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ON H -PROJECTIVE FG -MODULES
AND THE COMPLETE REDUCIBILITY
OF INDUCED FH -MODULES

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ABSTRACT

ON H-PROJECTIVE FG-MODULES
AND THE COMPLETE REDUCIBILITY
OF INDUCED FH-MODULES

By

Gloria Potter

Let H be a subgroup of an arbitrary group G and let F be a field of characteristic p . Let $\mathcal{C}$ denote the class of proper subgroups H of G for which the induced FG-module $N^G = N \otimes_{FH} FG$ is completely reducible for each irreducible FH-module N . Our first result shows that if $H \in \mathcal{C}$ then H must have finite index in G . Moreover, we show that the index is a unit in the base field F , thus proving that if $H \in \mathcal{C}$, then every FG-module is H-projective, i.e. that (G, H) is a projective pairing in the sense of Khatri.

For finite groups and normal subgroups it is known that these two properties are equivalent to a third property of the Jacobson radical namely that $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$, Rad denoting the Jacobson radical of the ring concerned. We will say that $H \in \mathcal{R}$ if this inclusion holds. Now the necessity of finite index for projective pairing prevents such a theorem from holding for algebras over infinite groups. However, we do give sufficient conditions for the three classes of subgroups to coincide for normal subgroups.

Now suppose that we have a group G for which $H \in \mathcal{C} \Leftrightarrow H \in \mathcal{R} \Leftrightarrow (G, H)$ is a projective pairing. Then we say that G is a θRC -group. We begin chapter two by showing that infinite θRC -groups do exist in the form of locally finite p -groups. Then we show that we can construct more θRC -groups by taking either extensions of locally finite p -groups by locally finite θRC -groups or extensions of locally finite θRC -groups by finite p' -groups. Next we show that quotients of infinite θRC -groups by p -groups are θRC -groups. Finally we extend the list of known θRC -groups to include the dihedral and dicyclic groups.

Chapter three builds up machinery on induced modules for finite dimensional group algebras and ends with a counterexample to Khatri's conjecture that extensions of θRC -groups by θRC -groups are θRC -groups.

In chapter four we investigate replacing the base field F by a commutative ring R . In the main theorem of the chapter, we give conditions under which $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$ implies $\text{Rad } RG \subseteq \text{Rad } RH \cdot RG$ for an F -algebra R .

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CHAPTER I

PRELIMINARY RESULTS ON PROJECTIVE PAIRING AND THE COMPLETE REDUCIBILITY OF INDUCED MODULES

§1. Introduction

Let F be a field and G a multiplicative group. Then the group ring $F[G]$ of G over F is an F -algebra with the elements of G as a basis. To be more precise, $F[G]$ consists of all formal finite sums $\alpha = \sum a_g g$ with $g \in G$, $a_g \in F$. Then $F[G]$ is a ring where the addition is given componentwise and multiplication is defined distributively using the multiplication of the group G .

If G is a finite group then $F[G]$ is a finite dimensional F -algebra and there are fairly strong structure theorems for studying such algebras. On the other hand, if G is infinite, then these methods are no longer available and the problems of studying the group ring are correspondingly more difficult. For this reason the first papers (with a few exceptions) on infinite group rings did not appear until the early 1950's. We will be studying infinite group rings as well as finite dimensional algebras.

A few simple but useful observations can be made immediately. Suppose H is a subgroup of G . Then the inclusion

$H \subseteq G$ gives rise to the obvious inclusion $F[H] \subseteq F[G]$. In fact, if $\{x_i\}$ is a set of right coset representatives for H in G , then $F[G] = \sum F[H]x_i$ exhibits $F[G]$ as a free left $F[H]$ -module.

Since group rings are rings after all, the questions we ask about them must necessarily be ring theoretic although the techniques involved often are strongly group theoretic in nature. One of the questions that frequently arises in the study of rings is "What does the radical of the ring look like?", where by radical we shall mean Jacobson radical, abbreviated $\text{Rad } FG$.

According to Passman, the semi-simplicity problem seems to be the most difficult of all the group ring problems. It is the problem of classifying all semi-simple group rings and it is the problem of determining the structure of the radical of a non-semi-simple group ring. The semi-simplicity problem for infinite group rings has been studied for nearly 21 years. For finite groups we do know:

Theorem 1.1. (Maschke) If G is a finite group, then $F[G]$ is semi-simple if and only if $\text{char } F = 0$ or $\text{char } F = p \nmid |G|$.

But very little is known about $\text{Rad } FG$ itself if $|G|$ is divisible by p . Therefore, if $H \leq G$ it could be useful to know when $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$. We will be studying this inclusion giving conditions under which it holds.

Since the radical is that ideal which annihilates every irreducible FG-module we are led to consider the irreducible FH-and irreducible FG-modules. This brings us to the notion of an induced module. Throughout this paper, by the way, all modules under consideration will be right modules unless specified otherwise. Let W be an irreducible right FH-module. Then W can be made into a module for FG by tensoring with FG . Namely, $W \otimes_{FH} FG$ is a right FG-module called the induced module. Although W is irreducible, the induced module $W \otimes_{FH} FG$ is rarely irreducible. But it is often completely reducible as we will see later. For now it will suffice to give the connection between these induced modules and the inclusion $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$.

Theorem 1.2. (Sinha and Srivastava [9]) If for every irreducible FH-module W , the induced module $W \otimes_{FH} FG$ is completely reducible, then $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$.

In keeping with Sinha's and Srivastava's terminology we will say that the pair (FG, FH) , or simply (G, H) if the field is clear from the context, has property ρ whenever $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$. It is easy to see that property ρ is equivalent to the following criterion: If $\{x_i \mid i \in I\}$ is a right transversal for H in G and if $\sum_i \rho_i x_i \in \text{Rad } FG$, with $\rho_i \in FH$, then $\rho_i \in \text{Rad } FH$.

As a result of Theorem 1.2 it is clear that we need to study induced modules. The following elementary results concerning induced modules will be crucial: (See for example [2]).

Lemma 1.3. If W is an FH -module and $W = W_1 \oplus W_2$ as FH -modules, then $W^G = W_1^G \oplus W_2^G$ as FG -modules.

Lemma 1.4. Let $\{x_i | i \in I\}$ be a set of right coset representatives for H in G and let W be an FH -module. Then $W^G = \oplus \sum_{i \in I} W \otimes x_i$ as vector spaces, where, for any $x \in G$, $W \otimes x = \{w \otimes x | w \in W\}$.

Lemma 1.5. If $H \leq K \leq G$ and W is an FH -module then $(W^K)^G = W^G$.

It is clear that $W \otimes x$ is an $F[x^{-1}Hx]$ -module and that $\dim_F(W \otimes x) = \dim_F W$. It is also clear from 1.4 that if H has finite index in G and if $\dim_F W < \infty$ then $\dim_F W^G = [G:H] \dim_F W$.

Hence for every FH -module W there is a corresponding FG -module W^G with the above properties. On occasion, however, we will be starting with an FG -module, M say. In this case if H is a subgroup of G , then M can be regarded naturally as an FH -module by simply restricting the operators to FH . The resulting FH -module will be denoted by M_H . It is clear that $(M_K)_H = M_H$ for $H \leq K \leq G$ and that $\dim_F M = \dim_F (M_H)$ and we shall use these properties of restricted modules freely throughout the remainder of this paper.

Also we will often be called upon to refer to the well-known theorem of Clifford (see [2]) concerning irreducible modules:

Theorem 1.6. (Clifford) Let M be an irreducible KG -module where K is an arbitrary field, and let $H \leq G$ be a normal subgroup of G of finite index. Then M_H is a completely reducible KH -module and the irreducible KH -submodules of M_H are all conjugates of each other.

Remark: In [2], Curtis and Reiner prove Clifford's theorem under the assumption that G is finite and the crucial step is the existence of an irreducible submodule of M_H . However, the added hypothesis of finite index on H makes M_H a finitely generated KH -module so we can prove the existence of an irreducible submodule of M_H . (I am grateful to Steven Gagola for his observation of this fact.) His proof goes as follows: Let $H \trianglelefteq G$ and $[G:H] = n < \infty$. Let M be an irreducible KG -module. Since M_H is a finitely generated KH -module there exists a maximal submodule, N , of M_H . Let V be the irreducible KH -module M_H/N and let $\varphi: M_H \rightarrow V$ be the natural map. Since $\text{Hom}_{KH}(M_H, V) \cong_K \text{Hom}_{KG}(M, V^G)$ there exists a non-zero map $f: M \rightarrow V^G$. From the irreducibility of M we must have $\ker f = 0$, i.e. f is one-to-one and M can be considered contained in the KG -module $V^G = \sum_{g \in G} V \otimes g$ since for any $g \in G$, $V \otimes g$ is a KH -module. In particular M_H contains some irreducible KH -submodule of the form $V \otimes g$.

Another concept which is closely related to property ρ is that of H -projective FG -modules. Let H be a subgroup of G . Then an FG -module M is said to be H -projective if every exact sequence $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$ of FG -modules for which the

associated sequence $0 \rightarrow N_H \rightarrow L_H \rightarrow M_H \rightarrow 0$ splits over FH , is itself split over FG . We observe that in a sense H -projectivity is a generalization of the usual concept of projectivity. For if we take $H = \{1\}$, the trivial subgroup, then M is H -projective if and only if M is projective in the usual sense. Many people including Gustafson [3], Khatri [5], Sinha [9], and Higman [4] have studied H -projective FG -modules, but Sinha and Srivastava were the ones responsible for noticing its connection with the radical. Higman [4] was responsible for characterizing H -projective FG -modules in terms of induced modules in the following theorem that bears his name:

Theorem 1.7. (Higman's Criteria) Let H be a subgroup of G of finite index. Then the following statements for an FG -module M are equivalent:

- (i) M is H -projective;
- (ii) M is a component (i.e. isomorphic to a direct summand) of $(M_H)^G$;
- (iii) there exists an FH -endomorphism η of M such that $\sum_{i=1}^n x_i^{-1} \eta x_i = 1_M$ where $\{x_i | i = 1, \dots, n\}$ are coset representatives for H in G and 1_M is the identity map on M .

