INVISCID COMPRESSIBLE FLUID FLOW IN A CURVED MEMBRANE SHELL

Dissertation for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

ROBERT JENNINGS WALDRON, JR.

1974

This is to certify that the

thesis entitled

INVISCID COMPRESSIBLE FLUID FLOW IN
A CURVED MEMBRANE SHELL

presented by

Robert Jennings Waldron, Jr.

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Charles J. Martin
Major professor

Date September 20, 1974

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ABSTRACT

INVISCID, COMPRESSIBLE FLUID FLOW IN A CURVED MEMBRANE SHELL

By
Robert J. Waldron, Jr.

A thin, membrane, cyclindrical shell of radius "a" is coiled into a torus of radius "R". The shell is filled with a compressible, inviscid fluid which is flowing axisymmetrically as it enters the coil. The Navier Stokes equations are used to describe the fluid motion while the dynamic form of Sanders' shell equations describes the motion of the membrane shell. All variables are expanded in a series about δ small, $\delta = aR^{-1}$. Wave solutions are sought giving variables the general form

$$X(\psi, \theta, r, t) = \{X_O(\psi, r) + \delta X_1(\psi, r) + \ldots\} \exp\{i(s\,\theta\,\delta^{-1} - \beta t)\}$$

where ψ and r form polar coordinates in any cross section of the tube, θ is the angle around the torus, β is the frequency and

$$s = s_O + \delta s_1 + \delta^2 s_2 + \ldots$$

is the real valued wave number.

The object of the analysis is to find the first non-zero correction to the frequency equation $s_O = f(\beta)$ where we interpret the zero order terms as representing the straight tube flow. The work revealed that $s_1 = O$ forcing the determination of the second order correction $s_2 = g(s_O, \beta)$.

The correction, s_2, has been determined in closed form for the special cases of wave motion of a fluid in a rigid tube, the free flowing jet of fluid, the vibrations of an empty tube, and the interaction problem of a fluid flowing in an elastic shell. The corrected phase velocity is compared to the straight tube phase velocity for all four cases with comments made on any significant differences.

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Robert J. Waldron, Jr.

A special case is examined in which the first correction, s_1, is not zero followed by a comparison of the present work with experimental findings. Finally comments are made on further areas of study.

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INVISCID COMPRESSIBLE FLUID FLOW IN A CURVED MEMBRANE SHELL

By

Robert Jennings Waldron, Jr.

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Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1974

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to my wife for her patience and support.

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ACKNOWLEDGMENTS

I would like to give my thanks to Mary Reynolds and Kathy Trebilcott who bore the burden of manuscript preparation. Special thanks must go to Jeff Gorney for the plots and graphs. My appreciation must also be extended to the Department of Mathematics for their support during this process.

Finally, I must indicate my appreciation to Professor Charles Martin who guided and encouraged me through the completion of this degree. His efforts were never ending. Again, many thanks.

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CHAPTER I

INTRODUCTION

In recent years there has been much work completed relating the use of applied and engineering mathematics to problems originating out of biological or medical situations. Examples include the use of stress analysis in the study of orthopedic protheses, the development of artificial heart valves and the design of electronic equipment for determining various parameters in the blood. All of these developments signal the arrival of mathematics and engineering in basic biomedical problems.

This thesis is concerned with a theoretical problem which has its origin or motivation in a variety of biological situations, namely fluid flow in curved, elastic tubes. In section 1.1 we will discuss the basic biological facts which are pertinent to our problem. In section 1.2 we will concern ourselves with a description of the mathematical analysis to be employed in the thesis.

Section 1.3 contains an outline of the development of the present problem, tracing the improvement and variation of analyses employed in studying our situation. Finally, in section 1.4 we will preview the analysis to be used in the remaining chapters of the thesis.

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1.1 Biological Considerations

Our concern in this thesis is with a fundamental problem common to a variety of biological fluid flows, namely wave propagation in fluid-filled, elastic tubes. An obvious example is the circulation of blood. Other examples include the peristaltic motion of the urinary tract (particularly in regard to reflux in small children) as well as the movement of fluid in the semi-circular canals of the inner ear.

Needless to say, there is a wealth of effort being directed toward various aspects of these larger problems. Since the physical situations are so complex, investigators are forced to consider one or two pertinent situations hoping to delineate their contribution to the total behavior of the system. Our concern will be with the particular problem of wave propagation in a curved, fluid-filled elastic tube, hoping to gain some understanding as to the importance of curvature in wave propagation. The motivation for this analysis springs from the circulatory system, particularly that portion containing the aortic arch.

We find in the mammalian circulation that blood is collected and returned via the veins to the right side of the heart. From here it is pumped into the lungs to have carbon dioxide removed and oxygen added. From the lungs blood is collecte

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the blood is collected in the left chambers of the heart and ejected periodically into the rest of the ciruclation. The first vessel to receive the ejected blood is the aorta which curves upward to the left and then downward again. The curvature is rather pronounced and the aorta exhibits branching midway through the curve. In this region of the aorta, called the aortic arch, the diameter of the vessel is from 1 cm. to 3 cm. and the radius of curvature is 3 cm. to 4 cm. It is apparent that the curvature here is rather pronounced.

Most efforts at modeling blood flow have followed more conventional engineering approaches, that is, the models have been based on a variety of straight tube conditions. It is apparent that some attention should be given to the problem of curvature, if only to determine to what extent straight tube models effectively forecast the behavior of the curved flows.

The physical parameters essential to modeling such a problem are known in physiological circles, but they will be detailed in part for the sake of completeness. The density of blood vessel walls is a variable ranging from 1.0 to 1.2g/cm^3. Since the density of blood itself is nearly the same it is reasonable to conclude that the ratio of desities is one.

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Elastic deformations take place at nearly constant volume so Poisson's ratio can be taken as .5. In addition the tube wall is composed of various fibers so that the elastic modulus may vary, depending on the degree to which the wall is stretched. Certain fibers are not stretched until high internal pressure is applied so that in the femoral artery of the dog the value of Young's modulus may vary from 1.2×10^{6} to 20.4×10^{4} dynes/cm^2.

The blood is also a composite, being a suspension of various cells in plasma. The cells form 45% to 50% of the whole blood volume and contribute to the viscosity. For flow in small diameter vessels, the viscosity is variable, while in vessels which are larger, the viscosity will not be altered significantly. For the problems we will consider, the viscosity will be treated as a constant, making the flow Newtonian.

There is still much discussion concerning the sizes of these various parameters. The difficulties accompanying experimental determination of these parameters are manifold since excising a sample of tissure from the living subject alters its condition. On the other hand, examination within the subject restricts the experimenter to only those vessels and tissues which are readily accessible. As can be seen there is great need for experimental techniques which can better cope with the problems inherent to testing living subjects. Further information details of th

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1.2 Wave Propagation Analyses

Since the intent of this work is to study the propagation of waves in a curved, fluid filled tube, we pause to consider some of the basic approaches to the study of waves propagating through a medium. The first step is to consider the propagation of waves in a medium which is vibrating freely. This means that we find the medium in a state of motion without benefit of any external force to maintain the vibration. We assume the movement of the vibration waves in some direction and assume harmonic motion in this direction. Hence we might write all our variables in separated form, the wave portion denoted by $\exp\{i[kz - \omega t]\}$ where z is the direction of propagation, k is a wave number related to the inverse of the wave length, ω is the frequency of vibration and the ratio $\omega/\mathrm{Re}(k)$ [RE denoting Real part] is called the phase velocity associated with the wave.

The substitution of the wave form given above into the equations of motion for the system results in an equation $k = f(\omega)$ which we call the frequency or dispersion equation. From this relation we can tell the wave number, wave length and phase velocity once we know at what frequency, ω, the system is vibrating.

If the problem is such that the direction of propagation, e.g. z, is not limited, we find that $k = f(\omega)$ gives a continuous representation for k in terms of ω.

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It may develop that under certain conditions, e.g. long waves or small wall inertia, the relation $k = f(\omega)$ reduces to a particular form which we call a mode of vibration. This mode characterizes the vibration we find under those certain conditions, i.e. long waves or small wall inertia, and is a special case of the larger situation given by $k = f(\omega)$.

Other problems associated with free vibrations are concerned with situations in which the direction of vibration is finite, say along a string of length L. Imposing boundary conditions at the ends of the string leads to a situation where the relationship between k and ω exists at an integral number of points. A solution to a problem like this, e.g. $y,_{zz} = c^{-2}y,_{tt}$ with ends L units apart and fixed, is given by

$$y(x,t) = \sum_{n=1}^{\infty}[A_n \cos(n\pi cL^{-1}t) + B_n \sin(n\pi cL^{-1}t)] \quad \sin n\pi L^{-1}z.$$

For each integer n the term inside the sum is a solution and superposition allows us to say the sum is a solution given the proper convergence.

If now we are to consider a system in which there is a function forcing the vibration to continue, then we would expand the forcing term in a Fourier series like that above and solve a series of problems in which the solution also has such an expansion.

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The problem under consideration in this work is that associated with the free vibrations of a medium which is unbounded in one direction. Here we find a continuous dependence of the wave number on the frequency. If the problem were to be altered later so as to have a finite domain in the direction of propagation we find only minor variations from the original problem. Love [2] discusses such a case in section 199 of his text. He finds the frequency relation for the longitudinal vibrations of an infinite circular cylinder. For the finite case the new frequency relation is satisfied by the original equation under the assumption that the length of the tube is long compared to its radius. Hence it is possible to relate the problem of an infinite medium to that of the finite case, which in turn gives rise to solutions of forced vibrations of the same system.

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1.3 History of the Problem

We begin the discussion by stating that the bulk of the analysis centers on various problems associated with vibrations of straight tubes. There are two basic approaches to the straight tube problem as outlined by Rudinger [3]. In one approach the problem is treated as being non-linear, but generally there is only one dimension studied. In the other approach the equations are linearized and wave solutions are sought. In general all three dimensions are studied in this approach. Our analysis will follow the latter procedure and so we leave the discussion of non-linear theories to the article by Rudinger as well as one by Skalak [4].

There are good physiological reasons for assuming a linear set of equations to describe the wave motion of a fluid in a tube. For blood flow in the aorta the average flow velocity is about 25 cm/sec while the pulse wave velocity may be 800 cm/sec. Next, the distension of the arteries, as measured by Poiseuille, is not large, being about 1.04 times normal values. Third, the blood, though non-Newtonian at the low shear rates of the capillaries can be treated as Newtonian in the faster motion associated with larger vessels. Finally, the tube wall, though non-linear, can be approximated by a linear, visco-elastic material if need be.

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So, we may conclude that a linear model although not a perfect assumption, does not grossly distort the system. The advantages are numerous as now we are able to find closed form solutions as well as utilize superposition of solutions which is essential to vibration analyses.

We begin our discussion of prior work with Thomas Young [5] who made one of the first attempts at a mathematical determination of the phase velocity associated with fluid filled tubes. Young's phase velocity can be written as

$$c_{p0} = (hE/2a\rho_0)^{1/2}$$

where h and E are the thickness and Young's modulus for the tube respectively, a is the tube radius and ρ_0 is the fluid density. Later, other investigators found this velocity after assuming the fluid to be inviscid, incompressible and the waves to have long wave length.

Korteweg [6] considered a compressible fluid and studied the velocity of sound in a fluid filled tube. He found his phase velocity to be

$$c_{p1} = \left(\frac{2a\rho_0}{hE} + \frac{\rho_0}{K}\right)^{-1/2} .$$

The new parameter, K, is the bulk modulus of the fluid and the ratio $(K/\rho_0)^{1/2}$ is the velocity of sound in the fluid. Hence Korteweg found that the phase velocity was less than the velocity of sound. For an incompressible fluid, becomes inf

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(K/ρ_0) becomes infinite and c_{p1} is then equivalent to c_{p0}, Young's mode.

Later, Lamb [7] studied the modes of free vibration for the linearized fluid in a membrane shell. He found two modes, one being c_{p0}, Young's mode, and the other given by

$$c_{p2} = (E/\rho(1-\nu^2))^{1/2}$$

where E is as before and ρ and ν are the tube density and Poisson's ratio for the tube respectively. This mode was different from the earlier modes in that it was associated with only tube parameters.

From this point the analyses became quite numerous with others improving Lamb's work by the addition of viscosity, tapering, branching, etc. Useful analogies were drawn to better known electrical concepts such as resistance and impedence. Electrical transmission line theory became a tool for some investigators.

In the 1950's, several reports, collected in one larger report, were written by J. R. Womersley [8]. His work was extensive and one of the most complete treatments of biological wave propagation. He examined an infinitely long rigid tube which had a region of steady pressure. He solved the linearized Navier Stokes equations and the continuity equation under the conditions of fluid viscosity, incompressibility and axial symmetry.

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Assuming traveling wave solutions Womersley found the pressure and two fluid velocities up to an arbitrary constant. He computed average flow velocity, quantity of flow and a phase velocity. His values for average flow and quantity of flow compared quite well with experimental results.

Womersley then proceeded to study the elastic tube filled with a viscous fluid. Under several simplifying assumptions, Womersley arrived at a frequency equation which gave the phase velocity directly. When the assumptions of Lamb were introduced, his value reduced to Lamb's mode given by c_{p2}. Lamb's mode is independent of the frequency of vibration but with viscous terms added, Womersley found the phase velocity of Lamb was altered significantly by chances in frequency.

In addition to the work mentioned, Womersley considered other effects: a thin boundary layer; added mass to the elastic tube; corrections for the non-linearity of the tube; and corrections for the approximate boundary conditions he used.

There were many aspects to Womersley's report and so his work prompted many new investigations, dealing with portions of the problems he discussed. Numerous refinements were made to his models with allowance being made for the pre-stressed condition of the tube and for its visco-elastic nature.

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One of the more complete analyses of the infinite fluid filled tube was advanced by Rubinow and Keller [9]. They considered a complex model involving a homogeneous, isotropic, elastic tube filled with a viscous, compressible fluid. The linearized elasticity equations and linearized equations of motion were used for the tube and fluid respectively and they allowed for a complex impedance matrix to describe various boundary conditions at the outer wall.

The main effort of Rubinow and Keller was extended to the discussion of axisymmetric vibrations of a thin walled elastic tube containing an inviscid fluid. The results we shall obtain in latter chapters duplicate many of their results, under slightly different conditions. For this reason we will discuss a few of the Rubinow and Keller results here.

For Poisson's ratio equal to zero Rubinow and Keller found that the tube had a mode in which the non-dimensional phase velocity was one. In dimensional terms it was found to be c_{p2}, Lamb's mode. They also found for Poisson's ratio equal to zero that there was another mode for the tube which did not exist for non-dimensional frequencies greater than one. This mode for small frequencies was merely the mode found by Korteweg, c_{p1}. Since they assumed a compressible fluid Rubinow and Keller found other modes which depended on the acoustic speed of the fluid. For small acoustic speeds these modes were similar to those of e tube. However

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the tube. However, for large acoustic speeds the acoustic modes were distinct from the tube modes of Lamb and Korteweg. As larger values of the acoustic speed were considered only the tube modes remained. Their paper considered various other problems connected with viscoelasticity of the tube wall as well as different boundary conditions.

We have presented a few of the many results which have been obtained in the study of linear theories. We will find later in the study of curved flows that the results of straight tube flows play an integral part. The reader is referred to the articles by Rudinger and Skalak (mentioned above) for further discussion of linear theories in straight tubes.

A feature common to most of these models is the assumption of a straight, cylindrical tube. However, it is apparent that there are important regions of fluid flows which are not straight. The situations of the aortic arch and the semi-circular canals of the inner ear mentioned earlier are particular examples. So we now discuss the work which has been done on curved tube flows.

One of the first analyses directed to the problem of curvature was put forth by Dean in two papers, [10], [11]. The object of Dean's work, as well as those who followed his lead, was to examine the secondary flow in the cross section of the curved tube, determining what effects were introduced by the curvature. This meant that the streamlines and particle paths were examined in the cross section.

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Dean began by studying the steady motion of a viscous, incompressible fluid flowing in a rigid torus. He examined the non-linear equations of fluid motion by expanding in a parameter $D = 2(Re)^2 a/R$ where Re is the Reynolds number, a is the tube radius and R is the radius of the torus. Dean analyzed his problem for the parameter D small. Subsequent investigations by McConalogue and Srivastava [12] and Barua [13] considered the size of D to be of moderate and large values respectively. A feature common to all of these works was the assumption of steady motion.

It was then the interest of others to investigate the unsteady motion in the cross section of a curved rigid tube. In the first work, by Lynne [14], we find an analysis of the unsteady, secondary motion in a cross section of a curved tube. The fluid is viscous, and subject to an applied pressure gradient. The fluid is assumed incompressible and the variables are all expanded in a series about the ratio (a/R) small, a and R taken to be the same as in Dean's analysis. Lynne utilized two basic parameters

$$\epsilon = W(a/R)^{1/2}/a\beta \quad \text{and} \quad R_s = W^2 a/\nu R\beta$$

where W is a typical velocity along the pipe, β is the frequency of applied pressure and ν is the kinematic viscosity of the fluid. The parameter ϵ is always small and R_s is a type of Reynolds number. Lynne found that

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when $2\epsilon^2/R_s$ is small, the viscous effects are confined to a boundary layer near the wall while the remainder of the fluid is inviscid.

Lynne found in his analysis, which involves matched asymptotic expansions between the boundary layer and core, that another boundary layer is formed at the edge of the first layer in addition to a free layer existing on the horizontal diameter of the pipe. For unsteady motion Lynne determined that the motion of the cross section is opposite to that predicted by steady motion, i.e. unsteady motion is from the outside of the curve to the inside.

In an analysis similar to Lynne's, Zalosh and Nelson [15] also studied pulsating flow in the curved tube. This study was part of the work done by Zalosh in his 1970 Ph.D. dissertation at Northeastern University. This work solved for the stream functions associated with the secondary flow in a cross section of the tube, the fluid being under the influence of a sinusoidally varying pressure gradient. As with Lynne's work, the tube was assumed to be rigid. All variables are expanded in a power series for the ratio (a/R). Then equations are found for the first term in the stream function expansion and for the first two terms in the expansion of the longitudinal velocity. Unfortunately, there seems to be a lack of consistency in their expansion of the differential operators, leaving one to suspect the accuracy of their equations for higher order terms in the expansion.

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The first term of the stream function expansion had two parts, one steady, the other varying at twice the frequency of the applied pressure gradient. The first term of the longitudinal velocity is found and these two first order terms together become inhomogeneous terms for the differential equation for the second term in the expansion of longitudinal velocity. As stated earlier some doubt is placed on the accuracy of this expansion. It is also unfortunate that the second term in the stream function was not computed since the zero order system taken by itself represents the straight tube, not the curved tube.

In all of the analysis for the curved tube presented to date no information had been gathered as to the influence of the wall on the motion, i.e. no work had been done to include an elastic wall in the model. In addition, these previous works concentrated on the streamlines and particle paths, therefore excluding wave propagation analyses.

In 1971 Daras [16] investigated wave propagation in a curved, fluid filled elastic tube. Acknowledging the difficulty associated with wave propagation in toroidal shells and flow in rigid tubes, Daras proposed to consider several salient features of the problem, hoping to shed some light on the essential aspects.

The first task Daras undertook was an expansion of the wave number k as

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Daras concluded that the effort involved in determining k_0 for a general problem was too great, since k_0 is the wave number associated with the straight tube problem and solving for k_0 involved equations for a fluid and a membrane. The next term in the expansion of k is interpreted as being a correction for the curvature and Daras was able to conclude that $k_1 = 0$ so that any corrections for the straight tube wave number k_0 are of order δ^2. He then found k_0 for some special cases (axisymmetric axial and pressure waves plus flexural waves) and obtained some agreement with the experimental work of Van Buskirk [17].

Daras then reviewed work previously completed on fluid filled curved tubes. He examined an axisymmetric straight tube in which curvature effects are introduced through a modified stress law, and effects of secondary flow are introduced as an equivalent viscous stress in the axial momentum equation. Finally, he considered basic problems associated with entrance effects in a curved tube. In all, he considered the highlights of several problems leaving more in depth analysis for others.

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1.4 Outline of Present Work

The present analysis proposes to investigate the first problem which Daras only touched upon, namely a determination of the zero order wave number and the first non-zero correction for wave propagation in a fluid-filled, curved elastic tube, coiled in a torus. We consider linearized Navier Stokes equations for a compressible fluid and for the tube we utilize the linear, membrane shell equations of Sanders [18] augmented with acceleration terms necessary for non-steady problems.

In Chapter 2 we begin by establishing the geometry and equations of motion for the conditions stated above. After non-dimensionalization we expand a typical variable as

$$X(\psi,\theta,r,t) = \{X_0(\psi,r) + \delta X_1(\psi,r) + \cdots\}\exp\{i[s\theta\delta^{-1}-\beta t]\}$$

where ψ and r are polar coordinates in a cross section of the tube, θ is a coordinate down the tube, t is the time variable, β is a non-dimensional frequency, $\delta = aR^{-1}$, a is the tube radius, R is the radius of the torus and $s = s_0 + \delta s_1 + \delta^2 s_2 + \cdots$ is the non-dimensional wave number. We note that if $\exp\{i[s\theta - \beta t]\}$ is used instead of the exponential mentioned above we obtain for our zero order problem a flow which is essentially two dimensional and which exists only in a plane cross section of the tube. This would be the expansion used, for example, by Lynne in studying secondary flow patterns.

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Inserting the expansion in δ for all variables and differential operators results in a series of problems labeled zero, first and second order accordingly as δ^0, δ^1 or δ^2 is a common term to the equation. The zero order equations are appropriate to straight tube flow and their solutions are the inhomogeneous terms of higher order equations. Chapter two contains all the equations and boundary conditions for the zero, first and second order problems.

In chapter three we consider the situation in which the tube is vibrating but without the fluid inside it. We solve the equations of motion appropriate to such a system and then determine a frequency equation written as $s_0 = f(\beta)$. This equation is the condition necessary for the existence of non-trivial displacements in the shell. We examine this frequency equation and then proceed to the determination of the first correction to s_0, namely s_1. We find, as did Daras, that $s_1 \equiv 0$ and so we determine the next correction, s_2.

A requisite for the value of s_2 is the value of all first order displacements. These we determine, as well as their effect on the displacements found in the zero order. We are then able to make some conclusions as to the effectiveness of the straight tube model in predicting the motion of a curved tube.

Chapter four contains an analysis of the system in which the fluid is of prime importance. This situation ices to two cas

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reduces to two cases. In the first case we consider the tube to be rigid and in the second case we consider the fluid to be flowing without the tube. In each situation we find a frequency relation by satisfying boundary conditions appropriate to the particular case. For the rigid tube we require that the fluid velocities vanish at the fluid-tube interface. For the fluid flowing without the container we demand that the fluid stresses vanish at the outer edge of the fluid medium.

Analysis of the two frequency equations indicates that the acoustic speed of the fluid is essential to the behavior of the waves. We find that as the acoustic speed increases the vibrations propagate at only high frequencies with the wave number being purely imaginary when the fluid is incompressible.

The remainder of chapter four is concerned with determining that $s_1 \equiv 0$, with the solution to the first order velocities and with the second order correction s_2. After determining each of these quantities we pause to examine their effect on lower order terms, commenting on the comparison between straight tube vibration and vibration which has been corrected for curvature.

In chapter five we move to a more difficult problem, namely the interaction problem in which the fluid and tube vibrate together. Utilizing the zero order fluid terms which we have found we solve for the shell displacements

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in terms of the fluid stresses acting of the shell. The boundary conditions require that we match the fluid velocities with the velocities of the shell. The result is a set of homogeneous equations and for a non-trivial solution leads to a frequency equation, $s_0 = f(\beta)$.

Analysis of this equation reveals that it is composed of those equations we investigated earlier in the empty tube, rigid tube and stress free cases. The new frequency equation is a transcendental equation and so we analyze it by expanding the wave number s_0, in several power series about the other frequency equations which we have previously investigated.

The next step is determining s_1, the first order fluid velocities, and shell displacements, and the second order correction, s_2. We examine the effects of these higher order terms on their zero order counterparts and are able to make some comments on straight tube and curved tube vibrations.

The first portion of chapter 6 is devoted to a special case which was not considered in the previous chapters. We have found a non-dimensional parameter, m, which appears in our equations and all our expansions about δ small have been based on the assumption that m is of order one compared to δ. We consider the very real possibility that m is $O(\delta)$ and then proceed to analyze the resulting equations. We find that, contrary to the earlier ation, the firs

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situation, the first order wave number, s_1, is not identically zero. Using this fact we find the remaining first order terms, i.e. fluid velocities and shell displacements, pointing out differences and similarities with earlier situations.

We conclude chapter six with some notes on recent experimental works which have dealt expressly with wave propagation in curved elastic tubes. We point out any similarities with our current efforts, as well as the differences. We then close the discussion with some possibilities for future consideration.

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CHAPTER II

MATHEMATICAL FORMULATION OF THE PROBLEM

2.1 Introduction

In this chapter the complete mathematical formulation of the thesis problem is presented. We begin in section 2.2 with equations which describe the motion of a thin elastic shell. These equations were originally derived by J.L. Sanders, Jr. [18]. We formulate them in a curvilinear coordinate system erected on the middle surface of the shell.

Section 2.3 contains the appropriate equations for a viscous, compressible fluid again written in terms of a curvilinear coordinate system. Section 2.4 is concerned with the specific geometry appropriate for final formulation of the equations. In section 2.5 the general shell and fluid equations presented in sections 2.2 and 2.3 are stated in terms of the specific geometry discussed in section 2.4. Concurrent with this discussion is the introduction of non-dimensional quantities for these equations.

Finally in section 2.6 we apply the perturbation expansion which is utilized in the remainder of the thesis. The first three problems resulting from the expansion are presented at this point also.

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2.2 Field Equations for the Thin Shell

2.2.1 Geometric Concepts

We assume the existence of a geometry on the middle surface of the shell specified by coordinates θ_1, θ_2 related to Cartesian coordinates by

$$x^i = x^i(\theta_1, \theta_2) \quad i = 1,2,3 \tag{2.2.1}$$

or in terms of the position vector $\bar{r}$ by

$$\bar{r} = \bar{r}(\theta_1, \theta_2) \tag{2.2.2}$$

where all quantities with a bar denote vectors.

An orthogonal triple of vectors is erected on the shell mid-surface by defining

$$\begin{aligned} \bar{e}_1 &= \alpha_1^{-1} \frac{\partial \bar{r}}{\partial \theta_1} & \alpha_1^2 &= \frac{\partial \bar{r}}{\partial \theta_1} \cdot \frac{\partial \bar{r}}{\partial \theta_1} \\ \bar{e}_2 &= \alpha_2^{-1} \frac{\partial \bar{r}}{\partial \theta_2} & \alpha_2^2 &= \frac{\partial \bar{r}}{\partial \theta_2} \cdot \frac{\partial \bar{r}}{\partial \theta_2} \\ \bar{e}_n &= \bar{e}_1 \times \bar{e}_2 . \end{aligned} \tag{2.2.3}$$

The element of arc length on the midsurface is given by

$$ds^2 = \alpha_1^2 \, d\theta_1^2 + \alpha_2^2 \, d\theta_2^2 . \tag{2.2.4}$$

The principal radii of curvature, R_1 and R_2, can be obtained from

$$\frac{\partial \bar{e}_n}{\partial \theta_1} = \alpha_1 R_1^{-1} \bar{e}_1 \qquad \frac{\partial \bar{e}_n}{\partial \theta_2} = \alpha_2 R_2^{-1} \bar{e}_2 . \tag{2.2.5}$$

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We let z denote a thickness variable with direction along the normal $\bar{e}_n$ with $-h/2 \leq z \leq h/2$ where h is the thickness of the shell. We say a shell is thin if $\max_R(\frac{h}{R}) \leq 1/20$ where R is a radius of curvature. Having described the geometry of the shell, we now proceed to the equations of motion for the shell.

2.2.2 Sanders' Thin Shell Theory - Membrane Equations

Sanders linear theory is based on the following three assumptions:

i) Straight fibers perpendicular to the middle surface of the shell before deformation remain perpendicular to the deformed middle surface.

ii) Normal stresses acting on planes parallel to the middle surface are neglected in comparison to other stresses.

iii) Membrane forces are much larger than bending forces.

With respect to our coordinates θ_i, we define components of the stress resultant tensor to be N_{ij} where the stress vector is acting on the line θ_i constant and in the direction of θ_j.

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(2.2.6) $$\bar{U} = U_1\bar{e}_1 + U_2\bar{e}_2 + U_3\bar{e}_n$$

while the body force vector $\bar{q}$ has components

(2.2.7) $$\bar{q} = q^{(1)}\,\bar{e}_1 + q^{(2)}\,\bar{e}_2 - q^{(3)}\,\bar{e}_n.$$

Denoting $\partial f/\partial\theta_j$ by f,j, $\partial f/\partial t$ by f,t and tube density by ρ the membrane equations of motion are written

(2.2.8) $$(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{12})_{,2} + N_{12}\alpha_{1,2} - N_{22}\alpha_{2,1} + \alpha_1\alpha_2 q^{(1)} = \rho h\alpha_1\alpha_2\, U_{1,tt}$$

(2.2.9) $$(\alpha_2 N_{12})_{,1} + (\alpha_1 N_{22})_{,2} + N_{12}\alpha_{2,1} - N_{11}\alpha_{1,2} + \alpha_1\alpha_2 q^{(2)} = \rho h\alpha_1\alpha_2\, U_{2,tt}$$

(2.2.10) $$-(N_{11}R_1^{-1} + N_{22}R_2^{-1})\alpha_1\alpha_2 - \alpha_1\alpha_2 q^{(3)} = \rho h\alpha_1\alpha_2\, U_{3,tt}$$

The stress resultants N_{ij} can be written in terms of strains ϵ_{ij}, Young's Modulus E, and Poisson's ratio ν, as

(2.2.11) $$N_{11} = \frac{Eh}{1-\nu^2}\,(\epsilon_{11} + \nu\epsilon_{22})$$

(2.2.12) $$N_{22} = \frac{Eh}{1-\nu^2}\,(\epsilon_{22} + \nu\epsilon_{11})$$

(2.2.13) $$N_{12} = \frac{Eh\,\epsilon_{12}}{1+\nu}\,.$$

The strains are then formulated in terms of displacements by

(2.2.14) $$\epsilon_{11} = \alpha_1^{-1}\,U_{1,1} + (\alpha_1\alpha_2)^{-1}U_2\alpha_{1,2} + U_3R_1^{-1}$$

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(2.2.15) $$\epsilon_{22} = \alpha_2^{-1} U_{2,2} + (\alpha_1\alpha_2)^{-1} U_1\alpha_{2,1} + U_3R_1^{-1}$$

(2.2.16) $$\epsilon_{12} = (2\alpha_1\alpha_2)^{-1}\{\alpha_2U_{2,1} + \alpha_1U_{1,2} - U_1\alpha_{1,2} - U_2\alpha_{2,1}\}.$$

The boundary conditions for this theory require that for θ_i = constant on an edge of the membrane, N_{ii} or U_i and N_{ij} or U_j $(i \neq j)$ be specified.

Substitution of the strain displacement relations (2.2.14), (2.2.15) and (2.2.16) via the stress resultants N_{ij} into the shell equations of motion (2.2.8), (2.2.9) and (2.2.10) produces three, coupled second order equations in terms of the three shell displacements and the three components of the body force $\bar{q}$. The actual substitution described above will be presented later, once non-dimensional variables and our particular geometry have been introduced.

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2.3 Field Equations for the Fluid

2.3.1 Geometric Concepts

The space occupied by the fluid will be described with a three dimensional set of orthogonal curvature coordinates $\{\varphi^j\}$ $j = 1,2,3$. We assume there is a relationship between the curvilinear coordinates and Cartesian coordinates given by

$$x^i = x^i(\varphi^k) \quad k,i = 1,2,3 \tag{2.3.1}$$

with the inverse relationship given by

$$\varphi^j = \varphi^j(x^m) \quad j,m = 1,2,3. \tag{2.3.2}$$

Denoting the position vector to a point in the fluid by $\bar{R}(\varphi^m)$ we define covariant base vectors $\bar{g}_j$ by

$$\bar{g}_j = \frac{\partial \bar{R}}{\partial \varphi^j} \quad j = 1,2,3 \tag{2.3.3}$$

and the element of arc length by

$$ds^2 = d\bar{R} \cdot d\bar{R}\,. \tag{2.3.4}$$

This in turn defines the covariant metric tensor $g_{k\ell}$ by

$$ds^2 = g_{k\ell}\, d\varphi^k\, d\varphi^\ell \tag{2.3.5}$$

where now $k,\ell = 1,2,3$. Indices repeated across the diagonal are summed over all values.

Contravariant base vectors $\bar{g}^i$ can be derived by solving the equations

$$\bar{g}^i \cdot \bar{g}_j = \delta^i_j \tag{2.3.6}$$

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δ^i_j being the Kronecker delta. Contravariant metric tensors can be defined by

(2.3.7) $$g^{k\ell} = \bar{g}^k \cdot \bar{g}^\ell \quad \text{or by}$$

(2.3.8) $$g^{k\ell} = \frac{\text{cofactor of } g_{k\ell}}{g}$$

where $g = |g_{ij}|$.

In order to formulate our equations we have need of the Christoffel symbols of the first and second kind defined respectively as

(2.3.9) $$[ij,k] = 1/2(g_{ij,j} + g_{jk,i} - g_{ij,k})$$
$$\left\{ {m \atop kj} \right\} = g^{mn}\, [kj,n]$$

where as in section (2.2) $f_{,j}$ denotes $\frac{\partial f}{\partial \varphi^j}$. This notation will prevail in the remainder of this work.

Finally we require covariant derivatives of tensors of the first and second orders, f^i and $f^i_{\ j}$. These derivatives are given respectively as

(2.3.10) $$f^i_{\ ;j} = f^i_{\ ,j} + \left\{ {i \atop nj} \right\} f^n$$
$$f^i_{\ m;k} = f^i_{\ m,k} + \left\{ {i \atop nk} \right\} f^n_{\ m} - \left\{ {n \atop mk} \right\} f^i_{\ n}$$

2.3.2 <u>Field Equations of the Fluid</u>

The field equations which govern fluid motion at coordinate φ^i and time t are written in terms of the stress tensor $t^k_{\ m}(\varphi^i,t)$, the rate of deformation tensor $d^i_{\ n}(\varphi^i,t)$, the velocity vector $\bar{v}(\varphi^i,t)$, the body force ctor $\bar{f}(\xi^i$

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vector $\bar{f}(\varphi^i,t)$, the fluid pressure $p(\varphi^i,t)$, a scalar quantity, and the fluid density $\rho_1(\varphi^i,t)$, also a scalar.

The tensor component, $t^k{}_m(\varphi^i,t)$, denotes the component of the stress vector acting on the surface φ^k constant and in the direction of φ^m at coordinate φ^i and time t. The component $v^k(\varphi^i,t)$ of vector $\bar{v}(\varphi^i,t)$ is that component acting in the direction of φ^k. The components f^k and $d^k{}_m$ are described in the same fashion as v^k and $t^k{}_m$.

We are interested in those equations which constitute an appropriate continuum theory for the study of a compressible, viscous Newtonian fluid undergoing small amplitude motion. Those equations are conservation of mass; balance of momenta, both linear and angular; constitutive equations; equation of state and kinematic equations. For a complete problem we require specification of boundary conditions on appropriate surfaces.

We will list these equations and refer the reader to Eringen [19] for their derivation. They are

<u>Conservation of Mass</u>

(2.3.11) $$\rho_{1,t} + (\rho_0 v^k)_{;k} = 0 \qquad *$$

<u>Balance of Angular Momentum</u>

(2.3.12) $$t^k{}_\ell = t_\ell{}^k$$

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*The notation $f_{,t}$ has been defined previously.

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Balance of Linear Momentum

$$(2.3.13) \qquad t^k_{m;k} + \rho_0 g_{mn}(f^n - v^n_{,t}) = 0 \qquad m = 1,2,3.$$

Constitutive Equations (Linear)

$$(2.3.14) \qquad t^k_m = (-p+\lambda d^n_n)\,\delta^k_m + 2\mu d^k_m \qquad k,m = 1,2,3.$$

Equation of State

$$(2.3.15) \qquad p = \rho_1 c_0^2.$$

Kinematic Equations (Rate of deformation/velocity relations)

$$(2.3.16) \qquad 2d^k_m = v^k_{;m} + g^{kn}\, g_{im} v^i_{;n} \qquad k,m = 1,2,3.$$

The constants ρ_0 and c_0 are the constant density and acoustic speed of the fluid while μ and λ are the shear and dilatational viscosities of the fluid respectively.

To complete the boundary value problem we assume that the bounding surface S may be written as $S = S_t \cup S_v$ $S_t \cap S_v = \emptyset$ where on S_t the fluid stresses are prescribed while on S_v the fluid velocities are known. Denoting the surface coordinates by $\{\varphi_s^i\}$ and the components of the outward normal to S by $n^k(\varphi_s^i)$ we write

$$(2.3.17) \qquad t^j_k(\varphi_s^i, t)\, n^k(\varphi_s^i) = \tilde{t}^j(\varphi_s^i) \qquad j = 1,2,3$$

for all φ_s^i on S_t and

$$(2.3.18) \qquad v^k(\varphi_s^i, t) = \tilde{v}^k(\varphi_s^i) \qquad k = 1,2,3$$

for all φ's on S_v. The $\tilde{t}^j$ and $\tilde{v}^k$ are known on their respective surfaces.

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These fluid equations completely describe the fluid motion, however other forms are often more useful in the work to follow. For this reason we rework these equations into a more desirable form.

To begin we make some preliminary remarks concerning the components and base vectors utilized in the above formulations. Since the base vectors $\bar{g}_i$ are not necessarily unit vectors, we introduce physical components by normalizing the base vectors. We define $v^{(k)}$ to be a physical component of $\bar{v}$ allowing $\bar{v} = v^k \bar{g}_k$ to be written as $\bar{v} = v^{(k)} \bar{e}_k$ where $\bar{e}_k = \bar{g}_k/(g_{\underline{kk}})^{1/2}$ and

$$v^{(k)} = v^k (g_{\underline{kk}})^{1/2}, \tag{2.3.19}$$

the notation $g_{\underline{kk}}$ indicating no sum on the index k. Physical components of the stress tensor $t^k{}_m$ are given by

$$t^{(k)}{}_{(m)} = (g_{\underline{kk}}/g_{\underline{mm}})^{1/2}\, t^k{}_m \, . \tag{2.3.20}$$

Equations (2.3.13) now become

$$\begin{aligned} \sum_{k=1}^{3} \{ & g^{-1/2}[t^{(k)}{}_{(m)} (g/g_{\underline{kk}})^{1/2}]_{,k} \\ & + t^{(k)}{}_{(m)} (g_{\underline{kk}} g_{\underline{mm}})^{-1/2} [(g_{\underline{mm}})^{-1/2}]_{,k} \\ & - t^{(k)}{}_{(k)} (g_{\underline{kk}} g_{\underline{mm}})^{-1/2} [(g_{\underline{kk}})^{1/2}]_{,m} \\ & + \rho_0 g_{km} [f^{(k)} - v^{(k)}{}_{,t}]\} = 0 \qquad m = 1,2,3. \end{aligned} \tag{2.3.21}$$

Another useful form is obtained when the stresses in the linear momentum equation (2.3.13) are eliminated by the constitutive equation (2.3.14) resulting in equations of rtion writt

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motion written in terms of the velocities. These equations are the Navier Stokes equations and have the form

$$(2.3.22)\qquad (\mu+\lambda)v^{k}{}_{;km} + \mu g_{mn}g^{ik}\, v^{n}{}_{;ik} - p_{,m} + \rho_0 g_{mn}(f^{n}-v^{n}{}_{,t}) = 0 \quad m = 1,2,3.$$

In vector notation, denoting the gradient operator by $\bar{\nabla}$, we have for the Navier Stokes equations

$$(2.3.23)\qquad (2\mu+\lambda)\bar{\nabla}(\bar{\nabla}\cdot\bar{v}) - \mu(\bar{\nabla} \times (\bar{\nabla} \times \bar{v})) - \bar{\nabla}p + \rho_0(\bar{f}-\bar{v}_{,t}) = 0.$$

The continuity equation or conservation of mass (2.3.11) takes on the form

$$(2.3.24)\qquad \rho_{1,t} + \rho_0(\bar{\nabla}\cdot\bar{v}) = 0.$$

Eliminating ρ_1 from (2.3.24) by the equation of state (2.3.15) and applying the divergence to the Navier Stokes equation (2.3.23) we obtain the pressure equation involving p alone and having the form

$$(2.3.25)\qquad (\lambda+2\mu)\nabla^2(p_{,t}) + \rho_0 c_0^2\, \nabla^2 p = \rho_0 p_{,tt}\ .$$

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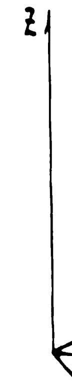
2.4 Geometry of the System

Having specified the equations of motion as well as the boundary conditions for both the membrane and the fluid it is necessary to specify the coordinate system appropriate to our problem and then determine the various metric tensors needed for the specific formulation of our equations.

We view the problem as one composed of a fluid flowing in a curved membrane shell. The shell is circular in cross section of outer radius $a+h$ and inner radius a. The shell itself is coiled on a larger circle of radius R which is constant. The relation between the tube and a set of Cartesian coordinates is shown in Figure 1a. A point in the fluid satisfies the following relationship

(2.4.1)
$$x = (R + r \sin \psi)\cos \theta$$
$$y = (R + r \sin \psi)\sin \theta$$
$$z = r \cos \psi$$
$$0 \leq r \leq a \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \psi \leq 2\pi .$$

We note that if $R = 0$ the coordinates are spherical polar while if R approaches infinity the tube appears straight. We will utilize this last concept later in the discussion.



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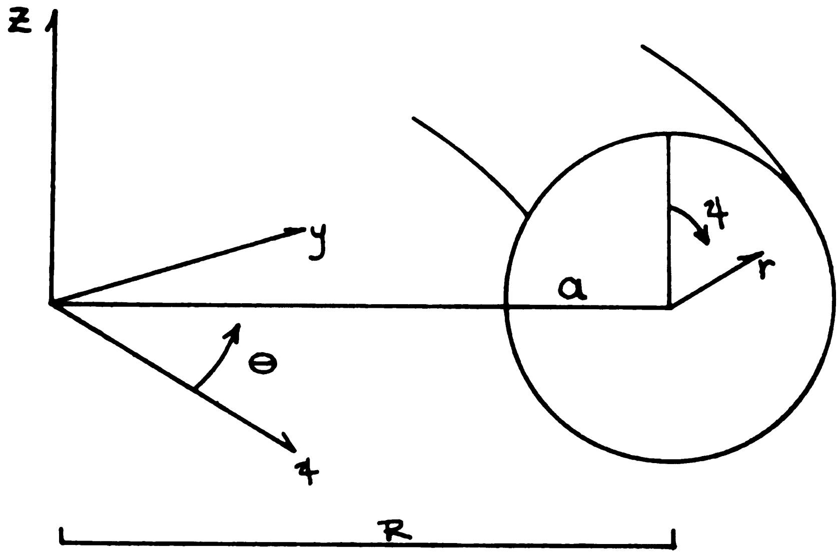

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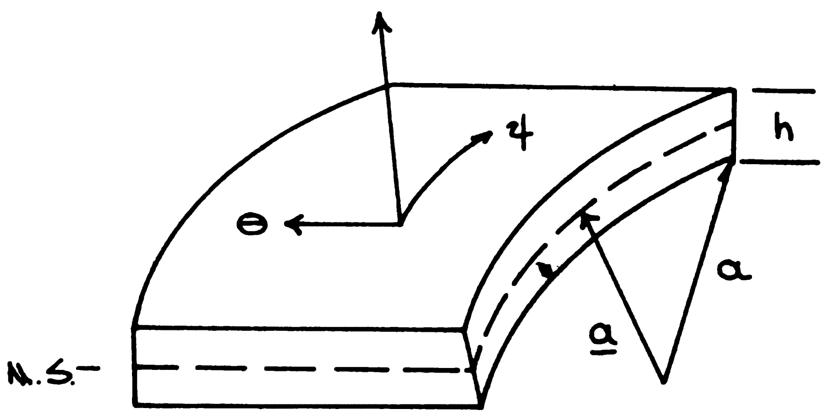

b.

FIGURE #1
N.S. - MIDDLE SURFACE

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On the middle surface of the tube we establish a coordinate system as seen in Figure 1b. Using (2.4.1), a point on this middle surface is given by

$$
\begin{aligned}
x &= (R + \underline{a} \sin \psi)\cos \theta \\
y &= (R + \underline{a} \sin \psi)\sin \theta \\
z &= \underline{a} \cos \psi
\end{aligned}
\tag{2.4.2}
$$

$$\underline{a} = a+h/2 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \psi \leq 2\pi .$$

From (2.2.3) to (2.2.5) we see

$$
\begin{aligned}
\alpha_1 &= \underline{a} & \alpha_2 &= R + \underline{a} \sin \psi \\
R_1 &= \underline{a} & R_2 &= (R + \underline{a} \sin \psi)/\sin \psi
\end{aligned}
\tag{2.4.3}
$$

where we have identified θ_1 with ψ and θ_2 with θ. From (2.2.1) to (2.2.4) we find

$$
\begin{aligned}
&g_{11} = r^2 \quad g_{22} = (R + r \sin \psi)^2 \quad g_{33} = 1 \\
&g_{kj} = 0 \quad k \neq j
\end{aligned}
\tag{2.4.4}
$$

where 1 corresponds to the ψ direction, 2 corresponds to the θ direction and 3 corresponds to the r direction.

2.5 Fluid

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2.5 Fluid Shell Problem - Complete Equations

Having found the metric tensors and principal radii of curvature we are now able to write the shell and fluid equations in a form appropriate to our geometry. We first introduce non-dimensional coordinates, velocities, displacements, stresses and pressure which will allow us to isolate an important parameter for our analysis.

We set

$$r^* = r/a \qquad t^* = \omega_0 t \tag{2.5.1}$$

where we define a frequency ω_0 by

$$\omega_0^2 = E/\rho \underline{a}^2(1-\nu^2). \tag{2.5.2}$$

We also define

$$\begin{aligned}
c^* &= c_0/a\omega_0 \qquad 1/\eta = (\lambda+\mu)/\rho_0 a^2\omega_0 \\
1/\eta_1 &= \mu/\rho_0 a^2\omega_0 \qquad p^* = p/\rho_0 a^2\omega_0^2 \\
u_\alpha &= v^{(\alpha)}/a\omega_0 \qquad w_\alpha = U_{(a)}/\underline{a} \\
S_{\alpha\beta} &= t^{(\alpha)}{}_{(\beta)}/\rho_0 a^2\omega_0^2 \qquad N^*_{\alpha\beta} = N_{\alpha\beta}/\rho h\underline{a}^2\omega_0^2 \qquad \alpha,\beta = 1,2,3 \\
m &= \rho h/\rho_0 a \qquad \delta = a/R \qquad \underline{\delta} = \underline{a}/R
\end{aligned} \tag{2.5.3}$$

The ratio m will prove to be a convenient parameter later in the analysis.

As before we identify subscripts 1,2,3 on field variables with ψ, θ, r respectively. Hence f_{12} is replaced by f_{i}. T

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by $f_{\psi\theta}$. The fluid-shell interaction problem can now be described in terms of our specific geometry. We begin with the shell equations of motion in section 2.2 identifying the body force components q_ψ, q_θ, q_r respectively with the fluid stresses $S_{r\psi}$, $S_{r\theta}$, S_{rr} evaluated at $r^* = 1$.

Utilizing (2.5.1) to (2.5.3) in (2.2.3) to (2.2.5) the shell equations of motion assume the form

(2.5.4) $$\frac{\underline{\delta} \cos\psi (N^*_{\psi\psi} - N^*_{\theta\theta})}{1 + \underline{\delta} \sin\psi} + N^*_{\psi\psi,\psi} + \frac{\underline{\delta} N^*_{\theta\psi,\theta}}{1 + \underline{\delta} \sin\psi}$$
$$+ \frac{a}{m\underline{a}} S_{r\psi}\Big|_{r^*=1} = w_{\psi,t^*t^*}$$

(2.5.5) $$\frac{2\underline{\delta} \cos\psi \, N^*_{\theta\psi}}{1 + \underline{\delta} \sin\psi} + N^*_{\theta\psi,\psi} + \frac{\underline{\delta} N^*_{\theta\theta,\theta}}{1 + \underline{\delta} \sin\psi}$$
$$+ \frac{a}{m\underline{a}} S_{r\theta}\Big|_{r^*=1} = w_{\theta,t^*t^*}$$

(2.5.6) $$-N^*_{\psi\psi} - \frac{\underline{\delta} \sin\psi \, N^*_{\theta\theta}}{1 + \underline{\delta} \sin\psi} - \frac{a}{m\underline{a}} S_{rr}\Big|_{r^*=1} = w_{r,t^*t^*} \,.$$

In like manner the stress-strain relations (2.2.6) through (2.2.8) and the strain-displacement relations (2.2.9) through (2.2.11) become

(2.5.7) $$N_{\psi\psi}^* = \epsilon_{\psi\psi} + \nu\epsilon_{\theta\theta}$$

(2.5.8) $$N_{\theta\theta}^* = \epsilon_{\theta\theta} + \nu\epsilon_{\psi\psi}$$

(2.5.9) $$N_{\theta\psi}^* = (1-\nu)\,\epsilon_{\psi\theta}$$

(2.5.10) $$\epsilon_{\psi\psi} = w_{\psi,\psi} + w_r$$

(2.5.11)

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(2.5.11) $$\epsilon_{\theta\theta} = \frac{\delta}{1+\delta \sin\psi}[w_{\theta,\theta} + w_{\psi}\cos\psi + w_r \sin\psi]$$

(2.5.12) $$2\epsilon_{\theta\psi} = w_{\theta,\psi} + \frac{\delta}{1+\delta \sin\psi}[w_{\psi,\theta} - w_{\theta}\cos\psi].$$

We are now able to apply (2.5.1) through (2.5.3) to the fluid equations given in section 2.3. We assume there is no body force on the fluid. We have then

Conservation of Mass

(2.5.13) $$\frac{\delta\cos\psi}{1+\delta r^*\sin\psi}u_{\psi} + \frac{1}{r^*}u_{\psi,\psi} + \frac{\delta}{1+\delta r^*\sin\psi}u_{\theta,\theta} + u_{r,r^*}$$
$$+ \left[\frac{1}{r^*} + \frac{\delta\sin\psi}{1+\delta r^*\sin\psi}\right]u_r = -\frac{1}{(c^*)^2}p^*_{,t^*}$$

Balance of Angular Momentum (2.3.12)

(2.5.14) $S_{\alpha\beta} = S_{\beta\alpha}$ α,β meaning ψ,θ,r

Balance of Linear Momentum (2.3.13)

(2.5.15) $$\frac{1}{r^*}S_{\psi\psi,\psi} + \frac{\delta\cos\psi}{1+\delta r^*\sin\psi}S_{\psi\psi} + \frac{\delta}{1+\delta r^*\sin\psi}S_{\psi\theta,\theta}$$
$$+ S_{\psi r,r^*} + \left[\frac{2}{r^*} + \frac{\delta\sin\psi}{1+\delta r^*\sin\psi}\right]S_{\psi r}$$
$$- \frac{\delta\cos\psi}{1+\delta r^*\sin\psi}S_{\theta\theta} = u_{\psi,t^*}$$

(2.5.16) $$\frac{2\delta\cos\psi}{1+\delta r^*\sin\psi}S_{\psi\theta} + S_{\theta r,r^*} + \left[\frac{1}{r^*} + \frac{2\delta\sin\psi}{1+\delta r^*\sin\psi}\right]S_{\theta r}$$
$$+ \frac{1}{r^*}S_{\theta\psi,\psi} + \frac{\delta}{1+\delta r^*\sin\psi}S_{\theta\theta,\theta} = u_{\theta,t^*}$$

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$$\frac{1}{r^*} S_{\psi\psi} + \frac{\delta}{1+\delta r^* \sin\psi} S_{\theta r,\theta} + \frac{\delta\cos\psi}{1+\delta r^*\sin\psi} S_{\psi r}$$

$$+ \frac{1}{r^*} S_{\psi r,\psi} + S_{rr,r^*} + \left[\frac{1}{r^*} + \frac{\delta\sin\psi}{1+\delta r^*\sin\psi}\right] S_{rr} \tag{2.5.17}$$

$$- \frac{\delta\sin\psi}{1+\delta r^*\sin\psi} S_{\theta\theta} = u_{r,t^*}$$

Constitutive Equations [(2.3.14), (2.3.16)]

$$S_{\psi\psi} = -p^* + \frac{2}{\eta_1 r^*}[u_{\psi,\psi} + u_r] + \left(\frac{1}{\eta} - \frac{1}{\eta_1}\right) D \tag{2.5.18a}$$

$$S_{\theta\theta} = -p^* + \frac{2\delta}{\eta_1[1+\delta r^*\sin\psi]}[u_{\theta,\theta} + u_\psi\cos\psi + u_r\sin\psi] + \left(\frac{1}{\eta} - \frac{1}{\eta_1}\right) D \tag{2.5.18b}$$

$$S_{rr} = -p^* + \frac{2}{\eta_1} u_{r,r^*} + \left(\frac{1}{\eta} - \frac{1}{\eta_1}\right) D \tag{2.5.18c}$$

$$S_{\psi\theta} = \frac{1}{\eta_1}\left[\frac{1}{r^*} u_{\theta,\psi} + \frac{\delta}{1+\delta r^*\sin\psi}(u_{\psi,\theta} - u_\theta\cos\psi)\right] \tag{2.5.18d}$$

$$S_{\psi r} = \frac{1}{\eta_1}\left[u_{\psi,r^*} + \frac{1}{r^*} u_{r,\psi} - \frac{1}{r^*} u_\psi\right] \tag{2.5.18e}$$

$$S_{\theta r} = \frac{1}{\eta_1}\left[u_{\theta,r^*} + \frac{\delta}{1+\delta r^*\sin\psi}(u_{r,\theta} - u_\theta\sin\psi)\right] \tag{2.5.18f}$$

$$D = \frac{1}{r^*}[u_{\psi,\psi} + u_r] + \frac{\delta}{1+\delta r^*\sin\psi}[u_{\theta,\theta} + u_\psi\cos\psi + u_r\sin\psi] + u_{r,r^*} \tag{2.5.18g}$$

It will be remembered that in addition to the above equations, we obtained the Navier Stokes equation (2.3.23) and the pressure equation (2.3.25) through various manipulations.

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Applying the same non-dimensional changes as before we have the Navier Stokes equation in component form.

Navier Stokes Equation (2.3.23)

$$\frac{1}{\eta_1}\{\frac{1}{(r^*)^2}[u_{\psi,\psi\psi} - u_\psi + 2u_{r,\psi}] + \frac{\delta\cos\psi}{r^*(1+\delta r^*\sin\psi)}[u_{\psi,\psi}$$
$$+ u_r] + \frac{\delta^2}{(1+\delta r^*\sin\psi)^2}[-u_\psi\cos^2\psi + u_{\psi,\theta\theta}$$
$$- 2\cos\psi\, u_{\theta,\theta} - \sin\psi\cos\psi\, u_r] \tag{2.5.19}$$
$$+ [\frac{1}{r^*} + \frac{\delta\sin\psi}{1+\delta r^*\sin\psi}]u_{\psi,r^*} + u_{\psi,r^*r^*}\}$$
$$- u_{\psi,t^*} = \frac{1}{r^*}\{p^*_{,\psi} + \frac{1}{\eta}\frac{p^*_{,\psi t^*}}{(c^*)^2}\}$$

$$\frac{1}{\eta_1}\{\frac{u_{\theta,\psi\psi}}{(r^*)^2} + \frac{\delta\cos\psi}{r^*(1+\delta r^*\sin\psi)}u_{\theta,\psi} +$$
$$+ \frac{\delta^2}{(1+\delta r^*\sin\psi)^2}[2\cos\psi\, u_{\psi,\theta} + u_{\theta,\theta\theta} - u_\theta$$
$$+ 2\sin\psi\, u_{r,\theta}] + [\frac{1}{r^*} + \frac{\delta\sin\psi}{1+\delta r^*\sin\psi}]u_{\theta,r^*} + u_{\theta,r^*r^*}\} \tag{2.5.20}$$
$$- u_{\theta,t^*} = \frac{\delta}{(1+\delta r^*\sin\psi)}\{p^*_{,\theta} + \frac{1}{\eta}\frac{p^*_{,\theta t^*}}{(c^*)^2}\}$$

$$\frac{1}{\eta_1}\{\frac{1}{(r^*)^2}[u_{r,\psi\psi} - u_r - 2u_{\psi,\psi}] + \frac{\delta\cos\psi}{r^*(1+\delta r^*\sin\psi)}[u_{r,\psi}$$
$$- u_\psi] + \frac{\delta^2}{(1+\delta r^*\sin\psi)^2}[u_{r,\theta\theta} - \cos\psi\sin\psi\, u_\psi \tag{2.5.21}$$
$$- 2\sin\psi\, u_{\theta,\theta} - \sin^2\psi\, u_r] + [\frac{1}{r^*} + \frac{\delta\sin\psi}{1+\delta r^*\sin\psi}]u_{r,r^*}$$
$$+ u_{r,r^*r^*}\} - u_{r,t^*} = p^*_{,r^*} + \frac{1}{\eta}\frac{p^*_{,r^*t^*}}{(c^*)^2}$$

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Finally we express the pressure equation as

Pressure Equation (2.3.25)

$$\left\{1 + \frac{1}{(c^*)^2}\left[\frac{1}{\eta} + \frac{1}{\eta_1}\right]\frac{\partial}{\partial t}\right\}\left\{\frac{1}{(r^*)^2} p^*_{,\psi\psi} + \frac{1}{r^*} p^*_{,r^*}\right.$$

$$+ p^*_{,r^*r^*} + \frac{\delta}{1+\delta r^* \sin\psi}\left[\frac{\cos\psi}{r^*} p^*_{,\psi} + \frac{\delta}{(1+\delta r^* \sin\psi)} p^*_{,\theta\theta}\right. \qquad (2.5.22)$$

$$\left.\left. + \sin\psi\, p^*_{,r^*}\right]\right\} = \frac{1}{(c^*)^2} p^*_{,t^*t^*} \,.$$

To complete the boundary value problem we specify the boundary conditions appropriate to the complete fluid-shell interaction problem as well as some special cases to be examined for the fluid alone and for the shell alone.

Fluid-Shell Interaction Boundary Conditions

$$r^* = 1 \quad \begin{aligned} u_\psi &= w_{\psi,t^*} \\ u_\theta &= w_{\theta,t^*} \\ u_r &= w_{r,t^*} \end{aligned} \qquad (2.5.23)$$

(2.5.24) $\quad r^* = 0 \quad p^*,\ u_\psi, u_\theta, u_r$ are finite.

Fluid Boundary Conditions

A. Rigid Shell

$$r^* = 1 \qquad u_\psi = u_\theta = u_r = 0$$

B. <u>No Shell</u> (stress free)

$$r^* = 1 \quad S_{r\psi} = S_{r\theta} = S_{rr} = 0$$

For both (A) and (B) (2.5.24) applies.

Empty Shell Boundary Condition

Displacements have period 2π in ψ.

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2.6 Traveling Wave Solutions

2.6.1 Introductory Remarks

A careful examination of the complete mathematical problem presented in section 2.5 reveals the enormous task one has in solving these equations in their most general form. The appearance of the parameters δ and $\underline{\delta}$ suggest a method of perturbation for the dependent variables allowing us to pose a series of problems, each simpler than the original, yet contributing to the solution of the original problem.

We recall that δ and $\underline{\delta}$ are defined by

$$\delta = a/R \quad \text{and} \quad \underline{\delta} = (a+h/2)R^{-1}$$

and that from the thin shell theory $h/a < .05$. Hence the ratios $\underline{\delta}/\delta$ and $\underline{a}/a$ are approximately 1. We also note that when δ is small we have the situation of a slightly curved tube with R large. Conversely, the case of δ large can be interpreted as R very small, yielding the description of a flow inside a sphere of radius nearly a.

Our choice is to examine the situation of δ small expanding all dependent variables in a perturbation about small δ. Once this decision is made two possibilities present themselves for our consideration.

If our tube were a straight cylinder, then a dependent variable X would have a wave solution of the form

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(2.6.1) $$X(r, \psi, \zeta, t) = \tilde{X}(r, \psi) \exp[i(k\zeta - \omega t)]$$

where r, ψ, ζ represent cylindrical coordinates ζ being the coordinate along the axis of the cylinder. The parameters k and ω are the wave number and frequency respectively.

Recognizing that in our curved tube ζ could be taken to be $R\theta$, an appropriate form for us to consider for the field variable X is

(2.6.2) $$X(r^*, \psi, \theta, \delta, t) = \hat{X}(\psi, r^*, \delta) \exp[i(\frac{s\theta}{\delta} - \beta t)]$$

where s and β are the wave number and frequency respectively. Since $\hat{X}$ depends on δ which is small, it also seems appropriate to expand $\hat{X}$ as a series about δ small, i.e.

(2.6.3) $$X(\psi, \theta, r^*, \delta, t) = \{X_0(\psi, r) + \delta X_1(\psi, r^*) + \cdots\}E$$
$$E = \exp\{i[\frac{\theta}{\delta}(s_0 + \delta s_1 + \delta^2 s_2 + \cdots) - \beta t]\}$$

The form (2.6.3) yields as its lowest terms the form (2.6.1) while the form

(2.6.4) $$X(\psi, \theta, r^*, \delta, t) = \{X_0^*(\psi, r^*) + \delta X_1^*(\psi, r^*) + \cdots\}E_1$$
$$E_1 = \exp\{i[\theta(s_0 + \delta s_1 + \cdots) - \beta t]\}$$

yields as its lowest terms solutions appropriate to plane flow, i.e., flow independent of distance down the tube. In this situation curvature effects are found in second or higher order terms. The form presented in (2.6.4) is utilized when one is concerned with the secondary flow patterns in the cross section of the tube.

