embankments on very sensitive clay foundations where the change is not only in strength but also in elasticity properties of the clay. The modulus of elasticity will decrease with an increase in deviatoric stress (or shear strain) for clay. For sand the effect of deviatoric stress is very small in contrast with that of the mean normal stress or the stress invariant $(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3})$. With an increase of the stress invariant, the modulus of elasticity of sand will also increase. The effects mentioned above can induce local failure in a slope

embankment, on sensitive clay, which changes the stress pattern in the soil mass. The changes of stress state in the soil mass in addition to the decrease in modulus of elasticity in the failure zone (for sensitive clays) can develop a progressive failure zone in the soil mass. Lower elastic modulus in the area means larger deformation of the zone which induces great change in the state of stresses in the adjacent area and further complete failure zones can be developed. Slope failure can occur by the effect of local strength reduction.

In general, the stress-strain behavior of any soil is obtained from the triaxial test. The stress path in the conventional triaxial test can be any of those shown on Figure 3-15.

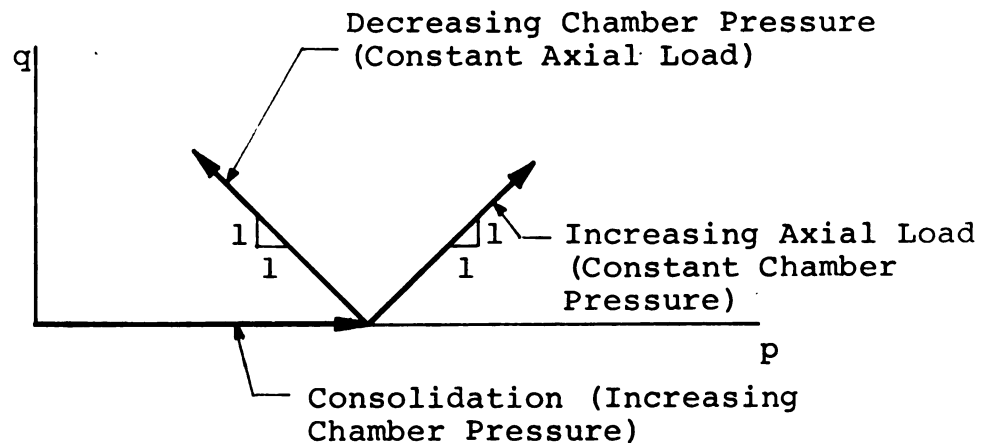


Figure 3-15.--Stress Path in the General Triaxial Test.

Each stress path will produce a certain stress-strain relationship. For example, in drained-shear, consolidation of soil takes place in two stages: first during the addition of the confining stress and second during the addition of axial stress that produces shear. In a conventional consolidation test where $\epsilon_2 = \epsilon_3 = 0$, the lateral stress is a constant fraction of the vertical stress. Although the mechanisms are similar, the stress-void ratio curve is different because the stress fields are different. The stress paths of the two processes are shown on Figure 3-16. The stress-strain curves are shown on Figure 3-17 for different stress paths in drained-triaxial tests.

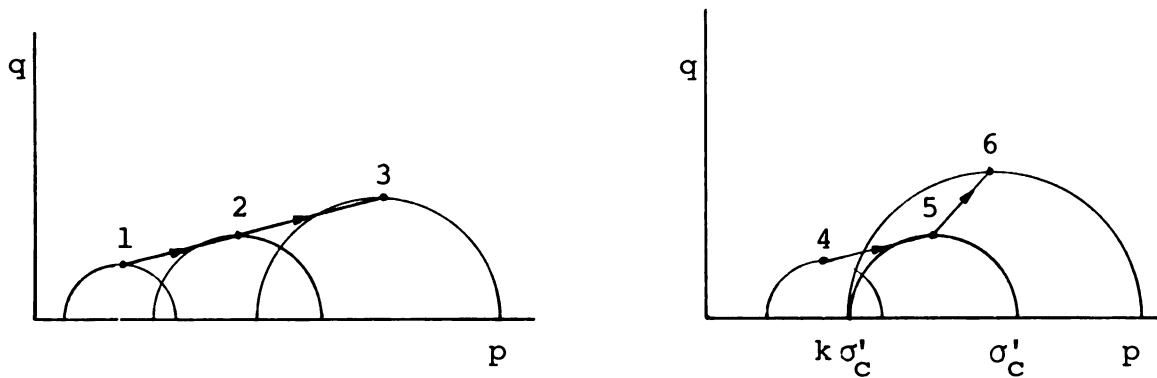
Consider the constitutive equation of the form

$$E_{ij} = -\frac{\nu}{E} T'_{kk} \delta_{ij} + \frac{1+\nu}{E} T'_{ij}. \quad (3-19)$$

If the major principal axis coincides with the z-axis, the equation can be written in engineering notation

$$\begin{aligned} \epsilon_x &= \frac{1}{E} [\sigma'_x - \nu(\sigma'_y + \sigma'_z)], \\ \epsilon_y &= \frac{1}{E} [\sigma'_y - \nu(\sigma'_x + \sigma'_z)] \end{aligned} \quad (3-20)$$

and
$$\epsilon_z = \left[\frac{1}{E} \sigma'_z - \nu(\sigma'_x + \sigma'_y) \right].$$



a) Successive Stresses for Increasing Load with a Constant Ratio of Axial to Confining Stress, 1-2-3 (as in Simple Consolidation Test).

b) Increasing Load with Constant Ratio of Axial to Confining Stress (4-5) Followed by Increasing Axial Stress at Constant Confining (5-6).

Figure 3-16.--Stress Path for Plane of Maximum Shear Stress in Triaxial Compression.

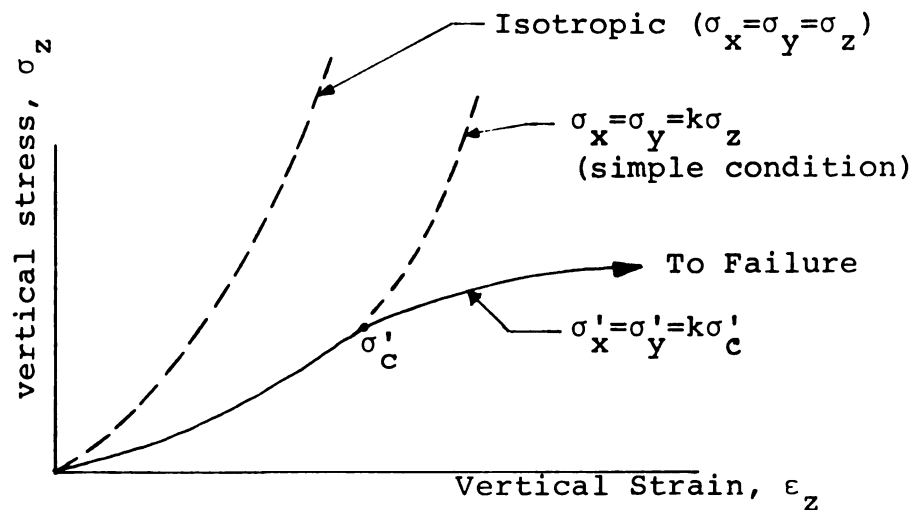


Figure 3-17.--Stress-strain in Drained Shear for Different Loading Systems.

In the triaxial test $\sigma'_x = \sigma'_y$, and the axial strain can be reduced to the form

$$\epsilon_z = \frac{1}{E}[\sigma'_z - 2\nu\sigma'_y]. \quad (3-21)$$

With the assumption of incompressible soil ($\nu = 0.5$), we have

$$\epsilon_z = \frac{1}{E}[\sigma'_z - \sigma'_y] \quad (3-22)$$

$$\text{or} \quad E = \left(\frac{\sigma'_z - \sigma'_y}{\epsilon_z} \right). \quad (3-23)$$

The relationship can be obtained from the test as a typical result for elastic behavior of the soil during the increment of axial stress. It can be seen from equation (3-23) that the modulus of elasticity depends on two parameters, $(\sigma'_z - \sigma'_y)$ and ϵ_z . The effect of chamber pressure (σ'_y) on E has been proposed for both clays and sands (Lambe et al., 1969). In the real situation σ'_z and σ'_y change simultaneously during the loading process, with the ratio depending very much on the loading system. The relationship can be of any type depending on the behavior of the loading. In plane strain problems, the effect of the intermediate stress (σ'_x , $\sigma'_x \neq \sigma'_y$) makes the relationship rather complicated and unpredictable. The only reasonable method is to investigate the stress path during the loading process at each point, and apply to those points the E 's obtained experimentally according to the stress path.

For the conventional consolidation test $\epsilon_x = \epsilon_y = 0$, and $\sigma_x = \sigma_y$. So equation (3-20) becomes

$$\sigma'_x = \sigma'_y = \frac{\nu}{(1-\nu)} \sigma'_z \quad (3-24)$$

and

$$\epsilon_z = \frac{1}{E} \left[\frac{1-\nu-2\nu^2}{(1-\nu)} \right] \sigma'_z. \quad (3-25)$$

For incompressible materials ($\nu = 0.5$), we have

$$\sigma'_x = \sigma'_y = \sigma'_z$$

and

$$\epsilon_z = \frac{1}{E} \left[\frac{0}{0.5} \right] \sigma'_z = 0, \quad (3-26)$$

which cannot be identified. For compressible material ($\nu < 0.5$), and the relationship can be identified by equation (3-25).

Experimental results show that failure cannot occur (unless stresses are very high) under consolidation stresses. For compressible material ($\sigma'_x = \sigma'_y \neq \sigma'_z$) a certain amount of deviatoric stress ($\sigma'_z - \sigma'_x$) may develop, but not large enough to produce failure.

It will be shown in the discussions that, development of stress paths at different points in the slope body during an increment of load are quite different. Only two categories of stress path (conventional consolidation;

Figure 3-16a; and triaxial-test; Figure 3-16b) are described here. The consequence of the variation in stress path at different locations within the soil body is a variation in stress-strain relationship or the modulus of elasticity. The effect is quite complicated for calculation purposes, and will not be considered in this paper.

3.6 Pore Pressure

Development of pore pressure during application of load is considered only for clay. With time this fluid stress will dissipate, transferring an increasing amount of the applied stress to the soil skeleton, with resulting increasing settlements with time. The process is called "consolidation."

It is assumed in this derivation that the structural skeleton of the soil is linearly elastic in its behavior, and is homogeneous, nondilatant and isotropic. The fluid in the pore space is also assumed to have a linear relationship between volume change and stress.

At the instant of applying load (before drainage can begin), the volume change of the soil skeleton is

$$\Delta V = V \left(\frac{1-2\nu}{E} \right) [\Delta\sigma'_1 + \Delta\sigma'_2 + \Delta\sigma'_3] \quad (3-27)$$

where $\Delta\sigma'_1$, $\Delta\sigma'_2$, $\Delta\sigma'_3$ are the increments of the effective principal stresses due to the applied load, V = the initial volume, E = modulus of elasticity, and ν = Poisson's ratio of the soil skeleton.

The decrease in volume of the soil skeleton is almost entirely due to the decrease in the volume of the voids. If n is the initial porosity, and C_w the compressibility of the fluid in the pore space, this volume change is related to the pore-pressure change, if no drainage occurs, by the expression

$$-\Delta V = nVC_w \Delta u. \quad (3-28)$$

It follows, therefore, that

$$nC_w \Delta u = \frac{1-2\nu}{E} [\Delta \sigma'_1 + \Delta \sigma'_2 + \Delta \sigma'_3]. \quad (3-29)$$

Equation (3-29) can be modified for the case of triaxial tests, where $\Delta \sigma_2 = \Delta \sigma_3$ as follows:

$$\Delta u = \frac{1}{1+n(C_w/C_c)} [\Delta \sigma_3 + \frac{1}{3}(\Delta \sigma_1 - \Delta \sigma_3)], \quad (3-30)$$

where $C_c = \frac{3(1-2\nu)}{E}$ = the compressibility of the soil skeleton, or in terms of empirical parameters (A and B) by

$$\Delta u = B[\Delta \sigma_3 + A(\Delta \sigma_1 - \Delta \sigma_3)]. \quad (3-31)$$

For plane strain problems $\Delta \sigma_2 \neq \Delta \sigma_3$, so the modified equation can be expressed as follows:

$$\Delta u = \frac{1}{(1+nC_w/C_c)} \left[\frac{\Delta \sigma_1 + \Delta \sigma_2 + \Delta \sigma_3}{3} \right], \quad (3-32)$$

or in terms of the empirical parameter, B,

$$\Delta u = B \left[\frac{\Delta \sigma_1 + \Delta \sigma_2 + \Delta \sigma_3}{3} \right] = B \cdot \Delta \sigma_{\text{oct}}. \quad (3-33)$$

CHAPTER IV

ELASTO-PLASTIC INCREMENTAL APPROACH

Very few investigations have been made into the behavior of combined elastic and plastic zones (Figure 4-1) within a body subjected to load. Exact solutions have been obtained only in a few simple cases, for example axisymmetric (Mendelson, 1968). A few numerical solutions have been obtained (Höeg, Christian, and Whitman, 1968; Dunlop and Duncan, 1970; and Clough and Woodward, 1967). Many uncontained plastic-flow problems (extrusion, drawing, punch, etc.) have been solved by the slip line method, but the solutions are not unique because of the ununiqueness of the slip line field. Behavior of contained plastic zones in connection with incrementally increasing load has not evidently been investigated by any experimental mean. Höeg, Christian and Whitman (1968) state that even after a point yields, according to the Tresca yield criterion, the stress condition continues to change at the point. These stress changes are the consequence of changes in stress at the surrounding elastic points. The sum of the principal stresses can change, thus causing elastic volume changes. The direction of the principal

stresses can change, thus altering the elastic portion of the shear distortion. However, the maximum shear stress at the point cannot increase.

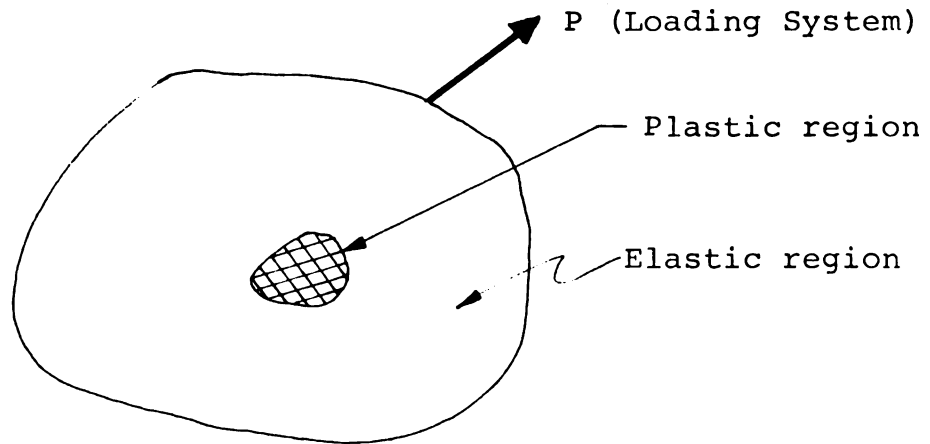


Figure 4-1.--Elasto-plastic System.

The analytical procedure described here will investigate the stresses and displacements within a soil slope, beginning from the initial elastic condition, continuing with the development of local yield, and finally spreading of the yield or plastic zone up to the critical failure condition. The stress-strain relationship of the soil is assumed to be ideal elastic-perfectly plastic or ideal elastic with linear strain hardening in the plastic portion (see Figure 4-2). The former is more applicable for contained plastic behavior (Malvern, 1969).

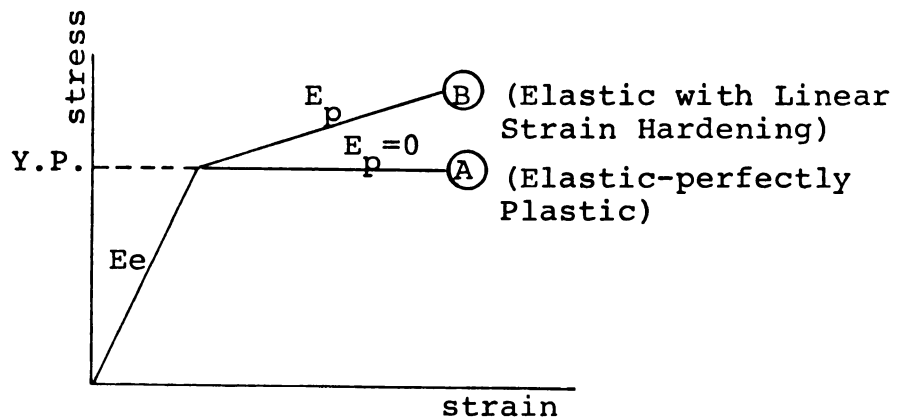


Figure 4-2.--Idealized Stress-strain Curves Applied in Elasto-plastic Problems.

The contained plastic principle is applied to represent the plastic zone (or zones) developing within the elastic regions of the soil slope. The actual stress and deformation fields are calculated at different stages of loading. Large deformations are allowed within the plastic regions, but compatibility of strains at the boundary between the surrounding elastic regions and the plastic zones is still applied. However, with one technique to be described later the strain compatibility condition can be violated in some particular locations. With these combined restrictions, the deformations will be restricted by the surrounding elastic material, while stress deviators within the plastic zone change in strain hardening material assumption with drained strength criterion and remain constant in perfectly plastic material assumption with undrained strength criterion.

The plastic zone will propagate as the load increases, while some local yields may develop in other locations. The yield condition may result from stress states of pure compression, pure tension (tensile strength is very low for soil), or a combined state. The assumed contained plastic flow condition is applied up to the limit of equilibrium (equilibrium will be defined later in this chapter). Beyond this limit, kinematic effects must be considered in analyzing stress and deformation-rate fields. At this stage, the problem becomes one of the uncontained plastic flow category, and can be analyzed by the slip line method (Mendelson, 1968). Since the uncontained plastic flow problem is beyond the scope of this paper, no further presentations will be made here.

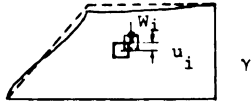
4.1 Energy Approach in Elasto-plastic Incremental Behavior

The energy approach is the most popular method used in analyzing the behavior of a continuous medium under load. Elastic perfectly-plastic or elastic-strain hardening behaviors can be assumed for the material. Gravitational forces and surface forces can be applied to the body. For static conditions, the energy approach yields equilibrium at any location in the body. The equilibrium requirement is independent of material behavior, i.e., whether the material is elastic or plastic.

The material applied in this analysis is assumed to be isotropic and ideal elastic-perfectly plastic. The problem is considered only for conditions of increasing load. Ideal elastic-perfectly plastic behavior can be approximated as a piece-wise linear material. The mechanical process will be explained according to different processes of construction (i.e., different loading systems) as follows:

4.1.1 Single-Lift Construction

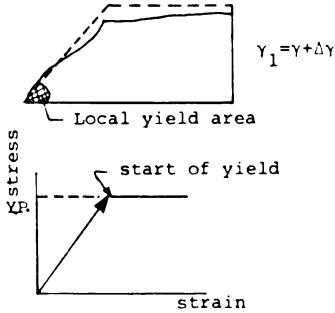
Stress and deformation behavior of a single lift embankment is shown in corresponding steps in Figure 4-3. The height of embankment is assumed to be great enough to cause collapse, and load increase is simulated by considering the soil unit weight to increase incrementally. The soil mass will first deform elastically (Figure 4-3a) storing internal energy within the soil mass due to the external work done by a part of the soil weight. Graphical presentation of elastic stored energy is shown on Figure 4-4a. During this range of deformation the stresses everywhere within the soil body are below the limiting failure condition. As the body force increases homogeneously, the external work done is increased, along with the internal energy. At a certain stage, local yield (or local yields) occurs (Figure 4-3b), and the internal energy is partly dissipated in the yield or plastic zone. The states

Mathematical Presentation

$$\text{Work done by body force} = \int_V \frac{\{x\}^T \{U\}}{2} dv = \int_V \frac{(u_x) \gamma}{2} dv$$

$$\text{Internal energy stored} = \int_V \{\sigma\}^T \frac{\{\delta \epsilon\}}{2} dv$$

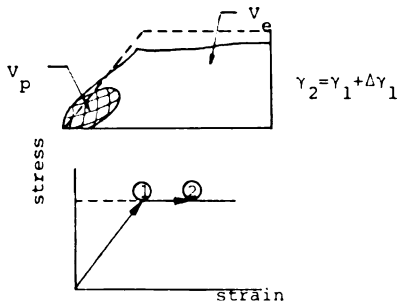
a. Elastic Deformation.



$$\Delta W = \int_{V=V_e} \delta u_z^e \gamma dv + \int_{V=V_e} \frac{\delta u_z^e \Delta \gamma}{2} dv$$

$$\Delta E = \int_{V=V_e} \sigma^T \delta \epsilon^e dv + \int_{V=V_e} \frac{\delta \sigma^T \delta \epsilon^e}{2} dv$$

b. Start of Local Failure.

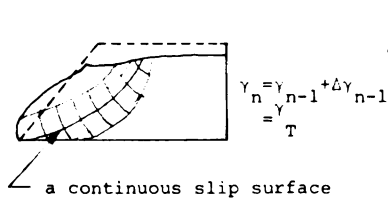


$$\Delta W_1 = \left[\int_{V_e} \delta u_{z1}^e \gamma_1 dv + \int_{V_e} \frac{\delta u_{z1}^e \Delta \gamma_1}{2} dv \right] + \left[\int_{V_P} \delta u_{z1}^P \gamma_1 dv \right]$$

$$\Delta E_1 = \left[\int_{V_e} \sigma_1^T \delta \epsilon_1^e dv + \int_{V_e} \frac{\delta \sigma_1^T \delta \epsilon_1^e}{2} dv \right] + \left[\int_{V_P} \sigma_1^T \delta \epsilon_1^P dv \right]$$

- ① - Condition of new yield points
② - Condition of formerly yield points

c. Spreading of Plastic Zone.



$$\Delta W_n = \left[\int_{V_e} \delta u_{zn}^e \gamma_n dv + \int_{V_e} \frac{\delta u_{zn}^e \Delta \gamma_n}{2} dv \right] + \left[\int_{V_P} \delta u_{zn}^P \gamma_n dv \right]$$

$$\Delta E_n = \left[\int_{V_e} \sigma_n^T (\delta \epsilon_n^e) dv + \int_{V_e} \frac{\delta \sigma_n^T \delta \epsilon_n^e}{2} dv \right] + \left[\int_{V_P} \sigma_n^T (\delta \epsilon_n^P) dv \right]$$

d. Equilibrium State (Limit of Static Condition).

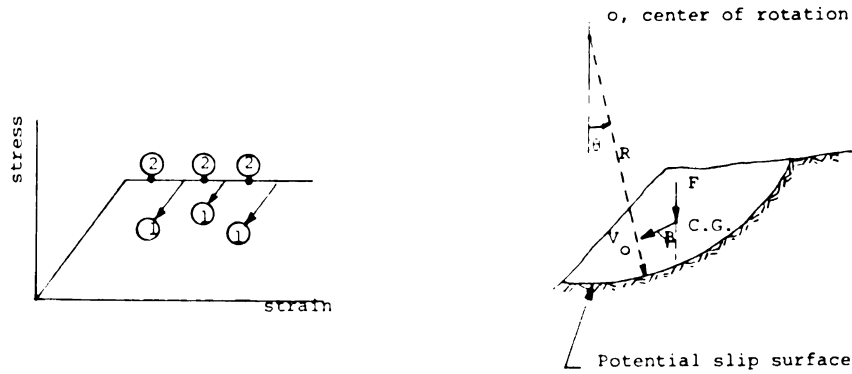


$$W^k/\text{unit time} = (F \cos \beta) V_o$$

$$E^k/\text{unit time} = \int_S \sigma_{ij} \dot{\epsilon}_{ij} ds$$

Part II is assumed to be rigid,
no effect due to
traction-forces.

Part I is also assumed to be a
rigid motion body.



① = drop of stress and strain states from plastic to elastic of points in the plastic area, but not on the critical slip surface.

② = stress and strain states of points on the critical slip surface.

$$W^k \geq E^k + \text{Frictional dissipation}$$

e. Collapse of Slope.

f. Notations:

$X = \{X\} = \{0, 0, \gamma\}$ = body force vector

$\dot{u} = \{\dot{u}\} = \{\dot{u}_x, \dot{u}_y, \dot{u}_z\}$ = displacement vector

$\sigma = \{\sigma\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}\}$ = stress vector

$\dot{\epsilon} = \{\dot{\epsilon}\} = \{\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\epsilon}_{zz}, \dot{\epsilon}_{xy}, \dot{\epsilon}_{yz}, \dot{\epsilon}_{zx}\}$ = strain vector

ϵ_{ij} = strain rate tensor

V = total volume, V^e = volume of elastic part, V^p = volume of plastic part

γ = unit weight of soil, γ^T = total unit weight of soil

ΔW = increment of external work done

ΔE = increment of internal energy

n = total steps of body force increment, i = number of step

e = elastic, p = plastic, K = kinetic, T = transposed, s = surface

V_0 = average velocity of the total velocity field

F = body force of the soil mass I

β = angle between direction of body force and direction of average velocity

Figure 4-3.--Elasto-plastic Incremental Behavior under Gradually Increasing Body Force, Single-Lift Embankment.

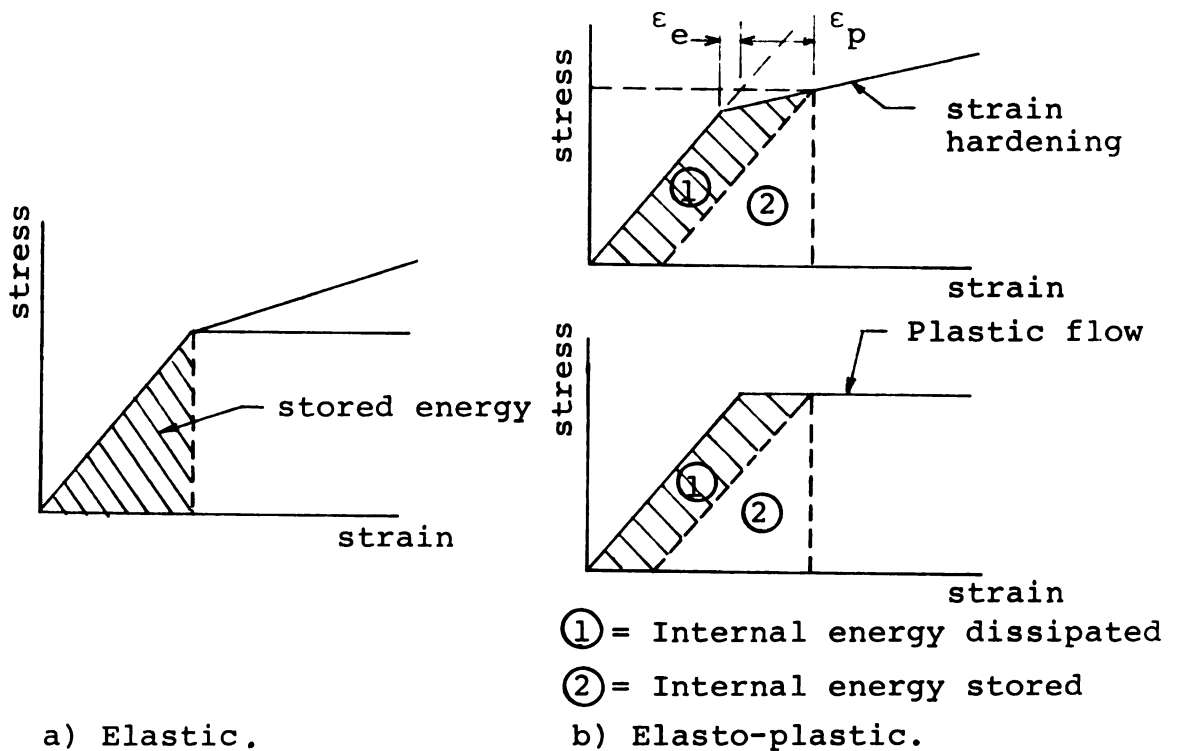
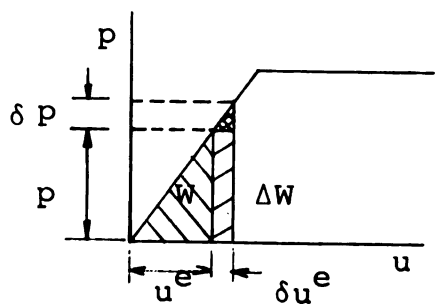


Figure 4-4.--Graphical Presentation of Internal Energy Development.

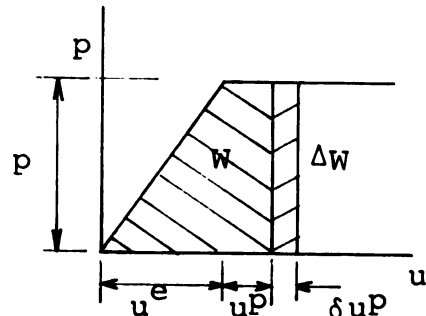
of stress of points within the plastic zone has now reached the yield condition (Mohr's envelope). At this stage, slip lines develop within the plastic zone, but the slope is still in a statically admissible stage, and compatibility and equilibrium conditions are not violated anywhere in the body. Additional storage of the developing internal energy is no longer admitted within the plastic region (for perfectly plastic case). The additional part of the energy absorbed by this zone will be transformed into other forms of energy, e.g., heat (Figure 4-4b). So the total incremental energy will be partly stored in the elastic part and dissipated in the plastic zone. The

state of stresses in the plastic zone can be only slightly changed. The deviatoric stresses in this zone will increase slightly in work hardening material, but must remain constant in perfectly plastic material.



$$W = \frac{1}{2} p u^e; \Delta W = p \delta u^e + \frac{1}{2} \delta p \delta u^e$$

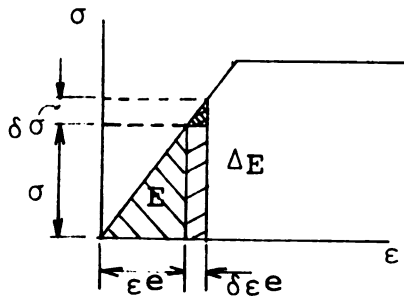
Elastic Portion



$$W = W^e + W^p = \frac{1}{2} p u^e + p u^p; \Delta W = p \delta u^p$$

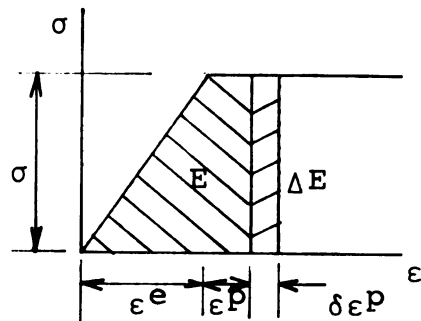
Plastic Portion

a) External Work-done.



$$E = \frac{1}{2} \sigma \epsilon^e; \Delta E = \sigma \delta \epsilon^e + \frac{1}{2} \delta \sigma \delta \epsilon^e$$

Elastic Portion



$$E = E^e + E^p = \frac{1}{2} \sigma \epsilon^e + \sigma \epsilon^p; \Delta E = \sigma \delta \epsilon^p$$

Plastic Portion

b) Internal Energy.

Figure 4-5.--External Work-done and Internal Energy Applied for Case Shown on Figure 4-3.

As more external work is done by further increases in the body forces, the plastic zone (or zones) will enlarge (Figure 4-3c). The soil slope is still in a static state as long as the elastic areas of the soil slope are not completely separated by a plastic zone as shown on Figure 4-3d. When this stage is reached the soil slope is said to be in the limiting equilibrium condition. The slope analysis methods discussed in Chapter II assume the existence of this stage but neglect geometrical changes due to elasto-plastic deformation up to this stage. It is appropriate that an analysis at this stage should be called the "limit-equilibrium-analysis."

A contradiction can be observed between the analysis presented here and the method of characteristics by Sokolovski (1960). The later method determines the limit load that causes failure throughout the soil body, designated as Part I shown on Figure 4-3e, and collapse under this condition. Actually, the true plastic zone does not develop in such fashion and the true limit load will also be different. Any loading system obtained from the characteristic method should be on the upper bound side, and the computed collapse load will be greater than that of the actual collapse load. A collapse mechanism will exist at this stage if there is a continuous slip surface between any two boundaries of the body (Figure 4-3d). If this slip surface has not developed, the

plastic zone will spread further with an additional increase in soil unit weight until a complete slip surface occurs. The slope is now in a kinematically admissible state. It is the optimum possible state that the slope can remain in static condition. After that, movement of the soil body (Part I, Figure 4-3e) can occur any time if only a small additional amount of external work is applied to the body. The soil body (Part I, Figure 4-3e) will move as a rigid body along the slip surface. It is possible that the state of stress of any point in the plastic zone, but not on the slip surface, may drop below the yield envelope during the movement (Figure 4-3e). Further analysis can be done to obtain an upper bound solution. This analysis is concerned with kinetic effects and the equations of motion are used instead of equilibrium equations. A certain amount of power is developed as this soil body I moves in the direction of the average velocity of the velocity field. In this unstable stage the load power must be equal to or greater than the power dissipated along the slip surface.

4.1.2 Incremental Construction

For the case of incremental construction a similar explanation applies, only now the applied external work is due to the additional surface force from additional thin layers of soil.

4.2 Strength and Failure Concepts

In the elasto-plastic incremental process, the development and propagation of the plastic zone (or zones) depends heavily on the strength envelope of the material. As described in Chapter III, the strength characteristics of clay are very dependent on history and on behavior of loading. For saturated clays two strength envelopes, drained and undrained, are normally obtained. In the embankment the application of load is fast enough so that drainage of clays during the construction period can be ignored. The concept of undrained strength ($\phi = 0$) will be applied as the criterion for strength.

However, the orientation of slip lines in the plastic zone cannot be determined from a value of $\phi = 0$, which in general gives the angle α equal to 45° . The true slip angle, α , must be obtained from the drained strength concept (ignoring differences between ϕ' and the true angle of internal friction), and $\alpha = 45 - \phi'/2$. The failure surface will be picked from among those constructed according to the true slip angle. Different types of clays may give similar configurations of plastic zones because of the similar strength concept, but their failure surfaces may be different. In analyzing the stability of a slope, a relationship between the soil strength and the modulus of elasticity is required. Such a relationship

was presented by Skempton and Henkel (1957), who investigated various samples of London clay. They found a linear relationship between the two parameters

$$E = 140 C_U \quad (4-1)$$

This relationship will be used in the stability analysis.

4.3 Stress Conditions during Loading

Stress conditions at a typical point in the slope during construction by lifts up to failure is discussed in this section. It is assumed that remolded soil is compacted in place to full saturation, and that drainage is not allowed during construction. Compaction is assumed to be equal for each layer and the applied work done is assumed to influence only the depth of the increasing layer. Thus, the initial effective stress (σ'_0) is assumed to be homogeneous throughout the soil body, and this establishes the shear strength of the soil.

Figure 4-6 shows the relative location of a typical point, A, for initial, intermediate, and yield conditions during construction. Stress-strain conditions for this point during construction are illustrated on Figure 4-7. The letters a, b and c represent conditions at progressive stages of construction, with yield occurring at c. Figure 4-8 shows Mohr's circles of total stress at these stages of construction. Figure 4-9 shows Mohr's circles for total and for effective stress at failure for point A.

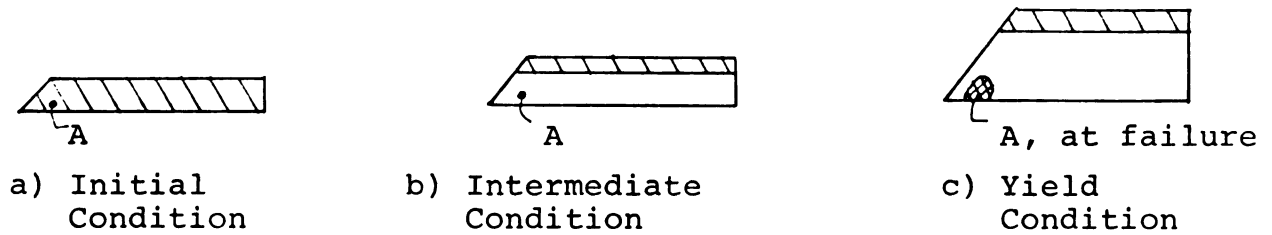


Figure 4-6.--Relative Location of a Typical Point in an Embankment During Construction.

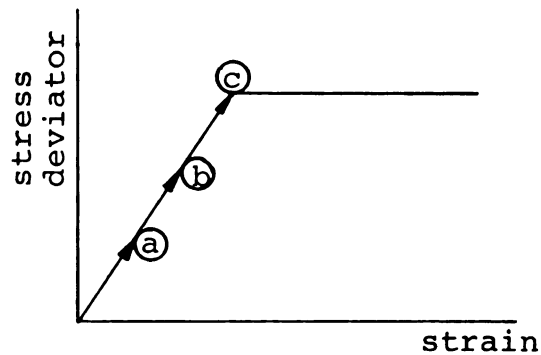


Figure 4-7.--Stress-strain Behavior of the Typical Point.

Stress conditions for the various stages of construction are summarized in Table 4-1.

It is obvious that in the present case the state of effective stresses at failure is independent of the development of neutral stresses. Since the total stress state can be used to specify failure, it is not necessary to recalculate the effective stresses. The effect of neutral stresses will not be considered in this paper.

TABLE 4-1.--Tabular Form of Stress Conditions at Different Stages of Embankment Loading.