Remark: When Higman originally proved the above theorem, he did so in the more general setting of a group ring RG where R is a commutative ring. We shall use this fact later on.

If H is a subgroup of G and R is a commutative ring we say that the pair (RG, RH) , or simply (G, H) , is a projective pairing if and only if every RG -module is H -projective. Therefore, (G, H) is a projective pairing over R if every exact sequence $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$ of RG -modules for which the exact sequence $0 \rightarrow N_H \rightarrow L_H \rightarrow M_H \rightarrow 0$ splits over FH , is itself split over RG . With this definition Higman [4] and Gustafson [3] showed

Theorem 1.8. Let R be a commutative ring with identity, G a group, and H a subgroup of finite index. If $[G:H] = n$ is a unit in R , then (RG, RH) is a projective pairing.

In the course of our investigations the notion of a Frobenius group will come into play. Therefore we offer

Definition 1.9. A finite group G is called Frobenius with kernel M and complement K if $G = MK$, $M \triangleleft K$, $K \cap M = 1$, and $K \cap K^x = 1$ for all $x \in G - K$.

Finally we will make frequent use of the following easy lemma concerning radicals, whose proof is essentially that of Lemma 16.5 in [6].

Lemma 1.10. Let H be a subgroup of G and let R be a ring with identity. Then $\text{Rad } RG \cap RH \subseteq \text{Rad } RH$.

§2. Relations between projective pairing,
property ρ , and the complete
reducibility of induced modules

Let R denote a commutative ring with identity. Let θ denote the class of proper subgroups H of G such that (G, H) is a projective pairing over R . Let $\mathcal{R}$ denote the class of proper subgroups H of G for which the pair (G, H) has property ρ , and finally let $\mathcal{C}$ be the class of proper subgroups H of G for which the induced modules N^G are completely reducible for every irreducible RH -module N . With this notation, Theorem 1.2 says that $\mathcal{C} \subseteq \mathcal{R}$. In this section, we will show that if either $H \in \mathcal{C}$ or $H \in \theta$ then H has finite index in G and the index is a unit in the base ring. This will yield as a corollary the inclusion $\mathcal{C} \subseteq \theta$.

Theorem 1.11. Let R be a semi-simple artinian ring and let G be a group with subgroup H . If $H \in \mathcal{C}$ ($H \in \theta$) then H has finite index in G . Furthermore, consider the following three conditions:

- a) $H \in \mathcal{C}$ ($H \in \theta$)
- b) $R \otimes_{RH} RG = R^G$ is a completely reducible RG -module
(the trivial module, R , is H -projective)
- c) $[G:H]$ is a unit in R .

Then $a) \Rightarrow b) \Rightarrow c)$. Further, $c) \Rightarrow a)$ if $H \trianglelefteq G$.

Proof: $a) \Rightarrow b)$ If $H \in \theta$ then all modules, including the trivial module R , are H -projective and there is nothing to prove. So suppose $H \in \mathcal{C}$. Since R is semi-simple artinian

it can be written as the direct sum of fields; $R = F_1 \oplus \dots \oplus F_n$. Now each F_i is an irreducible RH -module and hence $F_i \otimes_{RH} RG$ is a completely reducible RG -module. Thus, $R \otimes_{RH} RG = \bigoplus_{i=1}^n F_i \otimes_{RH} RG$ is completely reducible.

b) $\Rightarrow$ c) Since $R \otimes_{RH} RG$ is completely reducible the sequence

$$0 \rightarrow \text{Ker } f_1 \rightarrow R \otimes_{RH} RG \xrightarrow[f_2]{f_1} R \rightarrow 0$$

splits where f_1 is the RG -module epimorphism defined by $f_1(x \otimes y) = x \cdot y$ with the trivial action of G . Hence there exists an RG -module homomorphism $f_2: R \rightarrow R \otimes_{RH} RG$ with $f_1 \circ f_2 = 1_R$. (If $H \in \mathcal{O}$ then the trivial module is H -projective and the sequence splits since it splits as RG -modules using the map $x \rightarrow x \otimes 1$.) Let $\{g_i | i \in I\}$ be a right transversal for H in G and assume that $g_1 = 1$. Then we can write $f_2(1) = \sum_i \rho_i(1 \otimes g_i)$ where $\rho_i \in RH$. Writing $\rho_i = \sum_j \lambda_{ji} h_{ji}$ with $\lambda_{ji} \in R$ we have $f_2(1) = \sum_i \sum_j \lambda_{ji} h_{ji} \otimes g_i = \sum_i \sum_j \lambda_{ji} (1 \cdot h_{ij} \otimes g_i) = \sum_i \sum_j \lambda_{ji} (1 \otimes g_i) = \sum_i r_i \otimes g_i$ where we let $r_i = \sum_j \lambda_{ji} \in R$. Since $f_1 \circ f_2(1) = 1$ there exists some i for which $r_i \neq 0$. Therefore, assume g_k is a coset representative for which $r_k \neq 0$. Then $f_2(1) = f_2(1 \cdot g_k^{-1}) = f_2(1) g_k^{-1} = r_k(1 \otimes g_1) + \sum_{i \neq k} (r_i \otimes g_i g_k^{-1})$. Hence the coefficient of $1 \otimes g_1$ is non-zero. Now if g_j is any coset representative for H in G , we have $f_2(1) = f_2(1 \cdot g_j) = f_2(1) g_j = r_1(1 \otimes g_j) + \sum_{i \neq 1} r_i(1 \otimes g_i g_j)$. By uniqueness of representation of $f_2(1)$ we know $r_1 = r_k$ and hence the coefficient of $1 \otimes g_j$ is

non-zero. In this way we see that every coset representative appears in the finite sum $f_2(1) = \sum r_i(1 \otimes g_i)$ thus proving H has finite index in G . Let this index be n . Then

$f_2(1) = \sum_{i=1}^n r_i(1 \otimes g_i)$ where $r_i = r_k \forall i$. Applying f_1 to both sides of this equation we get $1 = f_1 \circ f_2(1) = \sum_{i=1}^n r_i \cdot g_i = \sum_{i=1}^n r_i = n \cdot r_k$. Hence n is a unit in R .

It only remains now to prove $c) \Rightarrow a)$ for $H \trianglelefteq G$. By virtue of Theorem 1.8 it suffices to prove $\theta \subseteq \mathcal{C}$ for normal subgroups. So let $H \in \theta$ and let $x_1, \dots, x_n$ be a set of coset representatives for H in G . If M is an irreducible RH -module then $M^G = \sum_{i=1}^n M \otimes x_i$ where $M \otimes x_i$ is an irreducible $R[x_i^{-1}Hx_i] = RH$ -module. Hence M^G is completely reducible as an RH -module. Now let N be any submodule of M^G and form the exact sequence

$$0 \rightarrow N \rightarrow M^G \rightarrow M^G/N \rightarrow 0.$$

Since M^G is completely reducible as an RH -module the associated exact sequence

$$0 \rightarrow N_H \rightarrow (M^G)_H \rightarrow (M^G/N)_H \rightarrow 0$$

splits as RH -modules. By projective pairing we have the original sequence splitting as RG -modules. Hence N is a direct summand of M^G . Since N was arbitrary, M^G is completely reducible. This completes the proof of the theorem.

Remark: We remark that in proving $a) \Rightarrow b) \Rightarrow c)$ for $H \in \theta$ we did not use the hypothesis that R was semi-simple artinian.

Corollary 1.12. Let R be a commutative ring with identity. Then (G, H) is a projective pairing if and only if H has finite index equal to a unit in R .

Corollary 1.13. $\mathcal{C} \subseteq \theta$, $R \not\subseteq \theta$, $R \not\subseteq \mathcal{C}$.

Proof: That $\mathcal{C} \subseteq \theta$ is immediate from the theorem and 1.8. To see that $R \not\subseteq \theta$ and $R \not\subseteq \mathcal{C}$ let G be any infinite group for which FG is semi-simple. (For example G could be the additive group of integers.) Then the trivial subgroup $H = \{1\} \in R$ but since $[G:H] = \infty$, $H \notin \theta$ and $H \notin \mathcal{C}$.

Remarks: 1) In the theorem we cannot have $c) \Rightarrow a)$ in general for then $\mathcal{C} = \theta$. But Khatri showed in [5] that for the group A_5 and a field F of characteristic 3, FA_5 has a subgroup $H \cong A_4$ such that (A_5, A_4) is a projective pairing but $H \notin \mathcal{C}$.

2) In chapter 3 we will give an example to show that $c) \not\Rightarrow b)$ in Theorem 1.11.

As was shown in the above theorem if $H \trianglelefteq G$ and $H \in \theta$, then $H \in \mathcal{C}$. For purposes of future reference we shall give this result a number:

Theorem 1.14. If $H \trianglelefteq G$ and $H \in \theta$, then $H \in \mathcal{C}$ where G is any group and R is a commutative ring with 1.

§3. Normal subgroups

In [5], Khatri shows that the three classes of subgroups $\mathcal{C}, \mathcal{R}, \mathcal{O}$ are equivalent for normal subgroups if all groups considered are finite. From Corollary 1.13 it is clear that such a statement is false in general for infinite groups. Nevertheless, we can give sufficient conditions for the three classes to coincide for normal subgroups and our result yields Khatri's theorem as a corollary.

For the main result of this section we will need the following definition and a lemma of Passman.

Definition 1.15. If G is a group, let $\Delta G = \{x \in G \mid [G : C_G(x)] < \infty\}$ where $C_G(x)$ is the centralizer of x in G .