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2.6.2 Perturbation Expansion in Small δ

We expand all dependent variables in an expansion suggested by (2.6.3). We assume

$$s = s_0 + \delta s_1 + \delta^2 s_2 + \cdots$$

$$(w_\alpha, u_\alpha) = i\{(w_\alpha^0, u_\alpha^0) + \delta(w_\alpha^1, u_\alpha^1) + \delta^2(w_\alpha^2, u_\alpha^2) + \cdots\}E$$

α meaning ψ or r

$$(w_\theta, u_\theta) = \{(w_\theta^0, u_\theta^0) + \delta(w_\theta^1, u_\theta^1) + \delta^2(w_\theta^2, u_\theta^2) + \cdots\}E$$

$$N^*_{\alpha\alpha} = i\{N^0_{\alpha\alpha} + \delta N^1_{\alpha\alpha} + \delta^2 N^2_{\alpha\alpha} + \cdots\}E$$

$$S_{\xi\xi} = i\{S^0_{\xi\xi} + \delta S^1_{\xi\xi} + \delta^2 S^2_{\xi\xi} + \cdots\}E$$

α meaning ψ or θ, ξ being ψ, θ or r^*

$$N^*_{\theta\psi} = \{N^0_{\theta\psi} + \delta N^1_{\theta\psi} + \delta^2 N^2_{\theta\psi} + \cdots\}E$$

$$(S^0_{\theta\psi}, S^0_{\theta r}) = \{(S^0_{\theta\psi}, S^0_{\theta r}) + \delta(S^1_{\theta\psi}, S^1_{\theta r}) + \delta^2(S^2_{\theta\psi}, S^2_{\theta r}) + \cdots\}E$$

All fluid quantities such as u^k_α are functions of ψ and r^* while all shell variables are functions of ψ only.

It now remains to substitute (2.6.4) into the equation of section 2.5 collecting terms as coefficients of powers of δ and equating these coefficients to zero. This allows us to define zero, first and second order problems which are to be solved in the following subsections. We will now present the equations in the order we intend to solve them. We will also drop the * from all variables realizing that all quantities are non-dimensional.

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To facilitate the presentation of these equations we choose to define the following differential operators. The L operators are associated with fluid velocities while those designated with M operate on fluid stresses.

$$L_1(u^k{}_\psi, u^k{}_\theta, u^k{}_r) = \frac{1}{r} u^k{}_{\psi,\psi} + s_0 u^k{}_\theta + \frac{1}{r}(r u^k{}_r)_{,r}$$

$k = 0,1,2,3,\cdots$

$$L_2 = \frac{1}{r^2} \frac{\partial^2}{\partial\psi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) - s_0{}^2 \tag{2.6.5}$$

$$L_3 = L_2 - \frac{1}{r^2}$$

$$M_1(S^k_{\psi r}, S^k_{\psi\psi}) = \frac{1}{r}[S^k_{\psi r} + (r S^k_{\psi r})_{,r} + S^k_{\psi\psi,\psi}]$$

$$M_2(S^k_{\theta\psi}, S^k_{\theta r}) = \frac{1}{r}[S^k_{\theta\psi,\psi} + (r\ S^k_{\theta r})_{,r}] \tag{2.6.6}$$

$$M_3(S^k_{\psi r}, S^k_{\psi\psi}, S^k{}_{rr}) = \frac{1}{r}[S^k_{r\psi,\psi} - S^k_{\psi\psi} + (r S^k_{rr})_{,r}]$$

$k = 0,1,2,3,\cdots$

We also define

$$\begin{aligned} k_0^2 &= 1 - i\beta/c^2\left\{\frac{1}{\eta} + \frac{1}{\eta_1}\right\} \\ k_1{}^2 &= 1 - i\beta/\eta c^2 \end{aligned} \tag{2.6.7}$$

2.6.3 Zero Order Problems (straight cylinder)

Pressure Equation

$$\left(L_2 + \frac{\beta^2}{c^2 k_0^2}\right)(p_0) = 0 \tag{2.6.8}$$

(2.6.9)

(2.6.1

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Navier Stokes Equations

$$\frac{1}{\eta_1}\{L_3(u^o{}_\psi) + \frac{2}{r^2} u^o{}_{r,\psi}\} + i\beta u^o{}_\psi = \frac{1}{r} k_1^2 p_{o,\psi}$$

$$\frac{1}{\eta_1} L_2(u^o{}_\theta) + i\beta u^o{}_\theta = -s_o k_1^2 p_o \tag{2.6.9}$$

$$\frac{1}{\eta_1}\{L_3(u^o{}_r) - \frac{2}{r^2} u^o{}_{\psi,\psi}\} + i\beta u^o{}_r = k_1^2 p_{o,r}$$

Continuity Equation

$$L_1(u^o{}_\psi, u^o{}_\theta, u^o{}_r) = \frac{-i\beta}{c^2} p_o \tag{2.6.10}$$

Constitutive Equations

$$s^o{}_{\psi\psi} = -p_o + (\frac{1}{\eta} - \frac{1}{\eta_1}) D_o + \frac{2}{\eta_1}\frac{1}{r}\{u^o{}_{\psi,\psi} + u^o{}_r\}$$

$$s^o{}_{\theta\theta} = -p_o + (\frac{1}{\eta} - \frac{1}{\eta_1}) D_o + \frac{2}{\eta_1} s_o u^o{}_\theta$$

$$s^o{}_{rr} = -p_o + (\frac{1}{\eta} - \frac{1}{\eta_1}) D_o + \frac{2}{\eta_1} u^o{}_{r,r}$$

$$s^o{}_{\psi\theta} = \frac{1}{\eta_1}\{-s_o u^o{}_\psi + \frac{1}{r} u^o{}_{\theta,\psi}\} \tag{2.6.11}$$

$$s^o{}_{\psi r} = \frac{1}{\eta_1}\{u^o{}_{\psi,r} + \frac{1}{r}[u^o{}_{r,\psi} - u^o{}_\psi]\}$$

$$s^o{}_{\theta r} = \frac{1}{\eta_1}\{u^o{}_{\theta,r} - s_o u^o{}_r\}$$

$$D_o = L_1(u^o{}_\psi, u^o{}_\theta, u^o{}_r)$$

2.6.12)

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2.6.14

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Fluid Stress Equations of Motion

$$(2.6.12)\qquad \begin{aligned} M_1(S^o_{\psi r}, S^o_{\psi\psi}) + s_o S^o_{\psi\theta} &= -i\beta u^o_\psi \\ M_2(S^o_{\theta\psi}, S^o_{\theta r}) - s_o S^o_{\theta\theta} &= -i\beta u^o_\theta \\ M_3(S^o_{\psi r}, S^o_{\psi\psi}, S^o_{rr}) + s_o S^o_{r\theta} &= -i\beta u^o_r \end{aligned}$$

Shell Stress - Displacement Equations

$$(2.6.13)\qquad \begin{aligned} N^o_{\psi\psi} &= w^o_{\psi,\psi} + w^o_r + \nu s_o w^o_\theta \\ N^o_{\theta\theta} &= s_o w^o_\theta + \nu\{w^o_{\psi,\psi} + w^o_r\} \\ N^o_{\theta\psi} &= \frac{1}{2}(1-\nu)\{w^o_{\theta,\psi} - s_o w^o_\psi\} \end{aligned}$$

Shell Equations of Motion

$$(2.6.14)\qquad \begin{aligned} N^o_{\psi\psi,\psi} + s_o N^o_{\theta\psi} + \frac{1}{m} S^o_{r\psi}\big|_{r=1} &= -\beta^2 w^o_\psi \\ N^o_{\theta\psi,\psi} - s_o N^o_{\theta\theta} + \frac{1}{m} S^o_{r\theta}\big|_{r=1} &= -\beta^2 w^o_\theta \\ - N^o_{\psi\psi} - \frac{1}{m} S^o_{rr}\big|_{r=1} &= -\beta^2 w^o_r \end{aligned}$$

Boundary Conditions at $r = 1$

A. Coupled Problem

$$(2.6.15)\qquad \begin{aligned} u^o_\psi(\psi,1) &= -i\beta w^o_\psi(\psi) \\ u^o_\theta(\psi,1) &= -i\beta w^o_\theta(\psi) \qquad 0 \le \psi \le 2\pi \\ u^o_r(\psi,1) &= -i\beta w^o_r(\psi) \end{aligned}$$

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C. St

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If the fluid is inviscid then (2.6.15) becomes

(2.6.16) $$u^0_r(\psi,1) = -i\beta w^0_r(\psi) \qquad 0 \leq \psi \leq 2\pi$$

B. Rigid Shell

(2.6.17) $$u^0_\psi(\psi,1) = u^0_\theta(\psi,1) = u^0_r(\psi,1) = 0 \qquad 0 \leq \psi \leq 2\pi$$

If the fluid is inviscid we replace (2.6.17) by

(2.6.18) $$u^0_r(\psi,1) = 0 \qquad 0 \leq \psi \leq 2\pi$$

C. Stress Free Fluid (No tube)

(2.6.19) $$S^0_{r\psi}(\psi,1) = S^0_{r\theta}(\psi,1) = S^0_{rr}(\psi,1) = 0 \qquad 0 \leq \psi \leq 2\pi$$

D. No Fluid

Shell displacements are periodic in ψ with period 2π.

Boundary Conditions at $r = 0$

All fluid variables are finite at $r = 0$.

2.6.4 First Order Problem

Pressure Equation

(2.6.20) $$\left(L_2 + \frac{\beta^2}{c^2 k_0^2}\right)p_1 = p_0\{2s_0 s_1 - 2rs_0^2 \sin\psi\} - \frac{1}{r}\, p_{0,\psi} \cos\psi - p_{0,r} \sin\psi$$

2.6.2

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Navier Stokes Equations

$$\frac{1}{\eta_1}\{L_3(u^1{}_\psi) + \frac{2}{r^2}u^1{}_{r,\psi}\} + i\beta u^1{}_\psi = \frac{1}{r}k_1^2 p_{1,\psi}$$
$$- \frac{1}{\eta_1}\{(2rs_o{}^2 u^o{}_\psi + u^o{}_{\psi,r})\sin\psi$$
$$+ \frac{1}{r}\cos\psi(u^o{}_{\psi,\psi} + u^o{}_r - 2rs_o u^o{}_\theta)\}$$

$$(2.6.21)\qquad \frac{1}{\eta_1}L_2(u^1_\theta) + i\beta u^1{}_\theta = -k_1^2 s_o p_1 - \frac{1}{\eta_1}\{[2rs_o{}^2 u^o{}_\theta$$
$$- 2s_o u^o{}_r + u^o{}_{\theta,r}]\sin\psi + \frac{1}{r}\cos\psi[u^o{}_{\theta,\psi}$$
$$- 2rs_o u^o{}_\psi]\} - rs_o p_o \sin\psi$$

$$\frac{1}{\eta_1}\{L_3(u^1{}_r) - \frac{2}{r^2}u^1{}_{\psi,\psi}\} + i\beta u^1{}_r = k_1^2 p_{1,r}$$
$$- \frac{1}{\eta_1}\{[2rs_o{}^2 u^o{}_r + u^o{}_{r,r} - 2s_o u^o{}_\theta]\sin\psi$$
$$+ \frac{1}{r}\cos\psi[u^o{}_{r,\psi} - u^o{}_\psi]\}$$

Continuity Equation

$$(2.6.22)\qquad L_1(u^1{}_\psi, u^1{}_\theta, u^1{}_r) = \frac{i\beta}{c^2}p_1 - s_1 u^o{}_\theta + \{rs_o u^o{}_\theta$$
$$- u^o{}_r\}\sin\psi - u^o{}_\psi\cos\psi$$

Constitutive Equations

$$s^1{}_{\psi\psi} = -p_1 + (\frac{1}{\eta} - \frac{1}{\eta_1})D_1 + \frac{2}{\eta_1}\frac{1}{r}\{u^1{}_{\psi,\psi} + u^1{}_r\}$$

12.6.2

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$$S^1_{\theta\theta} = -p_1 + \left(\frac{1}{\eta} - \frac{1}{\eta_1}\right) D_1 + \frac{2}{\eta_1} \{s_O u^1_\theta + s_1 u^O_\theta$$

$$+ u^O_\psi \cos \psi + (u^O_r - rs_O u^O_\theta)\sin \psi\}$$

$$S^1_{rr} = -p_1 + \left(\frac{1}{\eta} - \frac{1}{\eta_1}\right) D_1 + \frac{2}{\eta_1} u^1_{r,r}$$

(2.6.23)
$$S^1_{\psi r} = \frac{1}{\eta_1} \{u^1_{\psi,r} + \frac{1}{r} (u^1_{r,\psi} - u^1_\psi)\}$$

$$S^1_{\theta\psi} = \frac{1}{\eta_1} \{-s_O u^1_\theta - s_1 u^O_\theta + \frac{1}{r} u^1_{\theta,\psi} + rs_O u^O_\theta \sin \psi$$

$$- u^O_\theta \cos \psi\}$$

$$S^1_{\theta r} = \frac{1}{\eta_1} \{u^1_{\theta,r} + s_O u^1_r - u^O_\theta \sin \psi$$

$$+ (s_1 - rs_O \sin \psi) u^O_r\}$$

$$D_1 = L_1(u^1_\psi, u^1_\theta, u^1_r)$$

Fluid Stress Equations of Motion

$$M_1(S^1_{\psi r}, S^1_{\psi\psi}) + s_O S^1_{\psi\theta} = -i\beta u^1_\psi + \{S^O_{\theta\theta} - S^O_{\psi\psi}\} \cos \psi$$

$$+ \sin \psi \{rs_O S^O_{\psi\theta} - S^O_{\psi r}\} - s_1 S^O_{\psi\theta}$$

(2.6.24)
$$M_2(S^1_{\theta\psi}, S^1_{\theta r}) - s_O S^1_{\theta\theta} = -i\beta u^1_\theta - 2S^O_{\theta\psi} \cos \psi$$

$$- \{2S^O_{\theta r} + rs_O S^O_{\theta\theta}\} \sin \psi + s_1 S^O_{\theta\theta}$$

$$M_3(S^1_{r\psi}, S^1_{\psi\psi}, S^1_{rr}) + s_O S^1_{r\theta} = -i\beta u^1_r - S^O_{\psi r} \cos \psi$$

$$+ \{rs_O S^O_{r\theta} + S^O_{\theta\theta} - S^O_{rr}\} \sin \psi - s_1 S^O_{r\theta}$$

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N

(2.6.26) N

Shell Stress - Displacement Equations

$$N^1{}_{\psi\psi} = w^1{}_{\psi,\psi} + w^1{}_r + \nu\{(s_1 - s_0 \sin\psi) w^0{}_\theta + s_0 w^1{}_\theta + w^0{}_\psi \cos\psi + w^0{}_r \sin\psi\}$$

(2.6.25)
$$N^1{}_{\theta\theta} = s_0 w^1{}_\theta + \nu\{w^1{}_{\psi,\psi} + w^1{}_r\} + (s_1 - s_0 \sin\psi) w^0{}_\theta + w^0{}_\psi \cos\psi + w^0{}_r \sin\psi$$

$$N^1{}_{\theta\psi} = \frac{1}{2}(1-\nu)\{w^1{}_{\theta,\psi} - s_0 w^1{}_\psi - (s_1 - s_0 \sin\psi) w^0{}_\psi - w^0{}_\theta \cos\psi\}$$

Shell Equations of Motion

$$N^1{}_{\psi\psi,\psi} + s_0 N^1{}_{\theta\psi} + \frac{1}{m} S^1{}_{r\psi}\Big|_{r=1} = -\beta^2 w^1{}_\psi - s_1 N^0{}_{\theta\psi} - \{N^0{}_{\psi\psi} - N^0{}_{\theta\theta}\} \cos\psi + s_0 N^0{}_{\theta\psi} \sin\psi$$

(2.6.26)
$$N^1{}_{\theta\psi,\psi} - s_0 N^1{}_{\theta\theta} + \frac{1}{m} S^1{}_{r\theta}\Big|_{r=1} = -\beta^2 w^1{}_\theta + s_1 N^0{}_{\theta\theta} - 2N^0{}_{\theta\psi} \cos\psi - s_0 N^0{}_{\theta\theta} \sin\psi$$

$$-N^1{}_{\psi\psi} - \frac{1}{m} S^1{}_{rr}\Big|_{r=1} = -\beta^2 w^1{}_r + N^0{}_{\theta\theta} \sin\psi$$

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B. Rigid S

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C. Stress F

(2.6.31)

D. No Flui

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Boundary Conditions at $r = 1$

A. Coupled Problem

$$
\begin{aligned}
u^1_\psi(\psi,1) &= -i\beta w^1_\psi(\psi) \\
u^1_\theta(\psi,1) &= -i\beta w^1_\theta(\psi) \qquad 0 \le \psi \le 2\pi \\
u^1_r(\psi,1) &= -i\beta w^1_r(\psi)
\end{aligned}
\tag{2.6.27}
$$

If the fluid is inviscid (2.6.27) is replaced by

$$u^1_r(\psi,1) = -i\beta w^1_r(\psi) \quad 0 \le \psi \le 2\pi \tag{2.6.28}$$

B. Rigid Shell

$$u^1_\psi(\psi,1) = u^1_\theta(\psi,1) = u^1_r(\psi,1) = 0 \quad 0 \le \psi \le 2\pi \tag{2.6.29}$$

If the fluid is inviscid we replace (2.6.29) by

$$u^1_r(\psi,1) = 0 \quad 0 \le \psi \le 2\pi \tag{2.6.30}$$

C. Stress Free Fluid

$$S^1_{r\psi}(\psi,1) = S^1_{r\theta}(\psi,1) = S^1_{rr}(\psi,1) = 0 \quad 0 \le \psi \le 2\pi \tag{2.6.31}$$

D. No Fluid

Shell displacements are periodic in ψ with period 2π.

Boundary Conditions at $r = 0$

All fluid variables are finite at $r = 0$.

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2.6.5 Second Order Problem

For the second order problem it will not be necessary to find exact values of the pressure and velocity terms since the process for determining the second order correction eliminates these variables. We need to describe only the fluid stress equations of motion, the shell stress displacement relations, the stress equations of motion for the shell and the boundary conditions.

Fluid Stress Equations of Motion

$$M_1(S^2{}_{\psi r}, S^2{}_{\psi\psi}) + s_o S^2{}_{\psi\theta} = -i\beta u^2{}_{\psi} - s_1 S^1{}_{\psi\theta} - s_2 S^o{}_{\psi\theta}$$
$$+ \cos\psi\,\{S^1{}_{\theta\theta} - S^1{}_{\psi\psi}\} + \sin\psi\,\{r s_o S^1{}_{\psi\theta}$$
$$- S^1{}_{\psi r}\} + r\cos\psi\sin\psi\,\{S^o{}_{\psi\psi} - S^o{}_{\theta\theta}\}$$
$$+ r\sin^2{}_{\psi}\,\{S^o{}_{\psi r} - r s_o{}^2\, S^o{}_{\psi\theta}\}$$

(2.6.32) $$M_2(S^2{}_{\theta\psi}, S^2{}_{\theta r}) - s_o S^2{}_{\theta\theta} = -i\beta u^2{}_{\theta} + s_1 S^1{}_{\theta\theta} + s_2 S^o{}_{\theta\theta}$$
$$- 2S^1{}_{\psi\theta}\cos\psi - \sin\psi\}\, 2S^1{}_{\theta r} + r s_o S^1{}_{\theta\theta}$$
$$+ r s_1 S^o{}_{\theta\theta}\} + 2r\cos\psi\sin\psi\, S^o{}_{\psi\theta}$$
$$+ r\sin^2\psi\,\{2S^o{}_{\theta r} + r s_o{}^2\, S^o{}_{\theta\theta}\}$$

$$M_3(S^2{}_{\psi r}, S^2{}_{\psi\psi}, S^2{}_{rr}) + s_o S^2{}_{r\theta} = -i\beta u^2{}_{r} - s_1 S^1{}_{\theta r}$$
$$- s_2 S^o{}_{\theta r} - S^1{}_{\psi r}\cos\psi + \sin\psi\,\{r s_o S^1{}_{\theta r}$$

Shell

N

(2.6.33)

$$+ rs_1 S^0{}_{\theta r} + S^1{}_{\theta\theta} - S^1{}_{rr}\}$$

$$+ r \cos\psi \sin\psi\, S^0{}_{\psi r} + r \sin^2\psi\, \{S^0{}_{rr}$$

$$- S^0{}_{\theta\theta} + rs_0{}^2\, S^0{}_{\theta r}\}$$

Shell Stress - Displacement Relations

$$N^2{}_{\psi\psi} = w^2{}_{\psi,\psi} + w^2{}_r + \nu\{\cos\psi(w^1{}_\psi - w^0{}_\psi \sin\psi)$$

$$+ \sin\psi(w^1{}_r - w^0{}_r \sin\psi) + s_0 w^2{}_\theta + s_1 w^1{}_\theta + s_2 w^0{}_\theta$$

$$- (s_0 w^1{}_\theta + s_1 w^0{}_\theta)\sin\psi + s_0 w^0{}_\theta \sin^2\psi\}$$

(2.6.33) $$N^2{}_{\theta\theta} = \nu(w^2{}_{\psi,\psi} + w^2{}_r) + s_0 w^2{}_\theta + w^1{}_\theta(s_1 - s_0 \sin\psi)$$

$$+ w^0{}_\theta(s_2 - s_1 \sin\psi + s_0 \sin^2\psi)$$

$$N^2{}_{\theta\psi} = \frac{(1-\nu)}{2}\{w^2{}_{\theta,\psi} + s_0 w^2{}_\psi + w^1{}_\psi(s_1 - s_0 \sin\psi)$$

$$+ w^0{}_\psi(s_2 - s_1 \sin\psi + s_0 \sin^2\psi)$$

$$- \cos\psi(w^1{}_\theta - w^0{}_\theta \sin\psi)\}$$

Shell Equations of Motion

$$N^2{}_{\psi\psi,\psi} + s_0 N^2{}_{\theta\psi} + \frac{1}{m} S^2{}_{r\psi}\big|_{r=1} = -\beta^2 w^2{}_\psi - s_1 N^1{}_{\theta\psi}$$

$$- s_2 N^0{}_{\theta\psi} - \cos\psi\, \{N^1{}_{\psi\psi} - N^1{}_{\theta\theta}\}$$

$$+ \sin\psi\, \{s_0 N^1{}_{\theta\psi} + [s_1 - s_0 \sin\psi]N^0{}_{\theta\psi}\}$$

$$+ \sin\psi \cos\psi\, \{N^0{}_{\psi\psi} - N^0{}_{\theta\theta}\}$$

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$$(2.6.34)\quad N^2_{\theta\psi,\psi} - s_o N^2_{\theta\theta} + \frac{1}{m} S^2_{\theta r}\big|_{r=1} = -\beta^2 w^2_\theta$$

$$+ s_1 N^1_{\theta\theta} + s_2 N^o_{\theta\theta} - 2N^1_{\theta\psi} \cos\psi$$

$$- \sin\psi \, \{s_o N^1_{\theta\theta} + [s_1 - s_o \sin\psi] N^o_{\theta\theta}\}$$

$$+ 2\, N^o_{\theta\psi} \cos\psi \sin\psi$$

$$- N^2_{\psi\psi} - \frac{1}{m} S^2_{rr}\big|_{r=1} = -\beta^2 w^2_r + \sin\psi \, \{N^1_{\theta\theta}$$

$$- \sin\psi \, N^o_{\theta\theta}\}$$

Boundary Conditions at $r = 1$

A. Coupled Problem

$$(2.6.35)\quad \begin{aligned} u^2_\psi(\psi,1) &= -i\beta w^2_\psi(\psi) \\ u^2_\theta(\psi,1) &= -i\beta w^2_\theta(\psi) \qquad 0 \le \psi \le 2\pi \\ u^2_r(\psi,1) &= -i\beta w^2_r(\psi) \end{aligned}$$

If the fluid is inviscid (2.6.35) is replaced by

$$(2.6.36)\quad u^2_r(\psi,1) = -i\beta w^2_r(\psi) \qquad 0 \le \psi \le 2\pi$$

B. Rigid Shell

$$(2.6.37)\quad u^2_\psi(\psi,1) = u^2_\theta(\psi,1) = u^2_r(\psi,1) = 0 \qquad 0 \le \psi \le 2\pi$$

If the fluid is inviscid (2.6.37) is replaced by

$$(2.6.38)\quad u^2_r(\psi,1) = 0 \qquad 0 \le \psi \le 2\pi$$

C. Stress Free Fluid

(2.6.39) $\quad S^2_{r\psi}(\psi,1) = S^2_{r\theta}(\psi,1) = S^2_{rr}(\psi,1) = 0 \qquad 0 \le \psi \le 2\pi$

D. No Fluid

Shell displacements are periodic in ψ with period 2π.

Boundary Conditions at $r = 0$

All fluid variables are finite at $r = 0$.

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CHAPTER III

CURVATURE CORRECTION FOR THE EMPTY SHELL

3.1 Introduction

In Chapter two we displayed those equations necessary for the complete description of a fluid flowing in a thin, curved, elastic membrane shell. The solutions we seek are of the form of an amplitude function multiplied by a traveling wave. The wave portion of our solution is common to every solution and has the form $\exp i\{s\theta\delta^{-1} - \beta t\}$ where β is a non-dimensional frequency, t is the non-dimensional time, δ is the ratio aR^{-1}, θ is the angle down the tube and s is the non-dimensional wave number. We expanded s in the fashion of

(3.1.1) $$s = s_0 + \delta s_1 + \delta^2 s_2 + \ldots$$

as we did for all our variables except β.

We now assume that our fluid shell system has been set in motion somehow and now vibrates in some fashion, there being no outside forces interfering with the motion. Our task is to characterize this system by describing the modes of vibration. The modes are merely the various patterns which the vibratic

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the vibration waves exhibit as they travel along the tube and/or fluid.

It is our task to describe these modes by finding the relationship between the frequency, β and the wave number s. Once this relationship is established we may then indicate the frequency at which the tube and/or fluid is vibrating and then know the wave number s corresponding to the frequency indicated. From these two values we can determine the speed of the waves traveling down the tube and/ or fluid. The relationship between the frequency and wave number constitutes a characterization of our system and can be utilized in other problems involving the same system.

If we study the same model under the influence of outside forces during the vibration, we may utilize the modes to describe the outside force and assist in the solution of our new problem.

Our immediate concern is to find a relationship between the wave number s and the frequency. We begin by finding the relationship between s_0 and β following that with the corrections to s_0, namely s_1 and s_2. This is the procedure to be utilized in the remainder of the thesis.

We begin by considering the shell without the fluid, establishing the displacement equations of motion for the empty shell

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empty shell in section 3.2. In the next section, 3.3, we determine the relationships between s_0, s_1, s_2 and β for an arbitrary number of waves around the tube in the ψ direction.

In the remainder of Chapter three we concentrate on the fundamental case of axisymmetric motion for the zero order motion with section 3.4 containing the relationship between s_0 and β. In section 3.5 we discuss the effects of the first order corrections on the functions described in the zero order. Finally, in section 3.6 we establish the second order correction, s_2, and relate it to the zero order wave number s_0.

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3.2 Shell Displacement Equations of Motion

In section 2.6 we formulated the stress equations of motion for the shell in the zero, first and second orders, (2.6.14), (2.6.26) and (2.6.34) respectively. We also derived the stress displacement relations for the same three orders in equations (2.6.13), (2.6.25) and (2.6.33). We now make the substitution of appropriate stress displacement relations into the shell equations of motion (2.6.14), (2.6.26) and (2.6.34). To facilitate the notation we define three differential operations on the displacement components as:

$$
\begin{aligned}
F_1(w^k_\psi, w^k_\theta, w^k_r) &= w^k_{\psi,\psi\psi} + \{\beta^2 - \tfrac{1}{2} s_0^2(1-\nu)\} w^k_\psi \\
&\quad + \tfrac{1}{2} s_0(1+\nu) w^k_{\theta,\psi} + w^k_{r,\psi}, \\
F_2(w^k_\psi, w^k_\theta, w^k_r) &= -\tfrac{1}{2} s_0(1+\nu) w^k_{\psi,\psi} + \{\beta^2 - s_0^2\} w^k_\theta \\
&\quad + \tfrac{1}{2}(1-\nu) w^k_{\theta,\psi\psi} - \nu s_0 w^k_r, \\
F_3(w^k_\psi, w^k_\theta, w^k_r) &= -w^k_{\psi,\psi} - \nu s_0 w^k_\theta + (\beta^2 - 1) w^k_r, \\
k &= 0,1,2,3,\cdots.
\end{aligned}
\tag{3.2.1}
$$

We are now in a position to indicate the equations of motion for the shell, written in terms of the shell displacements. It should be repeated that the thin shell conditions result in the shell displacements being independent of r. Hence the w^k_ξ, ξ meaning θ, ψ, r, will be functions of ψ only.

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Zero Order Shell - Displacement Equations of Motion

$$
\begin{aligned}
&F_1(w^0{}_\psi, w^0{}_\theta, w^0{}_r) = -\frac{1}{m}\, S^0_{r\psi}\big|_{r=1}\,, \\
(3.2.2)\qquad &F_2(w^0{}_\psi, w^0{}_\theta, w^0{}_r) = -\frac{1}{m}\, S^0_{r\theta}\big|_{r=1}\,, \\
&F_3(w^0{}_\psi, w^0{}_\theta, w^0{}_r) = +\frac{1}{m}\, S^0_{rr}\big|_{r=1}\,.
\end{aligned}
$$

First Order Shell - Displacement Equations of Motion

$$
\begin{aligned}
(3.2.3)\qquad F_1(w^1{}_\psi, w^1{}_\theta, w^1{}_r) = {} & -\frac{1}{m}\, S^1_{r\psi}\big|_{r=1} - s_1\{\tfrac{1}{2}(1+\nu)w^0{}_{\theta,\psi} \\
& - (1-\nu)s_0 w^0{}_\psi\} + \cos\psi\{\tfrac{1}{2}(3-\nu)s_0 w^0{}_\theta - w^0{}_{\psi,\psi} - w^0{}_r\} \\
& + \sin\psi\{\tfrac{1}{2}\, s_0(1+\nu)w^0{}_{\theta,\psi} + w^0{}_\psi[\nu - s_0^2(1-\nu)] - \nu w^0{}_{r,\psi}\},
\end{aligned}
$$

$$
\begin{aligned}
(3.2.4)\qquad F_2(w^1{}_\psi, w^1{}_\theta, w^1{}_r) = {} & -\frac{1}{m}\, S^1_{r\theta}\big|_{r=1} + s_1\{\tfrac{1}{2}(1+\nu)w^0{}_{\psi,\psi} \\
& + 2s_0 w^0{}_\theta + \nu w^0{}_r\} + \frac{1}{2}\cos\psi\{s_0(3-\nu)w^0{}_\psi \\
& - (1-\nu)w^0{}_{\theta,\psi}\} + \frac{1}{2}\sin\psi\{2s_0(1-\nu)w^0{}_r \\
& - (1-\nu-4s_0^2)w^0{}_\theta - (1+\nu)s_0 w^0{}_{\psi,\psi}\},
\end{aligned}
$$

$$
\begin{aligned}
(3.2.5)\qquad F_3(w^1{}_\psi, w^1{}_\theta, w^1{}_r) = {} & +\frac{1}{m}\, S^1_{rr}\big|_{r=1} + \nu s_1 w^0{}_\theta + \nu w^0{}_\psi \cos\psi \\
& + \sin\psi\{s_0(1-\nu)w^0{}_\theta + 2\nu w^0{}_r + \nu w^0{}_{\psi,\psi}\}.
\end{aligned}
$$

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F_1

(3.2.6)

F_2

(3.2.7)

Second Order Shell - Displacement Equations of Motion

$$F_1(w^2{}_\psi, w^2{}_\theta, w^2{}_r) = -\frac{1}{m} S^2_{r\psi}\Big|_{r=1} - s_1\{\tfrac{1}{2}(1+\nu)w^1{}_{\theta,\psi}$$

$$- s_0(1-\nu)w^1{}_\psi - \tfrac{1}{2}s_1(1-\nu)w^o{}_\psi\} - s_2\{\tfrac{1}{2}(1+\nu)w^o{}_{\theta,\psi}$$

$$- s_0(1-\nu)w^o{}_\psi\} + \cos\psi\{\tfrac{1}{2}(3-\nu)[s_0 w^1{}_\theta + s_1 w^o{}_\theta]$$

$$- w^1{}_{\psi,\psi} - w^1{}_r\} + \sin\psi\{\tfrac{1}{2}(1+\nu)[s_0 w^1{}_{\theta,\psi} + s_1 w^o{}_{\theta,\psi}]$$

(3.2.6)
$$+ w^1{}_\psi[\nu - s_0^2(1-\nu)] - \nu w^1{}_{r,\psi} - 2s_0 s_1(1-\nu)w^o{}_\psi\}$$

$$+ \sin\psi\cos\psi\{w^o{}_{\psi,\psi} - s_0(3-\nu)w^o{}_\theta + w^o{}_r\}$$

$$+ \tfrac{1}{2}\sin^2\psi\{w^o{}_\psi[3s_0^2(1-\nu) - 2\nu] - s_0(1+\nu)w^o{}_{\theta,\psi}$$

$$+ 2\nu w^o{}_{r,\psi}\} + w^o{}_\psi\cos^2\psi\,,$$

$$F_2(w^2{}_\psi, w^2{}_\theta, w^2{}_r) = -\frac{1}{m} S^2_{r\theta}\Big|_{r=1}$$

$$+ s_1\{\tfrac{1}{2}(1+\nu)w^1{}_{\psi,\psi} + 2s_0 w^1{}_\theta + s_1 w^o{}_\theta + \nu w^1{}_r\}$$

$$+ s_2\{\tfrac{1}{2}(1+\nu)w^o{}_{\psi,\psi} + 2s_0 w^o{}_\theta + \nu w^o{}_r\}$$

(3.2.7)
$$+ \tfrac{1}{2}\cos\psi\{(3-\nu)[s_0 w^1{}_\psi + s_1 w^o{}_\psi] - (1-\nu)w^1{}_{\theta,\psi}\}$$

$$- \tfrac{1}{2}\sin\psi\{(1+\nu)[s_0 w^1{}_{\psi,\psi} + s_1 w^o{}_{\psi,\psi}] + [1-\nu+4s_0^2]w^1{}_\theta$$

$$- 8s_0 s_1 w^o{}_\theta - 2(1-\nu)[s_0 w^1{}_r + s_1 w^o{}_r]\}$$

$$+ \cos\psi\sin\psi\{\tfrac{1}{2}(1-\nu)w^o{}_{\theta,\psi} - s_0(3-\nu)w^o{}_\psi\}$$

F

(3.2.8)

$$+ \frac{1}{2}\sin^2\psi\{s_0(1+\nu)w^o{}_{\psi,\psi} + [1-\nu+6s_0^2]w^o{}_\theta$$

$$- 2s_0(2-\nu)w^o{}_r\} + \frac{1}{2}(1-\nu)w^o{}_\theta \cos^2\psi ,$$

$$F_3(w^2{}_\psi, w^2{}_\theta, w^2{}_r) = + \frac{1}{m} S^2_{rr}\big|_{r=1} + \nu\{s_1 w^1{}_\theta + s_2 w^o{}_\theta\}$$

$$+ \nu w^1{}_\psi \cos\psi + \sin\psi\{(1-\nu)[s_0 w^1{}_\theta + s_1 w^0{}_\theta]$$

(3.2.8)

$$+ \nu[2w^1{}_r + w^1{}_{\psi,\psi}]\} + (1-\nu)w^o{}_\psi \cos\psi \sin\psi$$

$$- \sin^2\psi\{\nu w^o{}_{\psi,\psi} + s_0(2-\nu)w^o{}_\theta - (1-2\nu)w^o{}_r\}.$$

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3.3 Corrections for Arbitrary n

In this section we develop the frequency equation as well as the first and second order correction equations for an arbitrary number of vibration waves around the tube. In subsequent sections we analyze these results for the fundamental case of axisymmetric flow in the zero order system.

In (2.5.3) we introduced the non-dimensional parameter $m = \rho h/\rho_0 a$, m being one-half the ratio of tube density per unit length to fluid density per unit length. If there is no fluid then $\rho_0 a = 0$ and m^{-1} is zero. In this situation the fluid stresses $S^k_{r\psi}$, $S^k_{r\theta}$ and S^k_{rr} $(k = 0,1,2,\ldots)$ evaluated on $r = 1$ have no influence on the shell displacements. The situation of $m^{-1} = 0$ is the case we are now investigating. We refer to it as the empty tube case.

We expect all shell displacements to have period 2π in ψ and we further expect that in the case of axisymmetric flow, displacement components are independent of ψ. These criteria suggest the following form for zero order displacements. It is

$$
\begin{aligned}
w^0_\psi &= W^0_\psi \sin(n\psi) \\
w^0_\theta &= W^0_\theta \cos(n\psi) \qquad n = 0,1,2,3,\ldots \\
w^0_r &= W^0_r \cos(n\psi)
\end{aligned}
\tag{3.3.1}
$$

For axisymmetric motion, we merely set $n = 0$ and the conditions mentioned previously are satisfied.

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Substitution of (3.3.1) into (3.2.2) gives the following system in homogeneous form. It is

(3.3.2)
$$\begin{pmatrix} \beta^2 - n^2 - \frac{1}{2} s_0^2(1-\nu) & -\frac{1}{2} ns_0(1+\nu) & -n \\ -\frac{1}{2} ns_0(1+\nu) & \beta^2 - s_0^2 - \frac{1}{2} n^2(1-\nu) & -\nu s_0 \\ -n & -\nu s_0 & \beta^2-1 \end{pmatrix} \begin{pmatrix} W^0{}_\psi \\ W^0{}_\theta \\ W^0{}_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In order that we have non-trivial solutions for this system we require that the determinant of the coefficient matrix vanish. It should be noted that replacing n by $-n$ does not alter the value of the determinant. Vanishing of the determinant leads to an equation of the form

(3.3.3)
$$s_0 = f(\beta, \nu, n)$$

which is known as the frequency equation. Given values of the frequency, Poisson's ratio and n we can compute a value for s_0, the zero order wave number. In the general case we find that s_0 can be obtained from

(3.3.4)
$$0 = s_0^4 \left(\frac{1-\nu}{2}\right)(\beta^2-1+\nu^2) - \beta^2 s_0^2\{(\beta^2-1)\tfrac{1}{2}(3-\nu) + \nu^2 - n^2(1-\nu)\} + \beta^4(\beta^2-1) - n^2\beta^2\{\tfrac{1}{2}(\beta^2-1-n^2)(1-\nu) + \beta^2\}.$$

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We will return to this equation when we investigate axisymmetric motion for the zero order.

Having found the zero order wave number, we proceed to our primary objective: determination of the first nonzero correction for the wave number. We begin with the first order shell equations, (3.2.3) through (3.2.5), since they contain s_1, the first correction to s_0. Examination of the right sides of these three equations indicates that the following forms are needed for our first order terms. They are

$$
\begin{aligned}
w^1_\psi &= a_\psi \cos(n+1)\psi + b_\psi \cos(n-1)\psi + c_\psi \sin(n+1)\psi \\
&\quad + d_\psi \sin(n-1)\psi + e_\psi \cos n\psi + h_\psi \sin n\psi \\
w^1_\theta &= a_\theta \sin(n+1)\psi + b_\theta \sin(n-1)\psi + c_\theta \cos(n+1)\psi \\
&\quad + d_\theta \cos(n-1)\psi + e_\theta \sin n\psi + h_\theta \cos n\psi \\
w^1_r &= a_r \sin(n+1)\psi + b_r \sin(n-1)\psi + c_r \cos(n+1)\psi \\
&\quad + d_r \cos(n-1)\psi + e_r \sin n\psi + h_r \cos n\psi .
\end{aligned}
\tag{3.3.5}
$$

These forms will allow for all possible trigonometric terms which might appear in the inhomogeneous terms of (3.2.3) through (3.2.5). All coefficients of trigonometric terms are constants.

Inserting (3.3.5) into (3.2.3) through (3.2.5) yields six systems of equations, three of which are homogeneous. The three homogeneous systems contain the $c_{(\cdot)}$, $d_{(\cdot)}$ and $e_{(\cdot)}$

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coefficients separately as column vectors. If we denote the coefficient matrix of (3.3.2) by $M_{(n)}$ then we may write the three homogeneous systems as

$$M_{(n+1)} \begin{pmatrix} c_\psi \\ c_\theta \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad M_{(n-1)} \begin{pmatrix} d_\psi \\ d_\theta \\ d_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(3.3.6)

$$M_{(-n)} \begin{pmatrix} e_\psi \\ e_\theta \\ e_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Our frequency equation is merely the result of $\det(M_{(n)}) = 0$ where det is the determinant. Since the determinants of the first two matrices in (3.3.6) are not identical to that of $M_{(n)}$ Cramer's rule allows us to conclude that the $c_{(\cdot)}$ and $d_{(\cdot)}$ coefficients are all zero. In the third system of (3.3.6) we conclude that the $e_{(\cdot)}$ coefficients are arbitrary and set them to zero.

The remaining systems are inhomogeneous and can be written as

$$(3.3.7) \quad M_{(-n-1)} \begin{pmatrix} a_\psi \\ a_\theta \\ a_r \end{pmatrix} = \begin{pmatrix} f_\psi \\ f_\theta \\ f_r \end{pmatrix} \qquad M_{(-n+1)} \begin{pmatrix} b_\psi \\ b_\theta \\ b_r \end{pmatrix} = \begin{pmatrix} g_\psi \\ g_\theta \\ g_r \end{pmatrix}$$

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$$(3.3.8) \qquad M_{(n)} \begin{pmatrix} h_\psi \\ h_\theta \\ h_r \end{pmatrix} = s_1 \begin{pmatrix} k_\psi \\ k_\theta \\ k_r \end{pmatrix}$$

$$(3.3.9) \qquad \begin{aligned} k_\psi &= \tfrac{1}{2}\, n(1+\nu) W^o{}_\theta + s_o(1-\nu) W^o{}_\psi \\ k_\theta &= \tfrac{1}{2}\, n(1+\nu) W^o{}_\psi + 2 s_o W^o{}_\theta + \nu\, W^o{}_r \\ k_r &= \nu\, W^o{}_\theta \end{aligned}$$

In (3.3.7) the inhomogeneous terms are all known in terms of zero order displacements. Again, the determinants of the coefficient matrices are non-zero and so we can solve for the $a_{(\cdot)}$ and $b_{(\cdot)}$ coefficients by Cramer's rule.

In (3.3.8) we cannot solve by Cramer's rule since the determinant is zero, this being our frequency equation. We adopt a different approach and multiply both sides of (3.3.8) by the row vector $(W^o{}_\psi, W^o{}_\theta, W^o{}_r)$ and take the transpose of the left most pair of terms. Symmetry of M_n allows us to write the result as

$$(3.3.10) \qquad \left(M_{(n)} \begin{pmatrix} W^o{}_\psi \\ W^o{}_\theta \\ W^o{}_r \end{pmatrix} \right)^t \begin{pmatrix} h_\psi \\ h_\theta \\ h_r \end{pmatrix} = s_1 (W^o{}_\psi, W^o{}_\theta, W^o{}_r) \begin{pmatrix} k_\psi \\ k_\theta \\ k_r \end{pmatrix}$$

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where the superscript t indicates the transpose. The quantity being transposed is the zero order homogeneous system (3.3.2) and is zero. Our conclusion is then that s_1 is zero as the product of the row and column vectors on the right side of (3.3.9) is not zero.

Knowing that $s_1 = 0$ is of modest importance so we continue with the same procedure to an analysis of the second order equations, hoping to isolate s_2 as we did s_1, but with a non-trivial result.

From (3.3.7) we can compute the first order displacements involving the coefficients $a_{(\cdot)}$ and $b_{(\cdot)}$. Since $s_1 = 0$ we conclude that the $h_{(\cdot)}$ coefficients are arbitrary and set them zero since the relation among them is already present in the zero order displacements. Having the first order terms computed we can now determine all the inhomogeneous terms in the second order shell equations, (3.2.6) through (3.2.8).

For second order velocities we should have terms involving $\cos(n\pm2)\psi$, $\cos(n\pm1)\psi$, $\cos n\psi$ as well as the sines of these same arguments. This gives rise to ten systems of equations, one-half of which are homogeneous. Our concern is only with those equations involving $\cos(n\psi)$ or $\sin(n\psi)$ since the second order correction, s_2, appears only with these trig

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these trigonometric functions. We designate these components of $w^2{}_\psi$ by $A_\psi \sin n\psi + B_\psi \cos n\psi$, these components of $w^2{}_\theta$ by $A_\theta \cos n\psi + B_\theta \sin n\psi$ and these components of $w^2{}_r$ by $A_r \cos n\psi + B_r \sin n\psi$. Again, there are more components for these second order displacements but our concern is only with these six.

We write our two systems as

$$\text{(3.3.11)} \qquad M_{(n)} \begin{pmatrix} A_\psi \\ A_\theta \\ A_r \end{pmatrix} = \begin{pmatrix} F_\psi \\ F_\theta \\ F_r \end{pmatrix} \qquad M_{(-n)} \begin{pmatrix} B_\psi \\ B_\theta \\ B_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

As before the $B_{(\cdot)}$ coefficients are arbitrary and set equal to zero. Our next task is to determine the $F_{(\cdot)}$ coefficients. This requires that products of the form $\cos\psi \cos(n\underline{+}1)\psi$ be rewritten as $\frac{1}{2}\{\cos(n\underline{+}2)\psi + \cos n\psi\}$. In fact we have to rewrite all such products, $\sin\psi \sin(n\underline{+}1)\psi$, $\sin\psi \cos(n\underline{+}1)\psi$ and $\cos\psi \sin(n\underline{+}1)\psi$, in a similar fashion so we can identify the coefficients of $\cos n\psi$ or $\sin n\psi$ in the inhomogeneous terms. We find then that

$$\begin{aligned} F_\psi &= s_2\{\tfrac{n}{2}(1+\nu)W^o{}_\theta + s_0(1-\nu)W^o{}_\psi\} \\ &\quad + \tfrac{1}{2}\{\tfrac{1}{2}\, s_0[(3-\nu)(a_\theta+b_\theta) + (1+\nu)(b_\theta(n-1) - a_\theta(n+1))] \\ \text{(3.3.12)} \qquad &\quad + (n+1)a_\psi + (n-1)b_\psi - (a_r+b_r) - \nu(a_r+b_r) \\ &\quad - (\nu-s_0^2(1-\nu))(a_\psi-b_\psi)\} \end{aligned}$$

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$$F_\theta = s_2\{\tfrac{1}{2}\, n(1+\nu)W^o{}_\psi + 2s_0W^o{}_\theta + \nu W^o{}_r\}$$

$$+ \tfrac{1}{4}\{(3-\nu)s_0(a_\psi+b_\psi) + s_0(1+\nu)[(n+1)a_\psi - (n-1)b_\psi]$$

$$- (1-\nu+4s_0^2)(a_\theta-b_\theta) - (1-\nu)[(n+1)a_\theta + (n-1)b_\theta] \tag{3.3.13}$$

$$- 2(1-\nu)s_0(a_r-b_r)\} + \tfrac{1}{4}\{ns_0(1+\nu)W^o{}_\psi$$

$$+ 2(1-\nu+3s_0^2)W^o{}_\theta - 2s_0(2-\nu)W^o{}_r\}$$

$$F_r = \nu s_2W^o{}_\theta + \tfrac{1}{2}\,\nu(a_\psi+b_\psi) + \tfrac{1}{2}\,\{\nu[(n-1)b_\psi - (n+1)a_\psi]$$

$$+ (1-\nu)s_0(a_\theta-b_\theta) + 2\nu(a_r-b_r) \tag{3.3.14}$$

$$- \nu nW^o{}_\psi - s_0(2-\nu)W^o{}_\theta + (1-2\nu)W^o{}_r\}.$$

The process used in determining s_1 is repeated in this case and the result is

$$0 = W^o{}_\psi F_\psi + W^o{}_\theta F_\theta + W^o{}_r F_r \,. \tag{3.3.15}$$

Since s_2 is not a common factor as s_1 was, it is not identically zero and we are able to write an expression of the form

$$s_2 = G(s_0, \beta, \nu, n) \tag{3.3.16}$$

which gives the second order correction to the zero order wave number s_0.

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We are now in a position to analyze the results of this section for the fundamental case of axisymmetric flow in the zero order tube motion. For this case we set $n = 0$ and proceed to the next section. In all that follows, the assumption of axisymmetric zero order flow will be made.

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3.4 Axisymmetric Frequency Equation

We are now considering axisymmetric vibration of our empty tube. We must therefore return to the frequency equation (3.3.4) and set $n = 0$. The result can be factored to give

$$\{s_0^2(\frac{1-\nu}{2}) - \beta^2\}\{s_0^2 - \frac{\beta^2(\beta^2-1)}{\beta^2-1+\nu^2}\}(\beta^2-1+\nu^2) = 0. \tag{3.4.1}$$

Setting the first term to zero gives us a frequency equation appropriate to torsion motion in which W^o_ψ is arbitrary and W^o_θ and W^o_r are zero. Setting the second factor zero is the frequency equation for axisymmetric flow. In this instance W^o_ψ is zero as can be seen from setting $n = 0$ in the first equation of (3.3.2). Hence

$$s_0 = \beta(\frac{\beta^2-1}{\beta^2-1+\nu^2})^{1/2} \tag{3.4.2}$$

is the frequency equation for the axisymmetric vibration of an empty, thin, elastic membrane.

The equation (3.4.2) has been determined by Rubinow and Keller [9] in a slightly different setting. The following analysis parallels theirs.

For frequencies in the range $(1-\nu^2)^{1/2} \leq \beta < 1$ s_0 is pure imaginary and the waves do not propagate. For frequencies outside this range the wave number s_0 is real and the waves move down the tube. When $\nu = 0$ the frequency equation reduces to $s_0 = \beta$.

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As β approaches $(1-\nu^2)^{1/2}$ s_0 becomes arbitrarily large. At $\beta = 1$ the wave number is zero but soon approaches the value of β as β increases. A graph of (3.4.2) is presented in Figure 2.

Two auxiliary functions which are of benefit to our discussion are the phase velocity c_p and group velocity c_g, defined as

$$c_p = \beta/\mathrm{Re}(s_0) \tag{3.4.3}$$

$$c_g = d\beta/d\ \mathrm{Re}(s_0). \tag{3.4.4}$$

The phase velocity gives an indication of the speed of a traveling wave as it travels down the tube. The group velocity denotes the speed at which the energy is propagated. It is the velocity at which a packet of waves is propagated, all waves having similar wave length. [Kolsky, 20]

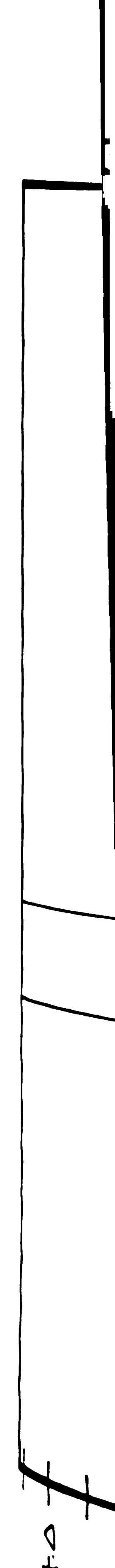
We find that

$$c_p = \left(\frac{\beta^2-1+\nu^2}{\beta^2-1}\right)^{1/2}, \quad \begin{array}{l} 0 < \beta < (1-\nu^2)^{1/2}, \\ 1 < \beta \end{array} \tag{3.4.5}$$

and that it is undefined outside these frequency ranges.

Differentiation of (3.4.2) with respect to $\mathrm{Re}\ s_0$ gives

$$c_g = c_p\left\{1 + \left(\frac{\nu\beta}{c_p(1-\beta^2)}\right)^2\right\}^{-1}. \tag{3.4.6}$$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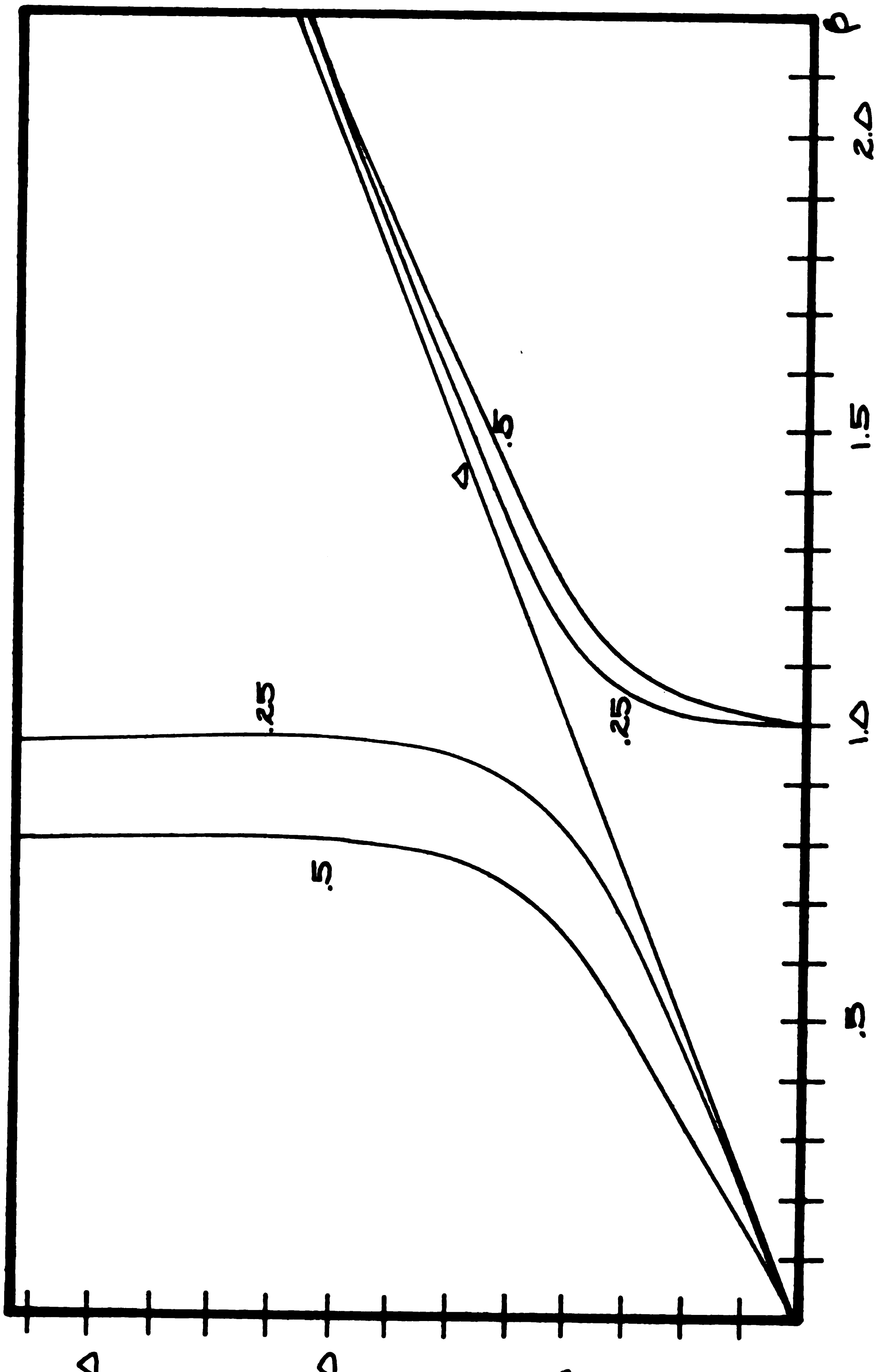

FIGURE #2
EMPTY TUBE FREQUENCY EVALUATION
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This function is zero at $\beta = (1-\nu^2)^{1/2}$ and at $\beta = 1$ and has no poles for real values of β. Graphs of c_p and c_g are given in Figure 3. If $\nu = 0$ then both c_p and c_g are 1. If the frequency is zero then the phase velocity is $(1-\nu^2)^{1/2}$, decaying to zero as the frequency approaches $(1-\nu^2)^{1/2}$. The same is true for the group velocity. For $\beta = 1$ the group velocity is zero while the phase velocity is infinite. Both approach the value one asymptotically as the frequency increases beyond one.

Let us now consider the ratio of radial displacements to longitudinal displacement. Solving (3.3.2) for $W^o{}_\theta$ and $W^o{}_r$ when $W^o{}_\psi = 0 = n$ we find

$$\left|\frac{W^o{}_r}{W^o{}_\theta}\right| = \left|\frac{\nu\beta}{(\beta^2-1)^{1/2}(\beta^2-1+\nu^2)^{1/2}}\right| . \tag{3.4.7}$$

When the frequency is near $(1-\nu^2)^{1/2}$ or 1 we see that the displacement ratio becomes large indicating primarily radial motion, while for small or large frequencies the motion is mainly longitudinal, i.e. in the direction of θ changing. The graph of (3.4.7) is presented in Figure 4 and demonstrates the behavior mentioned. All curves have a vertical asymptote at $\beta = 1$ while each has a separate asymptote at $\beta = (1-\nu^2)^{1/2}$ which is just to the left of $\beta = 1$.

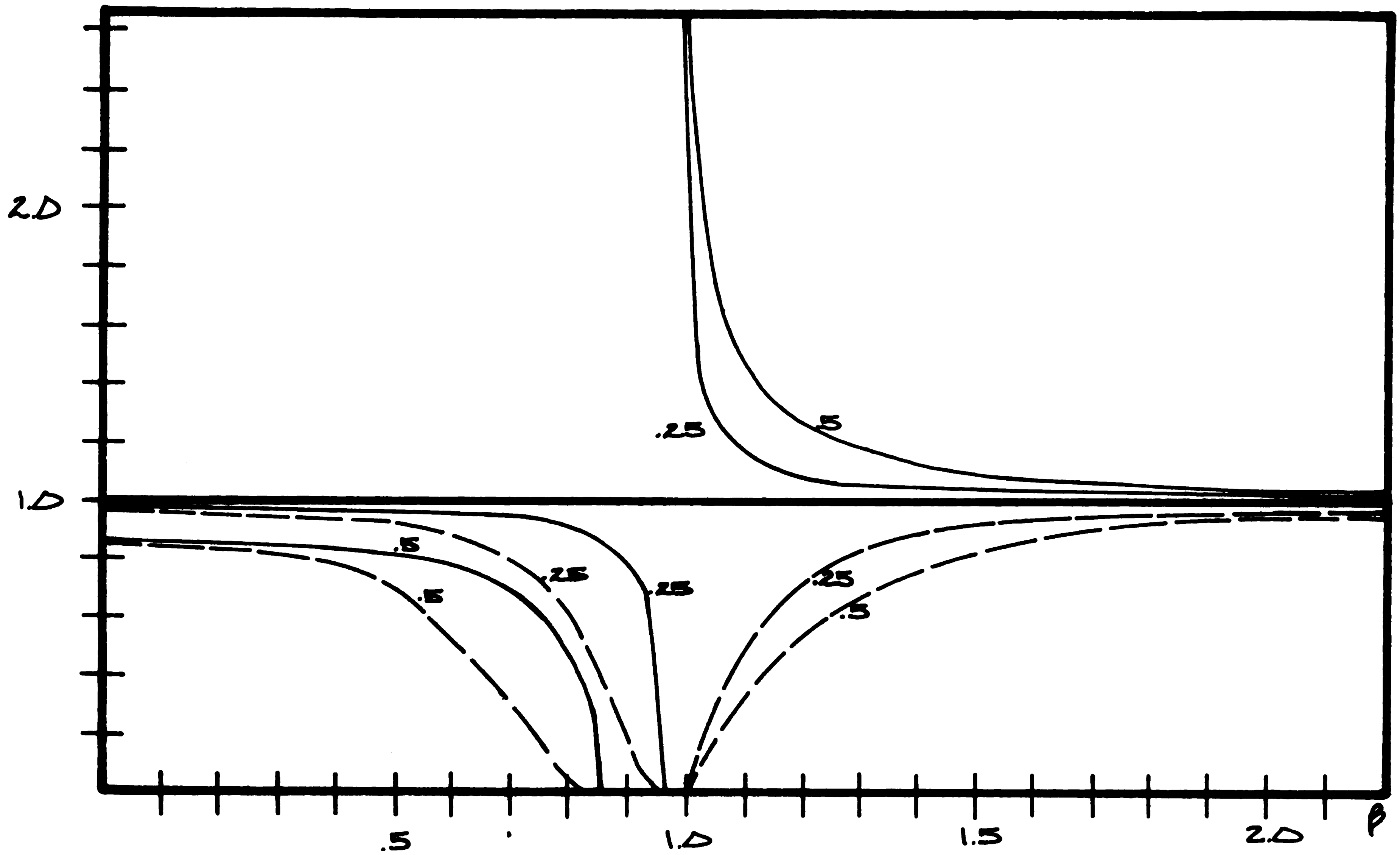

FIGURE #3
ζ_p(—) AND ζ_g(- - -) FOR THE EMPTY TUBE
ν VALUES SHOWN

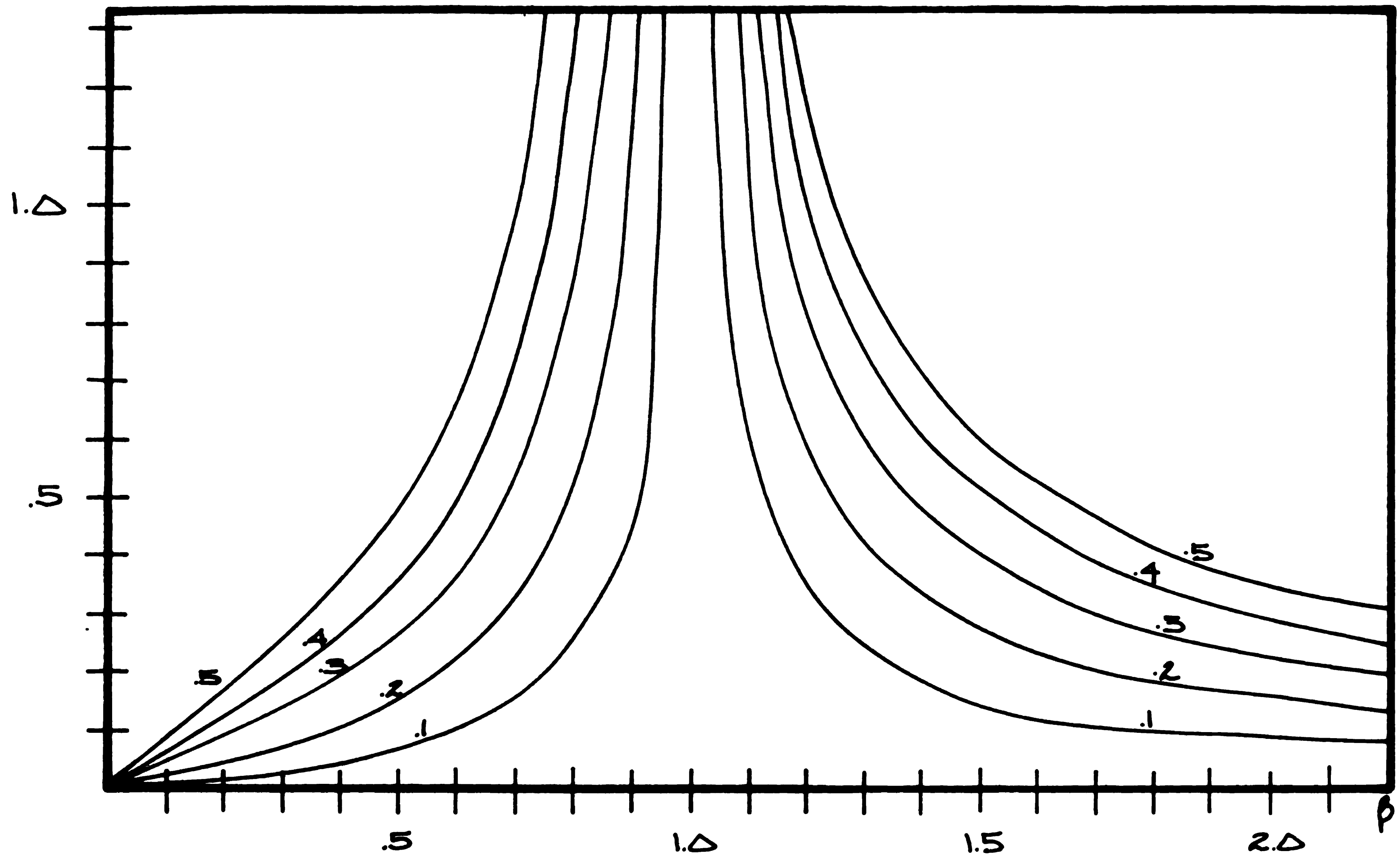

FIGURE #4
EMPTY TUBE DISPLACEMENT RATIO: ZERO ORDER
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3.5 First Order Displacements

In section 3.3 we demonstrated that for any value of n, $s_1 = 0$, but $s_2 \neq 0$. To determine the correction s_2 we must first obtain the first order displacements. We do so now.