Loading	Total Stresses	Neutral Stress (B=1)	Effective Stresses
Initial Stage, Figure 4-6a	$\sigma_1 = \sigma'_0 + \Delta\sigma_{1a}$	$u = \frac{\Delta\sigma_{1a} + \Delta\sigma_{2a} + \Delta\sigma_{3a}}{3}$	$\sigma'_1 = \sigma'_0 + \frac{2\Delta\sigma_{1a} - \Delta\sigma_{2a} - \Delta\sigma_{3a}}{3}$
	$\sigma_2 = \sigma'_0 + \Delta\sigma_{2a}$	$u = \frac{\Delta\sigma_{1a} + \Delta\sigma_{2a} + \Delta\sigma_{3a}}{3}$	$\sigma'_2 = \sigma'_0 + \frac{2\Delta\sigma_{2a} - \Delta\sigma_{1a} - \Delta\sigma_{3a}}{3}$
	$\sigma_3 = \sigma'_0 + \Delta\sigma_{3a}$	$u = \frac{\Delta\sigma_{1a} + \Delta\sigma_{2a} + \Delta\sigma_{3a}}{3}$	$\sigma'_3 = \sigma'_0 + \frac{2\Delta\sigma_{3a} - \Delta\sigma_{1a} - \Delta\sigma_{2a}}{3}$
Intermediate Stages, Figure 4-6b	$\sigma_1 = \sigma'_0 + \Delta\sigma_{1b}$	$u = \frac{\Delta\sigma_{1b} + \Delta\sigma_{2b} + \Delta\sigma_{3b}}{3}$	$\sigma'_1 = \sigma'_0 + \frac{2\Delta\sigma_{1b} - \Delta\sigma_{2b} - \Delta\sigma_{3b}}{3}$
	$\sigma_2 = \sigma'_0 + \Delta\sigma_{2b}$	$u = \frac{\Delta\sigma_{1b} + \Delta\sigma_{2b} + \Delta\sigma_{3b}}{3}$	$\sigma'_2 = \sigma'_0 + \frac{2\Delta\sigma_{2b} - \Delta\sigma_{1b} - \Delta\sigma_{3b}}{3}$
	$\sigma_3 = \sigma'_0 + \Delta\sigma_{3b}$	$u = \frac{\Delta\sigma_{1b} + \Delta\sigma_{2b} + \Delta\sigma_{3b}}{3}$	$\sigma'_3 = \sigma'_0 + \frac{2\Delta\sigma_{3b} - \Delta\sigma_{1b} - \Delta\sigma_{2b}}{3}$
Yield Stage, Figure 4-6c	$\sigma_1 = \sigma'_0 + \Delta\sigma_{1c}$	$u = \frac{\Delta\sigma_{1c} + \Delta\sigma_{2c} + \Delta\sigma_{3c}}{3}$	$\sigma'_1 = \sigma'_0 + \frac{2\Delta\sigma_{1c} - \Delta\sigma_{2c} - \Delta\sigma_{3c}}{3}$
	$\sigma_2 = \sigma'_0 + \Delta\sigma_{2c}$	$u = \frac{\Delta\sigma_{1c} + \Delta\sigma_{2c} + \Delta\sigma_{3c}}{3}$	$\sigma'_2 = \sigma'_0 + \frac{2\Delta\sigma_{2c} - \Delta\sigma_{1c} - \Delta\sigma_{3c}}{3}$
	$\sigma_3 = \sigma'_0 + \Delta\sigma_{3c}$	$u = \frac{\Delta\sigma_{1c} + \Delta\sigma_{2c} + \Delta\sigma_{3c}}{3}$	$\sigma'_3 = \sigma'_0 + \frac{2\Delta\sigma_{3c} - \Delta\sigma_{1c} - \Delta\sigma_{2c}}{3}$

"a" denotes initial condition
 "b" denotes intermediate conditions,
 "c" denotes yield condition.

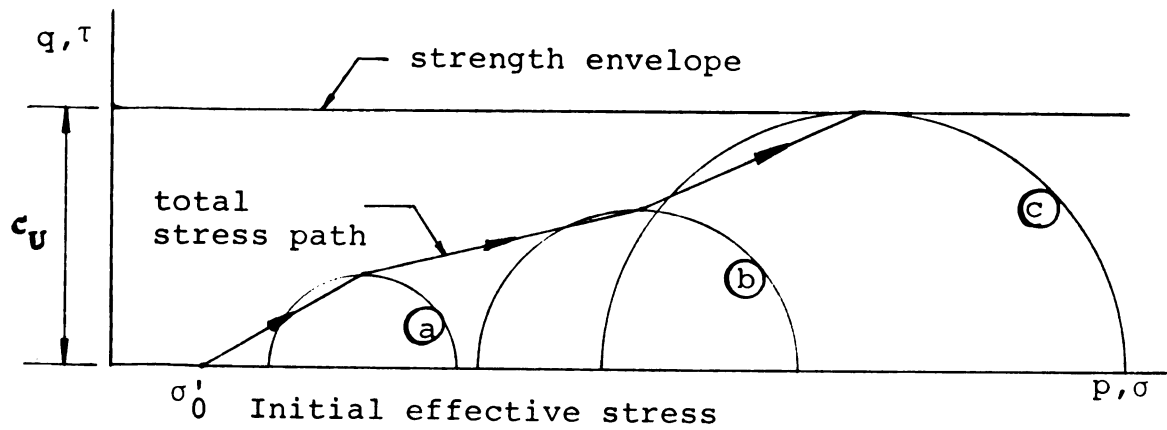


Figure 4-8.--Mohr's Circles Representing Total Stress Conditions for a Typical Point in an Embankment.

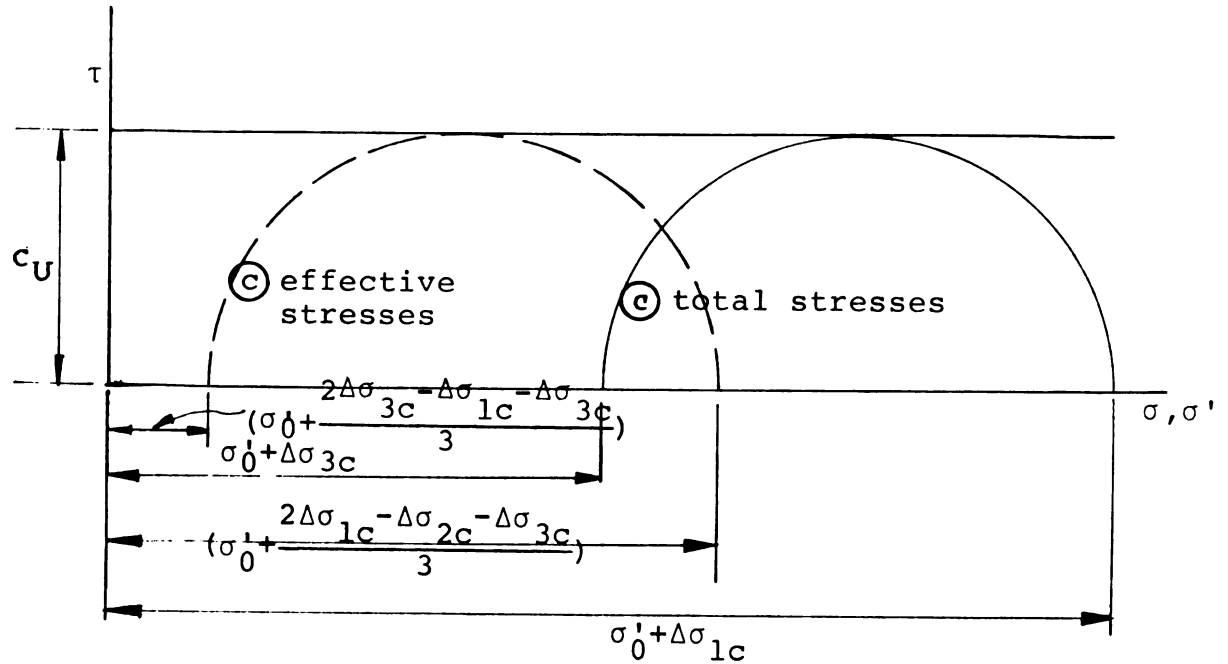


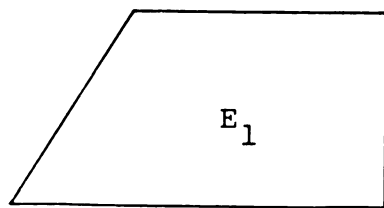
Figure 4-9.--Mohr's Circles at Yield Condition for the Typical Point.

4.4 Simulation of Contained Plastic Zones

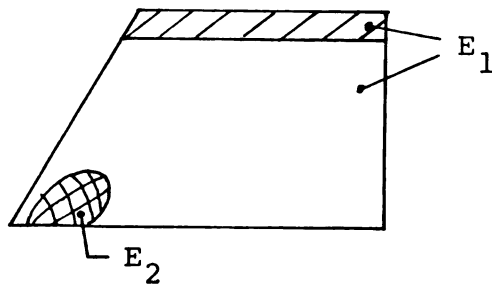
Three methods of simulation will be explained according to different failure assumptions as follows:

4.4.1 Different-E Method

In this simulation both equilibrium and continuity are enforced on the boundary between elastic and plastic zones. Equilibrium and continuity are also applied within the confined plastic zone. After a point yields, a very small value of modulus of elasticity, the plastic modulus, will be applied (see Figure 4-10).



a) Elastic.



b) Elastic and Plastic.

E_1 = Elastic modulus

E_2 = Plastic modulus

Figure 4-10.--Simulation by Different-E Method.

Changes in principal stresses (both direction and magnitude) occur slightly during an increment of load. The maximum shear stress at a point in the plastic zone is constant throughout the loading increments if perfectly plastic material is assumed, while for linear strain hardening material, the maximum shear stress may increase due to the change in yield surface. For this simulation, an undrained strength envelope is assumed, where the strength is independent of increasing normal stress, and this assumed condition applies until complete discontinuity of elastic zones occurs. For a strength envelope such that the shear strength increases linearly with an increase of normal stress, a change in magnitude of the principal stresses can change the point from plastic to elastic.

4.4.2 Truncated Plastic Zone

In this method the points within the plastic zone (or zones) can experience stress changes up to yield only. After the points yield, the stress states are kept constant. At the conclusion of any load increment the plastic part (or parts) is truncated and the existing stress is applied to the elastic boundary to calculate the stress field in the body for the next load increment (Figure 4-11).

In this case the stress-strain behavior can only be assumed as elastic-perfectly plastic. The strength must also be that of the undrained case. If the strength

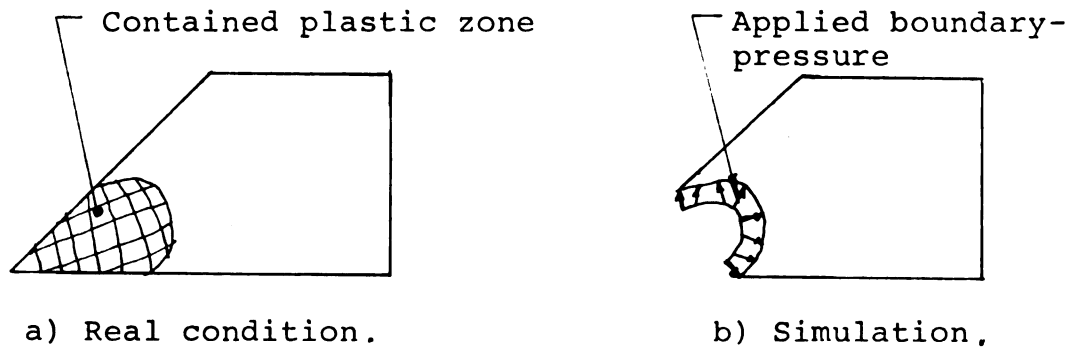


Figure 4-11.--Simulation by Truncated Plastic-zone Method.

is such that the shear strength increase linearly with normal stress, this method is inapplicable.

4.4.3 Propagation of a Slip Line

In the third simulation violation of compatability is allowed along a continuous surface. The surface is indicated by a selected slip surface according to the slip angle within the plastic region. The slip surface is not unique, but the most practical one will be selected. The stress states along the assumed surface are always at the limiting equilibrium, which means constant boundary pressure will be applied along the surface.

The simulation procedure is illustrated on Figure 4-12. When yield occurs over a small area, slip lines can be drawn through the area according to the known stress state. The slip surface will be simulated by applying a new external boundary surface along the fictious slip surface with boundary pressure according to the stress

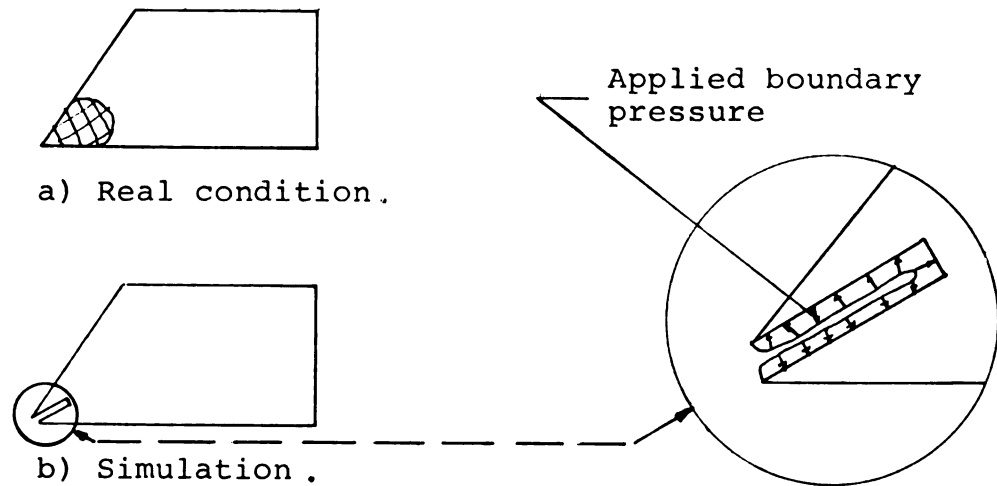


Figure 4-12.--Simulation by Propagation of a Slip Line.

state at the location. When a new load increment is applied, a larger area will yield. If the new yielded area is attached to the initial one, the next slip line can be drawn connecting from the original one. If the new yielded area is apart from the proceeding one, simulation of the new yield area is performed in the same fashion as the former one. In a situation in which there is more than one fictitious slip surface, the potential failure surface can be taken from the one that forms the first complete cut surface (the surface that connects any two slope boundaries).

For slope embankments the first yield area is quite predictable, and normally continuous spreading of this plastic zone occurs in embanking processes. Failure in tension may occur at some other points away from the initial compressive yield point. The tensile failures are

likely to be observed at the top surface, which produce tension cracks.

The disadvantage of this method is that when a discontinuity is applied along the slip surface, it is impossible to control the overlaps of the two fictitious boundary surfaces as deformation proceeds in further stages of loading.

Strain hardening material can be accommodated only in the first simulative method. In the second and third methods, the soil must be assumed to behave as an elastic-perfectly plastic material. Only the first method will be used to simulate contained plastic zones in this paper.

4.5 Stability Analysis

To facilitate discussion of the elasto-plastic incremental approach in the analysis of slope stability, two general theorems will be briefly stated:

(I) Lower Bound Theorem

The lower bound theorem states that the collapse load calculated from a statically admissible stress field is a lower bound to the actual collapse load. A statically admissible stress field is defined as a stress field which satisfies all stress boundary conditions, is in equilibrium, and nowhere violates the yield criterion.

(II) Upper Bound Theorem

The upper bound theorem states that the collapse load calculated from a kinematically admissible failure mechanism is an upper bound to the actual collapse load. A kinematically admissible failure mechanism is defined under the condition that collapse will occur if there is any compatible pattern of plastic deformation for which the external forces do work at a rate equaling or exceeding the rate of internal energy dissipation.

Most existing methods of analysis for slope stability require an assumed failure mechanism, and therefore, are all upper bound methods. The lower bound theorem has been applied to soil mechanics less frequently than the upper bound theorem since it is considerably easier to construct a good kinematically admissible failure mechanism than it is to construct a good statically admissible stress field. An upper bound solution is often a good estimate of the collapse load, but a lower bound solution is more valuable in engineering practice as it results in a safe design.

A lower bound solution combined with an upper bound solution can serve to bracket the actual collapse load. If one could guess the actual stress field that would be represented at collapse, then the predicted bounds would be equal.

In this paper the analysis of slope stability is done by developing actual stress fields at different stages of loading up to the collapse load. The stress fields are developed by a numerical technique (finite element method). The solution satisfies most of the requirements of the statically admissible stress field. The stress fields do not provide equilibrium of stresses along the element boundaries, but the nodal force resultants are in equilibrium. One additional property of the stress fields is the continuity of strains everywhere in the body. This property is not required for the statically admissible stress field. However, adding the property does not violate any restriction of the stress field.

The analysis yields actual stress fields at different stages of loading, and the true mode of failure is obtained. Consequently the true collapse load for the assumed material properties can be predicted. Lower bound loads are obtained at different stages prior to the limiting equilibrium condition. Since it is not the purpose of this paper in extending the investigation beyond the limit load, the kinematically admissible velocity field is not considered.

The factor of safety against failure can be obtained from the average factor of safety calculated by

$$\text{average F.S.} = \frac{\sum \tau \Delta L}{\sum C_U \Delta L} \quad (4.2)$$

where τ = shearing stress along any finite length, ΔL , along the potential sliding surface.

At any stage of loading, the factor of safety can be calculated. The minimum factor of safety (at the limiting condition) equals unity for this analytical method.

It should be noted that the shearing stress on the potential sliding surface (Figure 4-13) may not be the maximum shearing stress. The plane tangent to the surface at the point does not always orient in the plane of the maximum shearing stress. However, for the case of saturated clay ($\nu = .45$) the two planes will be proved to be close to each other.

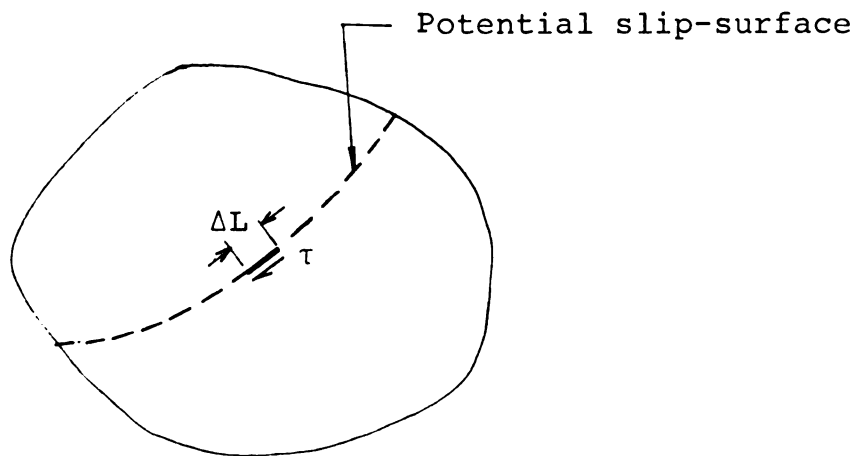


Figure 4-13.--A Potential Slip Surface in a Part of the Body.

CHAPTER V

PROCEDURE

The analysis of stresses and deformations in embankments is performed by a plane strain formulation of the finite element method. The calculation of stresses and strains at each stage of loading is accomplished by a computer program, "FEAST 1," by E. L. Wilson and J. T. Christian (1967). This computer program can be used to determine the deformations and stresses within certain types of stressed bodies. It will analyze problems of the following categories: axially symmetric, plane stress, and plane strain. Elastic, non-linear material properties are considered by a successive approximation technique. The effects of displacement or stress boundary conditions, concentrated loads, gravity forces and temperature changes can be included. Because of limited capacity of the computing machine, another computer program (NEWST) was constructed to calculate the stresses-addition (algebraic adding) process between the original stress field and the additional stress field resulting from the next load increment. The soil is assumed to behave as an ideal elastic-perfectly plastic material. This is a reasonable assumption for embankments where the soil is in a remolded

stage (Scott, 1963). It is also assumed that the soil is isotropic. Both homogeneous and layered cases are investigated. The analysis applies mainly to cohesive saturated soils, especially in embankment processes. The effect of pore pressure development is not considered.

Plane strain conditions are assumed in the calculation of stress fields developed according to the gravitational body force of the soil mass. The stress-strain relationship or modulus of elasticity is assumed to be homogeneous throughout the body. The ratio of

$\frac{E_p}{E_e}$ (E_p = plastic modulus, E_e = elastic modulus) is about $\frac{1}{1000}$ for elastic-perfectly plastic assumption and $\frac{1}{100}$ for elastic-strain hardening assumption. Poisson's ratio for saturated soil is about 0.45 ($\nu = 0.5$ gives perfect incompressibility). For the elastic problems, the material is assumed to have a linear stress-strain relationship, while a bi-linear relationship is applied to the elasto-plastic problems. The stress fields in the elastic range are directly obtained from a single step of calculation, while the super-position method is performed for the elasto-plastic stress fields.

5.1 Basic Concepts of Finite Element Method

The basic concept of the finite element method (Zienkiewicz et al., 1967; and Connor, 1967) is an

idealization of an actual elastic continuum as an assemblage of discrete elements interconnected at their nodal points. For the analysis of two-dimensional stress fields, it has been found convenient to use triangular and rectangular plate elements in the idealization. To maintain compatibility between the edges of adjacent elements, it is assumed that the deformations within each element vary linearly in the x and z-directions. On the basis of this assumption, it is possible to calculate the stiffness of the complete structural assemblage. This is obtained by merely superposing the appropriate stiffness coefficients of the individual elements connecting to each nodal point.

If the vector of all nodal point displacements in the complete assemblage is designated as $\{r\}$, and the vector of the corresponding nodal forces is $\{R\}$, the structure stiffness matrix, $[K]$, (which is obtained by superposing the finite element stiffnesses) expresses the relationship between these quantities as

$$\{R\} = [K] \{r\} \quad (5-1)$$

The order of these matrices is $2N$ if there are N nodes. Nodal displacement includes both x and z components. Where boundary conditions impose displacement constraints on any of the nodal points, the matrices may be reduced by eliminating the corresponding rows and columns.

In a standard elastic finite element analysis, these linear equations of equilibrium are solved for the nodal displacements resulting from the given nodal forces. Then the stresses in all of the elements, denoted by the matrix $\{\sigma\}$, are obtained from the nodal displacements by the matrix transformation

$$\{\sigma\} = [S] \{r\} \quad (5-2)$$

in which the stress transformation matrix, $[S]$, takes account of the assumed linear displacement patterns in the elements, as well as their given material properties.

Concerning the finite element plane strain analysis procedure, in general, it may be noted that (1) compatibility is satisfied everywhere in the system; (2) equilibrium is satisfied within each element; and (3) equilibrium of stresses is not satisfied along the element boundaries, in general, but the nodal force resultants are in equilibrium. This local discrepancy in stress equilibrium represents the type and extent of the approximation involved in the finite element method of analysis. For analysis by the finite-element method, the slope must be divided into a number of elements, as shown on Figure 5-1. For each nodal point in the system, displacements are obtained in two directions (x and z). After the displacements of all of the nodal points have been evaluated, the stresses within each element are evaluated and

represented by three stress parameters, σ_x , σ_z , σ_{xz} . The principal stresses are then calculated both in magnitude and direction. In plane strain analyses, the body is assumed to behave in the same manner with respect to the direction perpendicular to the area under consideration. The thickness considered in this direction is one unit.

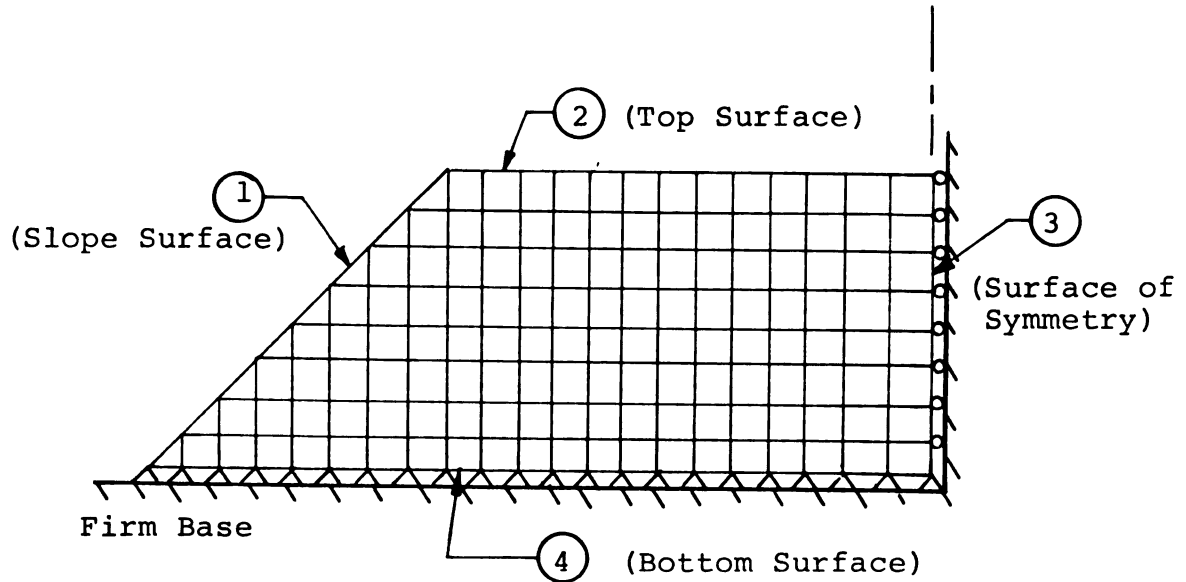


Figure 5-1.--Typical Slope Boundaries.

To simulate a slope embankment, boundary conditions must be comparable to those in the real conditions. In simple slopes (see Figure 5-1) the boundary conditions can be categorized as follows:

1. Boundaries (1) and (2) are free surfaces. No loads are applied along these boundaries, and points along these boundaries are also free to move in any direction.

2. Boundary ③, the center line of the slope, is simulated by requiring zero horizontal displacement and zero vertical load along the boundary. This simulation can also be used to represent any vertical plane that is far enough from face ① so that shear stresses along the plane can be assumed equal to zero.

3. Boundary ④ is the interface between the firm base and the bottom of the soil body under consideration. The simulation is performed by applying zero displacements in both directions (x and z).

5.2 Simulation of Embankment Stresses and Strains by the Superposition Method

The validity of the superposition method is discussed by Timoshenko and Goodier (1970). In applying the principle of superposition it is assumed that very small elastic deformations occur after each stage of loading.

The simulations are discussed under two categories according to the construction processes, single-lift construction, and incremental construction.

5.2.1 Single-Lift Construction

Although the assumption of a single lift is not strictly valid for a constructed embankment, the method is useful in studying cases where failure may be due to soil strength deterioration. The stresses and deformations in the soil body are obtained by direct application of the gravitational body forces on the completed structure.

If the embankment height is such that no plastic failures occur, the actual unit weight of the soil is applied and the stresses and deformations are calculated within a single step. If, however, the height of the single-lift embankment is great enough so that local yield (or yields) occurs, the calculations are performed by applying increments of unit weight. The total unit weight of the soil is divided into parts, as follows:

$$\gamma_{\text{total}} \left[\begin{array}{l} \gamma_1 = \text{(applied within elastic range up to the first local yield).} \\ \gamma_2 = \sum_{i=1}^n \Delta\gamma^i \text{ (divided into "n" increments).} \end{array} \right.$$

The first part, γ_1 , is obtained such that first yield just occurs at some point within the soil body. The second part, γ_2 , is divided into "n" fragments. The fragments do not necessarily have to be equal. The steps of calculation are illustrated by the simplified flow chart, Figure 5-2. In the elastic range of the first step, E_1 and γ_1 are applied throughout the body. In the second step, an increment of unit weight, $\Delta\gamma_2^1$, is applied throughout the body, while E_1 and E_2 are applied in the elastic and plastic regions respectively. In the third step, the calculated stress field must be checked so that the stress condition in the generating failure zone

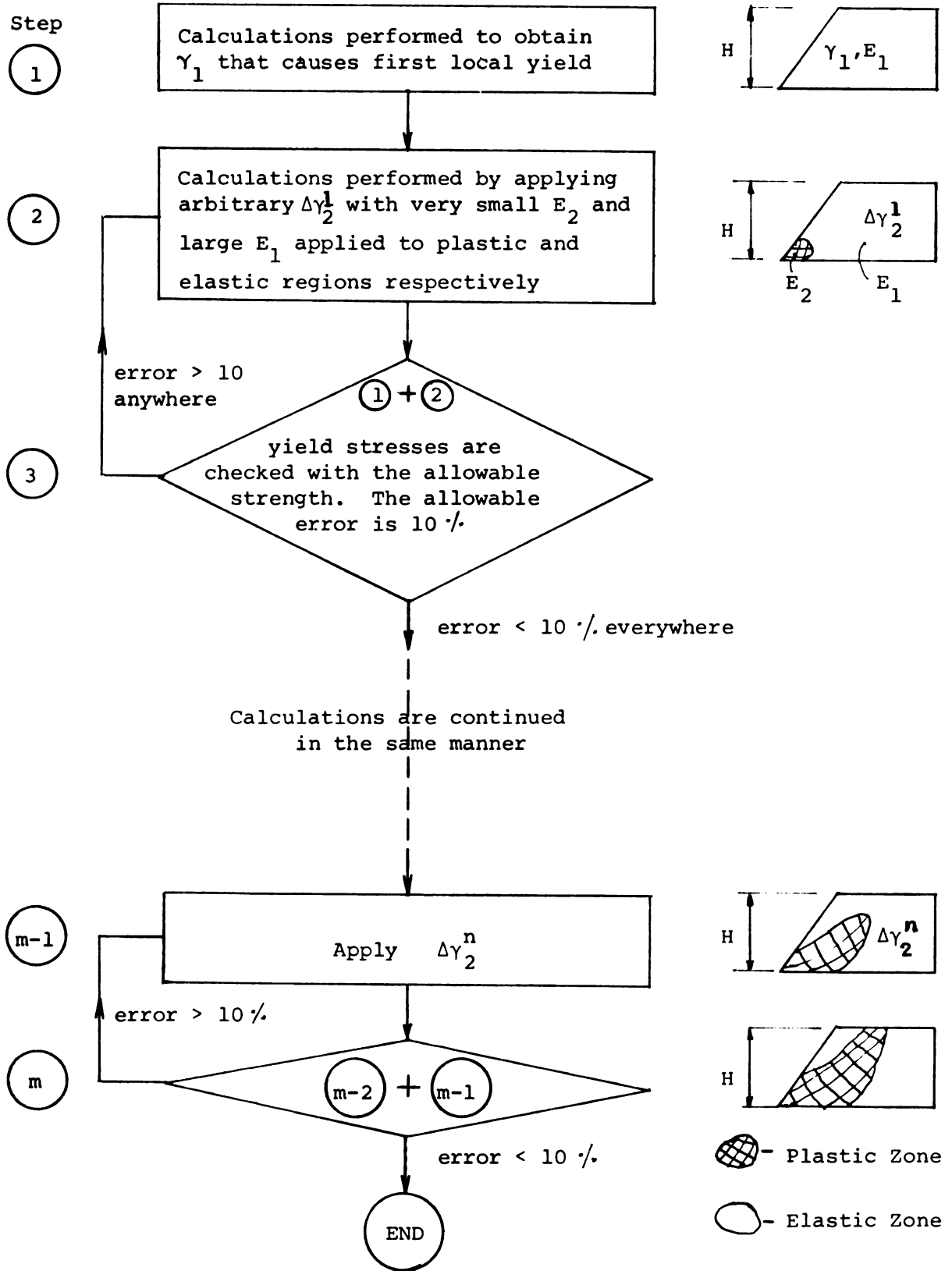


Figure 5-2.--Flow Diagram of Single Stage Construction Simulation.

will not deviate from the yield stress more than 10 per cent. A similar procedure is repeated until limiting equilibrium is reached.

The potential slip surface is drawn on the slope under the stress field at the limit equilibrium state. The start of the slip surface is selected as the location of the first-yield element and drawn continuously from this location to touch the tension zone. It was found that the slip surface normally touches the tension zone around the area where the maximum tensile stress occurs. Tension cracks were assumed to occur throughout the depth of the tension zone.

5.2.2 Incremental Construction

Incremental construction is intended to simulate stress and strain conditions in an embankment placed and compacted in lifts. The load increments result from the added weight of each lift as it is placed, and the soil unit weight is kept constant. The resulting stresses are again added algebraically. Complete presentation of the process is shown in the flow diagram (Figure 5-3). The initial height, H_1 , is obtained within the elastic range. At this stage γ_T and E_1 are applied throughout the body with height H_1 . In the second step the unit weight of the soil, γ_T , is applied only in the added layer, ΔH^1 , while the original body is considered to be weightless ($\gamma = 0$).

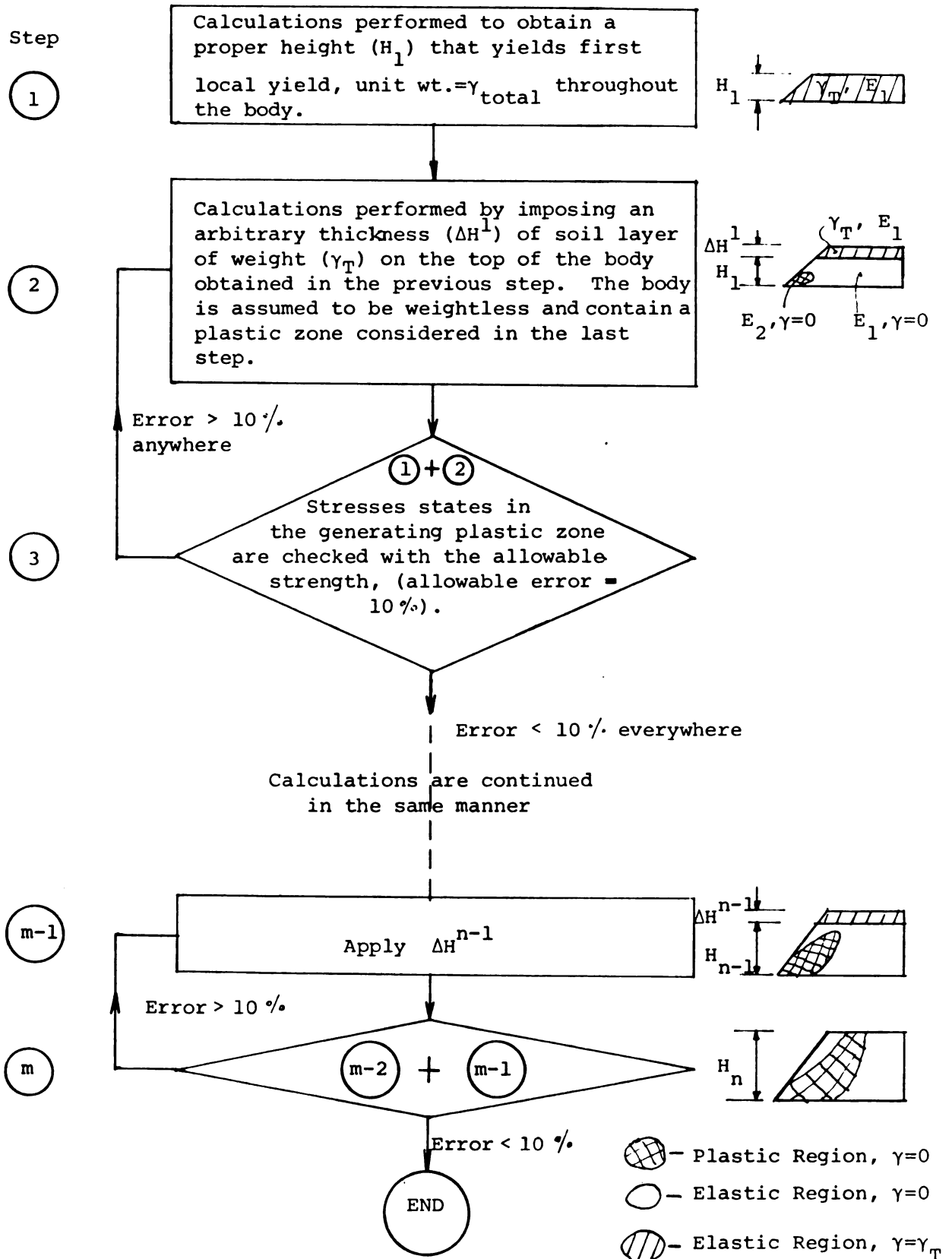


Figure 5-3.--Flow Diagram of Incremental Construction-Simulation.

E_2 is applied in the plastic region and E_1 is still used in all other parts. In the third step the calculated stress field must be checked so that the stress condition in the generating failure zone will not deviate from the yield stress more than 10 per cent. A similar procedure is repeated until limiting equilibrium is reached. The potential slip surface is drawn in the same manner as described in section 5.2.1.

5.3 Factor of Safety and Unused Shear Strength

Some comments regarding factor of safety and unused shear strength follow:

F = factor of safety

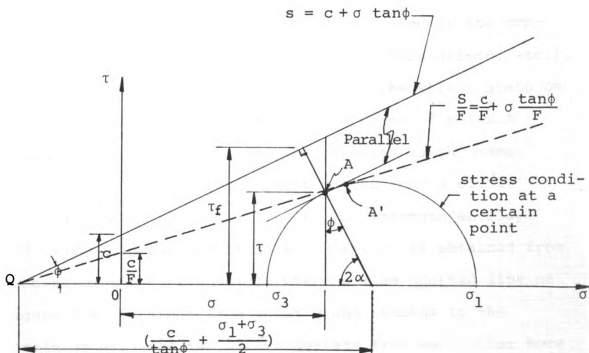


Figure 5-4.--Definition Sketch for Factor of Safety and Unused Shear Strength.

Referring to Figure 5-4, the Mohr's circle represents stress conditions below failure for some point, and the solid line represents the limiting strength. The factor of safety is defined as

$$FS = \frac{\tau_f}{\tau} \quad (5-3)$$

and unused shear strength is defined as

$$US = \tau_f - \tau \quad (5-4)$$

where τ = shearing stress on the potential slip plane (angle = α), and τ_f = shearing strength on the potential slip plane (angle = α).