Lemma 1.16. (Passman [6]) Let H be a torsion subgroup contained in ΔG such that $H \trianglelefteq G$. Then $\text{Rad } FH \subseteq \text{Rad } FG$.

Let H be a subgroup of G . We denote by $\mathfrak{U}_G(H)$ the ideal in FG which is generated by the set $\{1-h \mid h \in H\}$. It is well-known that if $H \trianglelefteq G$ and $\varphi: FG \rightarrow F(G/H)$ is the map obtained by extending the canonical map $G \rightarrow G/H$ to FG by linearity, then $\text{Ker } \varphi = \mathfrak{U}_G(H) = \mathfrak{U}_H(H) \cdot FG$ and so $\frac{FG}{\mathfrak{U}_H(H) \cdot FG} \cong F(G/H)$. Therefore $\frac{FG}{\mathfrak{U}(G)} \cong F$ and $\mathfrak{U}(G) \supseteq \text{Rad } FG$.

Lemma 1.17. Let H and H_1 be subgroups of G . Then $\mathfrak{U}(H_1) \cdot FG \cap FH \subseteq \mathfrak{U}(H \cap H_1) \cdot FH$.

Proof: Let $\{g_i | i \in I\}$ be a set of right coset representatives for H_1 in G , and let $x \in \mathcal{U}(H_1) \cdot FG \cap FH$. Then we can write $x = \sum_i \rho_i g_i$ where $\rho_i \in \mathcal{U}(H_1)$. Let $\rho_i = \sum_j f_{ij} h_{ij}$ where $f_{ij} \in F$, $h_{ij} \in H_1$, and $\sum_j f_{ij} = 0$. Since $x \in FH$, $h_{ij} g_i \in H \forall i, j$. Now write $\rho_i = \sum_{j=1}^{m_i} f_{ij} h_{ij} = f_{i1} h_{i1} + f_{i2} h_{i2} + \dots + f_{im_i} h_{im_i} = (f_{i1} + f_{i2} h_{i2} h_{i1}^{-1} + f_{i3} h_{i3} h_{i1}^{-1} + \dots + f_{im_i} h_{im_i} h_{i1}^{-1}) h_{i1}$. Since $h_{ij} h_{i1}^{-1} = h_{ij} g_i g_i^{-1} h_{i1}^{-1} = (h_{ij} g_i) (h_{i1} g_i)^{-1}$, the $h_{ij} h_{i1}^{-1} \in H \cap H_1 \forall j$ and therefore $\rho_i \in \mathcal{U}(H \cap H_1) \cdot FH \forall i$. This shows $x \in \mathcal{U}(H \cap H_1) \cdot FH$ as desired.

We can now state the main theorem of this section:

Theorem 1.18. Suppose $H \trianglelefteq G$ with $[G:H] < \infty$ and ΔG periodic. Also suppose that whenever $x \in G - H$ and $x^{p^\alpha} \in H$, $\alpha \neq 0$, then $x \in \Delta G$ where $\text{char } F = p$. Then the following are equivalent.

- a) $H \in \mathcal{O}$
- b) $H \in \mathcal{C}$
- c) $H \in \mathcal{R}$

Furthermore, none of the hypotheses can be dropped.

Proof. a) $\Rightarrow$ b) is just Theorem 1.14 and b) $\Rightarrow$ c) is always true by Theorem 1.2. So it suffices to prove c) $\Rightarrow$ a). By Gustafson's result it suffices to show that G/H is a p' -group. So let $x \in G - H$ with $x^{p^\alpha} \in H$. Then by assumption $x \in \Delta G$ and $(x^{p^\alpha})^n = 1$ for some integer n . Since x has

only a finite number of conjugates in G there exists a finite group $H' \trianglelefteq G$ which contains x . (Take H' to be the subgroup generated by x and its finitely many conjugates.)

Then

$$\begin{aligned} \text{Rad } FH' &\subseteq \text{Rad } FG \quad \text{by 1.16} \\ &\subseteq \text{Rad } FH \cdot FG \quad \text{by hypothesis} \\ &\subseteq \mathcal{U}(FH) \cdot FG. \end{aligned}$$

Hence $\text{Rad } FH' \subseteq \mathcal{U}(FH) \cdot FG \cap FH' \subseteq \mathcal{U}(F[H \cap H']) \cdot FH'$ by 1.17.

Therefore,

$$F\left(\frac{H'}{H \cap H'}\right) \cong \frac{FH'}{\mathcal{U}(F[H \cap H']) \cdot FH'}$$

is semi-simple since it is a homomorphic image of the finite dimensional semi-simple algebra $\frac{FH'}{\text{Rad } FH'}$. By Maschke's Theorem (1.1) $\frac{H'}{H \cap H'}$ cannot contain any elements of order p^α . But x^q is a p -element of H' where $np^\alpha = qp^\beta$ and $(q, p) = 1$, $\beta \geq \alpha$. Hence $x^q \in H$. Now $1 = aq + bp^\beta$ and therefore $x = x^{aq} \cdot x^{bp^\beta} = (x^q)^a (x^{p^\beta})^b \in H$. Hence $[G:H]$ is a unit in F which was to be proved.

Now suppose the hypothesis of normality is dropped.

Then Khatri's example of $G = A_5, H = A_4$, F a field of characteristic 3 gives a counterexample to $\theta \subseteq \mathcal{C}$. Nor can we drop the hypothesis that ΔG is periodic for in that case we could take $G = \mathbb{Z}$, the additive group of integers, and H the subgroup of $\mathbb{Z}$ consisting of all integers divisible by p where $\text{char } F = p$. Since $F\mathbb{Z}$ is semi-simple, $0 = \text{Rad } F\mathbb{Z} \subseteq \text{Rad } FH \cdot FG$ and $H \in \mathcal{R}$. But $[G:H] = p$ so that $H \notin \theta$.

The hypothesis of finite index cannot be dropped.

Let $\text{char } F = p$ and let G be an infinite locally finite abelian q -group where $q \neq p$. Then FG is semi-simple (see [6]). So $1 = H \in \mathcal{R}$ trivially but H does not have finite index in G so that $H \notin \theta$.

Finally, we cannot drop the hypothesis that $x^{p^\alpha} \in H \Rightarrow x \in \Delta G$. To see this let G be the wreath product $Z_q \wr \prod_{\bullet} Z_p$. Again from [6] (Theorem 21.2ii) we know that if F is a field of characteristic p then FG is semi-simple so that (G, H) has property ρ for any $H \leq G$. Let $W = \prod_{y \in \prod_{\bullet} Z_p} (Z_q)_y$. Then $W \trianglelefteq G$ and $G/W \cong \prod_{\bullet} Z_p$. Therefore there exists a one-to-one correspondence between the normal subgroups of G containing W and the subgroups of $\prod_{\bullet} Z_p$. In particular, there exists a normal subgroup H of G which has index p in G . Hence $H \notin \theta$ and therefore $\mathcal{R} \not\subseteq \theta$. On the other hand, G and H do not satisfy the hypothesis " $x^{p^\alpha} \in H \Rightarrow x \in \Delta G$." For let $x \in \prod_{\bullet} Z_p - H$. Then $x^p \in H$ but x cannot be contained in ΔG . Since $G/W \cong \prod_{\bullet} Z_p$ acts on W , clearly $C_W(x)$ consists of all those elements of W whose projections in the factors $(Z_q)_y$ are constant on the orbits of $\langle x \rangle$. Since $\prod Z_p$ is infinite, $[W : C_W(x)] = \infty$. Hence x has infinitely many distinct conjugates, i.e. $x \notin \Delta G$. This completes the proof of the theorem.

We have Khatri's result as a corollary:

Corollary 1.19. (Khatri) If G is a finite group
and $H \trianglelefteq G$, then $H \in \mathcal{O} \Leftrightarrow H \in \mathcal{C} \Leftrightarrow H \in \mathcal{R}$.

CHAPTER II

θRC -GROUPS

§2.1. Extensions of θRC -groups

In the previous chapter we studied property ρ and investigated its relationship to projective pairing and the complete reducibility of induced modules. We saw how the complete reducibility of induced modules implied both projective pairing and property ρ but that the reverse implications were false in general. However, there are many cases when the reverse implications do hold and in this chapter conditions will be given for which these three classes of subgroups do coincide with each other.

Following Khatri [5] we make the definition:

Definition 2.1. A group G is called a θRC -group over the field F if $\theta = \mathcal{C} = \mathcal{R}$.

As Khatri pointed out, examples of θRC -groups do exist for if F is a field of characteristic p and if $p \nmid |G|$ where G is a finite group, then trivially each class consists of all the subgroups of G . We also have

Theorem 2.2. Let G be a locally finite p -group and F a field of characteristic p . Then G is a θRC -group.

Proof: We will show that each of the three classes of subgroups $\theta, \mathcal{R}, \mathcal{C}$ is empty. First we claim that any subgroup H of G of finite index must have index a power of p . For suppose $H \leq G$ has finite index. Then there exists a normal subgroup $\bar{H}$ of finite index such that $\bar{H} \leq H$. Since $[G:\bar{H}] = [G:H][H:\bar{H}]$ it suffices to prove that the index of $\bar{H}$ is a power of p . But this is easy for if $x\bar{H} \in G/\bar{H}$ is an element of order $n = mp^\gamma$, say, with $(m,p) = 1$ then $x^{n\bar{H}} = \bar{H}$ or $x^n \in \bar{H}$. Since $\bar{H}$ is a locally finite p -group $(x^n)^{p^\alpha} = x^{np^\alpha} = 1$ for some integer α . If $m \neq 1$, then $(x^{p^{\alpha+\gamma}})^m = 1$ and $x^{p^{\alpha+\gamma}}$ is an element of G which is not a p -element. This contradiction proves the claim. Together with Theorem 1.11 this fact shows that the classes θ and $\mathcal{C}$ are empty.