For axisymmetric flow we set $n = 0$ in (3.3.7) and arrive at one inhomogeneous system in $\cos\psi$ and $\sin\psi$. It is

$$\begin{pmatrix} \beta^2-1-\frac{1}{2}s_0^2(1-\nu) & \frac{1}{2}s_0(1+\nu) & 1 \\ \frac{1}{2}s_0(1+\nu) & \beta^2-s_0^2-\frac{1}{2}(1-\nu) & -\nu s_0 \\ 1 & -\nu s_0 & \beta^2-1 \end{pmatrix} \begin{pmatrix} a_\psi \\ a_\theta \\ a_r \end{pmatrix} = \begin{pmatrix} f_\psi \\ f_\theta \\ f_r \end{pmatrix} . \tag{3.5.1}$$

From (3.3.2), with $n = 0$, we write

$$\begin{aligned} W_\theta^0 &= \nu s_0(\beta^2-s_0^2)^{-1}\, W_r^0 & \nu &\neq 0 \\ W_r^0 &= 0 & \nu &= 0 \end{aligned} \tag{3.5.2}$$

Utilizing these relations we find

$$\begin{aligned} f_\psi &= W_r^0\{\tfrac{1}{2}\,\nu s_0^2(3-\nu)(\beta^2-s_0^2)^{-1} - 1\}, \\ f_\theta &= W_r^0\{s_0(1-\nu) - \tfrac{1}{2}\nu s_0(1-\nu+4s_0^2)(\beta^2-s_0^2)^{-1}\}, \\ f_r &= W_r^0\,\nu\{2+s_0^2(1-\nu)(\beta^2-s_0^2)^{-1}\}, \end{aligned} \tag{3.5.3}$$

when $\nu \neq 0$ and when $\nu = 0$

$$\begin{aligned} f_\psi &= 3s_0W_\theta^0/2, \\ f_\theta &= \tfrac{1}{2}(4s_0^2-1)W_\theta^0 , \\ f_r &= s_0W_\theta^0 . \end{aligned} \tag{3.5.4}$$

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Knowing the values of f_ψ, f_θ and f_r given in (3.5.3), (3.5.4) we now solve (3.5.1) for a_ψ, a_θ and a_r by Cramer's rule. This procedure utilizes the determinant of the coefficient matrix in (3.5.1).

We find that the determinant does not vanish identically when s_0 satisfies (3.4.2), however for ν in the range $0 \leq \nu \leq 3 - (8)^{1/2}$ there are two values of the frequency, β, which cause the determinant to vanish. They are found as the solution to

(3.5.5) $$\beta^2 = \frac{1}{2}(\nu-1)\{\nu - 3 \pm [\nu^2-6\nu+1]^{1/2}\}.$$

If we allow the frequencies determined by (3.5.5) to occur, consistency requires that we set the zero order arbitrary constant, W_r^0, equal to zero. In this situation the zero order system is replaced by the first order system as being fundamental. Thus, allowing the determinant of the coefficient matrix in (3.5.1) to vanish is merely defining a frequency equation for the case of vibrations exhibiting one wave around the tube in the ψ direction. We thus restrict our frequencies to those for which the determinant of the coefficient matrix in (3.5.1) is non-zero.

Denoting the cofactors of elements a_{ij} in the coefficient matrix of (3.5.1) by A_{ij} we write

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$$a_\psi = \frac{W_r^o}{\mathrm{Det}} \{A_{11}\tilde{f}_\psi + A_{12}\tilde{f}_\theta + A_{13}\tilde{f}_r\},$$

$$(3.5.6) \qquad a_\theta = \frac{W_r^o}{\mathrm{Det}} \{A_{12}\tilde{f}_\psi + A_{22}\tilde{f}_\theta + A_{23}\tilde{f}_r\}, \quad \nu \neq 0$$

$$a_r = \frac{W_r^o}{\mathrm{Det}} \{A_{13}\tilde{f}_\psi + A_{23}\tilde{f}_\theta + A_{33}\tilde{f}_r\},$$

where the tildas reflect division of the functions f_ψ, f_θ and f_r by W_r^o and Det is the determinant of the coefficient matrix of (3.5.1). When $\nu = 0$ we designate the a_ψ, a_θ and a_r by tildas and find that

$$\tilde{a}_\psi = -W_\theta^o \{(\beta^2-1)(1+2\beta^2)-1\}/\beta(2-\beta^2),$$

$$(3.5.7) \qquad \tilde{a}_\theta = 2W_\theta^o \{\beta^4-4\beta^2+2\}/(2-\beta^2),$$

$$\tilde{a}_r = W_\theta^o (\beta^2+2)/\beta(2-\beta^2).$$

Before we proceed to the determination of the second order correction, s_2, it is beneficial to examine the corrected displacement ratio,

$$|(W_r^o + \delta a_r \sin\psi)/(W_\theta^o + \delta a_\theta \sin\psi)|,$$

as a function of the frequency for fixed values of ψ, δ and ν. Specifically, we choose $\psi = 30°$ and $\psi = -60°$ where both angles are measured from the north pole. Remembering that δ has value .15 to .3 in the human aorta, we choose for δ the values .005, .05 and .5 as being represen

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Case I.

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representative. Poisson's ratio for biological materials is nearly .5 but we consider ν over the range [0,.5] by considering various sub-intervals for ν.

Case I. $\nu = 0$

When $\nu = 0$ we find the corrected displacement ratio has the following value

$$(3.5.8) \qquad \left|\frac{\delta\tilde{a}_r \sin\psi}{W_\theta^0+\delta\tilde{a}_\theta \sin\psi}\right| = \frac{1}{\beta}\left|\frac{\delta(\beta^2+2)\sin\psi}{2-\beta^2+2\delta\sin\psi(\beta^4-4\beta^2+2)}\right|$$

In the zero order case the displacement ratio is zero for all frequencies. However, for the corrected displacement ratio we see some new behavior. When ψ is an integer multiple of π the motion is completely longitudinal, the ratio being zero. However, for other values of ψ the ratio exhibits both radial and longitudinal components. When the denominator is zero the motion is purely radial. If we designate $\delta \sin\psi$ by ϵ in (3.5.8) we find the zeroes of the denominator at frequencies given by

$$(3.5.9) \qquad \beta^2 = (4\epsilon)^{-1}\{1+8\epsilon \pm [1+32\epsilon^2]^{1/2}\}.$$

The approximate values are given by

$$(3.5.10) \qquad \beta \simeq \{(1+4\epsilon)/2\epsilon\}^{1/2} \quad \text{and} \quad \beta \simeq (2-2\epsilon)^{1/2}.$$

When the $\sin\psi$ is negative the first root exists only for imaginary frequencies and hence the only real zero of the denominator occurs at frequencies slightly greater

than $2^{1/2}$. If the sin ψ is positive, then one root is slightly less than $2^{1/2}$ and the other is given by the first approximation in (3.5.10).

The graphs of the ratio in (3.5.8) are given for the two cases of $\psi = -60°$ and $\psi = 30°$ in Figures 5 and 6 respectively. In Figure 5 we see two frequency ranges for which the motion is primarily radial, one near $\beta = 0$ and the other between $\beta = 2^{1/2}$ and $\beta = 2$. At other frequencies especially those greater than $\beta = 3$, the displacement is nearly zero though not exactly zero. In these regions we can be satisfied with the straight tube model which predicts only longitudinal motion for $\nu = 0$. At the two frequencies mentioned above we see that there is a marked difference between the zero order theory and the corrected one. For frequencies less than 2 the $\delta = .5$ exhibits large quantitative differences from the smaller δ values, even though the general behavior is the same. For frequencies greater than two the $\delta = .5$ curves are very close to the smaller δ value curves.

In Figure 6 we see the effect of the extra zero in the denominator of the displacement ratio. Now there are three verticle asymptotes for our curves. One is at $\beta = 0$, one is near $\beta = 2^{1/2}$ and the third is near $\beta = \{(1+4\epsilon)/2\epsilon\}^{1/2}$ where $\epsilon = \delta \sin \psi$. For small δ values this asymptote

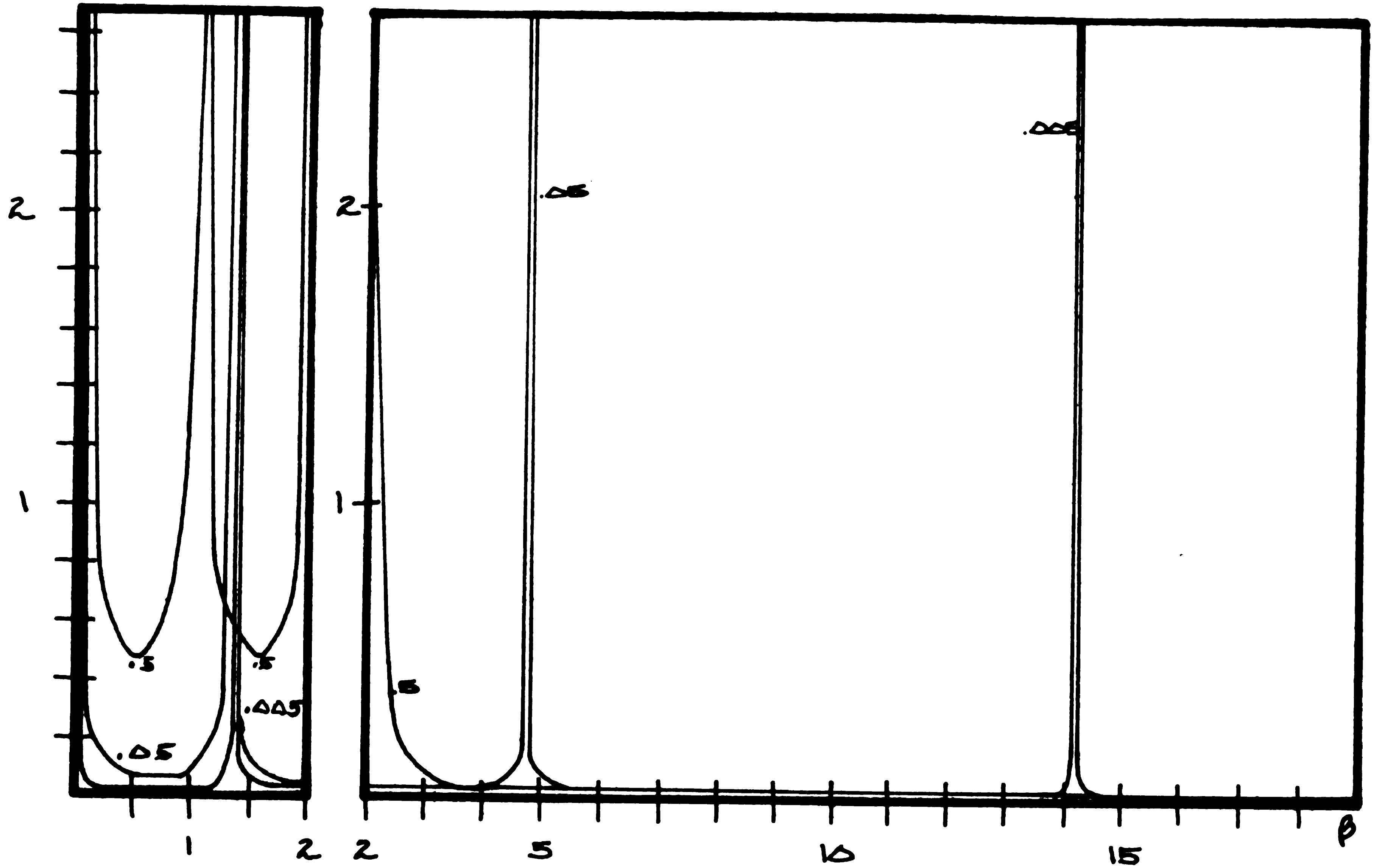

FIGURE #6
CORRECTED DISPLACEMENT RATIO
δ VALUES SHOWN $U = 0$ $\Psi = 30°$

occurs at a rather large frequency. For larger values of δ the asymptote moves to the left approaching the frequency $\beta = 2^{1/2}$. The change in abscissa at $\beta = 2$ was incorporated to show the radial motion present for $\beta = 0$ and β near $2^{1/2}$. The curves for $\delta = .005$ and $\delta = .05$ are almost identical especially at the asymptotes. The $\delta = .5$ curves is again different from the other δ values in degree though it too exhibit the three regions of radial motion. So we have stated before near those frequencies at which the motion is primarily radial the zero order straight tube model is insufficient and higher order terms are required. For large frequencies the two models give essentially the same results.

Case II. $\nu \neq 0$

From an examination of the displacement ratio using (3.5.6) we see that the ratio of radial to longitudinal displacement is like s_0^{-1} and so at $\beta = 0$ and $\beta = 1$ the ratio becomes unbounded indicating primarily radial motion. The same holds true for $\beta = (1-\nu^2)^{1/2}$. However, for large frequencies the corrected displacement ratio is like β^{-1} and hence becomes nearly zero for large frequencies. Again we find that the motion is longitudinal at high frequencies with radial motion predominating at selected smaller frequencies.

In Figures 7 and 8 we present graphs of the corrected displacement ratio for $\nu = .25$ and the two ψ values used previously, $\psi = -30°$ and $\psi = 60°$. In Figures 9 and 10 the value of Poisson's ratio is increased to .5. In neither case do we intend these curves to be representative of the entire spectrum of ψ and ν values. Instead, they are meant to shed some light on the nature of this corrected ratio of displacements.

It should be noted that the zeroes of the determinant (Det) are not the dominant influence in the ratio as the determinant is present in both numerator and denominator.

In comparing Figures 7 and 9 we notice little difference between the two graphs. Both approach zero as the frequency grows large indicating longitudinal motion. For $\nu = .25$ there are two regions for $\delta = .5$ which are off scale. Both are "parabolic" shaped, the first situated between $\beta = 1$ and $\beta = 1.02$, while the second occurs between 1.02 and 1.28. These portions of the $\delta = .5$ curve in Figure 7 are another indication of the differences between curves with $\delta = .5$ and those with lesser values of δ. The radial motion exhibited near $\beta = (1-\nu^2)^{1/2}$ and $\beta = 1$ is predicted by the zero order displacement ratio. Hence, the straight tube model is a rather accurate representation of the curved tube except for very small frequencies.

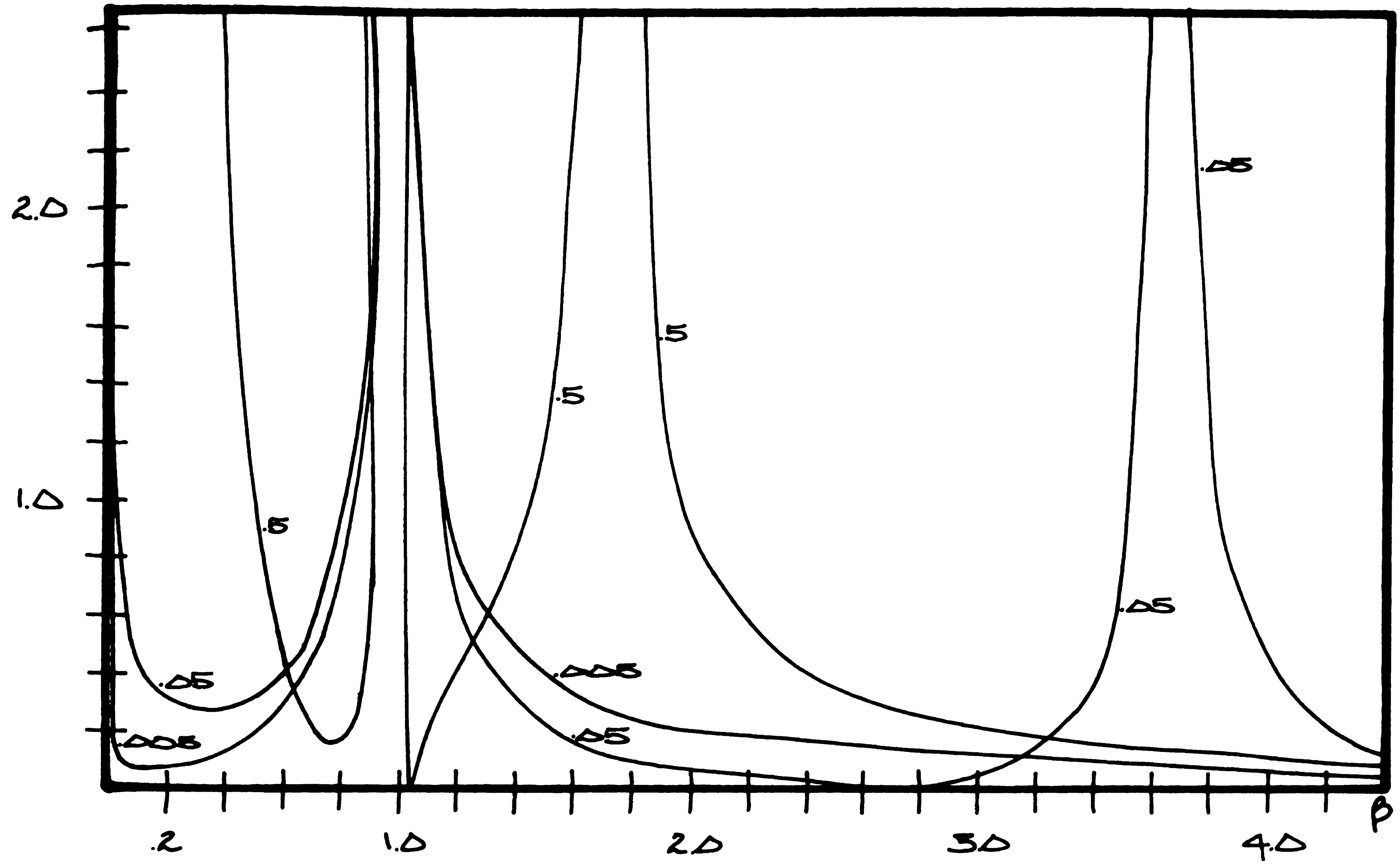

FIGURE #8
CORRECTED DISPLACEMENT RATIO
δ VALUES SHOWN $\nu = .25$ $\psi = -60°$

When $\psi = -60°$ we again notice little difference between ν values of .25 and .5, as seen in Figures 8 and 10. Both curves exhibit a section for $\delta = .5$ which is off scale in the frequency range $0 < \beta < .3$. For both of these cases the $\delta = .005$ curve rapidly approaches zero for frequencies larger than one. All curves approach zero for frequencies greater than two but for $\delta = .05$ and $\delta = .5$ we see one frequency region for each in which radial motion is predominate. These values are not found in the zero order case, indicating the unsuitability of the straight tube model in which it is assumed that the flow is axially symmetric. Indeed, from an examination of equations (3.5.1) and (3.3.2), one notices that if one were to set $n = -1$ in (3.3.2) we would obtain the same matrix in (3.5.1) and (3.3.2). This suggests that the motion predicted by our corrected system (3.5.1) is essentially the same that one would obtain in a straight tube model if one were to assume motion with one circumferential wave $(n = 1)$.

In summary we might say that on the whole the straight tube model can be used in some curved tube situations if the ratio $\delta = a/R$ is not too great and if certain select frequencies are avoided. In these last instances the need for higher order terms becomes apparent.

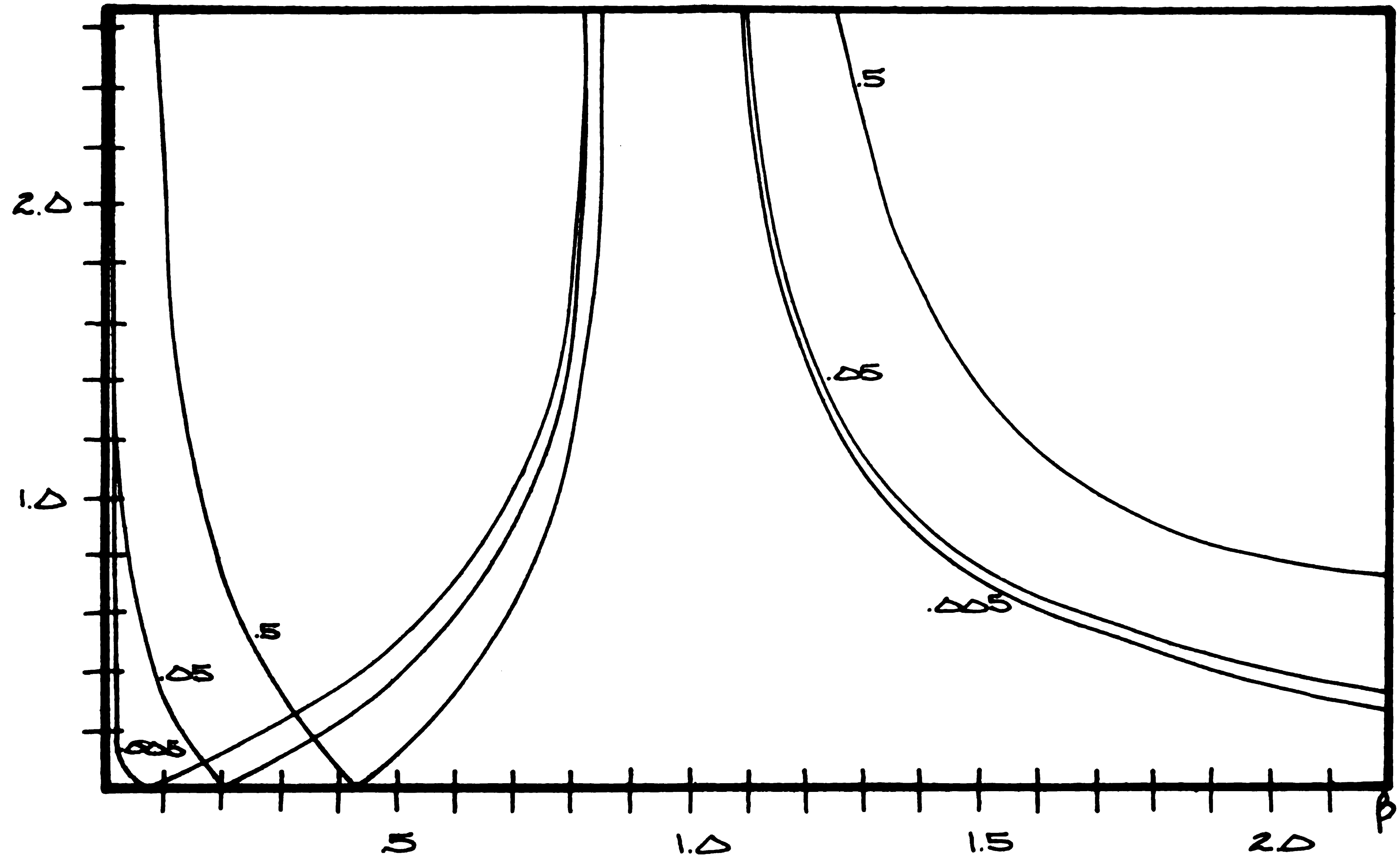

FIGURE #9
CORRECTED DISPLACEMENT RATIO
δ VALUES SHOWN $\nu = .5$ $\phi = 30°$

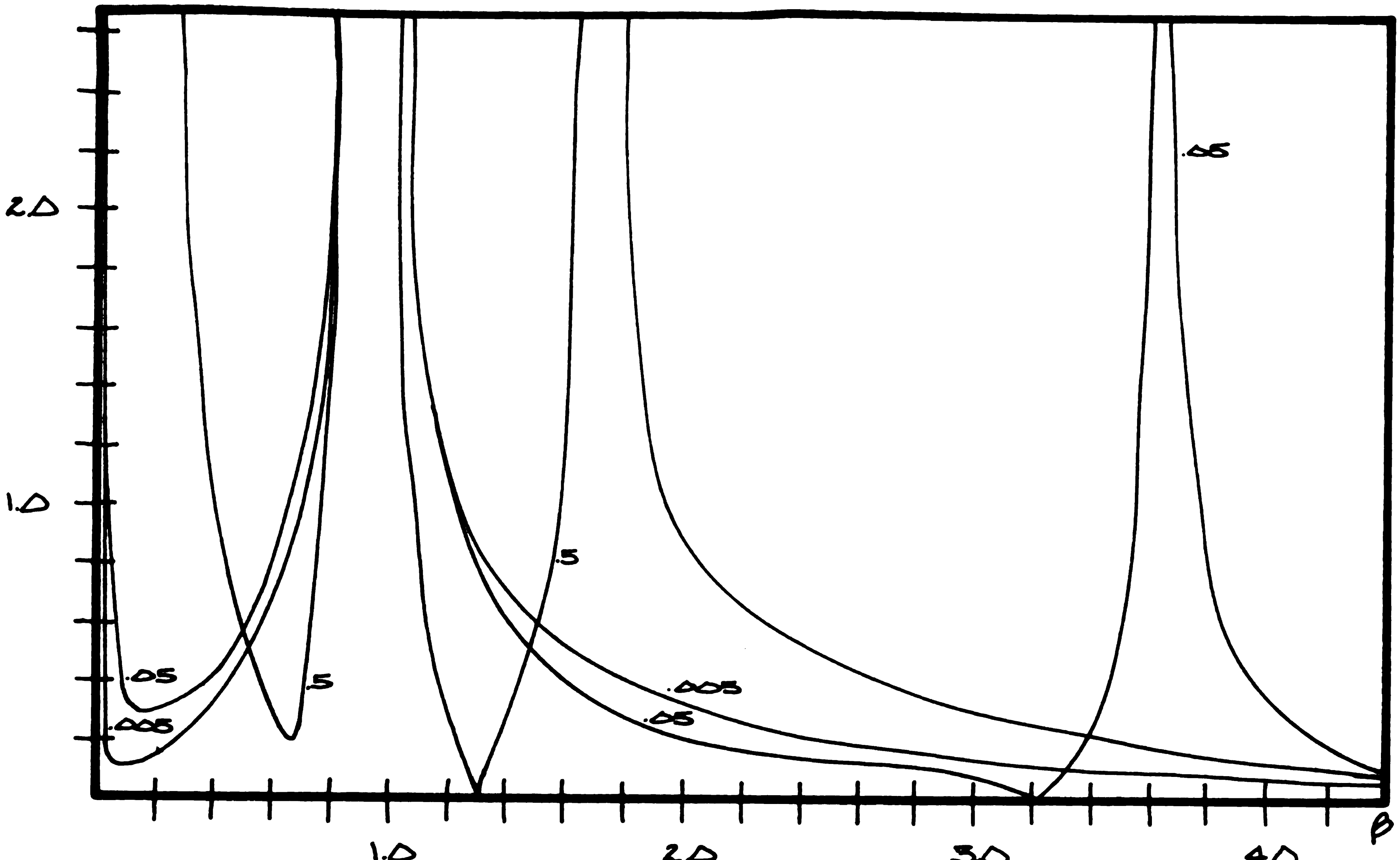

FIGURE #10
CORRECTED DISPLACEMENT RATIO
δ VALUES SHOWN $\nu = .5$ $\psi = -60°$

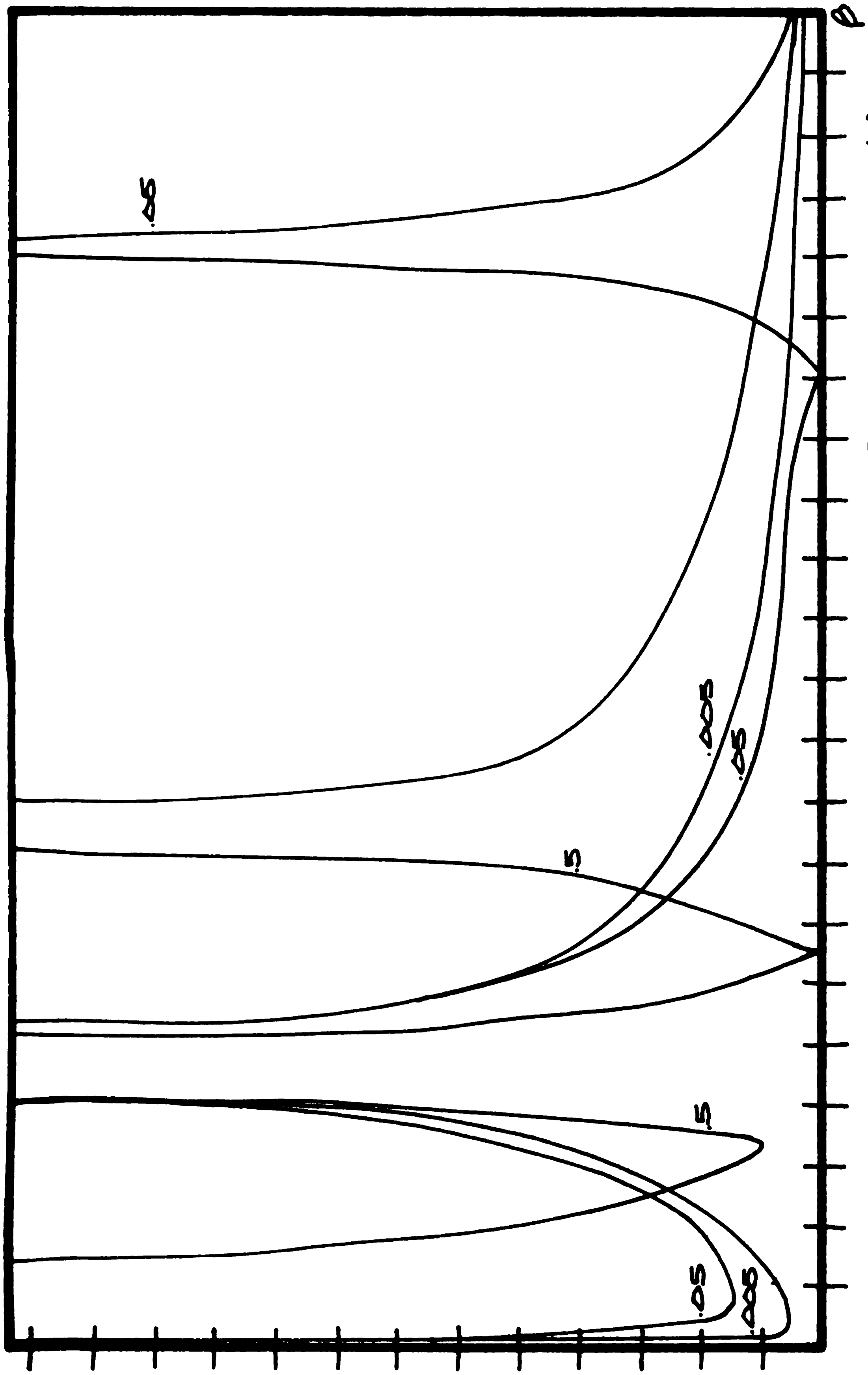

FIGURE #10
CORRECTED DISPLACEMENT RATIO
δ VALUES SHOWN 2-.5 4-.60°

3.6 Second Order Correction - Axisymmetric Case

We know from the first order displacements that they have the following form for axisymmetric flow up through the first order

$$
\begin{aligned}
& w_\psi = \delta a_\psi \cos\psi, \\
(3.6.1)\qquad & w_\theta = W_r^0 \nu s_0(\beta^2 - s_0^{\,2})^{-1} + \delta a_\theta \sin\psi, \quad \nu \neq 0, \\
& w_r = W_r^0 + \delta a_r \sin\psi,
\end{aligned}
$$

where a_ψ, a_θ and a_r are given by (3.5.6). For $\nu = 0$ we have

$$
\begin{aligned}
& w_\psi = \delta\tilde{a}_\psi \cos\psi, \\
(3.6.2)\qquad & w_\theta = W_\theta^0 + \delta\tilde{a}_\theta \sin\psi, \quad \nu = 0, \\
& w_r = \delta\tilde{a}_r \sin\psi,
\end{aligned}
$$

where $\tilde{a}_\psi$, $\tilde{a}_\theta$ and $\tilde{a}_r$ are given by (3.5.7). In either case we can proceed to the determination of the second order correction, s_2, as a function of s_0, β, and ν.

For axisymmetric flow $W_\psi^0 = 0$, and the correction can be found from (3.3.15).

Case I. $\nu = 0$.

In this case $W_r^0 = 0$ and the correction can be found by setting $F_\theta = 0$ along with n, W_ψ^0 and W_r^0. In addition, $s_0 = \beta$, and so we obtain

$$(3.6.3)\qquad s_2 = \frac{1}{4\beta(2-\beta^2)}\{4\beta^6 - 7\beta^4 - 6\beta^2 - 4\}, \quad \nu = 0.$$

As β nears zero or the value $2^{1/2}\, s_2$ becomes unbounded but when the frequency grows large the correction behaves like β^3 becoming larger than the zero order term it corrects. When the correction becomes larger than δ^{-2} we find we should include higher order terms.

We propose to present the corrected phase velocity $\tilde{c}_p$ which is given by

$$\tilde{c}_p = \beta(s_0+\delta^2 s_2)^{-1} . \tag{3.6.5}$$

It is instructive to compare this value to the zero order phase velocity which is 1 when $\nu = 0$. for $\nu \neq 0$ we will also consider the ratio $(s_0+\delta^2 s_2)/s_0$ which gives some idea of where the corrected wave number varies from the zero order wave number. When $\nu = 0$ this ratio is merely $\tilde{c}_p^{\,-1}$ and so it will not be discussed.

We present the corrected phase velocity for $\nu = 0$ in Figure 11. The reader should note the change in scale which allows us to examine the region $0 < \beta < 2$ in more detail. For the area designated by A in the lower left corner the curve for $\delta = .005$ is nearly a vertical line coinciding with the axis. In the upper portion B the curve for $\delta = .5$ is off scale though of the same shape as the other two curves. This is a further indication that $\delta = .5$ is much too large a parameter for our expansion. The curves for $\delta = .05$ and $\delta = .005$ are almost always near one in value which is the zero order phase velocity. At selected frequencies the corrected phase velocity differs from one up to $\beta = 2$. After a frequency of two the corrected phase velocities approach zero. The curve for $\delta = .005$ will gradually become zero, but at much higher frequencies. It is in this region that the zero order predictions are inadequate especially if $\delta = .05$ or $.5$. However, for frequencies less than two, the straight tube model is quite accurate.

Case II. $0 < \nu < 3 - (8)^{1/2}$

When $\nu \neq 0$ we have the correction coming from the expression

$$0 = W_\theta^0 F_\theta + W_r^0 F_r \tag{3.6.5}$$

which is merely (3.3.15) with $W_\psi^0 = 0$ for axisymmetric vibrations. In solving (3.6.5) for s_2 we find that s_2 has the following value

$$\begin{aligned} s_2 = -\frac{(\beta^2-s_0{}^2)^2}{4\nu^2 s_0 \beta^2 W_r^0}\{&2\nu s_0{}^2(\beta^2-s_0{}^2)^{-1}a_\psi \\ &+ s_0 a_\theta[1-\nu-\nu(\beta^2-s_0{}^2)^{-1}(2s_0{}^2+1-\nu)] \\ &+ \nu[2+s_0{}^2(1-\nu)(\beta^2-s_0{}^2)^{-1}]a_r + W_r^0[1-2\nu \\ &+ \nu s_0(\nu s_0-4s_0+\nu)(\beta^2-s_0{}^2)^{-1} \\ &+ \nu^2 s_0{}^2(3s_0{}^2+1-\nu)(\beta^2-s_0{}^2)^{-2}]\} \end{aligned} \tag{3.6.6}$$

in which we have used (3.5.2).

For ν in the range $0 < \nu < \{3-(8)^{1/2}\}$ the determinant of the matrix in (3.5.1) has two real zeroes, the positive zeroes being near $\beta = 1$ and $\beta = (2)^{1/2}$. Since the determinant appears in the denominator of the a_ψ, a_θ and a_r terms, the correction, s_2, becomes unbounded at these zeroes. For s_0 small, s_2 is like s_0^{-1} while for s_0 large s_2 acts like s_0^3. Hence, at some point the correction $\delta^2 s_2$ becomes larger than the term it is correcting, s_0. At this point we must bring in higher order terms for additional correction.

In Figure 12 we present the corrected phase velocity $\tilde{c}_p = \beta/(s_0+\delta^2 s_2)$. In the zero order the phase velocity was nearly one in value except in the neighborhood of $\beta = (1-\nu^2)^{1/2}$ and $\beta = 1$. For the corrected phase velocity in the range $0 < \nu < 3 - (8)^{1/2}$ we see in addition to the two frequencies $\beta = (1-\nu^2)^{1/2}$ and $\beta = 1$, the corrected phase velocity differs from one for very small frequencies and also near two other frequencies, $\beta \sim 1.03$ and $\beta \sim 1.14$. . The reader should note the change in scale at $\beta = 1$ and $\beta = 1.04$. This was done so that more of the behavior near $\beta = 1$ could be determined. In the regions marked A and B the curve for $\delta = .005$ is not plotted as it would appear only as a straight vertical line in this graph. In B, $\delta = .05$ is also not plotted for the same reason.

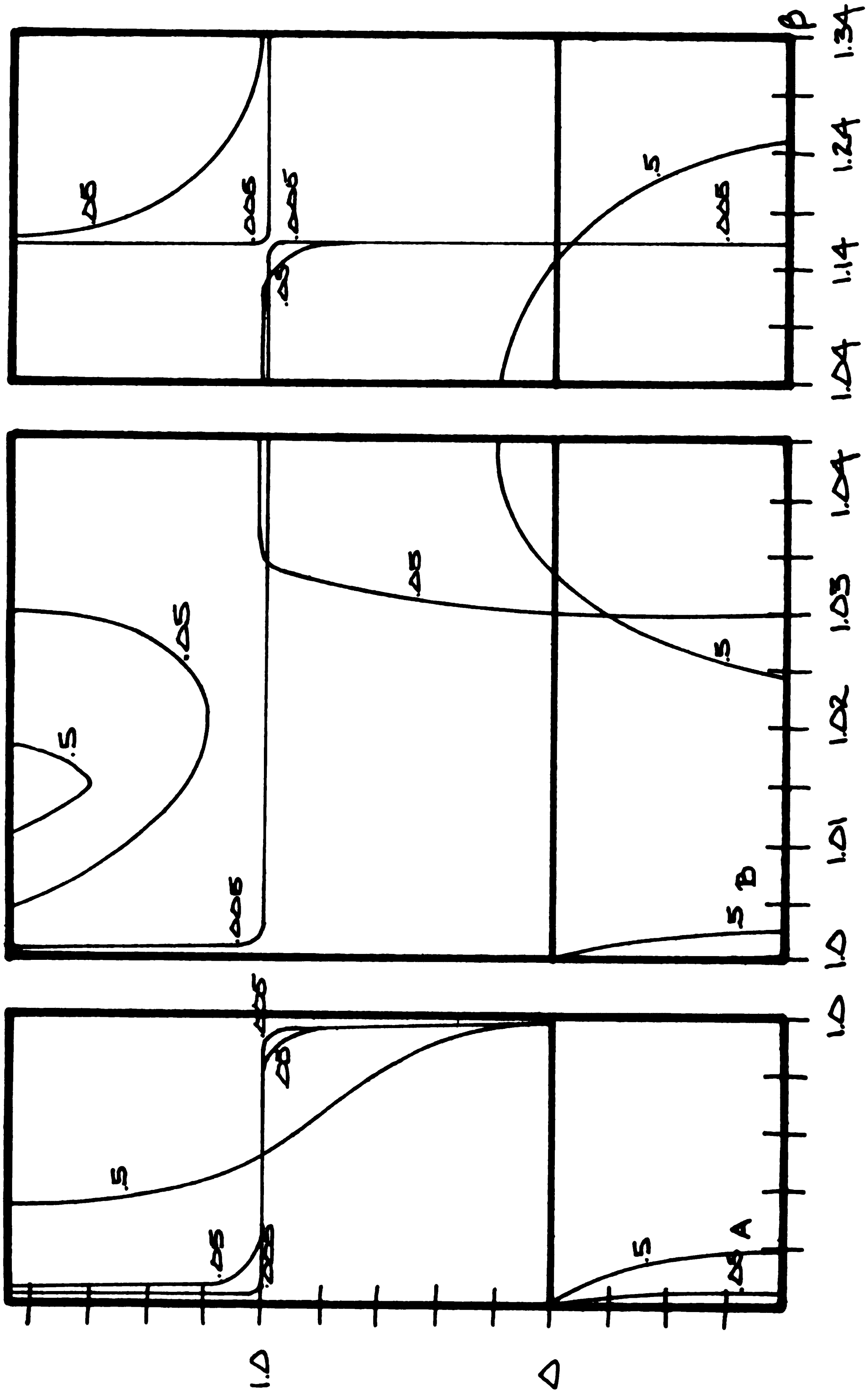

FIGURE #12
CORRECTED PHASE VELOCITY
δ VALUES SHOWN ν=.15

In Figure 13 we exhibit the behavior of the corrected wave number $(s_0+\delta^2 s_2)/s_0$. Again note the change of scale at $\beta = 1$. It should be noted that the correction is nearly one with exceptions occurring in the neighborhoods of $\beta = 0$, $\beta = (1-\nu^2)^{1/2}$, $\beta = 1$ and the two zeroes of the determinant. In these regions one would expect that the zero order wave number alone is in adequate and higher order terms are required.

In summary, we have shown that the corrected phase velocity agrees closely with the phase velocity predicted by the straight tube model in the regions $0 < \beta < \sqrt{1-\nu^2}$, $1 < \beta < 1.14$, $1.15 < \beta$, for $\delta = .005$, and $.05$. When $\delta = .5$ there is virtually no agreement at all.

The corrected wave number and the straight tube wave number show good agreement in the frequency ranges $0 < \beta < 1$, $1.04 < \beta < 1.15$, and $1.2 < \beta$.

In Figure 12 all curves eventually become zero while in the zero order situation the phase velocity approaches one for all ν values as the frequency increases. Conversely, the corrected wave number goes to infinity as the frequency increases in Figure 13. For this reason the straight tube model is inadequate for large frequencies and as the corrected wave number gets larger, higher order terms would be needed to add to those already computed.

Case II. $\nu = 3 - (8)^{1/2}$

When Poisson's ratio takes on the value $3 - (8)^{1/2}$, which is approximately .172, then the determinant (Det) has only one zero. We see then the situation of a more orderly behavior of the phase velocity and corrected wave number in the region of $\beta = 1$. From Figure 14 we see the behavior of the corrected phase velocity. In general the corrected phase velocity is near one for most values of the frequency, β. Again, for very small frequencies and for those near one, behavior foreign to the zero order phase velocity (Figure 3) is introduced. There are select values introduced for very limited frequency ranges. These are represented by the vertical asymptotes near $\beta = 0$, $\beta = 1$ and $\beta = 1.1$. The curve for $\delta = .005$ has not been plotted in the group of curves labeled A, B and C, nor has the curve for $\delta = .05$ for the C set. In each case these segments were off scale and they could not be accurately represented. Note also that there is a scale change at $\beta = 1$.

As before we see that the δ value of .5 gives results which vary significantly from smaller values of δ. This value gives a good indication of the general nature of the curves behavior and is included for this reason. In this case, it indicates that the corrected phase velocity approaches zero from above which is true regardless of the δ value. This is contrary to the zero order phase velocity which approaches one from above.

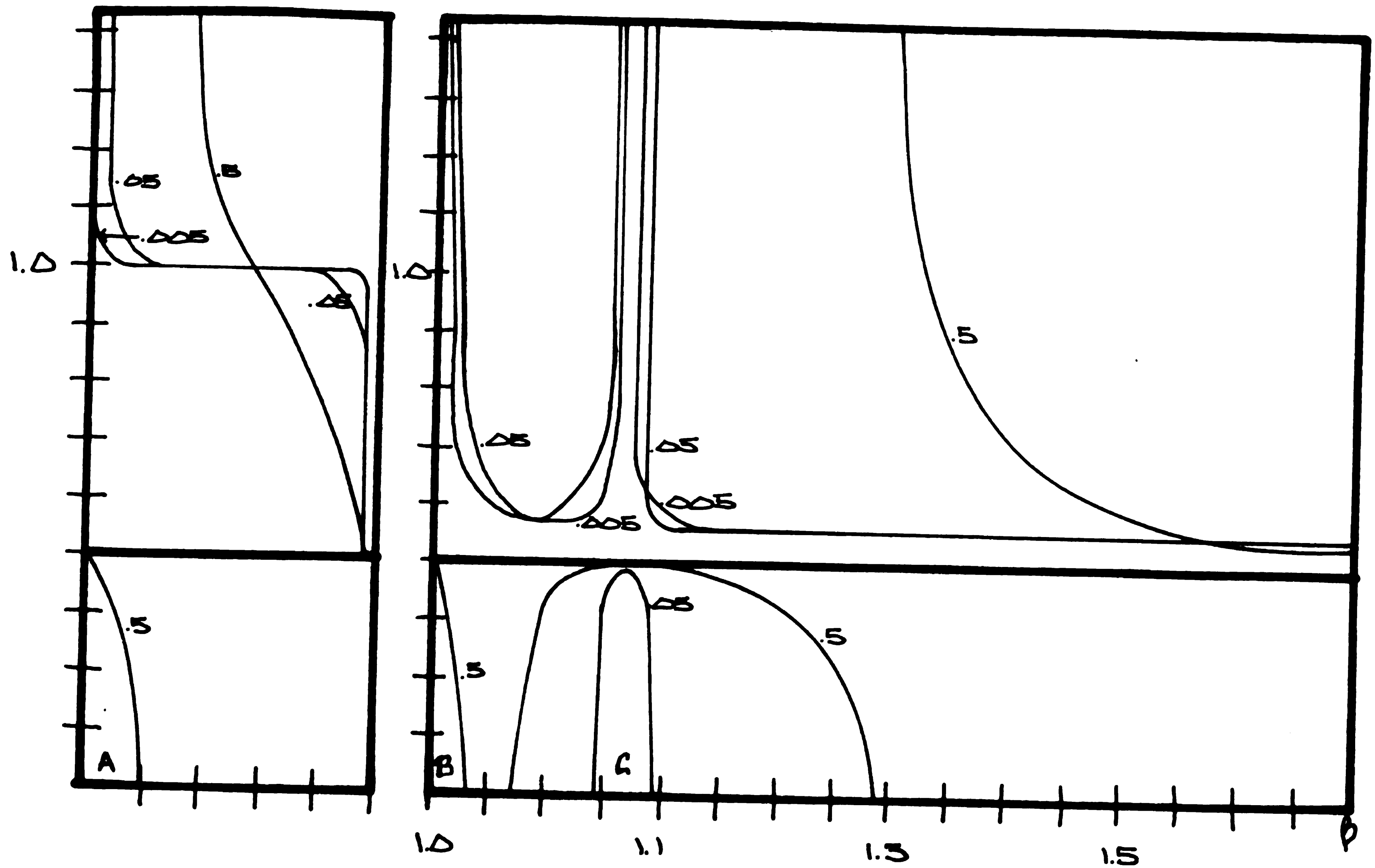

FIGURE #14
EMPTY TUBE CORRECTED PHASE VELOCITY
δ VALUES SHOWN $\nu = 3-\sqrt{8}$

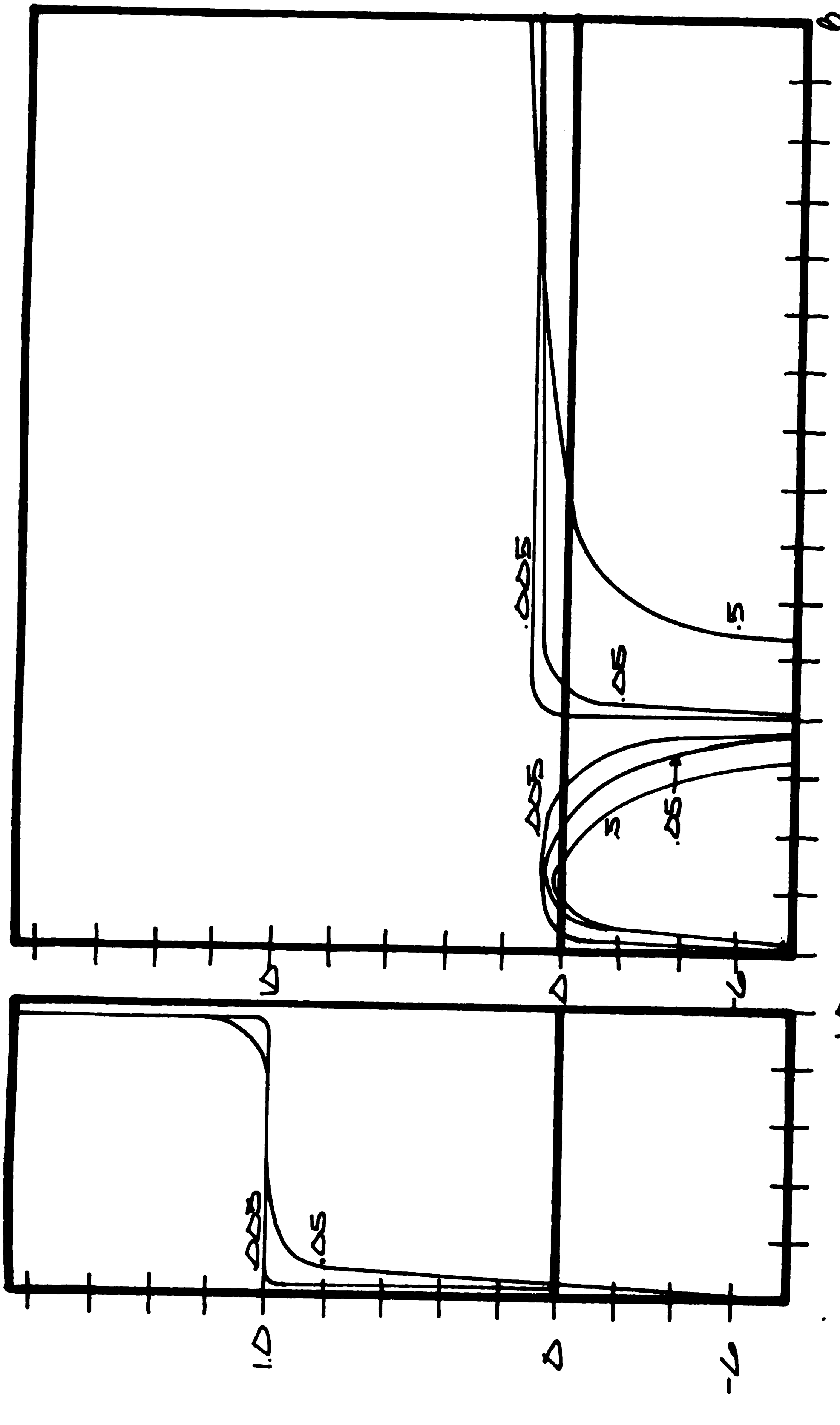

FIGURE #15
EMPTY TUBE CORRECTED WAVE NUMBER
δ VALUES SHOWN $D \cdot .3 \cdot \sqrt{B}$

The corrected wave number presented in Figure 15 exhibits the same general character found when $\nu = .15$. Agreement with the zero order wave number is good in the frequency range $0 < \beta < 1$, $1 < \beta < 1.1$ and $1.2 < \beta$.

The analysis for the corrected wave number $(1+\delta^2 s_2/s_0)$ follows that of the corrected phase velocity. There are portions of the curves in Figure 14 which have been omitted because of their existence in select frequency range nature. They correspond to the A, B and C portions of Figure 17 and are mostly off scale.

Case III. $\nu > 3 - (8)^{1/2}$

For $\nu > 3 - (8)^{1/2}$ we have a less segmented behavior for the correction, the corrected phase velocity and corrected wave number. Since the determinant does not have a real zero we do not obtain the great variation in behavior for our variables for frequencies between 1 and $\sqrt{2}$. The rapidly changing nature of the curve present near $\beta = 0$ is still apparent for $\nu > 3 - \sqrt{8}$. Except for $\delta = .5$ the behavior of all curves settles down rather quickly and follows that of the zero order system rather closely. This feature indicates that the straight tube model is farily accurate except for frequencies near zero. Here we find rather large values for our curves indicating use of higher order terms is required.

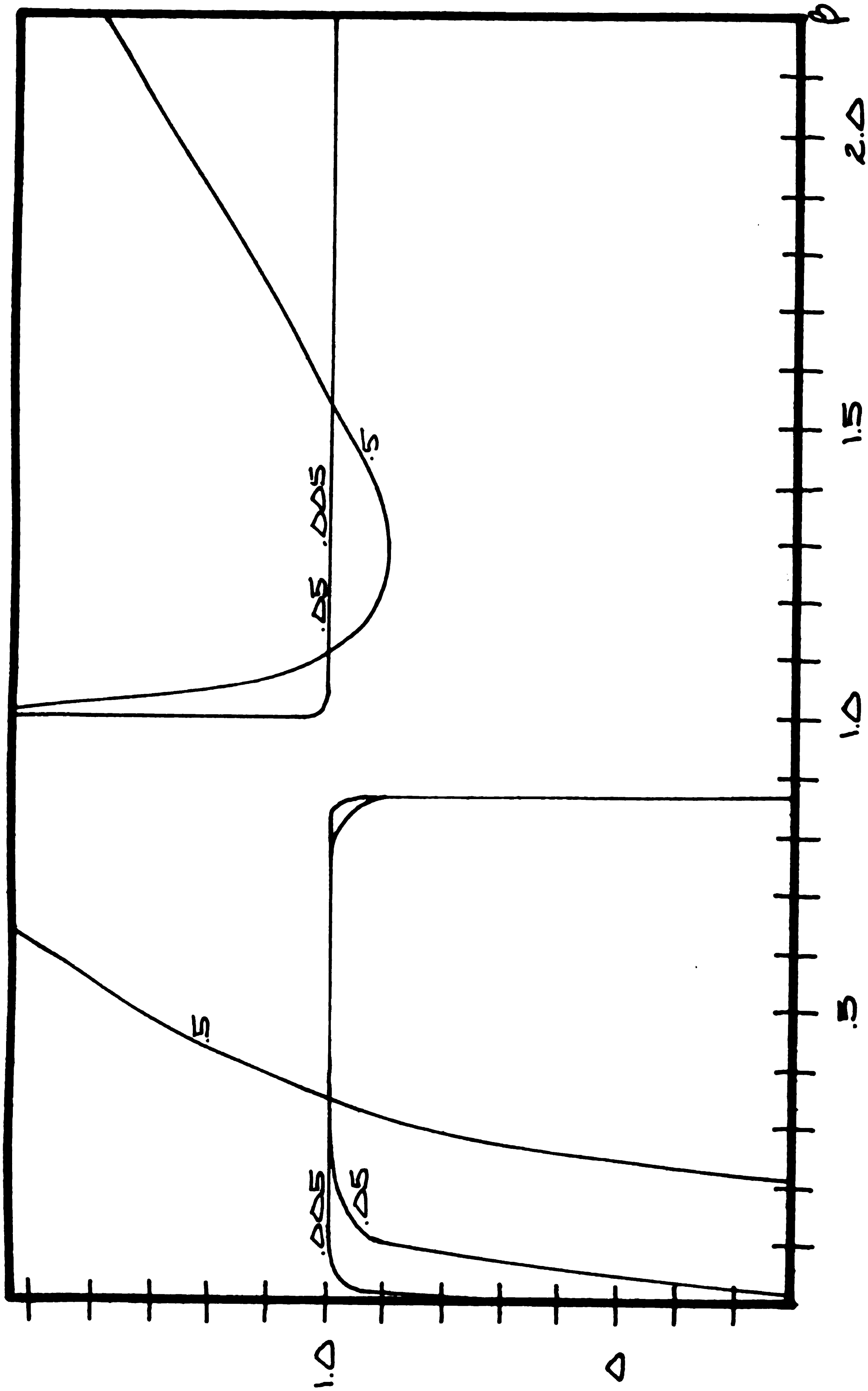

FIGURE #17
CORRECTED WAVE NUMBER
δ VALUES SHOWN ν=.5

The differences between the straight tube and curved tube expressions would become significant if large order scale factors were brought into play, thus accenting the differences. However, for frequency values less than two and away from zero the straight tube and curved tube models have good agreement. For $\delta = .5$ we notice a strange behavior for β near $(1-\nu^2)^{1/2}$. There is a vertical asymptote not present for the smaller values of δ which is further proof of the unacceptable size of δ at this value. It should also be noted that in Figure 19 the .005 curve for the a section of the curve is omitted due to its existence for only small frequencies. In both Figure 19 and Figure 20 the curves for $\delta = .005$ and $\delta = .05$ are nearly identical for β values larger than one. Although the curve for $\delta = .5$ differs greatly from the other two curves, it does indicate the general behavior of the curves for $\beta > 1$. We see that as the frequency becomes very large the phase velocity approaches zero and consequently the wave number grows large.

In conclusion we can say that except for narrow regions in the frequency spectrum, the corrected variables we've observed vary only slightly from the straight tube variables in the frequency range we've been considering. From our various figures we can conclude that the discrepency between the straight and curved models is slight until large frequencies are obtained. This information results from observing the behavior of $\delta = .5$ curves which are indicative of the general trend of the functions we are graphing. It is in these larger frequency ranges that one should resort to the curved tube model to gain some appreciation of the behavior of our tube.

CHAPTER IV

VIBRATIONS OF THE FLUID

4.1 Introduction

In Chapter three we investigated the free vibrations of an empty curved shell. We found and thoroughly examined a frequency equation relating s_0 to β as well as the correction expressions for s_0, namely s_1 and s_2.

We are now going to undertake a similar process for two problems in which the fluid is the primary factor. We begin in section 4.2 by solving the zero order pressure and Navier-Stokes equations. Section 4.3 contains a discussion which develops the necessary equations to find s_1 and s_2 in the general situation. In section 4.4 we analyze the fundamental problems of axisymmetric, inviscid flow in the zero order system by determining two frequency equations and investigating their properties.

Section 4.5 contains the first order terms associated with our fundamental problems and it develops the effect of these terms on zero order quantities. Finally in section 4.6 we find the second order correction for our two problems and make some comments on its effects and behavior.

4.2 Zero Order Pressure and Velocity Equations

4.2.1 Pressure Equation

Examination of the Navier-Stokes Equations (2.5.19) through (2.5.21) indicates the presence of the pressure on the right side of all equations. This requires our determining the pressure explicitly before we proceed to the velocity equations. We do this by solving the pressure equation (2.3.25).