The concept of factor of safety in this presentation is slightly different from the one used in the conventional stability analyses (Bishop's, Morgenstern, etc.). According to the conventional methods, the stress state on the slip plane will be at point A' instead of point A (Figure 5-4), while the shearing strength is the same. Point A is obtained from a point on the Mohr's circle where a line is drawn parallel to the strength envelope and tangent to the circle at A. Point A' is obtained from a point on the Mohr's circle where a line (dotted line on Figure 5-4) is drawn from point Q and tangent to the circle at A'. The two points deviate from each other more when the value of ϕ is larger. For small values of ϕ

the two points are close together, and the two concepts give about the same results.

In order to choose points at which failure is most eminent, or the yield priority, it is convenient to use the concept of unused shear strength. This can be explained with reference to Figure 5-5. The stress condition of (a) gives higher factor of safety than that of (b), while the two conditions have equal amounts of unused shear strength. With the principle of factor of safety, the stress state of (b) will indicate higher priority of yield which is not quite correct. Actually, the stress states have the same yield priority. For example, if yields are induced by increasing major

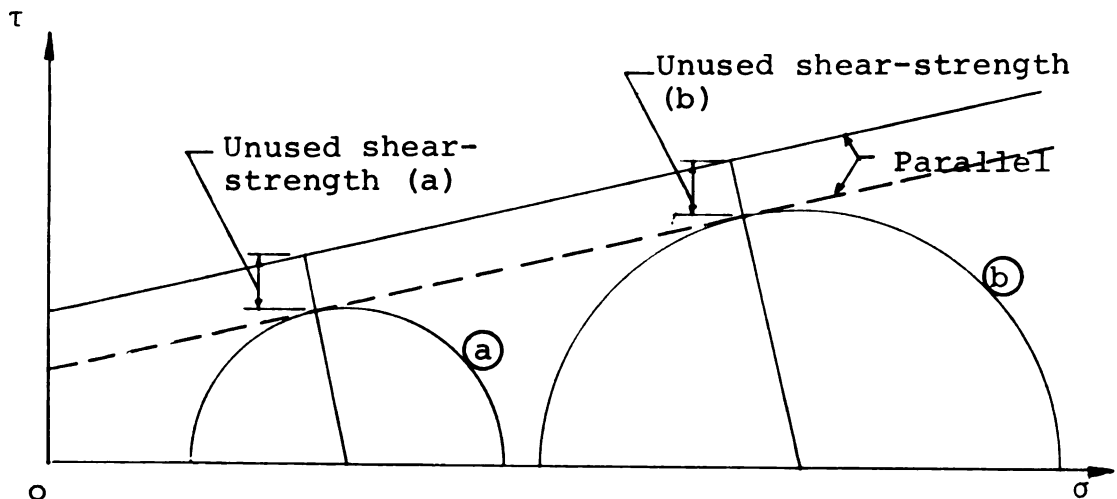


Figure 5-5.--Priority of Yield by Two Stability Concepts.

principal stress and decreasing minor principal stress as indicated in condition I (Figure 5-6), or increasing major principal stress while minor principal stress is kept constant as indicated in condition II (Figure 5-6), the amount of stress changes of (a) and (b) are equal in both cases. That is, $\Delta\sigma_{1a}^I = \Delta\sigma_{1b}^I$ and $\Delta\sigma_{3a}^I = \Delta\sigma_{3b}^I$, or $\Delta\sigma_{1a}^{II} = \Delta\sigma_{1b}^{II}$. The previous example shows that the two stress states, (a) and (b), have the same priority of yield while the factors of safety are different.

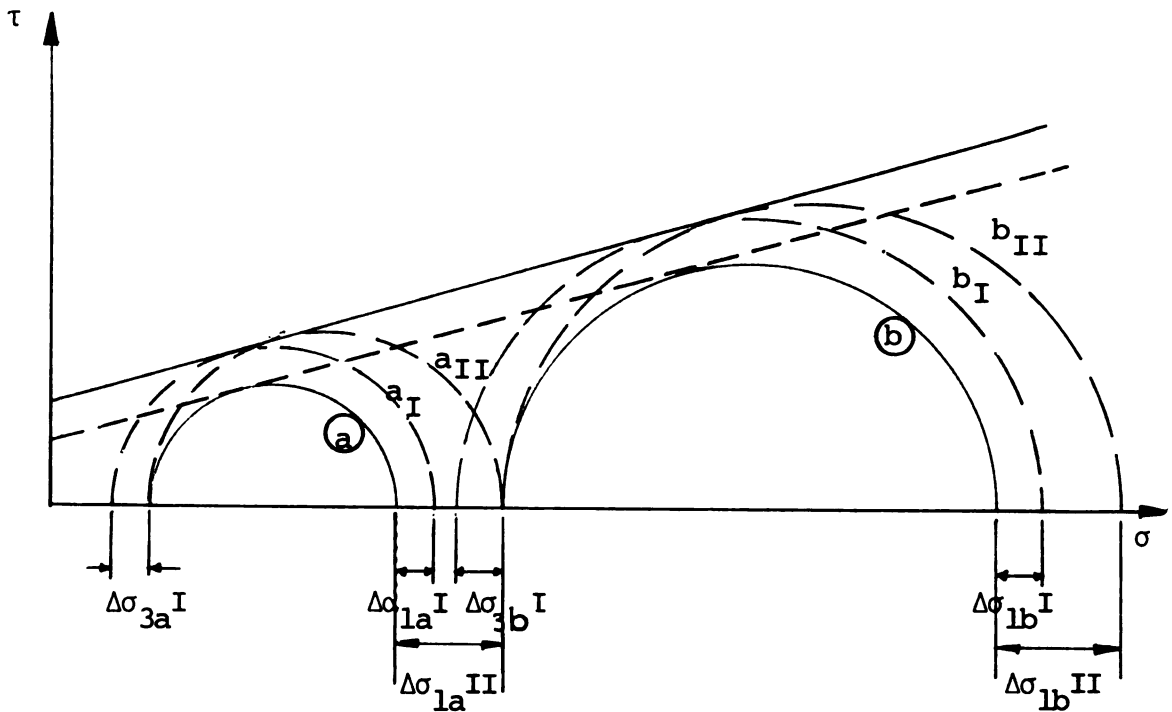


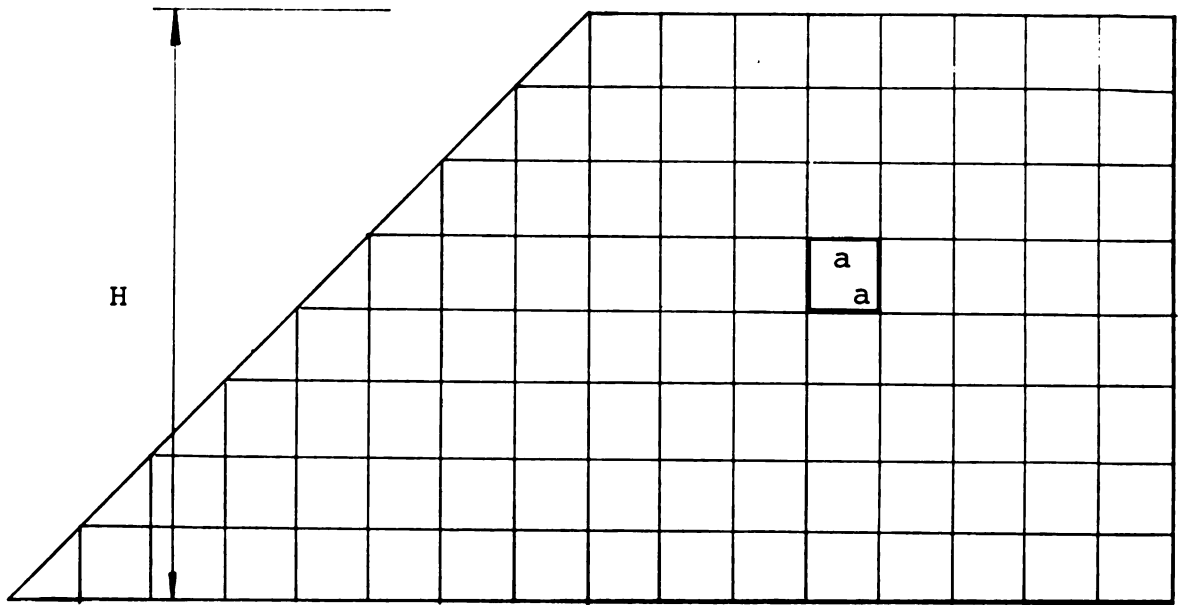
Figure 5-6.--Two Possible-failure Stress Conditions.

The above condition can affect the computed location of the critical failure surface. If, for some

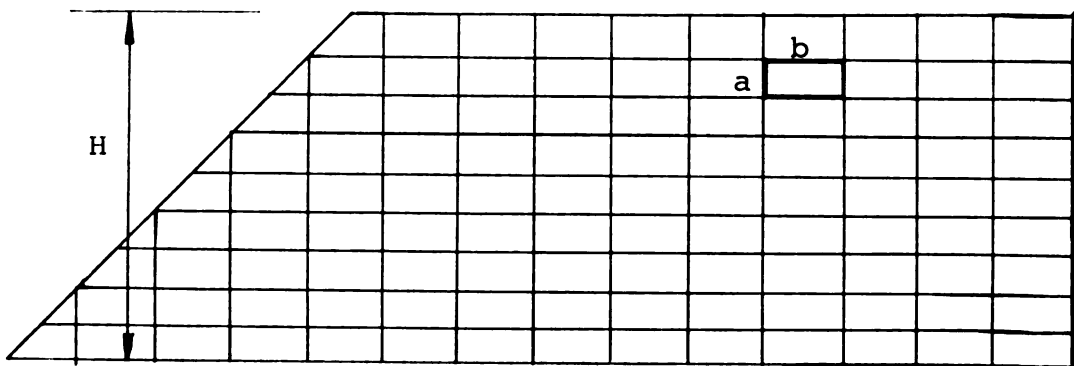
stress field, contour lines are plotted for the factor of safety field and the unused shear strength field, a difference in configuration of the contour fields will be observed. Consequently, if an analytical method is performed by selecting a continuous surface that gives the least average factor of safety or least average unused shear strength on each of the two contour fields, the surface with least average factor of safety and the one with least average unused shear strength will not locate at the same position. However, the difficulty occurs only when the drained strength envelope is used ($\phi \neq 0$). If the undrained strength envelope ($\phi = 0$) is used as the failure criterion, the difficulty does not occur.

In the elasticity analysis, the potential slip surface is drawn by the graphical method of slip-line. The start of the slip surface is selected from the location or element that gives least unused shear strength and drawn continuously to touch the tension zone.

Figure 5-7 is the typical finite element configurations of slope used in elasticity and elasto-plasticity investigations. Figure 5-8 is the result of an accuracy test to obtain appropriate element sizes applied in the investigations.

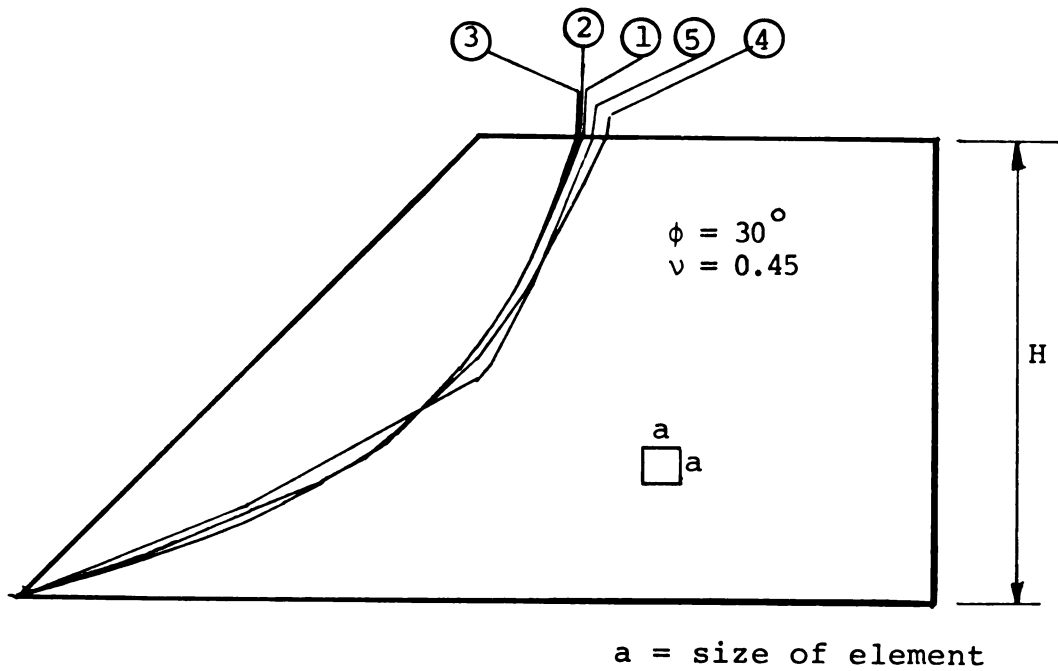


a) Elastic Investigations, $(\frac{a}{H} \times \frac{a}{H} = \frac{1}{8} \times \frac{1}{8})$.



b) Elasto-Plastic Investigations, $(\frac{a}{H} \times \frac{b}{H} = \frac{1}{9} \times \frac{2}{9})$.

Figure 5-7.--Typical Finite Element Configuration of a Slope.



	Normalized Element Size $(\frac{a}{H} \times \frac{a}{H})$
①	$\frac{1}{12} \times \frac{1}{12}$
②	$\frac{1}{8} \times \frac{1}{8}$
③	$\frac{1}{6} \times \frac{1}{6}$
④	$\frac{1}{3} \times \frac{1}{3}$
⑤	$\frac{1}{2} \times \frac{1}{2}$

Figure 5-8.--Comparison of Slip Surfaces for Different Element Sizes.

CHAPTER VI

PRESENTATION OF RESULTS AND DISCUSSIONS

The analytical results and general discussions will be presented in Chapter VI.

6.1 Homogeneous Soil Slopes

Results of analyses of homogeneous soil slopes will be presented under three headings: Elasticity Solutions; Elasto-plastic, Single-stage Construction; and Elasto-plastic Incremental Construction.

6.1.1 Elasticity Solutions

The results of analyses of slope embankments within the elastic range are presented for single-stage construction. The results indicate formation of a tensile stress zone (Figure 6-1). It was found that the behavior of this tensile stress zone depends only on Poisson's ratio, ν , and not on modulus of elasticity, E . As Poisson's ratio decreases, the depth intensity of the tensile stresses decreases. The size of the zone is less when Poisson's ratio is smaller. At Poisson's ratio of 0.3 no tensile stress zone is observed (slope angle = 45° , $\gamma = 110$ pcf, $E = 342000$ psi).

In general, for a slope with stresses within the elastic range, development of stresses due to gravity depends on the two elastic parameters, E and ν , on the unit weight of the soil, and on slope geometry. To explain the above phenomenon, one can observe that the modulus of elasticity and the unit weight can influence only the magnitude of the stresses, and that only Poisson's ratio can influence the principal stress directions at the various locations within the slope. For a simple soil slope, such as that shown on Figure 6-1, horizontal deformation is not restrained laterally at the slope surface (leftward, on Figure 6-1). Since the boundary is considered to be fixed at the bottom of the slope, the horizontal displacement of points along the slope face are greatest at about midheight. For incompressible material (ν approaches 0.5), the magnitude of these horizontal displacements is greatest, and decreases for smaller values of ν . Thus, the tensile stresses developing in the top part of the slope are of greater intensity as ν approaches 0.5.

Normally, clays possess values of Poisson's ratio varying from 0.4 to 0.5 for saturated clays and 0.1 to 0.3 for unsaturated clays, and sands have values varying from 0.2 to 0.4 (Bowles, 1968). Tensile cracks will occur in the locations where tensile stress states reach or exceed yield conditions. This will be observed mostly in saturated

clay rather than sand. Vertical cracks always precede failure in clay slopes. For sand, tensile failure will develop along slip surfaces.

Figure 6-2 shows several series of Mohr's stress circles drawn in corresponding order of the locations from left to right along the bottom horizontal layer in the slope bodies. The results shown are only the envelopes for points along the bottom layers, but the results are similar for any other horizontal layers. For small values of ν , the envelopes to these stress circles start as a straight line passing slightly above the origin and curve downward to approach a horizontal asymptote. If the value of ν is greater than about 0.35 the stress envelopes curve downward and form a peak. Also, for values of ν greater than about 0.35, the initial part of the stress envelope is curved. An explanation of these two characteristics of the strength envelopes is furnished in the following.

Consider the case of a semi-infinite medium with the surface parallel to the horizontal plane (x-y plane), and the principal stress directions parallel to the x, y and z directions (see Figure 6-3). With the gravity load, strain in the horizontal directions is assumed to be zero, and the horizontal stresses are equal,

$$\epsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_z + \sigma_x)] = 0 \quad . \quad (6-1)$$

With $\sigma_x = \sigma_y$ we have $\sigma_x = \left(\frac{\nu}{1-\nu}\right) \sigma_z$ (6-2)

For a value of $\nu = 0.5$, the magnitude of σ_x equals that of σ_z , which in turn gives the magnitude of deviatoric stress $(\sigma_1 - \sigma_3)$ equal to zero. (In the plane strain condition the magnitude is not exactly equal to zero.) As ν decreases, the value of $\left(\frac{\nu}{1-\nu}\right)$ decreases while the magnitude of $(\sigma_1 - \sigma_3)$ increases.

If we consider the soil in an area that is far from the slope surface so that the principal stress axes are parallel to the x , y and z directions, the magnitude of the deviatoric stresses will depend on magnitude of ν . In the cases of ν less than 0.5, the deviatoric stress equals to $\left(\frac{1-2\nu}{1-\nu}\right) \sigma_z$, which gives a definite value of the deviatoric stress for each ν . For ν less than the threshold values, around 0.35, the magnitude of $\left(\frac{1-2\nu}{1-\nu}\right) \sigma_z$ is greater in comparison to the deviatoric stresses of the points on the horizontal layer near the slope. The deviatoric stresses increase in magnitude for points at locations farther from the slope surface, and approach the maximum value of $\left(\frac{1-2\nu}{1-\nu}\right) \sigma_z$. For ν greater than the threshold value, the magnitude of $\left(\frac{1-2\nu}{1-\nu}\right) \sigma_z$ is not the maximum deviatoric stress for points along the horizontal layer. Now the magnitude of the deviatoric stresses increase to a maximum value at a certain location which is the peak of the envelope, and

then decrease approaching $(\frac{1-2\nu}{1-\nu}) \sigma_z$ for points far from slope surface.

The results (Figure 6-2) indicate the requirements of the strength characteristics for the slope to maintain a stable condition. Soils with low values of ν require low values of cohesion, or intercept of the τ -axis, and high values of ϕ , or slope of the strength envelope in the initial part to maintain a stable condition. As the soil property, ν , is larger, required values of the cohesion are greater, while the ϕ -values required are smaller. Maximum shear stress, $\tau_{\max}$ (Figure 6-2), developed is greatest at lowest ν , and decreases as ν increases.

The stress envelopes shown on Figure 6-2 can be used to indicate the type failure phenomena, and the location of first local failure. For dry soils (sand) with a drained shear strength envelope (Figure 6-4a), failures occur first at the points along the slope surface. The failure phenomena is known as skin failure. Experimental results show that development of skin failures for sand occur for slope angles greater than about the angle of internal friction (ϕ_D) of the sand (Lambe et al., 1969). In the case of wet soils (wet sand or clay), which possess a certain amount of cohesion, strength and stress envelopes are shown on Figure 6-5a. The location of first local yield is indicated by point A. The location of the

first local yield depends on the characteristics of the strength envelope, c and ϕ . A lower value of ϕ moves the location of the first local yield in the positive x -direction, and vice versa.

Figure 6-6 shows the effect of changes in Poisson's ratio on the position of slip surfaces. The slip surfaces are drawn according to the angle of internal friction (ϕ) equals to 30° , and for a 45° slope angle. It can be seen that the position of the slip surfaces moves toward the slope surface as ν decreases.

The position of slip surfaces for a given soil type changes also according to different ϕ angles. In Figure 6-7, the slip surfaces are drawn for $\phi = 30^\circ$ and 0° . Comparison between the slip surfaces obtained from an elastic stress field (a lower bound condition), and the potential rupture surfaces obtained from Bishop's analysis is established for saturated clay ($\nu = 0.45$). Slope angles are 45° , 38.6° , and 30° . Soil properties are, $\nu = 0.45$, $E = 56000$ psf, $\phi = 0^\circ$ and 30° . Three interesting points should be mentioned for the comparison.

1. The similarity in shape of the surfaces, i.e., the slip surface from the elastic solutions yields a shape very close to circular.

2. Correspondence in position of the surfaces, i.e., both cases give the position of the surfaces in such

a fashion that the position moves toward the slope surface as the value of ϕ increases.

3. Similarity in type of failure, i.e., both cases give toe failure for $\phi = 30^\circ$, and base failure for $\phi = 0^\circ$.

It is noticable that the rupture surface obtained by Bishop's analysis does not give any indication of tensile cracks, while in the elastic stress field, tensile cracks are evident (Figure 6-7a).

It appears that the position and shape of a potential slip surface for homogeneous saturated clays with homogeneous strength can be predicted by the slip surface obtained from elastic stress fields. (Since the potential rupture surface analyzed by Bishop's method has been shown able to predict the location of the true rupture surface in homogeneous soils [Sevaldson, 1956]).

6.1.2 Elasto-plastic, Single-lift Construction

Figure 6-8 shows the results of analysis of a single-lift embankment. The soil properties are assumed as $E = 56000$ psf, $\nu = 0.45$, $c = 400$ psf, $\phi = 0^\circ$. Height of the lift is 20 feet, and the gravitational body force of the soil was gradually increased from 96.5 pcf up to 128.75 pcf at the limiting equilibrium condition. With undrained strength criterion, the first local plastic yield occurred at a point on the firm base (stage 1), and not at the toe of the slope. Stages 1 through 4 show how the plastic zone

spread as the body force was increased (geometry and boundary conditions were assumed unchanged). At the final stage, when limiting equilibrium was reached, the plastic zone touched the tensile stress zone along the top surface. It should be observed that the behavior of the tensile stress zone changes as the plastic zone spreads. Maximum tensile-stress occurs at point (a) (on Figure 6-8a), in the elastic condition (stage 1). As the plastic zone spreads, occurrence of the maximum tensile stress condition shifts from point (a) toward others and closer to slope apex (point a→b→c). With this body shape and loading system, it is found that the plastic zone spreads continuously without development of plastic yield in other locations (disregarding the tensile stress zone). Spreading rate is faster in the later stages when a large plastic area has developed.

Comparison of potential slip surfaces ($\phi = 0^\circ$) is shown on Figure 6-8, stage number 4. Slip surfaces obtained from the elasto-plastic stress field, and from the elastic stress field are essentially the same as that predicted by Bishop's method. Locations and overall appearance are about the same for all analytical methods. The tensile stress zone near the potential slip surfaces is deeper for the elasto-plastic analysis than for the elasticity analysis.

6.1.3 Elasto-plastic, Incremental Construction

In practice the final shape of an embankment is attained by a building sequence, usually involving the placing of material in layers, one on top of another. In order to arrive at a practicable simulation process, an incremental approach is required, in contrast to the conventional solution that presumes the construction to be completed in a single stage and the gravity load to be withheld until the final shape is attained.

Results of elasto-plastic, incremental behavior by gradually increasing the height of slope embankment are shown on Figure 6-9. With the same material properties as those used in the single stage method, height at failure is found to be the same for both cases. Development of tensile stresses is less in this process of construction. Rate of spreading of the plastic zone is about the same at each stage. The general behavior of plastic zone development is similar for both simulations even though the configurations are not exactly alike.

Figure 6-10 shows results of including elastic-strain hardening soil behavior. Configurations of plastic region development are not much different from those obtained by the elastic-perfectly plastic assumption. The main difference is the creation of tensile stresses, which are less intense for this assumption. The slip surfaces

in the plastic region generated by the stress field at the final stage appear to be of the same form as those in the former cases.

The resulting location and configurations of slip surfaces presented in the previous sections depended on a $\phi = 0^\circ$ failure envelope. As discussed formerly in Chapter V, the slip angle should be obtained from the drained-strength envelope ($\phi \neq 0^\circ$), which means $\alpha = 45 - \phi_D/2$. Construction of slip-lines for different assumed ϕ_D (20° and 30°) are presented on Figure 6-11. The stress field used for this construction was the final stage of the incremental construction with elastic-perfectly plastic material assumption. It should be noted that in this presentation the strength criterion follows the undrained strength envelope while the failure angle depends on the drained strength envelope. The results show that base failure is still obtained for any value of ϕ_D , but the location of the potential sliding surface does change. Steeper slip surfaces are found with higher ϕ_D , which agrees very well with elasticity solutions and solutions according to Bishop's method in the sense of their orientation. The shape of the slip surfaces are also close to circular. It is obvious that application of Bishop's analysis in predicting the potential sliding surface is impossible in this case, because of the difference in strength and failure criteria mentioned above.

Figure 6-12 presents the results of elasto-plastic, incremental analysis for unsaturated soil, using the drained shear strength envelope for both strength and slip surface orientation. Elastic-strain hardening is assumed.

The results show a great effect of the strength characteristics in the generation of the plastic region. Local yield starts at the toe of the slope instead of at the base and continuously spreads out as the load increases. However, during an increment of load the stress condition of any point in the plastic region can shift from the yield condition (circle 1, Figure 6-13) to an elastic state of stress (circle 2, Figure 6-13). Location of the tensile stress zone also differs from the undrained analysis. The plastic region in the drained analysis develops closer to the slope surface so that the potential rupture surface can pass through the toe of the slope (shown in the final step of Figure 6-12), and is closer to the slope surface. Comparison of location and configuration of potential rupture surfaces analyzed by three procedures, elastic stress field, Bishop's Method, and elasto-plastic stress field, are shown in stages 5 and 6 of Figure 6-12. Good agreement is obtained among these solutions. The strength parameters are $c = 200$ psf, and $\phi = 30^\circ$. Other soil properties are shown on Figure 6-12.

A number of additional factors which could be considered are:

1. As formerly pointed out, Chapter III, soil strength is often reduced after soil is subjected to pressure sufficiently large to cause rearrangement of the soil skeleton.
2. Progressive formation of the failure surface can cause strength reduction in sensitive clays along part of the surface.
3. Other soil properties may also change during deformation, for example the modulus of elasticity.

6.2 Nonhomogeneous Soil Slopes

The results of analyses of nonhomogeneous soil slope embankments are presented in two parts, Elastic Range Analysis and Elasto-plastic Incremental Construction.

6.2.1 Elastic Range Analysis

Investigations were performed for embankments consisting of two layers of different soils. Only the case of hard soil overlaying soft material was considered. The materials were chosen to simulate saturated clays, and the following properties were assumed:

Hard clay; $E = 360,000$ psf, $\gamma = 140$ pcf.

Medium clay; $E = 170,000$ psf, $\gamma = 128$ pcf.

Soft clay; $E = 86,500$ psf, $\gamma = 110$ pcf.

Results of tension zone development are presented on Figure 6-14. Height ratio, $\frac{h}{H}$, is defined as the height of the bottom layer divided by the total height of the slope. Poisson's ratio values of 0.4, 0.45, and 0.49 are considered.

The results indicate that the height ratio has some effect on the configuration of the tension zone, but not as much effect as variation in Poisson's ratio. Also, softer soil in the bottom layer induces a longer magnitude of tensile stresses in the top layer.

Envelopes of required strength for different values of ν are essentially the same as those for homogeneous slopes, and are not shown.

6.2.2 Elasto-plastic, Incremental Construction

Only analyses of firm soil overlaying softer material are considered as before. The soil properties are:

top layer soil;	$E_1 = 210,000 \text{ psf},$
	$c_1 = 1500 \text{ psf},$
	$\nu_1 = 0.45,$
and bottom layer soil;	$E_2 = 56,000 \text{ psf},$
	$c_2 = 400 \text{ psf},$
	$\nu_2 = 0.45.$

The results (Figure 6-15) show similar behavior in the location of the first yield point, and the configuration and rate of plastic zone propagation as those obtained in the homogeneous soil cases. Characteristics and occurrence of tensile stress zone are also similar, except that the zone propagates deeper in the two-layer case. Configuration and location of the potential slip surfaces are also alike. Notice that the soil properties of bottom layer are the same as those of the homogeneous soil slope. Since higher quality soil is applied in the top layer, the height ($H_f = 22.5$ feet) at limit equilibrium in the two-layer case is greater than that of the homogeneous soil condition ($H_f = 20$ feet). Behavior and configuration of the failure surface tends to be in the same neighborhood as that of the homogeneous case. Nonhomogeneity does not appear to affect failure behavior greatly. Notice the discrepancy between the failure surfaces predicted for the two-layer case by Bishop's method and that predicted by the elasto-plasticity analysis. It appears that Bishop's method is not applicable in locating the actual failure surface, in nonhomogeneous soil slopes of this type. Many other factors, such as construction processes (excavation or embankment) and failure behavior (long term failure or sudden failure), could be considered. Each case must be investigated individually. An elasto-plastic incremental analysis with consideration of the

cause of failure, soil strength at failure, and development of pore pressures at failure could be of great value.

Some comments should be made about the difference in stress paths generated at different locations of the slope body during load increments. Results of investigation are presented on Figure 6-16. The results indicate that the stress paths for points closer to the slope surface more nearly approximate those of triaxial shear tests. The triaxial test (Chapter III) produces a shearing process which guarantees the occurrence of failure. The stress paths of locations (point D) quite far from the slope surface, give results similar to the consolidation test (Chapter III), which does not guarantee failure. It is obvious that failure can occur only in the slope vicinity and will not occur in the areas where stress states are similar to those of the semi-infinite body under gravitational body forces. As explained in Chapter III, the stress-strain relationships are different under different applied stress paths. Laboratory tests obtaining the stress-strain relationships (or E-values) must be performed to simulate stress conditions similar to those really occurring in the field. Different E-values should be applied to those corresponding locations in the computation of stresses and deformations. A similar situation is explained and performed by Huang (1968) in his work concerning the nonlinearity of soil media.

6.3 Stability Analysis

The results are presented in terms of a factor of safety field at different stages of construction. Only the incremental construction case is considered.

Figure 6-17 shows the factor of safety contour fields for homogeneous soil at various stages of construction. Conditions are the same as those represented on Figure 6-12. The magnitude of the factor of safety values in the fields decreases rapidly as the embankment height increases, especially in the potential failure zone. Spreading of the factor of safety equal one contour line is continuously toward the top surface. Since the factor of safety cannot go below one, the average gradient of the factor of safety field in the potential failure area decreases rapidly as construction progresses. In this case using drained strength criterion, the low factor of safety contours concentrate in the area close to the slope surface and the lowest factor of safety line starts at the toe of the slope.

In Figure 6-18 contour lines of the factor of safety for a two layer slope are shown. Since the undrained strength is used for the failure criterion, different configurations are found in this case comparing to that shown on Figure 6-17. Concentration of the low factor of safety contour lines is at the base area instead of the toe area. The factor of safety equal to one contour line

is within the slope body. The magnitude of the factors of safety and the gradient of the fields are lower in this case which indicates a more unstable condition. Built up slope embankments under undrained shear strength (sudden failure) will have higher potential to fail than those with drained shear strength (long term failure).

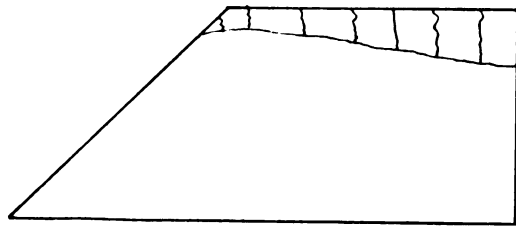
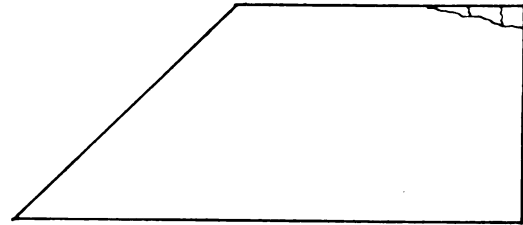
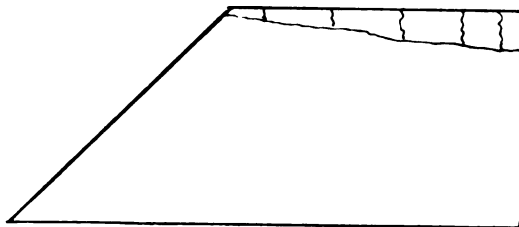
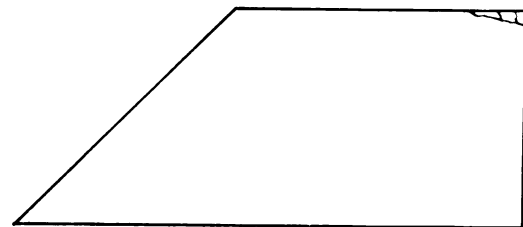
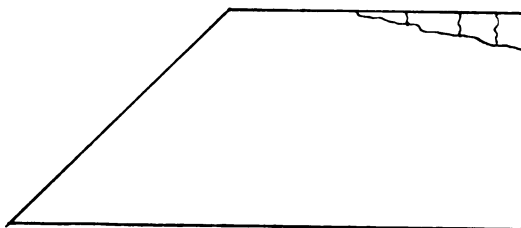
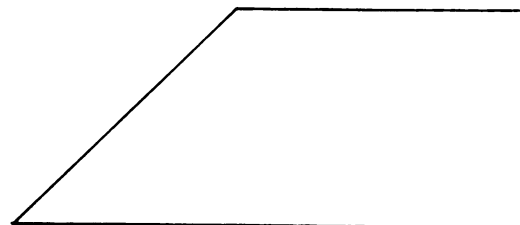
To illustrate the deformed shape of the slope at different stages, displacements of points along the boundaries of the slope are given in Table 6-1. It is quite obvious that the magnitudes of the displacements are much greater in the final stage than those in the first stages. The large displacements in the final stage are obtained because of development of the plastic zone in the soil body.

Average factor of safety values were calculated along the potential sliding surface predicted from the critical failure surface according to Bishop's method at different stages to indicate the stability of the slope. Undrained strength criterion is used in the analysis. These are presented on Table 6-2. Height of the slope is 20 feet for both single-stage and incremental constructions. The average factor of safety values were calculated, using the relationships for E and c_u given in Chapter III. These are compared with those given by Bishop's analysis. The results show good agreement between the two methods, and the two values converge more closely in the final stage.

Since comparison of the stability values as performed in this section requires knowledge of the relationship between the strength parameters (c_u for undrained, c' and ϕ' for drained) and the elastic parameters (E and ν), the author would like to see the development of a more complete definition of these relationships. Only a few investigations have been performed so far.

Figure 6-19 shows the potential sliding surfaces used in the calculation of the average factors of safety.