To see that the class $\mathcal{R}$ is also empty, suppose there exists an $H \subsetneq G$ with $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$. By Lemma 21.5 of [6] we know that $\text{Rad } FG = \{\sum a_g g \mid \sum a_g = 0\}$. If $x \in G - H$, then $1 \cdot 1 - 1 \cdot x \in \text{Rad } FG$. But $1, x$ can be made part of a transversal for H in G thereby putting 1 into the radical of FH by property ρ . This contradiction shows that $\mathcal{R}$ is also empty.

We can obtain further $\theta\mathcal{RC}$ -groups as follows

Theorem 2.3. An extension of a locally finite p -group by a locally finite $\theta\mathcal{RC}$ -group is a $\theta\mathcal{RC}$ -group.

We need two lemmas

Lemma 2.4. Let I be a nil ideal in a ring R and let $\tau: R \rightarrow R/I$ be the natural map. If $\text{Rad } R$ and $\text{Rad } R/I$ are both nil ideals, then $\tau(\text{Rad } R) = \text{Rad } R/I$. Moreover, if $\tau(r) = r + I \in \text{Rad } R/I$, then $r \in \text{Rad } R$.

Proof: We always have $\tau(\text{Rad } R) \subseteq \text{Rad } R/I$. For the reverse inclusion, suppose $r_1 + I \in \text{Rad } R/I$. Then for each $r_2 + I \in R/I$, $(r_1 + I)(r_2 + I) = r_1 r_2 + I$ is nilpotent, i.e. there exists an integer m such that $(r_1 r_2 + I)^m = (r_1 r_2)^m + I = I$. Therefore, $(r_1 r_2)^m \in I$. But I is nil, hence $r_1 r_2$ is nilpotent. Since r_2 was arbitrary, $r_1 \in \text{Rad } R$ and $r_1 + I = \tau(r_1) \in \tau(\text{Rad } R)$.

Lemma 2.5. If F is a field and G is a locally finite group, then $\text{Rad } FG$ is nil.

Proof: Let $x = a_1 g_1 + \dots + a_n g_n \in \text{Rad } FG$ and let H be the finite subgroup of G generated by the g_i , $i = 1, \dots, n$. Then $x \in \text{Rad } FG \cap FH \subseteq \text{Rad } FH$ which is nilpotent.

Proof of Theorem: Let G be an extension of a locally finite p -group P by a locally finite θRC -group. Since we always have $C \subseteq R$ by 1.2, we need only show that $\theta \subseteq C$ and $R \subseteq \theta$.

$\theta \subseteq C$: Let (G, H) be a projective pairing. Then since $[G:H] = n$ a unit in F , H must contain P as a normal subgroup. Now we use a result due to Woods [12] which says if $H \trianglelefteq G$ and G/H is locally finite, then $\text{Rad } FH \subseteq \text{Rad } FG$. Therefore,

$\text{Rad } FP = \{\sum a_p P \mid \sum a_p = 0\} \subseteq \text{Rad } FH$. So if N is an irreducible FH -module, then for every $n \in N$ and $p \in P$, $n \cdot 1 = n \cdot p$ since $1 - p \in \text{Rad } FH \subseteq \text{Annih } N$. This shows that we can regard N as an irreducible $F(H/P)$ -module. Now $(G/P, H/P)$ is a projective pairing and G/P is a θRC -group, so the induced $F(G/P)$ -module $N^{G/P}$ is completely reducible, say $N^{G/P} = \bigoplus_{i \in I} N_i$.

Since P acts trivially on N we can make $N^{G/P}$ and the N_i into FG -modules by defining $(n \otimes Pg_1)g_2 = (n \otimes Pg_1)Pg_2$. With this definition $N^{G/P}$ becomes isomorphic to N^G as an FG -module via the correspondence $n \otimes Pg \mapsto n \otimes g$. Now the N_i remain irreducible as FG -modules since if $M \subsetneq N_i$ then $M \subsetneq N_i$ also as an $F(G/P)$ -module. This proves the complete reducibility of N^G over FG .

$R \subseteq \theta$. Suppose that (G, H) has property ρ and suppose $\{x_i \mid i \in I\}$ is a set of coset representatives for H in G . Since $P \trianglelefteq G$, $\text{Rad } FP = \{\sum a_p P \mid \sum a_p = 0\} \subseteq \text{Rad } FG \subseteq \text{Rad } FH \cdot FG$ using Woods' result again. Then $p - 1 = hx_i - 1 \cdot 1 \in \text{Rad } FH \cdot FG$ implying $1 \in \text{Rad } FH$ unless $p \in H$. Thus $P \trianglelefteq H$.

Now let $\sum \bar{\rho}_i P x_i \in \text{Rad } F(G/P)$ where the $P x_i$ are right coset representatives for H/P in G/P , and $\bar{\rho}_i \in F(H/P)$. If $\varphi: FG \rightarrow F(G/P)$ is the natural map, then there exists $\rho_i \in FH$ such that $\varphi(\rho_i) = \bar{\rho}_i$, and $\varphi(\sum \rho_i x_i) = \sum \bar{\rho}_i P x_i \in \text{Rad } F(G/P)$. But $F(G/P) \cong FG/\mathcal{U}_G(P)$ and all groups in sight are locally finite so that $\mathcal{U}_G(P) = \text{Rad } FP \cdot FG$, $\text{Rad } F(G/P)$, and $\text{Rad } FG$ are nil by Lemma 2.5. Thus, we can apply Lemma 2.4 to conclude

$\sum \rho_i x_i \in \text{Rad } FG \subseteq \text{Rad } FH \cdot FG$, i.e. $\rho_i \in \text{Rad } FH$ and $\bar{\rho}_i = \varphi(\rho_i) \in \varphi(\text{Rad } FH) \subseteq \text{Rad } F(H/P)$. Since G/P is a θRC -group and $(G/P, H/P)$ has been shown to have property ρ , $[G/P : H/P] = [G : H]$ is a unit in F . This completes the proof of the theorem.

We also have

Theorem 2.6. Let F be a field of characteristic p . Then an extension G of a θRC -group L by a finite p' -group is a θRC -group.

Proof: Again we need only show that $\theta \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \theta$.

$\theta \subseteq \mathcal{C}$. Let (G, H) be a projective pairing and let N be an irreducible FH -module. Since $L \trianglelefteq G$, $H \cap L \trianglelefteq H$ and by Clifford's theorem $N_{H \cap L} = \oplus \sum N_i$ where the N_i are irreducible $F(H \cap L)$ -modules.

Now observe that $[G : H] = [G : LH][LH : H] = [G : LH][L : H \cap L]$ and therefore $[L : H \cap L]$ is a unit in F . By the hypothesis on L , each $(N_i)^L$ is completely reducible over FL . Since (G, L) is a projective pairing, $(N_i)^G = (N_i^L)^G$ are completely reducible over FG for all i . Since $(H, H \cap L)$ is also a projective pairing, a result of Higman [4] gives that N is a component of $(N_{L \cap H})^H$. This in turn implies that N^G is a component of the completely reducible FG -module $(N_{H \cap L})^G = ((N_{H \cap L})^H)^G$. Hence N^G is completely reducible over FG .

$R \subseteq \emptyset$. Let H be a subgroup of G such that (G, H) has property ρ . Then $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$. By Woods' result $\text{Rad } FL \subseteq \text{Rad } FG$ and therefore $\text{Rad } FL \subseteq \text{Rad } FH \cdot FG \cap FL$. Let $K = H \cap L$ and let $\{y_i \mid i \in I\}$ be a set of coset representatives for K in L , i.e. $L = \bigcup_{i \in I} Ky_i$. Then the y_i also belong to a set of coset representatives for H in G . For suppose $y_i = y_j h$ for some $h \in H$. Thus $y_j^{-1} y_i = h \in H \cap L = K$, or $y_i = y_j k$ for some $k \in K$, a contradiction. Thus, if $x \in \text{Rad } FL \subseteq \text{Rad } FH \cdot FG \cap FL$, then $x = \sum \rho_n y_n \in FL$ with $\rho_n \in FK \subseteq FH$. Since the y_i are coset representatives for H in G , $\rho_n \in \text{Rad } FH$. Therefore, $\rho_n \in \text{Rad } FH \cap FK \subseteq \text{Rad } FK$. We have shown then that $\text{Rad } FL \subseteq \text{Rad } FH \cdot FG \cap FL \subseteq \text{Rad } F(H \cap L) \cdot FL$, i.e. $(L, H \cap L)$ has property ρ . Since L is a θRC -group, $H \cap L$ has index a unit in L . Now $[G:H \cap L] = [G:H][H:H \cap L] = [G:L][L:H \cap L]$ which is a product of units in F . Thus $[G:H]$ is a unit in F and $H \in \emptyset$.

We will show in the next chapter that it is not true in general that extensions of θRC -groups by θRC -groups are θRC -groups, so perhaps Theorems 2.3 and 2.6 are the best possible.

§2.2. Quotients of θRC -groups

It is not known whether subgroups of θRC -groups are θRC -groups. Nor is it known in general if quotients of θRC -groups are θRC . However we do have some partial results in this direction.

Definition 2.7. If G is a group and p is a prime then G^p is the subgroup of G generated by the elements whose order is a power of p .

Theorem 2.8. Let F be a field of characteristic $p > 0$ and let G be a locally finite θRC -group. Let $A \trianglelefteq G$. If either A is a p -group or $A \supseteq G^p$, then G/A is a θRC -group.

Proof: The case $A \supseteq G^p$ is clear; for if G is a θRC -group then G^p must have finite index a unit in F since $\text{Rad } FG \subseteq \text{Rad } FG^p \cdot FG$ (see [6]). Thus G/A is a finite p' -group which is trivially a θRC -group.