We begin the zero order problem then with the zero order pressure equation (2.6.8) written as

$$\begin{aligned} &\frac{1}{r^2} p_{0,\psi\psi} + \frac{1}{r}(rp_{0,r})_{,r} - (s_0{}^2 - \frac{\beta^2}{c^2k^2})p_0 = 0 \\ (4.2.1) \quad &k^2 = 1-\{\frac{1}{\eta} + \frac{1}{\eta_1}\}i\beta/c^2. \end{aligned}$$

We expect that our solutions all have period 2π in ψ and so we separate variables for p_0 as

$$(4.2.2) \qquad p_0(\psi,r) = f(r)\cos n\psi \qquad n = 0,1,2,3,\ldots$$

Use of (4.2.2) in (4.2.1) leads to the following equation for $f(r)$ where the primes denote ordinary derivatives with respect to r.

$$(4.2.3) \qquad f''(r) + \frac{1}{r} f'(r) - (\frac{n^2}{r^2} + s_0{}^2 - \frac{\beta^2}{c^2k^2})f(r) = 0.$$

Denoting $s_0{}^2 - \frac{\beta^2}{c^2k^2}$ by α^2 we can write as our solution to (4.2.3),

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$$f(r) = AI_n(r\alpha) + BK_n(r\alpha) \tag{4.2.4}$$

where A and B are arbitrary constants and $I_n(r\alpha)$ and $K_n(r\alpha)$ are the modified Bessel functions of the first and second kinds respectively of order n. [Watson, 21]. Since $K_n(r\alpha)$ is not finite when $r = 0$ we set $B = 0$ allowing us to write

$$p_O(\psi, r) = AI_n(r\alpha)\cos n\psi. \tag{4.2.5}$$

4.2.2 Navier-Stokes Equations

We now turn our attention to the zero order Navier-Stokes equations (2.6.9). We separate variables as follows

$$\begin{aligned} u_\psi^O(\psi, r) &= \tilde{u}_\psi^O(r)\sin n\psi + \tilde{v}_\psi^O(r)\cos n\psi \\ u_\theta^O(\psi, r) &= \tilde{u}_\theta^O(r)\cos n\psi + \tilde{v}_\theta^O(r)\sin n\psi \\ u_r^O(\psi, r) &= \tilde{u}_r^O(r)\cos n\psi + \tilde{v}_r^O(r)\sin n\psi \end{aligned} \tag{4.2.6}$$

When $n = 0$ we have axisymmetric flow and there is no velocity in the ψ direction. Hence $\tilde{v}_\psi^O(r)$ must be zero in order to fulfill this condition.

In what follows the primes indicate differentiation with respect to the argument. Hence $f'(r)$ indicates $\frac{d}{dr} f(r)$ while $I_n(r\alpha)'$ means $\frac{d}{d(r\alpha)} I_n(r\alpha)$. We now drop the tildas from all variables as well as delete the arguments of the functions. All functions have r as the argument except when written out.

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Insertion of (4.2.5) and (4.2.6) into (2.6.9) results in the following systems of equations.

The first system is

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(4.2.7) \qquad
\begin{aligned}
&2nr^{-2}\, v_r^0 = 0, \\
&r^{-1}(rv_\theta^{0\prime})' - \{n^2r^{-2} + s_0{}^2 - i\beta\eta_1\}v_\theta^0 = 0 \\
&r^{-1}(rv_r^{0\prime})' - \{(n^2+1)r^{-2} + s_0{}^2 - i\beta\eta_1\}v_r^0 = 0,
\end{aligned}
$$

while the second system assumes the form below,

$$
(4.2.8) \qquad
\begin{aligned}
&r^{-1}(ru_\psi^{0\prime})' - \{(n^2+1)r^{-2} + s_0{}^2 - i\beta\eta_1\}u_\psi^0 - 2nr^{-2}\, u_r^0 \\
&\qquad = -\,Anr^{-1}\,\eta_1\{1-i\beta/\eta c^2\}I_n(r\alpha), \\
&r^{-1}(ru_\theta^{0\prime})' - \{n^2r^{-2}+s_0{}^2-i\beta\eta_1\}u_\theta^0 \\
&\qquad = -\,As_0\eta_1\{1-i\beta/\eta c^2\}I_n(r\alpha), \\
&r^{-1}(ru_r^{0\prime})' - \{(n^2+1)r^{-2}+s_0{}^2-i\beta\eta_1\}u_r^0 + 2nr^{-2}\, u_\psi^0 \\
&\qquad = A\alpha\eta_1\{1-i\beta/\eta c^2\}I_n'(r\alpha).
\end{aligned}
$$

The first equation of (4.2.7) indicates $v_r^0 = 0$ unless $n = 0$. If $n = 0$, the last equation of (4.2.7) yields $v_r^0 = A_1I_1(r\gamma) + B_1K_1(r\gamma)$ where $\gamma^2 = s_0{}^2 - i\beta\eta_1$, and, A_1 and B_1 are arbitrary constants. Finiteness at $r = 0$ requires $B_1 = 0$. [Note: The argument of finiteness for all variables at $r = 0$ will automatically be made in the remainder of this work. Hence only modified Bessel functions of the first kind, $I_n(r\alpha)$, or $I_n(r\gamma)$ will be given in our solutions.] From the second equation of (4.2.7) we write

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$$v_\theta^0(r) = A_2 I_n(r\gamma). \tag{4.2.9}$$

The system (4.2.8) exhibits coupling between the first and third equations while the second is completely independent. The coupled pair of equations can be written as

$$\begin{pmatrix} L_{33} & -2n \\ -2n & L_{33} \end{pmatrix} \begin{pmatrix} u_\psi^0 \\ u_r^0 \end{pmatrix} = A\eta_1\{1-i\beta/\eta c^2\} \begin{pmatrix} -nrI_n(r\alpha) \\ \alpha r^2 I_n'(r\alpha) \end{pmatrix} \tag{4.2.10}$$

where $L_{33} = r\,\frac{d}{dr}(r\,\frac{d}{dr}) - (n^2+1+r^2\gamma^2)$.

Since L_{33} and $2n$ are operators which commute we apply L_{33} to the first equation and $2n$ to the second and subtract. The result is

$$\{(L_{33})^2 - 4n^2\}u_\psi^0 = -An(\alpha^2-\gamma^2)r^3\eta_1\{i-i\beta/\eta c^2\}I_n(r\alpha) \tag{4.2.11}$$

The operator in (4.2.11) can be factored as $\{L_{33}-2n\}\{L_{33}+2n\}$. The homogeneous solutions for the first portion of the operator are $I_{n+1}(r\gamma)$ and $K_{n+1}(r\gamma)$, while the homogeneous solutions for the second part of the operator are $I_{n-1}(r\gamma)$ and $K_{n-1}(r\gamma)$.

We utilize these four functions through variation of parameters to find a particular solution for (4.2.11). We designate the particular solution by $u_{\psi p}^0(r)$ and find

$$u^{o}_{\psi p}(r) = \frac{1}{4} A\eta_1\{1-i\beta/\eta c^2\}(\alpha^2-\gamma^2)\{I_{n-1}(r\gamma)\int_0^r r^2 I_n(r\alpha)K_{n-1}(r\gamma)dr$$

$$- K_{n-1}(r\gamma)\int_0^r r^2 I_n(r\alpha)I_{n-1}(r\gamma)dr$$

$$(4.2.12) \qquad - I_{n+1}(r\gamma)\int_0^r r^2 I_n(r\alpha)K_{n+1}(r\gamma)dr$$

$$+ K_{n+1}(r\gamma)\int_0^r r^2 I_n(r\alpha)I_{n+1}(r\gamma)dr\}$$

$$= -Ar^{-1}n(\alpha^2-\gamma^2)^{-1}\ \eta_1\{1-i\beta/\eta c^2\}I_n(r\alpha)$$

We finally have as our general solution to (4.2.11) the velocity

$$(4.2.13) \qquad \begin{aligned} u^{o}_{\psi}(r) &= A_3 I_{n+1}(r\gamma) + A_4 I_{n-1}(r\gamma) \\ &- Anr^{-1}\ \eta_1\{1-i\beta/\eta c^2\}(\alpha^2-\gamma^2)^{-1}\ I_n(r\alpha) \end{aligned}$$

where A_3 and A_4 are arbitrary constants.

Substitution of (4.2.13) into the first equation of (4.2.10) gives us $u^{o}_{r}(r)$ as

$$(4.2.14) \qquad \begin{aligned} u^{o}_{r}(r) &= A_3 I_{n+1}(r\gamma) - A_4 I_{n-1}(r\gamma) \\ &+ A\alpha\eta_1\{1-i\beta/\eta c^2\}(\alpha^2-\gamma^2)^{-1}\ I'_n(r\alpha) \end{aligned}$$

The second equation of (4.2.8) can be solved directly by variation of parameters. We have then

$$(4.2.15) \qquad u^{o}_{\theta}(r) = A_5 I_n(r\gamma) - As_o\eta_1\{1-i\beta/\eta c^2\}(\alpha^2-\gamma^2)^{-1}\ I_n(r\alpha)$$

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4.2.3 Continuity Equation

We now have solutions for the velocity components and pressure for the zero order. Quick observation reveals the presence of six arbitrary constants. In a free vibrations problem in which we search for vibration modes there will always be one arbitrary constant. So we must determine the others remaining after designating A to be arbitrary.

It should be noted that we have solved the pressure equation and the Navier-Stokes equations without dealing with the continuity equation directly. Our first step in determining arbitrary constants will be to require that all velocities satisfy the zero order continuity equation (2.6.10).

This gives rise to a pair of equations

$$nr^{-1}\, u_\psi^0 + s_0 u_\theta^0 + r^{-1}(ru_r^0)' = (i\beta/c^2)AI_n(r\alpha) \tag{4.2.16}$$

$$s_0 v_\theta^0 + r^{-1}(rv_r^0)' = 0. \tag{4.2.17}$$

When $n \neq 0$ then $v_r^0 \equiv 0$ and by (4.2.9) and (4.2.17) we find $A_2 = 0$. If $n = 0$ then by (4.2.6) the v components disappear and (4.2.17) is satisfied identically. Substitution of (4.2.13) through (4.2.15) into (4.2.16) reveals that $A_5 = \gamma s_0^{-1}(A_3 - A_4)$.

The three zero order velocities and pressure are given by

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\begin{aligned}
p_0(\psi,r) &= AI_n(r\alpha)\cos n\psi \\
u^0_\psi(\psi,r) &= \left\{A_3 I_{n+1}(r\gamma) + A_4 I_{n-1}(r\gamma) - \frac{AMnI_n(r\alpha)}{r(\alpha^2-\gamma^2)}\right\} \sin n\psi, \\
u^0_\theta(\psi,r) &= \left\{\frac{\gamma}{s_0}(A_3-A_4)I_n(r\gamma) - \frac{AMs_0I_n(r\alpha)}{(\alpha^2-\gamma^2)}\right\} \cos n\psi \\
u^0_r(\psi,r) &= \left\{A_3 I_{n+1}(r\gamma) - A_4 I_{n-1}(r\gamma) + \frac{AM\alpha I_n'(r\alpha)}{(\alpha^2-\gamma^2)}\right\} \cos n\psi,
\end{aligned}
$$

where $M = \eta_1\{1-i\beta/\eta c^2\}$.

If the fluid is inviscid then $\eta_1^{-1} = \eta^{-1} = 0$, $k^2 = 1$ and (4.2.18) reduces to

$$
(4.2.19)\qquad
\begin{aligned}
p_0(r,\psi) &= AI_n(r\alpha)\cos n\psi\ , \\
u^0_\psi(\psi,r) &= \{-AnI_n(r\alpha)/ri\beta\} \sin n\psi, \\
u^0_\theta(\psi,r) &= \{-As_0I_n(r\alpha)/i\beta\} \cos n\psi, \\
u^0_r(\psi,r) &= \{A\alpha I_n'(r\alpha)/i\beta\} \cos n\psi\ .
\end{aligned}
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4.3 Discussion for Arbitrary n

As in Chapter three we present the procedures for determining the frequency equation and the higher order corrections for arbitrary n. In subsequent sections the analysis of these equations is restricted to the fundamental case of inviscid, axisymmetric flow. For the present, we will make no such assumptions while proceeding to detail the process for solving our problem.

4.3.1 Two Frequency Equations

The primary concern of this chapter is to discuss two problems which feature the fluid component of our model. In the first of these problems we postulate that the fluid is flowing in a tube which exhibits no movement of its own. We will refer to this as the "rigid tube" case. In the second problem we state that the fluid is flowing in a curved path without benefit of a container. We designate this situation as being "stress free" since no stresses are exerted on the boundary of the fluid.

In the rigid tube case we say that at the boundary of the fluid, which is the tube, the fluid velocities are all zero. Since the tube is not moving it is reasonable to expect that the fluid immediately next to the tube is also stationary. We should note that for inviscid flows there can be slippage at the wall so that only the radial velocity is zero.

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The vanishing of all three velocities at the boundary, $r = 1$, gives us three homogeneous equations in the three arbitrary constants A, A_3 and A_4. This system of equations is written

$$(4.3.1)\quad \begin{pmatrix} I_{n+1}(\gamma) & I_{n-1}(\gamma) & -\dfrac{MnI_n(\alpha)}{(\alpha^2-\gamma^2)} \\ \dfrac{\gamma I_n(\gamma)}{s_0} & -\dfrac{\gamma I_n(\gamma)}{s_0} & -\dfrac{Ms_0I_n(\alpha)}{(\alpha^2-\gamma^2)} \\ I_{n+1}(\gamma) & -I_{n-1}(\gamma) & \dfrac{M\alpha I_n'(\alpha)}{(\alpha^2-\gamma^2)} \end{pmatrix} \begin{pmatrix} A_3 \\ A_4 \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for viscous fluids. For inviscid fluids we set $u_r^0(1) = 0$ from (4.2.19) to get

$$(4.3.2)\qquad \alpha I_n'(\alpha)/i\beta = 0.$$

In the viscous case we seek non-trivial solutions and so require that the determinant of the coefficient matrix in (4.3.1) vanish. Since both γ and α involve s_0 and β the vanishing of the determinant leads to a complex transcendental relationship between s_0 and β which we have called the frequency equation.

In general it is quite difficult to find all the possible relationships between s_0 and β analytically. Normally, assumptions are made which simplify the Bessel functions using only one or two terms of an expansion. Doing this reduces the frequency equation from a transcendental equation to an algebraic one.

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Our concern, however, is to present the various equations for the general case. For inviscid flow equation (4.3.2) contains the relationship between s_O and β. This equation is simpler and a relationship can be established for s_O and β once the zeroes of $I_n'(\alpha)$ are known.

For the stress free case we find that we must set all fluid stresses acting at the fluid boundary equal to zero. These stresses are $S^O_{r\psi}$, $S^O_{r\theta}$ and S^O_{rr}. Returning to (2.6.11) and inserting (4.2.18) in those equations, we find after many algebraic manipulations that at $r = 1$

$$\frac{1}{\eta_1}\{A_3\gamma I_{n+2}(\gamma) + A_4\gamma I_{n-2}(\gamma) - \frac{2AMn}{(\alpha^2-\gamma^2)} [(n-1)I_n(\alpha) + \alpha I_{n+1}(\alpha)]\} = 0, \tag{4.3.3}$$

$$\frac{1}{\eta_1}\{A_3 s_O^{-1}[n\gamma I_n(\gamma) + (\gamma^2-s_O^2)I_{n+1}(\gamma)] + A_4 s_O^{-1}[n\gamma I_n(\gamma) - (\gamma^2-s_O^2)I_{n-1}(\gamma)] - \frac{2AMs_O}{(\alpha^2-\gamma^2)}\alpha I_n'(\alpha) = 0, \tag{4.3.4}$$

$$\frac{2A_3}{\eta_1}\gamma I_{n+1}(\gamma) - \frac{2A_4}{\eta_1}\gamma I_{n-1}(\gamma) - AI_n(\alpha)\{1 - \frac{i\beta}{c^2}(\frac{1}{\eta} - \frac{1}{\eta_1})\} + \frac{AM}{(\alpha^2-\gamma^2)}\{(n^2+\alpha^2-n)I_n(\alpha) - \alpha I_{n+1}(\alpha)\} = 0, \tag{4.3.5}$$

when the fluid is viscous. If the fluid is inviscid, then only the normal fluid stress, $S^O_{rr}(1)$, is zero, and we have then

$$AI_n(\alpha) = 0. \tag{4.3.6}$$

As in the system (4.3.1) we require the determinant of the coefficient matrix for the undetermined constants, A, A_3 and A_4, to vanish so as to have non-trivial solutions. As before, the viscous case leads to a complex, transcendental equation involving s_0 and β which in general is difficult to solve. For the inviscid case we merely prescribe the value of n we are considering and then we are able to find the zeroes of $I_n(\alpha)$. From this the frequency equation is written as

$$s_0^2 = \frac{\beta^2}{c^2} - z_{nm}^2 , \qquad (4.3.7)$$

where $J_n(z_{nm}) = 0$, $m = 1,2,3,4,\ldots,$ and J_n is the Bessel function of the first kind of order n.

4.3.2 First Order Correction

After determining the frequency equation appropriate to the zero order fluid equations, i.e., the straight cylinder case, we attempt now to find the first order correction, s_1.

We begin with the first order written in terms of the stresses rather than the Navier-Stokes equations. The procedure is straightforward. We multiply the stress equations by the zero order fluid velocities and integrate over the cross-sectional area occupied by the fluid. By an integration by parts we are able to eliminate the first order stresses, velocities and pressure in the region except for boundary integrals which will involve these terms. In this way we can isolate the contribution due to s_1.

From (2.6.2A) we write the first order stress equations of motion for the fluid without benefit of the operators M_1, M_2 and M_3. They are

$$\begin{aligned} r^{-1}S^1_{\psi\psi,\psi} + s_0 S^1_{\psi\theta} + r^{-1}S^1_{\psi r} + r^{-1}(rS^1_{\psi r})_{,r} &= -i\beta u^1_\psi \\ &- s_1 S^0_{\psi\theta} + \cos\psi\{S^0_{\theta\theta} - S^0_{\psi\psi}\} + \sin\psi\{rs_0 S^0_{\psi\theta} - S^0_{\psi r}\} \end{aligned} \tag{4.3.8}$$

$$\begin{aligned} r^{-1}S^1_{\theta\psi,\psi} + r^{-1}(rS^1_{\theta r})_{,r} - s_0 S^1_{\theta\theta} &= -i\beta u^1_\theta + s_1 S^0_{\theta\theta} \\ &- 2S^0_{\theta\psi}\cos\psi - \sin\psi\{2S^0_{\theta r} + rs_0 S^0_{\theta\theta}\} \end{aligned} \tag{4.3.9}$$

$$\begin{aligned} r^{-1}(S^1_{r\psi,\psi} - S^1_{\psi\psi}\} + r^{-1}(rS^1_{rr})_{,r} + s_0 S^1_{r\theta} &= -i\beta u^1_r \\ &- s_1 S^0_{r\theta} - S^0_{\psi r}\cos\psi + \sin\psi\{rs_0 S^0_{r\theta} + S^0_{\theta\theta} - S^0_{rr}\} \end{aligned} \tag{4.3.10}$$

We multiply (4.3.8) by u^0_ψ, (4.3.9) by u^0_θ and (4.3.10) by u^0_r and add all three together while integrating over the cross section of the tube. The result is an expression of the form

$$\begin{aligned} &\int_0^1 \{u^0_\psi S^1_{\psi\psi} + u^0_\theta S^1_{\theta\psi} + u^0_r S^1_{r\psi}\Big|_0^{2\pi} dr \\ &+ \int_0^{2\pi} r\{u^0_\psi S^1_{\psi r} + u^0_\theta S^1_{\theta r} + u^0_r S^1_{rr}\Big|_0^1 d\psi \\ &- \int_0^{2\pi}\int_0^1 r^{-1}\{S^1_{\psi\psi}u^0_{\psi,\psi} + S^1_{\psi\theta}u^0_{\theta,\psi} + S^1_{\psi r}u^0_{r,\psi}\} r\,dr\,d\psi \\ &- \int_0^{2\pi}\int_0^1 \{S^1_{\psi r}u^0_{\psi,r} + S^1_{\theta r}u^0_{\theta,r} + S^1_{rr}u^0_{r,r}\} r\,dr\,d\psi \\ &+ \int_0^{2\pi}\int_0^1 \{r^{-1}(u^0_\psi S^1_{\psi r} - u^0_r S^1_{\psi\psi}) + s_0(u^0_\psi S^1_{\psi\theta} - u^0_\theta S^1_{\theta\theta} + u^0_r S^1_{\theta r})\} r\,dr\,d\psi \end{aligned} \tag{4.3.11}$$

$$+ i\beta \int_0^{2\pi}\int_0^1 \{u_\psi^0 u_\psi^1 + u_\theta^0 u_\theta^1 + u_r^0 u_r^1\} r dr d\psi$$

$$+ \int_0^{2\pi}\int_0^1 \{s_1 - r s_0 \sin\psi\}\{u_\psi^0 S_{\psi\theta}^0 - u_\theta^0 S_{\theta\theta}^0 + u_r^0 S_{\theta r}^0\} r dr d\psi$$

$$+ \int_0^{2\pi}\int_0^1 \sin\psi\{u_\psi^0 S_{\psi r}^0 + 2u_\theta^0 S_{\theta r}^0 + u_r^0 (S_{rr}^0 - S_{\theta\theta}^0)\} r dr d\psi$$

$$+ \int_0^{2\pi}\int_0^1 \cos\psi\{u_\psi^0 (S_{\psi\psi}^0 - S_{\theta\theta}^0) + 2u_\theta^0 S_{\theta\psi}^0 + u_r^0 S_{r\psi}^0\} r dr d\psi = 0$$

For the general case u_ψ^0 is odd in ψ ($\sin n\psi$) while u_r^0, u_θ^0 and p_0 involve $\cos n\psi$ and are even functions of ψ. Examination of the fluid stresses reveals that $S_{\psi\psi}^0$, $S_{\theta\theta}^0$, S_{rr}^0 and $S_{\theta r}^0$ are even functions of ψ which $S_{\theta\psi}^0$ and $S_{r\psi}^0$ are odd functions of ψ. Utilizing the fact that the integral of an odd function over its period gives zero, the last two integrals in (4.3.11) are zero as well as the term involving $s_0 r \sin\psi$ in the third from the last integral in (4.3.11).

Our purpose is to eliminate all first order terms except s_1 by replacing them with known expressions involving zero order terms. To do so we replace the first order stresses by the first order constitutive equations (2.6.23) and integrate by parts any derivatives of first order velocities. The resulting expressions are written so as to have the first order fluid velocities multiplied by the zero order Navier-Stokes equations which are zero. We finally arrange the equation in the form

$$\int_0^{2\pi} r\{u^0_\psi s^1_{\psi r}+u^0_\theta s^1_{\theta r}+u^0_r s^1_{rr}-u^1_\psi s^0_{\psi r}-u^1_\theta s^0_{\theta r}-u^1_r s^0_{rr}\Big|_0^1 d\psi$$

$$+\int_0^1\{u^0_\psi s^1_{\psi\psi}+u^0_\theta s^1_{\theta\psi}+u^0_r s^1_{r\psi} s^1_{r\psi}-u^1_\psi s^0_{\psi\psi}-u^1_\theta s^0_{\theta\psi}-u^1_r s^0_{r\psi}\Big|_0^{2\pi} rdr$$

$$+\int_0^{2\pi}\int_0^1\{i\beta p_0 p_1/c^2-p_0[r^{-1}(u^1_{\psi,\psi}+[ru^1_r]_{,r}) + s_0 u^1_\theta$$

(4.3.12)

$$+ (s_1-rs_0 \sin \psi)u^0_\theta + u^0_\psi \cos \psi + u^0_r \sin \psi]\}rdrd\psi$$

$$-\int_0^{2\pi}\int_0^1 s^0_{\theta\theta}\{(s_1-rs_0 \sin \psi)u^0_\theta + u^0_\psi \cos \psi + u^0_r \sin \psi\}rdrd\psi$$

$$+ s_1\int_0^{2\pi}\int_0^1\{2u^0_\psi s^0_{\theta\psi}-u^0_\theta s^0_{\theta\theta}+2u^0_r s^0_{\theta r}\}rdrd\psi = 0$$

Due to the periodicity of all variables in ψ, the second integral of (4.3.12) is zero. In the third integral the terms in brackets, multiplying p_0, constitute the first order continuity equation (2.6.22) and can be replaced by $i\beta p_1/c^2$ thus making the entire integral zero. Finally in the fourth integral the only non-zero term multiplies s_1 so that (4.3.12) assumes the form

$$\int_0^{2\pi} r\{u^0_\psi s^1_{\psi r}+u^0_\theta s^1_{\theta r}+u^0_r s^1_{rr}-u^1_\psi s^0_{\psi r}-u^1_\theta s^0_{\theta r}-u^1_r s^0_{rr}\Big|_0^1 d\psi$$

(4.3.13)

$$+ 2s_1\int_0^{2\pi}\int_0^1\{u^0_\psi s^0_{\theta\psi}-u^0_\theta s^0_{\theta\theta}+u^0_r s^0_{\theta r}\}rdrd\psi = 0$$

The first integral in the above equation will make some non-trivial contribution to the general interaction problem. However, for the rigid tube all velocities are zero at $r = 1$ and for the stress free case all stresses are zero at $r = 1$. In either of these special cases the first integral is zero. Since the second integral is not zero, we conclude that $s_1 = 0$ in the stress free and rigid tube cases.

As in the empty tube problem we are not satisfied with this result so we proceed to a determination of the next correction, s_2. This procedure involves second order equations which require in turn the solution of the first order fluid velocities and pressure.

4.3.3 First Order Terms for Arbitrary n

We begin to determine the first order velocity fluid by seeking the solution of the first order pressure equation which we repeat here for convenience. We have that p_1 satisfies

$$
\begin{aligned}
& r^{-2}p_{1,\psi\psi} + r^{-1}(rp_{1,r})_{,r} - ({s_0}^2-\beta^2/c^2k^2)p_1 \\
(4.3.14) \quad & = -r^{-1}p_{0,\psi}\cos\psi - \sin\psi\{p_{0,r}+2r{s_0}^2p_0\}
\end{aligned}
$$

with k^2 defined by (4.2.1) and with α^2 defined by $\alpha^2 = ({s_0}^2-\beta^2/c^2k^2)$. The left hand side of (4.3.14) is of the same form as the left side of the zero order pressure equation (4.2.1). Substituting for $p_0(r,\psi)$ from the (4.2.5) the inhomogeneous terms may be written

$$
\begin{aligned}
& A\sin(n+1)\psi\{\tfrac{1}{2}\, r^{-1}nI_n(r\alpha)-r{s_0}^2I_n(r\alpha) - \tfrac{1}{2}\,\alpha I_n'(r\alpha)\} \\
(4.3.15) + \; & A\sin(n-1)\psi\{\tfrac{1}{2}\, r^{-1}nI_n(r\alpha)+r{s_0}^2I_n(r\alpha) + \tfrac{1}{2}\,\alpha I_n'(r\alpha)\} \\
+ \; & 2As_0s_1I_n(r\alpha)\cos n\psi
\end{aligned}
$$

Assuming that the first order pressure has the form

$$p_1(r,\psi) = f_1(r)\cos n\psi + f_2(r)\sin(n+1)\psi + f_3(r)\sin(n-1)\psi, \tag{4.3.16}$$

as its particular solution, we substitute (4.3.16) into (4.3.14) and arrive at three sets of equations. Designating an operator L_m to have the form

$$L_m = r^{-1}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \{m^2r^{-2}+\alpha^2\} \quad m \text{ an integer} \tag{4.3.17}$$

we can write the three equations as

$$L_n\{f_1(r)\} = 2As_0s_1I_n(r\alpha) \tag{4.3.18}$$

$$L_{n+1}\{f_2(r)\} = -\frac{1}{2}A\{\alpha I_{n+1}(r\alpha)+2rs_0{}^2I_n(r\alpha)\} \tag{4.3.19}$$

$$L_{n-1}\{f_3(r)\} = \frac{1}{2}A\{\alpha I_{n-1}(r\alpha)+2rs_0{}^2I_n(r\alpha)\} \tag{4.3.20}$$

To find the general solution for each of these equations we combine the homogeneous solution with a particular solution which can be determined through the variation of parameters. The particular solutions are

$$\begin{aligned} f_1(r) &= C_nI_n(r\alpha) + As_0s_1r\alpha^{-1}I_{n+1}(r\alpha) \\ f_2(r) &= D_{n+1}I_{n+1}(r\alpha) - \frac{1}{4}As_0{}^2r^2\alpha^{-1}\{I_{n+1}(r\alpha) \\ &\quad + 2n(r\alpha)^{-1}I_n(r\alpha)\} \\ f_3(r) &= D_{n-1}I_{n-1}(r\alpha) + \frac{1}{4}As_0{}^2r^2\alpha^{-1}\{I_{n-1}(r\alpha) \\ &\quad - 2n(r\alpha)^{-1}I_n(r\alpha)\} \end{aligned} \tag{4.3.21}$$

For inviscid flows we set $k^2 = 1$ changing the value of α^2 to $(s_O{}^2 - \beta^2 c^{-2})$.

To solve for the first order velocities we consider the first order fluid equations given in (2.6.21) and examine the inhomogeneous terms which are written in terms of the known zero order velocities and the first order pressure which is known up to the arbitrary constants C_n, C_{n+1} and D_{n-1}, plus the arbitrary constants arising from the homogeneous solutions, namely $D_n I_n(r\alpha)\sin n\psi$, $C_{n+1} I_{n+1}(r\alpha)\cos(n+1)\psi$, $C_{n-1} I_{n-1}(r\alpha)\cos(n-1)\psi$.

Knowing the first order pressure we can now establish the procedure for determining the first order velocities. The inhomogeneous terms for our system are composed of the first order pressure, its derivatives, the zero order velocities and pressure and their derivatives. Denoting the partial differential operators to be used by

$$
\begin{aligned}
L_{11} &= r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \psi^2} - (1 + r^2 \gamma^2) \\
L_{22} &= r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \psi^2} - r^2 \gamma^2 \qquad (4.3.22) \\
L_{13} &= 2 \frac{\partial}{\partial \psi}
\end{aligned}
$$

we write the first order system as

$$
(4.3.23) \quad \begin{pmatrix} L_{11} & L_{13} \\ -L_{13} & L_{11} \end{pmatrix} \begin{pmatrix} w_\psi^1 \\ w_r^1 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}
$$

(4.3.24) $$L_{22}w_\theta^1 = F_3$$

where

$$F_1 = r\eta_1(1+i\beta/\eta c^2)p_{1,\psi} - [rw^O_{\psi,\psi}-2r^2s_Ow^O_\theta+rw^O_r]\cos\psi$$
$$- \sin\psi[2r^3s_O{}^2w^O_\psi+r^2w^O_{\psi,r}]$$

(4.3.25) $$F_2 = r^2\eta_1(1+i\beta/\eta c^2)p_{1,r} + \cos\psi[rw^O_\psi-rw^O_{r,\psi}]$$
$$+ \sin\psi[2r^2s_Ow^O_\theta-2r^3s_O{}^2w^O_r-r^2w^O_{r,r}]$$

$$F_3 = -r^2\eta_1(1+i\beta/\eta c^2)s_Op_1 + \cos\eta[2r^2s_Ow^O_\psi-rw^O_\theta]$$
$$+ \sin\psi[2r^2s_Ow^O_r-r^2w^O_{\theta,r}-2r^3s_O{}^2w^O_\theta].$$

The inhomogeneous terms contained in the functions F_1, F_2 and F_3 are composed of the trigonometric terms $\cos(n \pm 1)\psi$, $\sin(n \pm 1)\psi$, $\cos n\psi$ and $\sin n\psi$. We must therefore write the first order velocities in such a fashion as to give us all the trigonometric possibilities. The result is six systems of coupled second order ordinary differential equations containing both the w^1_ψ and w^1_r components. In addition there are six second order equations involving only the w^1_θ velocity components. One half of these equations have inhomogeneous terms which are multiplied by the three arbitrary constants found in $p_1(r,\psi)$. The other equations involve these arbitrary constants with other known constants which make it possible to determine some of the arbitrary constants through the boundary conditions.

We will see later in the next section that only certain trigonometric functions give a non-zero value when integrated in the correction equation. We will see that only the $\cos m\psi$ portions of w^1_ψ and the $\sin m\psi$ portions of w^1_θ, w^1_r and p_1 give a non-zero value for the correction. Here we allow $m = (n-1)$, n or $(n+1)$.

In each system of coupled equations we find the same pattern as in the zero order system. By operating on one equation of the system with a differential operator and subtracting from it the other equations multiplied by a constant we arrive at a fourth order system which can be solved just as in section 4.2. The actual steps will not be produced here as they are cumbersome though of the same nature as those in section 4.2.

4.3.4 Second Order Correction

To find an expression for s_2 we begin with the second order stress equations of motion (2.6.32) for the fluid. They are

$$
\begin{aligned}
&r^{-1}\{S^2_{\psi r}+S^2_{\psi\psi,\psi}+(rS^2_{\psi r})_{,r}\} + s_0 S^2_{\psi\theta} = -i\beta u^2_\psi - s_2 S^0_{\psi\theta} \\
&\quad + \cos\psi\{S^1_{\theta\theta}-S^1_{\psi\psi}\} + \sin\psi\{rs_0 S^1_{\psi\theta}-S^1_{\psi r}\} \\
&\quad + r\cos\psi\sin\psi\{S^0_{\psi\psi}-S^0_{\theta\theta}\} + r\sin^2\psi\{S^0_{\psi r}-rs_0{}^2 S^0_{\psi\theta}\} \\
(4.3.26)\qquad &r^{-1}\{S^2_{\theta\psi,\psi}+(rS^2_{\theta r})_{,r}\} - s_0 S^2_{\theta\theta} = -i\beta u^2_\theta + s_2 S^0_{\theta\theta} \\
&\quad - 2S^1_{\theta\psi}\cos\psi - \sin\psi\{2S^1_{\theta r}+rs_0 S^1_{\theta\theta}\} \\
&\quad + 2r\cos\psi\sin\psi\, S^0_{\psi\theta} + r\sin^2\psi\{2S^0_{\theta r}+rs_0{}^2 S^0_{\theta\theta}\}
\end{aligned}
$$

$$r^{-1}\{S^2_{r\psi,\psi}-S^2_{\psi\psi}+(rS^2_{rr})_{,r}\} + s_oS^2_{r\theta} = -i\beta u^2_r$$
$$- s_2S^o_{\theta r} - S^1_{\psi r}\cos\psi + \sin\psi\{rs_oS^1_{\theta r}+S^1_{\theta\theta}-S^1_{rr}\}$$
$$+ rS^o_{\psi r}\cos\psi\sin\psi + r\sin^2\psi\{S^o_{rr}-S^o_{\theta\theta}+rs_o{}^2S^o_{\theta r}\}.$$

It should be noted that in the three preceding equations we have set $s_1 = 0$. The remaining procedure is exactly the same as that used in determining s_1. Rather than duplicate the derivation, the final expression is presented below. It is

$$\int_o^1\{u^o_\psi S^2_{\psi\psi}+u^o_\theta S^2_{\theta\psi}+u^o_rS^2_{r\psi}-u^2_\psi S^o_{\psi\psi}-u^2_\theta S^o_{\theta\psi}-u^2_rS^o_{r\psi}\Big|_o^{2\pi}\,dr$$
$$+\int_o^{2\pi} r\{u^o_\psi S^2_{\psi r}+u^o_\theta S^2_{\theta r}+u^o_rS^2_{rr}-u^2_\psi S^o_{\psi r}-u^2_\theta S^o_{\theta r}-u^2_rS^o_{rr}\Big|_o^1\,d\psi$$
$$+2s_2\int_o^{2\pi}\int_o^1\{u^o_\psi S^o_{\theta\psi}-u^o_\theta S^o_{\theta\theta}+u^o_rS^o_{\theta r}\}rdrd\psi$$
$$+\int_o^{2\pi}\int_o^1\cos\psi\{u^1_\theta S^o_{\theta\psi}+u^o_\psi(S^1_{\psi\psi}-S^1_{\theta\theta})+2u^o_\theta S^1_{\theta\psi}+u^o_rS^1_{r\psi}-2u^1_\psi S^o_{\theta\theta}\}rdrd\psi$$
$$-\int_o^{2\pi}\int_o^1 r\cos\psi\sin\psi\{3u^o_\theta S^o_{\theta\psi}+u^o_\psi(S^o_{\psi\psi}-2S^o_{\theta\theta})+u^o_rS^o_{r\psi}$$
$$-u^o_\psi S^o_{\theta\theta}\}rdrd\psi$$

(4.3.27)

$$+\int_o^{2\pi}\int_o^1\sin\psi\{u^o_\psi(S^1_{\psi r}-rs_oS^1_{\psi\theta})+u^o_\theta(rs_oS^1_{\theta\theta}+2S^1_{\theta r})$$
$$+u^o_r(S^1_{rr}-S^1_{\theta\theta}-rs_oS^1_{r\theta})-rs_ou^1_\psi S^o_{\theta\psi}+u^1_\theta(S^o_{\theta r}+2rs_oS^o_{\theta\theta})$$
$$-u^1_r(rs_oS^o_{\theta r}+2S^o_{\theta\theta})\}rdrd\psi$$
$$+\int_o^{2\pi}\int_o^1\sin^2\psi\{u^o_\psi(2r^2s_oS^o_{\theta\psi}-rS^o_{\psi r})-u^o_\theta(3r^2s_oS^o_{\theta\theta}+3rS^o_{\theta r})$$
$$+u^o_r(2r^2s_oS^o_{\theta r}-rS^o_{rr}+3rS^o_{\theta\theta})\}rdrd\psi = 0.$$

Since all variables have period 2π in ψ the first integral of (4.3.27) is zero. Evaluation of the remaining integrals is simplified if we note that all the integrals involve products of trigonometric functions. Hence, only integrals involving $\cos^2 m\psi$ or $\sin^2 m\psi$, m an integer, are non-zero. In fact the value of $\int_0^{2\pi} \cos^2 m\psi d\psi$ or $\int_0^{2\pi} \sin^2 m\psi d\psi$ is π unless $m = 0$ in which case the integrals have value 2π and zero respectively. We can conclude that only the $\cos(n \pm 1)\psi$ portions of u^1_ψ, $S^1_{\theta\psi}$ and $S^1_{r\psi}$ are non-zero when integrated over ψ. In a similar fashion only the $\sin(n \pm 1)\psi$ terms in the remaining velocities and stresses are non-zero when integrated. Since each of the first order terms contributes two terms which will be non-zero when integrated, it is more convenient to leave first order terms as they are and comment only that the fifth integral in (4.3.27) is also zero since the trigonometric functions for each term can be written as $\frac{1}{4}\{\cos^2(n-1)\psi - \cos^2(n+1)\psi\}$ which is zero when integrated on ψ over $0 \leq \psi \leq 2\pi$.

We finally write the correction as

$$s_2 = -\frac{1}{2}\{\int_0^{2\pi}\int_0^1 [u^0_\psi S^0_{\theta\psi} - u^0_\theta S^0_{\theta\theta} + u^0_r S^0_{\theta r}]rdrd\psi\}^{-1}$$

$$\times \{\int_0^{2\pi} r[u^0_\psi S^2_{\psi r} + u^0_\theta S^2_{\theta r} + u^0_r S^2_{rr} - u^2_\psi S^0_{\psi r} - u^2_\theta S^0_{\theta r} - u^2_r S^0_{rr}\Big|_0^1 d\psi$$

$$+ \int_0^{2\pi}\int_0^1 \cos\psi[u^1_\theta S^0_{\theta\psi} + u^0_\psi(S^1_{\psi\psi} - S^1_{\theta\theta}) + 2u^0_\theta S^1_{\theta\psi} + u^0_r S^1_{r\psi} - 2u^1_\psi S^0_{\theta\theta}]rdrd\psi$$

$$(4.3.28) \qquad + \int_0^{2\pi}\int_0^1 \sin\,\psi[u_\psi^0(S^1_{\psi r}-rs_0S^1_{\psi\theta})+u_\theta^0(rs_0S^1_{\theta\theta}+2S^1_{\theta r})$$
$$+ u_r^0(S^1_{rr}-S^1_{\theta\theta}-rs_0S^1_{r\theta})-rs_0u^1_\psi S^0_{\theta\psi}+u^1_\theta(S^0_{\theta r}+2rs_0S^0_{\theta\theta})$$
$$- u^1_r(rs_0S^0_{\theta r}+2S^0_{\theta\theta})]rdrd\psi + \int_0^{2\pi}\int_0^1 \sin^2\psi\{u^0_\psi(2r^2s_0S^0_{\theta\psi}$$
$$- rS^0_{\psi r})-u^0_\theta(3r^2s_0S^0_{\theta\theta}+3rS^0_{\theta r})+u^0_r(2r^2s_0S^0_{\theta r}-rS^0_{rr}$$
$$+ 3rS^0_{\theta\theta})\}rdrd\psi.$$

In the rigid tube case the first integral in the numerator of (4.3.28) is zero since all velocities are zero at $r = 1$, while in the stress free case this same integral vanishes since the stresses do. For inviscid flows all the shear stresses are identically zero and the normal stresses can be replaced by the negative of the pressure for that order. Hence, $S^k{}_{\theta\psi} = S^k{}_{r\psi} = S^k{}_{\theta r} = 0$ for all k and $S^k{}_{rr} = S^k{}_{\theta\theta} = S^k{}_{\psi\psi} = -p_k$ [Note that in this case, the integral vanishes.]

Now that we have determined the correction equation for the general case we turn to the discussion of the two cases we are most interested in, the stress free and rigid tube cases.

4.4 Frequency Equation For Inviscid Axisymmetric Flows

Having seen the general procedure for determining the pressures and velocities required to find the correction s_2, we propose to examine the situations of primary interest in this chapter. These situations are the stress free and rigid tube cases of an inviscid fluid exhibiting axisymmetric motion in the zero order.

Since the fluid is inviscid the non-dimensional viscosities, $1/\eta$ and $1/\eta_1$, are taken to be zero. The axisymmetric condition allows us to set $n = 0$ in our zero order system. From (4.2.19) we see that

$$p_0(r) = AI_0(r\alpha) \qquad u_\theta^0(r) = -As_0I_0(r\alpha)/i\beta \tag{4.4.1}$$
$$u_\psi^0(r) = 0 \qquad u_r^0(r) = A\alpha I_1(r\alpha)/i\beta$$

For the rigid tube boundary condition, $u_r^0(1) = 0$ so

$$\alpha I_1(\alpha) = 0. \tag{4.4.2}$$

Designating the zeroes of $J_1(z)$ by z_{1m} $m = 1,2,3,\ldots$ we find that $\alpha = -iz_{1m}$ is a root of (4.4.2). From the definition of α in (4.2.3) we see that the frequency equation corresponding to (4.4.2) is written

$$s_0{}^2 = \beta^2/c^2 - z_{1m}^2. \tag{4.4.3}$$

For the stress free boundary we have (4.3.6) which is

$$I_0(\alpha) = 0. \tag{4.4.4}$$

Designating the zeroes of $J_0(z)$ by z_{0m} $m = 1,2,3,\ldots$ we find the frequency equation for (4.4.4) to be

$$s_0{}^2 = \beta^2/c^2 - z_{0m}^2 . \tag{4.4.5}$$

Since the zeroes of both $J_1(z)$ and $J_0(z)$ are well known we are able to compute s_0 easily in either case. If the ratio β/c is less than the zero z_{1m} or z_{0m} we have no propagation as s_0 is imaginary. However, for frequencies greater than cz_{1m} or cz_{0m} we have propagation of waves in the θ direction.

As the frequency increases the value of s_0 approaches β/c asymptotically from below. This is true regardless of the boundary condition. To allow for the many possible values of the acoustic speed, c, we present the graph of the zero order wave number as a function of β/c for the first few modes of both cases. This is done in Figure 18.

Recalling the definition of the phase velocity c_p and group velocity c_g, as,

$$\begin{aligned} c_p &= \beta/\mathrm{Re}\ s_0 , \\ c_g &= d\beta/d\ \mathrm{Re}\ s_0, \end{aligned} \tag{4.4.6}$$

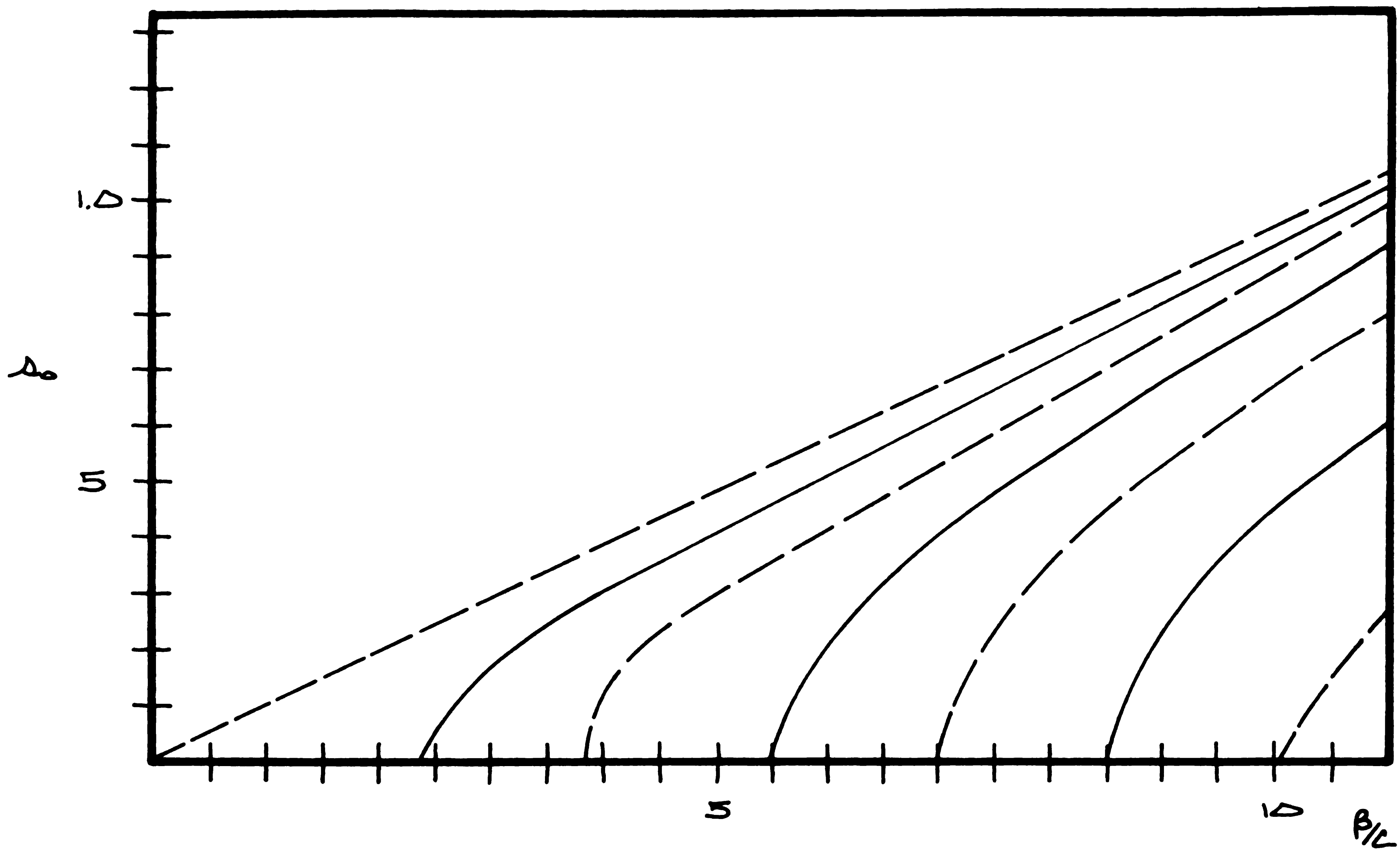


FIGURE #18
Δ_o VS β/c

RIGID TUBE (---) 1st FOUR NODES
STRESS FREE (—) 1st THREE NODES

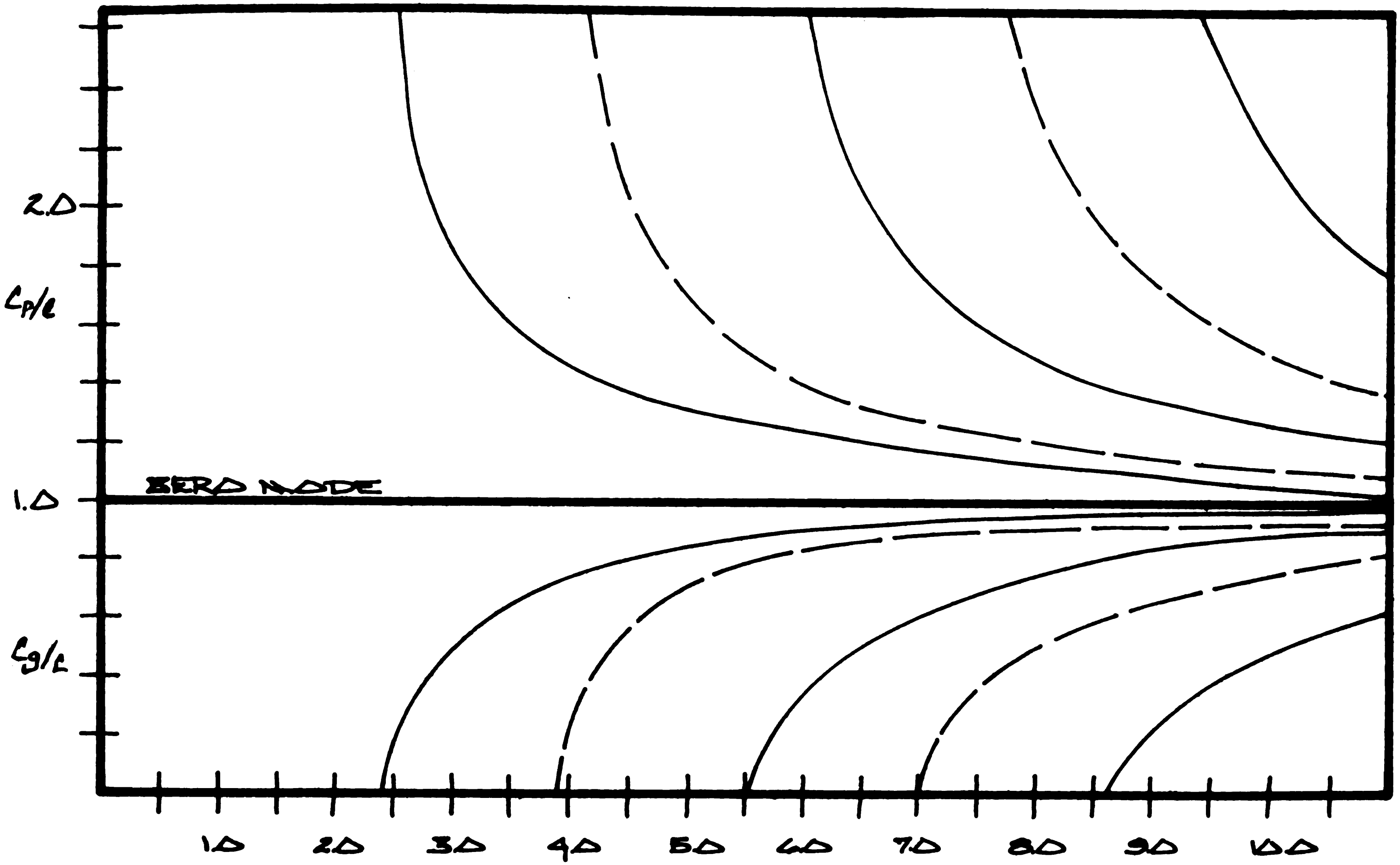


FIGURE #10
Cp/c AND Cg/c

RIGID TUBE (---)
STRESS FREE (—)
FIRST THREE MODES

we find immediately

$$\frac{c_p}{c} = (1-c^2 z_{1m}^2 \beta^{-2})^{-1/2} \quad \text{and} \tag{4.4.7}$$

$$\frac{c_g}{c} = \frac{c}{c_p} \tag{4.4.8}$$

for the rigid tube case. For stress free conditions we insert z_{Om} for z_{1m} and then perform the needed computations. From these equations we observe that for $\beta < cz_{Om}$ or $\beta < cz_{1m}$ there is no propagation in the respective cases. At $\beta = cz_{Om}$ or $\beta = cz_{1m}$ the value of s_O and c_g/c is zero while c_p/c is unbounded at the same value. As β increases s_O approaches β/c and both the group and phase velocity approach c in value.

The graphs of c_p/c and c_g/c are given in Figure 19 for three modes. We should note that the first zero in the series z_{1m} is 0 and when $z_{1m} = 0$ we have $s_O = \beta/c$ so that $c_p/c = c_g/c = 1$. This can also be seen in Figure 19.

Due to the interlacing of the zeroes of $J_O(z)$ and $J_1(z)$ we can easily assess the behavior of the velocity ratio (4.4.9). For example, consider the rigid tube case which corresponds to $\alpha = -iz_{1m}$ where m is the mth zero. Since the numerator will vanish at $r = 1$, it will also vanish at positions $r_O^{(n)} < 1$, corresponding to

$$r_O^{(n)} z_{1m} = z_{1n} \quad n = 1,2,\ldots,m-1. \tag{4.4.9}$$

Furthermore, at a zero of $J_1(z)$, $J_0(z)$ has a maximum or minimum value there and is nonzero. Thus at these points, (4.4.9), the ratio of velocities is longitudinal. Continuing, since the zeroes of $J_1(z)$ and $J_0(z)$ are interlaced, we find that the denominator vanishes m-1 times, or, between the locations (4.4.9) and, hence the motion is radial at these stations.

We note that the behavior just described for the rigid tube case holds as well for the stress free case if one interchanges z_{1m} with z_{0m} in (4.4.9).

As in the case of the empty tube, we find it instructive to examine the ratio of velocities as r increases from the center to the outer edge of the fluid. In the shell we considered a displacement ratio over a certain frequency range. For the fluid we have the additional variation of the variable r. We write

$$\left|\frac{u_r^0(r)}{u_\theta^0(r)}\right| = \left|\frac{\alpha I_1(r\alpha)}{(\beta^2/c^2+\alpha^2)^{1/2}\ I_0(r\alpha)}\right| \tag{4.4.10}$$

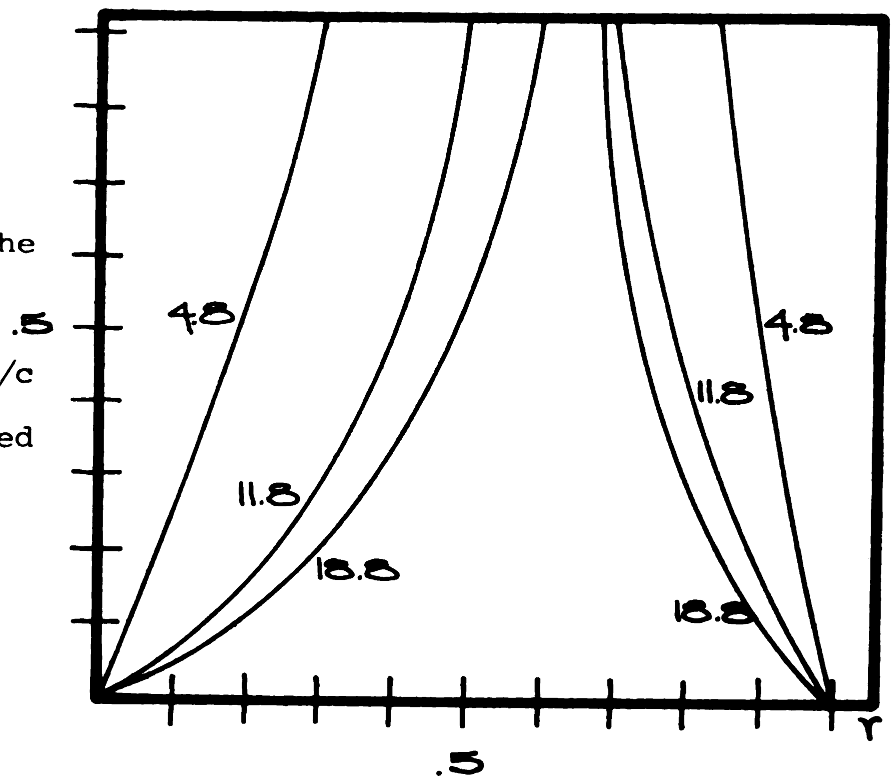
where $\alpha = -iz_{1m}$ for rigid tube flow and $\alpha = -iz_{0m}$ for stress free flow. When we have the rigid tube case the numerator will be zero at $r = 1$ while the denominator will be zero at some intermediate value of r. Just the opposite will occur in the stress free situation when the denominator is zero at $r = 1$ and the numerator is zero somewhere else.

All of this takes place for some fixed value of β, c and z_{1m} or z_{0m}. When $\beta = cz_{1m}$ or cz_{0m} then the denominator is zero regardless of value of r.

Graphs of the velocity ratio for various values of β, c and z_{jm}, $j = 0,1$ are presented in Figures 20 and 21 for r between 0 and 1. We see that in the rigid tube case for the second mode $r \cong 2.40/3.83$ is the value at which the ratio becomes unbounded, indicating radial motion. For small values of β/c the velocity ratio is nearly always large. However, as the value of β/c increases the motion is mostly longitudinal except near the value of r mentioned beforehand.

As we consider higher modes for the rigid tube case, we find there are more values of r, $0 \leq r \leq 1$ such that rz_{1m} is equal to a zero of the denominator, namely z_{0k} $k = 1,2,\ldots,m-1$. In Figure 20 a we indicate the behavior of the velocity ratio for the second mode $z_{12} = 3.83171$. [Note: We are considering $z_{11} = 0$ and for this value the velocity ratio is always zero.] In Figure 20 b we exhibit the velocity ratio for the next mode, $z_{13} = 7.01559$. Here we see the presence of two regions exhibiting primarily radial motion. From this it is easy to imagine that for very large modes the velocity ratio would be a series of verticle "spikes" indicating radial motion nearly everywhere. There are also $(m-2)$ zeroes since these are $(m-1)$ places where $rz_{1m} = z_{1j}$

Zero Order Velocity Ratio for the Rigid Tube. Values of β/c are Indicated $Z_{im}=3.8317$



Zim = 3.8317

Zero Order Velocity Ratio for the Rigid Tube Case Values of β/c Indicated $Z_{im}=7.01559$

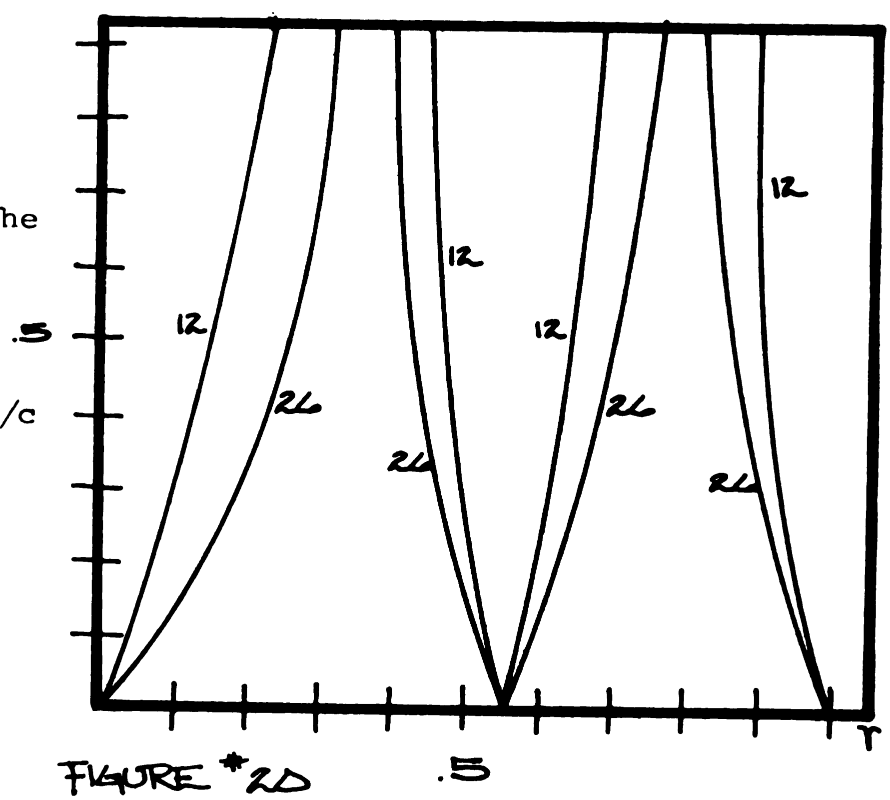
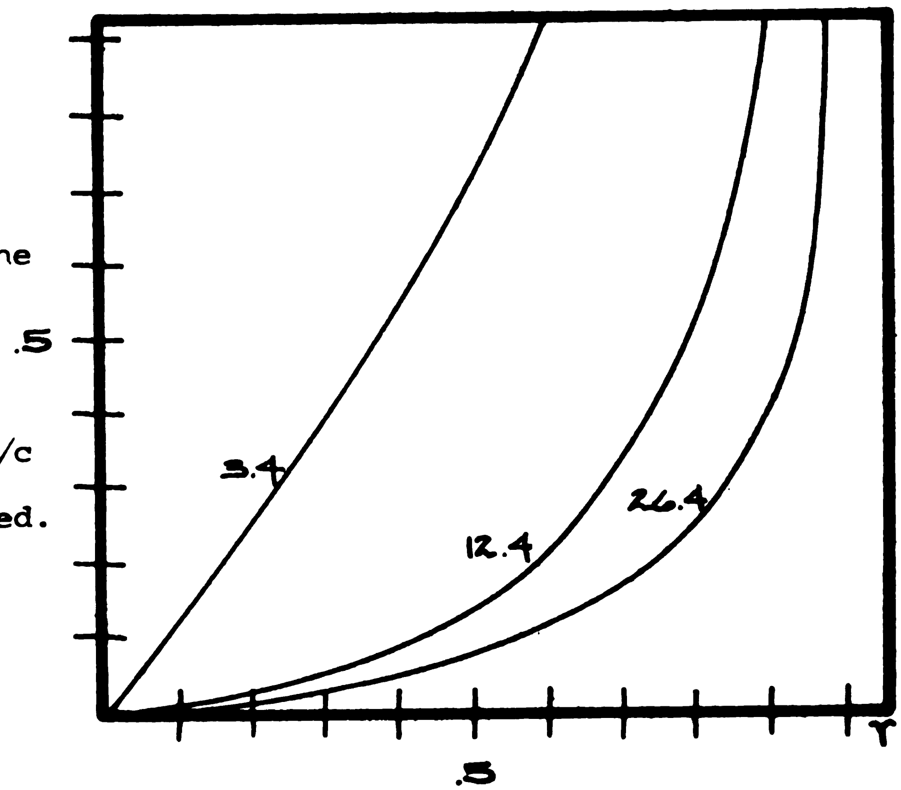


FIGURE #2D
Zim = 7.0156

$j = 1, 2, \ldots, m-1$. Hence there are $(m-1) + 1 = m$ places where the velocity ratio is zero and $m-1$ places where it is infinity for the rigid tube case. In general we might say that the motion is more often longitudinal, but that at higher modes this motion is fragmented by regions of radial motion.