I. HOMOGENEOUS--SOIL SLOPE

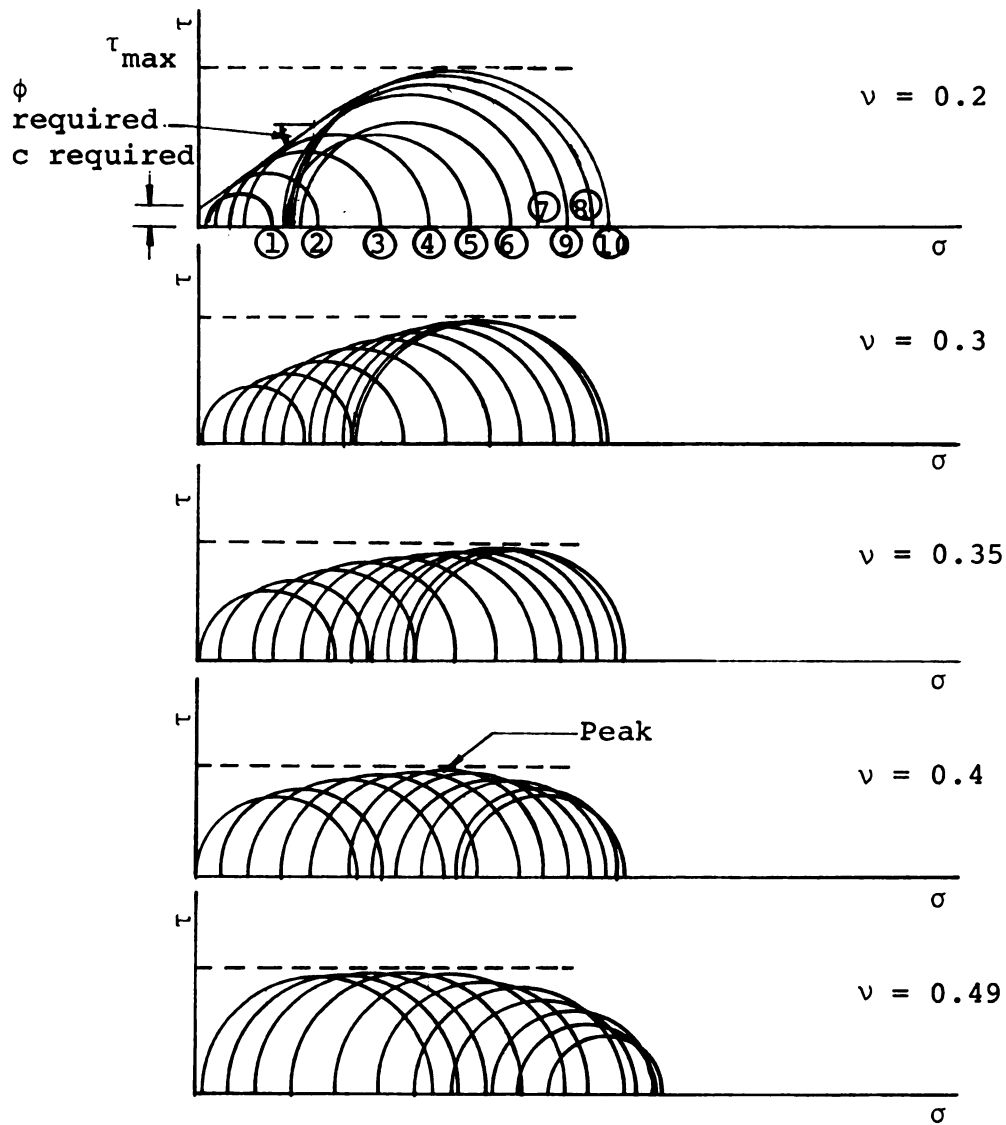
a) $\nu = 0.49$ d) $\nu = 0.35$ b) $\nu = 0.45$ e) $\nu = 0.325$ c) $\nu = 0.4$ f) $\nu = 0.3$

 Tension Zone

$E = 342,000 \text{ psf}$

$\sigma = 110 \text{ pcf}$

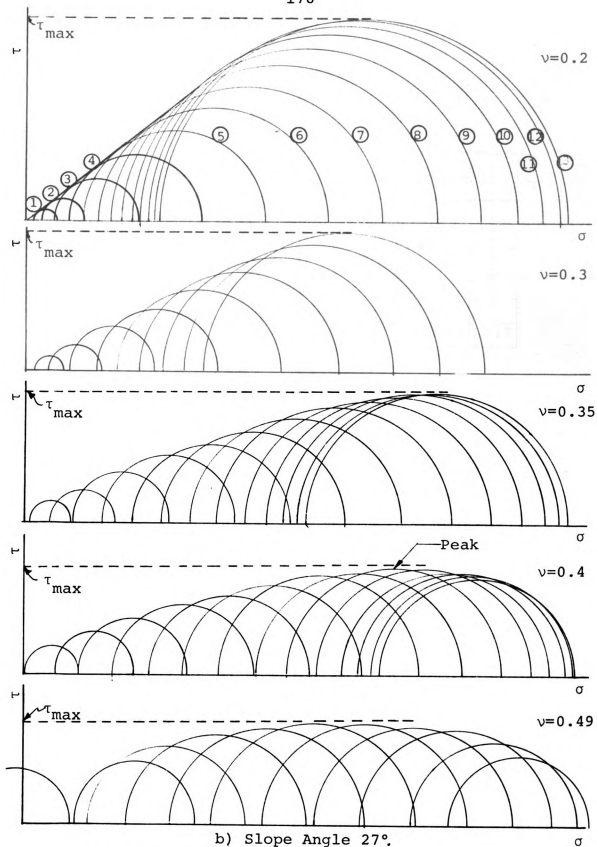
Figure 6.1.--Influence of ν on Tensile Stress Development.



a) Slope Angle 45° .

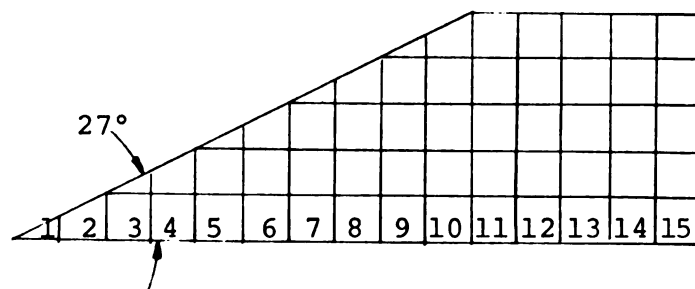
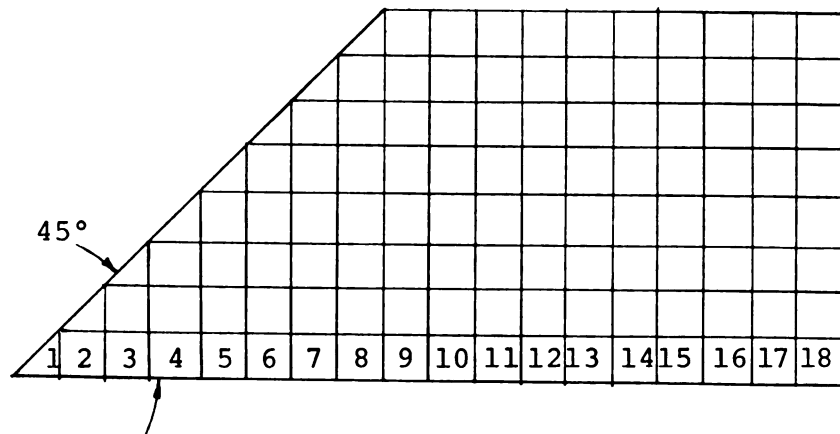
○-indicates location of points along bottom layer (see Figure 6.2c)

Figure 6.2.--Stress Envelopes for Points Along the Bottom Layer of a Slope.



b) Slope Angle 27° .

Figure 6.2.--Continued.



c) Location of Points Along Bottom Layer of Slope .

Figure 6.2.--Continued.

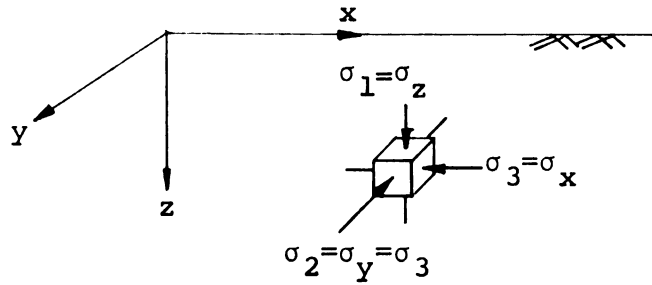
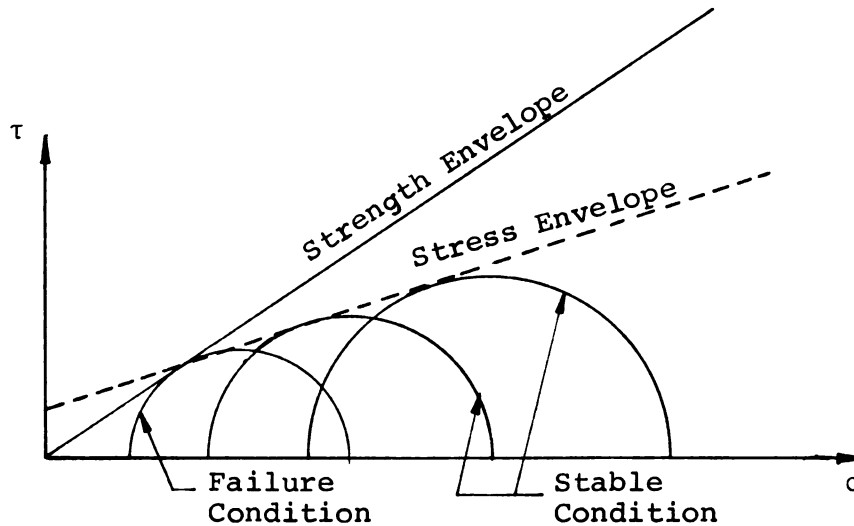
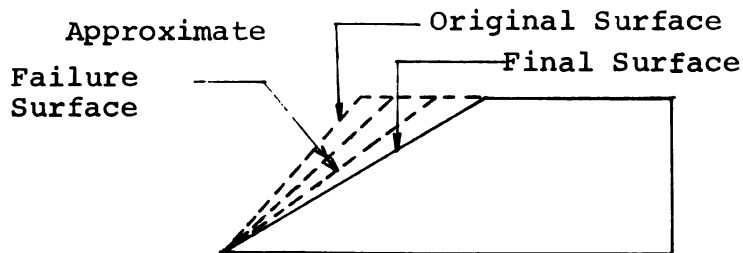


Figure 6-3.--Stresses in a Semi-Infinite Body.

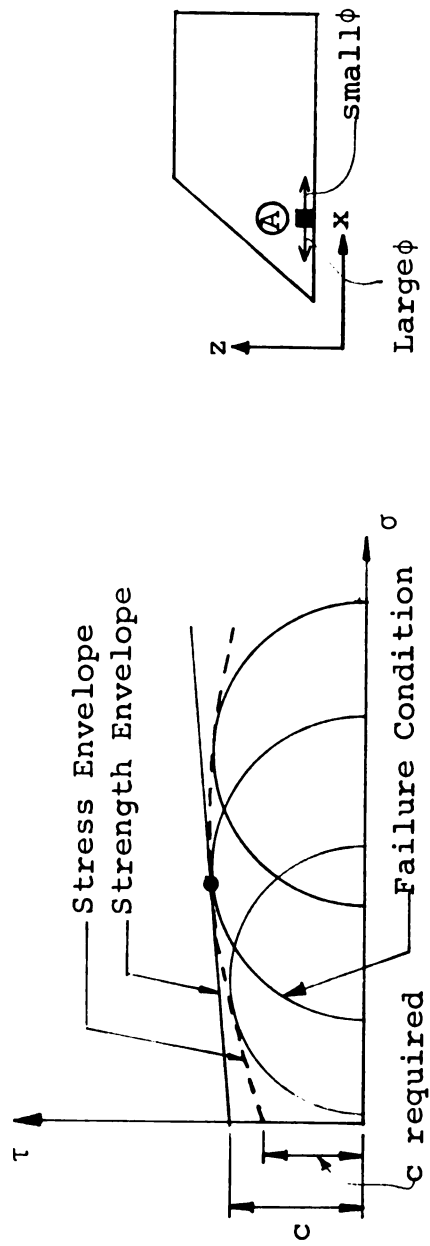


a) Stress Circles, and Strength Envelopes.



b) Failure Behavior.

Figure 6-4.--Effect of Strength Envelope on Failure Behavior of Cohesionless Soil Slopes.



a) Stress Circles, and Strength Envelopes, b) Failure Behavior.

Figure 6-5.--Effect of Strength Envelope on Failure Behavior of Cohesive Soil Slopes.

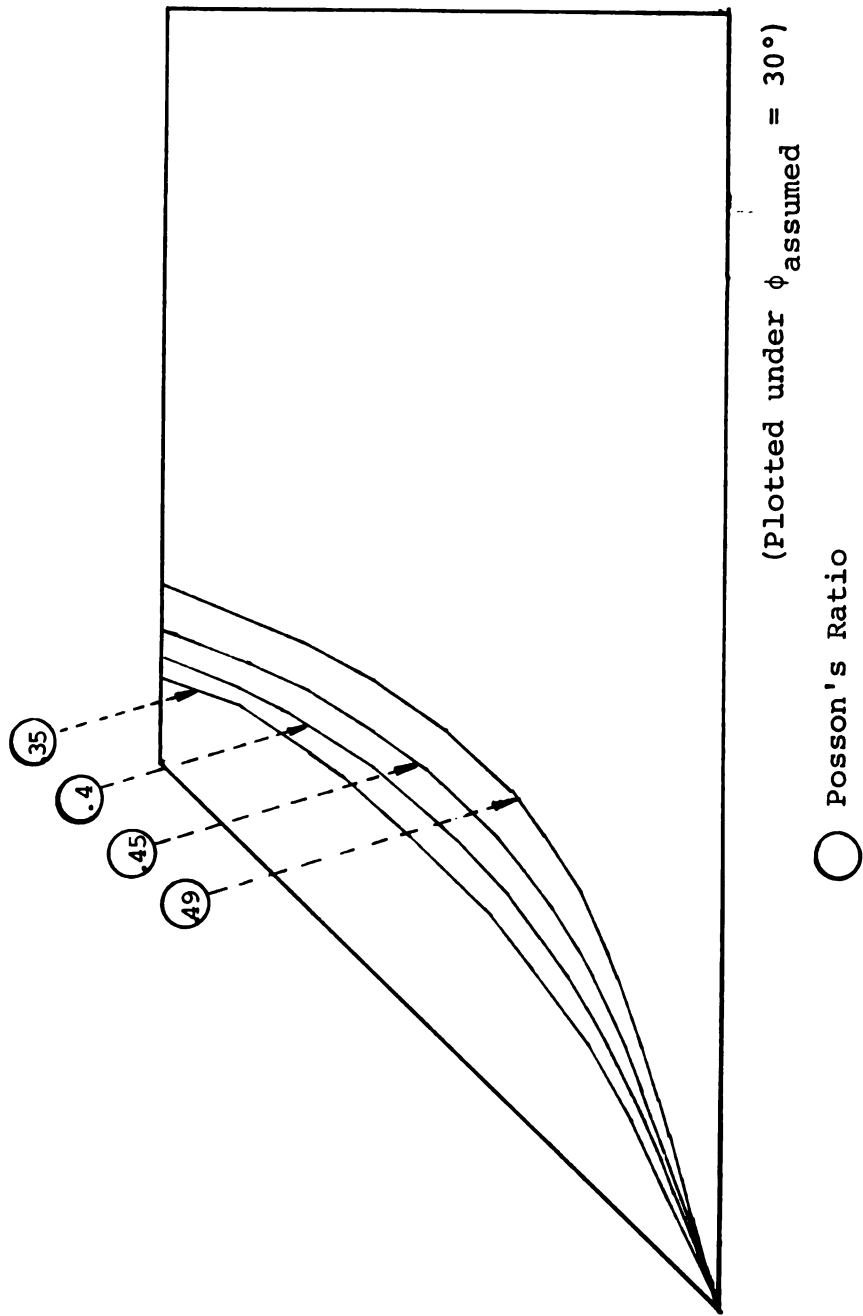


Figure 6.6.--Change of Slip Surfaces for Different ν -values.

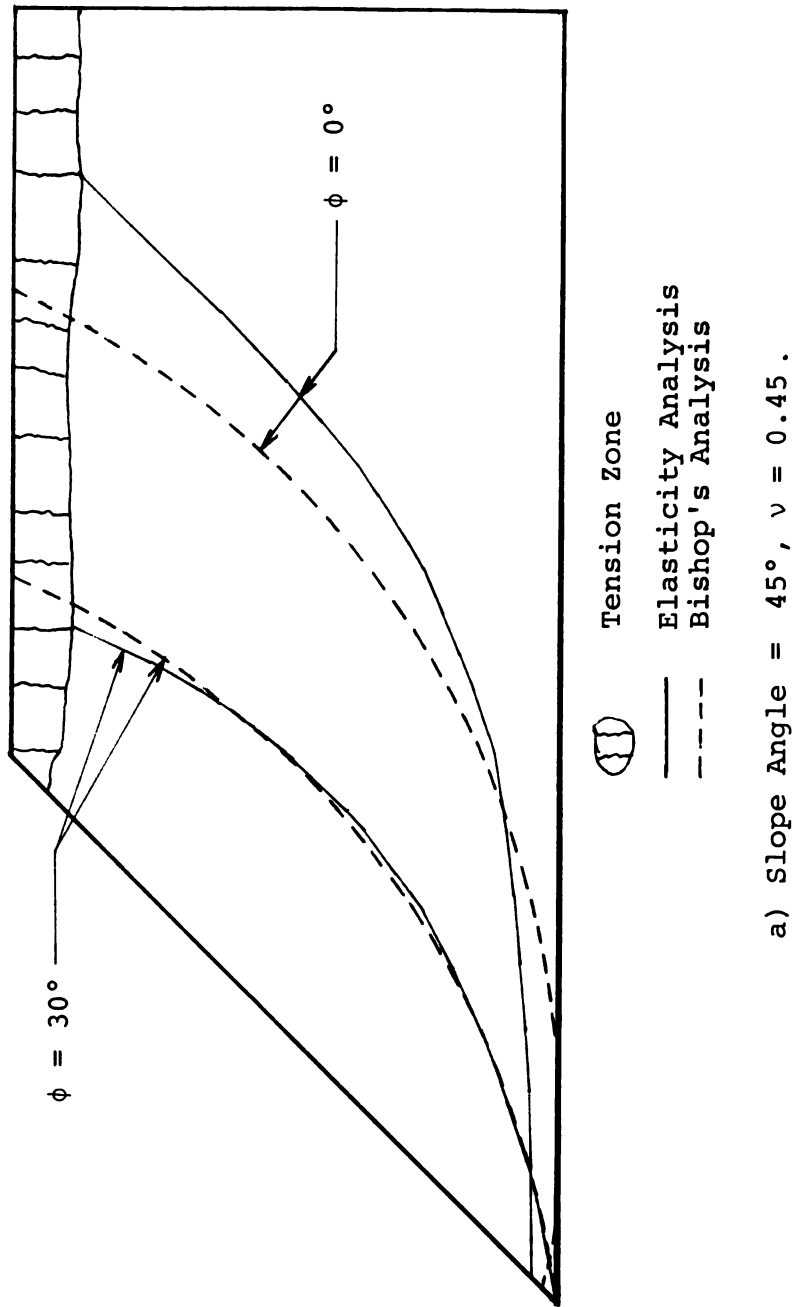
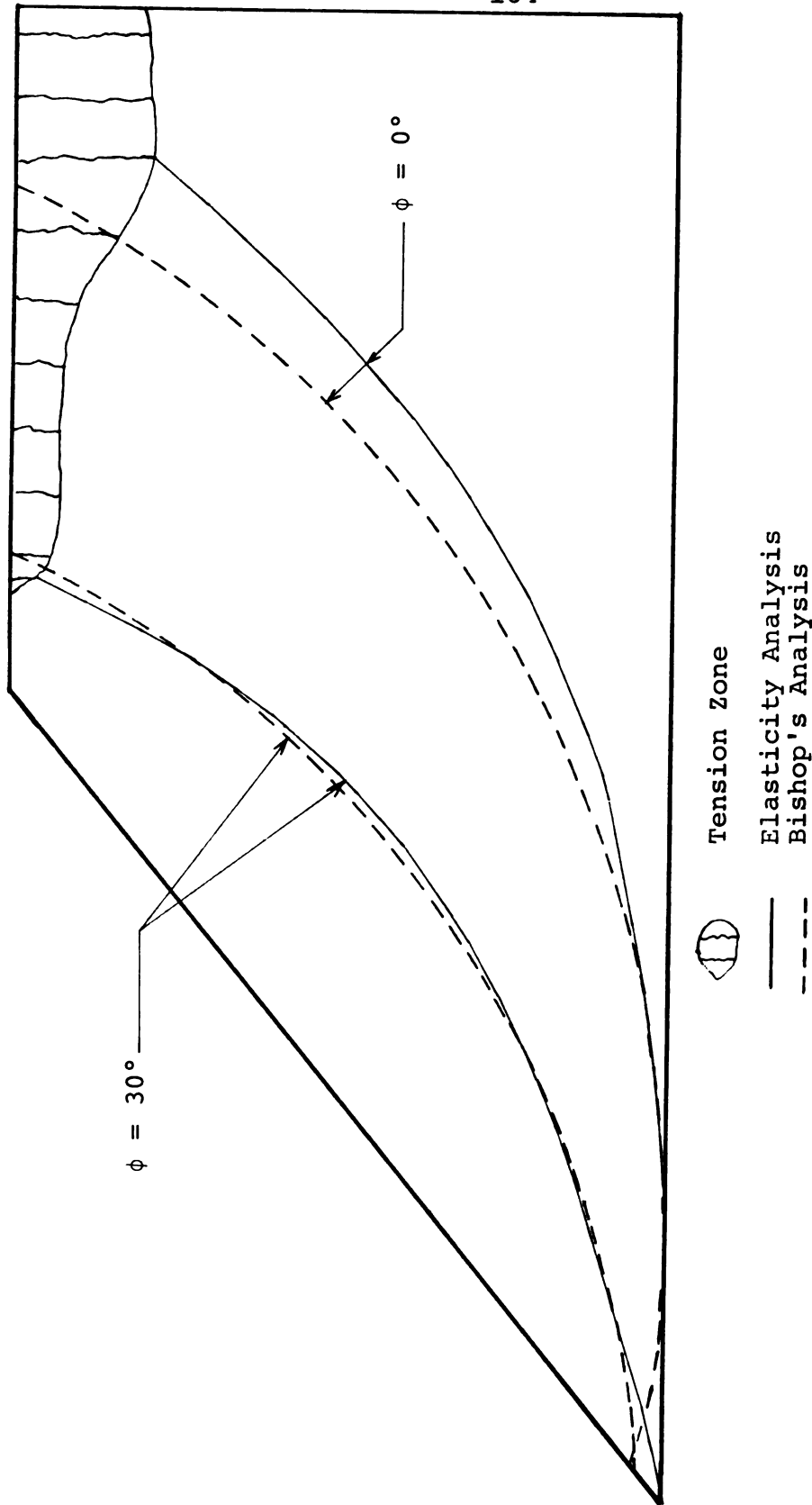
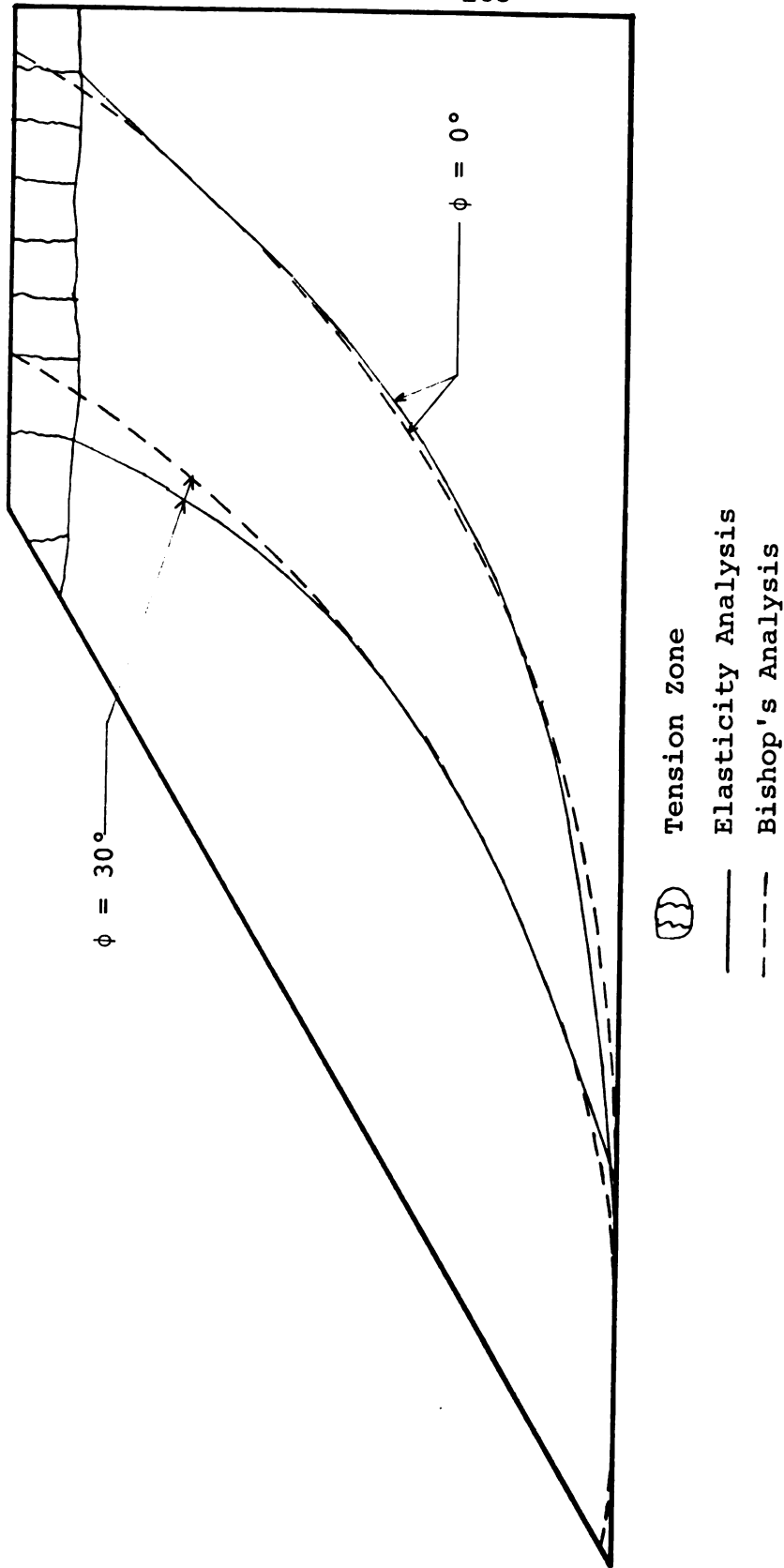


Figure 6.7.--Comparison of Potential Sliding Surfaces between Bishop's and Elasticity Analyses, for Different Slope Angles.



b) Slope Angle = 38.6° , $\nu = 0.45$.

Figure 6.7.--Continued.



c) Slope Angle = 30° , $\nu = 0.45$.

Figure 6.7.--Continued.

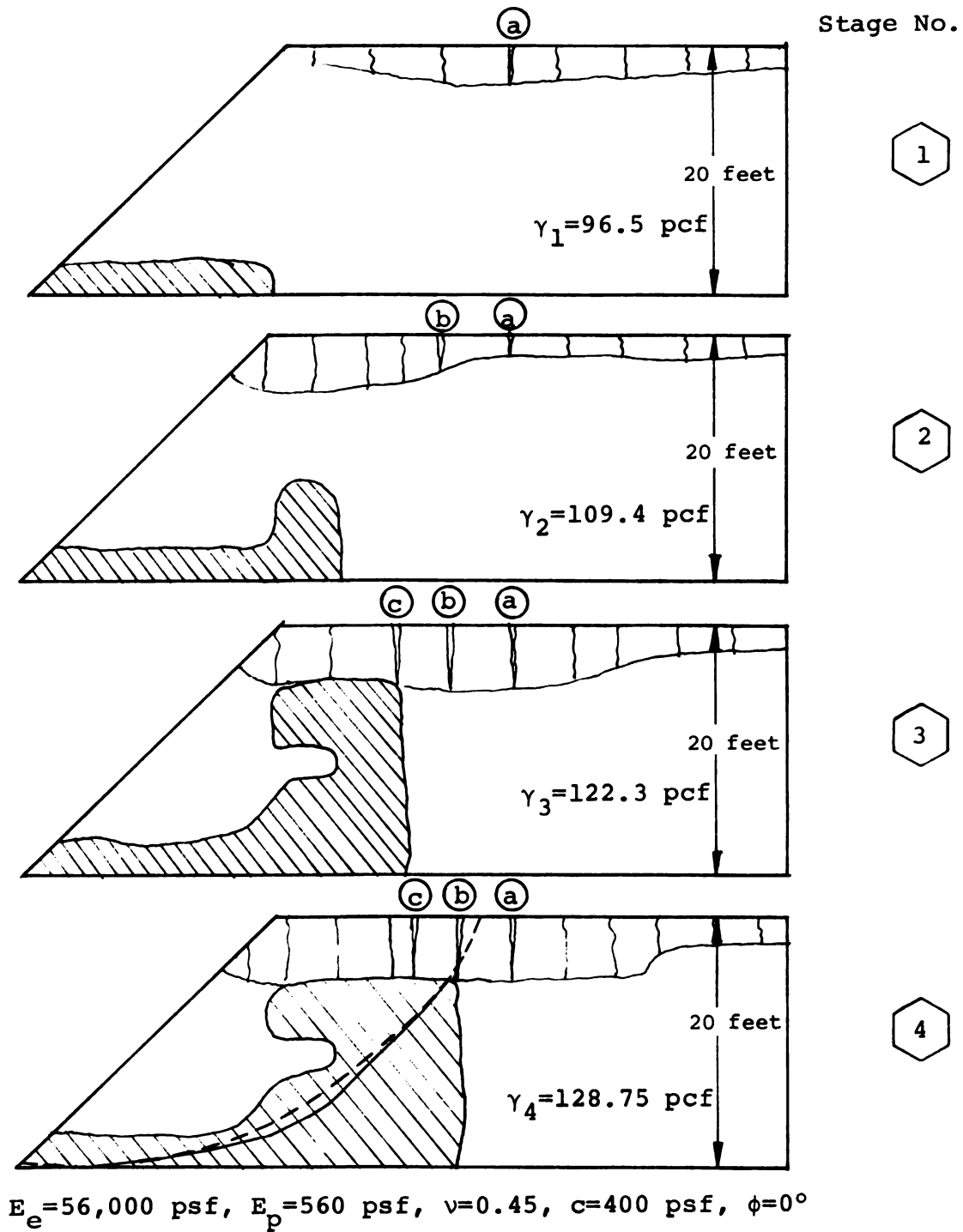
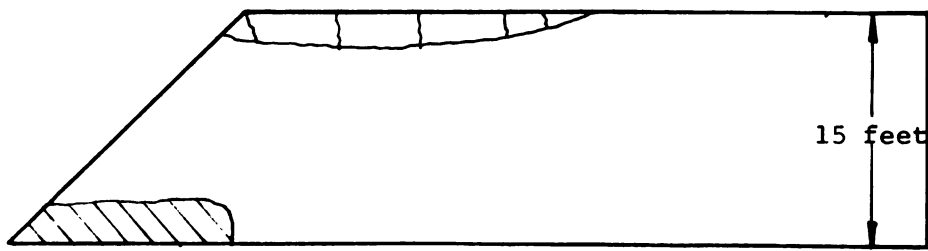


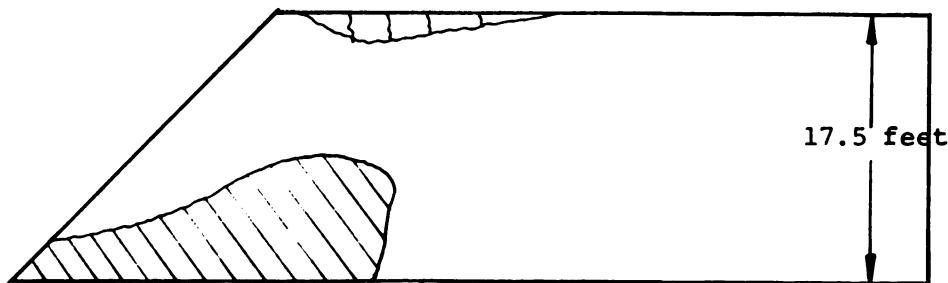
Figure 6-8.--Elasto-plasticity Analysis of Single-Lift Construction, $H = 20$ feet.

187

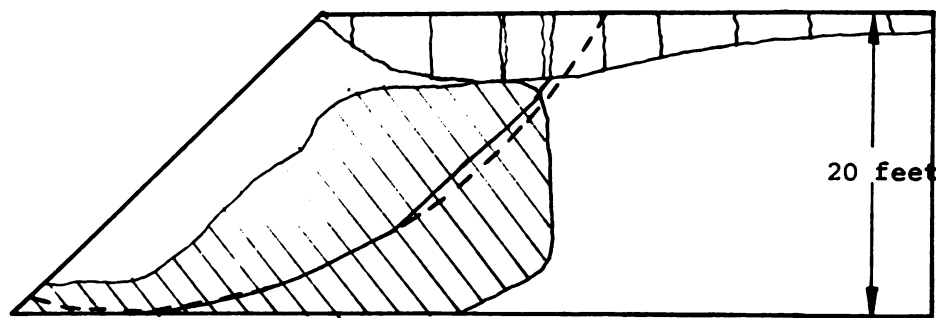
Stage No.



1



2



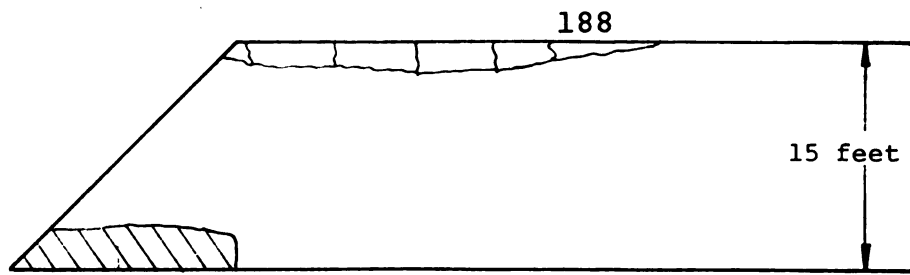
3

$\gamma=128$ pcf, $E_e=56,000$ psf, $E_p=56$ psf, $\nu=0.45$, $c=400$ psf, $\phi=0^\circ$

○ Elastic Zone ⊗ Plastic Zone ⊕ Tension Zone

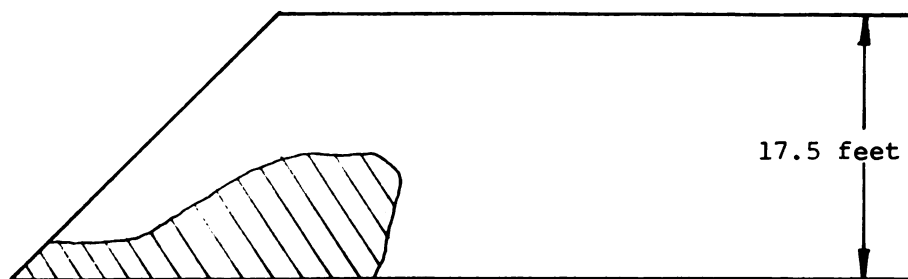
————— Elasto-Plasticity, ($\phi=0$) - - - - - Bishop's, ($\phi=0$)

Figure 6-9.--Elasto-Plasticity Analysis of Incremental Construction, Elastic Perfectly-Plastic Assumption.

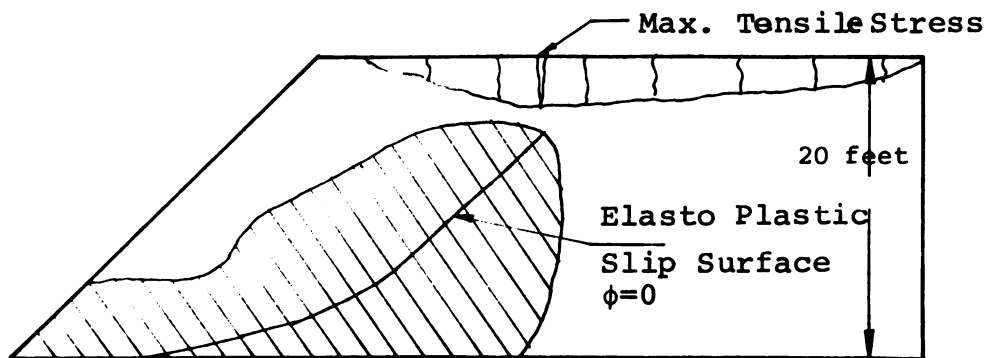


Stage No.

1



2



3

$E_e = 56,000$ psf, $E_p = 560$ psf, $\nu = 0.45$, $\gamma = 128$ pcf,
 $c = 400$ psf, $\phi = 0^\circ$

 Elastic Zone

 Plastic Zone

 Tension Zone

Figure 6-10.--Elasto-plasticity Analysis of Incremental Construction, with Elastic Strain Hardening Included.