Now suppose A is a p -group. We first show $\theta \leq \mathcal{C}$. Let H/A be a subgroup of G/A such that $(G/A, H/A)$ is a projective pairing and let N be an irreducible H/A -module. Note that N is also an irreducible FH -module for $n \cdot h = n(hA)$ for $n \in N$ and $h \in H$. But (G, H) is a projective pairing and G is a θRC -group, hence N^G is completely reducible over FG . As in the proof of Theorem 2.3 N^G is isomorphic to $N^{G/A}$ as FG -modules. Hence $N^{G/A}$ is completely reducible.

To show $\mathcal{R} \subseteq \mathcal{C}$, let H/A be a subgroup of G/A such that $\text{Rad } F(G/A) \subseteq \text{Rad } F(H/A) \cdot F(G/A)$. We will show that (G, H) also has property ρ . Let $\{x_i \mid i \in I\}$ be a right transversal for H in G and let $\sum \rho_i x_i \in \text{Rad } FG$ with $\rho_i \in FH$. Let $\varphi: FG \rightarrow F(G/A)$ be the canonical mapping. Then $\varphi(\sum \rho_i x_i) = \sum \varphi(\rho_i) Ax_i \in \text{Rad } F(G/A) \subseteq \text{Rad } F(H/A) \cdot F(G/A)$. Thus,

$\varphi(\rho_i) \in \text{Rad } F(H/A) \cong \text{Rad}(\frac{FH}{\mathfrak{U}_H(A)})$ and by Lemma 2.4, $\rho_i \in \text{Rad } FH$. Hence (G, H) has property ρ and therefore is a projective pairing, i.e. $[G/A:H/A] = [G:H]$ is a unit in F .

We have two corollaries, the first of which follows immediately from the proof of the theorem.

Corollary 2.9. If G is a θRC -group and $H \trianglelefteq G$, then G/H has the property that $\theta \subseteq \mathcal{C}$.

Corollary 2.10. Let G be a finite group with $H \trianglelefteq G$. Let F be a field of characteristic p and assume $|G/H|$ is divisible by p^α with $0 \leq \alpha \leq 1$. Then $\mathcal{R}(G/H) \subseteq \theta(G/H)$ and hence G/H is a θRC -group.

Proof: If $p \nmid |G/H|$ then G/H is a p' -group and there is nothing to prove. Suppose $p \mid |G/H|$ but no higher power of p divides $|G/H|$ then $\text{Rad } F(G/H)$ is non-zero by Maschke's theorem. So if $\text{Rad } F(G/H) \subseteq \text{Rad } F(A/H) \cdot F(G/H)$ then $\text{Rad } F(A/H) \neq 0$. Hence (A/H) is divisible by p and A/H contains a p -Sylow subgroup of G/H . This proves $\mathcal{R} \subseteq \theta$ and the second assertion follows immediately from Corollary 2.9.

Example 2.11. Let F be a field of characteristic 3 and let G be the linear group $SL(2,5)$; then G is not a θRC -group.

Proof: Recall that if $\text{char } F = 3$, then FA_5 is not a θRC -group. But $PSL(2,5) \cong A_5$ and $PSL = \frac{SL(2,5)}{Z(G)}$ so by Corollary 2.10, $SL(2,5)$ is not a θRC -group.

§2.3. Dihedral and dicyclic groups

Besides observing that finite p' -groups and finite p -groups were θRC -groups, Khatri was able to prove in [5] that p -nilpotent Frobenius groups are θRC -groups. Using this result we will extend the list of known θRC -groups to include the dihedral and dicyclic groups.

Theorem 2.12. Dihedral groups are θRC -groups over any field F .

Proof: Write $G = \{a, b \mid a^m = 1 = b^2, bab = a^{m-1}\}$. We first suppose that $\text{char } F \neq 2$. Let $H = \{1, a, a^2, \dots, a^{m-1}\}$. Then H is an abelian group (and therefore a θRC -group by Corollary 1.19) and G is an extension of H by a group of order 2. Therefore by Theorem 2.6 G is a θRC -group.

Hence we can assume $\text{char } F = 2$.

Case I: $|G| = 2m$, m odd.

In this case, if we let $M = \{1, a, a^2, \dots, a^{m-1}\}$ and $P = \{1, b\}$ then we easily have $G = PM$ and $P \cap M = \{1\}$. Moreover, since conjugation of b by powers of a yield elements of the form ba^k , $k \neq m$, each conjugate of P by an element outside of P intersects P trivially. This makes G a 2-nilpotent Frobenius group with kernel M . By Khatri's result, then, G is a θRC -group.

Case II: $|G| = 2^{\alpha} m'$, m' odd, $\alpha > 1$.

In this case consider the subgroup $H = \{a^{m'}, a^{2m'}, \dots, a^{2^{\alpha-1}m'} = 1\}$. H is a normal 2-group and G/H is a dihedral group of order $2m'$. Hence by Case I, G/H is a θRC -group. Therefore, by Theorem 2.3, we again have that G is a θRC -group.

One may then ask if the infinite dihedral group is a θRC -group. The answer is no.

Theorem 2.13. The infinite dihedral group is not a θRC -group over any field F .

Proof: Write $G = \{x, y \mid y^2 = 1, yxy = x^{-1}\}$. Let A be the torsion-free abelian subgroup generated by the element x . Then $A \trianglelefteq G$ and $C_G(A) = A$. Hence, by Theorem 21.2(iii) of [6], FG is semi-simple. By 1.11 it is clear that G cannot be a θRC -group. (Take $H = \{1\}$, then $H \in \mathcal{R}$, but $H \notin \theta$.)

Theorem 2.14. Dicyclic groups are θRC -groups over any field F .

Proof: We can write $G = \{a, b \mid a^{2m} = 1, a^m = b^2, b^{-1}aba = 1\}$ where $|G| = 4m$. If $\text{char } F \neq 2$, let H be the subgroup $\{1, a^2, a^4, \dots, a^{2m-2}\}$. Then H is a normal abelian subgroup of G and G/H is a group of order 4. Hence by Theorem 2.6 we are done.

If $\text{char } F = 2$, let $H = \{1, a^m\}$. Then H is a normal 2-group and G/H is isomorphic to a dihedral group of order $2m$. By Theorem 2.12 and Theorem 2.3 G is a θRC -group. This proves the theorem.

CHAPTER III

ANSWERS TO TWO QUESTIONS OF KHATRI

Let F be a field of characteristic p . Then in [5] Khatri asks whether extensions of θRC -groups by θRC -groups are θRC -groups and in particular if extensions of p' -groups by p -groups are θRC -groups. We will answer both of Khatri's questions with a counterexample. In the following, then, let G be a finite p -nilpotent group with p -Sylow subgroup P isomorphic to a cyclic group of order p . Let P have normal p -complement K . Notice that G is an extension of a p' -group and hence falls into the category of groups under discussion. We will also be assuming throughout this chapter that F is a finite splitting field for the group G .

Next we observe that the trivial module F is the unique (up to isomorphism) irreducible FG -module. We shall study F^G under the assumption that it is completely reducible.

Let T be a fixed irreducible FG -module such that T_K is also irreducible and G -invariant, i.e. for any $g \in G$, the FK -module $T_K \otimes_{FK} g$ is isomorphic to T_K .

Definition 3.1. Let G be a group and H a subgroup of G . If L is an FH -module, we say that the FG -module M is an extension of L if $M_H \cong L$.

Lemma 3.2. $F^G \cong FK$ as FK-modules.

Proof: Let $K = \{1, k_2, k_3, \dots, k_l\}$. Then, since $G = PK$ the k_i form a set of coset representatives for P in G . Therefore, $F^G = F \otimes 1 + \dots + F \otimes k_l$. Define $\lambda: FK \rightarrow F^G$ by $\lambda(k) = 1 \otimes k$ and extend linearly. Since $\lambda(k_{i1}k_{i2}) = 1 \otimes k_{i1}k_{i2} = (1 \otimes k_{i1})k_{i2} = \lambda(k_{i1})k_{i2}$, λ is an FK-module homomorphism which is easily seen to be one-to-one and onto.

The next lemma is a well-known fact about irreducible modules and we refer the reader to [10] for a proof.

Lemma 3.3. Let $H \trianglelefteq G$ and let $[G:H] = p^\alpha$. Let N be an irreducible FH-module such that $N \cong N \otimes g$ for all $g \in G$. Then N can be extended uniquely to an irreducible FG-module M .

We also need some results on homogeneous modules.

An FG-module V is said to be homogeneous if it is a direct sum of, say, d copies of an irreducible FG-module W . Still assuming that F is a splitting field for G , let $a = (a_1, a_2, \dots, a_d) \in F \times F \times \dots \times F$ where not all of the a_i are zero. Define W_a , a submodule of V , by $W_a = \{(a_1w, a_2w, \dots, a_dw) \mid w \in W\}$.

With this notation we have

Lemma 3.4. Every irreducible submodule of V has the form W_a for some a , and $W_a = W_{a'}$ if and only if $a = \lambda a'$ for some $\lambda \neq 0$ in F .

Proof: Let W be an irreducible FG-module and W_a as defined above. We must first show that W_a is an irreducible submodule of V . It is clear that W_a is a submodule so it suffices to prove W_a is irreducible. Let $N \subseteq W_a$ be a submodule and let $0 \neq (a_1 w_1, \dots, a_d w_1) \in N$. Since $0 \neq w_1$ and since W is irreducible, $FG \cdot w_1 = W$, i.e. for any $w \in W$, there exists $x \in FG$ such that $xw_1 = w$. Therefore $x(a_1 w_1, a_2 w_1, \dots, a_d w_1) = (a_1 w, a_2 w, \dots, a_d w) \in N$. Since w was arbitrary, $N = W_a$ and W_a is irreducible.