In the stress free case we see that at $r = 1$ the denominator of the velocity ratio is zero, it being a multiple of the pressure. At $r = 0$ the ratio is zero as in the rigid tube case. If we consider a higher mode, say z_{Om}, we have $(m-1)$ points at which the denominator is zero in addition to the value $r = 0$. There are also $(m-1)$ points at which the numerator is zero in addition to $r = 0$. Hence for z_{Om} we find that in the stress free case there are as many points where the motion is purely radial as there are places where it is purely longitudinal. Thus we find that the motion becomes more longitudinal as the frequency increases. This suggests that at very high frequencies the motion is nearly longitudinal except at a finite number of discrete points, these points being the zeroes of the denominator.

Zero Order Velocity Ratio for the Stress Free Case. Values of β/c are Indicated.



$Z_{om} = 2.4048$

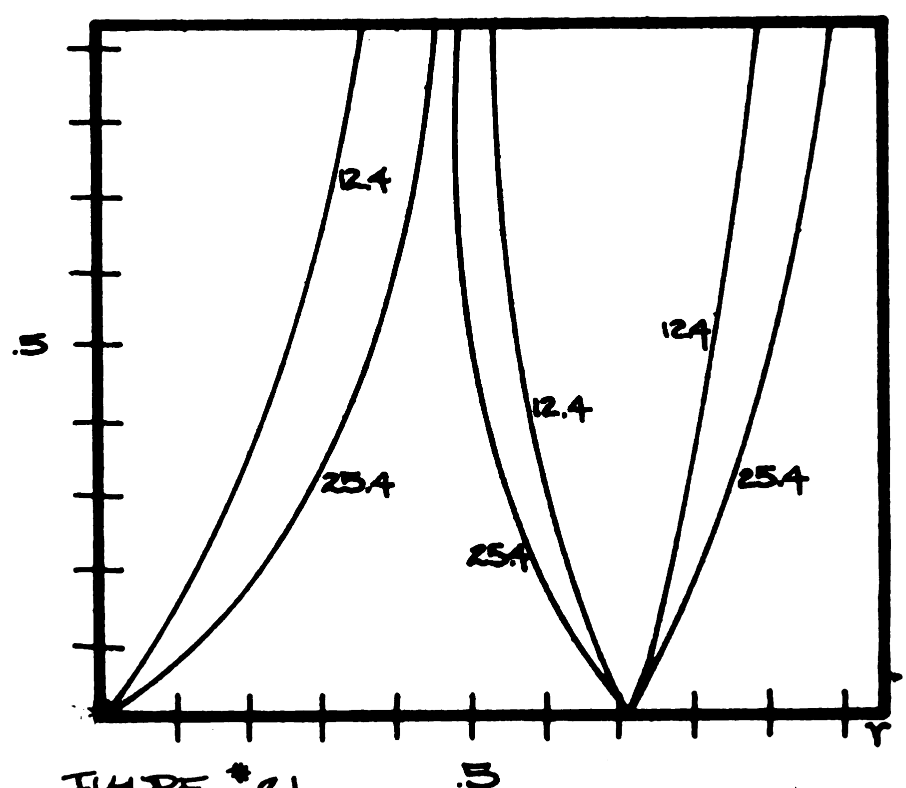


FIGURE #21
$Z_{om} = 5.52007$

4.5 Determination of First Order Functions

In 4.3 we indicated how one proceeds to the determination of the pressure and velocities of the first order system which are needed in the second order correction equation.

In the case of inviscid axisymmetric flow we set $n = 0$, $\frac{1}{\eta} = \frac{1}{\eta_1} = 0$, and $k^2 = 1$. In this case, by (4.3.16) and (4.3.21) we find

$$\begin{aligned} p_1(r,\psi) &= C_0 I_0(r\alpha) + I_1(r\alpha)\{C \cos\psi + D \sin\psi\} \\ &\quad - \tfrac{1}{2} A \sin\psi \{r^2 s_0{}^2 \alpha^{-1} I_1(r\alpha) + r I_0(r\alpha)\} \end{aligned} \tag{4.5.1}$$

where $C = C_1 - C_{-1}$ and $D = D_1 - D_{-1}$.

Returning now to (2.6.21) and inserting (4.5.1) as well as the inviscid assumptions mentioned above we find

$$\begin{aligned} u_\psi^1 &= p_{1,\psi}/ri\beta \\ u_\theta^1 &= s_0(rp_0 \sin\psi - p_1)/i\beta \\ u_r^1 &= p_{1,r}/i\beta \end{aligned} \tag{4.5.2}$$

We now proceed to the boundary conditions noting that the continuity equation (2.6.22) is satisfied identically.

For the rigid tube case, (2.6.30), the radial velocity component must vanish at $r = 1$. Hence, by (4.5.2),

$$p_1(r,\psi)_{,r}\big|_{r=1} = 0, \tag{4.5.3}$$

and inserting (4.5.1) we obtain,

$$(4.5.4)\quad \begin{cases} C_0\alpha I_1(\alpha) = 0, \\ C\alpha I_1'(\alpha) = 0, \\ D\alpha I_1'(\alpha) - \frac{1}{2} A[(2s_0{}^2\alpha^{-1}+\alpha)I_1(\alpha) + s_0{}^2 I_1'(\alpha) + I_0(\alpha)] \\ \qquad = 0. \end{cases}$$

Recall now that α must satisfy (4.4.2),

$$\alpha I_1(\alpha) = 0.$$

Although $\alpha = 0$ is a root satisfying (4.4.2), it is easy to see that in this case that (4.4.1), (4.5.1) and (4.5.4) would lead to the conclusion

$$u_\psi^0 = u_\theta^0 = u_r^0 = p^0 \equiv 0$$

and our first order system would become the fundamental system. Hence we conclude $\alpha \neq 0$, and proceed to the non-zero roots of (4.4.2), i.e., $I_1(\alpha) = 0$, $\alpha = -iz_{1m}$.

From (4.5.4) to (4.5.6) we find C_0 arbitrary,

$$(4.5.5)\qquad C = 0,$$

and

$$(4.5.6)\qquad D = \frac{1}{2} A\alpha^{-1}[s_0{}^2+1].$$

Knowing the value of the arbitrary constants we are now in a position to completely describe the first order pressure and velocities for the rigid tube case.

For stress free flow (2.6.19) requires $p_1(1,\psi) = 0$. Inserting (4.5.1) gives

$$(4.5.7)\qquad \begin{cases} C_0 I_0(\alpha) = 0 \\ C I_1(\alpha) = 0 \\ \{D - \frac{1}{2} A s_0{}^2/\alpha\} I_1(\alpha) - \frac{1}{2} A I_0(\alpha) = 0 \end{cases}$$

where now α satisfies (4.4.4), $I_0(\alpha) = 0$. As before we find C_0 is arbitrary, that $C = 0$, and

$$(4.5.8)\qquad D = \frac{1}{2} A s_0{}^2/\alpha.$$

In summary we have, for zero and first order,

$$(4.5.9)\qquad \begin{aligned} p_0(r) &= A I_0(r\alpha) & u^0_\psi(r) &= 0 \\ u^0_\theta(r) &= -A s_0 I_0(r\alpha)/i\beta & u^0_r(r) &= A\alpha I_1(r\alpha)/i\beta \end{aligned}$$

$$(4.5.10)\qquad \begin{aligned} p_1(r,\psi) &= \frac{1}{2} \sin\psi\{(2D\alpha - Ar^2 s_0{}^2) I_1(r\alpha)/\alpha - ArI_0(r\alpha)\} \\ u^1_\psi(r,\psi) &= \left(\frac{1}{ri\beta}\right) \frac{1}{2} \cos\psi\{(2D\alpha - Ar^2 s_0{}^2) I_1(r\alpha)/\alpha - ArI_0(r\alpha)\} \\ u^1_\theta(r,\psi) &= \left(\frac{1}{i\beta}\right) \frac{1}{2} s_0 \cos\psi\{3ArI_0(r\alpha) - (2D\alpha - Ar^2 s_0{}^2) \\ &\qquad\qquad I_1(r\alpha)/\alpha\} \\ u^1_r(r,\psi) &= \frac{1}{2} \sin\psi\{I_0(r\alpha)[2D\alpha - A(r^2 s_0{}^2+1)] \\ &\qquad - I_1(r\alpha)[2D\alpha + Ar^2(s_0{}^2 - z^2)]/r\alpha\}/i\beta \end{aligned}$$

where $D\alpha = \frac{1}{2} A s_0{}^2$ for stress free flow, $D\alpha = \frac{1}{2} A(s_0{}^2+1)$ for rigid tube flow and z is a zero of $J_1(z)$ or $J_0(z)$ depending on the boundary conditions. Note that since C_0 is arbitrary it has been incorporated into the arbitrary constant A.

Let us now consider the first order functions on the velocity ratio for various modes and frequencies within the two cases. We write the ratio as

$$\left| \frac{u_r^0(r) + \delta u_r^1(\psi,r)}{u_\theta^0(r) + \delta u_\theta^1(\psi,r)} \right|$$

and examine it for the cases presented in Figures 22 through 29. These curves are not meant to be an exhaustive display of the velocity ratio's behavior, but only indicative of the behavior as various parameters are changed.

The graphs are given as r varies from 0 to 1 each graph being for a fixed frequency and mode. In Figures 22 through 25 we present the stress free condition. The addition of the lower case "a" to a figure number will denote the upper graph for each figure will the lower case "b" will denote the lower graph.

In the zero order stress free case for the first mode $(z_{01} = 2.4048)$ the velocity ratio is zero at $r = 0$ gradually growing large as the boundary is approached. This indicates that the motion becomes radial as we approach the boundary. Increasing the frequency merely delayed the onset of the radial motion to larger r values. [See Figure 21a]. Examination of Figures 22a and b reveals the same behavior with the addition of their being more radial motion at $r = 0$

Corrected

Velocity

Ratio:

$\psi=30°$

$z_0=2.2048$

$\delta=.005\ (\cdots)$

$\delta=.05\ (\text{——})$

$\delta=.5\ (-\ -)$

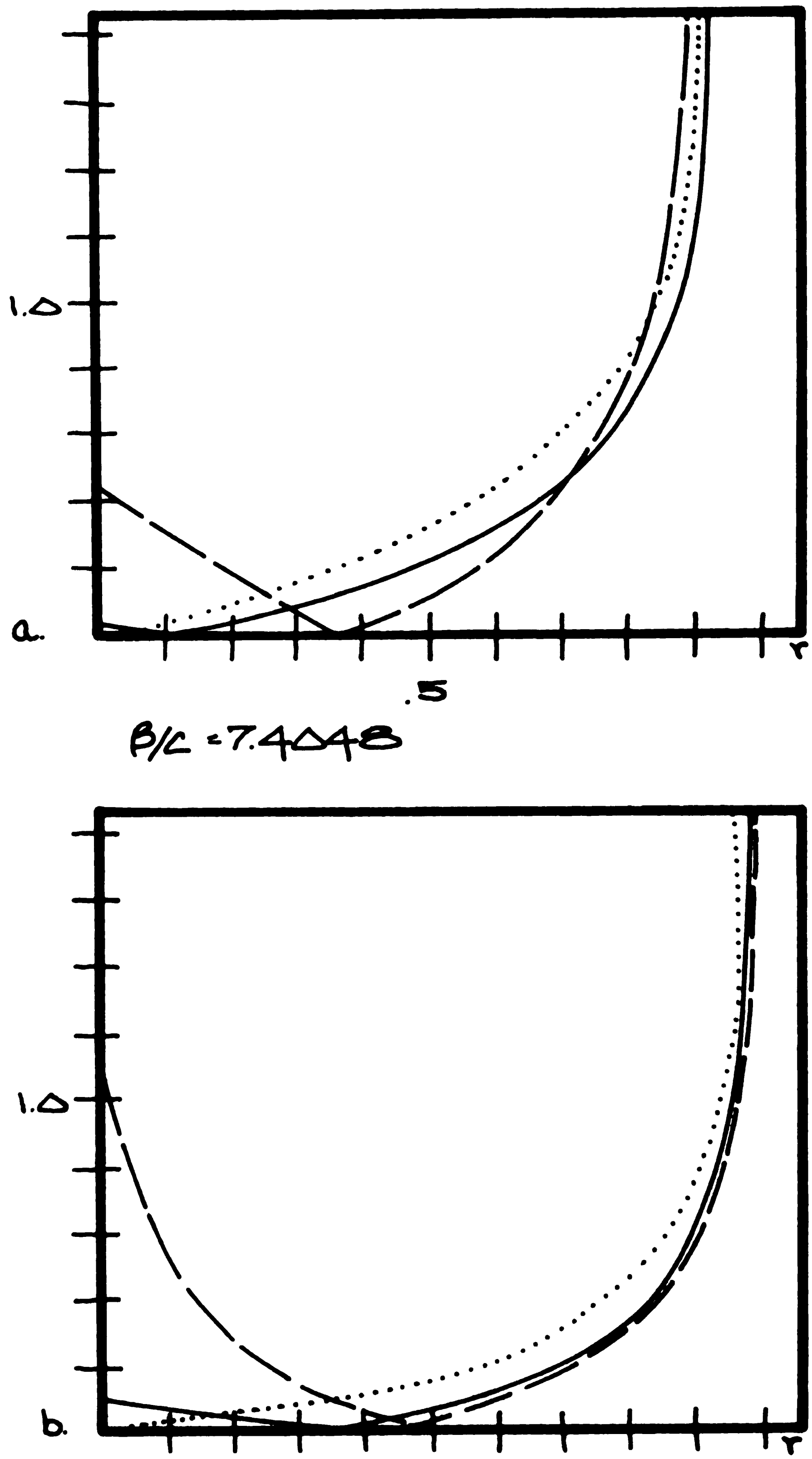


FIGURE #22 .5
β/c = 17.4048

in the corrected ratio. For r in the range $.3 < r < 1.0$ the zero order and first order ratios are quite close. Near the center we see that the first order ratio displays behavior not found in the zero order, indicating a need for higher order terms. We also see a greater variation for the largest value of δ indicating even more descrepency between the straight tube model and those which are highly curved.

In Figure 21 b we showed the behavior for the zero order ratio for the next mode ($z_0 = 5.52007$). The zero order indicated two regions which contained primarily radial motion. Examination of Figures 23a and b reveals the same behavior at approximately the same r values, .5 and 1. As in the zero order case the corrected ratios tended to become unbounded more abruptly as the frequency increased for a given mode. Again the $\delta = .5$ curves showed their variance with the zero order and smaller value delta curves near $r = 0$.

When we consider the case of $\psi = -60°$ we find that there is more variation from the zero order cases than when $\psi = 30°$. In the smaller frequencies, Figure 24 a, the $\delta = .5$ curve is the only one to offer significant difference from the zero order curve in Figure 21 a. However, increasing the frequency, as in Figure 24 b, caused the $\delta = .05$ to adopt the pattern of the larger δ value, exhibiting a new region of radial motion for $r < .4$.

Corrected

Velocity

Ratio: Stress

Free

$\psi=30°$

$z=5.52007$

$\delta=.005$ (· · ·)

$\delta=.05$ (——)

$\delta=.5$ (--)

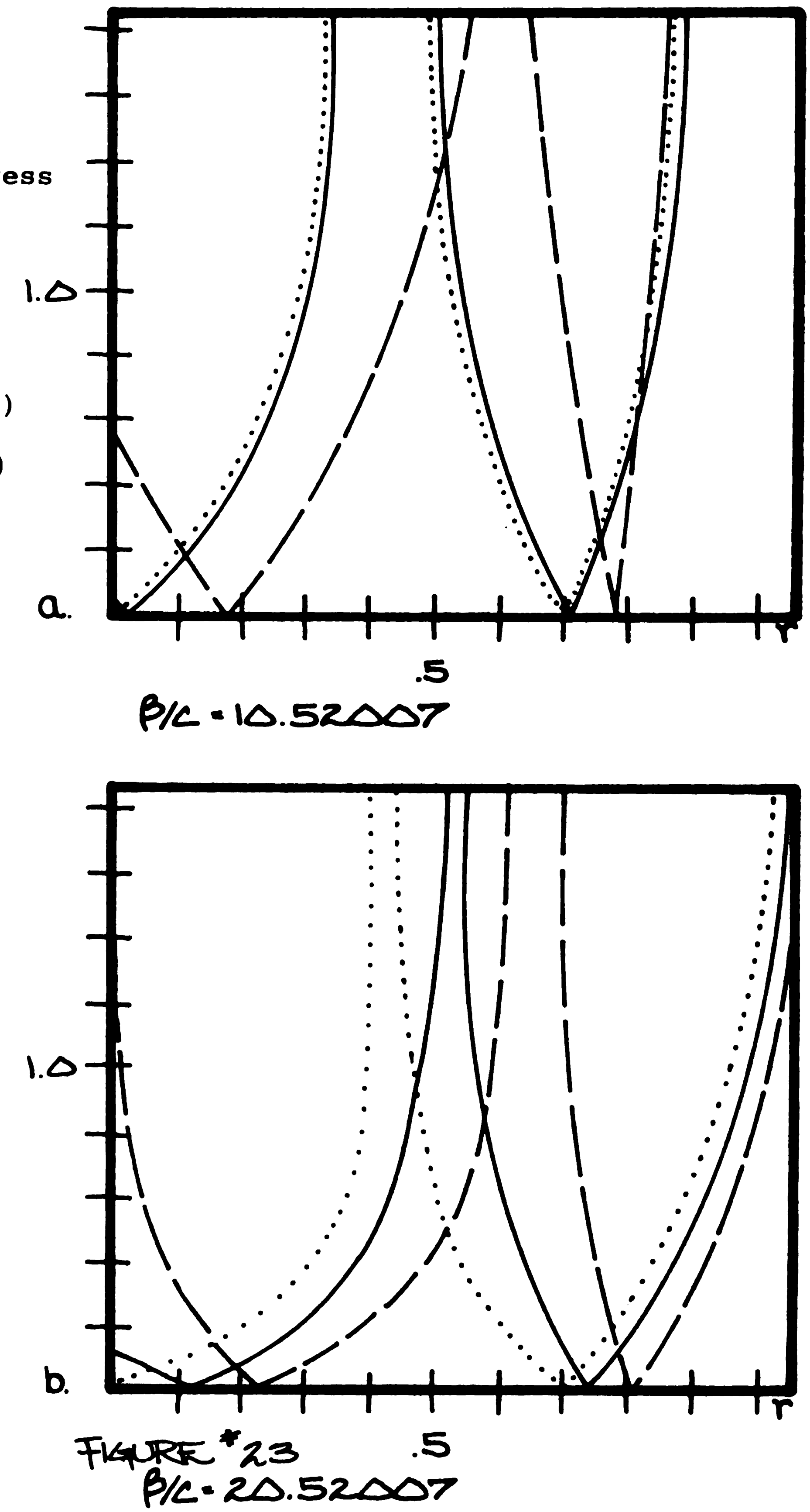


FIGURE #23

Corrected

Velocity

Ratio: Stress

Free

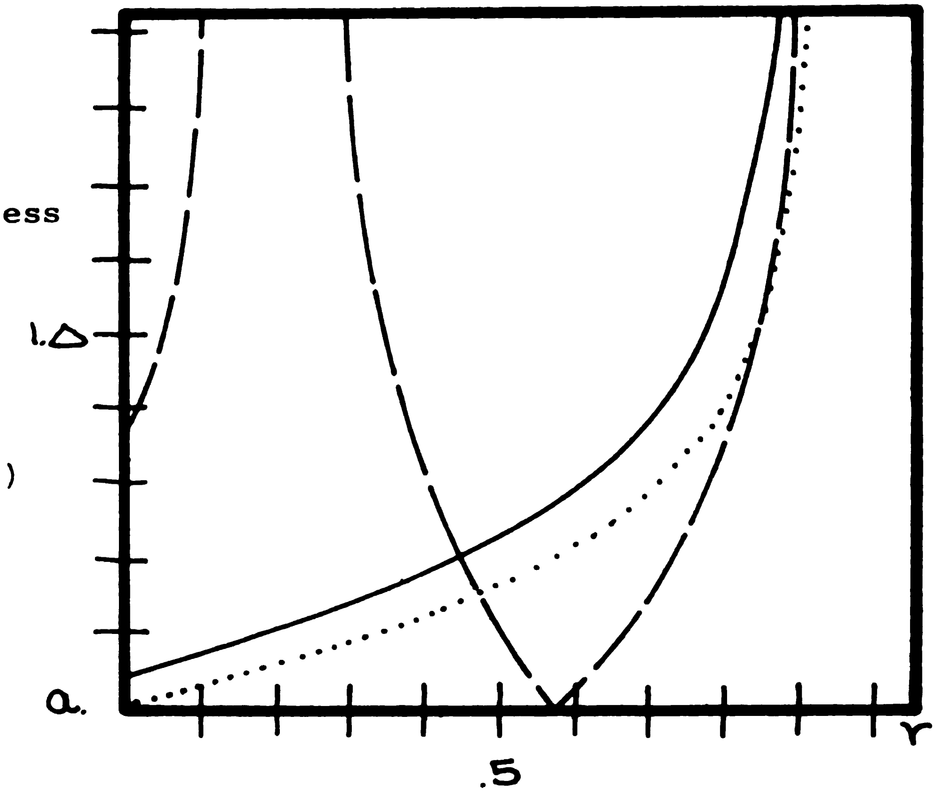
$\psi=-60°$

$z_0=2.2048$

$\delta=.005$ (· · · ·)

$\delta=.05$ (——)

$\delta=.5$ (– –)



$\beta/c = 7.4048$

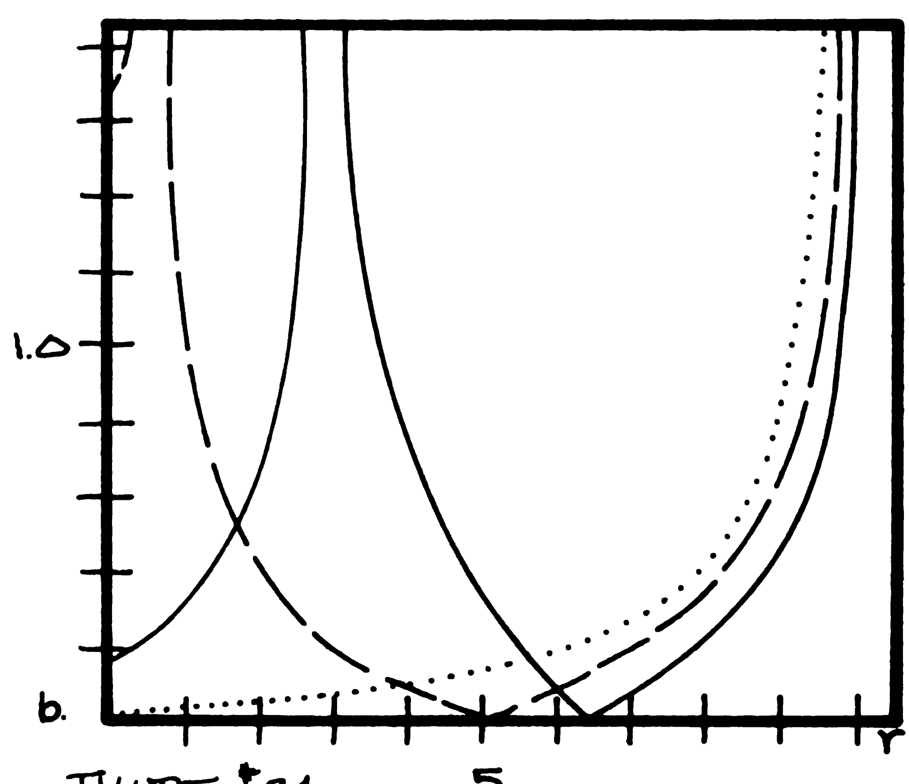


FIGURE #24
$\beta/c = 17.4048$

When we increase the mode to $z_0 = 5.52007$ as in Figures 21 a, 25 a and 25 b we see the presence of two regions of radial motion. When $\psi = 30°$ the corrected ratio is nearly the same as the zero order. However, for $\psi = -60°$ we see new behavior especially for $\delta = .5$. In Figure 25 a, the higher frequency curve, the $\delta = .5$ curve introduces a third region of radial motion near $r = .1$. This behavior is not found at all in the zero order curves of Figure 21 b.

Hence, when $\psi = -60°$ we find that the zero order model is less suitable for curved flow than when $\psi = 30°$. For the value of $\delta = .05$ the curves of $\psi = -60°$ in the corrected ratio are differing from the zero order more so than those of $\psi = 30°$ do.

For the rigid tube case the $\psi = 30°$ curves of Figures 26 and 27 do not differ substantially from the zero order system in Figure 20. Both curves tend to reach points of radial motion more abruptly as the frequency is increased. Both systems introduce new regions of radial motion as higher order zeroes are used. Only the $\delta = .5$ curves show marked differences from the zero order motion, these differences being present primarily for small values of r. Again, this merely suggests the unsuitability of the straight tube axially-symmetric model for tightly curved shells.

Corrected

Velocity

Ratio: Stress

Free

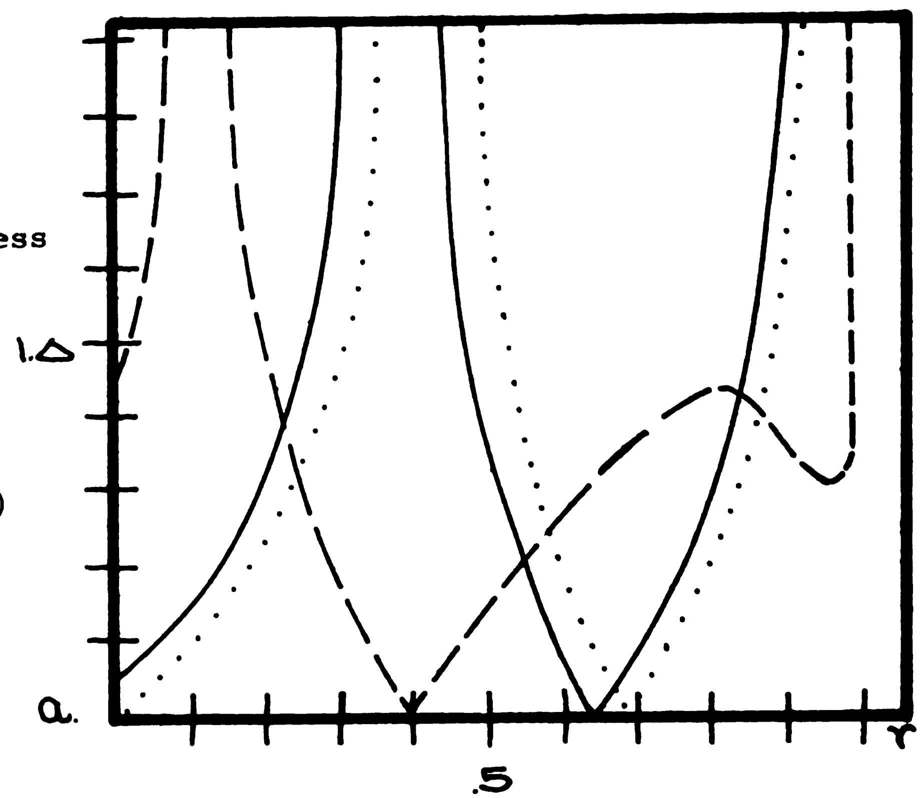
ψ=-60°

z=5.52007

δ=.005 (···)

δ=.05 (——)

δ=.5 (--)



β/c = 10.52007

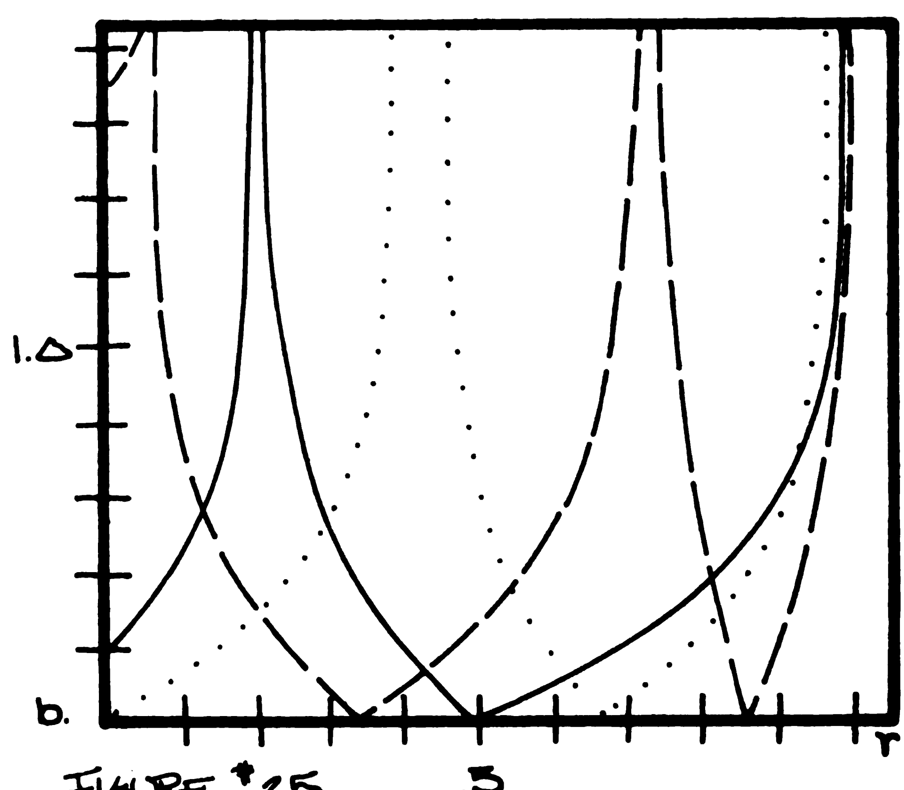


FIGURE #25

β/c = 20.52007

Corrected Velocity Ratio: Rigid Tube

$\psi=30°$

$z_1=3.83171$

$\delta=.005$ (···)

$\delta=.05$ (——)

$\delta=.5$ (--)

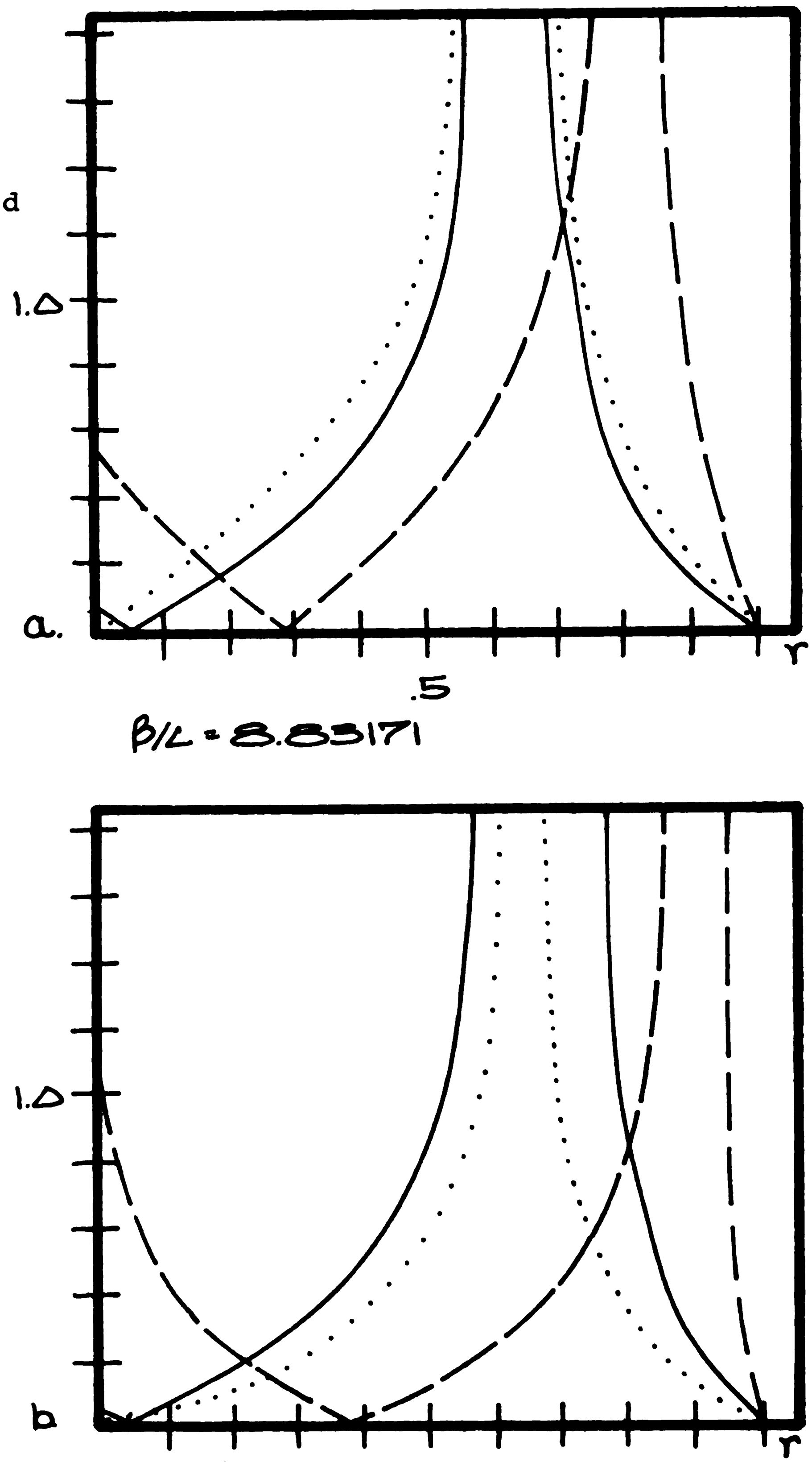


FIGURE #26
β/L = 18.83171

Corrected

Velocity

Ratio: Rigid

Tube

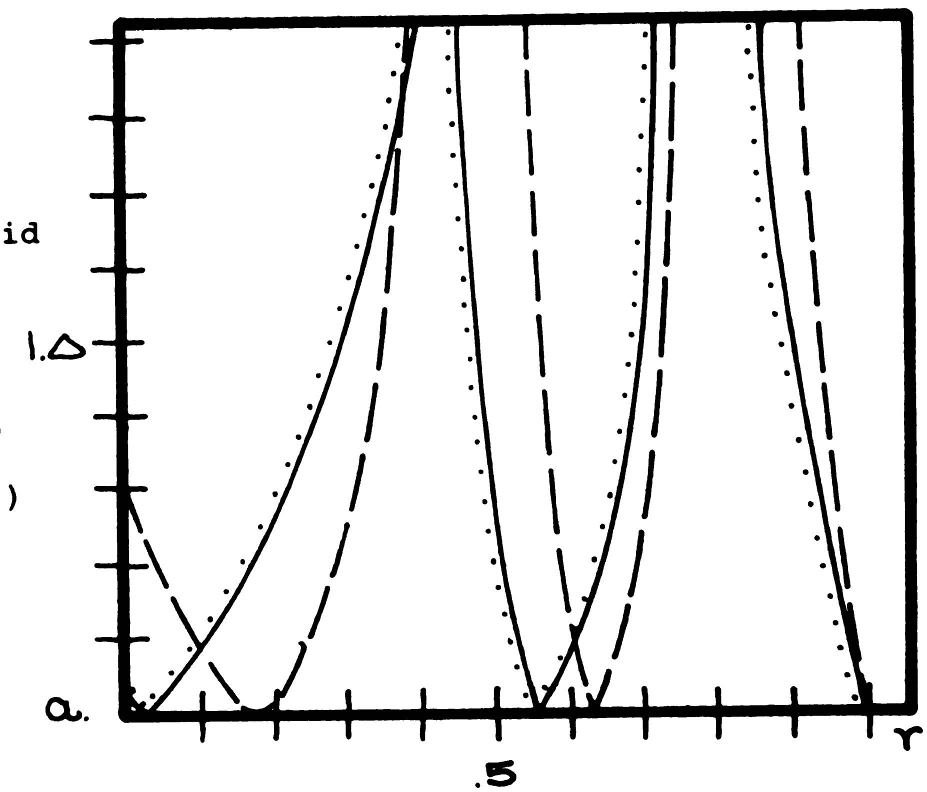
$\psi=30°$

$z_1=7.01559$

$\delta=.005\,(\cdots)$

$\delta=.05\,(—)$

$\delta=.5\,(--)$



$\beta/c = 12.01559$

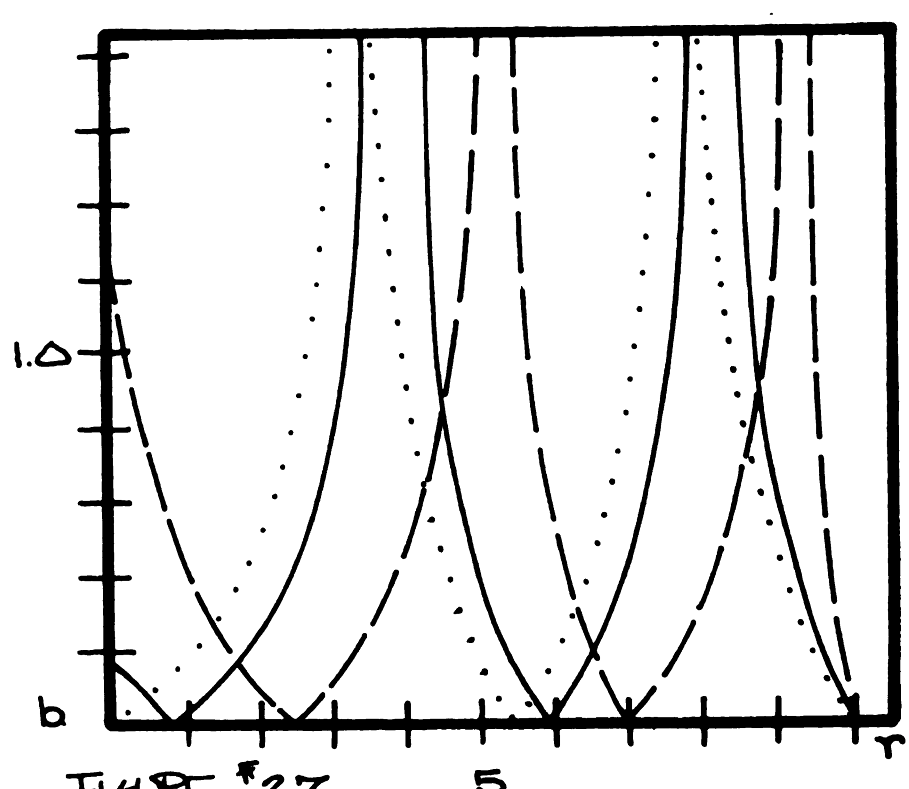


FIGURE #27

$\beta/c = 22.01559$

When $\psi = -60°$ the $\delta = .5$ curve for small frequencies, Figure 28 a, differs remarkably from the zero order case, Figure 20 a. When the frequency is increased as in Figure 28 b, then even the $\delta = .05$ curve has altered its behavior significantly from the zero order system. Only $\delta = .005$ retains the true behavior of the zero order system.

As the mode is increased as in Figure 20 b and Figure 29, the $\delta = .5$ curve is quite different from the zero order system. In this case of the higher mode number, the increase in frequency does not alter the $\delta = .05$ curve of Figure 29 b nearly as much as increasing the frequency did in 28 b, the lower mode. However, other data suggest that as the higher frequencies are reached for the second mode of Figure 29 the curve for $\delta = .05$ will change from the zero order case in greater degree.

Our conclusions from all of this indicate that the zero order model is a reasonable representation for the curved tube on the outside of the tube and for small values of δ. However, on the inside the straight tube becomes unsatisfactory for all but the smallest δ values and for all but the lowest frequencies. It is in this region that one must necessarily include higher order terms if the true velocity pattern is to be seen.

Corrected

Velocity

Ratio: Rigid

Tube

ψ=−60°

z=3.8317

δ=.005 (···)

δ=.05 (——)

δ=.5 (−−)

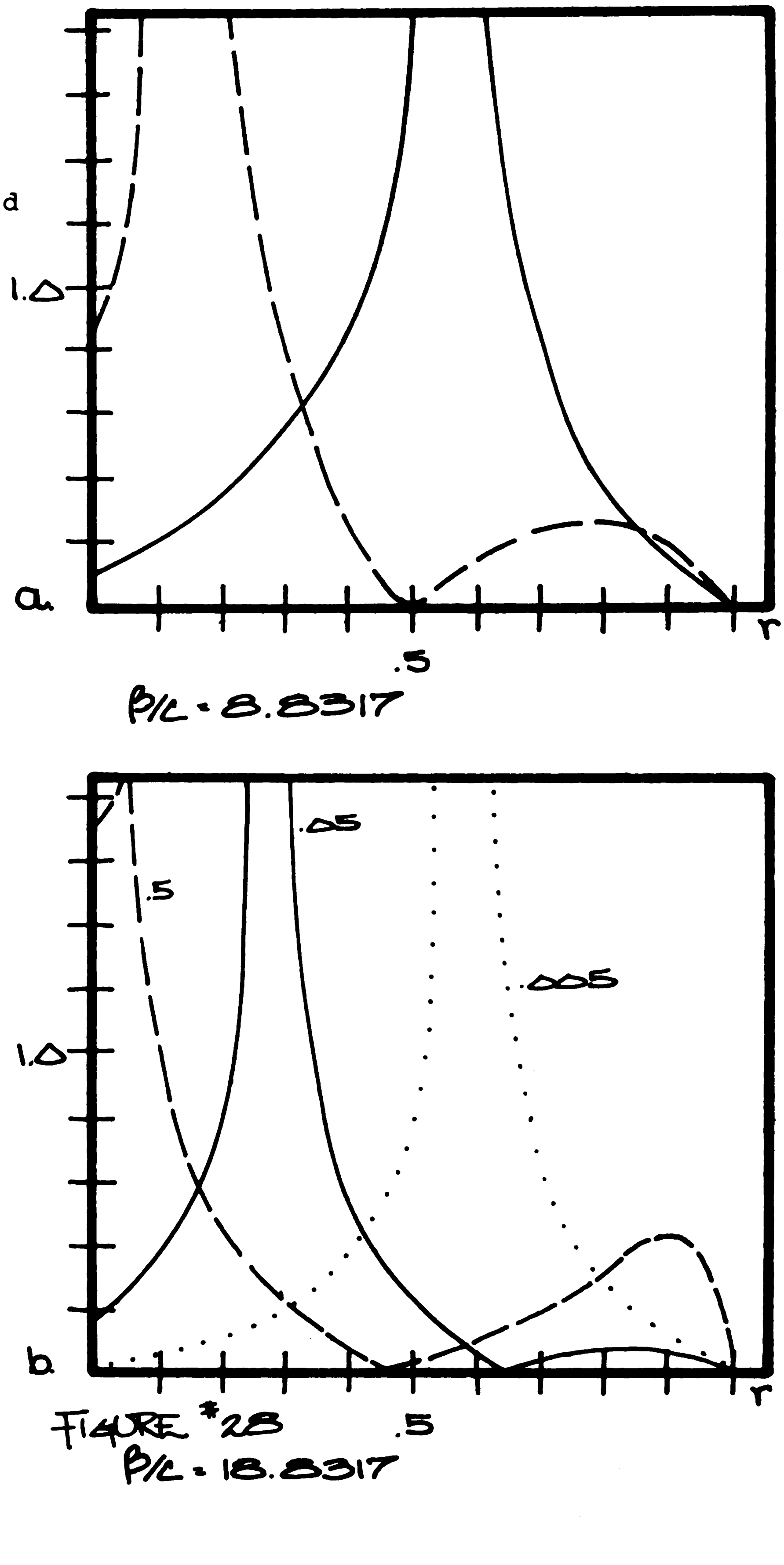
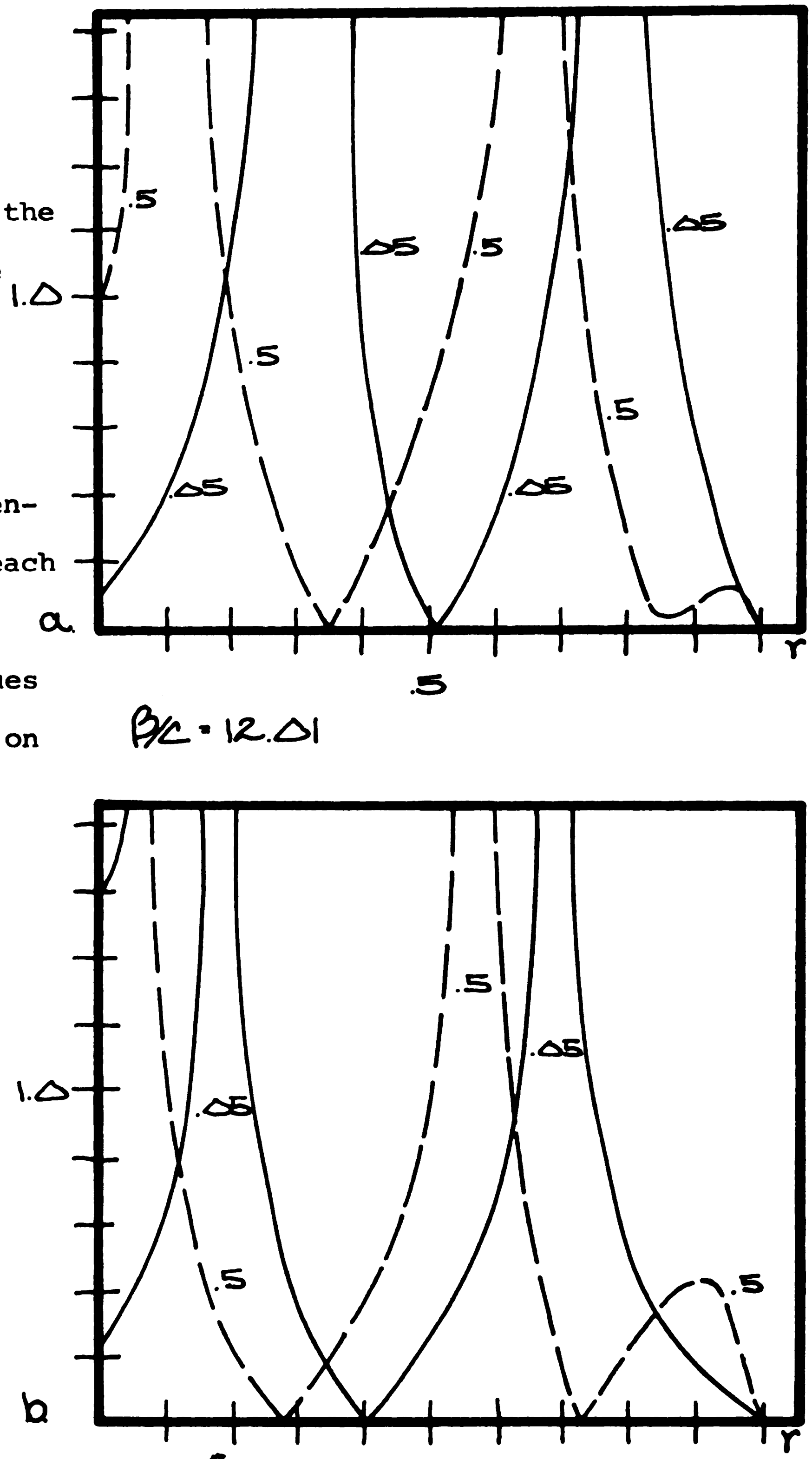


FIGURE #28



Corrected Velocity Ratio for the Rigid Tube Case $\phi=60°$ Two Modes Two Frequencies for each mode Delta values indicated on z=7.01559

FIGURE #29
β/C = 22.01

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4.6 Second Order Correction

We turn our attention to the correction equation (4.3.28) for s_2 developed in section 4.3. For inviscid flows, the shear stresses, $S^k_{r\psi}$, $S^k_{r\theta}$ and $S^k_{\psi\theta}$ are all zero since they depend on viscosity. The normal stresses are all that remain, being equal to the negative of the pressure for their respective order. For axisymmetric flows we know that $u^o_\psi = 0 = u^o_{\theta,\psi} = u^o_{r,\psi} = p_{o,\psi}$. Utilizing these facts in (4.3.28) we see that now s_2 has the value

$$s_2 = -\frac{1}{2}\{\int_0^{2\pi}\int_0^1 - u^o_\theta\, S^o_{\theta\theta}\, rdrd\psi\}^{-1}\{\int_0^{2\pi}\int_0^1(\cos\psi[-2u^1_\psi S^o_{\theta\theta}]$$

$$+ \sin\psi[rs_o u^o_\theta S^1_{\theta\theta} + 2rs_o u^1_\theta S^o_{\theta\theta} - 2u^1_r S^o_{\theta\theta}] \tag{4.6.1}$$

$$+ \sin^2\psi[-u^o_\theta 3r^2 s_o S^o_{\theta\theta} + ru^o_r(3S^o_{\theta\theta} - S^o_{rr})]rdrd\psi.$$

In (4.3.28) we have set the boundary integral zero since both stress free and rigid tube cases lead to this result. Only the $\cos\psi$ terms of u^1_ψ and the $\sin\psi$ portions of u^1_θ, u^1_r and $S^1_{\theta\theta}$ will make a non-zero contribution to the total.

Our first task is to perform the integration required in (4.6.1). In the first integral the ψ integration gives 2π as the value while in all of the others the result is π since we have non-zero results from the $\sin^2\psi$ or $\cos^2\psi$ terms. In order to integrate the remaining terms which are fun

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functions of r we require the following integrals

$$\int_0^r rI_0^2(r\alpha)dr = \frac{1}{2}r^2\{I_0^2(r\alpha) - I_1^2(r\alpha)\}$$

$$\int_0^r r^2I_0(r\alpha)I_1(r\alpha)dr = \frac{1}{2}r^2I_1^2(r\alpha)/\alpha$$

$$\int_0^r r^3I_0^2(r\alpha)dr = r^4\{3I_0^2(r\alpha) - 2I_1^2(r\alpha) - I_2^2(r\alpha)\}/12 \tag{4.6.2}$$

$$\int_0^r r^3I_1^2(r\alpha)dr = r^4\{I_1^2(r\alpha) - I_2^2(r\alpha)\}/6$$

$$\int_0^r r^4I_1(r\alpha)I_0(r\alpha)dr = r^4\{2I_1^2(r\alpha) + I_2^2(r\alpha)\}/6\alpha$$

After utilizing these results in (4.6.1) and dividing all terms by $\pi A^2/i\beta$ we find

$$\begin{aligned} s_2 = \frac{1}{2s_0}\{I_0^2(\alpha) - I_1^2(\alpha)\}^{-1}\{I_0^2(\alpha)[\frac{D\alpha}{A} - 1 - \frac{15s_0^2}{8}] \\ + I_1^2(\alpha)[\frac{D\alpha}{A}(\frac{3s_0^2}{2\alpha^2} - 1) + s_0^2(\frac{5}{4} - \frac{1}{\alpha^2} - \frac{s_0^2}{2\alpha^2}) - \frac{1}{2}] \\ + \frac{1}{4}I_2^2(\alpha)[\frac{5}{2} - \frac{s_0^2}{\alpha^2}]s_0^2\} \end{aligned} \tag{4.6.3}$$

In the rigid tube case we set $I_1(\alpha) = 0$ and $\frac{D\alpha}{A} = \frac{1}{2}(s_0^2+1)$. We also find $I_2(\alpha) = I_0(\alpha)$ giving

$$\begin{aligned} s_2 &= -\frac{1}{2s_0}\{\frac{3s_0^2}{4} + \frac{1}{2} + \frac{s_0^4}{4\alpha^2}\} \\ &= \frac{1}{8s_0z_{1m}^2}\{s_0^4 - z_{1m}^2(3s_0^2+2)\} \end{aligned} \tag{4.6.4}$$

where as before $J_1(z_{1m}) = 0 \quad m = 1,2,\cdots$.

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For the stress free case we set $I_0(\alpha) = 0$ and $\frac{D\alpha}{A} = \frac{1}{2} s_0^2$. We find $I_2(\alpha) = 2I_1(\alpha)/\alpha$. This allows us to write

$$s_2 = \frac{1}{8 s_0 z_{0m}^4} \{ s_0^4 [z_{0m}^2 + 4] + 3 s_0^2 z_{0m}^2 (2 - z_{0m}^2) + 2 z_{0m}^4 \} \tag{4.6.5}$$

We can see from (4.6.4) and (4.6.5) that for the situation of small values of s_0 the rigid tube and stress free corrections behave like $-1/4\, s_0^{-1}$ and $+1/4\, s_0^{-1}$ respectively. For large values of s_0 they are both of order s_0^3. At some point then the value of $\delta^2 s_2$ will become larger than s_0 indicating that we need to consider some higher order terms in the expansion. The same is true for values of s_0 which are very small.

In Figure 30 we show the corrected phase velocity, normalized by the non-dimensional acoustic speed, for both the rigid tube and stress free cases. Their behaviors are quite similar, both being zero at the point where $s_0 = 0$ and both indicating an asymptote very near this same point. After this initial behavior the phase velocity quickly approaches one. Eventually it becomes zero with large values of β/c. For $\delta = .5$ we see that in both cases the phase velocity becomes zero much faster than for smaller values of δ. For small delta the straight tube is an accurate

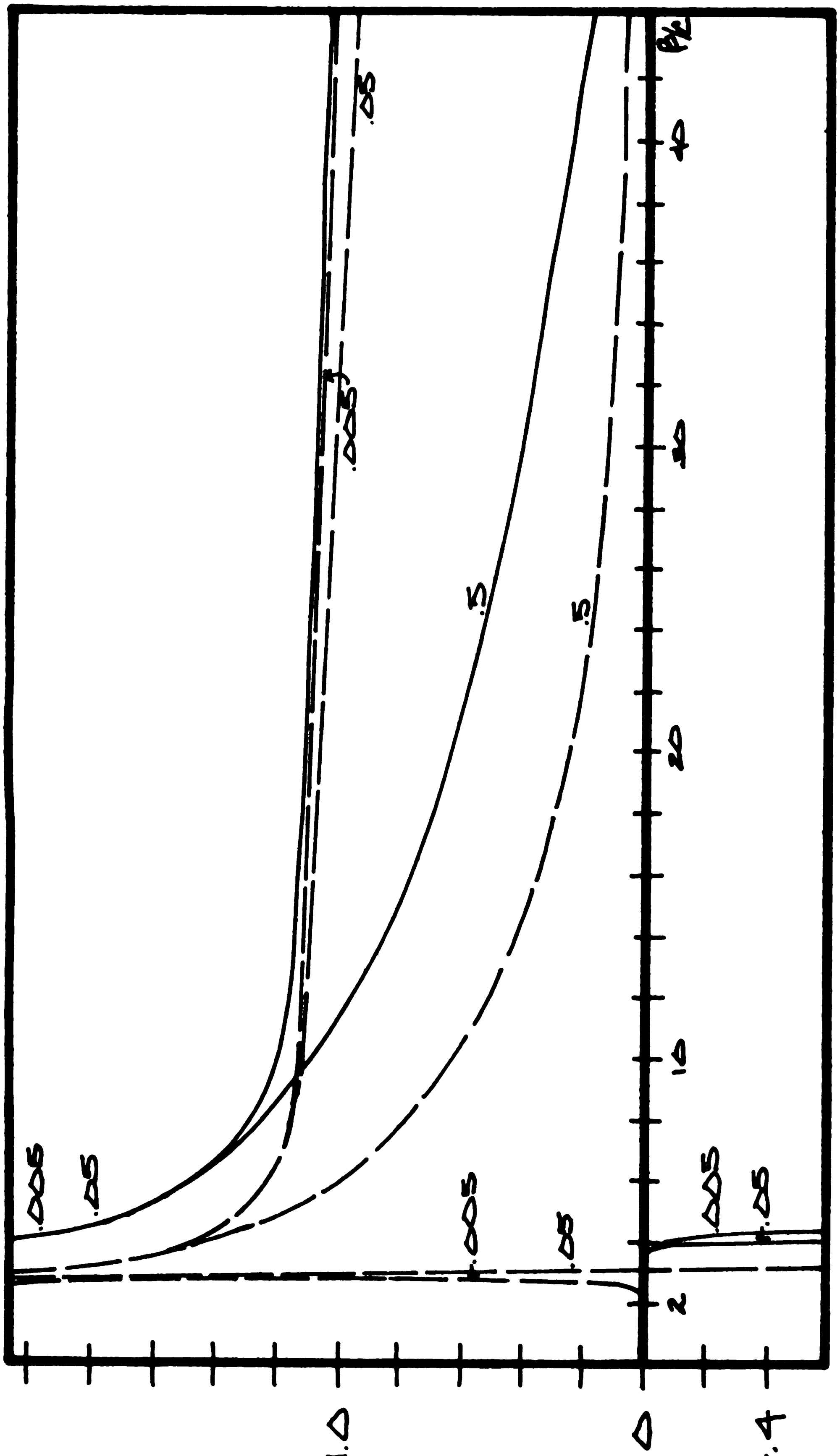

FIGURE 30
CORRECTED PHASE VELOCITY: RIGID TUBE CASE (——)
STRESS FREE CASE (- - - -)
FIRST NON-ZERO Z_{1m}

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representation for the curved tube since its phase velocity approaches one from above. Only when the frequency becomes quite large or the value of δ grows large does the straight tube mode cease to be an effective predictor of phase velocity.

The curves for the corrected wave number, $(s_0+\delta^2 s_2)/s_0$, have been omitted since they are essentially the reflection of the phase velocity about the line $\tilde{c}p/c = 1$. The correspondence is not exact but the behavior can be determined by noting that as the phase velocity approaches zero the wave number becomes infinite and as the phase velocity becomes unbounded the wave number approaches zero.

CHAPTER V

GENERAL INTERACTION PROBLEM

5.1 Introduction

We have presented analyses dealing with the vibration of the empty tube in Chapter 3 and with the vibration of the fluid in Chapter 4. It is now our task to analyze the interaction problem in which the fluid and membrane are allowed to vibrate together.

In section 5.2 we begin by establishing the frequency equation for an arbitrary number, n, of waves around the tube. We also discuss the correction equations appropriate to s_1 and s_2. Section 5.3 deals with the frequency equation appropriate to the fundamental case of axisymmetric inviscid flow which is the concern of following sections.

Section 5.4 contains the determination of first order velocity and displacement components for the fluid and shell respectively. As before we examine their effect on corresponding zero order terms. Finally in section 5.5 we examine the correction, s_2, and determine its effects on the zero order wave number s_0.

5.2 Discussion for Arbitrary n

In section 4.2 we established the zero order values for the fluid pressure and velocities for the viscous fluid flowing with n waves in the ψ direction. They are given by (4.2.18). Knowing the fluid velocities it is possible to write out the fluid stresses which are inhomogeneous terms of the shell equations.

The shell displacements can be written from (3.2.2) as

$$\begin{pmatrix} \beta^2-n^2-\frac{1}{2}s_o^{\,2}(1-\nu) & -\frac{1}{2}ns_o(1+\nu)-n & -n \\ -\frac{1}{2}ns_o(1+\nu) & \beta^2-s_o^{\,2}-\frac{1}{2}n^2(1-\nu) & -\nu s_o \\ -n & -\nu s_o & \beta^2-1 \end{pmatrix} \begin{pmatrix} W_\psi^o \\ W_\theta^o \\ W_r^o \end{pmatrix}$$

(5.2.1)

$$= -\frac{1}{m} \begin{pmatrix} \tilde{S}^o_{\psi r} \\ \tilde{S}^o_{\theta r} \\ -\tilde{S}^o_{rr} \end{pmatrix}_{r=1}$$

where the tildas reflect division by the appropriate trigonometric functions and where the three shell displacements have been written as

(5.2.2) $\quad w_\psi^o = W_\psi^o \sin n\psi \qquad w_\theta^o = W_\theta^o \cos n\psi \qquad w_r^o = W_r^o \cos n\psi$

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Unlike the empty tube case the mass ratio $m(=\rho h/\rho_0 a)$ is no longer zero and a solution of (5.2.1) is given by the three relationships

$$(5.2.3)\qquad w_\psi^0 = -\frac{1}{m}(\mathrm{Det})^{-1}\{A_{11}\tilde{S}_{r\psi}^0 + A_{12}\tilde{S}_{r\theta}^0 - A_{13}\tilde{S}_{rr}^0\}_{r=1}$$

$$(5.2.4)\qquad w_\theta^0 = -\frac{1}{m}(\mathrm{Det})^{-1}\{A_{12}\tilde{S}_{r\psi}^0 + A_{22}\tilde{S}_{r\theta}^0 - A_{23}\tilde{S}_{rr}^0\}_{r=1}$$

$$(5.2.5)\qquad w_r^0 = -\frac{1}{m}(\mathrm{Det})^{-1}\{A_{13}\tilde{S}_{r\psi}^0 + A_{23}\tilde{S}_{r\theta}^0 - A_{33}\tilde{S}_{rr}^0\}_{r=1}$$

where Det represents the determinant of the coefficient matrix in (5.2.1) and the A_{ij} are cofactors of elements a_{ij} in the same coefficient matrix.