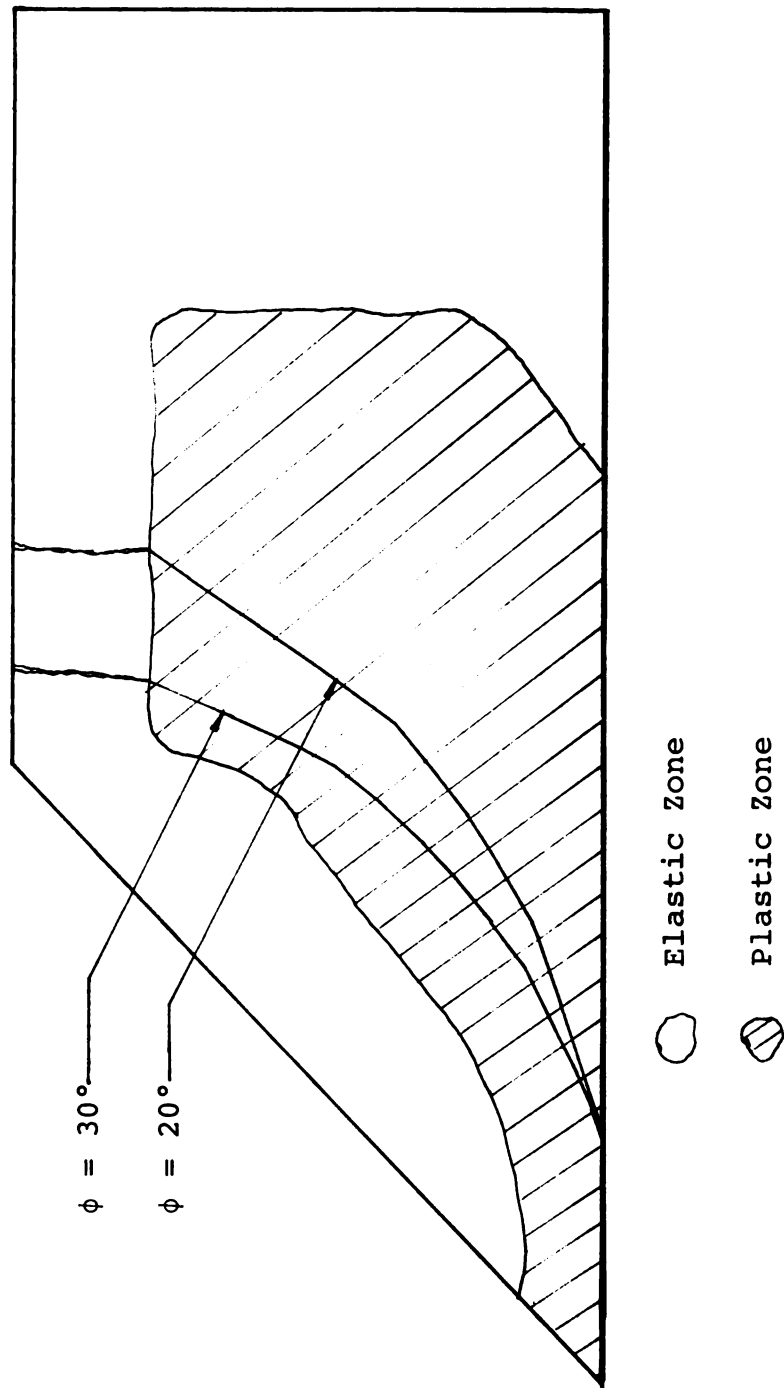
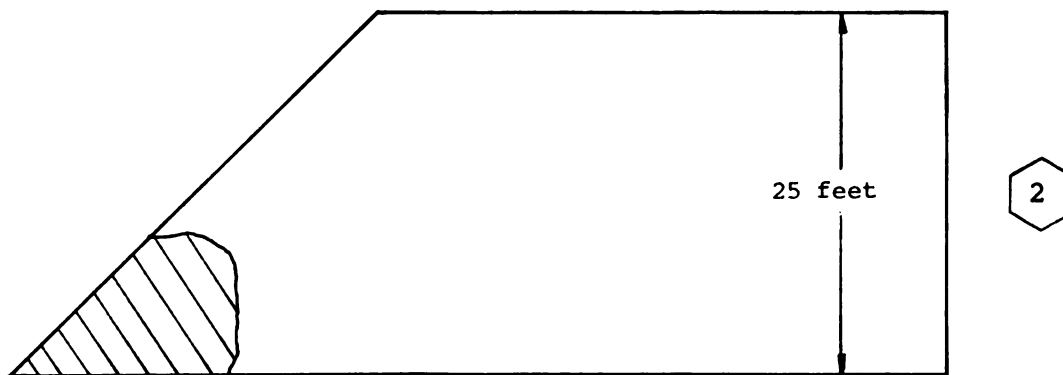
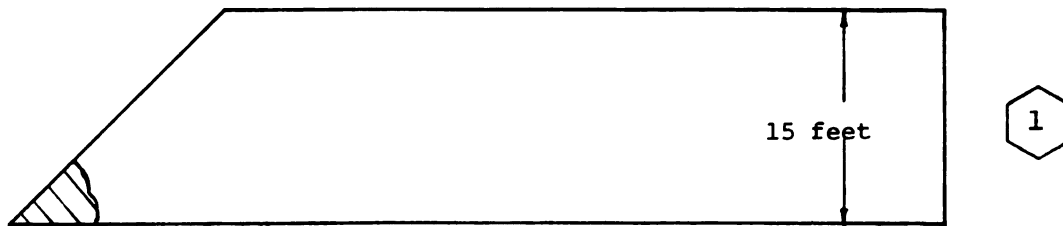


Figure 6.11.--Slip Surfaces Plotted According to the Stress Field of the Final Stage of Incremental Construction (Figure 6.9), for Different Assumed ϕ -values.

Stage No.



$E_e = 56,000$ psf, $E_p = 560$ psf, $\nu = 0.45$, $\gamma = 128$ pcf,

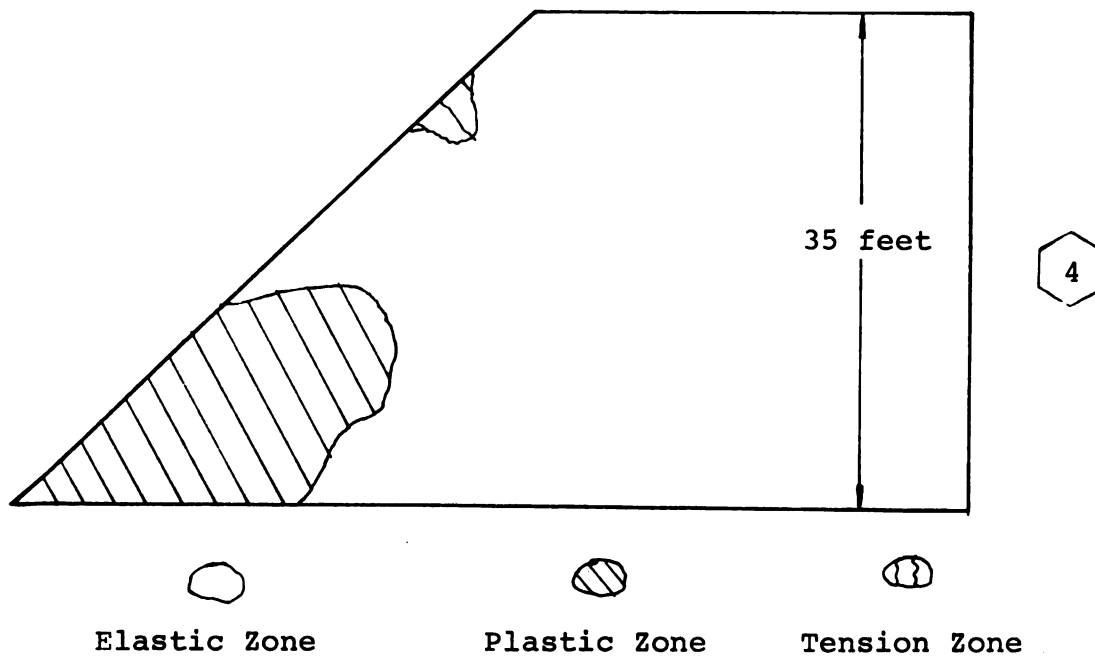
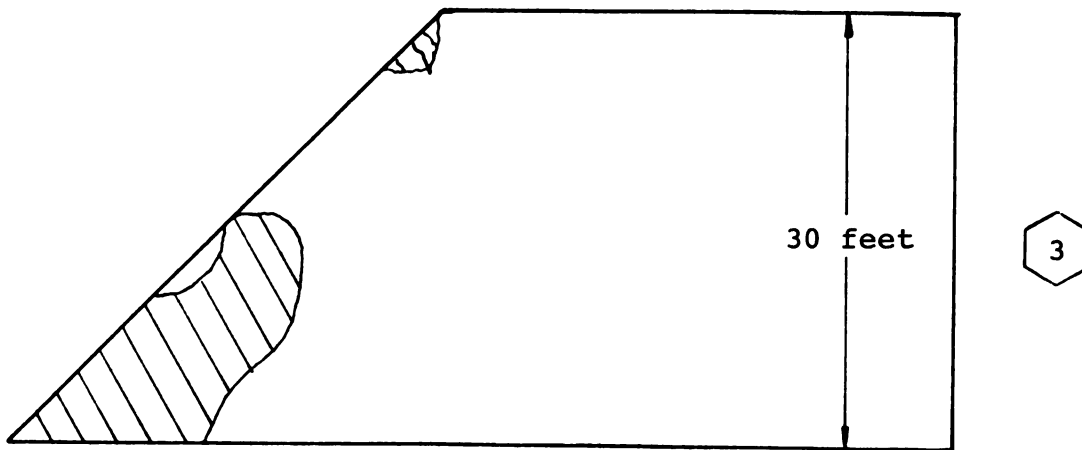
$c = 200$ psf, $\phi = 30$ degree

○ Elastic Zone

⊗ Plastic Zone

Figure 6-12.--Elasto-plasticity Analysis Under Drained Strength Criterion and Incremental Construction, Elastic Strain-Hardening Assumption.

Stage No.

Figure 6.12.--Continued.

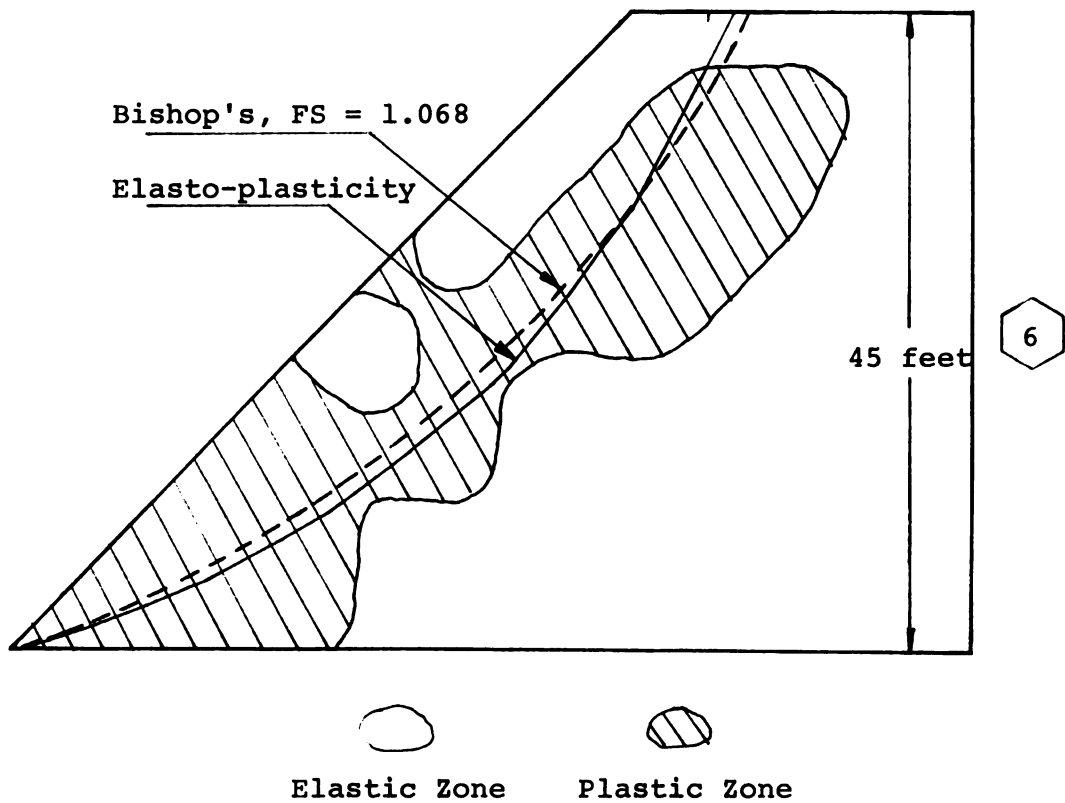
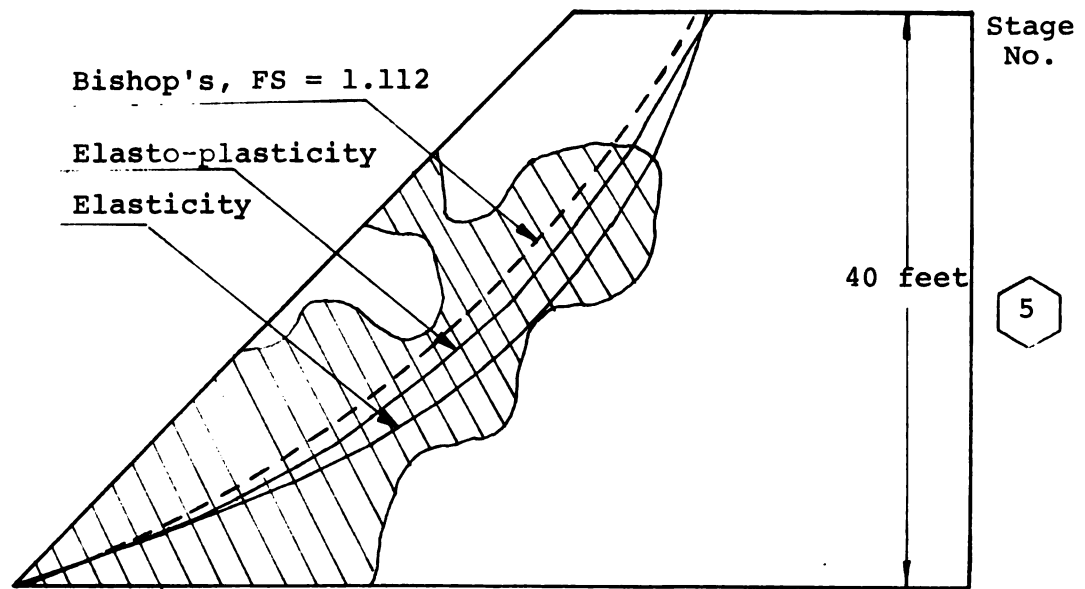


Figure 6.12.--Continued.

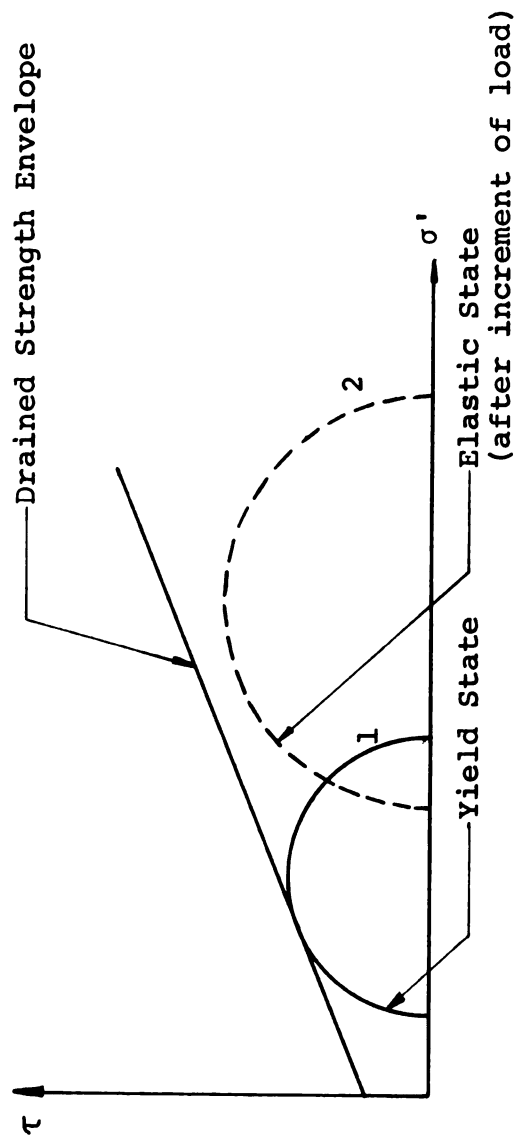
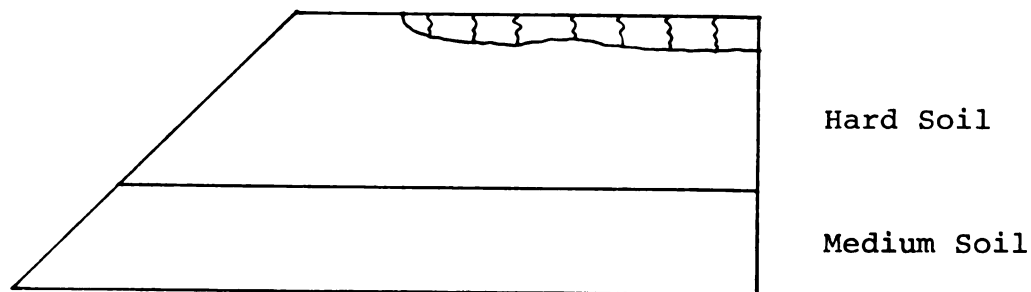
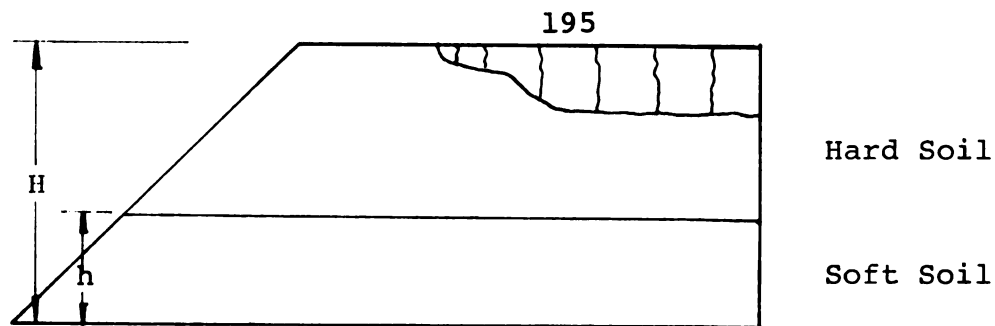
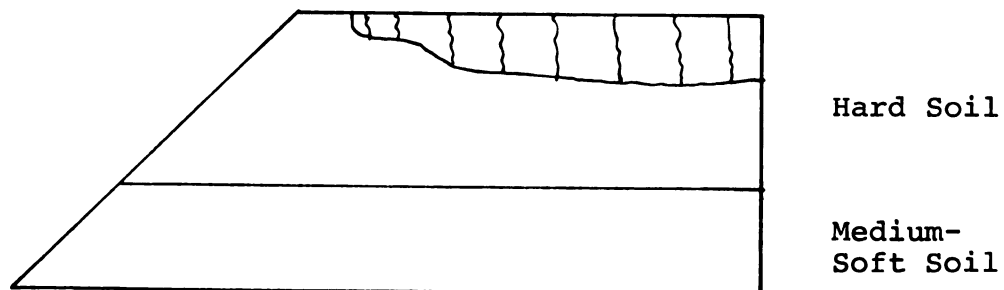
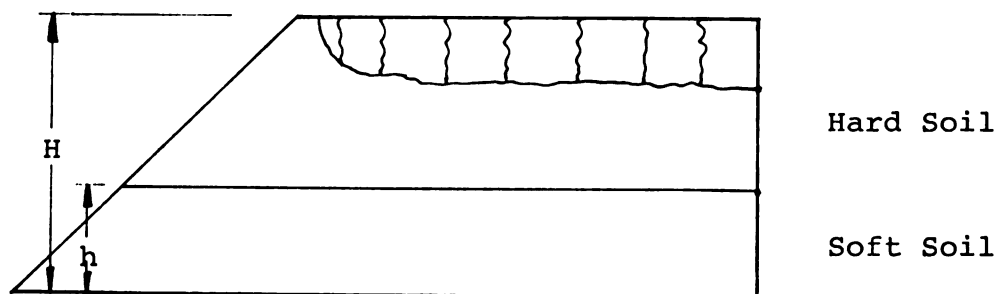


Figure 6.13.--Possible Stress Conditions During an Increment of Load.

II. NON-HOMOGENEOUS SOIL SLOPE



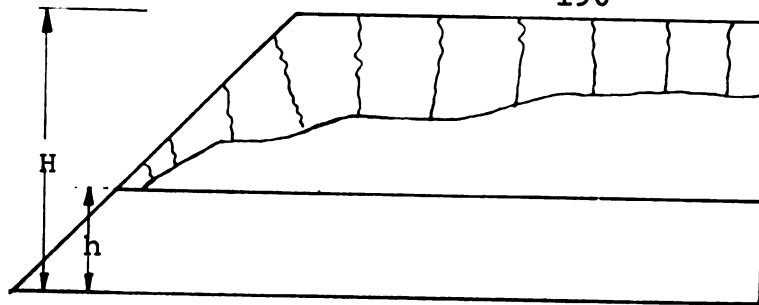
a) $\nu = 0.4, \frac{h}{H} = 0.375$



b) $\nu = 0.4, \frac{h}{H} = 0.375$

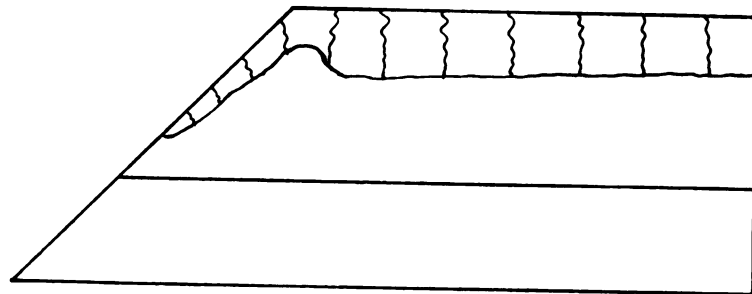
⊖⊖ Tension Zone

Figure 6.14.--Development of Tension Zone for Two-layer Case.



Hard Soil

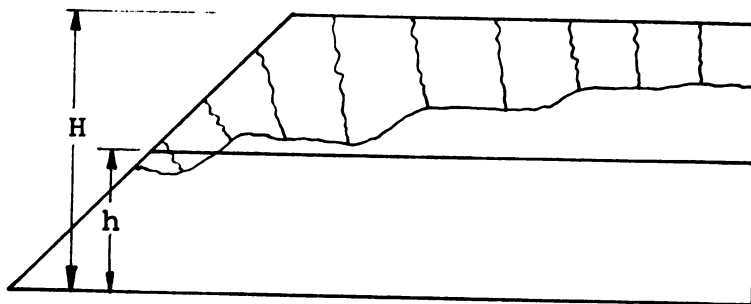
Soft Soil



Hard Soil

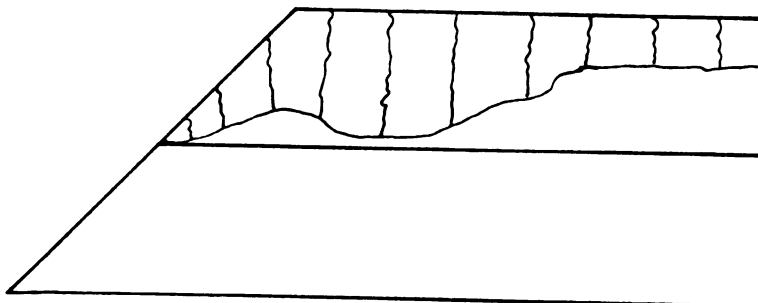
Medium Soil

$$c) \nu = 0.49, \frac{h}{H} = 0.375$$



Hard Soil

Soft Soil



Hard Soil

Very Soft Soil

$$d) \nu = 0.49, \frac{h}{H} = 0.5$$

⊖ ⊖ Tension Zone

Figure 6.14.--Continued.

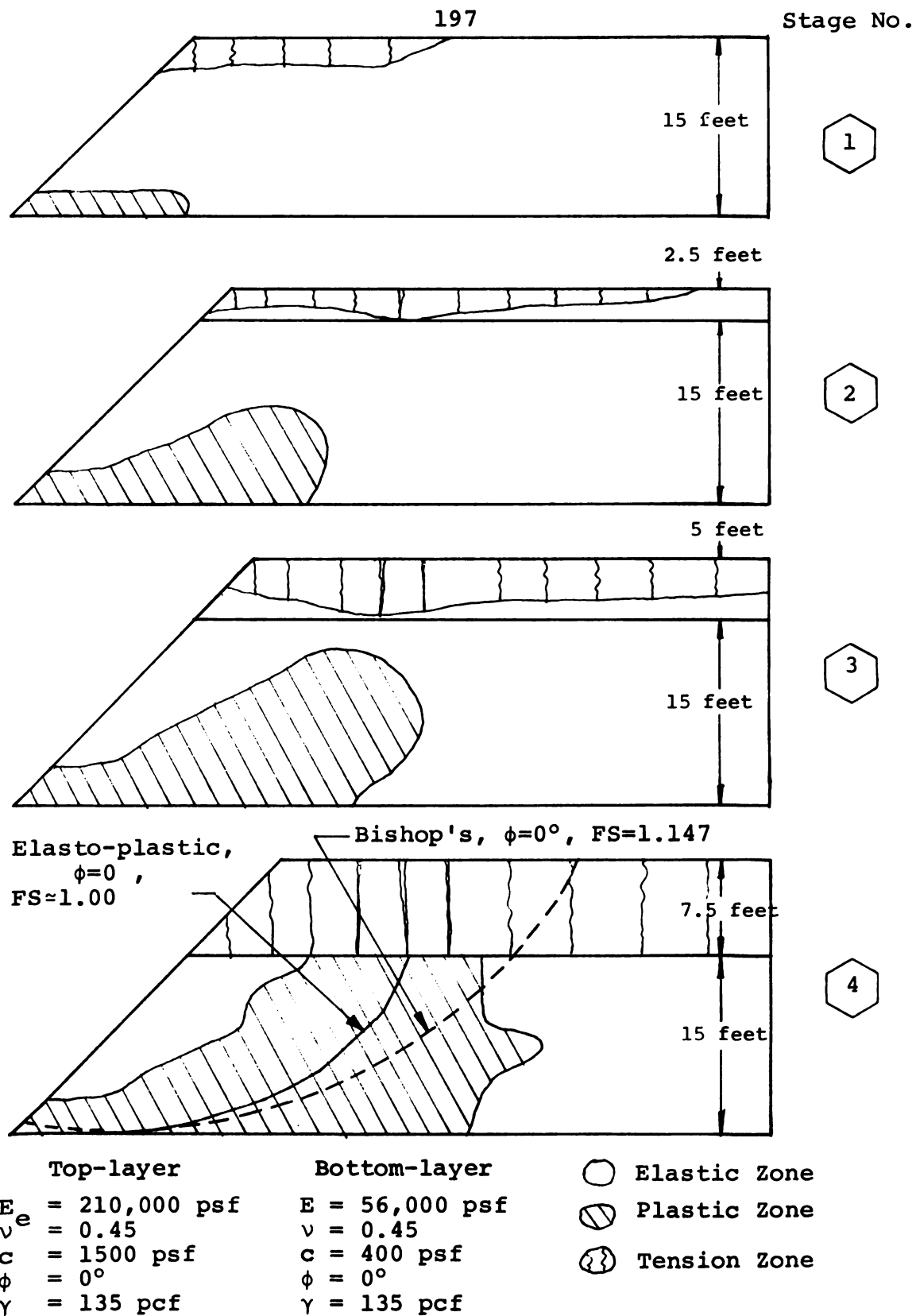


Figure 6-15.--Elasto-plasticity Analysis, Incremental Construction for Two-Layer Slope.

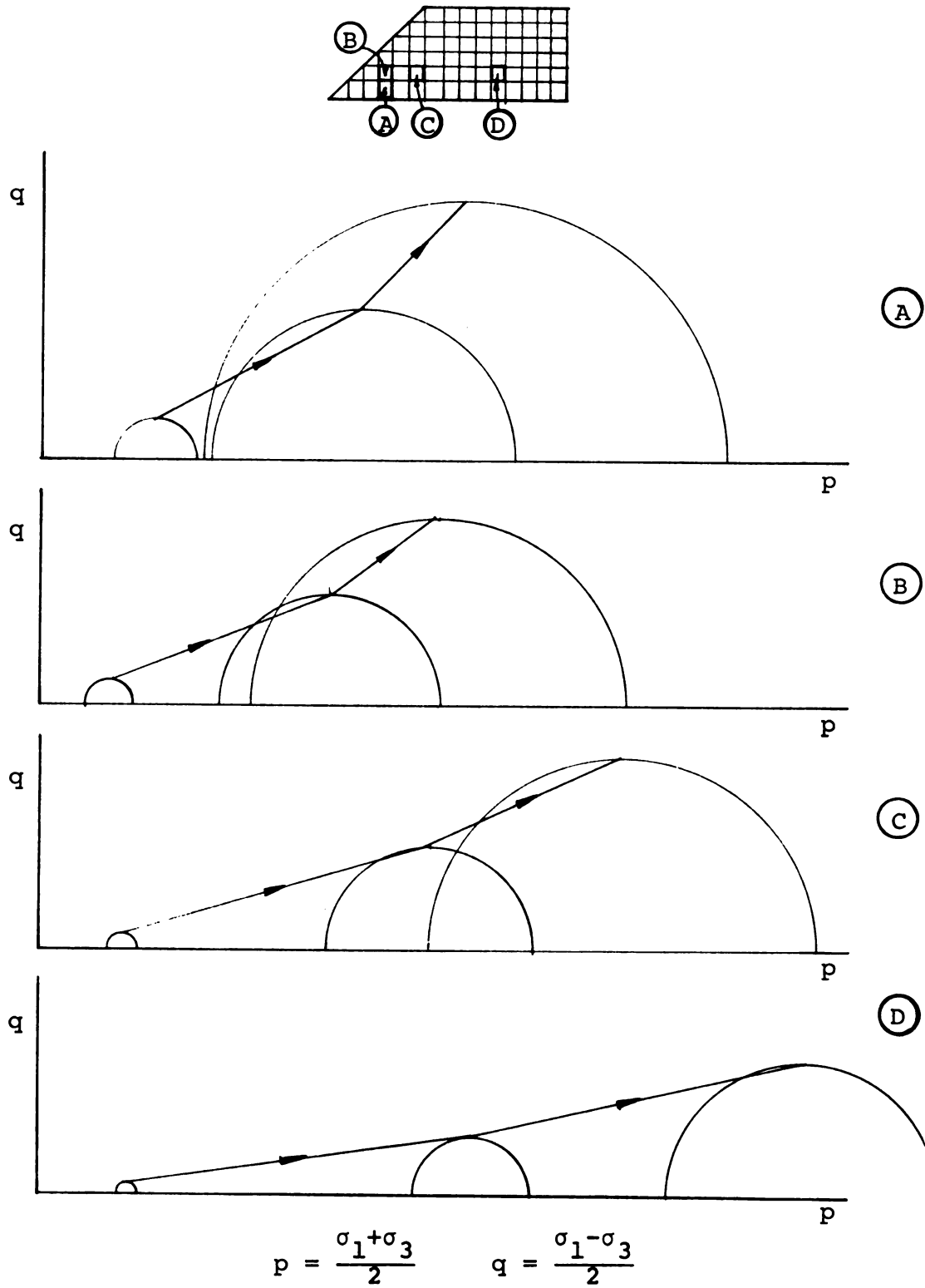
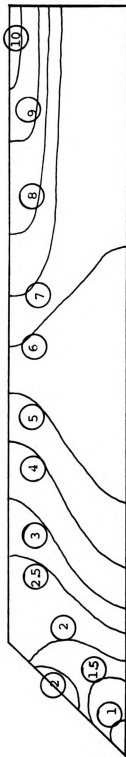


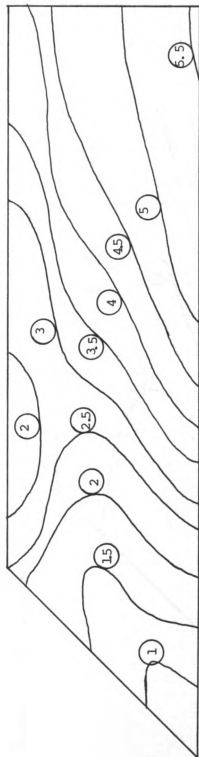
Figure 6.16.--Stress-path at Different Locations in Slope Body, 10 feet lifts, 45°-slope Angle.

III. STABILITY ANALYSIS

Stage No.



1

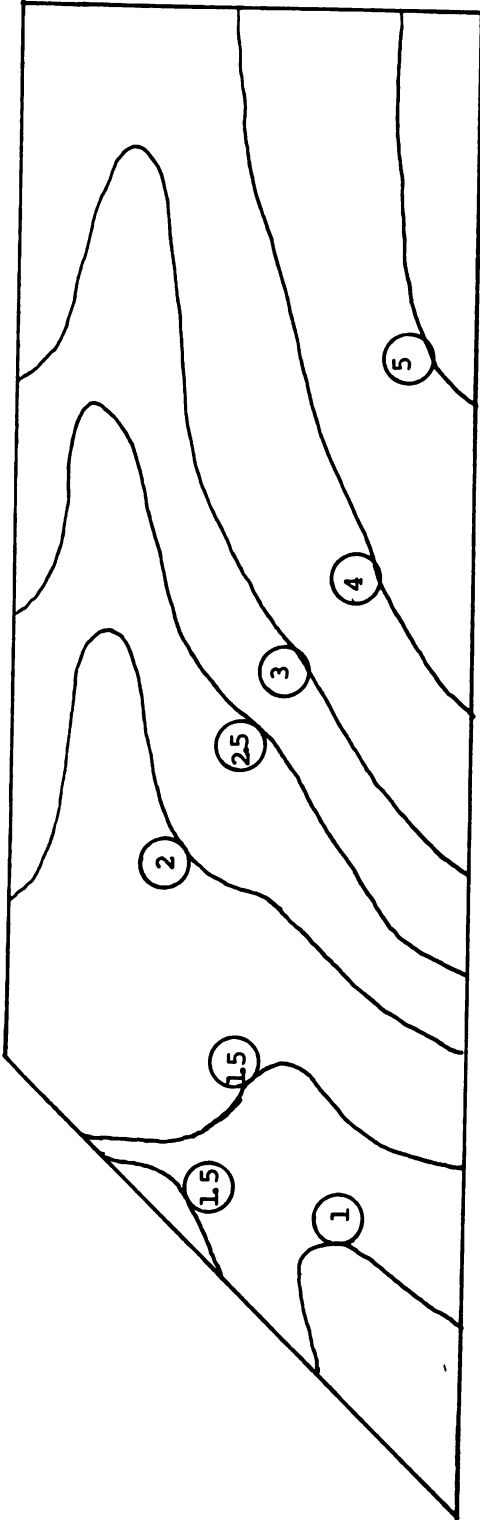


2

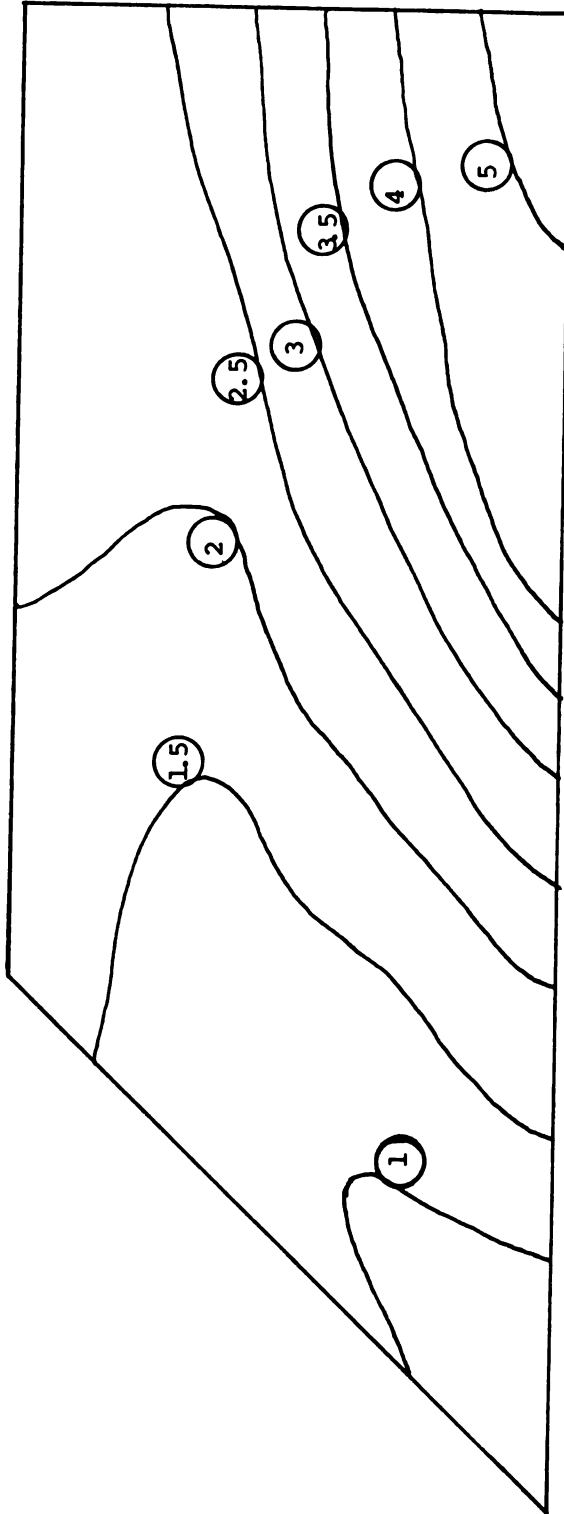
○ Indicates FS

Figure 6.17.--Factor of Safety Contours for a Homogeneous Soil Slope.

Stage No.



3



4

Figure 6.17.--Continued.

Stage No.

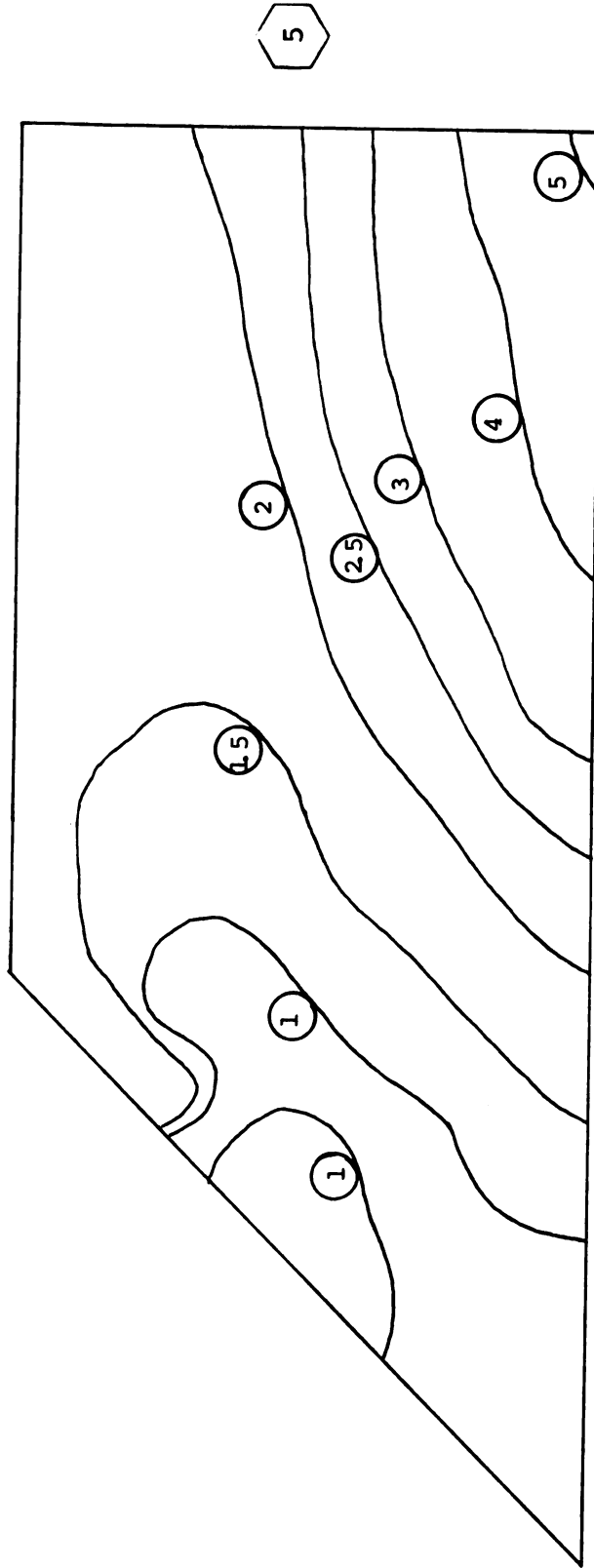


Figure 6.17.--Continued.