Next we show that any irreducible submodule N of V occurs in the form W_a . Let N be an irreducible submodule of V and let $0 \neq (w_1, \dots, w_d) \in N$. We need to show that $w_1, w_2, \dots, w_d \in W$ are in the same one-dimensional subspace of W . Without loss of generality assume $w_1 \neq 0$. Let $\pi_j: W \oplus \dots \oplus W \rightarrow W$ denote the projection onto the j^{th} coordinate. Then since $w_1 \neq 0$, $\pi_1|_N: N \rightarrow W$ is an isomorphism. Therefore $\pi_j \circ [\pi_1|_N]^{-1}: W \rightarrow W$ belongs to $\text{End}_{FG}(W) = F \cdot 1_W$ since F is a splitting field, i.e. $\pi_j \circ [\pi_1|_N]^{-1}$ is scalar multiplication. But $\pi_j \circ [\pi_1|_N]^{-1}(w_1) = w_j \in Fw_1$, i.e. there exists $a_j \in F$ such that $a_j w_1 = w_j$. We have shown, then, that there exists $1 = a_1, a_2, \dots, a_d \in F$ such that $(w_1, \dots, w_d) = (a_1 w_1, a_2 w_1, \dots, a_d w_1) \in W_a$.

Now suppose $W_a = W_{a'}$, where $a = (a_1, \dots, a_d)$ and $a' = (a'_1, a'_2, \dots, a'_d)$. Since $W_a = W_{a'}$, for any $w \in W$ there exists $w' \in W$ such that $(a_1 w, a_2 w, \dots, a_d w) = (a'_1 w', a'_2 w', \dots, a'_d w')$.

So if $0 \neq a_i w$, then $a_i w = a_i' w' \Rightarrow w' = (a_i / a_i') w = b w$, say.
Choose $\lambda = b$.

On the other hand, if $a = \lambda a'$ for some $\lambda \in F$ and if $(a_1 w, a_2 w, \dots, a_d w) \in W_a$, then $(\lambda a_1' w, \lambda a_2' w, \dots, \lambda a_d' w) = (a_1 w, a_2 w, \dots, a_d w) \in W_a$. But $(a_1'(\lambda w), a_2'(\lambda w), \dots, a_d'(\lambda w)) \in W_a$. Therefore, $W_a \subseteq W_a$. By the irreducibility of W_a , $W_a = W_a$. This completes the proof of the theorem.

As an easy consequence of this we get

Corollary 3.5. If F is a finite splitting field for G and V is a homogeneous FG -module containing d isomorphic constituents of an irreducible submodule W , then there are exactly $\frac{|F|^d - 1}{|F| - 1}$ distinct irreducible submodules of V .

Lemma 3.6. (Green) Let $N \trianglelefteq G$ and let V be an irreducible FG -module. Assume also that G/N is a p -group and $V|_N$ is homogeneous. Then $V|_N$ is irreducible.

Proof: Since V is irreducible and $N \trianglelefteq G$, V_N is completely reducible into (conjugate) $F[N]$ -modules by Clifford's theorem. Let $V_N = W_1 \oplus \dots \oplus W_d$ where all the W_i are isomorphic as $F[N]$ -modules, and $W_i = W_1 \otimes g$ for some $g \in G$. Since $V_N = \sum_{g \in G} W_1 \otimes g$ (not necessarily a direct sum), G acts on the irreducible submodules of V_N . But all the irreducible submodules of V_N are $F[N]$ -modules, so that G/N acts on the set of all irreducible submodules of V_N . Suppose that under this action no orbit is of size 1. Then, since the size of an

orbit divides the order of the group and since G/N is a p -group, all orbits will have size a power of p . This implies that the number of irreducible submodules of V_N is divisible by p , contradicting Corollary 3.5. Therefore some orbit has size 1 and some irreducible submodule, W_a , is fixed by G . Thus, W_a is an irreducible FG -module contained in V . Thus $W_a = V$ by the irreducibility of V . This proves the lemma.

Lemma 3.7. The multiplicity of T as a composition factor of F^G equals the multiplicity of T_K as a composition factor of $(F^G)_K$.

Proof: Let $0 = V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_N = F^G$ be a composition series for F^G and let $0 = U_1 \subsetneq U_2 \subsetneq \dots \subsetneq U_m = (F^G)_K$ be a composition series for $(F^G)_K$ refined from the composition series for F^G . Then either $\frac{U_{i+1}}{U_i} \cong \frac{V_{j+1}}{V_j} |_K$ for j or $\frac{U_{i+1}}{U_i}$ is a direct summand of $\frac{V_{j+1}}{V_j} |_K$ by Clifford's theorem. Suppose $\frac{V_{i+1}}{V_i} \cong T$ for some i . Then by hypothesis $\frac{V_{i+1}}{V_i} |_K \cong T_K$ remains irreducible so that for each T which occurs as a composition factor of F^G , there exists a composition factor of $(F^G)_K$ isomorphic to T_K . We now show that this is the only way T_K can occur as a composition factor of $(F^G)_K$.

So suppose $\frac{V_{i+1}}{V_i} \not\cong T$. Then there are two cases to consider; either $\frac{V_{i+1}}{V_i} |_K$ is irreducible or $\frac{V_{i+1}}{V_i} |_K$ is completely reducible into more than one constituent. First of

all if $\frac{V_{i+1}}{V_i} \downarrow_K$ is irreducible then it cannot be isomorphic to T_K by Lemma 3.3. Secondly, if $\frac{V_{i+1}}{V_i} \downarrow_K$ is completely reducible into more than one constituent then $\frac{V_{i+1}}{V_i} \downarrow_K$ cannot be homogeneous by Lemma 3.6. Thus there are at least two non-isomorphic conjugate constituents of $\frac{V_{i+1}}{V_i} \downarrow_K$. Hence, T_K cannot be one of the constituents since T_K is invariant under G and the constituents are not. This shows that T_K occurs as a composition factor of $(F^G)_K$ precisely the same number of times as T occurs in F^G , which was to be proved.

Lemma 3.8. (Frobenius, see [2]) Let $H \leq G$, W an FH -module, and V and F^G -module. Then $\text{Hom}_{F^G}(W^G, V) \cong_F \text{Hom}_{FH}(W, V_H)$.

Theorem 3.9. $\dim_F T = \dim_F (\text{Hom}_{FP}(F, T_P))$.

Proof: By the Frobenius relation we know $\text{Hom}_{FP}(F, T_P) \cong \text{Hom}_{FG}(F^G, T)$. Hence it suffices to prove $\dim_F T = \dim_F (\text{Hom}_{FG}(F^G, T))$. By Lemma 3.7 we have $\dim_F (\text{Hom}_{FG}(F^G, T)) = \dim_F (\text{Hom}_{FK}(F^G \downarrow_K, T_K))$. But K is a p' -group and by assumption F is a splitting field of characteristic p for G , so that $\dim_F (\text{Hom}_{FK}(F^G \downarrow_K, T_K)) = \dim (\text{Hom}_{FK}(FK, T_K)) = \dim_F T_K$ by Lemma 3.2 and the fact that in the regular representation of a semi-simple group algebra each constituent occurs as often as its dimension. Hence $\dim_F T = \dim_F T_K = \dim_F (\text{Hom}_{FP}(F, T_P))$ which proves the theorem.

Now we arrived at Theorem 3.9 under the assumptions that F^G was completely reducible as an FG -module, that T was an irreducible $F[g]$ -module, and that T_K was an irreducible, G -invariant FK -module. In the following example we will have all of these hypotheses holding except we will not know whether F^G is completely reducible or not. We will then show that Theorem 3.9 cannot hold in our example thus proving F^G is not completely reducible.

Theorem 3.10. An extension of a θRC -group by a θRC -group is not necessarily a θRC -group.

Proof: Let Q be the group of quaternions generated by i, j, k . Then Q is a group of order 8 and there exists a cyclic group of order 3 acting on Q by permuting i, j, k . That is, there exists $\langle g \rangle = \{1, g, g^2\}$ such that $g^{-1}ig = j$, $g^{-1}jg = k$, $g^{-1}kg = i$ in the holomorph of Q .

Let G be the extension of Q by $\langle g \rangle$ in the holomorph of Q and let F be a finite splitting field for G of characteristic 3. (E.g. F could be the field of 9 elements containing a fourth root of unity $\sqrt{-1}$.) Then Q is equivalent to K in the above discussion and $\langle g \rangle$ is equivalent to the p -Sylow P .

We construct a representation ρ of G by defining

$$\rho(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \rho(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(g) = \begin{pmatrix} -\sqrt{-1}+1 & \sqrt{-1}-1 \\ -\sqrt{-1}+1 & \sqrt{-1}+1 \end{pmatrix}.$$

One can easily check that $\rho(i)^4 = \rho(j)^4 = \rho(g)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and that $\rho(g^{-1}ig) = \rho(j)$. Moreover, we have $\det \rho(g) = 1$ and $\text{tr } \rho(g) = -1$. Hence, 1 is the only eigenvalue of the matrix for g . Therefore there is a non-singular matrix M such that

$$M^{-1} \begin{pmatrix} \sqrt{-1} + 1 & \sqrt{-1} - 1 \\ \sqrt{-1} + 1 & \sqrt{-1} + 1 \end{pmatrix} M = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and it can be checked that $a \neq 0$. Hence there is only one eigenvector corresponding to the eigenvalue 1.

Now let $\lambda \in \text{Hom}_{\text{FP}}(F, T_p)$ where T is the irreducible module corresponding to ρ . Since λ is an FP-map, $\lambda(1) = \lambda(1 \cdot g) = \lambda(1) \cdot g$ so that there is a correspondence between $\lambda \in \text{Hom}_{\text{FP}}(F, T_p)$ and the eigenvectors corresponding to the eigenvalue 1. Therefore, $1 = \dim_F(\text{Hom}_{\text{FP}}(F, T_p)) \neq 2 = \dim_F T$. This shows that Theorem 3.9 fails to hold.