5.2.1 Frequency Equation

The boundary conditions require that at $r = 1$ the fluid velocities match the shell velocities which are time derivatives of shell displacements. We see then

$$(5.2.6)\qquad -i\beta w_\xi^0 = u_\xi^0\big|_{r=1} \qquad \xi = \psi, \theta, r$$

Inserting the fluid velocities (4.2.18) into the fluid constitutive equations (2.6.11) we find from (5.2.6) that

$$(5.2.7)\qquad \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} A_3 \\ A_4 \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where

$$C_{11} = -\frac{\beta^2}{m\ \mathrm{Det}}\frac{1}{\eta_1}\{A_{11}\gamma I_{n+2}(\gamma) + A_{12}s_0^{-1}[\gamma^2 I_n'(\gamma) - s_0^2 I_{n+1}(\gamma)]$$

$$- 2A_{13}\gamma I_{n+1}'(\gamma)\} - I_{n+1}(\gamma),$$

$$C_{12} = -\frac{\beta^2}{m\ \mathrm{Det}}\frac{1}{\eta_1}\{A_{11}\gamma I_n(\gamma) - A_{12}s_0^{-1}[\gamma^2 I_n'(\gamma) - s_0^2 I_{n-1}(\gamma)]$$

$$+ 2A_{13}\gamma I_{n-1}'(\gamma)\} - I_{n-1}(\gamma),$$

$$C_{13} = -\frac{M\beta^2}{(\alpha^2-\gamma^2)m}\frac{1}{\mathrm{Det}}\frac{1}{\eta_1}\{-2nA_{11}[\alpha I_n'(\alpha) - I_n(\alpha)$$

$$-2\alpha s_0 A_{12} I_n'(\alpha)\} + \frac{\beta^2}{m\ \mathrm{Det}}\{A_{13}[1 + \frac{i\beta}{c^2}(\frac{1}{\eta} - \frac{1}{\eta_1})]I_n(\alpha)$$

$$- A_{13}M\alpha^2 I_n''(\alpha)\} - \frac{M\alpha I_n'(\alpha)}{\alpha^2-\gamma^2},$$

$$C_{21} = -\frac{\beta^2}{\eta_1 m\ \mathrm{Det}}\{A_{12}\gamma I_{n+2}(\gamma) + A_{22}s_0^{-1}[\gamma^2 I_n'(\gamma) - s_0^2 I_{n+1}(\gamma)]$$

(5.2.8) $$+ 2A_{23}\gamma I_{n+1}'(\gamma)\} - \gamma s_0^{-1} I_n(\gamma),$$

$$C_{22} = -\frac{\beta^2}{\eta_1 m\ \mathrm{Det}}\{A_{12}\gamma I_n(\gamma) - s_0^{-1}A_{22}[\gamma^2 I_n'(\gamma) - s_0^2 I_{n-1}(\gamma)]$$

$$+ 2A_{23}\gamma I_{n-1}'(\gamma)\} + \gamma s_0^{-1} I_n(\gamma),$$

$$C_{23} = -\frac{\beta^2}{m\ \mathrm{Det}}\{-\frac{2M}{\eta_1(\alpha^2-\gamma^2)}[nA_{12}(\alpha I_n'(\alpha) - I_n(\alpha))$$

$$+ \alpha s_0 A_{22} I_n'(\alpha)] + A_{23} I_n(\alpha)[1 + \frac{i\beta}{c^2}(\frac{1}{\eta} - \frac{1}{\eta_1})]$$

$$- \frac{M\alpha I_n''(\alpha)}{(\alpha^2-\gamma^2)} A_{23}\},$$

$$C_{31} = -\frac{\beta^2}{\eta_1 m\ \mathrm{Det}}\{A_{13}\gamma I_{n+2}(\gamma) + s_0^{-1}A_{23}[\gamma^2 I_n'(\gamma) - s_0^2 I_{n+1}(\gamma)]$$

$$- 2A_{33}\gamma I_{n+1}'(\gamma)\} - I_{n+1}(\gamma),$$

$$C_{32} = -\frac{\beta^2}{m\eta_1 \text{ Det}}\{A_{13}\gamma I_n(\gamma) - s_0^{-1}A_{23}[\gamma^2 I_n'(\gamma) - s_0^2 I_{n-1}(\gamma)]$$

$$+ 2\gamma A_{33}I_{n-1}'(\gamma)\} + I_{n-1}(\gamma),$$

$$C_{33} = -\frac{\beta^2}{m \text{ Det}}\{-\frac{2M}{\eta_1(\alpha^2-\gamma^2)}[nA_{13}(\alpha I_n'(\alpha) - I_n(\alpha))$$

$$+ \alpha s_0 A_{23} I_n'(\alpha)] + A_{33}[(1 + \frac{i\beta}{c^2}[\frac{1}{\eta} - \frac{1}{\eta_1}])I_n(\alpha)$$

$$- \frac{M\alpha^2 I_n''(\alpha)}{\alpha^2-\gamma^2}]\} - \frac{M\alpha I_n'(\alpha)}{\alpha^2-\gamma^2}\}.$$

In order that we have non-trivial solutions for our problem we require that the determinant of the coefficient matrix in (5.2.7) vanish. This defines an implicit relationship between s_0 and β which we designate as the frequency equation.

5.2.2 First Order Correction

In Chapter 4 we found the first order equation for s_1 and were able to verify that in the stress free and rigid tube cases, $s_1 = 0$. We now consider this equation (4.3.13) for the interaction problem. The line integral of (4.3.13) is repeated here as it is no longer zero. It is

$$I = \int_0^{2\pi} \{u_\psi^0 S_{\psi r}^1 + u_\theta^0 S_{\theta r}^1 + u_r^0 S_{rr}^1 - u_\psi^1 S_{\psi r}^0 - u_\theta^1 S_{\theta r}^0 - u_r^1 S_{rr}^0\}d\psi \tag{5.2.9}$$

where all arguments are evaluated at $r = 1$. The first order fluid stresses at $r = 1$ may be replaced by the first order shel

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shell terms with which they appear, (2.6.26). We replace zero order fluid stresses in a similar fashion using (2.6.14) and replace the fluid velocities at $r = 1$ by the shell displacements. Shell stresses are then replaced by the shell displacements using (2.6.13) and (2.6.25). Noting that the integral of an odd function in ψ is zero over the interval we are considering we write

$$(5.2.10)\quad \begin{aligned} I = {} & i\beta m s_1 \int_0^{2\pi} \{\tfrac{1}{2} w_\psi^0 N_{\theta\psi}^0 - w_\theta^0 N_{\theta\theta}^0 + \nu w_\psi^0 w_{\theta,\psi}^0 \\ & - \tfrac{1}{2} s_0 (1-\nu)(w_\psi^0)^2 - s_0 (w_\theta^0)^2 - \tfrac{1}{2}(1-\nu) w_\theta^0 w_{\psi,\psi}^0 \\ & - \nu w_r^0 w_\theta^0\} d\psi - i\beta m \int_0^{2\pi} \{w_\psi^1 w_{\psi,\psi\psi}^0 - w_\psi^0 w_{\psi,\psi\psi}^1 \\ & - \tfrac{1}{2}(1-\nu)(w_\theta^1 w_{\theta,\psi\psi}^0 - w_\theta^0 w_{\theta,\psi\psi}^1) + \tfrac{1}{2} s_0 (1+\nu)(w_\theta^0 w_\psi^1 \\ & - w_\psi^0 w_\theta^1)_{,\psi} - (w_r^0 w_\psi^1 - w_r^1 w_\psi^0)_{,\psi}\} d\psi. \end{aligned}$$

Examination of (5.2.10) indicates that the second integral of (5.2.10) contains two basic terms. The first has the form

$$(5.2.11)\quad I_2 = \int_0^{2\pi} \{f(\psi) g(\psi)_{,\psi\psi} - g(\psi) f(\psi)_{,\psi\psi}\} d\psi$$

which when integrated twice by parts in the first term yields

$$\begin{aligned} I_2 = {} & \{f(\psi) g(\psi)_{,\psi} - g(\psi) f(\psi)_{,\psi}\Big|_0^{2\pi} \\ & + \int_0^{2\pi} \{g(\psi) f(\psi)_{,\psi\psi} - g(\psi) f(\psi)_{,\psi\psi}\} d\psi. \end{aligned}$$

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The first term is zero by periodicity and the integral is identically zero.

The second term to be found in (5.2.10) is

$$I_3 = \int_0^{2\pi} \{f(\psi)g(\psi)\}_{,\psi} d\psi = f(\psi)g(\psi)\Big|_0^{2\pi} = 0 \tag{5.2.12}$$

by periodicity. Hence (5.2.10) reduces to

$$\begin{aligned} I = i\beta m s_1 \int_0^{2\pi} \{ & \tfrac{1}{2} w_\psi^0 N_{\theta\psi}^0 - w_\theta^0 N_{\theta\theta}^0 + \nu w_\psi^0 w_{\theta,\psi}^0 \\ & - \tfrac{1}{2} s_0 (1-\nu)(w_\psi^0)^2 - s_0 (w_\theta^0)^2 - \tfrac{1}{2}(1-\nu) w_\theta^0 w_{\psi,\psi}^0 \\ & - \nu w_r^0 w_\theta^0 \} d\psi. \end{aligned} \tag{5.2.13}$$

Using (5.2.12) as the boundary integral in (4.3.13) we find the equation (4.3.13) can be written as

$$s_1 F(s_0, \beta) = 0. \tag{5.2.14}$$

Since $F(s_0,\beta) \neq 0$, we conclude that $s_1 = 0$.

If we are to have any sort of correction we must find s_2 and this requires the solution to first order velocities and displacements. The first order pressure and velocities can be found as in section 4.3. Using the boundary conditions we match first order fluid velocities with first order shell velocities at $r = 1$. The result is a determination of the arbitrary constants acquired in solving the first order problem. This having been done our next step is the solution of the second order correction.

5.2.3 Second Order Correction for Arbitrary n

The second order correction equation for s_2, (4.3.27) developed for the fluid alone must be modified to account for the fluid-shell interaction. We begin with the boundary terms in (4.3.26) and repeat here those integrals which are not identically zero at $r = 1$ due to periodicity in ψ. The integrals just mentioned, in (4.3.26), are designated as

$$I_1 = \int_0^{2\pi} [u_\psi^0 s_{r\psi}^2 + u_\theta^0 s_{r\theta}^2 + u_r^0 s_{rr}^2 - u_\psi^2 s_{r\psi}^0 - u_\theta^2 s_{r\theta}^0 - u_r^2 s_{rr}^0] d\psi$$

where the integrand has argument $r = 1$.

We eliminate the fluid stresses by means of (2.6.14) and (2.6.34), and we eliminate the fluid velocities by the boundary conditions (2.6.15), (2.6.35).

After some simple algebraic manipulations it is easy to show that only certain components of the integrand will make a non-zero contribution. We designate by tilda, ~, the $\sin(n+1)\psi$ components of $N_{\psi\psi}^1$ and $N_{\theta\theta}^1$ and the $\cos(n+1)\psi$ term of $N_{\theta\psi}^1$. We use a hat, $\wedge$, to designate the $\sin(n-1)\psi$ portions of $N_{\psi\psi}^1$ and $N_{\theta\theta}^1$ as well as the $\cos(n-1)\psi$ term in $N_{\theta\psi}^1$. Denoting the constant portion of the zero order terms by $*$ we write

$$
\begin{aligned}
I_1 = {} & i\beta m s_2 \int_0^{2\pi} \{w_\psi^0 N_{\theta\psi}^0 - w_\theta^0 N_{\theta\theta}^0\} d\psi \\
& + i\beta m \int_0^{2\pi} \{w_\psi^0 [N_{\psi\psi,\psi}^2 + s_0 N_{\theta\psi}^2] \\
& + w_\theta^0 [N_{\theta\psi,\psi}^2 - s_0 N_{\theta\theta}^2] - w_r^0 N_{\psi\psi}^2 + w_r^2 N_{\psi\psi}^0 \\
& - w_\psi^2 [N_{\psi\psi,\psi}^0 + s_0 N_{\theta\psi}^0] - w_\theta^2 [N_{\theta\psi,\psi}^0 - s_0 N_{\theta\theta}^0]\} d\psi \\
& - i\beta m \frac{\pi}{2} \{-W_\psi^0 [\tilde{N}_{\psi\psi}^1 + \hat{N}_{\psi\psi}^1 - \tilde{N}_{\theta\theta}^1 - \hat{N}_{\theta\theta}^1 \\
& + s_0 (\tilde{N}_{\theta\psi}^1 - \hat{N}_{\theta\psi}^1) + s_0 {}^{*}N_{\theta\psi}^0] \\
& + W_\theta^0 [s_0 ({}^{*}N_{\theta\theta}^0 - \tilde{N}_{\theta\theta}^1 + \hat{N}_{\theta\theta}^1) - 2(\tilde{N}_{\theta\psi}^1 + \hat{N}_{\theta\psi}^1)] \\
& + W_r^0 [{}^{*}N_{\theta\theta}^0 - \tilde{N}_{\theta\theta}^1 + \hat{N}_{\theta\theta}^1]\}
\end{aligned} \tag{5.2.14}
$$

The second order shell terms are now eliminated by means of (2.6.33) and we finally achieve

$$
\begin{aligned}
I_1 = {} & i\beta m s_2 \int_0^{2\pi} \{w_\psi^0 N_{\theta\psi}^0 - w_\theta^0 N_{\theta\theta}^0\} d\psi \\
& + i\beta m \frac{\pi}{2} W_\psi^0 \{-n\nu(\tilde{w}_\psi^1 + \hat{w}_\psi^1 + \tilde{w}_r^1 + \hat{w}_r^1 + W_r^0 \\
& + 2s_2 W_\theta^0 - s_0(\tilde{w}_\theta^1 - \hat{w}_\theta^1) + s_0 W_\theta^0) \\
& + \frac{1}{2} s_0 (1-\nu)(s_0[\tilde{w}_\psi^1 - \hat{w}_\psi^1 + w_\psi^0] + 2s_2 W_\psi^0 - \tilde{w}_\theta^1 + \hat{w}_\theta^1)\} \\
& + i\beta m \frac{\pi}{2} W_\theta^0 \{\frac{1}{2} (1-\nu)[n s_0 (\hat{w}_\psi^1 - \tilde{w}_\psi^1) + 2n s_2 W_\psi^0 + n s_0 W_\psi^0 \\
& - n(\tilde{w}_\theta^1 - \hat{w}_\theta^1)] + s_0{}^2 (\tilde{w}_\theta^1 - \hat{w}_\theta^1 + W_\theta^0) + 2 s_0 s_2 W_\theta^0\} \\
& - i\beta m \frac{\pi}{2} W_\theta^0 \nu \{\tilde{w}_\psi^1 + \hat{w}_\psi^1 + \tilde{w}_r^1 - \hat{w}_r^1 - s_0(\tilde{w}_\theta^1 - \hat{w}_\theta^1)
\end{aligned} \tag{5.2.15}
$$

$$- w_r^o + (2s_2+s_o)w_\theta^o\} - i\beta m \frac{\pi}{2} \{-w_\psi^o[\tilde{N}_{\psi\psi}^1 + \hat{N}_{\psi\psi}^1$$

$$- \tilde{N}_{\theta\theta}^1 - \hat{N}_{\theta\theta}^1 + s_o(\tilde{N}_{\theta\psi}^1 - \hat{N}_{\theta\psi}^1) + s_o {}^*N_{\theta\psi}^o]$$

$$+ w_\theta^o[s_o({}^*N_{\theta\theta}^o - \tilde{N}_{\theta\theta}^1 + \hat{N}_{\theta\theta}^1) - 2(\tilde{N}_{\theta\psi}^1 + \hat{N}_{\theta\psi}^1)]$$

$$+ w_r^o[{}^*N_{\theta\theta}^o - \tilde{N}_{\theta\theta}^1 + \hat{N}_{\theta\theta}^1]\}$$

$$- i\beta m\int_o^{2\pi} \{w_\psi^o w_{\psi,\psi\psi}^2 - w_\psi^2 w_{\psi,\psi\psi}^o + \frac{1}{2}(1-\nu)[w_\theta^2 w_{\theta,\psi\psi}^o$$

$$- w_\theta^o w_{\theta,\psi\psi}^2] + \frac{1}{2} s_o(1+\nu)[w_\psi^o w_\theta^2 - w_\theta^o w_\psi^2]_{,\psi}$$

$$+ (w_r^2 w_\psi^o - w_r^o w_\psi^2)_{,\psi}\}d\psi.$$

We recognize the integrand of the last integral above is zero due to terms of the form found in (5.2.11) and (5.2.12). Coupling this equation, which has no second order terms except s_2, with that developed in section 4.3, for the fluid we have an equation for the second order correction in terms of known zero and first order quantities.

5.3 Frequency Equation for Inviscid, Axisymmetric Flow

We repeat, for convenience, from (4.4.1), the pressure and velocities in the inviscid, axisymmetric case:

$$p_0(r) = AI_0(r\alpha), \qquad u^0_\psi(r) = 0,$$

(5.3.1) $$u^0_\theta(r) = -As_0I_0(r\alpha)/i\beta, \qquad u^0_r(r) = A\alpha I_1(r\alpha)/i\beta,$$

$$\alpha^2 = s_0^2 - \beta^2/c^2,$$

where A is arbitrary.

In the inviscid case only the normal fluid stresses are non-zero and hence the only body force on the shell is $S^0_{rr\ r=1} = -p_0(1)$. The shell displacements then satisfy the following

(5.3.2) $$\{\beta^2 - \tfrac{1}{2}(1-\nu)s_0^2\}w^0_\psi = 0$$

(5.3.3) $$\begin{pmatrix} \beta^2-s_0^2 & -\nu s_0 \\ -\nu s_0 & \beta^2-1 \end{pmatrix} \begin{pmatrix} w^0_\theta \\ w^0_r \end{pmatrix} = -\frac{A}{m} I_0(\alpha) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solving for displacements we see that for axisymmetric flow $w^0_\psi = 0$ and

$$w^0_\theta = -\nu s_0 AI_0(\alpha)m^{-1}\{(1-\nu^2-\beta^\beta)s_0^2 - \beta^2(1-\beta^2)\}^{-1}$$

(5.3.4)

$$w^0_r = -(\beta^2-s_0^2)AI_0(\alpha)m^{-1}\{(1-\nu^2-\beta^2)s_0^2 - \beta^2(1-\beta^2)\}^{-1}$$

where $m = ph/p_0a$. In the inviscid problem we match velocities in the radial direction giving us the frequency equation for the interaction problem. It is

(5.3.5) $m\alpha I_1(\alpha)\{(\beta^2-1+\nu^2)s_0^2-\beta^2(\beta^2-1)\}-\beta^2(\beta^2-s_0^2)I_0(\alpha) = 0$

This equation has been determined by Rubinow and Keller [9] in a different situation. As before we utilize portions of their analysis as the basis for our discussion.

Our first task is to recognize those portions of (5.3.5) which have appeared before in special cases. We shall refer to those modes which reduce to the empty tube frequency equation (3.4.2) as <u>tube</u> <u>modes</u>. Those modes which reduce to either of the two modes associated with the fluid will be called <u>acoustic</u> <u>modes</u>, due to their dependence on the non-dimensional acoustic speed.

When m approaches zero we may interpret this as meaning that tube density is small as in the stress free case. In fact setting $m = 0$ produces three possible equations; $I_0(\alpha) = 0$, $s_0 = \beta$ or $\beta = 0$. The first equation is the stress free boundary condition (4.4.4). However, the next mode is a new mode which results from the interaction. It is the tube mode (3.4.2) when $\nu = 0$. The last equation, $\beta = 0$, is of no interest since s_0 is undefined. Hence letting m approach zero gives us a mode which we recognized from before, the stress free mode. It also produced a new mode, $s_0 = \beta$, not normally associated with small tube density.

If we let m grow unbounded in (5.3.5) we find that $\alpha I_1(\alpha) = 0$ or $s_0^2 = \beta^2(1-\beta^2)(1-\nu^2-\beta^2)^{-1}$. The first mode is the rigid tube mode (4.4.2) which we could associate with ρh growing large (and hence m also). The second mode we found to be the empty tube mode (3.4.2) which we could associate with $\rho_0 a$ becoming small (and hence m becoming large.) In this situation setting $m^{-1} = 0$ results in two modes, both recognizable from before.

<u>5.3.1 Frequency Equation for $\nu = 0$</u>

If we set $\nu = 0$ in (5.3.5) we see that the equation can be written as

$$(5.3.6) \qquad (\beta^2 - s_0^2)\{m\alpha I_1(\alpha)(\beta^2-1) + \beta^2 I_0(\alpha)\} = 0$$

We see that $s_0 = \beta$ is a frequency equation for all values of m when $\nu = 0$ (and for all values of ν when $m = 0$ as in (5.3.5)). Let us now examine the term in braces in (5.3.5) utilizing an interesting and informative approach of Rubinow and Keller [9].

<u>Case I:</u> $\alpha^2 < 0$

We set $\alpha = ix$ where x is real. We then write

$$(5.3.7) \qquad -x\,\frac{J_1(x)}{J_0(x)} = \frac{\beta^2}{m(1-\beta^2)} .$$

If we differentiate the left side of (5.3.6) with respect to x we find

$$(5.3.8) \qquad \frac{d}{dx}\left(-x\,\frac{J_1(x)}{J_0(x)}\right) = -x\left(1 + \frac{J_1^2(x)}{J_0^2(x)}\right)$$

Hence for positive x the left side of (5.3.7) is always a decreasing function of x. If $\beta > 1$ then the right side of (5.3.7) is an increasing function of m. If $\beta < 1$ then the same expression is decreasing in m. Since the left side of (5.3.7) exhibits the same behavior as the right side we can say that for $\beta > 1$, x is a decreasing function of m and for $\beta < 1$, x is an increasing function of m. Since

$$(5.3.9) \qquad s_0^2 = \beta^2/c^2 - x^2$$

we have that for real values of s_0, s_0 is a decreasing function of m when $\beta < 1$ and an increasing function of m when $\beta > 1$. If s_0 is imaginary the conclusions are just the opposite.

Knowing the qualitative behavior of s_0 we are now in a position to understand the behavior of the computations we are about to make. We know how the acoustic modes behave in the extremes of $m = 0$ and $m = \infty$. Hence we expand in a MacLauren series about ϵ small where $\epsilon = m$ or m^{-1}. We say

$$(5.3.10) \qquad s_0 = s_0\,|_{\epsilon=0} + \epsilon \frac{ds}{d\epsilon}\,|_{\epsilon = 0} + \cdots$$

We see then that when m is small we have

$$(5.3.11) \quad s_0 = (\beta^2/c^2 - z_{0n}^2)^{1/2} + m z_{0n}^2 (\beta^2-1)/\beta^2 (\beta^2/c^2 - z_{0n}^2)^{1/2}$$

where $J_0(z_{0n}) = 0$ $n=1,2,3\ldots$ If we rewrite the term in braces of (5.3.6) so that ϵ is m^{-1} then we find

(5.3.12) $s_0 = (\beta^2/c^2 - z_{1n}^{\ 2})^{1/2} + \beta^2/\{m(1-\beta^2)(\beta^2/c^2 - z_{1n}^{\ 2})^{1/2}\}$

where $J_1(z_{1n}) = 0$, $n = 1,2,3$.

We plot the frequency equation for these acoustic modes by using (5.3.11) and (5.3.12). We know that the curves for $m = 0$ or $1/m = 0$ are the extremes for any intermediate values and we know that at $\beta = 1$ there is a change in s_0 from a decreasing function in m for $\beta < 1$ to an increasing function of m for $\beta > 1$.

These curves are presented in Figure 31 for $c = 0.1$. These curves are the lower set of curves which exhibit an abrupt change at $\beta = 1$. Real values of s_0 dictate that $\beta \geq cz_{1n}$ or $\beta \geq cz_{0n}$ depending on the mode. Hence if c is large enough, cz_{1n} or cz_{0n} will be larger than one and the acoustic curves will start further to the right, displaying no switch from decreasing to increasing functions of m. In fact, if the fluid is incompressible (c very large) the acoustic modes exist only for extremely high frequencies.

If we examine the phase velocity $c_p = \beta/s_0$ in Figure 32 we see the obvious sort of behavior for the curves near $\beta = 1$. These curves are plotted for $c = .1$ and if c is large enough the acoustic modes exist only when $\beta > 1$ and hence do not exhibit the switch from increasing to decreasing functions of m. [Note: Since $c_p = \beta/s_0$, c_p is increasing in m when s_0 is decreasing and vice versa.]

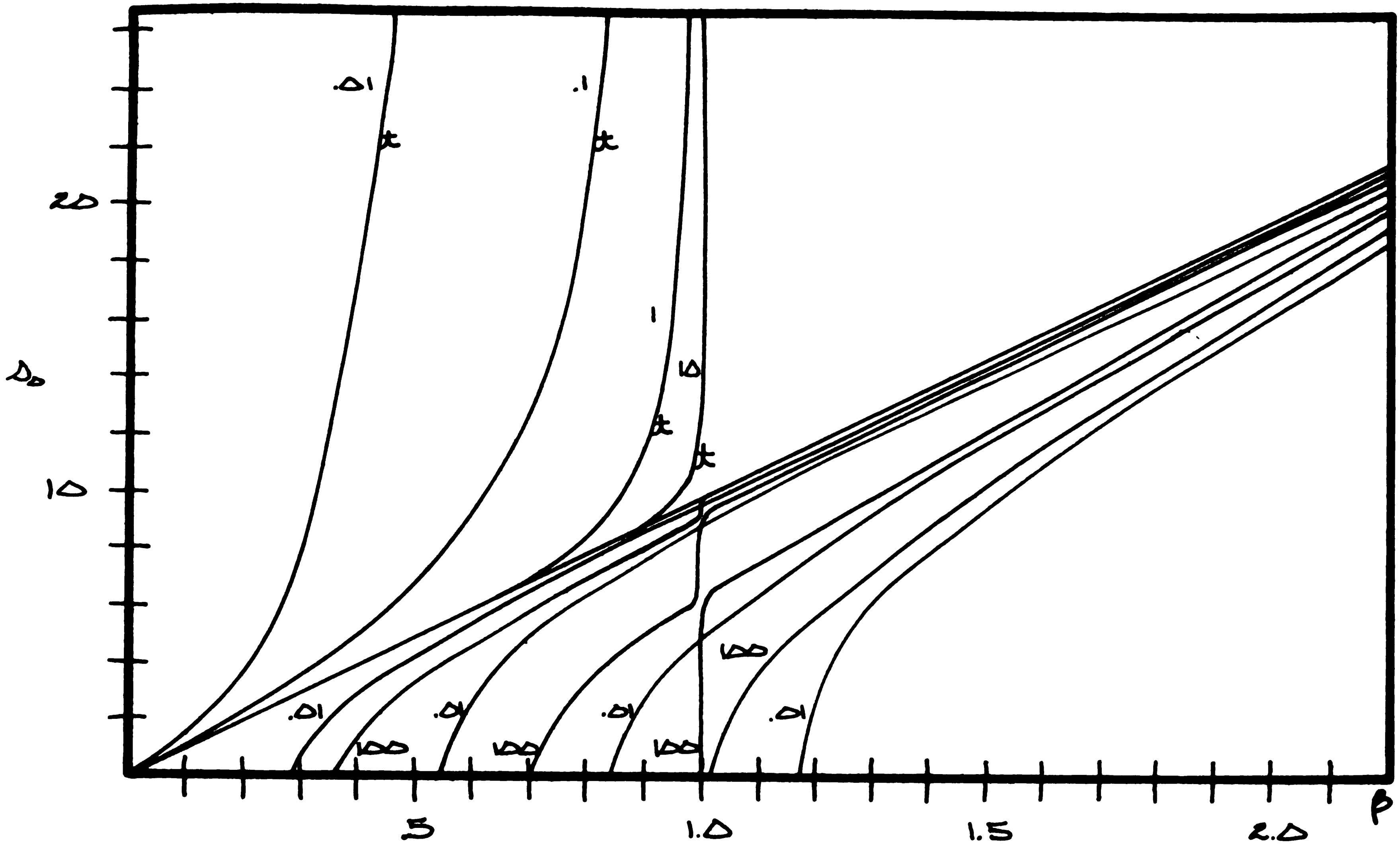


FIGURE #31 Ω_0 FOR INTERACTION PROBLEM VALUES FOR m SHOWN
$\nu = 0$, $L = .1$ THOSE LABELED t ARE TUBE MODES
ACOUSTIC MODES (5,3,11) (5,3,12) TUBE MODES (5,3,15) (5,3,16)

<u>Case II</u> $\alpha^2 > 0$ (Tube Modes)

If we return to the expression in braces in (5.3.6) we may rewrite it as

$$m\alpha I_1(\alpha)(1-\beta^2) = \beta^2 I_0(\alpha) \tag{5.3.13}$$

or as

$$\frac{m(1-\beta^2)}{\beta^2} = \frac{I_0(\alpha)}{\alpha I_1(\alpha)} \quad . \tag{5.3.14}$$

When α is real the right side of (5.3.13) is a positive decreasing function of α which behaves like $2/\alpha^2$ for small α, i.e. s_0 near β/c, and like $1/\alpha$ for α large. If we replace the right side of (5.3.14) by $2/\alpha^2$ then we see for small α that

$$s_0^2 = \beta^2\left(\frac{2}{m(1-\beta^2)} + \frac{1}{c^2}\right) \tag{5.3.15}$$

while for α large we find

$$s_0^2 = \beta^2\left(\frac{\beta^2}{m^2(1-\beta^2)^2} + \frac{1}{c^2}\right) \quad . \tag{5.3.16}$$

In plotting the curve for this mode we utilized (5.3.15) until $\beta^2 = 2m(2m+1)^{-1}$ at which point both representations are equal. Then we plotted (5.3.16) for the remainder of the β values. Since the right side of (5.3.14) is positive, β is restricted to values less than or equal to 1. We also see that the left side of (5.3.13) is increasing in m $(\beta \leq 1)$ and that the right side of the same equation is decreasing in α. Hence α must be decreasing in m and therefore $s_0^2 (= \alpha^2+\beta^2/c^2)$ is also decreasing in m.

The curves for this mode are plotted in Figure 31, and are labeled with t. We see that the first acoustic modes are close to the m large tube modes for $\beta < 1$. When the acoustic modes exist only for $\beta > 1$ then there is no similarity.

The phase velocity associated with the mode of (5.3.15) and (5.3.16) is given in Figure 32 and it exhibits the behavior we expect from looking at the frequency equation. If one returns all variables to dimensional form in (5.3.15) and sets $\beta = 0$, the resulting phase velocity is that of Korteweg mentioned in the introduction as c_{pl}. At $\beta = 0$ the phase velocity for this mode has value $(mc^2/[2c^2+m])^{1/2}$. If the acoustic speed is small then the phase velocity starts near zero and becomes zero at $\beta = 1$. If the fluid is nearly incompressible, $c \sim \infty$, then the phase velocity begins near $(m/2)^{1/2}$. If instead we allow m to become large then the phase velocity starts near the acoustic speed c.

Having examined the two acoustic modes (5.3.11) and (5.3.12) as well as the tube mode given by (5.3.15) and (5.3.16) we examine the displacement and velocity ratios associated with these modes.

From (5.3.1) we have

$$(5.3.17) \quad |u_r^0(r)/u_\theta^0(r)| = |-\alpha I_1(r\alpha)/s_0 I_0(r\alpha)| \, .$$

Case I Acoustic Modes

The velocity ratio (5.3.17) for the acoustic modes is plotted in Figures 38 and 40. The curves labeled $\delta = 0$ in these figures represent the behavior of (5.3.17). When $m = 100$ (Figure 38) we are near the rigid tube mode and find that the $\delta = 0$ curve of Figure 38 is very nearly the same as the rigid tube case of Figure 26. Increasing the mode number to the second mode for the interaction gives us similar behavior to the second mode of the rigid tube case, i.e. the presence of a second region of radial motion as in Figure 27. Thus examination of the interaction problem near the rigid tube case indicates that there is little change from the rigid tube itself.

The acoustic mode for m small given in Figure 40 exhibits behavior nearly the same as the stress free mode of Figure 22. As for the stress free case further computations for the interaction problem show that increasing the order of the mode increases the number of regions of radial motion.

In examining the displacement ratio $|w_r^0/w_\theta^0|$ for these two acoustic modes we see from (5.3.4) that $w_\theta^0 = 0$ and the tube dispacements are purely radial.

Case II Tube Modes

Having examined the acoustic modes and their effect on the velocity and displacement ratios we turn to the tube modes associated with $\nu = 0$. The first mode is $s_0 = \beta$

and the argument of the Bessel functions is $\alpha^2 = \beta^2(1-c^{-2})$. We note that if $c > 1$, then α is real and $I_0(\alpha)$, $I_1(\alpha)$ are real. However if $c < 1$, then $I_0(\alpha)$, $I_1(\alpha)$ must be replaced by $J_0(\alpha)$, $J_1(\alpha)$ which are, of course, oscillatory. Hence if $c < 1$ then the v locity ratio (5.3.17) will have zeroes when $r\alpha = z_{1n}$ and will be unbounded when $r\alpha = z_{0n}$. Therefore for any fixed $c < 1$ increasing β increases $|\alpha|$ and hence $r\alpha$ is equal to more of the zeroes z_{1n} or z_{0n} as r increases from zero to one.

For $s_0 = \beta$ we find that the displacement w_r^0 is determined in (5.3.4) but that w_θ^0 is arbitrary. Examination of the next order displacement equations indicates that both w_θ^0 and the arbitrary constant A are present. Since A is the arbitrary constant for this problem, we set $w_\theta^0 = 0$ obtaining only radial motion in the shell.

Finally we examine the ratios for the tube mode given in (5.3.15) and (5.3.16). We approximate the modified Bessel functions for $r\alpha$ large and $r\alpha$ small finding that the velocity ratio is like $|-r\alpha^2/2s_0|$ for $(r\alpha)$ small and it is like $|-\alpha/s_0|$ for $(r\alpha)$ large. The velocity ratio is then zero at the center of the tube growing to some value $|-\alpha/s_0|$ at an intermediate point.

We note that since $s_0 \neq \beta$ (from (5.3.15) or (5.3.16)) we find that (5.3.4) yields $w_\theta^0 \equiv 0$ so that the motion is purely radial in the tube.

5.3.2 Poisson's Ratio is Non-Zero

To examine the frequency equation when Poisson's ratio is non-zero we expand s_0 in a series about m small or m^{-1} small using (5.3.10). In this section only we shall denote s_0 as a function of m by the symbol $\tilde{s}_0$ and we shall reserve the symbol s_0 to designate the value of s_0 corresponding to $m = 0$ or $m^{-1} = 0$ where appropriate.

For m small we find from (5.3.5) and (5.3.10) for the acoustic mode $I_0(\alpha) = 0$, we have

$$(5.3.18) \qquad \tilde{s}_0 = s_0 + \frac{m\ z_{0n}{}^2\{(1-\nu^2-\beta^2)s_0^2-\beta^2(1-\beta^2)\}}{\beta^2(\beta^2-s_0^2)s_0}$$

where

$$s_0^2 = \beta^2/c^2-z_{0n}{}^2, \quad J_0(z_{0n}) = 0.$$

The expansion about the tube mode, $s_0 = \beta$, is from (5.3.5) and (5.3.10)

$$(5.3.19) \qquad \tilde{s}_0 = \beta-m\nu^2\alpha I_1(\alpha)/2\beta I_0(\alpha)$$

where

$$\alpha^2 = \beta^2(1-1/c^2)$$

For m large we have the acoustic mode, $I_1(\alpha) = 0$, and we obtain

$$(5.3.20) \qquad \tilde{s}_0 = s_0 + \frac{\beta^2(s_0^2-\beta^2)}{ms_0\{(1-\nu^2-\beta^2)s_0^2-\beta^2(1-\beta^2)\}}$$

where

$$s_0^2 = \beta^2/c^2 - z_{1n}{}^2, \qquad J_1(z_{1n}) = 0.$$

For m large we have the tube mode

$$(5.3.21) \qquad \tilde{s}_0 = s_0 + \frac{\beta^2(s_0^2 - \beta^2)I_0(\alpha)}{2ms_0(1-\nu^2-\beta^2)\alpha I_1(\alpha)}$$

where

$$s_0^2 = \beta^2(1-\beta^2)(1-\nu^2-\beta^2)^{-1}$$

and

$$\alpha^2 = s_0^2 - \beta^2/c^2.$$

We now analyse the behavior of $\tilde{s}_0$ for the cases we have given.

<u>Case 1.</u> <u>Acoustic Modes (5.3.18) and (5.3.20)</u>.

For $\nu = 0$ we found a change in the behavior of the acoustic modes as β changed from less than one to greater then one. We expect a similar behavior for $\tilde{s}_0$ when $\nu \neq 0$.

We write the acoustic mode (5.3.18) for m small as

$$(5.3.22) \qquad \tilde{s}_0 = s_0 - \frac{mz_{0n}{}^2(1-\nu^2-\beta^2)}{s_0(s_0{}^2-\beta^2)} \left\{\frac{s_0^2}{\beta^2} - \frac{1-\beta^2}{1-\nu^2-\beta^2}\right\}$$

where $s_0^2 = \beta^2/c^2 - z_{0n}^2$, $J_0(z_{0n}) = 0$ $n = 1,2,\cdots$. In what follows we will assess the behavior of $\tilde{s}_0$ given in (5.3.22) as a function of m, i.e., we will determine the regions in which $\tilde{s}_0$ is an increasing (or decreasing) function of m. For this purpose we examine the sign of the three terms which form the coefficient of m in (5.3.22) - the curly bracket term, $1-\nu^2-\beta^2$, and $s_0^2-\beta^2$.

Starting with the curly bracket term in (5.3.22) and noting that for $c = 0.1$, $cz_{On} < 1$ for $n=1,2,3$, define

$$\text{(5.3.23)} \qquad \begin{aligned} y_1 &= s_O^2/\beta^2 , \\ y_2 &= (1-\beta^2)(1-\nu^2-\beta^2)^{-1}. \end{aligned}$$

The functions y_1, y_2 are given in Figure 33 and we note that $y_1 - y_2$ can be > 0, $= 0$, or < 0 depending upon the value of cz_{On}. If $cz_{On} < 1$, as in Figure 33a it is easy to show that $y_1 = y_2$ at two frequencies, say a_1 and a_2. It is found that $y_1 - y_2 > 0$ for $a_1 < \beta < a_2$ and is negative otherwise.

In the same manner the term, $s_O^2 - \beta^2$, for $cz_{On} < 1$, vanishes at a value $\beta = b_1 = cz_{On}(1-c^2)^{-1/2}$ and is negative in the interval $cz_{On} < \beta < b_1$ and positive otherwise. Simple computation shows that

$$cz_{On} < b_1 < a_1 < a_2 < (1-\nu^2)^{1/2}, \text{ for } n = 1,2.$$

Utilizing this information we conclude that for

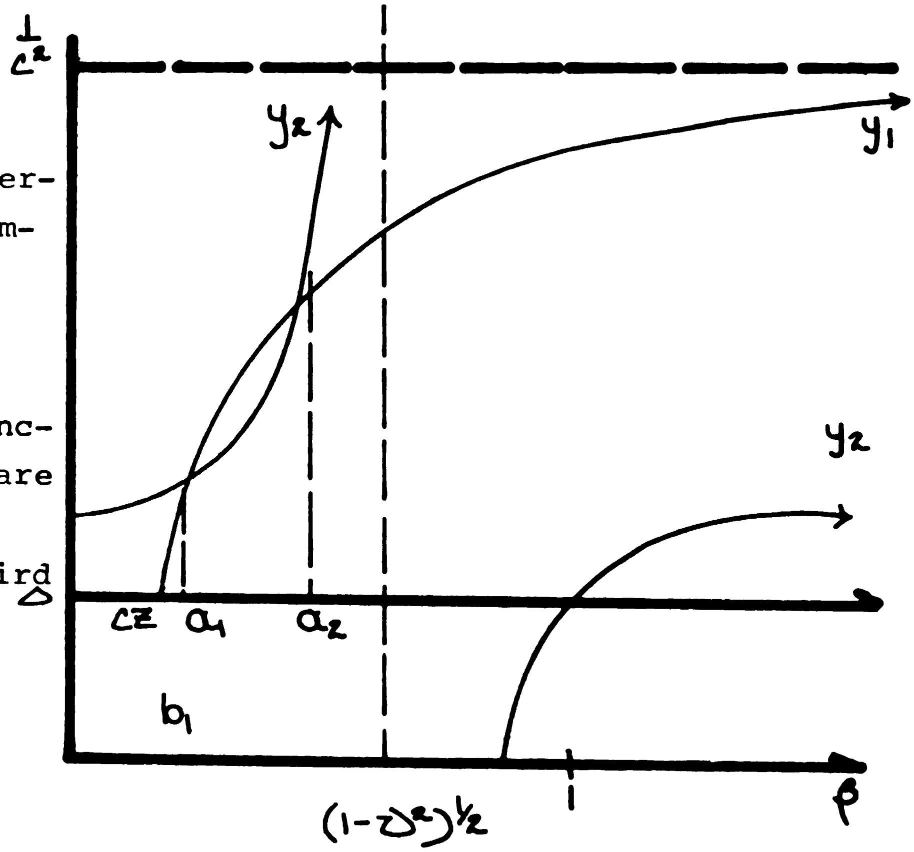
$$\text{(5.3.24)} \qquad \begin{matrix} cz_{On} < \beta < b_1 \\ \\ a_1 < \beta < a_2 \end{matrix} \qquad \tilde{s}_O \text{ is decreasing in } m$$

and is an increasing function of m otherwise.

When $n = 3$ we find $b_1 > (1-\nu^2)^{1/2}$ and $y_1 - y_2$ is never zero for real values of β. Hence for $cz_{O3} < \beta < b_1$

Representation of possible interaction of components in frequency equation (5.3.20). Functions y_1, y_2 are defined in (5.3.21). Third possibility places b_1 to left of $\beta=1$ in 32b. Curves are not to scale.

a.



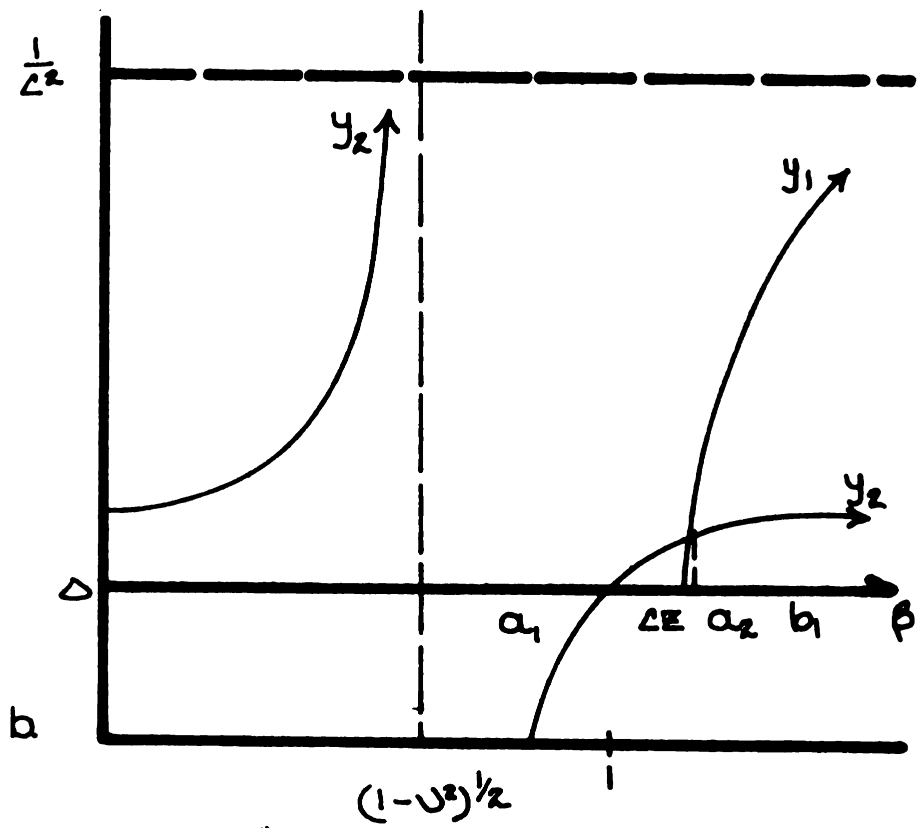


FIGURE #33

$\tilde{s}_0$ is decreasing in m and increasing in m for $\beta > b$.

Finally for $n \geq 4$ $cz_{0n} > 1$ for $c = .1$ and we find $a_1 < cz_{0n} < a_2 < b_1$ as in Figure 33b. Hence s_0 is decreasing in m for $a_2 < \beta < b_1$ and increasing for $cz_{0n} < \beta < a_2$ and for $\beta > b_1$.

Let us now rewrite the acoustic mode, $I_1(\alpha) = 0$, for m large, (5.3.19) in the form

$$\tilde{s}_0 = s_0 + \frac{(s_0^2-\beta^2)}{ms_0(1-\nu^2-\beta^2)} \left\{\frac{s_0^2}{\beta^2} - \frac{1-\beta^2}{1-\nu^2-\beta^2}\right\}^{-1},$$

(5.3.25)

$$s_0^2 = \frac{\beta^2}{c^2} - z_{1n}{}^2, \quad J_1(z_{1n}) = 0.$$

It is easy to note the similarity between (5.3.22) and (5.3.25), the major difference being the root z_{1n} rather than z_{0n}. For the smallest root, $z_{10} = 0$, $\tilde{s}_0$ is decreasing in m when $b^2 < 1 + \frac{\nu^2}{c^2-1}$ and increasing when the inequality is reversed.

Since the roots z_{1n} satisfy the inequality $cz_{1n} < 1$, if $n=1,2$, the behavior of $\tilde{s}_0$ given in (5.3.25) is the same as that indicated by (5.3.22) if z_{1n} replaces z_{0n} there, and a_1, a_2, b_1 retain their meaning with respect to the functions $s_0^2 - \beta^2$ and y_1, y_2 defined in (5.3.23).

When $n \geq 3$, we find that in the intervals $cz_{1n}<\beta<a_2$, and $\beta>b_1$ $\tilde{s}_0$ is an increasing function of m and is a decreasing function of m when $a_2<\beta<b_1$.

Although we have shown that the behavior of $\tilde{s}_0$ given by (5.3.22) and (5.3.25) changes from increasing to decreasing functions in the intervals determined by the points a_1, a_2, b_1 their actual locations are different for the two formulas due to the expansions used. In general, the locations a_1, and b_1 are near the value cz_{0n} or cz_{1n} depending on whether (5.3.22) or (5.3.25) is used, and in both cases a_2 is near $(1-\nu^2)^{1/2}$. However it is obvious that $cz_{0n} \neq cz_{1n}$ and these very small intervals are difficult to show in our graphs of the frequency expansions. In general we state that $\tilde{s}_0$ is decreasing in m for $\beta < a_2$ and increasing for $\beta > a_2$. When $cz_{jn} > 1$, $j=0,1$, then $\tilde{s}_0$ is increasing in m for $\beta > b_1$ and decreasing in the interval $cz_{jn} < \beta < b_1$, $j=0,1$.

We present the graphs of the frequency equation for m large, (5.3.25), in Figure 34. Only one situation in which the $\tilde{s}_0$ changes from decreasing to an increasing function of m is presented. This is due to the fact that the other changes occur very close together on the frequency axis. We have designated these regions by an asterisk, *, and give as an example of the behavior of $\tilde{s}_0$ in these regions the curves in Figure 35. This set of curves shows the behavior of the second acoustic mode for m large near the starting frequency cz_{12}. Here we see the function $\tilde{s}_0$ is a decreasing function of m for $cz_{12} < \beta < b_1$, $\beta > a_1$ and increasing in m elsewhere.

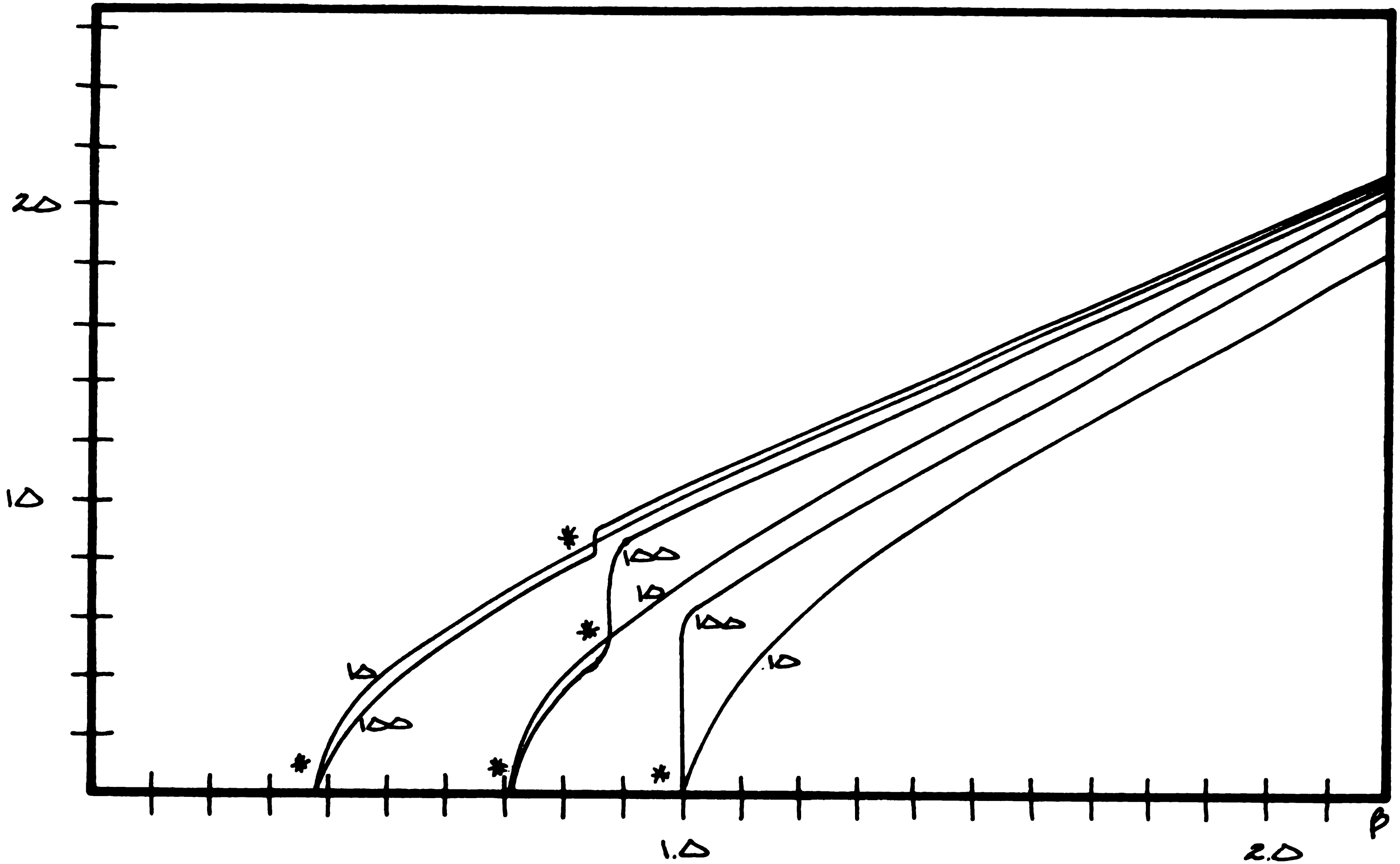


FIGURE #34 Ω_0 FOR ACOUSTIC MODES
m LARGE MODES ν=.5
* DENOTES REGION FROM CZ TO a_1 WHERE TWO SWITCHES OCCUR

Case II. Tube Modes

We have examined the behavior of $\tilde{s}_0$ for the acoustic modes, noting their similarities. We now turn our attention to the tube modes (5.3.19) and (5.3.21). In all that follows z_{1n} will be a zero of $J_1(z)$ while z_{0n} will be a zero of $J_0(z)$.

Examination of (5.3.19) indicates that for $c > 1$ the modified Bessel functions, $I_0(\alpha)$ and $I_1(\alpha)$, have real argument and are themselves real. We find then that for $c > 1$ $\tilde{s}_0$ is a decreasing function of m. If, on the other hand $c < 1$ the arguments of the Bessel functions are imaginary and they behave like Bessel functions of the first kind. These Bessel functions, $J_0(\alpha_1)$, $J_1(\alpha_1)$, $(\alpha_1^2 = -\alpha^2)$ oscillate between positive and negative values. We find that $I_0(\alpha) = J_0(\alpha_1)$ and that $\alpha I_1(\alpha) = -\alpha_1 J_1(\alpha_1)$. Making these changes in (5.3.19) we find that for $z_{1n} < \alpha_1 < z_{0(n+1)}$ $\tilde{s}_0$ is an increasing function of m and decreasing for other values of α_1. The value of $\tilde{s}_0$, (5.3.19), is plotted as the dashed line in Figure 36. To utilize the graph for other values of m small one merely picks a frequency and a value of c. From these values one can determine the value of α and hence α_1. Once α_1 is determined its location between the zeroes of the Bessel functions can be ascertained. One then knows if the function $\tilde{s}_0$ is increasing or decreasing in m at that point which in turn determines whether $\tilde{s}_0$ lies above or below the curve given in Figure 36 for $m = .01$.

Further examination of (5.3.19) in more detail reveals that when $\beta = cz_{1n}(1-c^2)^{-1/2}$, $\tilde{s}_0 = \beta$, while at $\beta = cz_{0n}(1-c^2)^{-1/2}$, $\tilde{s}_0$ is unbounded. We find that when $c < 1$ the fluid is compressible and we have the vertical asymptotes given in the graph.

The second tube mode is given by (5.3.21) for m large. For the frequency range $(1-\nu^2) < \beta^2 < 1$ we find that s_0 and $\tilde{s}_0$ are pure imaginary and there is no wave propogation. Examination of α^2 indicates that of the frequencies for which s_0 is real, α^2 is positive only in the regions

$$1 - \nu^2/(1-c^2) < \beta^2 < 1-\nu^2 \qquad c < 1$$

or

$$1 < \beta^2 < 1 + \nu^2/(c^2-1) \qquad c > 1 .$$

In either case α^2 is positive in only a limited frequency range so that in general the modified Bessel functions have imaginary argument and therefore behave like J_0 and J_1. When β^2 is near $(1-\nu^2)$, s_0 and $\tilde{s}_0$ both become unbounded. When β approaches 1 then s_0 is near zero and $\tilde{s}_0$ is arbitrarily large. When $\alpha^2 = -z_{1n}{}^2$ then $\tilde{s}_0$, given by (5.3.21), becomes unbounded while when $\alpha^2 = -z_{0n}{}^2$ then $\tilde{s}_0 = \beta$. Utilizing the values of s_0 and α^2 given with (5.3.21) we find that there are two possible frequencies when $\alpha^2 = -z_{1n}{}^2$ or $\alpha^2 = -z_{0n}{}^2$.

The graph of $\tilde{s}_0$ is presented in Figure 36 as the solid line for one value of m, namely 100. Defining $\alpha_0^2 = -\alpha^2$ we find from (5.3.21) that for $z_{1n} < \alpha_0 < z_{0(n+1)}$, $\tilde{s}_0$ is increasing in m while it is decreasing in m for other values of α_0. Hence, if one is to determine the behavior of $\tilde{s}_0$ as given by (5.3.21) for some m large other than 100 , one would have to find the relationship of the frequency to the zeroes and determine then whether the curve were above or below that given for $m = 100$.

The phase velocities for these two modes, (5.3.19) and (5.3.21) are given in Figure 37 over a larger frequency range. We must indicate that no comparison between the m large and m small curves regarding increasing or decreasing properties has been made.

Let us now consider the velocity ratio, u_r^0/u_θ^0, and the displacement ratio, w_r^0/w_θ^0, obtained from (5.3.1) and (5.3.4), respectively. The behavior of these ratios is determined by the solution $\tilde{s}_0$ of the frequency equation (5.3.5). To show how the various modes, acoustic and tube, alter the amplitude ratios we present the ratios using $\tilde{s}_0$ found in (5.3.18) to (5.3.21).

The velocity ratio for the acoustic mode, (5.3.20), is presented in Figures 38 and 39, and for the mode (5.3.18), in Figures 40 and 41. The appropriate curves are the $\delta = 0$ curves in each figure.

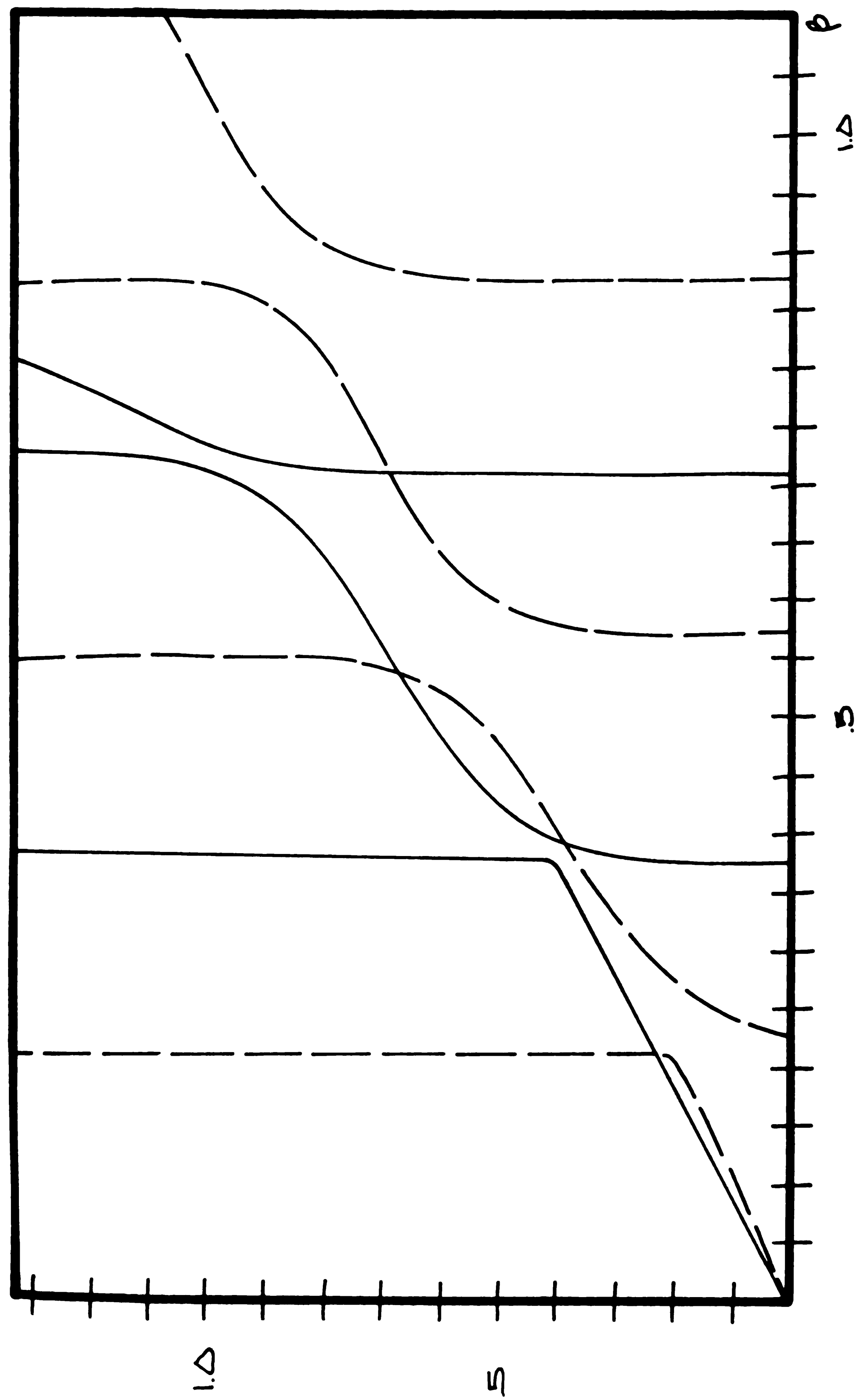


FIGURE #36 TUBE MODES ν=.5 ε=.1
N = 100 (—) N = .01 (– –)
(5.3.21) (5.3.19)

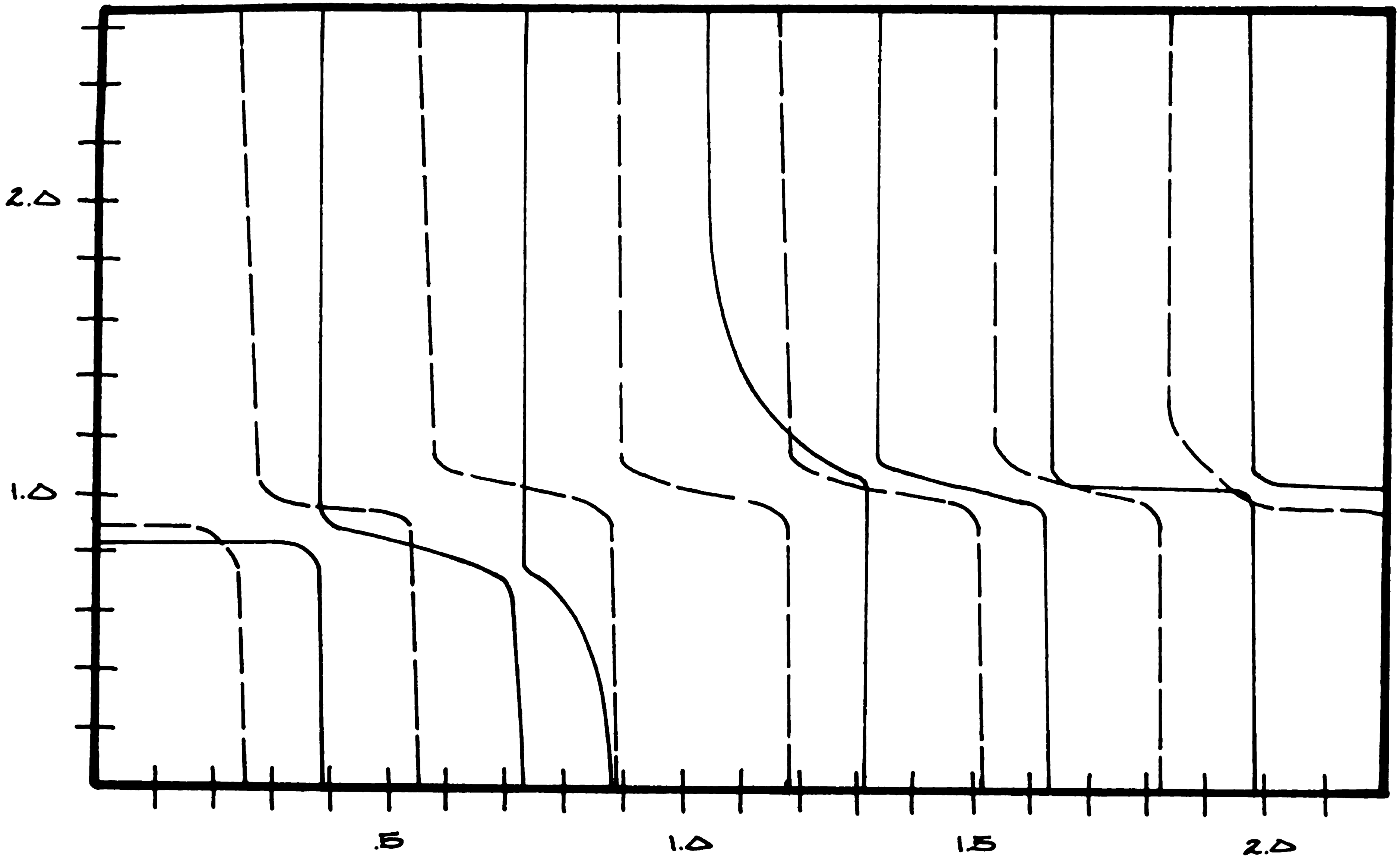


FIGURE #37 PHASE VELOCITY FOR TUBE MODES
N = 100 (—) N = 01 (- -)
(5.3.21) (5.3.19)

A comparison of Figure 38 with 39 or 40 with 41 shows that the effect of ν is insignificant for these modes. Furthermore, comparing the velocity ratio found here with the ratio found for the rigid tube or stress free tube (in Chapter 4) shows that the fluid behavior is only slightly altered. Hence, the elasticity of the tube wall is insignificant if the motion of the fluid is governed by an acoustic mode.