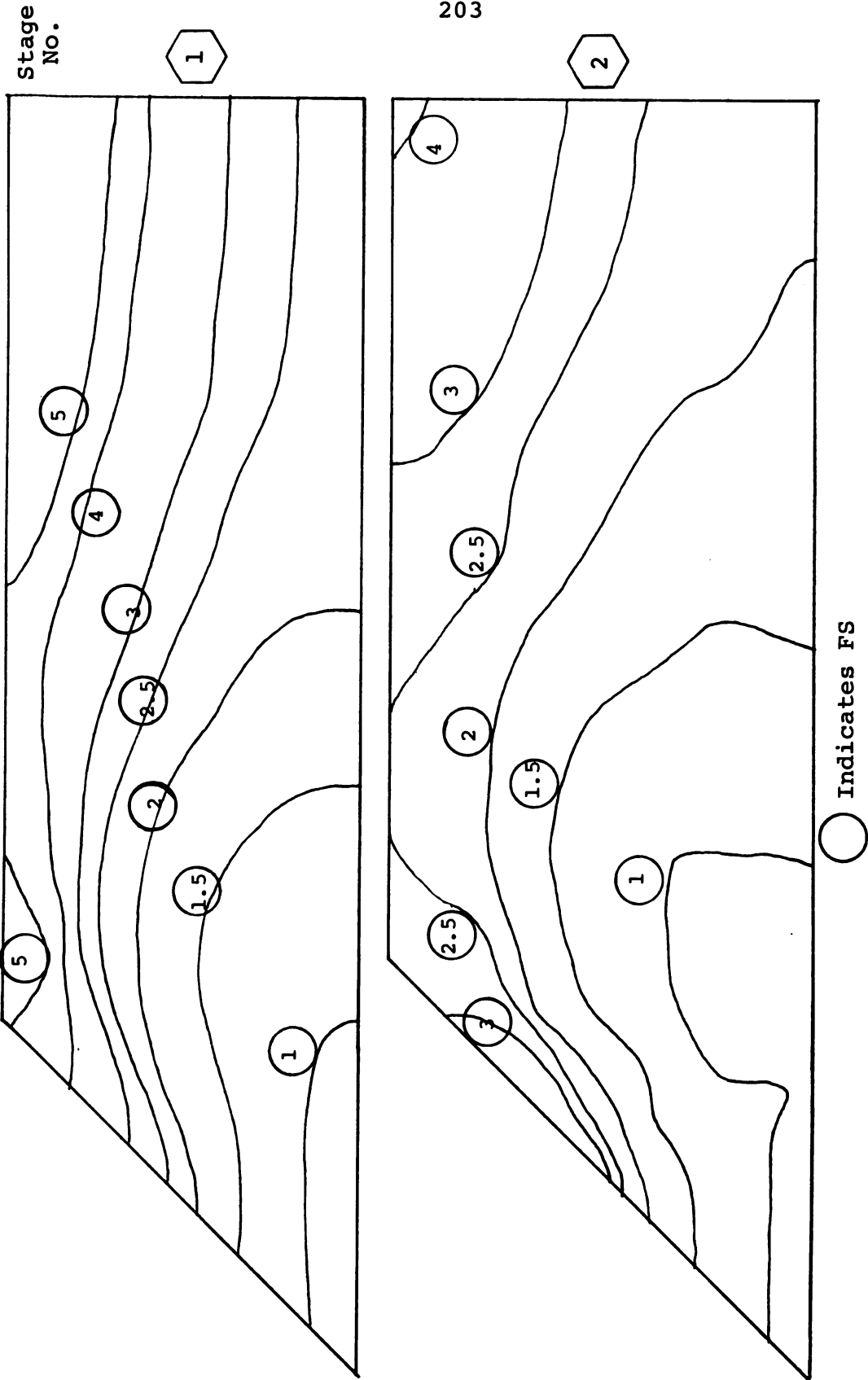
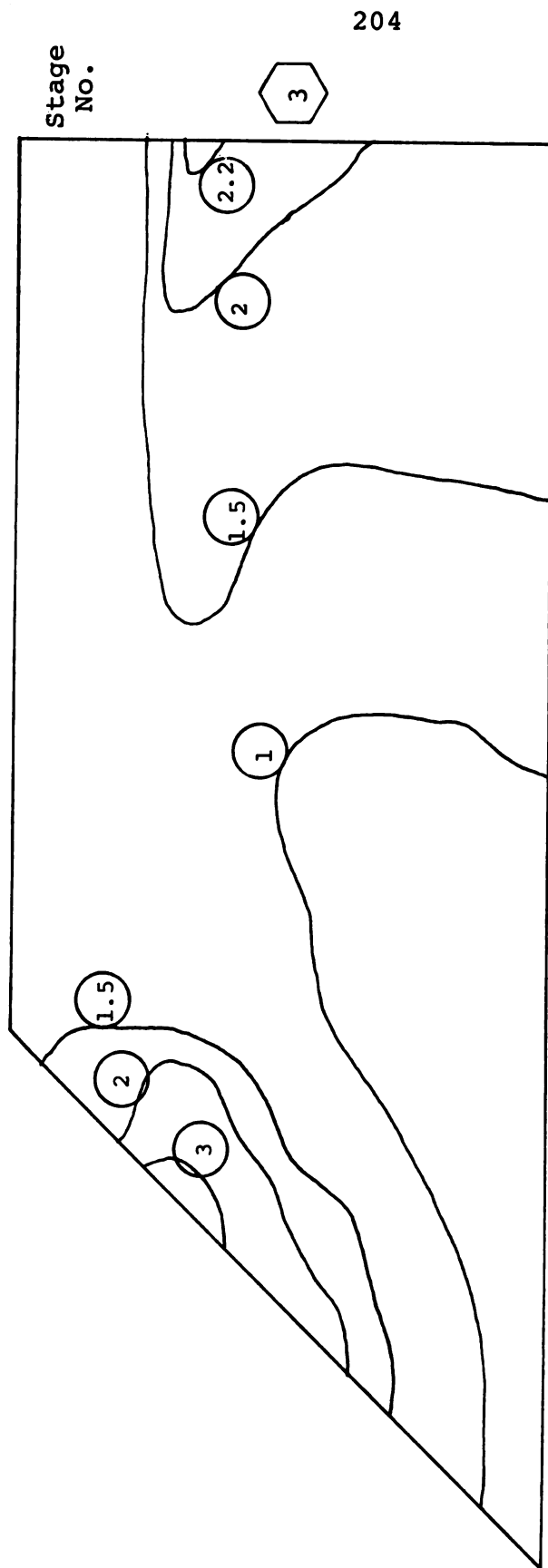


Figure 6.18.--Factor of Safety Contours for a Two-layer Slope.



204

Figure 6.18.--Continued.

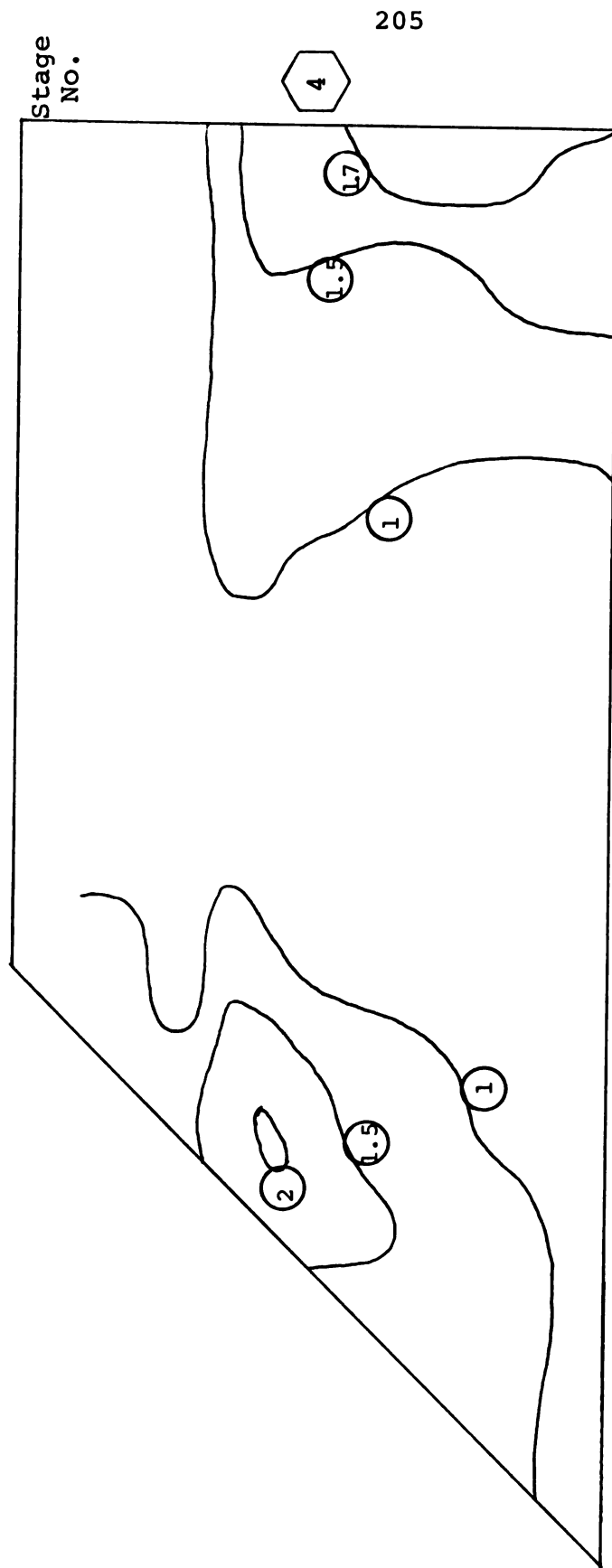
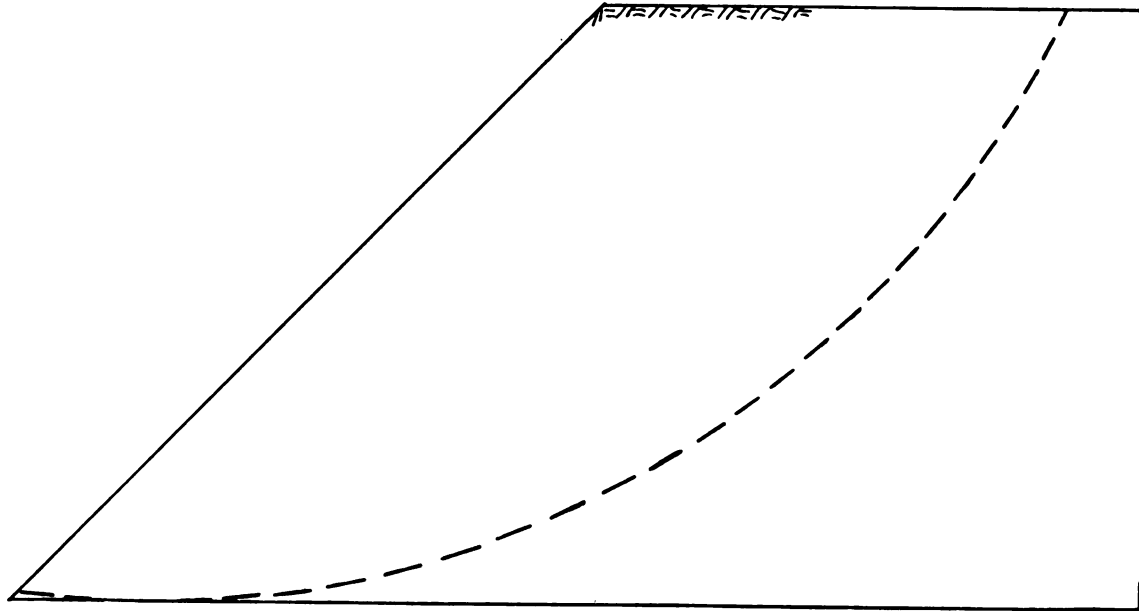
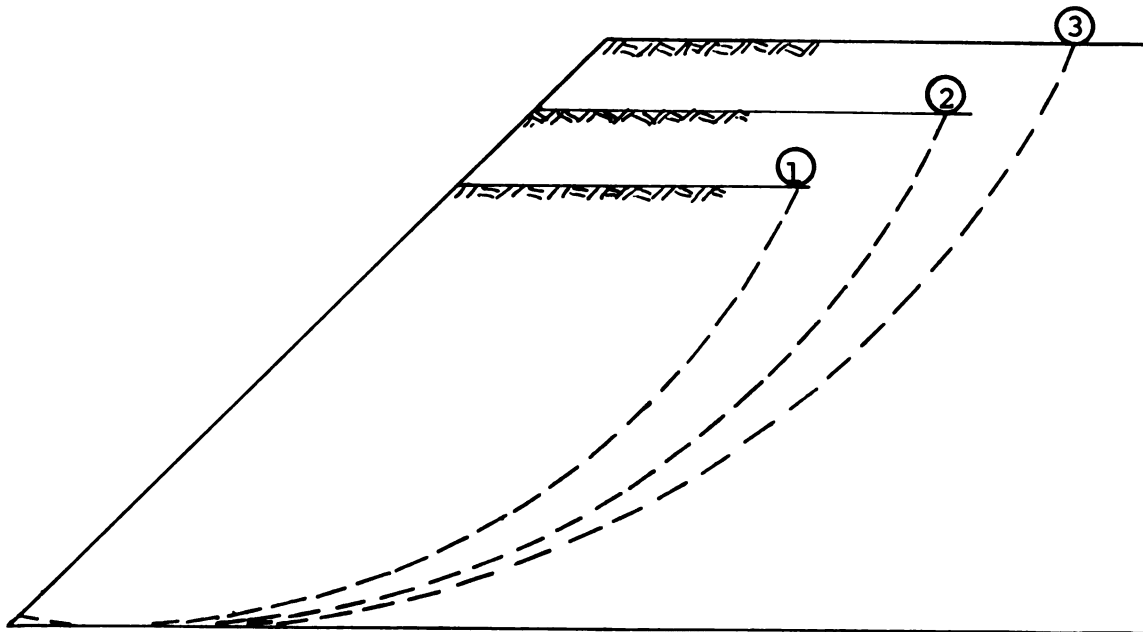


Figure 6.18.---Continued.



a) Single-lift Construction.



b) Incremental Construction.

Figure 6.19.--Potential Sliding Surfaces Used in Calculation of Average FS.

110	109	108	107	106	105	104	
100	99	97	96	95	94	93	
	89	88	87	86	85	84	83
							82
78	77	76	75	74	73	72	71
							70
66	65	64	63	62	61	60	59
							58
54	53	52	51	50	49	48	47
							46
41	40	39	38	37	36	35	34
							33
							32
28	27	26	25	24	23	22	21
							20
14	13	12	11	10	9	8	7
							6
							5
							4

c) Element Number.

Figure 6.19.--Continued.

TABLE 6-1.--Movement of Points Along the Slope Boundary During the Loading Process of Two Layer Embankment (in feet).

Step	Pt	15	58	96	130	127	124
x-dir.	1	0	-0.1049128	-0.06968299			
	2	0	-0.1641989	-0.12756272			
	3	0	-0.2434115	-0.20764625			
	4	0	-0.2819562	-0.26863423	-0.1304171	-0.1188424	-0.05431757
z-dir.	1	0	-0.02701218	-0.0972165			
	2	0	-0.06979308	-0.1316914			
	3	0	-0.17273888	-0.2481547			
	4	0	-0.49401118	-0.6058491	-0.3063074	-0.1577108	-0.04800492

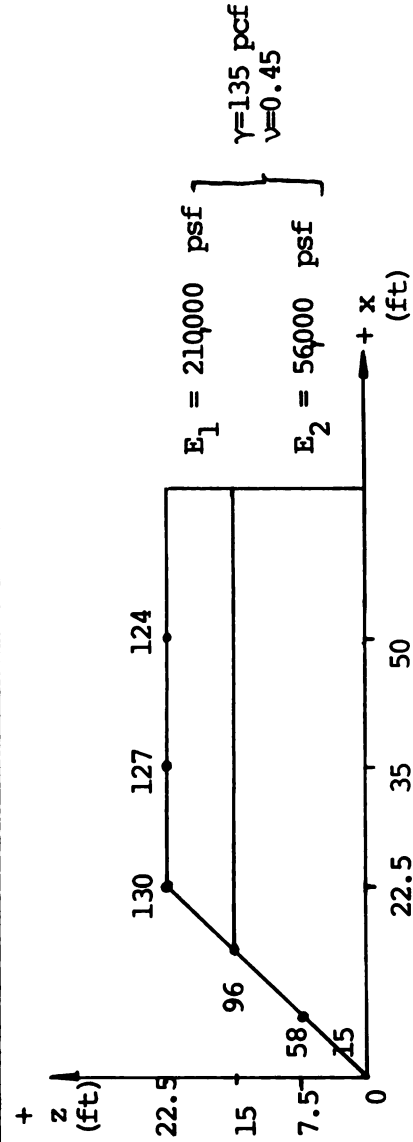


TABLE 6-2.--Computed Average Factor of Safety Values

a. Single-stage Construction

H = 20 feet, E = 56,000 psf, $\nu = 0.45$, $c_U = 400$ psf.

Elastic-perfectly plastic material

Stage	γ (pcf)	F S (Bishop's Anal.)	Av. F S
1	96.5	1.25	1.622
2	109.4	1.102	1.292
3	122.3	.994	1.052
4	128.75	.947	1.0

b. Incremental Construction

 $\gamma = 128$ pcf., E = 26,000 psf, $\nu = 0.45$, $c_U = 400$ pcf.

Elastic-perfectly plastic material

Stage	H (ft)	F S (Bishop's Anal.)	Av. F S
1	15	1.26	1.635
2	17.5	1.081	1.5
3	20	.947	1.11

CHAPTER VII

SUMMARY AND CONCLUSIONS

Presentation of the results and discussions have been made in the preceeding chapter. These results will be summarized under the headings: Homogeneous Soil Slope, Nonhomogeneous Soil Slope, and Stability Analysis. Items covered under each heading are intended to reflect the findings of this investigation and are limited to the methods used, under the assumptions described previously.

7.1 Homogeneous Soil Slope

Elasticity and elasto-plasticity approaches have been used in the investigations of homogeneous soil slopes. Two construction processes, single-step construction and incremental construction, were assumed to simulate loading behaviors.

Elasticity analyses were found to be useful for design purposes. Results obtained from the elasticity investigations are:

1. development of a tension zone in the top part of the slope.
2. required soil strength characteristics to prevent slope failure.
3. location and configuration of the potential failure surface.

The behavior of tension zone development was found to depend on Poisson's ratio (ν). Values of ν closer to 0.5 (incompressible, $\nu = 0.5$) give a larger tension zone area. The tensile stresses are normally observed along the top surface and slightly appear along the slope surface. Magnitude of the tensile stresses developed in this zone increase with an increasing value of modulus of elasticity or unit weight of the material.

The required strength characteristics, as investigated by elasticity analysis, are also found to be dependent on the value of the soil parameter, ν . The required angle of shearing resistance to prevent slope failure increases as the ν -value decreases, while the required cohesion increases with an increase in ν . The required soil strength, or the stress envelope developed in the elastic stress field can be used to indicate the location of the first yield point on a slope, if the actual strength envelope of the soil is known. Under the elastic stress field, a slip surface can be drawn according to the orientations computed from the soil strength envelope, which predicts the location of the critical failure surface. For saturated or dense soils and simple slope geometry, the predicted failure surface was found to agree well, in location and configuration, with the potential sliding surface predicted by the classical method of Bishop.

The results obtained from elasto-plasticity analyses show the development of a plastic zone and a tension zone during the various stages of loading. Different plastic behavior assumptions, strain hardening and perfectly plastic, were applied in the investigation. It was found that the behavior of the plastic zone development is only slightly influenced by the construction sequence and by the material behavior assumption. The plastic, or failure zone develops closer to the slope surface if the soil has a higher angle of internal friction (dry soil). The general appearance of the potential sliding surface obtained from a slip surface was found to be the same for all cases using the same assumed soil strength. The potential sliding surface was also found to have good agreement with the one given by Bishop's method (and elasticity analysis).

The tension zone development appears to be greatly influenced by two factors, construction sequence and material behavior. A smaller area and of less intensity was observed for incremental construction. The location of the maximum tensile stresses, developing during the collapse stage, gave good continuity with the possible sliding surface predicted by the slip line field in plastic zone.

7.2 Nonhomogeneous Soil Slope

Only the case of a two layer soil slope with a stiff upper layer was considered. It was found from the elasticity analysis that development of a tensile stress zone depends on the following:

1. ν values (more intense and larger area appears as ν approaches 0.5).
2. Modulus of elasticity of the soil in the bottom layer (higher values of E in the bottom layer produce less tensile stress development).
3. The relative thickness between the bottom layer and the top layer.

From the elasto-plasticity solutions, it was found that the potential slip surface of the two layer soil slope does not deviate much from that of the homogeneous soil slope. The potential sliding surface located by Bishop's method does not agree with that by the elasto-plasticity results. In this analysis, where the soil in the top layer is stiffer than the bottom one, it was found that tensile stresses developed have a marked influence on the cause of slope failure. Maximum tensile stresses with deep propagation develop near the area where possible sliding surfaces in the plastic region are found.

7.3 Stability Analysis

Slope stability was presented in terms of the average factor of safety along the potential sliding

surface given by Bishop's analysis (or elasticity analysis for homogeneous slopes). Contours of factor of safety were plotted to show the potential failure region and the decreasing stability as the load increases.

It was found that under undrained strength analysis the calculated average factor of safety and the factor of safety given by Bishop's method at each stage were in fair agreement. At the failure stage for both the undrained and the drained strength analysis, the factor of safety given by Bishop's analysis was in close agreement with the one from the elasto-plasticity analysis.

With the undrained strength analysis, the Bishop's factor of safety is about 1.25 when first local failure occurred. Bishop (1952) concluded that local overstress occurs where the factor of safety lies below a value of about 1.8.

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APPENDICES

APPENDIX A

PROGRAM FEASTX(T,OUT,OUTPUT,TAPF8,TAPF9,TAPF60=INPUT,TAPF61=OUTPUT	2	FEASTX
1,PUNCH,TAPF 62=PUNCH)	3	FEASTX
COMMON F(4,4,12),RO(12),XXNN(12),R(300),Z(300),UR(300),	4	FEASTX
1 IJ(300),CODE(300),T(300),IRC(100),JHC(100),PR(100),ANGLE(4),	5	FEASTX
2ACELZ,ANGFO,TFMP,Q,HED(20),NUMNP,NUMEL,NUMMAT,	6	FEASTX
3 NUMPC,MTYPE,NP,SIG(10)	7	FEASTX
COMMON /ARG/ RRR(5),ZZZ(5),S(10,10),P(10),TT(4),LM(4),DD(3,3),	8	FEASTX
1 HH(6,10),RR(4),Z/(4),C(4,4),H(6,10),D(6,6),F(6,10),TP(6),XI(10)	9	FEASTX
2,FF(7),TX(300,5),EPS(300)	10	FEASTX
COMMON /HARAPG/ MRRND,NUMRRLK,H(77),A(77,39)	11	FEASTX
COMMON /PLANE/ NPP	12	FEASTX
COMMON SIGMA(2,300),FS(300),FANG(2,300),SOIL(12,4),FSMIN,NELFSM	13	FEASTX
1,UIJSS(300)	1	FIX3
C	14	FEASTX
C*****	15	FEASTX
C READ AND PRINT OF CONTROL INFORMATION AND MATERIAL PROPERTIES	16	FEASTX
C*****	17	FEASTX
50 READ (60,1000) HED,NUMNP,NUMEL,NUMMAT,NUMPC,ACELZ,ANGFO,Q,NP,NPP,	18	FEASTX
1 IJPLT	19	FEASTX
WRITE(61,2000) HED,NUMNP,NUMEL,NUMMAT,NUMPC,ACELZ,ANGFO,Q,NP	20	FEASTX
C	21	FEASTX
IF (NPP) 55,56,54	22	FEASTX
55 WRITE (61,2012)	23	FEASTX
GO TO 56	24	FEASTX
54 WRITE (61,2008)	25	FEASTX
56 DO 54 M=1,NUMMAT	26	FEASTX
READ (60,1001) MTYPE,NUMTC,RO(MTYPE),XXNN(MTYPE)	27	FEASTX
WRITE (61,2011) MTYPE,NUMTC,RO(MTYPE),XXNN(MTYPE)	28	FEASTX
READ (60,1005) ((E(I,J,MTYPE),J=1,8),I=1,NUMTC)	29	FEASTX
WRITE (61,2010) ((E(I,J,MTYPE),J=1,8),I=1,NUMTC)	30	FEASTX
DO 58 I= NUMTC,8	31	FEASTX
DO 58 J=1,8	32	FEASTX
58 E(I,J,MTYPE)=F(NUJ,IC,1,MTYPE)	33	FEASTX
59 CONTINUE	34	FEASTX
C	35	FEASTX
C*****	36	FEASTX

C	READ AND PRINT OF NODAL POINT DATA	FEASTX	37
1	C*****	FEASTX	38
2	WRITE (61,2004)	FFASTX	39
3	L=0	FEASTX	40
4	60 READ (60,1002) N.CODE(N),R(N),Z(N),UR(N),UZ(N),T(N)	FEASTX	41
5	NI=L+1	FEASTX	42
6	IF(L.FO.0) GO TO 70	FEASTX	43
7	ZX=N-L	FEASTX	44
8	DP=(R(N)-R(L))/ZX	FEASTX	45
9	DZ=(Z(N)-Z(L))/ZX	FEASTX	46
10	DT=(T(N)-T(L))/ZX	FEASTX	47
11	70 L=L+1	FEASTX	48
12	IF(N-L) 100,90,90	FEASTX	49
13	80 CODE(L)=0.0	FFASTX	50
14	R(L)=R(L-1)+DP	FEASTX	51
15	Z(L)=Z(L-1)+DZ	FEASTX	52
16	UR(L)=0.0	FEASTX	53
17	UZ(L)=0.0	FEASTX	54
18	T(L)=T(L-1)+DT	FEASTX	55
19	GO TO 70	FEASTX	56
20	90 WRITE (61,2002) (K.CODE(K),R(K),Z(K),UR(K),UZ(K),T(K),K=NL,N)	FEASTX	57
21	IF(NUMNP-N) 100,110,60	FEASTX	58
22	100 WRITE (61,2009) N	FEASTX	59
23	110 CONTINUE	FEASTX	60
24	C*****	FEASTX	61
25	C READ AND PRINT OF ELEMENT PROPERTIES	FEASTX	62
26	C*****	FEASTX	63
27	WRITE (61,2001)	FEASTX	64
28	N=0	FEASTX	65
29	130 READ (60,1003) M,(IX(1,I),I=1,5)	FEASTX	66
30	140 N=N+1	FEASTX	67
31	IF(4-N) 170,170,150	FEASTX	68
32	150 IX(N,1)=IX(N-1,1)+1	FEASTX	69
33	IX(N,2)=IX(N-1,2)+1	FEASTX	70
34	IX(N,3)=IX(N-1,3)+1	FEASTX	71
35	IX(N,4)=IX(N-1,4)+1	FEASTX	72

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IX(N,5)=IX(N-1,5)
170 WRITE (61,2003) N,(IX(N,I),I=1,5)
IF (M-N) 180,180,140
180 IF (NUMFL-N) 190,190,130
190 CONTINUE
C*****
C READ AND PRINT OF PRESSURE BOUNDARY CONDITIONS
C*****
IF (NUMPC) 290,310,290
290 WRITE (61,2005)
DO 300 L=1,NUMPC
  READ (60,1004) IRC(L),JRC(L),PR(L)
300 WRITE (61,2007) IRC(L),JRC(L),PR(L)
310 CONTINUE
C*****
C DETERMINE BAND WIDTH
C*****
J=0
DO 340 N=1,NUMEL
  DO 340 I=1,4
    DO 325 L=1,4
      KK=IAHS(IX(N,I))-IX(N,L)
      IF (KK-J) 325,325,320
320 J=KK
325 CONTINUE
340 CONTINUE
  MRAND=2*J+2
C*****
C SOLVE NON-LINEAR STRUCTURE BY SUCCESSIVE APPROXIMATIONS
C*****
DO 350 N=1,NUMEL
350 EPS(N)=0.0
C
DO 3005 L1=1,NUMMAT
  READ(60,3006) SOIL(L1,I),SOIL(L1,2),SOIL(L1,3)
3006 FORMAT(3F10.0)

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FEASTX 73

FEASTX 74

FEASTX 75

FEASTX 76

FEASTX 77

FEASTX 78

FEASTX 79

FEASTX 80

FEASTX 81

FEASTX 82

FEASTX 83

FEASTX 84

FEASTX 85

FEASTX 86

FEASTX 87

FEASTX 88

FEASTX 89

FEASTX 90

FEASTX 91

FEASTX 92

FEASTX 93

FEASTX 94

FEASTX 95

FEASTX 96

FEASTX 97

FEASTX 98

FEASTX 99

FEASTX 100

FEASTX 101

FEASTX 102

FEASTX 103

FEASTX 104

FEASTX 105

FEASTX 106

FEASTX 107

FEASTX 108

100

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1205 3005 CONTINUE
1210 DO 500 NNN=1,NP
1211 C
1212 C FORM STIFFNESS MATRIX
1213 C
1214 C CALL STIFF
1215 C
1216 C SOLVE FOR DISPLACEMENTS
1217 C
1218 C CALL HANSOL
1219 C
1220 C WRITE (61,2006) (N,R(2*N-1),R(2*N),N=1,NUMNP)
1221 C
1222 C COMPUTE STRESSES
1223 C
1224 C IF (IPLOT.LF.N) GO TO 501
1225 C IF (NNN.LT.NP) GO TO 501
1226 C IPLOT=-IPLOT
1227 C 501 CALL STRESS(IPLOT)
1228 C
1229 C 500 CONTINUE
1230 C WRITE(61,3001)
1231 C 3001 FORMAT(1H1,* FL.NO. SIGMA1 SIGMA3 FS FAILANG1 FAILANG FIX3
1232 C 12 UNUSDFD SHEAR STRENGTH*)
1233 C DO 3002 N=1,NUMEL
1234 C WRITE(61,3003) N,SIGMA(1,N),SIGMA(2,N),FS(N),FANG(1,N),FANG(2,N)
1235 C 1,UUSS(N)
1236 C 3003 FORMAT(17,6F10.3)
1237 C 3002 CONTINUE
1238 C 3001 WRITE(61,3010) FMIN,NELFSM
1239 C 3010 FORMAT(* MINIMUM FACTOR OF SAFETY IS*,F10.4,* AT ELEMENT NUMBE
1240 C 1R*,I7)
1241 C *****
1242 C GO TO 50
1243 C *****
1244 C 1000 FORMAT (20A4/4I5,3F10.2,3I5)

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1001	FORMAT (2I5,2F10.0)	FEASTX	144
1002	FORMAT (I5,F5.0,5F10.0)	FEASTX	145
1003	FORMAT (6I5)	FEASTX	146
1004	FORMAT (2I5,F10.0)	FEASTX	147
1005	FORMAT (8F10.0)	FEASTX	148
2000	FORMAT (I4I,2044/ 1 30H0 NUMBER OF NODAL POINTS----- I3 / 2 30H0 NUMBER OF ELEMENTS----- I3 / 3 30H0 NUMBER OF DIFF. MATERIALS----- I3 / 4 30H0 NUMBER OF PRESSURE CARDS----- I3 / 5 30H0 AXIAL ACCELERATION----- F12.4/ 6 30H0 ANGULAR VELOCITY----- F12.4/ 7 30H0 REFERENCE TEMPERATURE----- F12.4/ 8 30H0 NUMBER OF APPROXIMATIONS----- I3)	FEASTX	149
		FEASTX	150
		FEASTX	151
		FEASTX	152
		FEASTX	153
		FEASTX	154
		FEASTX	155
		FEASTX	156
		FEASTX	157
2001	FORMAT (49H1FLEMENT NO. I J K L MATERIAL)	FEASTX	158
2002	FORMAT (I12,F12.2,2F12.3,2E24.7,F12.3)	FEASTX	159
2003	FORMAT (I1I3,4I6,I1I2)	FEASTX	160
2004	FORMAT (I08H1NODAL POINT TYPE R-ORDINATE Z-ORDINATE R LO FEASTX IAD OR DISPLACEMENT Z LOAD OR DISPLACEMENT TEMPERATURE)	FEASTX	161
2005	FORMAT (29H0PRESSURE BOUNDARY CONDITIONS/ 24H I J PRES FEASTX IURE)	FEASTX	162
		FEASTX	163
		FEASTX	164
2006	FORMAT (I2H1N.P. Number 18X 2HUP 18X 2HUZ / (I1I2,2E20.7))	FEASTX	165
2007	FORMAT (2I6,F12.3)	FEASTX	166
2008	FORMAT (23H0PLANE STRESS STRUCTURE)	FEASTX	167
2009	FORMAT (26H0NODAL POINT CARD ERROR N= I5)	FEASTX	168
2010	FORMAT (I5H0 TEMPERATURE 10X 5HE(R7) 9X 6HNU(RZ) 11X 4HE(T) 1 10X 5HNU(T) 6X 9HALPHA(RZ) 7X 8HALPHA(T) 15H YIELD STRESS / 2 (F15.2,7F15.5))	FEASTX	169
		FEASTX	170
		FEASTX	171
2011	FORMAT (I7H0MATERIAL NUMBER= I3, 30H, NUMBER OF TEMPERATURE CARDS= 1 I3, 15H, MASS DENSITY= E12.4 ,16H,MODULUS RATIO= E12.4)	FEASTX	172
2012	FORMAT (23H0PLANE STRAIN STRUCTURE)	FEASTX	173
		FEASTX	174
		FEASTX	175
		FEASTX	176
		STIFF	2
	END SUBROUTINE STIFF	STIFF	3
	COMMON E(8,8,12),RO(12),XXNN(12),R(300),Z(300),UR(300),	STIFF	4

IF (IX(N,I)-NL) 80.70.70	STIFF	41
70 IF (IX(N,I)-NM) 90.90.80	STIFF	42
80 CONTINUE	STIFF	43
GO TO 210	STIFF	44
C	STIFF	45
90 CALL QUAD(N,VOL)	STIFF	46
C	STIFF	47
IF(VOL) 142.142.144	STIFF	48
142 WRITE (61,2003) N	STIFF	49
STOP=1.0	STIFF	50
144 IF (IX(N,3)-IX(N,4)) 145.145.145	STIFF	51
145 DO 150 II=1,9	STIFF	52
CC=S(II,10)/S(10.10)	STIFF	53
P(II)=P(II)-CC*P(10)	STIFF	54
DO 150 JJ=1,9	STIFF	55
150 S(II,JJ)=S(II,JJ)-CC*S(10,JJ)	STIFF	56
C	STIFF	57
DO 160 II=1,8	STIFF	58
CC=S(II,9)/S(9,9)	STIFF	59
P(II)=P(II)-CC*P(9)	STIFF	60
DO 160 JJ=1,8	STIFF	61
160 S(II,JJ)=S(II,JJ)-CC*S(9,JJ)	STIFF	62
C	STIFF	63
ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS	STIFF	64
C	STIFF	65
165 DO 166 I=1,4	STIFF	66
166 LM(I)=2*IX(N,I)-2	STIFF	67
C	STIFF	68
DO 200 I=1,4	STIFF	69
DO 200 K=1,2	STIFF	70
IT=LM(I)+K-KSHIFT	STIFF	71
KK=2*I-2+K	STIFF	72
R(II)=R(II)+P(KK)	STIFF	73
DO 200 J=1,4	STIFF	74
DO 200 L=1,2	STIFF	75
JJ=LM(J)+L-II+1-KSHIFT	STIFF	76

LI=2*J-2+L	STIFF	77
IF(JJ) 200,200,175	STIFF	78
175 IF(N0-JJ) 180,195,105	STIFF	79
180 WRITE (61,2004)N	STIFF	80
STOP=1.0	STIFF	81
GO TO 210	STIFF	82
195 A(II,JJ)=A(II,JJ)+S(KK,LL)	STIFF	83
200 CONTINUE	STIFF	84
210 CONTINUE	STIFF	85
C	STIFF	86
C	STIFF	87
C	STIFF	88
DO 250 N=NL,NM	STIFF	89
IF (N.GT.NUMNP) GO TO 251	STIFF	90
K=2*N-KSHIFT	STIFF	91
R(K)=R(K)+U7(N)	STIFF	92
250 R(K-1)=R(K-1)+U7(N)	STIFF	93
251 CONTINUE	STIFF	94
C	STIFF	95
C	STIFF	96
C	STIFF	97
C	STIFF	98
1. PRESSURE R.C.	STIFF	99
C	STIFF	100
IF (NUMPC) 260,310,260	STIFF	101
260 DO 300 L=1,NUMPC	STIFF	102
T=IAC(L)	STIFF	103
J=JAC(L)	STIFF	104
PP=PR(L)/6.	STIFF	105
07=(Z(I)-Z(J))*PP	STIFF	106
DR=(R(J)-R(I))*PP	STIFF	107
RX=2.0*R(I)+R(J)	STIFF	108
ZX=R(I)+2.0*R(J)	STIFF	109
IF (NPP) 262,264,262	STIFF	110
262 RX=3.0	STIFF	111
ZX=3.0	STIFF	112
264 II=2*I-KSHIFT	STIFF	112