To see that ρ and $\rho|_K$ are indeed irreducible representations we observe that if ρ were reducible then there would be a non-singular matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A\rho(i) = \rho(i)A$ and $A^{-1}\rho(j)A = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. But

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow b = c = 0.$$

Then $A^{-1}\rho(j)A = \begin{pmatrix} 1/a & 0 \\ 0 & 1/d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & a/d \\ -d/a & 0 \end{pmatrix} \neq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}.$

Finally, we must verify that T_K is a G -invariant module.

That is, we must show that $T_K \otimes_{FK} g \cong T_K$ and $T_K \otimes_{FK} g^2 \cong T_K$. In terms of representations we must show that ρ_K is equivalent to the representations ρ_K^g and $\rho_K^{g^2}$ where $\rho_K^g(x) = \rho_K(g^{-1}xg)$.

To prove that ρ_K and ρ_K^g are equivalent we must produce a matrix T such that $\rho_K^g(k) = \rho_K(g^{-1}kg) = T^{-1}\rho_K(k)T$ for all $k \in K$. We claim that $T = \begin{pmatrix} \sqrt{-1} + 1 & \sqrt{-1} - 1 \\ \sqrt{-1} + 1 & \sqrt{-1} + 1 \end{pmatrix}$ makes ρ_K and ρ_K^g equivalent, and we omit all the tedious matrix multiplications involved in the proof of this fact. Similarly ρ_K and $\rho_K^{g^2}$ are equivalent using $T = \begin{pmatrix} 1 + \sqrt{-1} & 1 + \sqrt{-1} \\ -1 + \sqrt{-1} & 1 - \sqrt{-1} \end{pmatrix}$.

We have shown, then, that T and T_K are FG and FK modules respectively with all of the required properties. Hence by the remark prior to this theorem, F^G cannot be completely reducible and therefore $P = \langle g \rangle \notin \mathcal{C}$. But P is a p -Sylow subgroup, so clearly $P \in \theta$. Thus G is not a θRC -group.

We have actually shown

Corollary 3.11. An extension of a p' -group by a p -group is not necessarily a θRC -group.

Remark: The above example also shows that c) $\neq$ b) in Theorem 1.11 by taking $H = P = \langle g \rangle$.

CHAPTER IV

FURTHER RESULTS ON PROPERTY ρ

§4.1. Introduction

In this chapter we would like to give some results concerning group algebras over arbitrary rings. In Chapter One we saw that projective pairing depended on the group rather than the coefficients, i.e. the coefficients could come from an arbitrary commutative ring R with 1 . Also it seems likely that when talking about the complete reducibility of induced modules $N \otimes_{RH} RG$ the most general ring we will be able to use will have to be semi-simple artinian. However, the picture is not so clear with property ρ . For instance, it is not even known when $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$ implies $\text{Rad } KG \subseteq \text{Rad } KH \cdot KG$ for an arbitrary field extension K of F . Therefore, we would like to give conditions under which we do have property ρ with the coefficients coming from commutative rings.

In [11], Wallace proves that if F is a field of characteristic p and G is an abelian group with p -Sylow subgroup P , then $\text{Rad } FG \subseteq \text{Rad } FP \cdot FG$. Assuming G is a locally finite group, we will generalize Wallace's result to the cases where the coefficients come from a principal ideal

domain, a semi-perfect commutative ring, a semi-local commutative ring, and then finally an arbitrary F -algebra.

In [7] Passman shows that if F is a field of characteristic p , if G is a locally finite group, and if H is a normal subgroup of G with $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$ then $H \supseteq \bigcap^p (G)$ [see page 43]. In this direction we shall prove:

Theorem. Let G be a locally finite group satisfying $O_p(G) = 1$ for all primes p and let $\bigcap (G) \leq H \trianglelefteq G$. Then if $\text{Rad } FG \subseteq \text{Rad } F \bigcap (G) \cdot FG$ for all prime fields F , $\text{Rad } RG \subseteq \text{Rad } RH \cdot RG$ for any commutative semi-simple ring R .

We will need the following five known results.

Theorem 4.1. (Connell [1]) If R is a ring with identity then $\text{Rad } RG \cap R \subseteq \text{Rad } R$ with equality if either

- i) R is artinian, or
- ii) G is locally finite.

Theorem 4.2. (Woods [12]) Let R be a ring, G a group, and H a normal subgroup of G . If G/H is locally finite, then $\text{Rad } RH \subseteq \text{Rad } RG$.

Theorem 4.3. (Passman [6]) Let A be an algebra over a field K and let F be a field extension of K of finite degree, say $(F:K) = n$. Then $(\text{Rad}(F \otimes A))^n \subseteq F \otimes \text{Rad } A \subseteq \text{Rad}(F \otimes A)$.

Theorem 4.4. (Passman [6]) Let A be an algebra over a field K and let F be a field extension of K . Then $\text{Rad}(F \otimes A) \cap A \subseteq \text{Rad } A$.

Theorem 4.5. (Passman [6]) Let A be an algebra over a field K and let F be a purely transcendental field extension of K . Then $\text{Rad}(F \otimes_K A) = F \otimes_K (\text{Rad}(F \otimes_K A) \cap A)$. If in addition $F \neq K$, then $\text{Rad}(F \otimes_K A) \cap A$ is a nil ideal of A .

§4.2. Property ρ and locally finite abelian groups

We begin this section with the following theorem

Theorem 4.6. Let G be a locally finite abelian group and R a commutative ring with Jacobson radical J such that $\bar{R} = R/J$ has the descending chain condition. Let $H \leq G$ and suppose $\text{Rad } \bar{R}G \subseteq \text{Rad } \bar{R}H \cdot \bar{R}G$, then $\text{Rad } RG = \text{Rad } RH \cdot RG$.

Proof: Let $x \in \text{Rad } RG$. We can write $x = \rho_1 \cdot 1 + \rho_2 g_2 + \dots + \rho_n g_n$ where the g_i belong to a set of coset representatives for H in G . Let W be the finite group generated by $\text{supp } \rho_1$. Let $\varphi: RG \rightarrow \bar{R}G$ be the natural map. Then $\varphi(x) \in \text{Rad } \bar{R}G \subseteq \text{Rad } \bar{R}H \cdot \bar{R}G$, and $\varphi(\rho_1) \in \text{Rad } \bar{R}H \cdot \bar{R}G \cap \bar{R}W \subseteq \text{Rad } \bar{R}W$. Now $|W| = m < \infty$ and $\bar{R}$ has the descending chain condition so that $\text{Rad } \bar{R}W$ is nilpotent. Hence $(\rho_1)^m \in \ker \varphi = JG$. But by Theorem 4.1, $J \subseteq \text{Rad } RG$ since G is locally finite. Hence $JG \subseteq \text{Rad } RG$.

Now $\rho_1^m \in \text{Rad } RG$ or ρ_1^m is in every maximal ideal of RG . But maximal ideals are prime ideals in commutative rings and therefore ρ_1 is in every maximal ideal. Thus, $\rho_1 \in \text{Rad } RG \cap RH \subseteq \text{Rad } RH$. Similarly we can show all the $\rho_i \in \text{Rad } RH$. This proves $\text{Rad } RG \subseteq \text{Rad } RH \cdot RG$. The reverse inclusion holds by 4.2.

Remark: In the above theorem the only place we used the fact that G was abelian was in the claim that in commutative rings, maximal ideals are prime. If RG was a ring for which this condition is satisfied, then we can drop the hypothesis that G is abelian.

Using the same notation for $\bar{R}$ as above we get the immediate corollary:

Corollary 4.7. Let R be either a principal ideal domain, a commutative semi-perfect ring, or a commutative semi-local ring, and let G be a locally finite abelian group. If for any $H \leq G$, $\text{Rad } \bar{R}G \subseteq \text{Rad } \bar{R}H \cdot \bar{R}G$, then $\text{Rad } RG = \text{Rad } RH \cdot RG$.

The next theorem not only plays an important role in the proof of our final result but also yields as a corollary the desired generalization of Wallace's theorem.

Theorem 4.8. Let R be any ring and let H be a locally finite normal subgroup of a group G . Let $\{I_\nu \mid \nu \in \Gamma\}$ be a family of ideals of R such that

- i) $\cap I_v = 0$,
- ii) $R/I_v = R_v$ has the descending chain condition
 $\forall v \in \Gamma$,
- iii) $\text{Rad } R_v G \subseteq \text{Rad } R_v H \cdot R_v G$.

Then $\text{Rad } RG = \text{Rad } RH \cdot RG$.

Proof: Let $\{g_j | j \in J\}$ be a set of coset representatives for H in G . Then for any $x \in RG$ we can write $x = \sum \rho_j g_j$ where $\rho_j \in RH$. Let $\pi: RG \rightarrow RH$ be the projection map, i.e. $\pi(x) = \rho_1$ where we are assuming $g_1 = 1$. Then $\pi(\text{Rad } RG)$ is an ideal of RH . We'll show that $\pi(\text{Rad } RG)$ is in fact a nil ideal of RH . Let $x \in \text{Rad } RG$ and let W be the finite group generated by $\text{supp } \pi(x)$ in H . Let $\varphi_v: RG \rightarrow R_v G$ be the canonical map. Since $\varphi_v(x) \in \text{Rad } R_v G \subseteq \text{Rad } R_v H \cdot R_v G$ we have $\varphi_v(\pi(x)) \in \text{Rad } R_v H \cdot R_v G \cap R_v W \subseteq \text{Rad } R_v W$. But $|W| = n < \infty$ and R_v has the descending chain condition so that $\text{Rad } R_v W$ is nilpotent. Hence $[\pi(x)]^n \in \text{Ker } \varphi_v \forall v$. Therefore $[\pi(x)]^n \in \cap I_v G = 0$ and $\pi(\text{Rad } RG)$ is a nil ideal in RH . Hence $\rho_1 \in \text{Rad } RH$. Now suppose ρ_l occurs in the sum $\sum \rho_j g_j = x$. Then xg_l^{-1} is another element of $\text{Rad } RG$ and repeating the above argument with x replaced by xg_l^{-1} we get $\pi(xg_l^{-1}) = \rho_l \in \text{Rad } RH$. Thus $\text{Rad } RG \subseteq \text{Rad } RH \cdot RG$ which was to be proved. The reverse inclusion follows from Theorem 4.2.