Let us now consider the velocity ratio for the tube modes, (5.3.19), (5.3.21). The ratio is presented in Figures 42 through 44. The velocity ratio for this case is now almost a linear function of the radial distance becoming very radial in character at the edge of the tube. Note that the two expansions in m yield almost the same ratio values, for $\delta = 0$.

The graphs of the displacement ratios are given in Figures 45 through 48 where again the $\delta = 0$ curves are the zero order curves. In Figures 45 and 46 we present the first two displacement ratios for the acoustic modes. We see that for both modes the displacement ratio grows larger as the frequency is increased, indicating that the displacements in the tube become more radial at higher frequencies when the frequency relation is near the acoustic modes.

In Figure 47 we present the displacement ratio for the tube mode (5.3.19). This ratio exhibits behavior radically different from the empty tube mode and demonstrates the effect of the compressibility of the fluid. The displacement ratio $\left|w_r^O/W_\theta^O\right|$ obtained from (5.3.4) is given by

$$\frac{\beta^2 - \tilde{s}_O^2}{\tilde{\nu} s_O}\ .$$

Since $c = 0.1$, (5.3.19) becomes

$$\tilde{s}_O = \beta + \frac{m\nu^2 z J_1(z)}{2\beta J_O(z)}$$

if we use $z = i\alpha$ there. Thus when $J_1(z) = 0$, $z = z_{im}$, the ratio is zero whereas when $J_O(z) = 0$, the ratio is infinite. It is apparent that for $c > 1$, the behavior is altered since $I_1(\alpha)$ and $I_O(\alpha)$ do not vanish.

When we examine the displacement ratio for the tube mode associated with the empty tube case (5.3.21) we find the displacement ratio behaving nearly identically to the displacement ratio for the empty tube. Figure 48 gives the displacement ratio for the interaction problem, near the empty tube mode. There is no ratio given for $(1-\nu^2)^{1/2} \leq \beta < 1$. Near these values of $\beta[=(1-\nu^2)^{1/2}$ and $1]$ the displacement ratio indicates primarily radial motion, while for other values of β the motion is longitudinal.

5.4 First Order Pressure, Velocities and Displacements

From (4.5.1) and (4.5.2) we have the first order pressure and velocities, determined up to their arbitrary constants. The first order pressure influences the first order displacements so we write our displacements in the shell as

$$\begin{aligned} w^1_\psi &= a_\psi \cos\psi + b_\psi \sin\psi + c_\psi \\ w^1_\theta &= a_\theta \sin\psi + b_\theta \cos\psi + c_\theta \qquad (5.4.1) \\ w^1_r &= a_r \sin\psi + b_r \cos\psi + c_r \, . \end{aligned}$$

If we designate the matrix of (5.2.1) by $M_{(n)}$ then the first order displacement system, upon using (5.4.1), (4.5.1) and (4.5.2), becomes

$$M_{(-1)} \begin{pmatrix} b_\psi \\ b_\theta \\ b_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \qquad (5.4.2)$$

$$M_{(+1)} \begin{pmatrix} a_\psi \\ a_\theta \\ a_r \end{pmatrix} = \begin{pmatrix} f_\psi \\ f_\theta \\ f_r \end{pmatrix} , \qquad (5.4.3)$$

$$\{\beta^2 - \tfrac{1}{2}s_0^2(1-\nu)\}c_\psi = 0, \qquad (5.4.4)$$

$$\begin{pmatrix} \beta^2 - s_0^2 & -\nu s_0 \\ -\nu s_0 & \beta^2 - 1 \end{pmatrix} \begin{pmatrix} c_\theta \\ c_r \end{pmatrix} = -\frac{1}{m} C_0 I_0(\alpha) \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad (5.4.5)$$

where

$$f_\psi = \tfrac{1}{2}s_0(3-\nu)w_\theta^0 - w_r^0 ,$$

$$(5.4.6) \qquad f_\theta = -\tfrac{1}{2}(1-\nu+4s_0^2)w_\theta^0 + s_0(1-\nu)w_r^0 ,$$

$$f_r = 2\nu w_r^0 - \tfrac{1}{m}\{DI_1(\alpha) - \tfrac{1}{2}A[s_0^2\alpha^{-1}I_1(\alpha) + I_0(\alpha)]\}$$

$$+ s_0(1-\nu)w_\theta^0 .$$

The constants C_0, D and C are to be determined by matching the radial velocities at $r = 1$. Hence, we have, in addition to (5.4.2) to (5.4.6), the boundary conditions

$$-i\beta c_r = C_0\alpha I_1(\alpha)/i\beta,$$

$$(5.4.7) \qquad -i\beta b_r = C\alpha I_1'(\alpha)/i\beta,$$

$$-i\beta a_r = \{D\alpha I_1'(\alpha) - \tfrac{1}{2}A[2s_0^2\alpha^{-1}I_1(\alpha) + s_0^2 I_1'(\alpha)$$

$$+ I_0(\alpha) + \alpha I_1(\alpha)]\}/i\beta.$$

The procedure is rather straightforward. Recognizing that s_0 is related to β by means of (5.3.5), the frequency equation, we have the determinant $M_{(-1)}$ is non zero so that

$$(5.4.8) \qquad b_\psi = b_\theta = b_r = 0.$$

Since the determinant of (5.4.3) is the same as the determinant of (5.4.2) we can solve for a_ψ, a_θ, and a_r. Finally, (5.4.4) and (5.4.5) yield c_ψ, c_θ and c_r.

Substituting c_r into the first of (5.4.7) reveals C_0 is arbitrary provided (5.3.5) is satisfied. We take

$$C_0 = c_r = 0. \tag{5.4.9}$$

The second of (5.4.7), using (5.4.8), yields

$$C = 0. \tag{5.4.10}$$

A solution a_r of (5.4.3) involves D, and so the third of (5.4.7) may be used to determine D. We write this symbolically,

$$\begin{aligned} &D\{\alpha I_0(\alpha) - I_1(\alpha) + m^{-1}\beta^2 A_{33} I_1(\alpha)/\mathrm{Det}\} \\ &= \frac{\beta^2}{\mathrm{Det}}\{A_{13}f_\psi + A_{23}f_\theta + A_{33}[2\nu w_r^0 + s_0(1-\nu)w_\theta^0 \\ &+ \tfrac{1}{2}m^{-1}A(s_0^2\alpha^{-1}I_1(\alpha) + I_0(\alpha))]\} \\ &+ \tfrac{1}{2}A\{\alpha^{-1}(2s_0^2+\alpha^2)I_1(\alpha) + s_0^2 I_1'(\alpha) + I_0(\alpha)\} \end{aligned} \tag{5.4.11}$$

where Det is the determinant of $M_{(+1)}$, A_{ij} is the cofactor of the element a_{ij} in $M_{(+1)}$, and the f_ψ, f_θ, f_r are defined in (5.4.6).

Incorporating all of this, (5.4.8) to (5.4.11), determines the first order fluid velocities, pressure and shell displacements subject to the relation (5.3.5)

Let us now examine the corrected velocity and displacement ratios, $|u_r/u_\theta|$ and $|w_r/w_\theta|$, respectively,

Velocity
Ratios
m=100
Acoustic
Mode #1.
ν=0.
δ=0 (—)
δ=.05 (--)
δ=.5 (···)
ν=0
z=3.832
β/c=8.832
c=.1

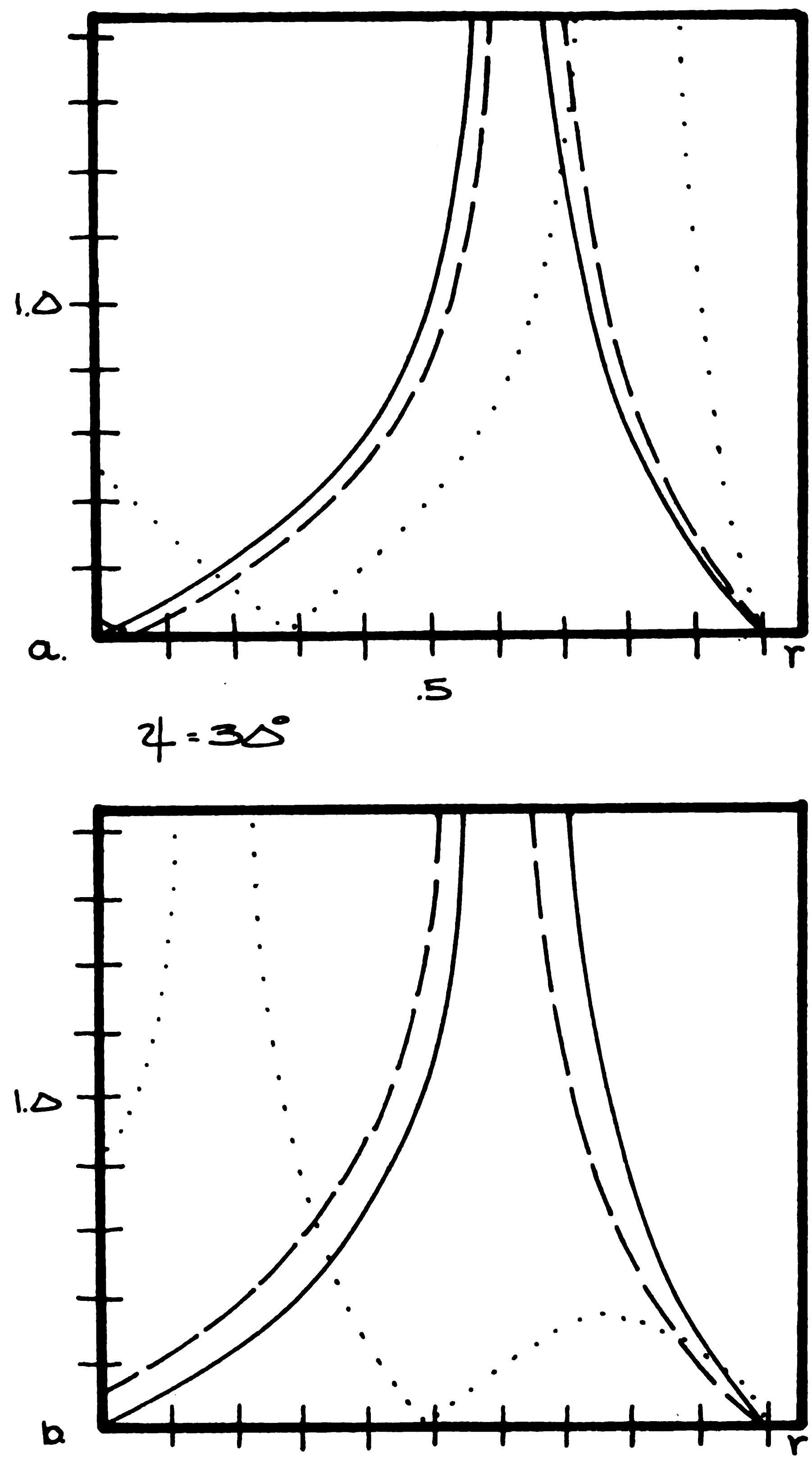


FIGURE* 38
ψ = 60°

Velocity
Ratios
m=100
Acoustic
Mode # 2.
$\nu=.5$
$\delta=0$ (—)
$\delta=.05$ (--)
$\delta=.5$ (···)
z=3.832
$\beta/c=8.832$

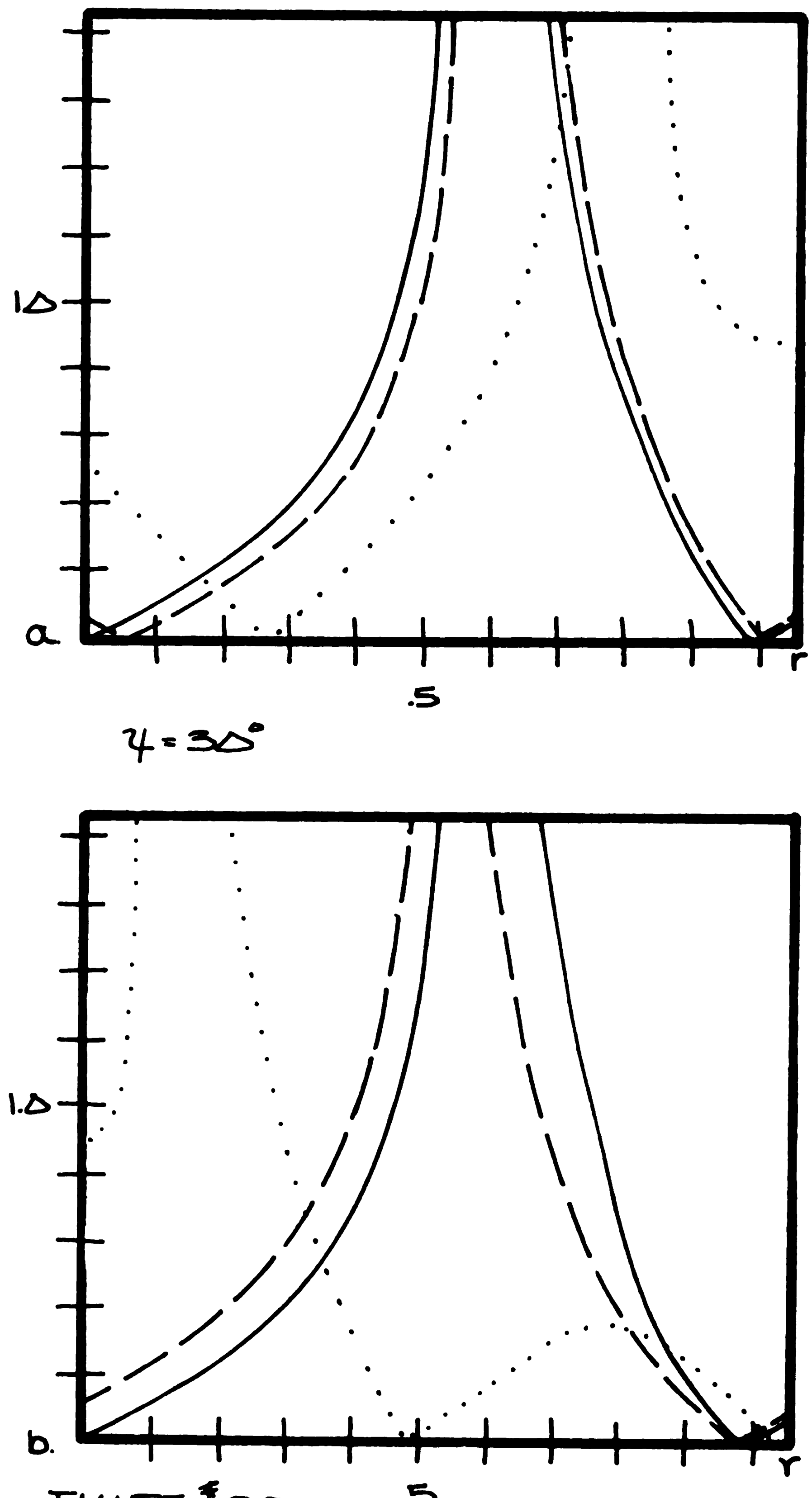


FIGURE # 39
ψ = -60°

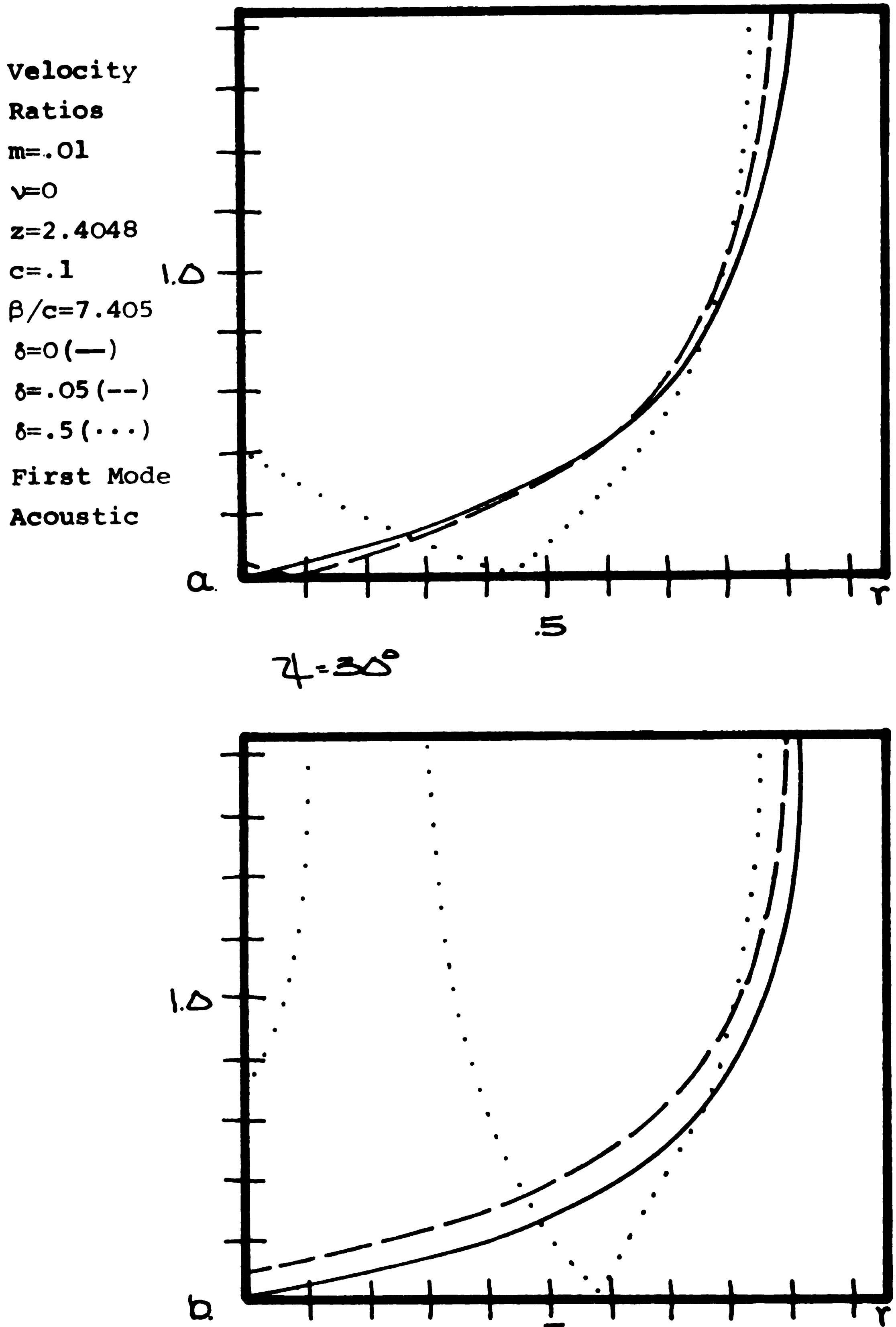


FIGURE #40
ψ=-60°

Velocity
Ratios
m=.01
ν=.5
z=2.4048
c=.1
β/c=7.405
Acoustic
First Mode
δ=0 (—)
δ=.05 (--)
δ=.5 (···)

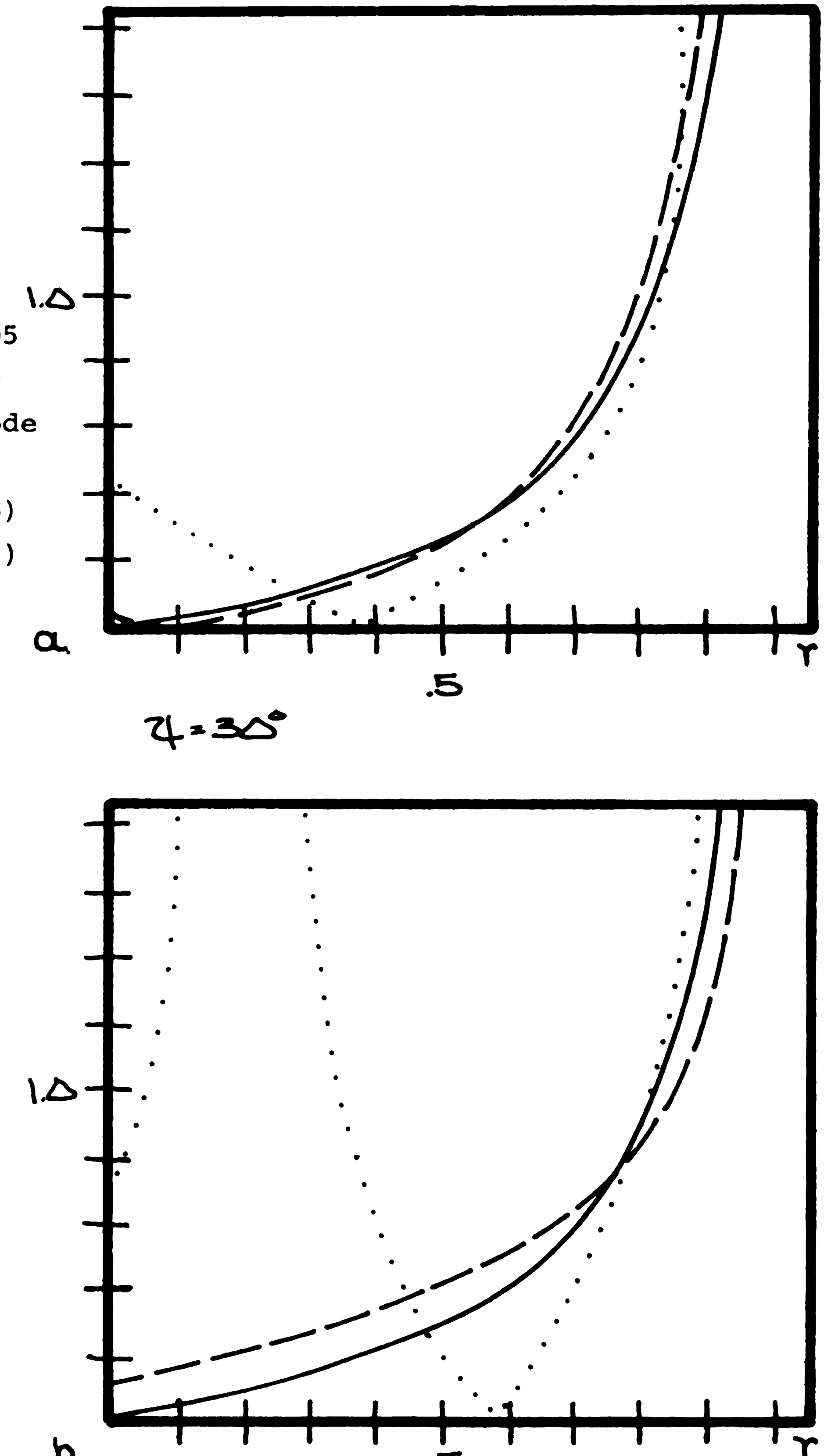


FIGURE #41
ψ=-60°

for the effect created by δ, Poisson's ratio ν, and the polar angle ψ. We concentrate our attention on $\nu = 0.5$ as representative of biological materials and we allow δ to be 0.005, 0.05 or 0.5 again banding the situation presented in the aortic arch. Two angles, $\psi = 30°$, $\psi = -60°$, have been chosen so that their effect can be measured as well.

The graphs of the velocity ratio for the different modes of motion are presented in Figures 38 to 43, and the displacement ratio is graphed in Figures 44 to 48.

Case I Acoustic Modes

We first examine the corrected ratios for the acoustic modes. The acoustic mode for m small is given by (5.3.11) for $\nu = 0$ and by (5.3.18) for $\nu \neq 0$ and the velocity ratio evaluated for this mode is graphed in Figures 40 and 41 respectively. First, we notice that changing ν from 0.0 to 0.5 does not alter the behavior significantly. Second, we see that the behavior of the zero order case ($\delta = 0$) and the corrected value for $\delta = .05$ are quite close, with the $\delta = .5$ curve showing significant differences from the lower δ values, especially for $\psi = -60°$. The behavior of this ratio is remarkably similar to the stress free velocity ratio presented in Figures 22 through 25.

When we consider the acoustic mode for m large given by (5.3.12) for $\nu = 0$ and by (5.3.20) for $\nu \neq 0$

we find the corrected velocity ratios given in Figures 38 and 39. As before, little change is introduced by allowing ν to be 0.0 or 0.5; the $\delta = .05$ curve is similar to the zero order case ($\delta = 0$); as we change angle we can expect to introduce even more significant deviation of the $\delta = .5$ curve from the zero order curve.

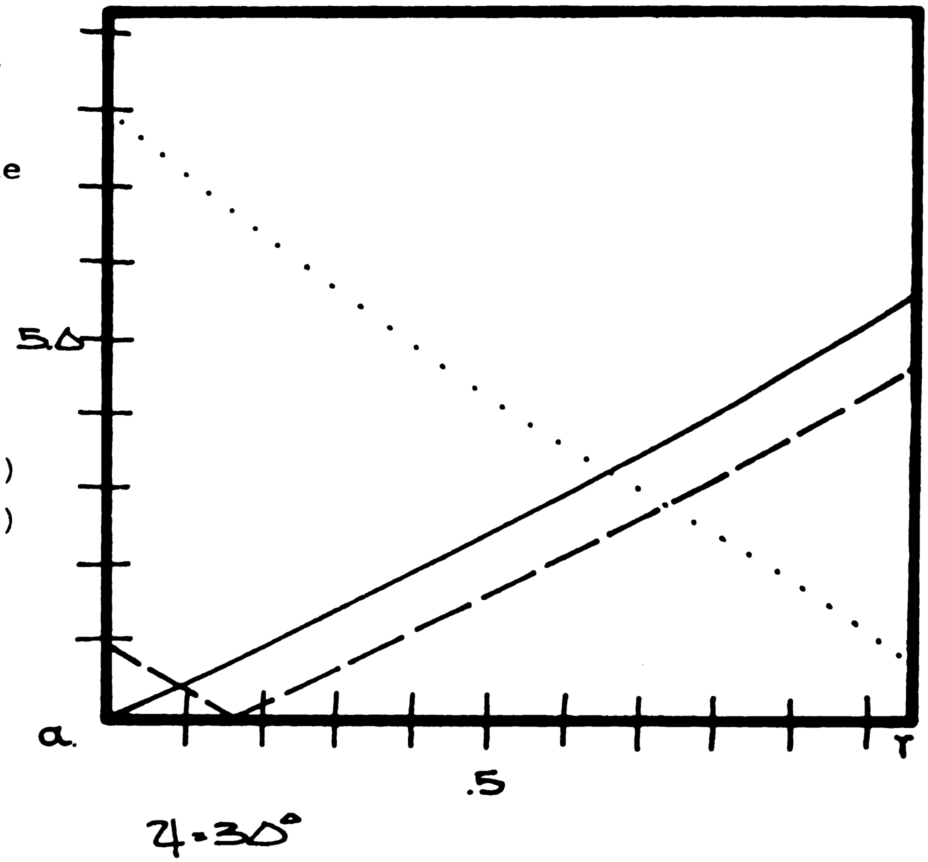
The displacement ratios corresponding to these acoustic modes are given in Figures 45 and 46 for $\nu = .5$. In general the corrected ratios follow the zero order ratios rather closely except for the asymptote in the $m = .01$ curve. In general the motion in the tube becomes more radial as higher frequencies are encountered.

For $\nu = 0$ the corrected ratio is not given. In the zero order the motion is purely radial for these modes when $\nu = 0$. The corrected ratios are quite large indicating generally radial motion.

Case II Tube Modes

The tube mode for m small is given by the $\tilde{s}_0$ expansion (5.3.19) and the velocity ratio for this mode is graphed in Figure 42. Note that the $\delta = 0$ curve in this figure is the plot of the straight tube velocity ratio and we have already remarked upon its increasing radial character. The character of the ratio for $\delta = 0.005$ and 0.05 is also radial from the point $r = 0.2$ to $r = 1$ and is more radial than the straight tube case. However

Velocity
Ratios
Tube Mode
m=.01
ν=.5
β/c=1.
c=.1
δ=0 (—)
δ=.05 (--)
δ=.5 (···)



δ=.5
off scale

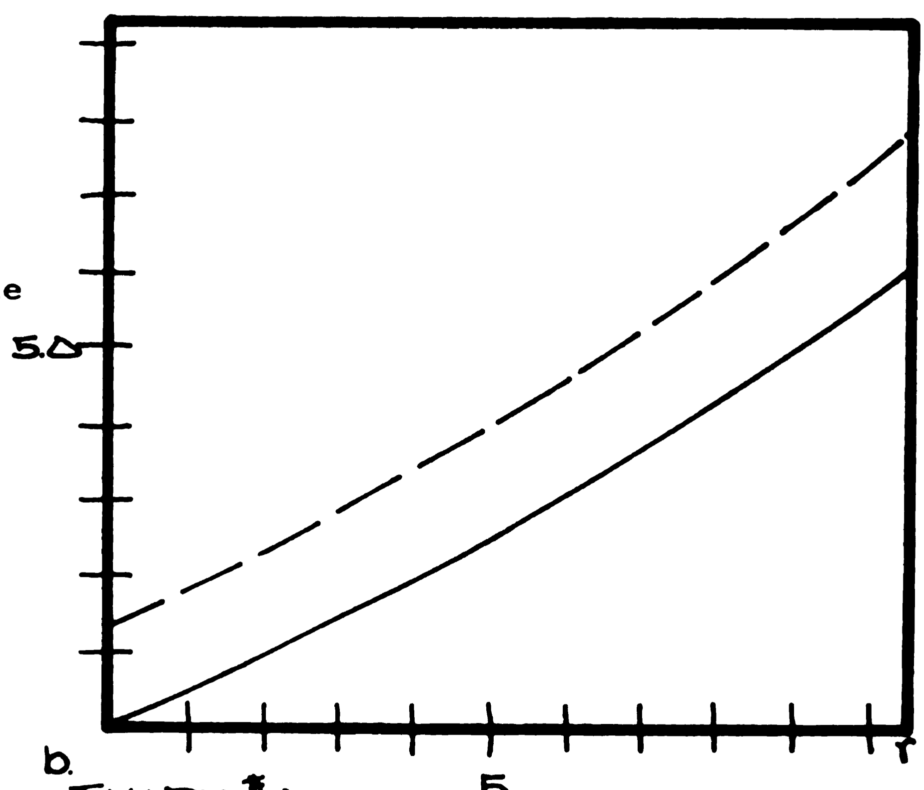


FIGURE #42
ψ=-60°

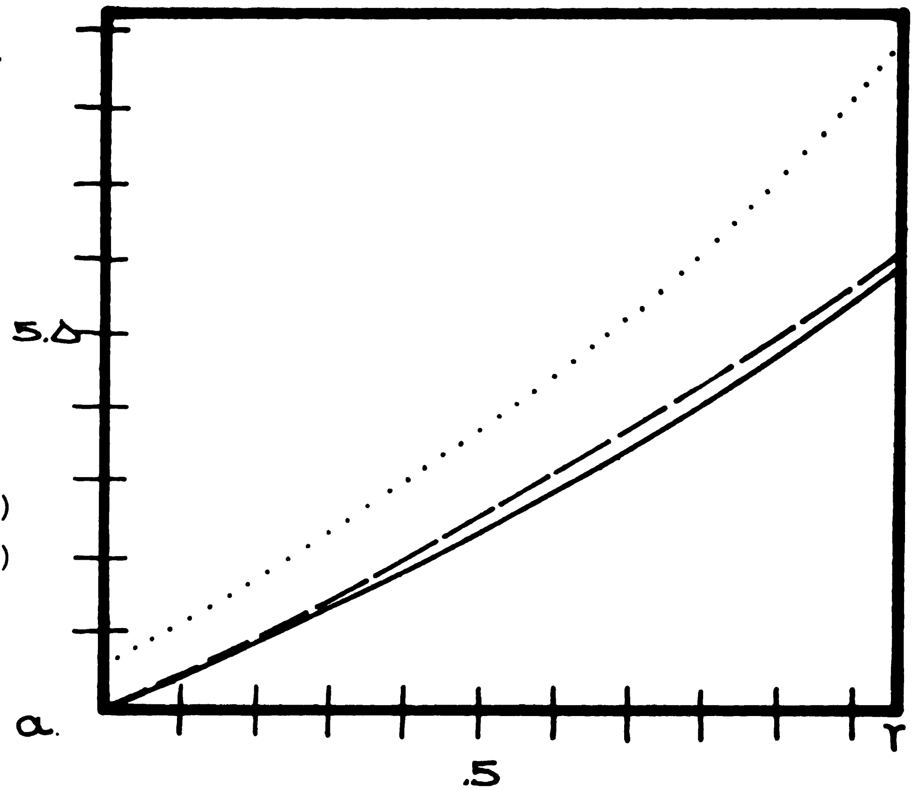
when $\delta = 0.5$, the ratio, apparently, reverses the trend for small δ and goes from predominantly radial at $r = 1$ to almost longitudinal there. It is felt that this is probably a spurious result due to the size of δ.

If we examine the velocity ratios for the m large tube modes given by (5.3.21) given in Figures 43 and 44 we see that for $\nu = 0$ all δ values are rather close, the velocities being mostly longitudinal near the center and more radial near the outer edge. When $\nu = 0$ we have essentially that $\tilde{s}_0 = \beta$.

For $\nu = .5$ the $\delta = .05$ curves in Figure 44 behave much differently from the $\delta = 0$ curves, indicating further restrictions on the use of the straight tube to model the curved tube for this flow.

To examine the displacement ratios for these two tube modes we turn to Figures 47 and 48. Figure 47 represents the m small mode displacement ratio plotted for $\nu = 0.5$. The figure contains the results for $\delta = 0$ and $\delta = 0.05$ and they are almost identical. Plotted against β we note several regions of radial motions and this is due primarily to the compressibility of the fluid. If $c > 1.0$, then $\tilde{s}_0$ of (5.3.19) does not become infinite and the regions of radial motion do not exist.

Velocity
Ratios.
Tube
Mode.
m=100.
$\nu=0$
$\beta/c=1.$
c=.1
$\delta=0$ (—)
$\delta=.05$ (--)
$\delta=.5$ (···)



$\psi=30°$

$\delta=.005$
curves
are nearly
equal to
$\delta=.05$
curves.

$\delta=.05$
off
scale

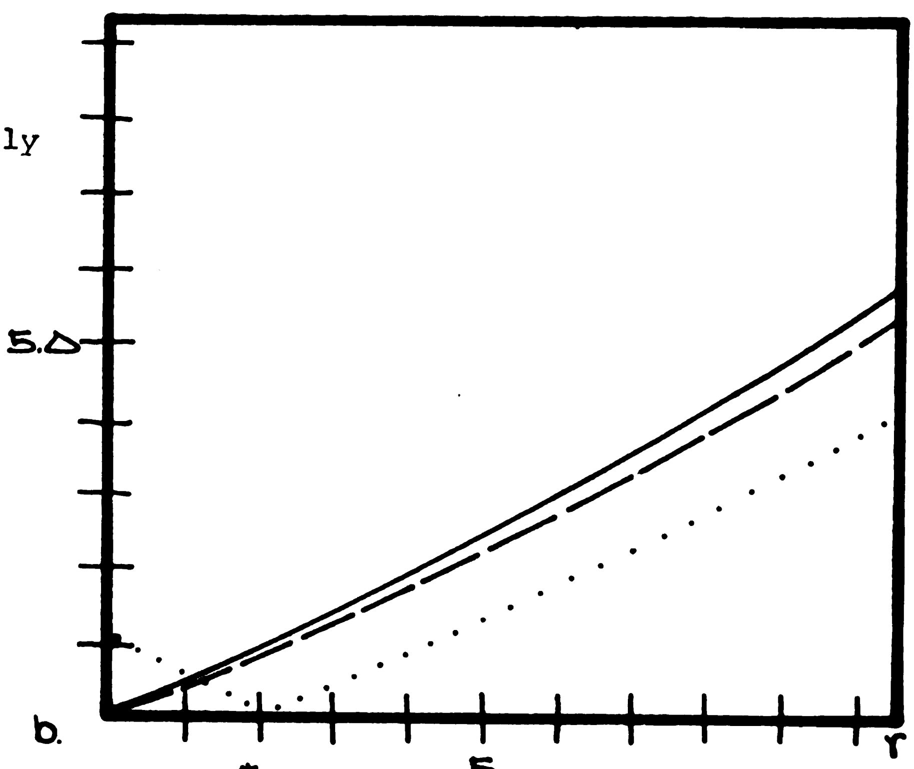
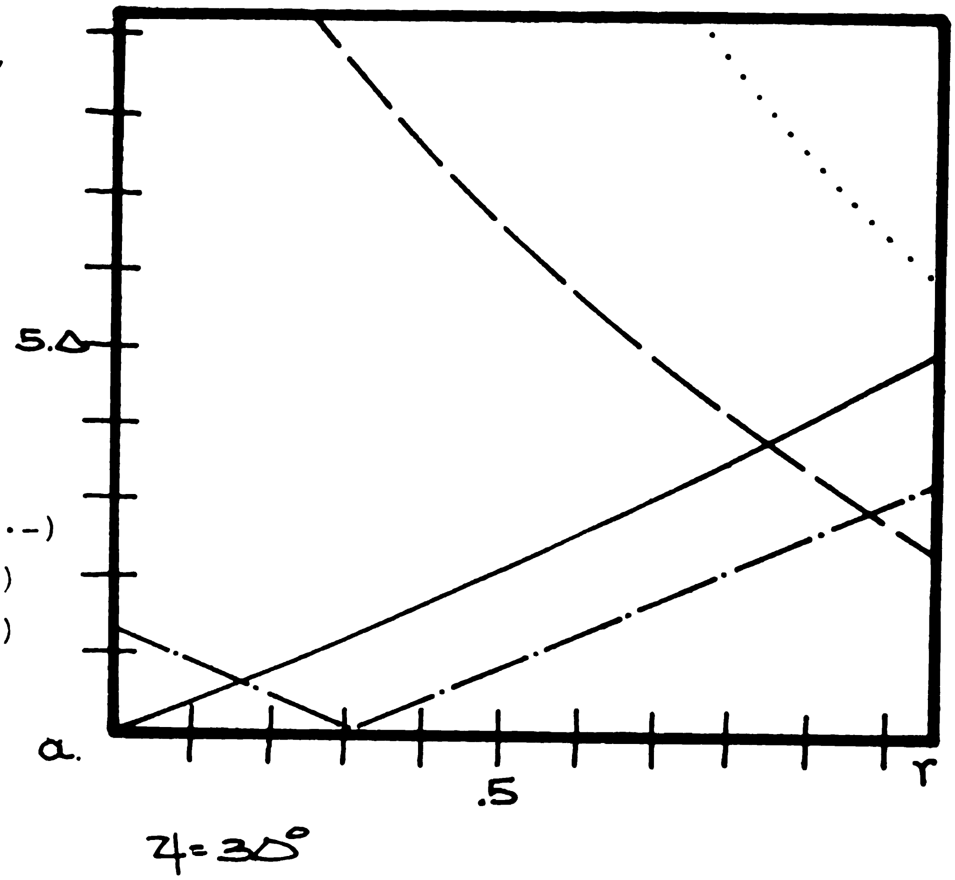


FIGURE #43
$\psi=-60°$

Velocity
Ratios.
Tube
Mode.
m=100
ν=.5
β/c=1.
c=.1
δ=0 (—)
δ=.005 (-·-)
δ=.05 (--)
δ=.5 (···)



δ values not
shown
are off
scale.

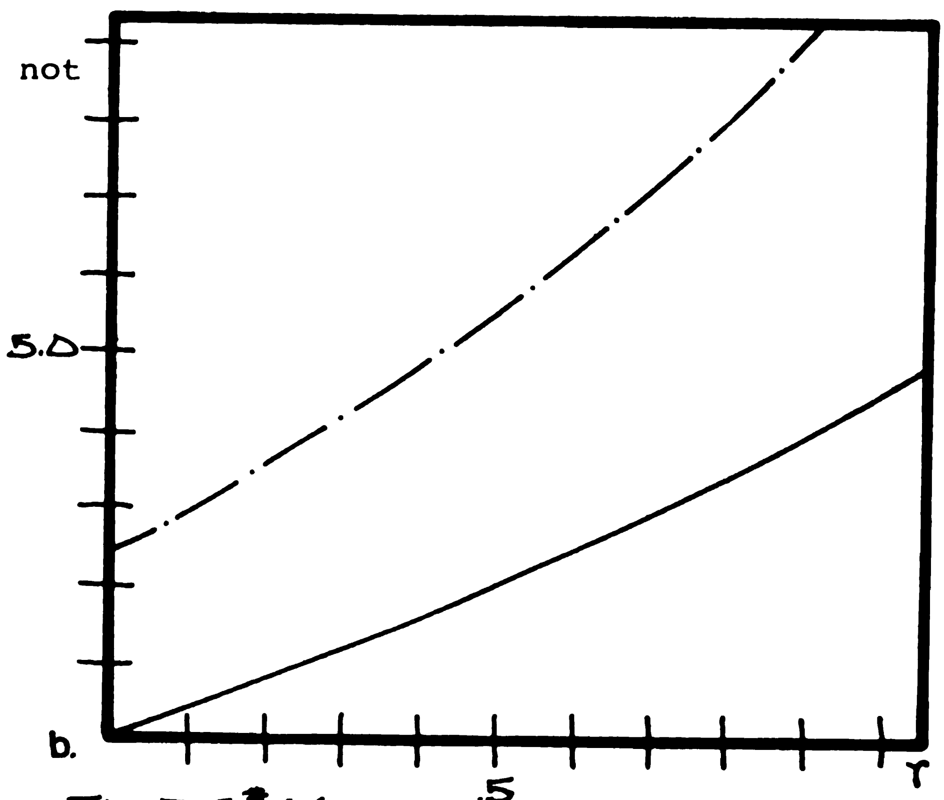


FIGURE #44
ψ=-60°

Figure 48 gives a representation of the displacement ratio for the tube mode of m large (5.3.21). This mode is nearly the same as the empty tube mode so we expect that the displacement ratio will behave like the empty tube displacement ratio examined previously. Note that the ratio evaluated at $\psi = 30°$ and at $\psi = -60°$ is given on the same graph and indicates little sensitivity to the ψ location.

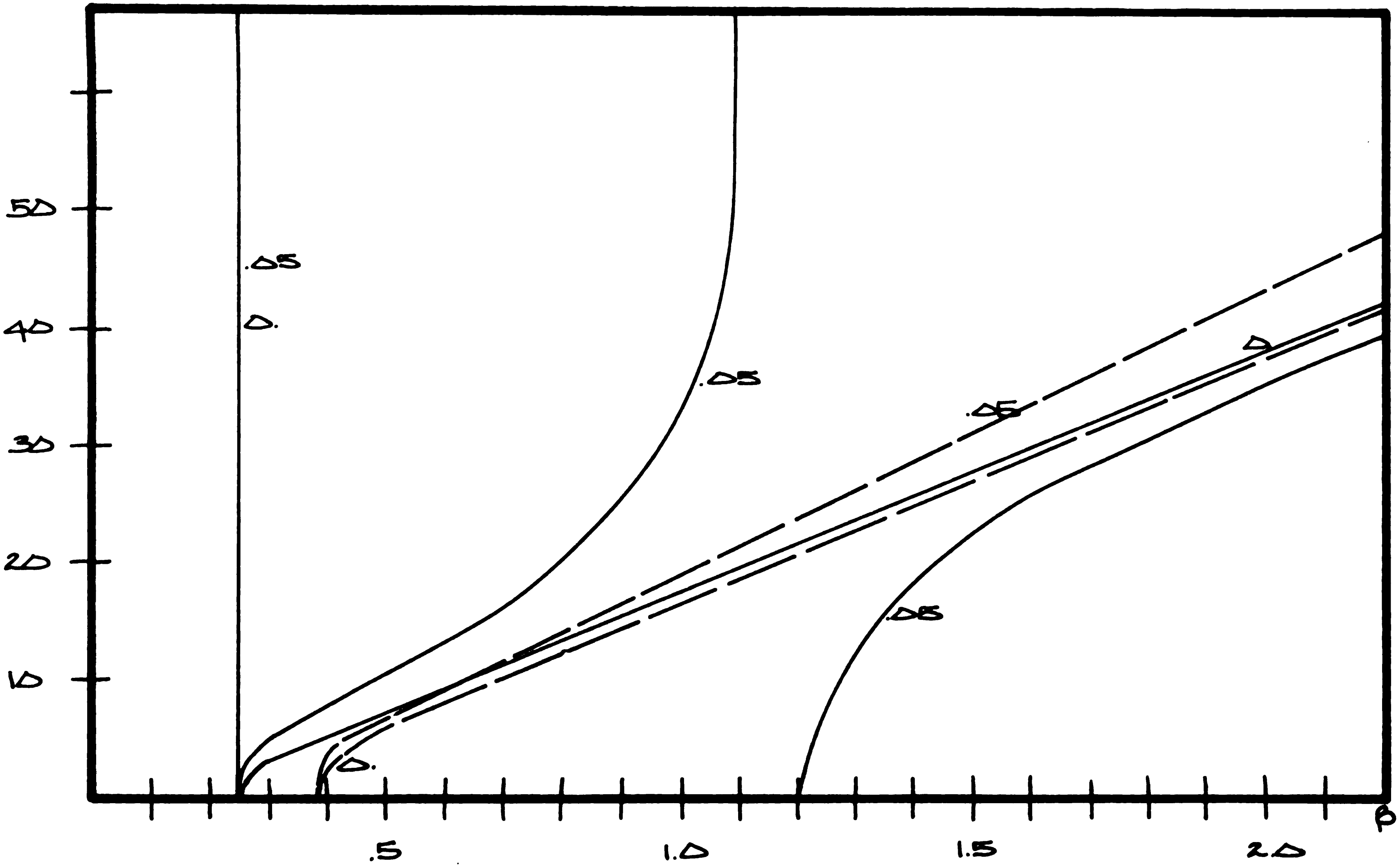


FIGURE #45 DISPLACEMENT RATIO 1st ACOUSTIC MODES
$\psi = -60°$ $\nu = .5$ δ SHOWN $\mu = .01$ (—) $\mu = 100$ (— —)
$\delta = 0$ IS ZERO ORDER CASE $\epsilon = .1$

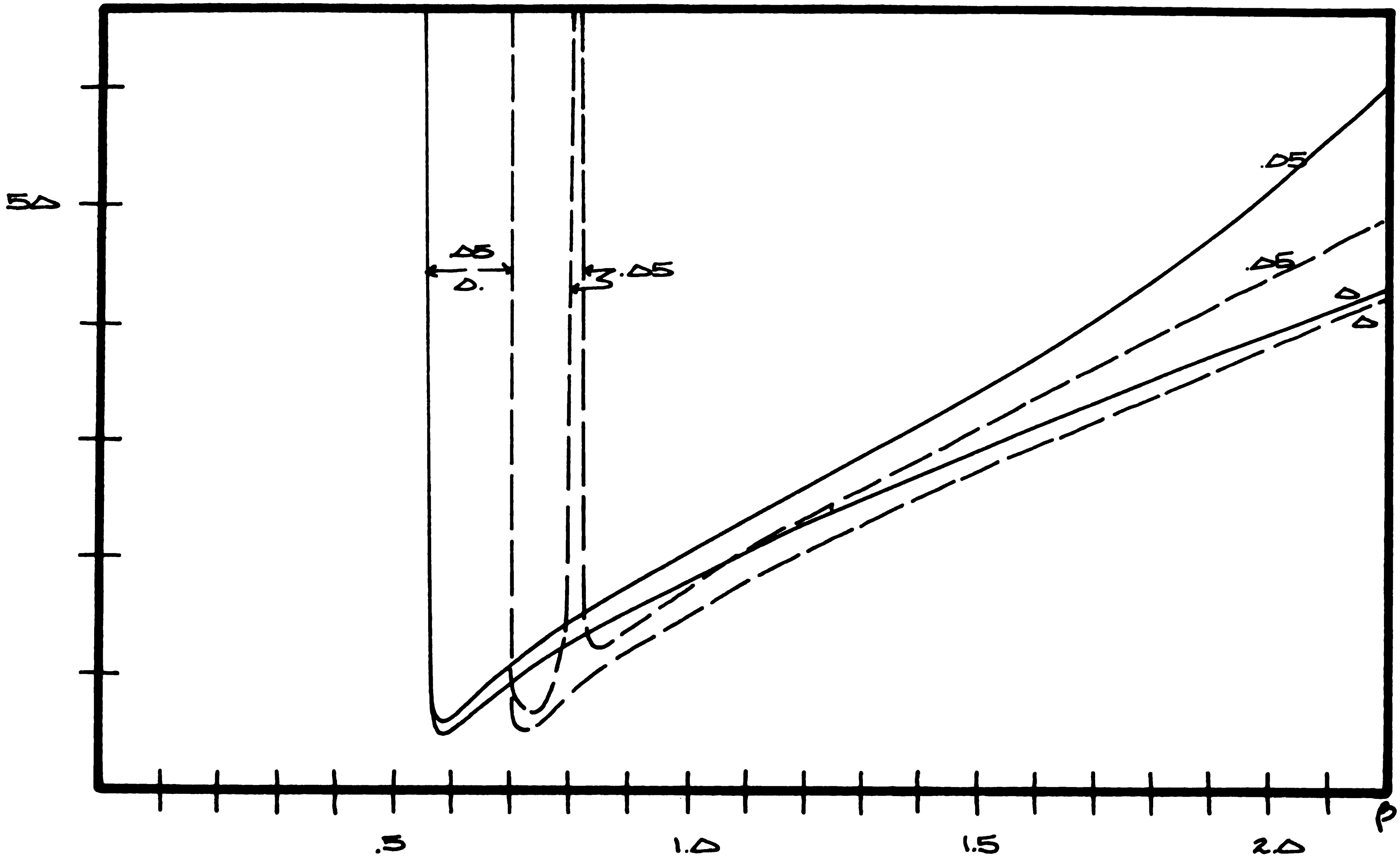


FIGURE #46 DISPLACEMENT RATIO 2ND ACOUSTIC MODES
$\psi = -60°$ $\nu = .5$ δ SHOWN $N = .01$ (—) $N = 100$ (-- -)
$\delta = 0$ IS ZERO ORDER $\epsilon = .1$

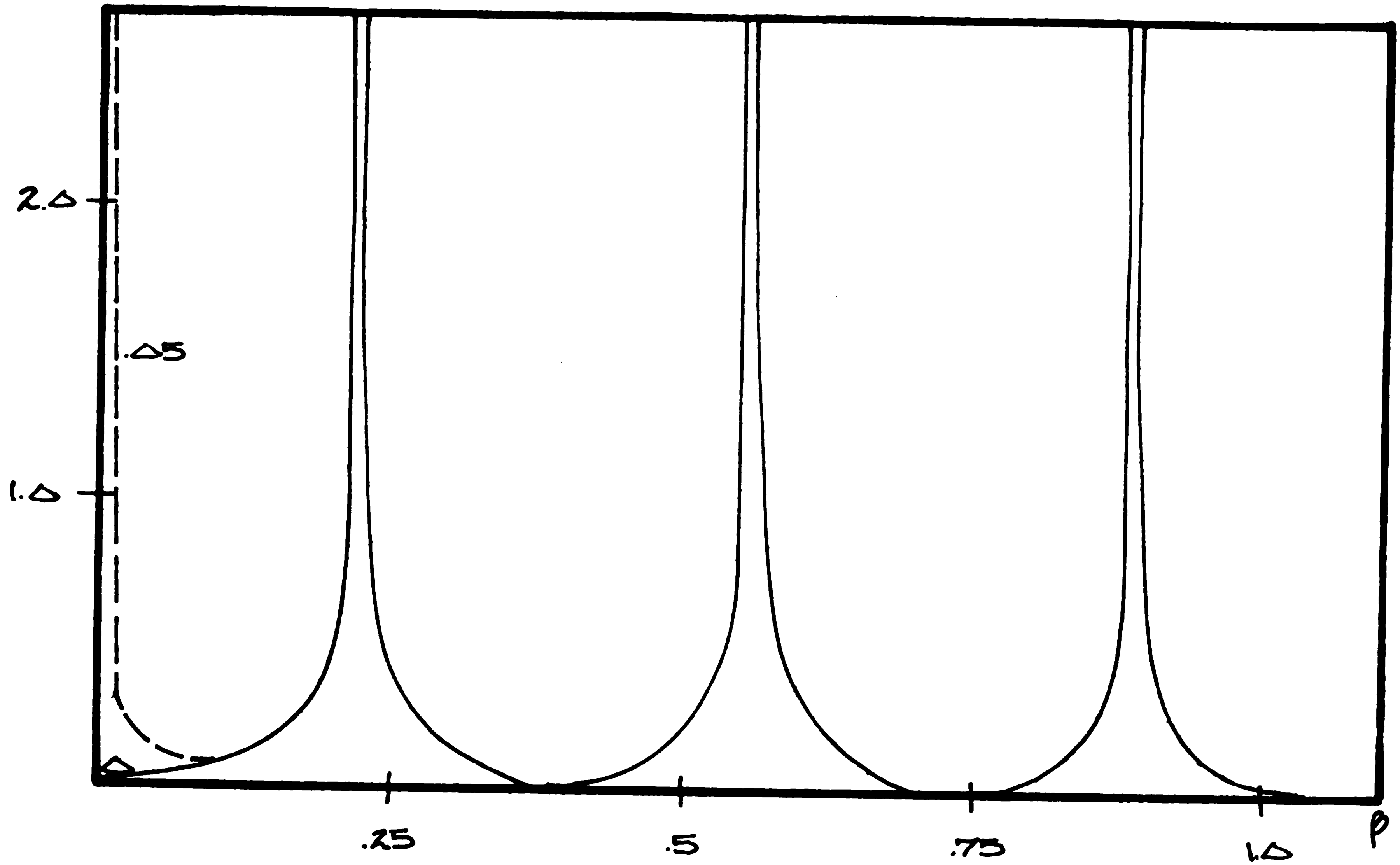


FIGURE #47 DISPLACEMENT RATIO TUBE MODE $N=.01$ $\Omega=.5$ $L=.1$
CURVES FOR $\delta=.05$ AND $\delta=0$ ARE NEARLY IDENTICAL EXCEPT
FOR β SMALL (---)

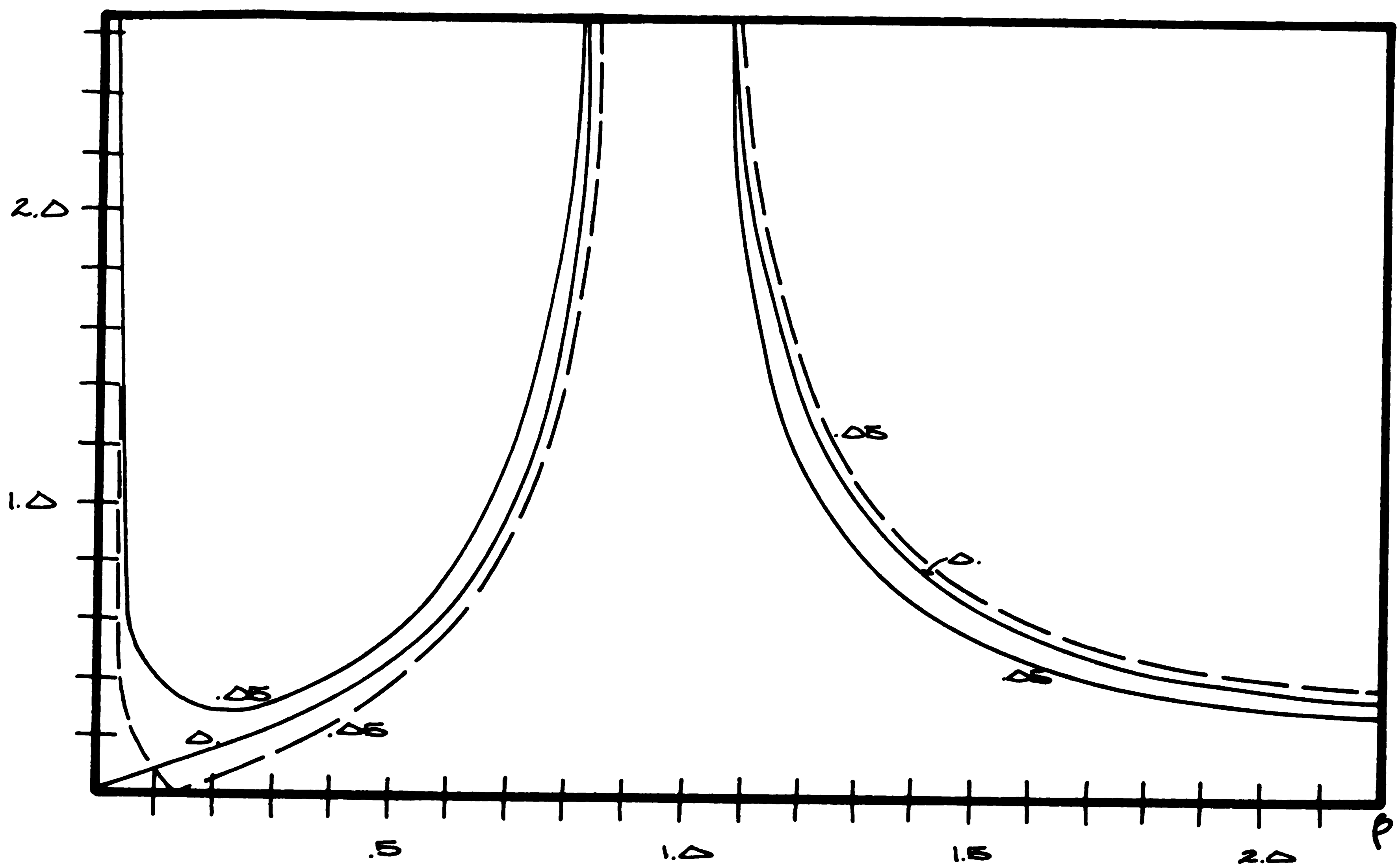


FIGURE #4B DISPLACEMENT RATIO TUBE MODE N=100 ν=.5
ψ=30° (– –) ψ=-60° (—)
δ VALUES SHOWN

5.5 Second Order Correction

In section 5.2.3 we gave, in integral form, the expression that allows us to determine the s_2 correction to the wave number. Equation (5.2.15) when substituted into (4.3.27) and (4.3.28) yields the equation for s_2. For the inviscid fluid this equation is simplified since the shear stresses $S_{r\psi}$ and $S_{r\theta}$ vanish. Further, in the case of zero order axisymmetric flow, $u_\psi^O = u_{\theta,\psi}^O = u_{r,\psi}^O = O$. Employing these restrictions we obtain

$$
\begin{aligned}
& s_2\{-4m\beta^2[s_O w_\theta^O + \nu w_r^O]w_\theta^O + 2A^2 s_O[I_O^2(\alpha) \\
& - I_1^2(\alpha)]\} = m\beta^2\{2s_O a_\psi w_\theta^O - a_\theta[(2s_O^2 + 1-\nu)w_\theta^O \\
& - s_O(1-\nu)w_r^O] + a_r[s_O(1-\nu)w_\theta^O + 2\nu w_r^O] \\
& + w_\theta^O[(3s_O^2 + 1-\nu)w_\theta^O - s_O(2-\nu)w_r^O + s_O(\nu-2)w_\theta^O] \\
& + w_r^O w_\theta^O(1-2\nu)\} + A^2\{I_O^2(\alpha)[\frac{D\alpha}{A} - 1 - \frac{15s_O^2}{8}] \\
& + I_1^2(\alpha)[\frac{D\alpha}{A}(\frac{3s_O^2}{2\alpha^2} - 1) - \frac{1}{2} - \frac{s_O^2}{\alpha^2}(1 + \frac{1}{2}s_O^2) \\
& + (5/4)s_O^2] + I_2^2(\alpha)\frac{s_O^2}{4}[\frac{3}{2} - \frac{s_O^2}{\alpha^2}]\}
\end{aligned}
\tag{5.5.1}
$$

where $\alpha^2 = s_O^2 - \beta^2/c^2$

and

$$w_\theta^0 = -\nu s_0 A I_0(\alpha) m^{-1}\{(1-\nu^2-\beta^2)s_0^2 - \beta^2(1-\beta^2)\}^{-1}$$

(5.5.2)

$$w_r^0 = -(\beta^2-s_0^2) A I_0(\alpha) m^{-1}\{(1-\nu^2-\beta^2)s_0^2 - \beta^2(1-\beta^2)\}^{-1}$$

The a_ψ, a_θ and a_r terms are those of the first order shell displacements which are solutions of (5.4.3). Eventually we are able to write the entire expression in terms of the modified Bessel functions $I_0(\alpha)$ and $I_1(\alpha)$.