JJ=2*J-KSHIFT	STIFF	113
IF (II) 280.280.265	STIFF	114
265 IF (II-ND) 270.270.280	STIFF	115
270 SINA=0.0	STIFF	116
COSA=1.0	STIFF	117
IF (CODE(I)) 271.272.272	STIFF	118
271 SINA=SIN(CODE(I))	STIFF	119
COSA=COS(CODE(I))	STIFF	120
272 R(II-1)=R(II-1)+RX*(COSA*DZ+SINA*DR)	STIFF	121
R(II)=R(II)-RX*(SINA*DZ-COSA*DR)	STIFF	122
280 IF (JJ) 300.300.285	STIFF	123
285 IF (JJ-ND) 290.290.300	STIFF	124
290 SINA=0.0	STIFF	125
COSA=1.0	STIFF	126
IF (CODE(J)) 291.292.292	STIFF	127
291 SINA=SIN(CODE(I))	STIFF	128
COSA=COS(CODE(I))	STIFF	129
292 R(JJ-1)=B(JJ-1)+ZX*(COSA*DZ+SINA*DR)	STIFF	130
R(JJ)=B(JJ)-ZX*(SINA*DZ-COSA*DR)	STIFF	131
300 CONTINUE	STIFF	132
C	STIFF	133
C	STIFF	134
C	STIFF	135
310 DO 400 M=NL,NH	STIFF	136
IF (M-NUMNP) 315.315.400	STIFF	137
315 U=UR(M)	STIFF	138
N=2*M-1-KSHIFT	STIFF	139
IF (CODE(M)) 390.400.316	STIFF	140
316 IF (CODE(M)-1.) 317.370.317	STIFF	141
317 IF (CODE(M)-2.) 318.390.318	STIFF	142
318 IF (CODE(M)-3.) 390.390.390	STIFF	143
370 CALL MODIFY(A,R,ND2,MBAND,N,U)	STIFF	144
GO TO 400	STIFF	145
380 CALL MODIFY(A,R,ND2,MBAND,N,U)	STIFF	146
390 U=UZ(M)	STIFF	147
N=N+1	STIFF	148

CALL MODIFY(A,H,ND2,MHAND,N,U)	STIFF	149
400 CONTINUE	STIFF	150
C	STIFF	151
C WRITE HLOCK OF EQUATIONS ON TAPE AND SHIFT UP LOWER BLOCK	STIFF	152
C	STIFF	153
C WRITE (R) (R(N) * (A(N,M) * M = 1, MHAND), N = 1, ND)	STIFF	154
C	STIFF	155
DO 420 N = 1, ND	STIFF	156
K = N + ND	STIFF	157
R(N) = R(K)	STIFF	158
R(K) = 0.0	STIFF	159
DO 420 M = 1, ND	STIFF	160
A(N,M) = A(K,M)	STIFF	161
420 A(K,M) = 0.0	STIFF	162
C	STIFF	163
C CHECK FOR LAST BLOCK	STIFF	164
C	STIFF	165
IF (NM - NUMNP) 60, 480, 480	STIFF	166
480 CONTINUE	STIFF	167
C *****	STIFF	168
500 RETURN	STIFF	169
C	STIFF	170
2003 FORMAT (26H0NEGATIVE AREA ELEMENT NO. 14)	STIFF	171
2004 FORMAT (29H0RAND WIDTH EXCEEDS ALLOWABLE 14)	STIFF	172
END	STIFF	173
SUBROUTINE QUAD(N,VOL)	QUAD	2
C	QUAD	3
COMMON E(H,R,12),RO(12),XXNM(12),W(300),Z(300),JUR(300),	QUAD	4
1 UZ(300),CODE(300),T(300),IRC(100),JRC(100),PR(100),ANGLE(4),	QUAD	5
2 ACFLZ,ANGFO,TEMP,0,HEN(20),NUMNP,NUMEL,NUMMAT,	QUAD	6
3 NUMPC,MTYPE,NP,SIG(10)	QUAD	7
COMMON /ARG/ RRO(5),Z77(5),S(10,10),P(10),TT(4),LM(4),DD(3,3),	QUAD	8
1 H4(6,10),H2(4),Z7(4),C(4,4),H(6,10),D(6,6),F(6,10),TP(6),XI(10)	QUAD	9
2,FF(7),IX(300,5),FPS(300)	QUAD	10
COMMON /RANARG/ HAND,NUMBLK,R(77),A(77,39)	QUAD	11
COMMON /PLANE/ NPP	QUAD	12

	QUAD	13
C 90 I=IX(N.1)	QUAD	14
J=IX(N.2)	QUAD	15
K=IX(N.3)	QUAD	16
L=IX(N.4)	QUAD	17
MTYPE=IX(N.5)	QUAD	18
IX(N.5)=-IX(N.5)	QUAD	19
C	QUAD	20
C FORM STRESS-STRAIN RELATIONSHIP	QUAD	21
C	QUAD	22
TFMP=(T(I)+T(J)+T(K)+T(L))/4.0	QUAD	23
DO 103 M=2,8	QUAD	24
IF (E(M.1,MTYPE)-TEMP) 103.104.104	QUAD	25
103 CONTINUE	QUAD	26
104 RATIO=0.0	QUAD	27
DEFN=E(M.1,MTYPE)-E(M-1.1,MTYPE)	QUAD	28
IF (DEFN) 70.71.70	QUAD	29
70 RATIO=(TEMP-E(M-1.1,MTYPE))/DEFN	QUAD	30
71 DO 105 KK=1,7	QUAD	31
105 EF(KK)=E(M-1.1,MTYPE)+RATIO*(E(M.1,MTYPE)-E(M-1.1,MTYPE))	QUAD	32
TFMP=TEMP-Q	QUAD	33
C	QUAD	34
C	QUAD	35
EPSR=EE(7)/EE(1)	QUAD	36
IF (EPSR-EPS(N)) 106.109.108	QUAD	37
106 RATIO=(FF(7)/EPS(N)*FF(1))*(1.0-XXNN(MTYPE))+XXNN(MTYPE)	QUAD	38
EF(1)=FF(1)*RATIO	QUAD	39
EF(3)=FF(3)*RATIO	QUAD	40
108 CONTINUE	QUAD	41
C	QUAD	42
C	QUAD	43
IF (NPP) 85.86.84	QUAD	44
85 C(1,1)=1./EF(1)	QUAD	45
C(1,2)=-FE(4)/EF(3)	QUAD	46
C(1,3)=-FF(2)/FF(1)	QUAD	47
C(2,1)=C(1,2)	QUAD	48
C(2,2)=1./EF(3)	QUAD	

C(2,3)=C(1,2)	QUAD	49
C(3,3)=C(1,1)	QUAD	50
C(3,1)=C(1,3)	QUAD	51
C(3,2)=C(2,3)	QUAD	52
CALL SYMINV(C,3)	QUAD	53
C(4,4)=.5*EE(1)/(EE(1)/EE(3)+EE(4))	QUAD	54
DO R1 M=1,3	QUAD	55
R1 TT(M)=(C(M,1)*FF(5)+C(M,2)*FF(6))*TFMP	QUAD	56
DO R3 M=1,3	QUAD	57
R3 C(3,M)=0.	QUAD	58
C(1,3)=0.	QUAD	59
C(2,3)=0.	QUAD	60
GO TO 87	QUAD	61
R4 XX=EE(1)/EE(3)	QUAD	62
COMM=FF(1)/(XX-FF(2)**2)	QUAD	63
C(1,1)=COMM*XX	QUAD	64
C(1,2)=COMM*EF(2)	QUAD	65
C(1,3)=0.0	QUAD	66
C(2,1)=C(1,2)	QUAD	67
C(2,2)=COMM	QUAD	68
C(2,3)=0.0	QUAD	69
C(3,1)=0.0	QUAD	70
C(3,2)=0.0	QUAD	71
C(3,3)=0.0	QUAD	72
C(4,4)=.5*FF(1)/(XX+EE(2))	QUAD	73
GO TO 88	QUAD	74
C	QUAD	75
R6 C(1,1)=1.0/EE(1)	QUAD	76
C(1,2)=-EE(2)/EE(1)	QUAD	77
C(1,3)=-EE(4)/EE(3)	QUAD	78
C(2,1)=C(1,2)	QUAD	79
C(2,2)=C(1,1)	QUAD	80
C(2,3)=C(1,3)	QUAD	81
C(3,1)=C(1,3)	QUAD	82
C(3,2)=C(2,3)	QUAD	83
C(3,3)=1.0/EE(3)	QUAD	84

	CALL SYMINV(C,3)	QUAD	85
	C(4,4)=FF(1)/(2.0+2.0*FF(2))	QUAD	86
C		QUAD	87
	88 DO 110 M=1,3	QUAD	88
	110 IT(M)=((C(M,1)+C(M,2))*EE(5)+C(M,3)*EE(6))*TEMP	QUAD	89
C		QUAD	90
C	FORM QUADRILATERAL STIFFNESS MATRIX	QUAD	91
C		QUAD	92
	87 RRR(5)=(R(I)+R(J)+R(K)+R(L))/4.0	QUAD	93
	ZZZ(5)=(Z(I)+Z(J)+Z(K)+Z(L))/4.0	QUAD	94
	DO 94 M=1,4	QUAD	95
	MM=IX(N,M)	QUAD	96
	IF(NPP) 93,99,93	QUAD	97
	89 IF (R(MM)) 93,91,93	QUAD	98
	91 R(MM)=.01*RRR(5)	QUAD	99
	IF (CODE(MM)) 93,92,93	QUAD	100
	92 CODE(MM)=1.0	QUAD	101
	93 RRR(M)=R(MM)	QUAD	102
	94 ZZZ(M)=Z(MM)	QUAD	103
C		QUAD	104
	DO 100 II=1,10	QUAD	105
	P(II)=0.0	QUAD	106
	DO 95 JJ=1,6	QUAD	107
	95 HH(JJ,II)=0.0	QUAD	108
	DO 100 JJ=1,10	QUAD	109
	100 S(II,JJ)=0.0	QUAD	110
	DO 119 II=1,4	QUAD	111
	JJ=IX(N,II)	QUAD	112
	119 ANGLE(II)=CODE(JJ)/57.3	QUAD	113
C		QUAD	114
	IF (K-L) 125,120,125	QUAD	115
	120 CALL TRISIF(1,2,3)	QUAD	116
	RRR(5)=(RRR(1)+RRR(2)+RRR(3))/3.0	QUAD	117
	ZZZ(5)=(ZZZ(1)+ZZZ(2)+ZZZ(3))/3.0	QUAD	118
	VOL=XI(1)	QUAD	119
	GO TO 130	QUAD	120

125	VOL=0.0	QUAD	121
	CALL TRISIF(4,1,5)	QUAD	122
	VOL=VOL+XI(1)	QUAD	123
	CALL TRISIF(1,2,5)	QUAD	124
	VOL=VOL+XI(1)	QUAD	125
	CALL TRISIF(2,3,5)	QUAD	126
	VOL=VOL+XI(1)	QUAD	127
	CALL TRISIF(3,4,5)	QUAD	128
	VOL=VOL+XI(1)	QUAD	129
	DO 140 II=1,6	QUAD	130
	DO 140 JJ=1,10	QUAD	131
	140 HH(II,JJ)=HH(II,JJ)/4.0	QUAD	132
C		QUAD	133
	130 RETURN	QUAD	134
C		QUAD	135
	END	QUAD	136
C		QUAD	137
	SUBROUTINE TRISIF(II,JJ,KK)	TRISIF	2
	COMMON F(8,8,12),RO(12),XXNN(12),R(300),Z(300),UR(300),	TRISIF	3
	1 UZ(300),CODE(300),T(300),IRC(100),JBC(100),PR(100),ANGLE(4),	TRISIF	4
	2 ACCELZ,ANGFO,TEMP,Q,HFD(20),NUMNP,NUMEL,NUMMAT,	TRISIF	5
	3 NUMPC,MTYPE,NP,SIG(10)	TRISIF	6
	COMMON /ARG/ RRP(5),ZZ(5),S(10,10),P(10),IT(4),LM(4),DD(3,3),	TRISIF	7
	1 HH(6,10),PR(4),ZZ(4),C(4,4),H(6,10),D(6,6),F(6,10),IP(6),XI(10)	TRISIF	8
	2,FF(7),IX(300,5),EPS(300)	TRISIF	9
	COMMON /PLANE/ NPB	TRISIF	10
C		TRISIF	11
C	1. INITIALIZATION	TRISIF	12
C		TRISIF	13
	LM(1)=II	TRISIF	14
	LM(2)=JJ	TRISIF	15
	LM(3)=KK	TRISIF	16
C		TRISIF	17
	RR(1)=RRR(II)	TRISIF	18
	RR(2)=RRR(JJ)	TRISIF	19
	RR(3)=RRR(KK)	TRISIF	20

RR(4)=RRR(II)	TRISTF	21
Z7(1)=Z7Z(II)	TRISTF	22
Z7(2)=Z7Z(JJ)	TRISTF	23
Z7(3)=Z7Z(KK)	TRISTF	24
Z7(4)=Z7Z(II)	TRISTF	25
C	TRISTF	26
85 DO 100 I=1.6	TRISTF	27
DO 90 J=1.6	TRISTF	28
F(I,J)=0.0	TRISTF	29
90 H(I,J)=0.0	TRISTF	30
DO 100 J=1.6	TRISTF	31
100 D(I,J)=0.0	TRISTF	32
C	TRISTF	33
3. FORM INTEGRAL(G)*T*(C)*T(G)	TRISTF	34
C	TRISTF	35
CALL INTER(XI,RR,ZZ)	TRISTF	36
C	TRISTF	37
D(2,6)=XI(1)*(C(1,2)+C(2,3))	TRISTF	38
D(3,5)=XI(1)*C(4,4)	TRISTF	39
D(5,5)=XI(1)*C(4,4)	TRISTF	40
D(6,6)=XI(1)*C(2,2)	TRISTF	41
IF (NPP) 104,106,104	TRISTF	42
104 D(2,2)=XI(1)*C(1,1)	TRISTF	43
D(3,3)=XI(1)*C(4,4)	TRISTF	44
GO TO 108	TRISTF	45
106 D(1,1)=XI(3)*C(2,3)	TRISTF	46
D(1,2)=XI(2)*(C(1,3)+C(3,3))	TRISTF	47
D(1,3)=XI(5)*C(3,3)	TRISTF	48
D(1,6)=XI(2)*C(2,3)	TRISTF	49
D(2,2)=XI(1)*(C(1,1)+2.0*C(1,3)+C(3,3))	TRISTF	50
D(2,3)=XI(4)*(C(1,3)+C(3,3))	TRISTF	51
D(3,3)=XI(6)*C(3,3)+XI(1)*C(4,4)	TRISTF	52
D(3,6)=XI(4)*C(2,3)	TRISTF	53
C	TRISTF	54
108 DO 110 I=1.6	TRISTF	55
DO 110 J=1.6	TRISTF	56

110	D(J,I)=D(I,J)	TRISF	57
C		TRISF	58
C	4. FORM COEFFICIENT-DISPLACEMENT TRANSFORMATION MATRIX	TRISF	59
C		TRISF	60
	COMM=RR(2)*(ZZ(3)-ZZ(1))+RR(1)*(ZZ(2)-ZZ(3))+RR(3)*(ZZ(1)-ZZ(2))	TRISF	61
	DN(1,1)=(RR(2)*ZZ(3)-RR(3)*ZZ(2))/COMM	TRISF	62
	DN(1,2)=(RR(3)*ZZ(1)-RR(1)*ZZ(3))/COMM	TRISF	63
	DN(1,3)=(RR(1)*ZZ(2)-RR(2)*ZZ(1))/COMM	TRISF	64
	DN(2,1)=(ZZ(2)-ZZ(3))/COMM	TRISF	65
	DN(2,2)=(ZZ(3)-ZZ(1))/COMM	TRISF	66
	DN(2,3)=(ZZ(1)-ZZ(2))/COMM	TRISF	67
	DN(3,1)=(RR(3)-RR(2))/COMM	TRISF	68
	DN(3,2)=(RR(1)-RR(3))/COMM	TRISF	69
	DN(3,3)=(RR(2)-RR(1))/COMM	TRISF	70
C		TRISF	71
	DO 120 I=1,3	TRISF	72
	J=2*LM(I)-1	TRISF	73
	H(1,J)=DN(1,I)	TRISF	74
	H(2,J)=DN(2,I)	TRISF	75
	H(3,J)=DN(3,I)	TRISF	76
	H(4,J+1)=DN(1,I)	TRISF	77
	H(5,J+1)=DN(2,I)	TRISF	78
	120 H(6,J+1)=DN(3,I)	TRISF	79
C		TRISF	80
C	ROTATE UNKNOWNNS IF REQUIRED	TRISF	81
C		TRISF	82
	DO 125 J=1,2	TRISF	83
	I=LM(J)	TRISF	84
	IF (ANGLE(I) 122,125,125	TRISF	85
122	SINA=SIN(ANGLE(I))	TRISF	86
	COSA=COS(ANGLE(I))	TRISF	87
	I,J=2*I	TRISF	88
	DO 124 K=1,6	TRISF	89
	TFM=H(K,I,J-1)	TRISF	90
	H(K,I,J-1)=TFM*COSA+H(K,I,J)*SINA	TRISF	91
124	H(K,I,J)= -TFM*SINA+H(K,I,J)*COSA	TRISF	92

125	CONTINUE	TRISIF	93
C		TRISIF	94
C	5. FORM ELEMENT STIFFNESS MATRIX (H)I*(D)*(H)	TRISIF	95
C		TRISIF	96
	DO 130 J=1,10	TRISIF	97
	DO 130 K=1,6	TRISIF	98
	IF (H(K,J)) 128,130,128	TRISIF	99
	128 DO 129 I=1,6	TRISIF	100
	129 F(I,J)=F(I,J)+D(I,K)*H(K,J)	TRISIF	101
	130 CONTINUE	TRISIF	102
C		TRISIF	103
	DO 140 I=1,10	TRISIF	104
	DO 140 K=1,6	TRISIF	105
	IF (H(K,I)) 138,140,138	TRISIF	106
	138 DO 139 J=1,10	TRISIF	107
	139 S(I,J)=S(I,J)+H(K,I)*F(K,J)	TRISIF	108
	140 CONTINUE	TRISIF	109
C		TRISIF	110
C	6. FORM THERMAL LOAD MATRIX	TRISIF	111
C		TRISIF	112
	IF (NPP) 145,150,145	TRISIF	113
	145 TT(3)=0.0	TRISIF	114
	150 COMM=RO(MTYPE)*ANGFN**2	TRISIF	115
	TP(1)=COMM*XI(7) + XI(2)*TT(3)	TRISIF	116
	TP(2)=COMM*XI(9) + XI(1)*(TT(1)+TT(3))	TRISIF	117
	TP(3)=COMM*XI(13) + XI(4)*TT(3)	TRISIF	118
	COMM=-RO(MTYPE)*ACFL7	TRISIF	119
	TP(4)=COMM*XI(1)	TRISIF	120
	TP(5)=COMM*XI(7)	TRISIF	121
	TP(6)=COMM*XI(8) + XI(1)*TT(2)	TRISIF	122
C		TRISIF	123
	DO 160 I=1,10	TRISIF	124
	DO 160 K=1,6	TRISIF	125
	160 P(I)=P(I)+H(K,I)*TP(K)	TRISIF	126
C		TRISIF	127
C		TRISIF	128

IC	FORM STRAIN TRANSFORMATION MATRIX	TRISF	129
C		TRISF	130
400	DO 410 I=1,6	TRISF	131
	DO 410 J=1,10	TRISF	132
410	HH(I,J)=HH(I,J)+H(I,J)	TRISF	133
C	RETURN	TRISF	134
C		TRISF	135
		TRISF	136
	END	TRISF	137
	SUBROUTINE SYMJEV(A,NMAX)	SYMIN	2
C		SYMIN	3
	DIMENSION A(4,4)	SYMIN	4
C		SYMIN	5
	DO 200 N=1,NMAX	SYMIN	6
C		SYMIN	7
	D=A(N,N)	SYMIN	8
	DO 100 J=1,NMAX	SYMIN	9
100	A(N,J)=-A(N,J)/D	SYMIN	10
C		SYMIN	11
	DO 150 I=1,NMAX	SYMIN	12
	IF(N-I) 110,150,110	SYMIN	13
110	DO 140 J=1,NMAX	SYMIN	14
	IF(N-J) 120,140,120	SYMIN	15
120	A(I,J)=A(I,J)+A(I,N)*A(N,J)	SYMIN	16
140	CONTINUE	SYMIN	17
150	A(I,N)=A(I,N)/D	SYMIN	18
C		SYMIN	19
	A(N,N)=1.0/D	SYMIN	20
C		SYMIN	21
200	CONTINUE	SYMIN	22
C		SYMIN	23
	RETURN	SYMIN	24
C		SYMIN	25
	END	SYMIN	26
	SUBROUTINE INTER(XI,RP,ZZ)	INTER	2
	DIMENSION RP(1),ZZ(1),XI(1),XM(6),R(6),Z(6),XX(6)	INTER	3

C		INTER	4
	COMMON /PLANE/ NPP	INTER	5
	DATA XX/3*I.0.3*3.0/	INTER	6
C		INTER	7
	COMM=RR(2)*(ZZ(3)-ZZ(1))+RR(1)*(ZZ(2)-ZZ(1)-ZZ(2))	INTER	8
	COMM=COMM/24.0	INTER	9
	R(1)=RR(1)	INTER	10
	R(2)=RR(2)	INTER	11
	R(3)=RR(3)	INTER	12
	R(4)=(R(1)+R(2))/2.	INTER	13
	R(5)=(R(2)+R(3))/2.	INTER	14
	R(6)=(R(3)+R(1))/2.	INTER	15
C		INTER	16
	Z(1)=ZZ(1)	INTER	17
	Z(2)=ZZ(2)	INTER	18
	Z(3)=ZZ(3)	INTER	19
	Z(4)=(Z(1)+Z(2))/2.	INTER	20
	Z(5)=(Z(2)+Z(3))/2.	INTER	21
	Z(6)=(Z(3)+Z(1))/2.	INTER	22
C		INTER	23
	IF (NPP) 10,30,10	INTER	24
	10 DO 20 I=1,6	INTER	25
	20 XM(I)=XX(I)	INTER	26
	GO TO 40	INTER	27
	30 DO 35 I=1,6	INTER	28
	35 XM(I)=XX(I)*R(I)	INTER	29
C		INTER	30
	40 DO 50 I=1,10	INTER	31
	50 XI(I)=0.0	INTER	32
C		INTER	33
	DO 100 I=1,6	INTER	34
	XI(1)=XI(1)+XM(I)	INTER	35
	XI(7)=XI(7)+XM(I)*R(I)	INTER	36
	XI(8)=XI(8)+XM(I)*Z(I)	INTER	37
	XI(9)=XI(9)+XM(I)*R(I)**2	INTER	38
	XI(10)=XI(10)+XM(I)*R(I)*Z(I)	INTER	39

IF (NPP.NE.0) GO TO 170	INTER	40
XT(2)=XT(2)+XM(I)/R(I)	INTER	41
XT(3)=XI(3)+XM(I)/(R(I)**2)	INTER	42
XT(4)=XI(4)+XM(I)*Z(I)/R(I)	INTER	43
XT(5)=XI(5)+XM(I)*Z(I)/(R(I)**2)	INTER	44
XT(6)=XI(6)+XM(I)*Z(I)**2/(R(I)**2)	INTER	45
100 CONTINUE	INTER	46
C	INTER	47
DO 150 I=1,10	INTER	48
150 XT(I)=XI(I)*COMP	INTER	49
C	INTER	50
RETURN	INTER	51
C	INTER	52
END	INTER	53
SUBROUTINE MODIFY(A,R,NFQ,MRANI),N,I)	MODIFY	2
C	MODIFY	3
DTMFNSION A(77,39),R(77)	MODIFY	4
C	MODIFY	5
DO 250 M=2,MRAND	MODIFY	6
K=N-M+1	MODIFY	7
IF(K) 235,235,230	MODIFY	8
230 R(K)=R(K)-A(K,M)*I	MODIFY	9
235 K=N-M-1	MODIFY	10
IF(NEQ-K) 250,240,240	MODIFY	11
240 R(K)=R(K)-A(N,M)*I	MODIFY	12
250 CONTINUE	MODIFY	13
A(N,I)=1.0	MODIFY	14
R(N)=U	MODIFY	15
RETURN	MODIFY	16
C	MODIFY	17
END	MODIFY	18
SUBROUTINE STRESS(I,PL0T)	MODIFY	19
C	MODIFY	20
COMMON E(8,8,12),R0(12),XXNN(12),R(300),Z(300),UR(300),	STRESS	2
	STRESS	3
	STRESS	4

1	U(300).CODE(300).T(300).IRC(100).JBC(100).PR(100).ANGLE(4).	STRESS	5
2	ACELZ,ANGFO,TFMP,Q,HFN(20).NUMNP,NUMEL,NUMMAT.	STRESS	6
3	NUMPC,MTYPE,NP,SIG(10)	STRESS	7
	COMMON /ARG/ RRR(5),ZZZ(5),S(10,10).P(10).TT(4).LM(4).DN(3,3).	STRESS	8
1	HH(6,10).RR(4).ZZ(4).C(4,4).H(6,10).D(6,6).F(6,10).IP(6).XI(10)	STRESS	9
2	FE(7).IX(300,5).FPS(300)	STRESS	10
	COMMON /BANARG/ MBAND,NUMRLK,R(77).A(77,39)	STRESS	11
	COMMON /PLANE/ NPP	STRESS	12
	COMMON SIGMA(2,300).FS(300).FANG(2,300).SOIL(12,4).FSMIN,NELFSM	STRESS	13
1	UUSS(300)	FIX3	5
	FSMIN=100.	STRESS	14
	*****	STRESS	15
C	COMPUTE ELEMENT STRESSES	STRESS	16
	*****	STRESS	17
	XKE=0.0	STRESS	18
	XPE=0.0	STRESS	19
	MPRINT=0	STRESS	20
C	DO 300 M=1,NUMEL	STRESS	21
		STRESS	22
C		STRESS	23
	N=M	STRESS	24
	IX(N,5)=IARS(IX(M,5))	STRESS	25
	MTYPE=IX(N,5)	STRESS	26
C		STRESS	27
	CALL QUAD(N,VOL)	STRESS	28
	IX(N,5)=MTYPE	STRESS	29
C		STRESS	30
	DO 120 I=1,4	STRESS	31
	IT=2*I	STRESS	32
	JJ=2*IX(N,I)	STRESS	33
	P(II-1)=R(JJ-1)	STRESS	34
	120 P(II)=R(JJ)	STRESS	35
C		STRESS	36
	DO 150 I=1,2	STRESS	37
	R(I)=P(I+R)	STRESS	38
	DO 150 K=1,8	STRESS	39

150	RR(I)=RR(I)-S(I+R,K)*D(K)	STRESS	40
C		STRESS	41
	COMM=S(9,9)*S(10,10)-S(9,10)*S(10,9)	STRESS	42
	IF (COMM) 155,160,155	STRESS	43
155	P(9)=(S(10,10)*RR(I)-S(9,10)*RR(2))/COMM	STRESS	44
	P(10)=(-S(10,9)*RR(1)+S(9,9)*RR(2))/COMM	STRESS	45
C		STRESS	46
160	DO 170 I=1,4	STRESS	47
	TP(I)=0.0	STRESS	48
	DO 170 K=1,10	STRESS	49
170	TP(I)=TP(I)+HH(I,K)*D(K)	STRESS	50
C		STRESS	51
C		STRESS	52
	RR(1)=TP(2)	STRESS	53
	RR(2)=TP(4)	STRESS	54
	RR(3)=(TP(1)+TP(2)*RR(5)+TP(3)*77(5))/RR(5)	STRESS	55
	RR(4)=TP(3)+TP(5)	STRESS	56
C		STRESS	57
176	DO 180 J=1,2	STRESS	58
	SIG(I)=-TT(I)	STRESS	59
	DO 180 K=1,3	STRESS	60
180	SIG(I)=SIG(I)+C(I,K)*Q~(K)	STRESS	61
	SIG(3)=-TT(3)	STRESS	62
	IF (NPP) 181,182,182	STRESS	63
182	DO 183 K=1,3	STRESS	64
183	SIG(3)=SIG(3)+C(3,K)*RR(K)	STRESS	65
	GO TO 185	STRESS	66
181	COMM=EE(1)/((1.+FF(2))* (FF(1)/EE(3)*(1.-EF(2))-2.*FF(4)*FE(4)))	STRESS	67
	SIG(3)=SIG(3)+FF(2)*(1.+FF(2))*(RR(1)+RR(2))*COMM	STRESS	68
185	SIG(4)=C(4,4)*RR(4)	STRESS	69
C		STRESS	70
C	CALCULATE ENERGY TERMS	STRESS	71
C		STRESS	72
	DO 250 I=1,10	STRESS	73
	COMM=0.0	STRESS	74
	DO 200 K=1,10	STRESS	75

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200 COMM=COMM4+S(I,K)*P(K)
250 XPE=XPE+COMM*P(I)
    XKE=XKF+VOL*RO(MTYPE)*(P(I)**2+P(10)**2)
C
C    CALCULATE EFFECTIVE STRAIN
C
    IF (NPP) 253,252,251
253 RR(3)=0.
    GO TO 252
251 RR(3)=-((SIG(1)+SIG(2))*FF(2)/FE(1)
252 CC=(RR(1)+RR(2))/2.0
    CR=SQRT(((RR(2)-RR(1))/2.0)**2+(RR(4)/2.0)**2)
    RR(1)=CC+CR
    RR(2)=CC-CR
    EPS(N)=SQRT((RR(1)-RR(2))**2+(RR(1)-RR(3))**2)
    I *.707/(1.0+FF(2))
C
C*****
C    OUTPUT STRESSES
C*****
C*****
C    CALCULATE PRINCIPAL STRESSES
C
    CC=(SIG(1)+SIG(2))/2.0
    RR=(SIG(1)-SIG(2))/2.0
    CR=SQRT(RR**2+SIG(4)**2)
    SIG(5)=CC+CR
    SIG(6)=CC-CR
    IF (RR.NE.0.) GO TO 100
    IF (SIG(4).LT.0.) GO TO 101
    SIG(7)=45.
    ARG=.78540
    GO TO 102
101 SIG(7)=-45.
    ARG=-.78540
    GO TO 102

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STRESS 76
 STRESS 77
 STRESS 78
 STRESS 79
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 STRESS 108
 STRESS 109
 STRESS 110
 STRESS 111

100	ARG=ATAN2(SIG(4),HR)/2.	STRESS	112
	SIG(7)=57.2958*ARG	STRESS	113
102	CONTINUE	STRESS	114
C		STRESS	115
C	STRESSES PARALLEL TO LINE I-J	STRESS	116
C	I=IX(N,1)	STRESS	117
	J=JX(N,2)	STRESS	118
	ANG=2.*ATAN2(7(J)-7(I),2(I)-2(J))	STRESS	119
	COS2A=COS(ANG)	STRESS	120
	SIN2A=SIN(ANG)	STRESS	121
	CX=.5*(SIG(1)-SIG(2))	STRESS	122
	SIG(8)=CX*COS2A+SIG(4)*SIN2A+CC	STRESS	123
	SIG(9)=2.*CC-SIG(8)	STRESS	124
	SIG(10)=-CX*SIN2A+SIG(4)*COS2A	STRESS	125
	SIGMA(1,N)=-SIG(6)	STRESS	126
	SIGMA(2,N)=-SIG(5)	STRESS	127
	R1=(SIGMA(1,N)-SIGMA(2,N))/2.0	STRESS	128
	NSOIL=IX(N,5)	STRESS	129
	IF(SOIL(NSOIL,3).GT.1.0E-5) GO TO 3006	STRESS	130
	XI=SOIL(NSOIL,2)	FIX1	1
	UISS(N)=XI-R1	FIX1	2
	GO TO 3007	FIX3	6
3006	TANPHI=TAN(SOIL(NSOIL,3)*3.14159/180.)	FIX1	3
	SINPHI=SIN(SOIL(NSOIL,3)*3.14159/180.)	FIX1	4
	CENTER=(SIGMA(1,N)+SIGMA(2,N))/2.0	STRESS	132
	XI=SINPHI*(CENTER+SOIL(NSOIL,2)/TANPHI)	FIX3	7
	COSPHI=COS(SOIL(NSOIL,3)*3.14159/180.)	FIX3	8
	UISS(N)=(XI-R1)*COSPHI	FIX3	9
	FS(N)=XI/R1	FIX3	10
3007	IF(FS(N).GT.FSMIN) GO TO 3003	FIX1	5
	FSMIN=FS(N)	STRESS	135
	NFLFSM=N	STRESS	136
3003	CONTINUE	STRESS	137
	ALPHA=45.0+SOIL(NSOIL,3)/2.0	STRESS	138
	FANG(1,N)=SIG(7)+ALPHA	STRESS	139
		FIX2	1

	FIX2	2
FANG(2,N)=SIG(7)-ALPHA	STRESS	142
IF(FANG(1,N).LT.90.0) GO TO 3004	STRESS	143
FANG(1,N)=140.0-FANG(1,N)	STRESS	144
3004 CONTINUE		
IF(FANG(2,N).LT.90.0) GO TO 3005	STRESS	145
FANG(2,N)=180.0-FANG(2,N)	STRESS	146
3005 CONTINUE	STRESS	147
C	STRESS	148
104 IF (MPRINT) 110.1,5.110	STRESS	149
105 WRITE (61,2000)	STRESS	150
MPRINT=50	STRESS	151
110 MPRINT=MPRINT-1	STRESS	152
C	STRESS	153
305 WRITE (61,2001) N,RRR(5),ZZZ(5),(SIG(1),I=1,10)	STRESS	154
IF(IPL0T.GE.0) GO TO 300	STRESS	155
WRITE(62,2002)N,RRR(5),ZZZ(5),SIG(5),SIG(6),ARG	FIX5	1
C		
WRITE(62,2001)N,RRR(5),ZZZ(5),(SIG(1),I=1,10)	FIX5	2
300 CONTINUE	STRESS	157
IF (IPL0T.LT.0) WRITE(62,2003) (N,RR(N),Z(N),B(2*N-1),R(2*N),N=1,NU	STRESS	158
1MNP)	STRESS	159
2003 FORMAT (15,4E12.5)	STRESS	160
C	STRESS	161
IF (XRF) 310,320,310	STRESS	162
310 W=SQRT(XPE/XKE)	STRESS	163
WRITE (61,2006) W	STRESS	164
C	STRESS	165
320 RETURN	STRESS	166
C	STRESS	167
2000 FORMAT (7H1EL.N0. 7X 1HR 7X 1HZ 4X 8HR-STRESS 4X 8HZ-STRESS 4X	STRESS	168
1 8HT-STRESS 3X 9HRZ-STRESS 2X 10HMAX-STRESS 2X 10HMIN-STRESS	STRESS	169
2 3TH ANGLE 1J-STRESS JK-STRESS SHFAR)	STRESS	170
2001 FORMAT (17,2F8.2,1P6E12.4,0P1F7.2,1P3E10.2)	STRESS	171
2002 FORMAT(15,5E12.5)	STRESS	172
2006 FORMAT (35HCAPOXIMATE FUNDAMENTAL FREQUENCY = E12.5)	STRESS	173
C	STRESS	174
END	STRESS	175

SUBROUTINE BANSOL	BANSOL	2
COMMON/BAVAR67MM,NUMBLK,3(77),A(77,39)	BANSOL	3
C	BANSOL	4
C	BANSOL	5
NN=34	FIX4	2
NI=NN+1	BANSOL	7
NH=NN+NN	BANSOL	8
REWIND 9	BANSOL	9
REWIND 8	BANSOL	10
NR=0	BANSOL	11
GO TO 150	BANSOL	12
C*****	BANSOL	13
C*****	BANSOL	14
C*****	BANSOL	15
C*****	BANSOL	16
C*****	BANSOL	17
C 1. SHIFT BLOCK OF EQUATIONS	BANSOL	18
100 NR=NR+1	BANSOL	19
DO 125 N=1,NN	BANSOL	20
NM=NN+N	BANSOL	21
R(N)=R(NM)	BANSOL	22
R(NM)=0.0	BANSOL	23
DO 125 M=1,MM	BANSOL	24
A(N,M)=A(NM,M)	BANSOL	25
125 A(NM,M)=0.0	BANSOL	26
C	BANSOL	27
C 2. READ NEXT BLOCK OF EQUATIONS INTO CORE	BANSOL	28
C	BANSOL	29
IF (NUMBLK-NR) 150,200,150	BANSOL	30
150 READ (R) (R(N), (A(N,M), M=1,MM), N=NL, NH)	BANSOL	31
IF (NR) 200,100,200	BANSOL	32
C	BANSOL	33
C 3. REDUCE BLOCK OF EQUATIONS	BANSOL	34
C	BANSOL	35
200 DO 300 N=1,NN	BANSOL	36
IF (A(N,1)) 225,300,225	BANSOL	37