Corollary 4.9. Let G be a locally finite abelian group with p -Sylow subgroup P . Also assume R is either a principal ideal domain, a commutative semi-perfect ring, or a commutative semi-local ring of characteristic p . Then $\text{Rad } RG = \text{Rad } RP \cdot RG$.

Proof: Let $\bar{R} = R/\text{Rad } R$ and let $\{I_\nu\}$ be the collection of maximal ideals of $\bar{R}$. Then hypotheses (i) and (ii) of the theorem are easily satisfied. Since R/I_ν are fields of characteristic p , hypothesis (iii) is satisfied by Wallace's theorem. Hence the theorem yields $\text{Rad } \bar{R}G \subseteq \text{Rad } \bar{R}P \cdot \bar{R}G$, and therefore $\text{Rad } RG \subseteq \text{Rad } RP \cdot RG$ by Theorem 4.6. The reverse inclusion holds by Theorem 4.2 again.

More generally we have

Theorem 4.10. Let G be a locally finite group and H a normal subgroup of G such that G/H has no elements of order p^α if $\text{char } F = p$. Let R be an arbitrary commutative K -algebra. Then $\text{Rad } RG = \text{Rad } RH \cdot RG$.

Proof: Let $x = \sum_{i=1}^n r_i g_i \in \text{Rad } RG$ where $r_i \in R$ and $g_i \in G$. Let K be the subgroup of G generated by H and the g_i , $i = 1, \dots, n$. Then $x \in \text{Rad } RG \cap RK \subseteq \text{Rad } RK$. But K/H is finite since G is locally finite and $[K:H]$ is not divisible by p since G/H has no elements of order p^α . Therefore, by Theorem 1.8, (RK, RH) is a projective pairing. Applying Theorem 3.3 of [9], (RK, RH) has property ρ ; i.e. $\text{Rad } RK \subseteq \text{Rad } RH \cdot RK \subseteq \text{Rad } RH \cdot RG$. Hence $x \in \text{Rad } RH \cdot RG$ as desired. Conversely, $\text{Rad } RH \cdot RG \subseteq \text{Rad } RG$ by Woods' result.

Thus we have the promised generalization of Wallace's theorem:

Corollary 4.11. Let F be a field of characteristic p and let G be a locally finite abelian group with p -Sylow subgroup P . Then for any commutative F -algebra R , $\text{Rad } RG = \text{Rad } RP \cdot RG$.

§4.3. Property ρ and the subgroup $\int^p(G)$

In this section we would like to consider the case where the group G is not necessarily abelian. We first need a few preliminary definitions.

Definition 4.12. If G is a group then $O_p(G)$ is the maximal normal p -subgroup of G .

Definition 4.13. If G is a locally finite group and A is a subgroup of G , then A is locally subnormal in G if

- i) A is finite, and
- ii) A is subnormal in all finite subgroups of G containing it.

In [7] Passman defines the characteristic subgroups

$$\int(G) = \langle A \mid A \text{ is locally subnormal in } G \rangle \text{ and}$$

$$\int^p(G) = \left\langle A \mid \begin{array}{l} A \text{ is locally subnormal in } G \text{ and} \\ A \text{ is generated by elements of order } p^\alpha \end{array} \right\rangle.$$

These subgroups turned out to have many interesting properties in the study of the radical of a group ring over a locally finite group. For instance, in [7] Passman proved

Theorem 4.14. If K is a field of characteristic $p > 0$ and if G is a locally finite group with $O_p(G) = 1$, then $\text{Rad } K[\int^p(G)] \cdot KG$ is a semi-prime ideal.

He also showed

Theorem 4.15. If G is a locally finite group and if $H \trianglelefteq G$ such that $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$, then $H \supseteq \int^p(G)$ where $\text{char } F = p > 0$.

Moreover, there are no known examples of locally finite groups G and normal subgroups H where $\text{Rad } FG \subseteq FH \cdot FG$ and H does not contain $\int^p(G)$. Therefore, it makes sense to study subgroups H containing $\int^p(G)$. With this background we are ready to prove the main result of this section.

Theorem 4.16. Let K be a field of characteristic $p > 0$. Let G be a locally finite group and $\int^p(G) \subseteq H \trianglelefteq G$. Also assume $O_p(G) = 1$ and $\text{Rad } KG \subseteq \text{Rad } K[\int^p(G)] \cdot KG$. If R is a semi-simple K -algebra, then $\text{Rad } RG = \text{Rad } RH \cdot RG$.

Proof: First note that it suffices to show that $\text{Rad } RG \subseteq \text{Rad } R[\int^p(G)] \cdot RG$. This is so by virtue of Theorem 4.2 and the fact that $\int^p(G)$ is a characteristic subgroup of G and therefore a normal subgroup of H .

Let F be a field extension of K of degree n . By Theorem 4.3 we have $(\text{Rad } FG)^n \subseteq F \otimes_K \text{Rad } KG \subseteq F \otimes_K \text{Rad } K[\int^p(G)] \cdot KG \subseteq \text{Rad } F[\int^p(G)] \cdot FG$. Since $F[\int^p(G)] \cdot FG$ is a semi-prime

ideal, $\frac{FG}{\text{Rad } F[\int^P(G)] \cdot FG}$ has no nilpotent ideals. Therefore $\text{Rad } FG \subseteq \text{Rad } F[\int^P(G)] \cdot FG$.

Now let F be a purely transcendental extension of K . By Theorems 4.4 and 4.5, $\text{Rad } FG = F \otimes_K \text{Rad } FG \cap KG \subseteq F \otimes_K \text{Rad } KG$. Now by hypothesis $\text{Rad } KG \subseteq \text{Rad } K[\int^P(G)] \cdot KG$ so again we have $\text{Rad } FG \subseteq F \otimes_K \text{Rad } K[\int^P(G)] \cdot KG \subseteq \text{Rad } F[\int^P(G)] \cdot FG$.

Finally, let F be an arbitrary field extension of K . If $x \in \text{Rad } FG$, then there exists a field L with $K \subseteq L \subseteq F$ such that L is finitely generated over K , and $x \in LG$. Using Theorem 4.4 again we have $x \in \text{Rad } FG \cap LG \subseteq \text{Rad } LG$. But L is a finite extension of a purely transcendental extension of K so that the first two steps yield $x \in \text{Rad } LG \subseteq \text{Rad } L[\int^P(G)] \cdot LG \subseteq F \otimes_L \text{Rad } L[\int^P(G)] \cdot LG \subseteq \text{Rad } F[\int^P(G)] \cdot FG$.

Thus, $\frac{KG}{\text{Rad } K[\int^P(G)] \cdot KG}$ is a classically separable K -algebra. Since R is assumed to be semi-simple $R \otimes_K \frac{KG}{\text{Rad } K[\int^P(G)] \cdot KG}$ is semi-simple. Therefore, $\text{Rad } RG \subseteq R \otimes_K \text{Rad } K[\int^P(G)] \cdot KG \subseteq \text{Rad } R[\int^P(G)] \cdot RG$ and this completes the proof of the theorem.

Corollary 4.17. Let G be a locally finite group with $O_p(G) = 1$ for all primes p and let $\int(G) \leq H \trianglelefteq G$. Suppose that for all prime fields K of characteristic $p > 0$, $\text{Rad } kG \subseteq \text{Rad } K[\int^P(G)] \cdot KG$. Then for any semi-simple commutative ring R , $\text{Rad } RG = \text{Rad } RH \cdot RG$.

Proof: First note that $\int(G)$ contains $\int^p(G)$ for any prime p by definition. Therefore by the proof of the theorem we have $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$ for any field F of characteristic $p > 0$. If F is a field of characteristic zero then FG is semi-simple by 18.7 of [6] so trivially $\text{Rad } FG \subseteq \text{Rad } FH \cdot FG$. Now take $\{I_\nu \mid \nu \in \Gamma\}$ to be the collection of all the maximal ideals of R . Then all the hypotheses of Theorem 5.8 are satisfied and the result follows.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Connell, I.G., On the Group Ring, Canadian Journal of Mathematics, Vol. 15 (1963), pp.650-685.
2. Curtis, C., and Reiner, I., Representation Theory of Finite Groups and Associative Algebras, Interscience Publishers, 1962.
3. Gustafson, W.H., Remarks on Relatively Projective Modules, Mathematics Japonicae, Vol. 16, No. 1, 1971, pp.21-24.
4. Higman, D.G., Indecomposable Representations at Characteristic p, Duke Mathematical Journal, 21 (1964), pp.377-381.
5. Khatri, D.C., Relative Projectivity, the Radical and Complete Reducibility in Modular Group Algebras, Transactions American Math Society, 186, 1973.
6. Passman, D.S., Infinite Group Rings, Marcel Dekker, Inc., New York, 1971.
7. Passman, D.S., Radical Ideals in Group Rings of Locally Finite Groups, to appear.
8. Rotman, J.J., The Theory of Groups, An Introduction, Allyn and Bacon, Inc., 1971.
9. Sinha, I., and Srivastava, J., Relative Projectivity and a Property of the Jacobson Radical, Publications Mathematicae, 1971, pp.37-41.
10. Srinivasan, B., On the Indecomposable Representations of a Certain Class of Groups, Proceedings London Math Society, 10, 1960, pp.497-513.
11. Wallace, D.A.R., The Jacobson Radicals of the Group Algebras of a Group and of Certain Normal Subgroups, Mathematische Zeitschrift, Vol. 100 (1967), pp.282-294.
12. Woods, S.M., Some Results on Semi Perfect Group Rings, Canadian Journal of Mathematics, Vol. 26, No. 1, 1974, pp.121-129.

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