We should note that $m = \rho h/\rho_0 a$ where ρh is associated with the tube and $\rho_0 a$ is connected with the fluid. When there is no tube, ρh is zero, and since there are no shell displacements we interpret (5.5.2) so as to have $w_\theta^0 = 0 = w_r^0$. This necessarily means that $I_0(\alpha) = 0$, which is the stress free boundary condition. Examination of (5.5.1) shows that we arrive at the stress free correction if $w_\theta^0 = w_r^0 = I_0(\alpha) = 0$ and $\frac{D\alpha}{A} = \frac{1}{2}s_0^2$.

For the case when $\rho_0 a = 0$ we see that $m^{-1} = 0$ and so if we interpret (5.5.2) so that $\{(1-\nu^2-\beta^2)s_0^2 - \beta^2(1-\beta^2)\} \to 0$ as $m^{-1} \to 0$ then w_θ^0 and w_r^0 are not zero. Understanding that A must be zero if there is no fluid we find that we have $\{(1-\nu^2-\beta^2)s_0^2 - \beta^2(1-\beta^2)\} = 0$ as the frequency equation and we have found the correction for the empty tube.

Finally if ρh grows infinite in size we interpret this case as the rigid tube. Then $m^{-1} = 0$ and $w_\theta^0 = w_r^0 = 0$. Understanding that $I_1(\alpha) = 0$ we find the correction for the rigid tube.

Hence the correction (5.5.1) contains within it all the special cases we have been considering and is therefore a composite of all those situations we considered earlier.

To measure the effect of s_2 we now concentrate our attention on the phase velocity

$$(5.5.3) \qquad c_p = \frac{\beta}{s_0+\delta^2 s_2} \, .$$

Equation (5.5.3) has been evaluated for the four main modes considered previously. For example, using the acoustic mode for m small, i.e., (5.3.18), we obtain the corresponding value for s_2 from (5.5.1). Now choosing a δ value and a ν value, we evaluate (5.5.3) versus β where we have standardized c at 0.1. Our results are given in Figures 49 through 53.

Case I Acoustic Modes

In Figure 49 we present the phase velocity (5.5.3) in which we have used the acoustic mode expansion (5.3.18) for s_0 for $m = 0.01$. We have plotted the curves for (5.5.3) for the first two zeros of $J_0(z_{0n})$; $n = 1,2$ and labeled the curves corresponding to n as 1 or 2.

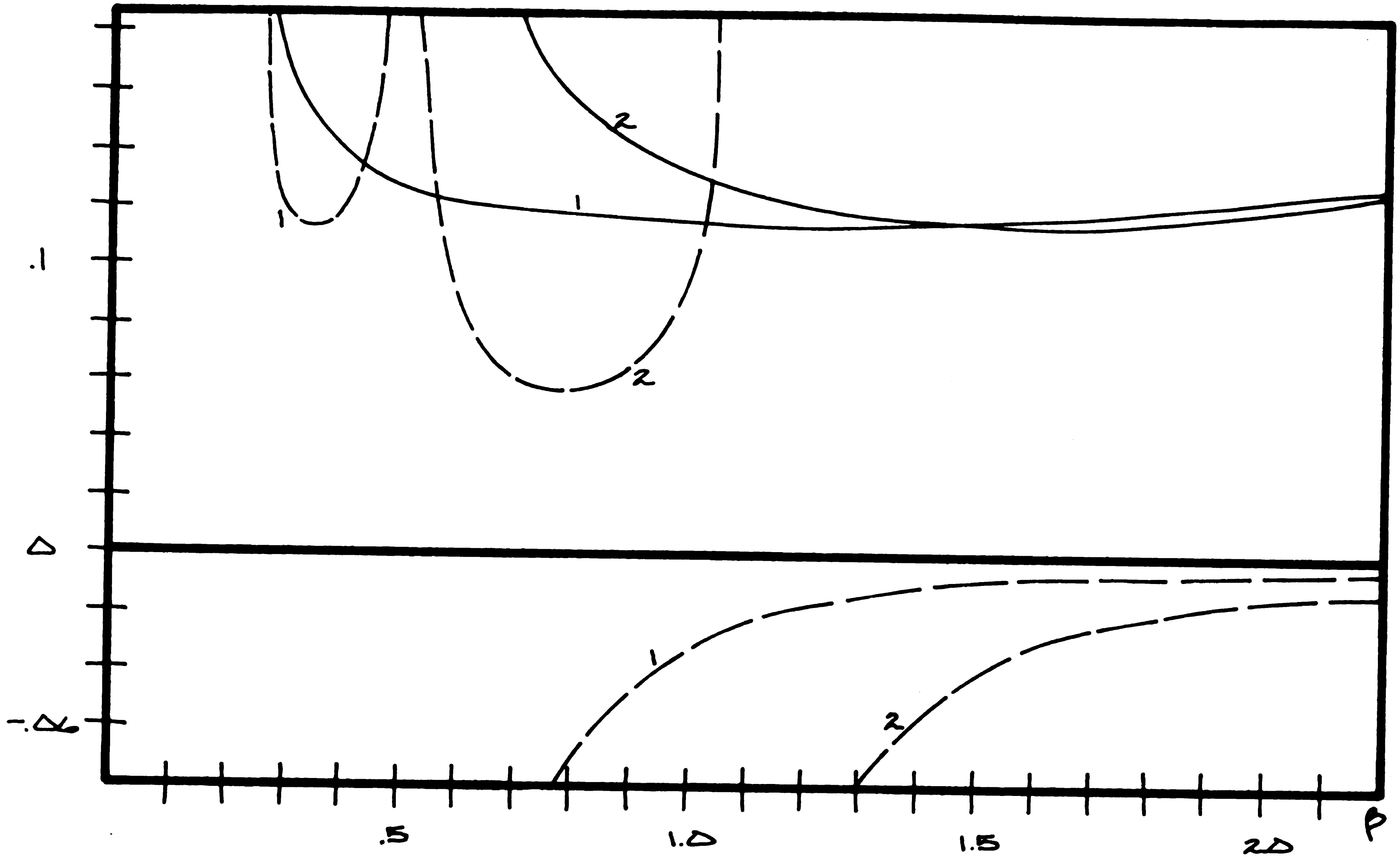


FIGURE #49 CORRECTED PHASE VELOCITY
ACOUSTIC MODE M=.01 D=.5
MODE NUMBERS GIVEN δ=.05 (—) δ=.5 (– –)

The curves for $\delta = 0.005$ and 0.05 were extremely close together and their distinction on the plot was not possible. Thus we show only the $\delta = 0.05$ curves as the solid lines.

A comparison of the phase velocity obtained from (5.5.3) and that corresponding to $\delta = 0$ (previously plotted in Figure 31) shows extremely good agreement over the interval in β to about $\beta = 1.5$ at which point (5.5.3) starts to rise monotonically. The agreement range between the $\delta = 0$ c_p and the corrected c_p expands in β to 2.0 for the second mode $n = 2$, and it is expected that a larger interval of agreement would be obtained for higher modes.

For the sake of comparison we include the results predicted for $\delta = 0.5$ and for this value of δ we find no agreement with the straight tube case ($\delta = 0$).

Let us now consider the phase velocity for the second acoustic mode using (5.3.20) and $n = 1,2$. This case has been plotted in Figure 50 wherein the two modes corresponding to $n = 1,2$ are labeled as such and where $\delta = 0.05$ is the solid curve. We note that the two δ curves for $n = 1$ are close until β reaches 1.1 and the δ curves for $n = 2$ agree until β is near 2.0. However, and more importantly, the agreement between the $\delta = 0.0$ curves of Figure 31 ($m = 100$, there) and those in Figure 50 is very good for β nearly 2.0.

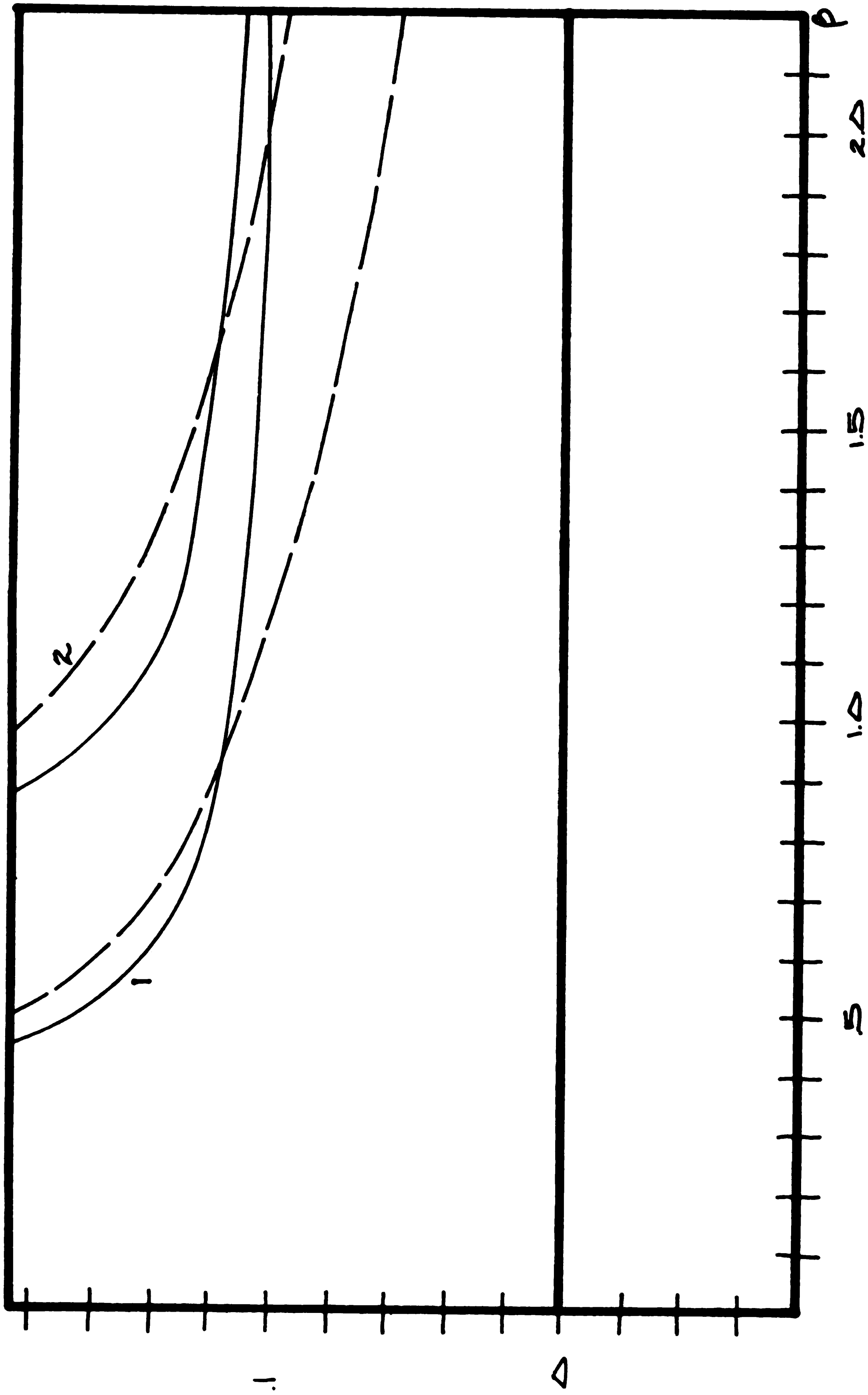


FIGURE #50 CORRECTED PHASE VELOCITY
ACOUSTIC MODE N=100 D=.5
MODE NUMBERS GIVEN S=.05 (—) S=.5 (- -)

<u>Case II</u> Tube Modes

The phase velocity of the straight cylinder interaction problem is given by the tube mode $s_o = \beta$ or, $c_p = 1$, in the case where Poisson's ratio is 0.0. For ν nonzero, however, we have found that c_p, predicted by (5.3.19) and (5.3.21), exhibits asymptotes due to the compressibility of the fluid. Figure 37 is a graph of c_p for $\nu = 0.5$ and for m large and small, 100.0, 0.01, respectively. While the m small mode exhibits the general behavior pattern, $c_p = 1.0$, except for the asymptotes, the c_p curve for $m = 100.0$ shows a cutoff in the interval $\sqrt{1-\nu^2} < \beta < 1$. The character of this mode is much more akin to the empty tube mode previously examined.

Let us now examine the effect of curvature on c_p by inserting (5.3.19) or (5.3.21) into (5.3.3) for $\nu = 0.5$. The results are illustrated in Figure 51 for the value $\delta = 0.05$.

A direct comparison with the zero order empty tube mode, $m = \infty$, Figure 3, shows remarkable agreement with the c_p value obtained from (5.5.3) using (5.3.21) with $m = 100.0$. The agreement is good over the entire frequency range $\beta > 0$, except near $\beta = 0$. In the interval $0 \leq \beta \leq 0.02$, the c_p obtained from (5.5.3) becomes negative and approaches zero through negative values due

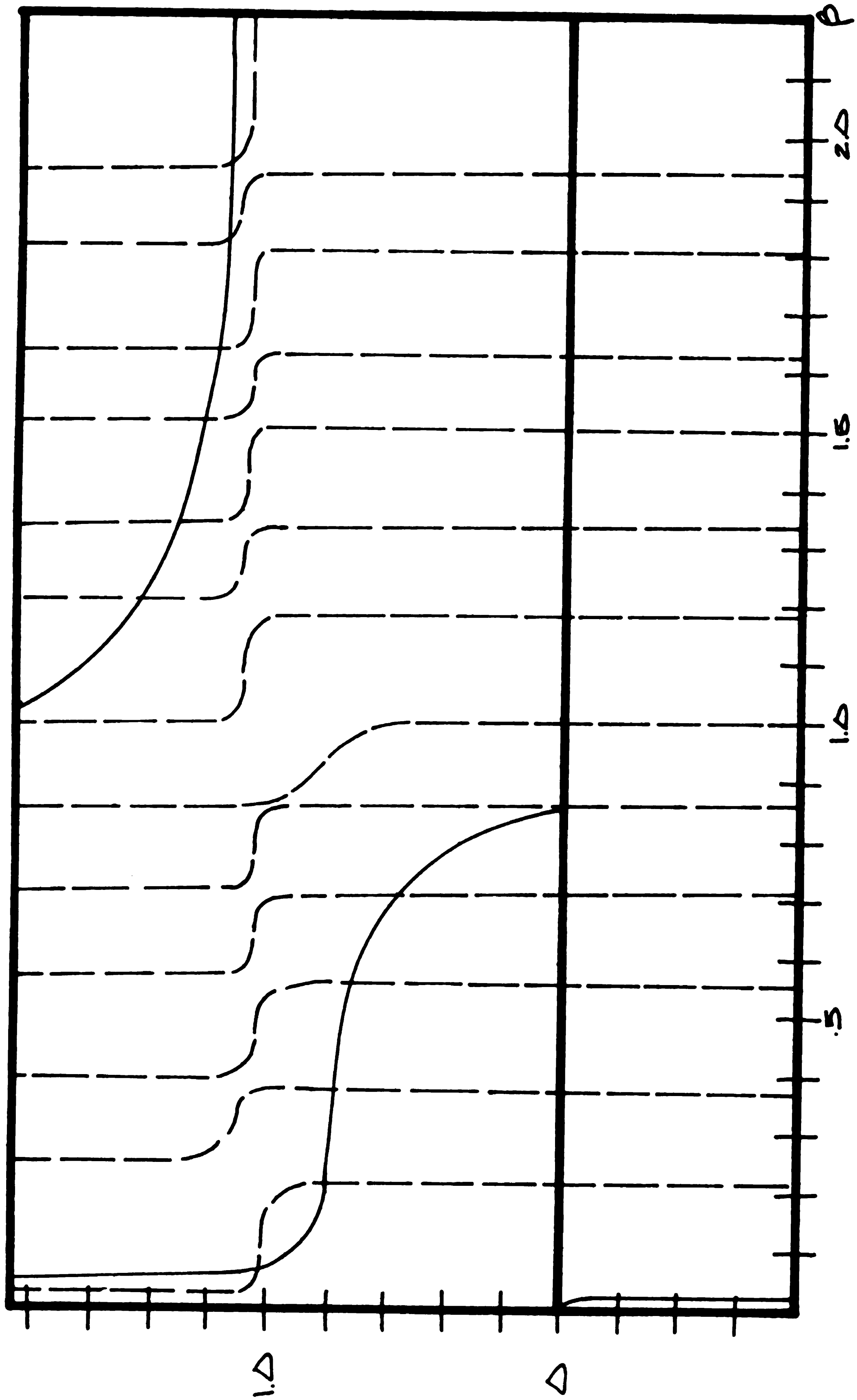


FIGURE #51 CORRECTED PHASE VELOCITY
TUBE MODES $\omega = .5$ $\delta = .05$ $N = 100$ (—) (5.3.21) $N = .01$ (- - -) (5.3.20)

to the asymptote at about 0.02. The corrected velocity differs from the velocity, Figure 37, in that the compressibility of the fluid is suppressed now when m is large.

Turning our attention to the m small case we note good agreement with the straight cylinder phase velocity plotted in Figure 37. Compressibility of the fluid is responsible for the asymptotes and in general c_p is near 1.0 in value as in Figure 37.

We present further results concerning the m small mode in Figures 52 and 53. The pertinent data for these cases is $\nu = 0.0$ and $\delta = 0.05$ (Figure 52) and $\delta = 0.005$ (Figure 53). Drastic behavior changes in c_p are found in these cases and especially in the $\delta = 0.05$ curve. The phase velocity is influenced primarily by the fluid compressibility especially in the neighborhood of $\beta = 1$. For the smaller curvature, c_p is again near 1.0.

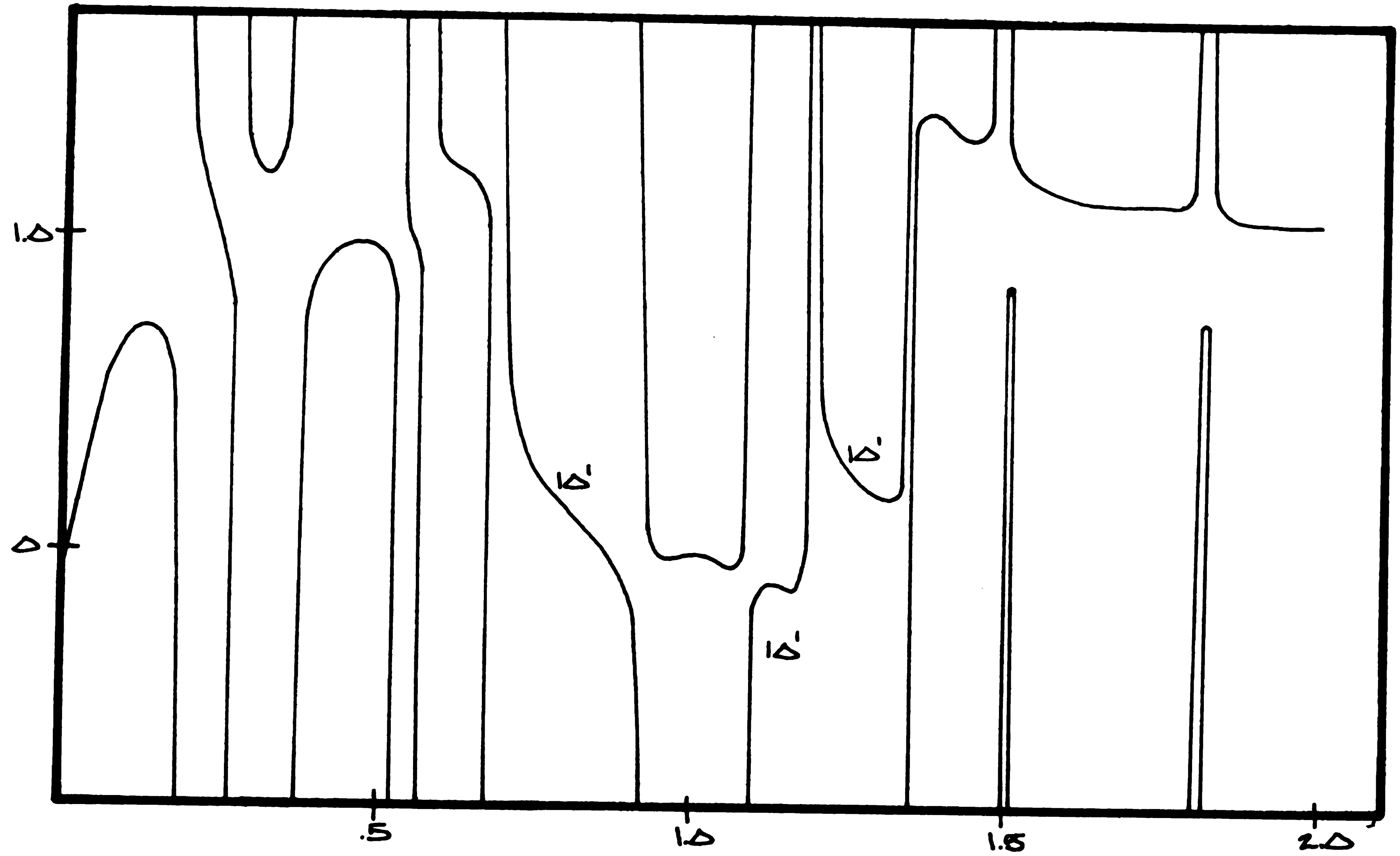


FIGURE #52 CORRECTED PHASE VELOCITY
TUBE MODES $\nu = 0$ $\mu = .01$
$\delta = .05$ (5.3.19) & (5.3.21) ARE THE SAME $\nu = 0$

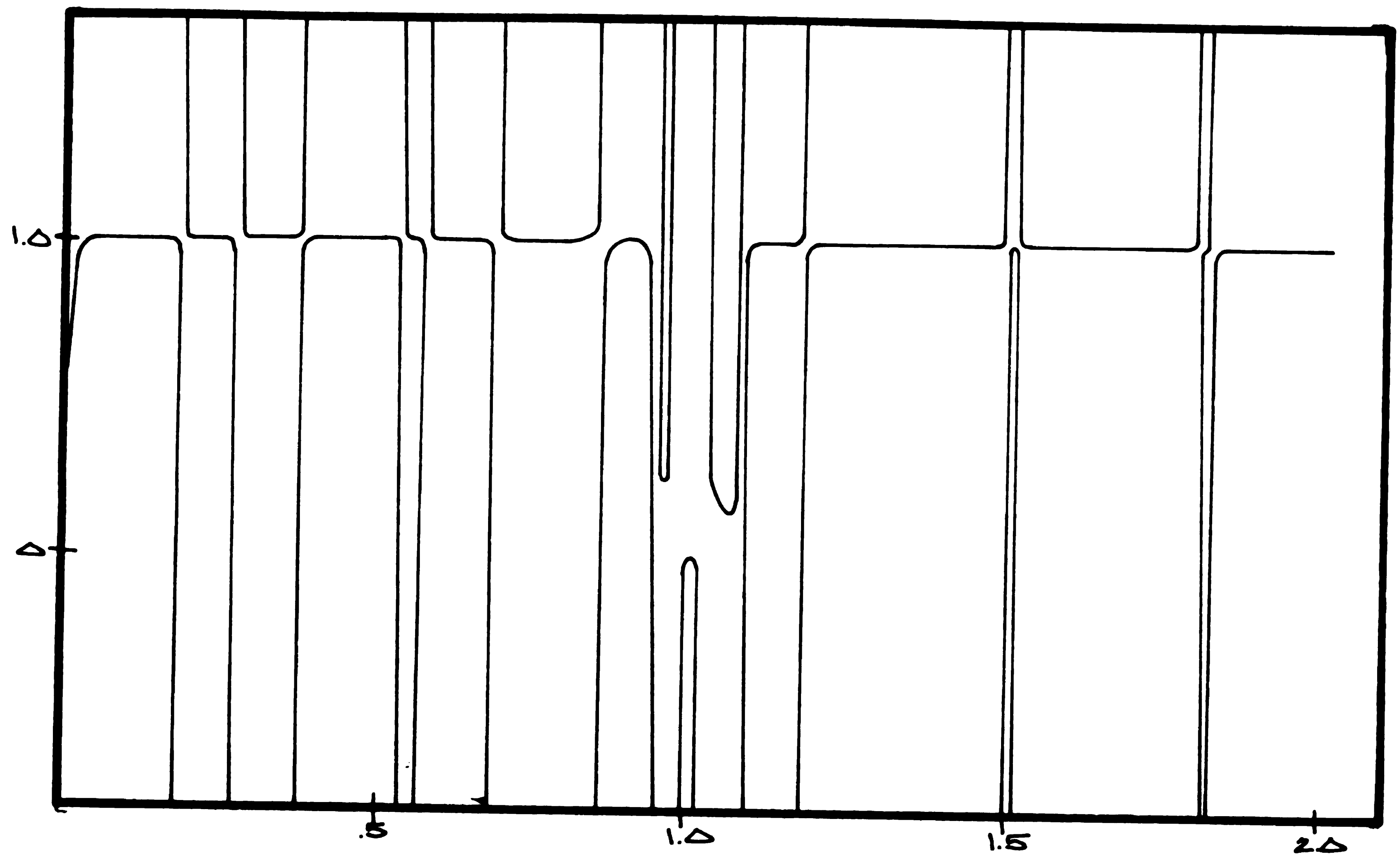


FIGURE #53 CORRECTED PHASE VELOCITY $\nu=0$ $\mu=.01$
TUBE MODE $\delta=.005$

CHAPTER VI

COMMENTS AND CONSIDERATIONS

6.1 Introduction

In this chapter we conclude the present work with some comments on what has been presented in the preceeding chapters. In section 6.2 we consider the situation in which the mass parameter, m, is the same size as the perturbation parameter δ.

In section 6.3 we present some restricted results and make a comparison to available experimental data.

Finally in section 6.4 we summarize the present contributions and indicate what further studies are contemplated. We are also able to make some statements regarding areas of analysis which must be mastered before further generalizations can be contemplated.

6.2 Comments on the Size of m

Implicit in the development of all the equations in Chapter II is the assumption that the mass parameter, m, is of order 1 in comparison to the perturbation parameter, δ. We recall that $m = \rho h/\rho_0 a$ and that in biological cases ρ/ρ_0 is nearly one. We also note that h/a is required to be small in the thin shell theory so that it is possible for m to be small. We therefore choose to consider $m = Q\delta/\rho_0 a$ where Q and $(\rho_0 a)$ are of order one compared to δ. If we examine typical terms from the shell equations (2.2.8) we find that there are three basic terms written as

$$\rho h \underline{a}^2 \omega_0^2 N^*_{\alpha\beta}, \quad \rho_0 a^2 \underline{a} \omega_0^2 S_{\alpha\beta}, \quad \rho h \underline{a}^2 \omega_0^2 \beta^2 w$$

where we have non-dimensionalized using (2.5.3) and set $q_\xi = S_{r\xi}|_{r=1}$. Here $N^*_{\alpha\beta}$ is a typical shell stress resultant, $S_{\alpha\beta}$ is a typical fluid stress evaluated at $r = 1$ and w is a typical shell displacement. If we replace ρh by $Q\delta$ then the zero order shell equations are reduced to setting the fluid stresses (at $r = 1$) equal to zero. We see then that if we are to consider the shell displacements of order k $(k \geq 1)$ then the fluid stresses are of order $(k+1)$ in these equations.

For the zero order inviscid fluid the axisymmetric velocities and pressure are given in (4.4.1). Setting the fluid stress equal to zero at $r=1$ gives

$$(6.2.1) \qquad s^{0}_{rr}(1) = -p_0(1) = AI_0(\alpha) = 0$$

where

$$\alpha^2 = s_0^2 - \beta^2/c^2.$$

We proceed to the first correction equation (4.3.13) written as follows in the inviscid case.

$$(6.2.2) \quad \int_0^{2\pi} u_r^0(1)\, s^1_{rr}(1)\, d\psi = 2s_1 \int_0^{2\pi}\int_0^1 u_\theta^0(r)\, s^0_{\theta\theta}(r)\, r\,dr\,d\psi.$$

We replace $s^1_{rr}(1)$ by the remainder of the first order shell equation (3.2.2), i.e.,

$$s^1_{rr}(1) = +\frac{Q}{\rho_0 a}\{(\beta^2-1)w_r^0 - w^0_{\psi,\psi} - \nu s_0 w_\theta^0\}.$$

Since only radial velocities must match at the boundary we assume that w_ψ^0 and w_θ^0 are not necessarily axisymmetric as are their counterparts in the velocity field of the fluid. We assume

$$w_\psi^0 = a_\psi \cos\psi + b_\psi \sin\psi + c_\psi$$

$$(6.2.3) \qquad w_\theta^0 = a_\theta \sin\psi + b_\theta \cos\psi + c_\theta$$

$$w_r^0 = c_r$$

where the a's, b's and c's are constants. The correction is written as

$$- \frac{A\alpha I_1(\alpha)}{i\beta} \frac{Q}{\rho_0 a} \int_0^{2\pi} \{ (\beta^2 - 1)\; w_r^0 - w_{\psi,\psi}^0$$

(6.2.4) $$-\nu s_0 [a_\theta \sin\psi + b_\theta \cos\psi + c_\theta]\} d\psi$$

$$= \frac{4\pi s_1 A^2}{i\beta} \int_0^1 s_0 I_0^2(r\alpha) r dr.$$

The integral of $w_{\psi,\psi}^0$ is zero by periodicity as is true of the $\sin\psi$ and $\cos\psi$ terms. Hence we have

$$- \frac{A\alpha I_1(\alpha)}{i\beta} \frac{Q}{\rho_0 a} \{ (\beta^2-1) c_r - \nu s_0 c_\theta \}$$

(6.2.5) $$= \frac{s_0 s_1}{i\beta} A^2 \{ I_0^2(\alpha) - I_1^2(\alpha) \} .$$

We must solve for c_r and c_θ if we are to evaluate the correction. The equations for the zero order shell displacements involve $p_1(1)$. We find the first order pressure from equation (2.6.20) and determine

$$p_1(r,\psi) = C_0 I_0(r\alpha) + I_1(r\alpha)\{C \cos\psi + D \sin\psi\}$$

(6.2.6) $$- \tfrac{1}{2}A \sin\psi \{ r^2 s_0^2 \alpha^{-1} I_1(r\alpha) + r I_0(r\alpha) \}$$

$$+ A s_0 s_1 \alpha^{-1} r I_1(r\alpha).$$

We should note that for our earlier work in chapter IV $s_1 = 0$ so that the last term in (6.2.6) was not present. [cf. (4.5.1)].

Substitution of (6.2.3) into the first order shell equations (3.2.3) through (3.2.5) gives

$$(6.2.7)\quad \begin{pmatrix} \beta^2-1-\frac{1}{2}s_0^2(1-\nu) & \frac{1}{2}s_0(1+\nu) \\ \frac{1}{2}s_0(1+\nu) & \beta^2-s_0^2-\frac{1}{2}(1-\nu) \end{pmatrix} \begin{pmatrix} a_\psi \\ a_\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(6.2.8)\quad \begin{pmatrix} \beta^2-1-\frac{1}{2}s_0^2(1-\nu) & -\frac{1}{2}s_0(1+\nu) \\ -\frac{1}{2}s_0(1+\nu) & \beta^2-s_0^2-\frac{1}{2}(1-\nu) \end{pmatrix} \begin{pmatrix} b_\psi \\ b_\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(6.2.9)\quad \{\beta^2 - \tfrac{1}{2}s_0^2(1-\nu)\}c_\psi = 0$$

$$(6.2.10)\quad \begin{pmatrix} \beta^2 - s_0^2 & -\nu s_0 \\ -\nu s_0 & \beta^2 - 1 \end{pmatrix} \begin{pmatrix} c_\theta \\ c_r \end{pmatrix} = -\frac{\rho_0 a}{Q}\,\frac{As_0 s_1}{\alpha}\, I_1(\alpha) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(6.2.11)\quad \begin{aligned} -b_\psi - \nu s_0 b_\theta &= -\frac{\rho_0 a}{Q} C I_1(\alpha) \\ a_\psi - \nu s_0 a_\theta &= -\frac{\rho_0 a}{Q}\{D - \tfrac{1}{2}As_0^2/\alpha\} I_1(\alpha) \end{aligned}$$

By (6.2.1) the frequency equation is $I_0(\alpha) = 0$, i.e., $J_0(i\alpha) = 0$. Since the determinants of the coefficient matrices in (6.2.7) through (6.2.10) are not identically zero we can conclude a_ψ, a_θ, b_ψ, b_θ and c_ψ are all zero and that

$$c_\theta = -\frac{A\nu s_0^2 s_1 I_1(\alpha)\rho_0 a}{\alpha Q\ \mathrm{Det}},$$

(6.2.12)
$$c_r = -\frac{A(\beta^2-s_0^2)s_0 s_1 I_1(\alpha)\rho_0 a}{\alpha Q\ \mathrm{Det}},$$

$$\mathrm{Det} = (\beta^2-s_0^2)(\beta^2-1) - \nu^2 s_0^2 \ .$$

From (6.2.11) we find $C = 0$ and $D = \frac{1}{2}As_0^2/\alpha$ which is the value of D for the stress free condition. Matching radial velocities we find that

$$c_r = A\alpha I_1(\alpha)/\beta^2$$

and therefore

(6.2.13)
$$s_1 = \frac{\alpha^2 Q\,\mathrm{Det}}{s_0\beta^2(s_0^2-\beta^2)\rho_0 a}$$

We have then found s_1 by matching velocities, but we also have the correction equation (6.2.5). Substitution of (6.2.13) into (6.2.5) satisfies the latter relationship exactly.

When $m = 0$, as in the stress free case, we interpret this to mean that Q is zero and hence $s_1 = 0$. Therefore the situation in which m is small reduces to the case $m = 0$ with s_1 becoming zero just as in the stress free case.

Use of (6.2.13) in (6.2.12) leads to the following relations for the zero order shell displacements:

$$w_\psi^0 = 0,$$

$$\text{(6.2.14)} \qquad w_\theta^0 = A\nu s_0 \alpha I_1(\alpha)/\beta^2(\beta^2 - s_0^2),$$

$$w_r^0 = A\alpha I_1(\alpha)/\beta^2.$$

We are also able to write

$$p_1(r,\psi) = C_0 I_0(r\alpha) + \tfrac{1}{2}A\,\sin\psi\{s_0^2\alpha^{-1}(1-r^2)I_1(r\alpha)$$

$$- rI_0(r\alpha)\} - \frac{AQ}{\rho_0 a}\,\frac{\alpha(\mathrm{Det})\,rI_1(r\alpha)}{\beta^2(\beta^2 - s_0^2)}\ .$$

For inviscid flow we have from (4.5.2) that

$$u_\psi^1(r,\psi) = p_{1,\psi}/ri\beta,$$

$$\text{(6.2.15)} \qquad u_\theta^1(r,\psi) = s_0\{rp_0\,\sin\psi - p_1\}/i\beta,$$

$$u_r^1(r,\psi) = p_{1,r}/i\beta\ .$$

We see that u_ψ^1 does not depend on Q and hence s_1 since the derivative with respect to ψ eliminates the Q term from $p_1(r,\psi)$. One can note then that the first order correction has no influence on the circumferential velocity, its effects being felt in the pressure and other velocities.

6.3 Comparison with Experimental Work

There has been some experimental work reported in the literature regarding the wave propagation characteristics of fluid filled curved elastic tubes. This work was performed at Stanford University and is summarized in a report by Anliker and Van Buskirk [22].

In general a fluid filled flexible tube was coiled in a semi-circle and suspended by flexible strips from another rigid tube. A disc was inserted in one end of the tube and vibrated in a controlled manner. Three separate types of waves were generated during the course of the experiments. The three waves were denoted by the terms axial, pressure and flexural respectively.

Daras, in his thesis, compares his analytical results with those reported by Anliker and Van Buskirk. All investigations assume an incompressible fluid so that $\alpha^2 = s_o^2 - \beta^2/c^2$ reduces to $\alpha = s_o$.

<u>Case I</u>. Flexural Waves (one wave around the tube)

Daras assumed in this case that $\beta << 1$ and that the zero order terms have the form

$$(6.3.1) \quad \begin{cases} w_\psi^o = W_\psi^o \sin\psi, \quad w_\theta^o = W_\theta^o \cos\psi, \quad w_r^o = W_r^o \cos\psi \\ p_o(r,\psi) = A I_1(r s_o) \cos\psi \end{cases}$$

Inserting these expressions into the shell equations (5.2.1) with $n = 1$ results in a system of equations in W_ψ^o, W_θ^o, W_r^o and A. If we ignore the β^2 terms we find

$$(6.3.2)\quad \begin{pmatrix} -1-\frac{1}{2}s_o^2(1-\nu) & -\frac{1}{2}s_o(1+\nu) & -1 \\ -\frac{1}{2}s_o(1+\nu) & -s_o^2-\frac{1}{2}(1-\nu) & -\nu s_o \\ -1 & -\nu s_o & -1 \end{pmatrix} \begin{pmatrix} W_\psi^o \\ W_\theta^o \\ W_r^o \end{pmatrix} = -\frac{A}{m} I_1(s_o) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solving for W_r^o and matching velocities at $r = 1$ gives us

$$(6.3.3)\quad -i\beta W_r^o = u_r^o(1) = p^o(r,\psi),r/i\beta\Big|_{r=1} .$$

The result of (6.3.3) is a frequency equation of the form

$$(6.3.4)\quad \beta^2 I_1(s_o)\{s_o^4+2s_o^2+1\} = m(1-\nu^2)s_o^5 I_1'(s_o)$$

If we make the further restriction that s_o is small, writing $I_1(s_o) \cong s_o/2$ and $I_0(s_o) \cong 1$, then the result takes the form

$$(6.3.5)\quad s_o^4\{\beta^2-m(1-\nu^2)\} + 2\beta^2 s_o^2 + \beta^2 = 0.$$

We present the graph of the phase velocity, $c_p(=\beta/s_o)$, in Figure 54. We converted the data of the Stanford experiments into our notation using $h/a = .0967$, $\rho = 1$, $\rho_o = 1.248$ and $\nu = 0$ or $\nu = .5$. These curves compare well with the experimental data presented in the

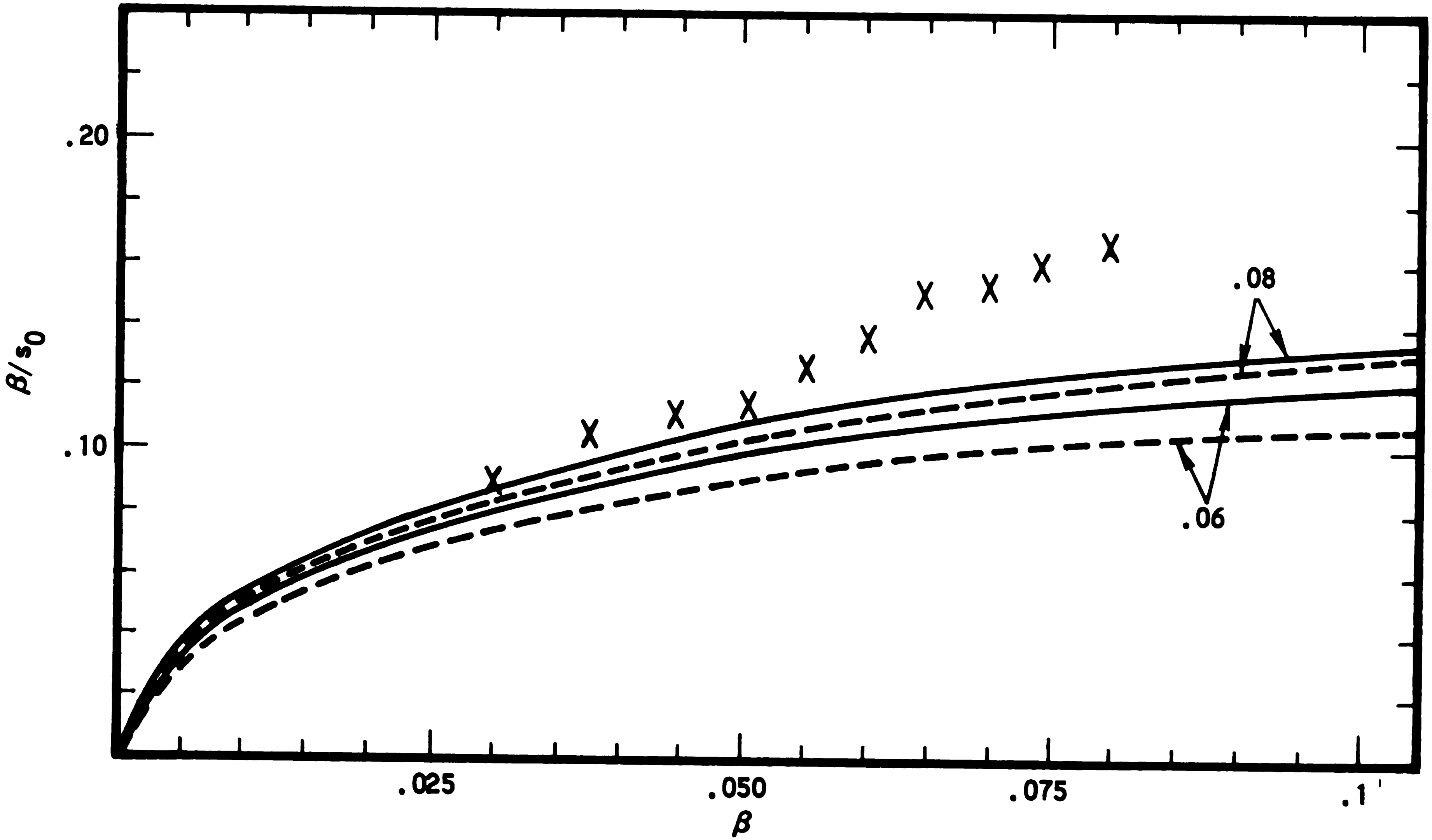


Figure No. 54. Phase Velocity
FLEXURAL WAVES M Values Shown
$\nu = 0$ (–) $\nu = .5$ (– –)
EQUATION (6.3.5)

Stanford reports and indicated by crosses on the graph.

Daras solved (6.3.5) by the quadratic formula, expanding the denominator in a geometric series. Taking the lowest order term he finds

(6.3.6) $$s_0^2 = \beta\{m(1-\nu^2)\}^{-1/2}$$

Computing c_p from this value also gives reasonable agreement with experimental data.

Case II. Axial Waves.

The second situation Daras considers is that of axisymmetric axial waves. In this situation the flow in the zero order system is assumed to be inviscid and axisymmetric. Daras set $w_\psi^0 = 0$, $w_{r,\psi}^0 = 0$ and $w_{\theta,\psi}^0 = 0$. If one makes a long wave length assumption, i.e., s_0 is small and $m << 1$, then the zero order shell equations take the form

(6.3.7) $$\begin{pmatrix} \beta^2 - s_0^2 & -\nu s_0 \\ -\nu s_0 & \beta^2 - 1 \end{pmatrix} \begin{pmatrix} w_\theta^0 \\ w_r^0 \end{pmatrix} = -\frac{A}{m} I_0(s_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Daras notes that under these conditions Anliker and Maxwell [23] have found w_r^0 small for small β. Setting $w_r^0 = 0$ forces us to conclude that

$$s_0 = \beta$$

$$w_\theta^0 = A/\nu s_0 m$$

where we have replaced $I_0(s_0)$ by 1.

If we now pursue this problem further by searching for the correction s_2 we find that the first order system assumes the form

$$\text{(6.3.8)}\quad \begin{pmatrix} -1+\frac{3}{4}s_0^2 & \frac{3}{4}s_0 & 1 \\ \frac{3}{4}s_0 & -\frac{1}{4} & -s_0/2 \\ 1 & -s_0/2 & s_0^2-1 \end{pmatrix} \begin{pmatrix} a_\psi \\ a_\theta \\ a_r \end{pmatrix} = \begin{pmatrix} f_\psi \\ f_\theta \\ f_r \end{pmatrix}$$

where we have utilized the fact that

$$w_\psi^1 = a_\psi \cos\psi, \quad w_\theta^1 = a_\theta \sin\psi, \quad w_r^1 = a_r \sin\psi, \quad \nu = .5,$$

and

$$f_\psi = 5A/2m,$$

$$\text{(6.3.9)}\quad f_\theta = -A(1+8s_0^2)/2s_0 m,$$

$$f_r = A/m - \{\frac{Ds_0}{2} - \frac{A}{2}[s_0^2/2+1]\}/m.$$

Using Cramer's rule we solve for a_r and set it equal to $\tilde{u}_r^1(1)/\beta^2$, with the tilda denoting the $\sin\psi$ portion of $\tilde{u}_r^1$.

If we keep only those terms which involve m^{-1} we find that

$$\frac{Ds_0}{2} = \frac{7A}{2}$$

and from that we find

$$p_1(r,\psi) = 3Ar\sin\psi.$$

This value for p_1 corresponds to that found by Daras.

Solving for a_r by setting $a_r = \frac{\tilde{p}_{1,r}}{\beta^2}$ we

we find for the lowest order terms in s_0 that $a_r = 3A/\beta^2$ which also corresponds to the value obtained by Daras.

Returning to the shell equations of first order we find by Cramer's rule that

$$a_\theta = \frac{2A}{m\beta}$$

which agrees with Daras. Finally we solve for a_ψ by Cramer's rule again, obtaining

$$a_\psi = \frac{-17A}{8m} .$$

Daras obtains $3A/\beta^2$ which he finds by setting $a_\psi = \frac{\tilde{u}^1_\psi}{\beta^2}$, i.e. from matching velocities in the ψ direction. This procedure is not applicable in the inviscid fluid since there are no shear stresses in the fluid and hence no coupling of the fluid and shell velocities in the circumferential direction.

Utilizing the values we have found for a_ψ, a_θ, a_r, w_θ^0 and D we examine (5.5.1) and find s_2 by taking those terms which are of lowest order in β. We find

(6.3.10) $$s_2 = -3m/16\beta$$

so that the corrected phase velocity is given by

$$c_p = \{1 - 3m\delta^2/16\beta^2\}^{-1} \tag{6.3.11}$$

Daras finds a different value for c_p although it is of the same form. He expands the right side in a geometric series which is not correct as the expression $3m\delta^2/16\beta^2$ is not less than one for β near zero. In fact there is a vertical asymptote for $\beta = \delta\sqrt{3m}/4$. The graph of c_p is given in Figure 55. We have utilized the following values for the parameters appearing in (6.3.11): $m = 0.075$, $\delta^2 = 0.01$ and $0.0 < \beta < 0.1$. The agreement between our theoretical values and the experimental results are quite good.

<u>Case III</u> Pressure Waves

In this case $\nu = .5$ and the acceleration terms are neglected in comparison to other terms. As in the axial waves the flow is inviscid, incompressible and axisymmetric in the zero order. The wave lengths are long so that s_0 is small.

The shell terms are written as

$$\begin{pmatrix} -s_0^2 & -\nu s_0 \\ -\nu s_0 & -1 \end{pmatrix} \begin{pmatrix} w_\theta^0 \\ w_r^0 \end{pmatrix} = -\frac{A}{m} I_0(s_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6.3.12}$$

while the fluid velocity in the radial direction is

$$u_r^0 = s_0 I_1(s_0)/i\beta ,$$

and the boundary condition takes the form

$$-i\beta w_r^0 = u_r^0 .$$

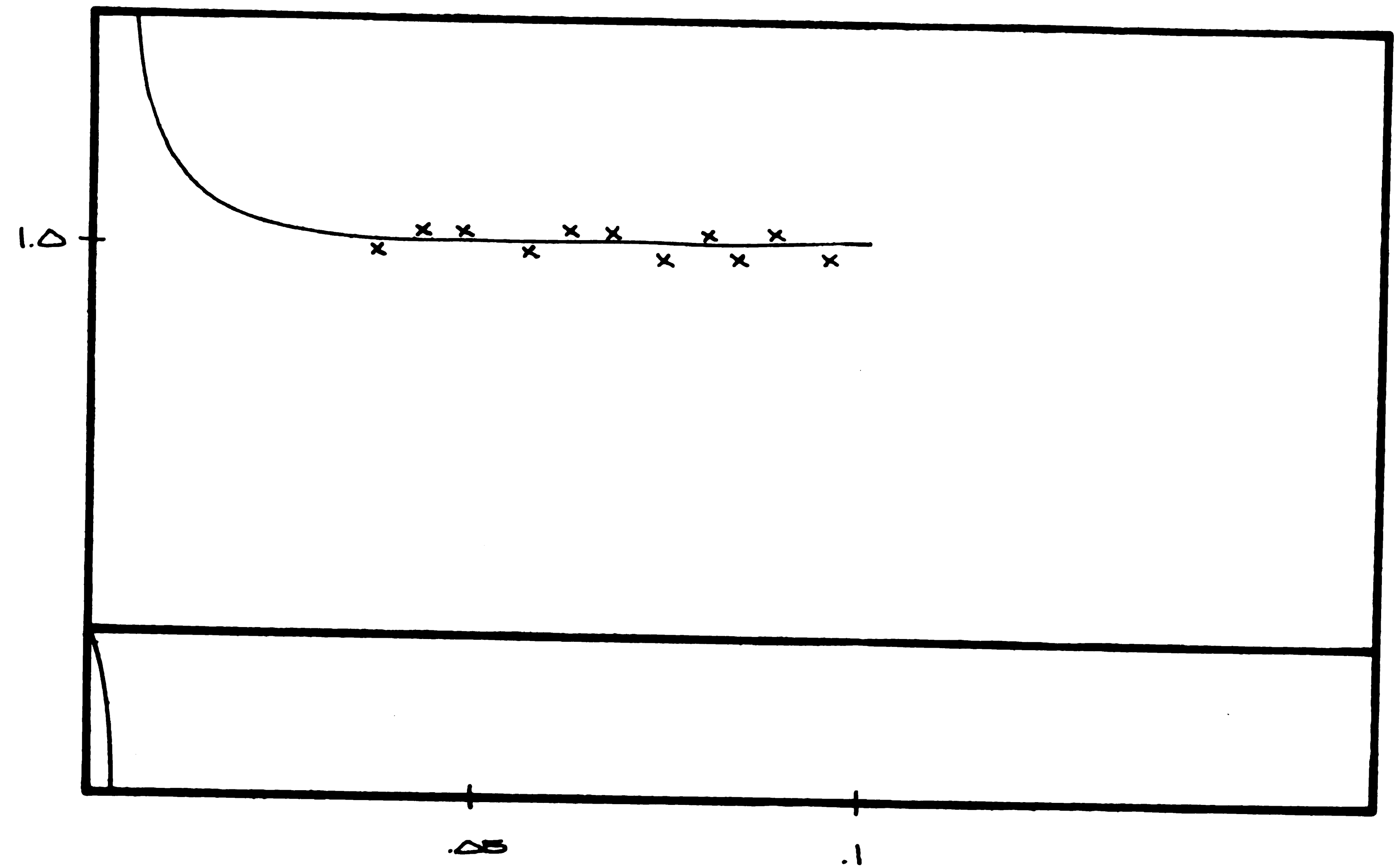


FIGURE # 55 CORRECTED PHASE VELOCITY
AXIAL WAVES $\delta = .033$ $\mu = .075$
EQUATION (6.3.11)

Solving for w_r^0 and satisfying the boundary condition we find

$$s_0 = \beta\sqrt{8/3m},$$

$$w_r^0 = \frac{4A}{3m}, \quad w_\theta^0 = -\frac{2A}{3m\beta}\sqrt{\frac{3m}{8}}, \quad w_\psi^0 = 0.$$

The first order displacement equations satisfy

$$\text{(6.3.13)} \quad \begin{pmatrix} -1 - \frac{1}{4}s_0^2 & \frac{3}{4}s_0 & 1 \\ \frac{3}{4}s_0 & -s_0^2 - \frac{1}{4} & -s_0/2 \\ 1 & -s_0/2 & 1 \end{pmatrix} \begin{pmatrix} a_\psi \\ a_\theta \\ a_r \end{pmatrix} = \begin{pmatrix} f_\psi \\ f_\theta \\ f_r \end{pmatrix}$$

where the first order displacements are the same as those in the axial waves, and

$$f_\psi = -13w_r^0/8$$

$$\text{(6.3.14)} \quad f_\theta = \frac{w_r^0}{s_0}\left\{\frac{3s_0^2}{2} + \frac{1}{8}\right\}$$

$$f_r = \frac{3}{4}w_r^0 - \frac{1}{m}\left\{\frac{Ds_0}{2} - \frac{A}{2}(s_0^2/2+1)\right\}$$

Solving for a_r and satisfying the boundary conditions we find $\frac{Ds_0}{A} = -1$ and that

$$a_r = -A/\beta^2$$

$$\text{(6.3.15)} \quad a_\theta = -5A/3s_0 m$$

$$a_\psi = -A/ms_0^2 .$$

As before we have some disagreement with Daras due to his use of boundary conditions to determine other than the radial velocity.

Utilizing these values for the zero and first order velocities we find from (5.5.1) that keeping only terms of order $(ms_0^2)^{-1}$ we find

(6.3.16) $$s_2 = 3/16s_0$$

and the corrected phase velocity is given by

(6.3.17) $$c_p = \sqrt{\frac{3m}{8}}\,\{1 + \frac{9m\delta^2}{128\beta^2}\}^{-1}$$

The corrected phase velocity is plotted in Figure 56 and the comparison with experimental data is rather close.

All of the above cases have shown close agreement between theoretical and experimentally predicted phase velocities. The results plotted by Daras are different analytically from those presented here yet they also agree fairly well with experimental data. As can be seen the experimental data exists only for a limited frequency range and the values given are essentially the first term of the phase velocity.

Further experimental work needs to be done on the very small frequencies in order to determine the exact behavior in this region of small frequencies.

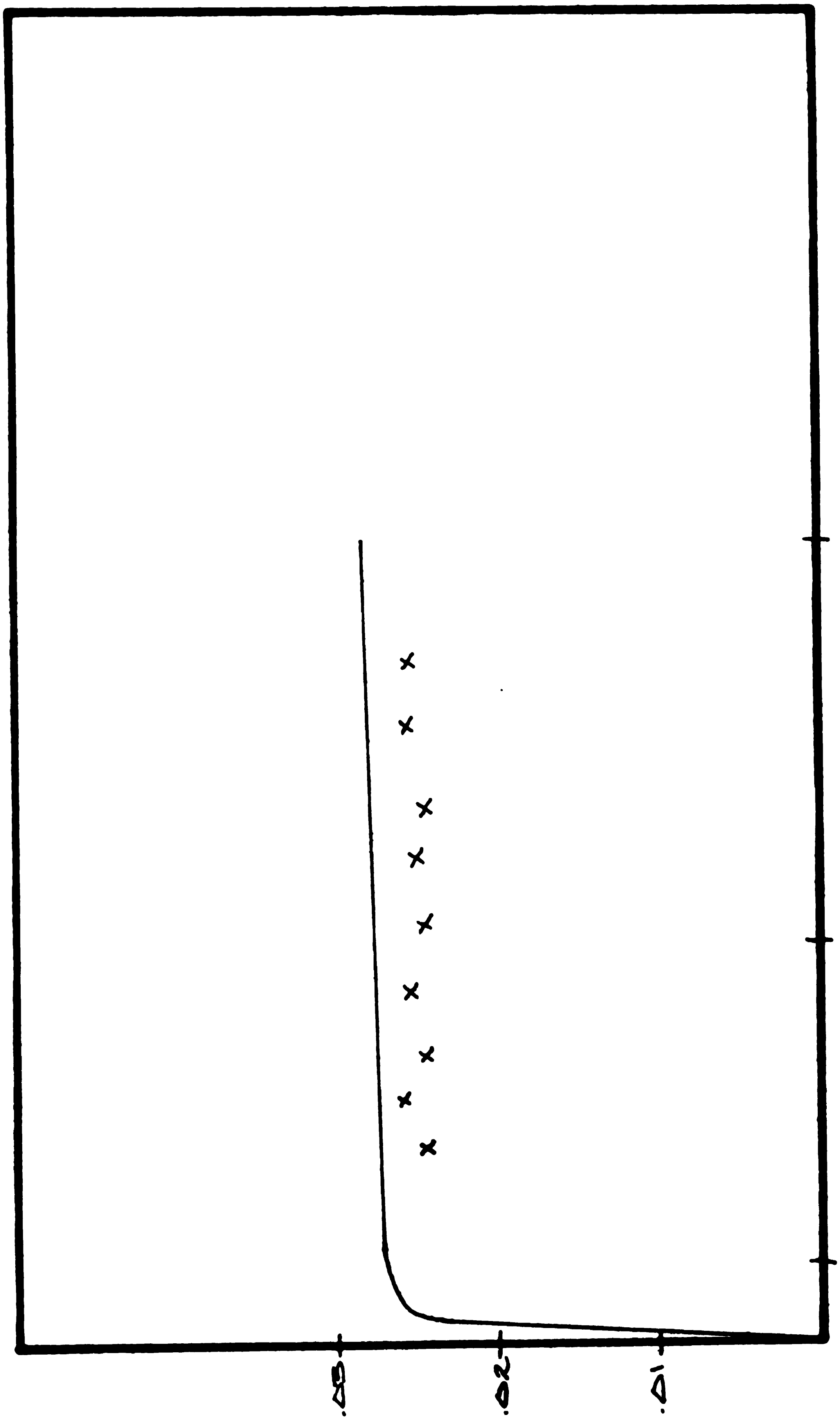


FIGURE #51 PRESSURE CORRECTED WAVES PHASE VELOCITY δ=.033 h=.075

6.4 Final Considerations

We are now in a position to reflect on what has gone before in this treatise. We established the equations for a fluid flowing in a curved membrane shell. We attacked the problem through the use of a power series expansion in a small parameter for all our variables. The result of this procedure was a series of problems starting with a problem appropriate to a straight tube flow. We analysed the frequency equation resulting from the straight tube problem and then proceeded to find corrections for our zero order variables.

We found for a particular variable, the wave number, that the first correction was zero and that the second order correction was required. We interpreted this correction as accounting for curvature effects not found in the straight tube. Various ratios of velocities and displacements were calculated and these values compared to those of the straight tube. We found some regions where there were differences between the straight and curved tubes. In other frequency ranges the straight and curved results are nearly identical.

We have also found that when the parameter m is of order δ, then the first correction to the zero order wave number is not zero. This fact has not been noticed by previous investigators and therefore deserves further study.

Finally we should comment on the basic problem present in such an analysis as we have presented. The fundamental problem is analysis of the frequency or dispersion relation developed for the zero order system. This equation is transcendental in nature and in general quite difficult to solve. A recent paper by Scarton and Rouleau [24] presents a computer analysis of a frequency equation similar to ours. In this analysis the complex roots are sketched. Unfortunately it is difficult to find an analytic relationship among these roots which could be used in computing corrections for lower order terms.

In conjunction with the last comments we should point out that there are further problems which require solution. For example the bending terms might be introduced into the shell equations or the shell might be described as anisotropic. Another problem would involve the introduction of viscous terms into the solutions. We have presented the solutions but the complete analysis of the frequency equation for these more general models is lacking.

What we have been able to do is solve a basic problem, fundamental to the field of biological fluid flows. We have analysed the zero order situation and then proceeded to write an expression which accounts for curvature corrections. In addition to demonstrating the existence of such a correction we have also computed it and shown its effect on lower order terms.

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