225	B(N)=B(N)/A(N,1)	RANSOL	38
DO	275 I=2,M	RANSOL	39
IF	(A(N,L)) 230,275,230	RANSOL	40
230	C=A(N,1)/A(N,1)	RANSOL	41
I=N+L-1		RANSOL	42
J=0		RANSOL	43
DO	250 K=L,M	RANSOL	44
J=J+1		RANSOL	45
250	A(I,J)=A(I,J)-C*A(N,K)	RANSOL	46
R(I)=R(I)-A(N,L)*A(N)		RANSOL	47
A(N,L)=C		RANSOL	48
275	CONTINUE	RANSOL	49
300	CONTINUE	RANSOL	50
C		RANSOL	51
C	4. WRITE BLOCK OF REDUCED EQUATIONS ON TAPE 2	RANSOL	52
C		RANSOL	53
IF	(NUMRLK-NR) 375,400,375	RANSOL	54
375	WRITE (9) (R(N),(A(N,M),M=2,M),N=1,N)	RANSOL	55
GO	TO 100	RANSOL	56
C*****		RANSOL	57
C	BACK-SUBSTITUTION	RANSOL	58
C*****		RANSOL	59
400	DO 450 M=1,N	RANSOL	60
N=NN+1-M		RANSOL	61
DO	425 K=2,M	RANSOL	62
L=N+K-1		RANSOL	63
425	R(N)=R(N)-A(N,K)*R(L)	RANSOL	64
NM=N+NM		RANSOL	65
B(NM)=R(N)		RANSOL	66
450	A(NM,NR)=R(N)	RANSOL	67
NR=NR-1		RANSOL	68
IF	(NR) 475,500,475	RANSOL	69
475	BACKSPACE 9	RANSOL	70
READ	(9) (R(N),(A(N,M),M=2,M),N=1,N)	RANSOL	71
BACKSPACE 9		RANSOL	72
GO	TO 400	RANSOL	73

C***** C ORDER UNKNOWN'S IN P ARRAY C***** 500 K=0	BANSOL BANSOL BANSOL BANSOL	74 75 76 77
DO 600 NR=1,NUMPI K DO 600 N=1,NN NM=N+NN K=K+1	BANSOL BANSOL BANSOL BANSOL	78 79 80 81
600 R(K)=A(NM,NP) C	BANSOL BANSOL BANSOL	82 83 84
RETURN END	BANSOL BANSOL	84 85

APPENDIX B

```

PROGRAM NEWSTR(INPUT,OUTPUT,PUNCH)
DIMENSION SI(250,3),SN(250,3),S(250,3)
READ100,C,PHI,N,M
READ101,((SI(I,J),J=1,3),I=1,N)
READ101,((SN(I,J),J=1,3),I=1,N)
PRINT 102
100 FORMAT(2F10.3,2I5)
101 FORMAT(23X,2E12.4,12X,E12.4)
102 FORMAT(*ELEM-NO.   X-STRESS   Z-STRESS   XZ-STRESS   MAX-STRESS
1MIN-STRESS   ANGLE   F.S.   UNUSED-SHEAR   FAILANG1   FAILA
2NG2*)
103 FORMAT(I5,10F12.3)
NF=0
PI=(3.147*PHI)/180.
DO1I=1,N
DO2J=1,3
2 S(I,J)=SI(I,J)+SN(I,J)
TX=S(I,1)
TZ=S(I,2)
TXZ=-S(I,3)
T1=(TX+TZ)/2.+SQRT((TX+TZ)**2-4.*(TX*TZ-TXZ**2))/2.
T2=(TX+TZ)/2.-SQRT((TX+TZ)**2-4.*(TX*TZ-TXZ**2))/2.
TETA=(ATAN(2.*(-TXZ)/(TX-TZ)))/2.
TETA=(180.*TETA)/3.143
ANG1=TETA-45.-PHI/2.
ANG2=TETA+45.+PHI/2.
TA1=-T2
TA2=-T1
R=(TA1-TA2)/2.
RM=((TA1+TA2)/2.+(C/TAN(PI)))*SIN(PI)
RES=RM-R
REST=RES*COS(PI)
FS=RM/R
NF=NF+1
PRINT103,NF,S(I,1),S(I,2),S(I,3),T1,T2,TETA,FS,REST,ANG1,ANG2
1 CONTINUE
IF(M)4,4,3
3 DO5I=1,N
5 PUNCH101,(S(I,J),J=1,3)
4 CONTINUE
STOP
END

```

APPENDIX C

EL.NO.	SIGMA1	SIGMA3	FS	FAILANG1	FAILANG2	UNUSED	SHEAR STRENGTH
1	1787.640	1453.000	2.370	44.162	-45.838	232.478	
2	1787.640	1453.000	2.370	42.467	-47.533	232.274	
3	1788.970	1450.267	2.362	40.705	-44.295	230.659	
4	1791.583	1446.795	2.320	38.784	-51.216	227.606	
5	1796.958	1441.009	2.248	36.546	-53.454	222.026	
6	1806.117	1429.071	2.120	33.757	-56.243	211.877	
7	1818.753	1406.342	1.940	30.200	-59.800	193.794	
8	1829.888	1377.364	1.690	25.873	-64.127	163.250	
9	1829.443	1263.213	1.413	21.099	-68.901	116.860	
10	1789.911	1105.561	1.169	16.351	-73.649	57.825	
11	1679.277	875.470	.999	12.008	-77.992	-1.904	
12	1472.772	588.852	.905	7.980	-82.020	-41.960	
13	1187.242	288.880	.890	4.702	-85.298	-84.281	
14	791.440	13.230	.922	2.457	-87.043	-94.055	
15	1591.020	1242.520	2.393	44.127	-45.873	245.750	
16	1591.511	1240.431	2.576	42.388	-47.612	244.710	
17	1591.002	1237.252	2.230	40.567	-49.433	242.275	
18	1596.511	1231.734	2.463	38.698	-51.302	237.611	
19	1583.288	1221.125	2.342	36.733	-53.267	229.220	
20	1579.873	1213.264	2.191	34.864	-55.135	219.885	
21	1567.251	1171.201	1.923	32.354	-57.646	191.900	
22	1595.443	1113.043	1.800	29.743	-60.257	159.099	
23	1581.711	1018.926	1.422	26.554	-63.446	118.807	
24	1517.055	875.776	1.247	22.357	-67.641	74.356	
25	1377.617	687.923	1.180	17.154	-72.846	55.153	
26	1128.051	422.283	1.123	10.884	-79.116	43.756	
27	837.733	102.888	1.185	4.141	-85.859	82.488	
28	1314.151	1031.996	2.835	44.109	-45.891	238.922	
29	1314.739	1029.883	2.800	42.326	-47.674	237.382	
30	1316.442	1025.374	2.740	40.557	-49.443	234.449	
31	1320.280	1017.885	2.644	38.853	-51.147	228.712	
32	1327.246	1005.031	2.483	37.304	-52.696	228.894	
33	1331.820	993.137	2.257	36.002	-53.998	222.959	
34	1344.986	947.185	1.989	34.876	-55.124	194.910	
35	1322.623	855.375	1.613	31.287	-58.711	166.544	
36	1335.385	797.755	1.447	31.427	-58.573	131.082	
37	1280.444	678.264	1.355	27.582	-62.418	104.815	
38	1079.432	573.172	1.224	24.315	-68.684	104.321	
39	831.892	283.744	1.482	12.176	-77.824	128.448	
40	590.820	84.743	1.583	3.583	-88.417	147.380	
41	1078.879	822.629	2.142	44.114	-45.886	232.889	
42	1077.513	819.567	2.102	42.381	-47.619	231.057	
43	1079.185	814.125	2.018	40.688	-49.312	228.480	
44	1082.829	803.222	1.882	38.822	-51.178	221.648	
45	1095.722	791.142	1.784	37.143	-53.857	211.239	
46	1097.920	784.248	1.731	35.820	-55.179	205.833	
47	1108.111	774.281	1.642	34.822	-56.178	203.672	
48	1112.819	767.854	1.597	33.482	-57.518	198.390	
49	1092.192	596.417	1.814	38.044	-51.956	152.113	
50	1013.048	488.134	1.269	26.288	-63.712	128.843	
51	832.500	344.144	1.851	27.052	-62.948	157.742	
52	538.248	148.384	2.053	14.781	-75.219	205.158	
53	739.172	813.288	2.277	44.152	-45.848	237.224	
54	739.823	810.443	2.491	42.443	-47.557	235.410	
55	880.481	804.783	2.371	40.977	-49.023	231.581	
56	993.810	779.437	2.281	38.729	-51.271	223.674	
57	888.140	588.033	2.463	34.153	-55.847	203.921	
58	894.878	580.883	2.713	39.829	-50.171	222.558	
59	883.558	574.113	2.843	36.253	-57.747	233.513	
60	888.122	483.883	2.881	41.214	-48.786	207.780	
61	894.222	478.230	2.820	40.878	-49.122	198.989	
62	904.778	461.300	2.742	38.582	-51.418	178.181	
63	898.843	387.180	2.612	38.849	-51.151	214.188	
64	390.213	169.845	2.785	14.348	-75.654	205.441	
65	888.523	488.688	2.812	44.231	-45.769	202.184	
66	888.441	482.815	2.833	42.703	-47.297	200.817	
67	881.838	395.441	2.878	41.471	-48.529	200.882	
68	884.474	384.177	2.888	40.881	-49.119	200.831	
69	880.119	371.153	2.934	40.308	-49.692	202.317	
70	810.238	353.341	3.117	46.941	-43.059	271.851	
71	812.778	341.436	2.818	42.747	-47.253	257.430	
72	824.242	331.334	2.477	45.004	-44.996	238.324	
73	814.473	248.388	2.183	47.289	-42.711	218.143	
74	739.731	179.337	2.814	40.930	-49.070	234.277	
75	344.334	77.200	2.499	30.718	-59.284	288.433	
76	381.326	288.384	4.476	44.401	-45.599	319.599	
77	381.326	178.922	4.881	43.627	-46.373	317.488	
78	381.326	178.922	4.881	42.245	-47.755	313.203	
79	382.471	176.788	4.389	41.894	-48.106	307.149	
80	381.326	182.832	3.784	41.887	-48.113	294.588	
81	384.788	147.954	3.885	42.551	-47.449	291.444	
82	387.589	134.833	3.433	44.584	-45.416	283.582	
83	373.323	124.117	3.144	47.478	-42.522	274.911	
84	388.789	108.920	2.878	50.588	-39.414	281.815	
85	251.140	81.142	4.212	34.848	-55.154	305.826	
86	334.370	42.123	2.810	38.383	-51.617	274.141	
87	121.042	-3.224	8.438	44.718	-45.282	337.887	
88	121.042	-7.771	8.288	44.187	-45.813	335.343	
89	121.042	-18.473	7.743	43.781	-46.219	330.474	
90	121.020	-38.817	5.298	43.924	-46.071	324.477	
91	120.830	-45.173	4.813	43.888	-46.112	318.900	
92	120.262	-29.444	4.447	44.187	-45.811	318.815	
93	119.789	-28.142	4.383	45.387	-44.613	307.844	
94	122.042	-25.833	4.583	47.460	-42.540	311.181	
95	121.019	-23.144	5.213	37.173	-52.827	327.220	
96	118.383	15.771	8.448	38.953	-51.047	322.854	

MINIMUM FACTOR OF SAFETY IS 1.3000 AT ELEMENT NUMBER 13

1	-1884.130	-2232.620	-12.900	-1803.742	-2233.008	-1.722	2.028	220.558	-47.222	43.778
2	-1799.960	-2227.890	-38.388	-1796.544	-2231.306	-5.083	2.002	217.733	-50.583	40.417
3	-1794.790	-2219.140	-62.595	-1785.749	-2228.181	-8.215	1.967	213.777	-53.715	37.285
4	-1788.180	-2207.030	-83.874	-1772.009	-2223.201	-10.308	1.928	209.234	-56.408	34.592
5	-1790.260	-2203.290	-99.654	-1767.473	-2226.077	-12.874	1.896	205.514	-58.374	32.626
6	-1805.248	-2213.270	-108.690	-1778.093	-2240.417	-14.017	1.882	203.873	-59.517	31.483
7	-1867.850	-2290.980	-111.957	-1840.053	-2318.777	-13.937	1.823	196.900	-59.437	31.563
8	-1340.470	-2460.520	-177.692	-1901.453	-2499.537	-14.791	1.466	139.346	-60.291	30.709
9	-1975.910	-2755.528	-327.524	-1856.579	-2874.851	-20.010	1.857	-67.829	-65.510	25.490
10	-1817.286	-2416.652	-314.524	-1682.460	-2551.478	-23.188	1.006	2.439	-68.686	22.314
11	-1137.923	-1470.336	-365.804	-902.337	-1705.922	-32.768	1.052	20.943	-78.268	12.732
12	-945.777	-1196.228	-423.706	-629.179	-1512.826	-36.751	1.948	-23.157	-82.251	9.749
13	-713.502	-871.231	-441.845	-343.538	-1241.195	-39.922	1.322	-35.032	-85.422	5.578
14	-459.084	-561.147	-430.762	-76.342	-943.889	-41.603	1.943	-24.913	-87.103	3.837
15	-1518.770	-1939.990	-12.042	-1518.426	-1940.334	-1.635	2.039	219.185	-47.135	43.865
16	-1515.650	-1933.950	-36.168	-1512.546	-1937.054	-4.903	2.026	217.805	-50.403	40.597
17	-1510.700	-1923.620	-60.213	-1502.099	-1932.221	-8.126	1.999	214.865	-53.626	37.374
18	-1509.390	-1912.670	-83.660	-1492.724	-1929.336	-11.262	1.969	211.513	-56.762	34.238
19	-1514.290	-1903.780	-105.512	-1487.544	-1930.526	-14.218	1.941	208.294	-59.718	31.282
20	-1539.180	-1921.040	-126.227	-1501.227	-1958.993	-16.727	1.879	201.271	-62.227	28.773
21	-1573.180	-1983.300	-147.968	-1525.368	-2031.112	-17.899	1.705	178.120	-63.399	27.601
22	-1638.310	-2220.860	-151.743	-1601.154	-2258.016	-13.753	1.320	105.225	-59.253	31.747
23	-1418.970	-2541.660	-175.789	-1332.089	-2568.541	-8.630	1.739	-153.643	-54.190	36.810
24	-1102.667	-1789.859	-251.074	-1020.709	-1871.818	-18.070	1.999	-3.331	-63.570	27.430
25	-1093.321	-1470.339	-195.809	-1037.265	-1526.395	-19.779	1.727	177.756	-65.279	25.721
26	-737.648	-881.190	-319.293	-482.159	-1136.679	-38.648	1.265	86.816	-84.148	6.852
27	-431.231	-550.426	-323.736	-161.652	-820.005	-39.767	1.241	79.331	-85.267	5.733
28	-1237.247	-1645.170	-11.177	-1236.941	-1645.476	-1.588	2.081	220.833	-47.068	43.932
29	-1233.329	-1639.990	-34.004	-1230.505	-1642.814	-4.745	2.062	218.868	-50.245	40.755
30	-1228.467	-1630.980	-58.159	-1220.232	-1639.215	-8.055	2.028	215.410	-53.555	37.445
31	-1224.480	-1618.320	-84.038	-1207.298	-1635.502	-11.550	1.984	210.654	-57.050	33.950
32	-1228.440	-1612.290	-111.569	-1198.368	-1642.362	-15.079	1.913	202.742	-60.578	30.422
33	-1239.540	-1624.380	-139.654	-1194.202	-1669.718	-17.978	1.787	187.187	-63.478	27.522
34	-1264.870	-1704.990	-163.976	-1210.495	-1759.365	-18.337	1.552	151.441	-63.837	27.150
35	-1241.080	-1823.460	-154.263	-1202.742	-1861.798	-13.950	1.295	97.184	-59.450	31.463
36	-1284.050	-2304.430	30.000	-1283.169	-2305.311	1.682	1.844	-79.753	-43.818	47.182
37	-805.350	-1532.288	-173.827	-765.922	-1571.715	-12.774	1.043	17.473	-58.274	32.726
38	-696.694	-1095.979	-113.521	-666.676	-1125.997	-14.805	1.810	185.920	-60.305	30.695
39	-480.403	-585.131	-181.159	-344.192	-721.344	-36.923	2.170	220.644	-82.423	8.577
40	-324.429	-390.338	-214.300	-140.565	-574.202	-40.611	1.873	189.339	-66.111	4.889
41	-958.460	-1350.140	-10.107	-958.199	-1350.401	-1.476	2.142	223.984	-46.976	44.024
42	-953.725	-1345.560	-31.212	-951.254	-1348.031	-4.524	2.117	221.615	-50.024	40.976
43	-945.548	-1337.070	-54.855	-938.008	-1344.610	-7.823	2.065	216.557	-53.323	37.677
44	-937.062	-1327.560	-82.284	-920.432	-1344.190	-11.421	1.981	207.823	-56.921	34.079
45	-930.505	-1320.460	-113.202	-900.025	-1350.940	-15.063	1.861	194.128	-60.563	30.437
46	-929.911	-1337.900	-144.394	-883.979	-1383.832	-17.638	1.690	169.810	-63.138	27.862
47	-917.480	-1390.650	-161.434	-867.650	-1440.480	-17.146	1.467	133.680	-62.646	28.354
48	-907.210	-1528.620	-147.352	-874.040	-1561.790	-12.681	1.225	77.344	-58.181	32.819
49	-857.690	-1668.020	-64.640	-852.566	-1673.144	-4.530	1.029	11.726	-50.630	40.970
50	-972.290	-1564.870	128.520	-945.617	-1591.543	11.719	1.307	99.139	-33.781	57.219
51	-434.271	-730.310	-25.864	-432.028	-732.593	-4.994	2.729	259.817	-50.454	40.454
52	-227.561	-328.066	-67.540	-193.630	-361.998	-26.653	4.608	320.563	-72.163	18.837
53	-683.525	-1054.290	-6.657	-683.323	-1054.492	-1.136	2.237	229.510	-46.836	44.164
54	-676.927	-1050.980	-27.121	-674.971	-1052.936	-4.124	2.196	226.026	-49.624	33.736
55	-664.348	-1044.691	-48.976	-658.143	-1050.896	-7.218	2.113	218.468	-52.718	38.282
56	-645.894	-1036.455	-75.875	-632.638	-1050.711	-10.637	1.984	205.585	-56.137	35.863
57	-627.364	-1032.537	-107.521	-600.599	-1059.302	-13.972	1.807	185.069	-59.472	31.528
58	-603.284	-1041.582	-137.434	-563.755	-1081.111	-16.039	1.602	152.515	-61.539	23.451
59	-577.435	-1089.410	-150.915	-536.261	-1130.584	-15.254	1.395	117.330	-60.754	30.246
60	-528.702	-1188.220	-122.530	-506.673	-1210.249	-10.187	1.180	63.149	-55.687	35.313
61	-459.729	-1334.330	-28.109	-458.827	-1335.232	-1.838	1.948	-22.578	-47.338	43.652

62	-325.274	-1085.330	171.930	-288.191	-1122.413	12.166	1.988	-4.841	-33.334	57.666
63	-220.934	-509.471	99.610	-189.887	-540.518	17.304	2.318	230.373	-28.196	62.804
64	-57.041	-147.395	37.533	-43.484	-160.952	19.851	6.840	342.939	-25.649	65.351
65	-412.965	-756.456	-6.710	-412.834	-756.587	-1.118	2.386	238.248	-46.618	44.382
66	-404.088	-754.586	-21.343	-402.773	-755.801	-3.471	2.323	233.516	-48.971	42.029
67	-385.980	-750.586	-39.536	-381.742	-754.834	-6.115	2.197	223.294	-51.615	39.385
68	-354.463	-745.754	-64.189	-348.414	-755.803	-9.032	2.011	205.865	-54.532	36.498
69	-320.457	-742.106	-91.831	-301.325	-761.238	-11.763	1.780	179.243	-57.263	33.736
70	-274.427	-748.141	-119.404	-246.032	-776.536	-13.371	1.541	143.604	-58.871	32.429
71	-220.033	-775.942	-128.496	-191.763	-804.206	-12.400	1.334	102.410	-57.900	33.100
72	-151.275	-850.190	-89.926	-149.730	-861.735	-7.312	1.148	92.770	-52.812	38.158
73	-76.330	-934.460	39.063	-74.555	-936.235	2.600	1.949	-22.062	-42.900	48.100
74	-3.513	-742.580	198.980	46.653	-792.746	14.144	1.958	-13.237	-31.356	59.644
75	59.281	-259.517	175.600	137.039	-337.275	23.874	1.094	164.507	-21.626	69.374
76	-147.147	-456.447	-4.270	-147.088	-456.506	-7.790	2.819	250.468	-46.290	44.710
77	-136.229	-455.770	-13.783	-135.636	-456.313	-2.465	2.527	244.737	-47.965	43.035
78	-113.062	-454.380	-26.235	-111.057	-456.385	-4.368	2.345	232.200	-49.868	41.132
79	-74.469	-452.299	-43.227	-69.587	-457.181	-6.441	2.087	210.714	-51.941	33.099
80	-17.910	-450.793	-64.883	-8.394	-460.309	-8.340	1.788	178.051	-53.840	37.160
81	57.730	-451.625	-86.678	72.076	-465.971	-9.334	1.499	134.338	-54.894	35.106
82	143.130	-463.482	-93.987	157.358	-477.710	-8.605	1.268	85.192	-54.105	36.835
83	221.940	-495.487	-55.039	226.138	-499.685	-4.360	1.109	39.413	-49.860	41.140
84	258.150	-527.385	65.854	253.632	-532.867	4.757	1.010	4.042	-40.743	50.237
85	209.120	-408.755	197.250	266.720	-466.355	16.271	1.096	35.141	-29.229	61.771
86	214.140	-128.640	203.920	309.129	-223.629	24.966	1.499	132.792	-20.534	70.466
87	113.271	-153.711	-1.460	113.273	-153.719	-3.313	2.998	266.753	-45.813	45.187
88	125.328	-153.846	-4.807	125.411	-153.929	-7.986	2.805	260.479	-46.486	44.514
89	152.320	-154.200	-9.451	152.611	-154.491	-1.764	2.005	266.166	-47.264	43.716
90	199.830	-154.589	-16.231	200.572	-155.331	-2.615	2.245	221.558	-48.115	42.885
91	275.170	-154.758	-25.440	276.670	-156.258	-3.373	1.843	182.394	-48.873	42.127
92	383.010	-154.479	-35.694	385.370	-156.839	-3.781	1.468	126.817	-49.281	41.719
93	515.830	-151.937	-40.643	518.295	-154.402	-3.463	1.180	60.401	-48.969	42.031
94	628.870	-151.089	-22.273	629.506	-151.725	-1.634	1.013	5.147	-47.134	43.886
95	619.730	-119.569	47.861	622.816	-122.654	3.687	1.061	22.828	-41.813	43.187
96	354.630	-158.168	121.202	381.833	-185.371	12.645	1.404	114.602	-32.855	58.145

1	-1886.167	-2336.160	-15.091	-1885.664	-2336.663	-1.909	1.929	210.333	-47.409	43.591
2	-1881.804	-2330.990	-84.293	-1877.280	-2335.314	-9.973	1.907	207.713	-91.873	39.927
3	-1876.774	-2318.826	-70.012	-1865.951	-2329.649	-8.784	1.883	204.733	-54.284	36.716
4	-1873.844	-2306.785	-88.827	-1856.328	-2324.301	-11.150	1.865	202.465	-56.650	34.350
5	-1868.971	-2312.440	-97.763	-1849.983	-2333.828	-12.339	1.889	202.493	-97.839	33.109
6	-1927.680	-2359.100	-99.466	-1905.661	-2380.919	-12.367	1.841	199.749	-57.867	33.133
7	-2015.500	-2498.040	-111.447	-1991.004	-2522.536	-12.391	1.653	173.600	-57.691	33.109
8	-2123.980	-2804.450	-209.210	-2088.889	-2881.949	-19.941	1.114	49.883	-81.841	29.999
9	-1992.898	-2775.079	-326.229	-1874.698	-2893.279	-19.908	1.867	-67.664	-65.488	25.592
10	-1844.229	-2450.294	-313.204	-1711.457	-2583.066	-22.963	1.004	1.673	-68.463	22.537
11	-1989.900	-1908.484	-384.403	-937.009	-1740.798	-32.513	1.093	21.447	-78.013	12.987
12	-1981.366	-1239.522	-422.353	-666.807	-1552.081	-36.487	1.950	-22.281	-81.987	9.013
13	-753.273	-919.880	-440.483	-388.285	-1284.868	-39.628	1.925	-33.722	-85.128	5.872
14	-901.483	-612.902	-429.869	-123.927	-990.498	-41.288	1.948	-23.982	-88.788	4.212
15	-1582.144	-2031.013	-14.232	-1581.693	-2031.464	-1.813	1.919	206.605	-47.313	43.687
16	-1579.920	-2022.280	-42.447	-1575.884	-2026.316	-5.429	1.916	206.179	-50.929	40.071
17	-1378.074	-2007.988	-89.589	-1387.092	-2018.970	-8.989	1.909	209.315	-94.489	38.939
18	-1584.598	-1994.911	-93.826	-1564.161	-2015.348	-12.283	1.912	205.603	-57.783	33.217
19	-1605.936	-1996.560	-112.998	-1575.604	-2026.892	-15.019	1.912	205.753	-60.519	30.481
20	-1647.900	-2048.330	-127.978	-1610.492	-2085.919	-18.288	1.819	194.596	-61.786	29.214
21	-1691.360	-2202.700	-150.159	-1650.526	-2243.534	-15.206	1.464	137.452	-60.706	30.294
22	-1692.146	-2515.310	-149.756	-1656.046	-2541.410	-9.882	1.986	-6.051	-55.382	35.618
23	-1430.374	-2593.859	-174.891	-1403.850	-2980.379	-8.632	1.739	-153.479	-94.132	36.888
24	-1111.010	-1798.954	-251.844	-1028.670	-1881.295	-18.097	1.998	-9.937	-63.597	27.403
25	-1086.330	-1497.346	-143.553	-1041.157	-1542.519	-17.460	1.688	171.816	-62.960	28.040
26	-718.732	-909.437	-310.380	-488.093	-1138.078	-36.361	1.274	89.188	-89.188	9.139
27	-377.671	-598.320	-329.760	-140.270	-835.721	-35.735	1.175	60.735	-81.235	9.765
28	-1281.714	-1722.150	-13.579	-1281.296	-1722.568	-1.764	1.932	205.529	-47.264	43.736
29	-1278.857	-1714.305	-41.323	-1274.370	-1718.192	-5.371	1.923	204.461	-50.871	40.129
30	-1277.699	-1700.698	-70.556	-1266.241	-1712.156	-9.220	1.911	202.985	-54.720	36.280
31	-1282.703	-1684.290	-100.987	-1258.738	-1708.255	-13.344	1.895	201.085	-58.844	32.156
32	-1303.745	-1685.858	-124.638	-1254.475	-1725.128	-16.969	1.850	195.715	-62.469	28.531
33	-1332.744	-1735.120	-147.657	-1284.377	-1781.491	-18.130	1.710	177.172	-63.630	27.100
34	-1344.472	-1892.183	-146.259	-1306.927	-1929.925	-14.204	1.375	116.716	-59.704	31.296
35	-1274.770	-2110.110	-119.568	-1257.992	-2128.888	-7.984	1.989	-4.920	-53.484	37.516
36	-1283.888	-2302.445	31.916	-1282.889	-2303.484	1.792	1.845	-78.998	-43.708	47.292
37	-803.732	-1529.097	-173.972	-764.165	-1568.665	-12.807	1.045	18.078	-58.307	32.693
38	-780.433	-1107.278	-97.392	-716.180	-1131.531	-13.977	2.004	208.383	-59.477	31.923
39	-452.289	-561.824	-175.538	-323.173	-690.940	-36.320	2.223	224.886	-81.820	9.180
40	-379.517	-558.160	-229.088	-222.353	-714.724	-34.334	1.660	162.225	-79.834	11.166
41	-985.179	-1413.648	-12.713	-984.798	-1414.029	-1.897	1.961	206.263	-87.197	43.803
42	-981.053	-1407.171	-39.460	-977.430	-1410.794	-5.244	1.942	204.101	-50.744	40.256
43	-975.141	-1393.749	-69.927	-963.769	-1405.121	-9.233	1.906	199.939	-54.733	36.267
44	-973.269	-1381.227	-104.935	-947.880	-1408.836	-13.605	1.833	191.103	-59.105	31.895
45	-978.887	-1374.003	-140.576	-933.977	-1418.913	-17.709	1.734	178.011	-63.209	27.791
46	-1002.236	-1421.269	-157.254	-949.787	-1473.718	-18.437	1.608	159.133	-63.937	27.063
47	-996.361	-1384.590	-132.027	-956.710	-1374.241	-12.652	1.367	113.280	-68.152	32.888
48	-938.711	-1810.120	-84.899	-930.517	-1818.314	-5.511	1.955	-19.929	-51.811	39.989
49	-862.769	-1672.330	-63.151	-857.872	-1677.227	-4.432	1.030	12.419	-49.932	41.868
50	-981.498	-1923.358	92.272	-976.902	-1928.393	9.499	1.929	149.887	-40.842	50.958
51	-472.387	-704.430	68.932	-453.429	-723.388	15.360	3.039	275.204	-30.140	60.860
52	-204.912	-375.956	-70.582	-179.548	-401.320	-19.758	3.653	294.085	-65.258	25.742
53	-897.689	-1184.406	-84.381	-892.381	-1189.709	-2.018	2.008	209.882	-87.982	43.938
54	-685.498	-1099.437	-35.584	-682.462	-1102.474	-4.876	1.979	205.584	-58.376	48.624
55	-672.492	-1090.250	-65.082	-662.588	-1100.154	-8.649	1.898	196.534	-54.149	36.851
56	-855.411	-1076.975	-101.648	-852.152	-1100.205	-12.867	1.774	181.043	-58.367	32.633
57	-642.199	-1073.372	-141.916	-539.681	-1115.890	-16.671	1.608	156.807	-62.171	28.829
58	-632.366	-1079.437	-156.367	-577.291	-1134.912	-18.309	1.488	136.074	-63.809	27.191
59	-668.051	-1211.420	-118.595	-643.311	-1236.160	-11.782	1.405	119.924	-57.282	33.718
60	-558.471	-1471.410	-29.059	-557.546	-1472.335	-1.821	1.913	-39.706	-47.321	43.679
61	-464.795	-1339.130	-26.978	-463.363	-1339.961	-1.765	1.949	-22.288	-47.265	41.735

62	-327.860	-1087.756	171.240	-291.054	-1124.562	12.125	1.983	-4.440	-33.375	57.625
63	-112.430	-334.680	116.381	-62.641	-384.469	23.151	2.510	242.896	-22.349	68.651
64	-186.910	-453.330	43.919	-179.381	-460.259	9.042	2.894	265.356	-36.458	54.542
65	-405.495	-793.522	-8.993	-405.287	-793.730	-1.325	2.113	216.165	-46.825	44.175
66	-334.553	-790.400	-28.879	-332.499	-792.498	-4.148	2.051	210.249	-49.858	41.392
67	-372.077	-784.229	-54.308	-365.041	-791.285	-7.379	1.924	196.904	-52.879	38.121
68	-338.607	-777.351	-87.800	-321.689	-794.269	-10.902	1.734	173.177	-56.402	34.598
69	-292.524	-767.745	-127.805	-260.333	-799.939	-14.131	1.517	139.383	-59.631	31.369
70	-253.847	-770.403	-163.205	-206.634	-818.016	-16.127	1.338	103.189	-61.627	29.373
71	-236.016	-778.920	-143.412	-200.461	-814.475	-13.918	1.332	101.788	-59.418	31.582
72	-375.350	-1198.010	84.380	-370.342	-1203.018	8.446	1.994	-2.649	-41.094	49.940
73	-69.513	-925.470	40.261	-67.624	-927.758	2.685	1.950	-21.424	-42.815	48.185
74	3.613	-733.354	199.088	53.957	-783.637	14.185	1.970	-12.507	-31.315	59.685
75	171.130	-178.122	182.048	248.764	-299.796	23.088	1.908	147.717	-22.414	68.980
76	-124.272	-479.556	-5.850	-124.176	-479.652	-0.943	2.280	227.444	-46.443	64.557
77	-109.742	-478.378	-19.102	-108.755	-479.365	-2.957	2.186	219.741	-48.457	42.543
78	-73.700	-478.399	-36.997	-73.288	-479.811	-5.287	2.001	202.498	-90.787	40.233
79	-25.927	-472.735	-61.684	-17.568	-481.094	-7.714	1.744	172.508	-53.214	37.786
80	50.130	-470.207	-94.188	66.655	-486.732	-9.946	1.459	126.898	-55.446	35.554
81	192.632	-497.771	-130.012	179.170	-588.309	-11.932	1.214	70.898	-97.832	33.988
82	230.290	-454.621	-157.543	264.790	-589.121	-12.347	1.066	24.940	-57.847	33.153
83	199.647	-593.389	-89.028	209.519	-603.261	-6.324	1.993	-3.008	-51.824	39.176
84	22.930	-796.920	280.730	113.190	-847.140	17.882	1.848	-73.779	-28.618	63.382
85	328.890	-77.513	229.889	432.511	-181.131	24.252	1.298	90.908	-21.248	68.752
86	-23.020	-605.433	141.558	4.817	-638.267	13.054	1.261	83.921	-32.446	56.554
87	149.693	-162.021	-2.033	149.708	-162.034	-0.374	2.987	244.139	-49.874	49.128
88	168.371	-162.231	-4.806	168.512	-162.374	-1.186	2.432	235.424	-46.686	44.314
89	204.489	-162.724	-13.620	204.993	-163.229	-2.120	2.170	215.430	-47.620	43.380
90	272.377	-183.491	-23.793	273.872	-184.786	-3.114	1.820	179.731	-48.814	42.388
91	382.070	-163.337	-37.755	384.671	-165.938	-3.939	1.446	122.703	-44.439	41.561
92	542.360	-162.707	-58.097	547.115	-167.463	-4.677	1.110	39.325	-50.177	40.823
93	755.030	-150.709	-90.591	774.802	-169.478	-9.529	1.838	-77.481	-51.029	39.971
94	1012.090	-157.915	-83.961	1018.084	-163.909	-4.082	1.664	-198.494	-49.582	41.418
95	891.570	-92.393	216.250	936.999	-137.822	11.859	1.731	-144.435	-33.641	57.359
96	249.210	-192.461	123.448	277.827	-224.878	14.707	1.990	148.202	-38.793	60.207

APPENDIX D.--Sample Calculation of Average Factor of Safety (Presented in Table 6-2).

Elem. No.	ΔL	a. Single-lift Construction (H = 20 feet)					
		$\gamma = 96.5$ pcF		$\gamma = 109.4$ pcF		$\gamma = 122.3$ pcF	
		F S	(FS $\times\Delta L$)	F S	(FS $\times\Delta L$)	F S	(FS $\times\Delta L$)
14	1.9	0.922	1.75	0.94	1.785	0.943	1.79
13	2.02	0.89	1.8	0.919	1.855	0.922	1.86
12	2.07	0.905	1.87	0.946	1.96	0.948	1.96
11	1.19	0.995	1.185	1.051	1.25	1.052	1.25
25	0.95	1.16	1.1	1.642	1.56	1.727	1.64
24	1.25	1.247	1.56	0.997	1.245	0.999	1.25
37	1.1	1.355	1.47	1.042	1.15	1.1043	1.15
36	0.65	1.487	0.966	1.149	0.747	0.844	0.55
49	1.5	1.614	2.42	1.202	1.8	1.029	1.545
61	0.59	1.826	1.076	1.298	0.767	0.948	0.56
60	0.76	2.081	1.58	1.572	1.195	1.18	0.896
72	1.25	2.477	3.1	1.741	2.18	1.148	1.435
83	1.16	3.198	3.71	1.992	2.31	1.109	1.285
94	0.35	4.503	1.575	2.266	0.793	1.013	0.354
93	0.75	4.303	3.23	2.408	1.807	1.18	0.885
Σ		17.49	28.392	22.604	18.401	17.49	17.490
Av. F S		$\frac{28.392}{17.49} =$	1.622	$\frac{22.604}{17.49} =$	1.292	$\frac{18.401}{17.49} =$	1.052
						$\frac{17.49}{17.49} =$	1.00

b. Incremental Construction ($\gamma = 128$ pcf)

H = 15 feet				H = 17.5 feet				H = 20 feet			
Elem. No.	ΔL	F S	(FS $\times\Delta L$)	Elem. No.	ΔL	F S	(FS $\times\Delta L$)	Elem. No.	ΔL	F S	(FS $\times\Delta L$)
14	1.9	0.933	1.775	14	1.9	0.952	1.81	14	1.9	0.951	1.807
13	2.05	0.926	1.9	13	2.0	0.959	1.965	13	2.02	0.957	1.96
12	1.7	1.006	1.7	12	2.1	1.058	2.22	12	2.07	1.056	2.19
26	0.35	1.256	0.44	11	0.65	1.16	0.755	11	1.19	0.79	0.94
25	1.6	1.326	2.12	25	1.5	0.93	1.395	25	0.95	0.929	0.883
38	0.8	1.53	1.225	24	0.55	1.104	0.607	24	1.25	1.049	1.31
37	0.75	1.69	1.267	37	1.65	1.116	1.84	37	1.1	0.982	1.08
50	1.3	1.986	2.58	50	0.35	1.254	0.44	36	0.65	0.843	0.548
62	1.2	2.584	3.1	49	1.05	1.49	1.565	49	1.5	0.879	1.32
73	1.1	4.328	4.76	61	1.25	1.705	2.13	61	0.59	0.948	0.56
				73	1.0	2.032	2.032	60	0.76	1.08	0.82
				72	0.2	2.264	4.528	72	1.25	1.156	1.445
								83	1.16	1.609	1.865
								94	0.35	2.412	0.845
								93	0.75	2.496	1.87
Σ	12.75		20.867		14.20		21.3		17.49		19.443
Av. F S		$\frac{20.867}{12.75} = 1.635$				$\frac{21.3}{14.2} = 1.5$				$\frac{19.443}{17.49} = 1.11$	

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