STATISTICAL PROPERTIES OF SOME ALMOST ANOSOV SYSTEMS

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ABSTRACT

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We investigate the polynomial lower and upper bounds for decay of correlations of a class of two-dimensional almost Anosov diffeomorphisms with respect to their Sinai-Ruelle-Bowen measures (SRB measures), where the almost Anosov diffeomorphism is a system which is hyperbolic everywhere except for one point. At the indifferent fixed point, the Jacobian matrix is an identity matrix. The degrees of the bounds are determined by the expansion and contraction rates as the orbits approach the indifferent fixed point, and can be expressed by using coefficients of the third order terms in the Taylor expansions of the diffeomorphisms at the indifferent fixed points.

We discuss the relationship between the existence of SRB measures and the differentiability of some almost Anosov diffeomorphisms near the indifferent fixed points in dimensions bigger than one. The eigenvalue of Jacobian matrix at the indifferent fixed point along the one-dimensional contraction subspace is less than one, while the other eigenvalues along the expansion subspaces are equal to one. As a consequence, there are twice-differentiable almost Anosov diffeomorphisms that admit infinite SRB measures in two or three-dimensional spaces; there exist twice-differentiable almost Anosov diffeomorphisms with SRB measures in dimensions bigger than three. Further, we obtain the polynomial lower and upper bounds for the correlation functions of these almost Anosov maps that admit SRB measures.

To my family and friends for their love and support.

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Chapter 1

Introduction

The research of dynamical systems is motivated by the problems in classical physics, statistical mechanics and so on. Given a space X, a deterministic and discrete dynamical system is defined by a map $T : X \to X$, where X is a Riemannian manifold, and the map T preserves an invariant probability measure μ. This dynamical system is denoted by (X, T, μ). The orbit of an initial state $x \in X$ is denoted by $\{x, Tx, ..., T^n x, ...\}$, which represents the long term behaviour of the system.

In dynamical systems, there exist lots of simple maps T with complicated dynamics, which lead to interesting stories about chaos theory. We are concerned with ergodic theory, which can be seen as a quantitative description of the dynamics with the help of measure theory. The state space X should come with a σ-algebra $\mathcal{B}$ of measurable subsets.

The statistical properties of an observable function ϕ on X with respect to the map T is also an interesting problem. We introduce a sequence of random variables $X_n = \phi \circ T^n$, this sequence of random variables are identically distributed since the measure μ is invariant with respect to the map T. For the statistical properties of dynamical systems, there exist lots of interesting problems, for example, the existence of Sinai-Bowen-Ruelle measures (SRB measures), decay of correlations for some observable functions, central limit theorems, large deviation principles, almost sure invariance principles, and so on.

There exist lots of work on the study of the SRB measures. Given a twice-differentiable

Anosov diffeomorphism on a compact connected Riemannian manifold with the Riemannian measure, there is a unique invariant Borel probability measure with respect to this diffeomorphism such that the measure has absolutely continuous conditional measure on unstable manifolds, the map has positive Lyappunov exponents, the metric entropy is equal to the sum of the positive Lyapunov exponents, and the map has exponential decay of correlations for Hölder continuous observable functions [42]. For Axiom A attractors, similar results have been obtained by Bowen, Ruelle, and so on [4]. Pesin, Ledrappier, Young and others have extended the theory on nonuniformly hyperbolic sets [29, 45]. For Hénon attractors, Benedicks and Young showed that there exist SRB measures for certain parameters and good statistical properties [3]. For more information on Siani-Ruelle-Bowen measure, please refer to [47]. We will study the existence of SRB or infinite SRB measures for a class of almost Anosov diffeomorphisms in dimensions bigger than one.

The correlation function of a system is used to describe how fast the state of the system becomes uncorrelated with its future status, and to estimate this function is a very interesting problem in dynamical systems. To investigate the statistical properties, Young introduced a powerful tool "Young Tower", which has been successfully applied to study many systems [45]. In [46], Young applied the "coupling method" to obtain the polynomial upper bounds for the correlation functions of some systems. Later, Sarig introduced a powerful method, estimating the asymptotic norms of renewal sequences of bounded operators acting on Banach spaces, and gave the polynomial lower bounds for correlation functions [40]. And, Gouëzel sharped Sarig's results and obtained better estimates for some systems [8].

For the study of the correlation functions of the maps on two-dimensional spaces, Liverani and Martens investigated a class of area preserving maps on torus, and obtained the polynomial upper bounds. In [12], Hu showed that there exist either SRB measures or infinite SRB measures for almost Anosov diffeomorphisms with non-degeneracy conditions, where the decomposition of the tangent space of the almost Anosov systems is discontinuous at the indifferent fixed points. It is an interesting problem to investigate the statistical properties of almost Anosov systems, since this kind of systems can be thought of as the generalization of the map $x \to x + x^{1+s}$ [13], which has polynomial lower bounds for correlation functions. We will show that some almost Anosov diffeomorphisms have both the polynomial upper and lower bounds.

For the study of the large deviation principles, there are lots of interesting results. Kifer provided a unified method to establish large deviation principles based on the existence of a pressure functional and on the uniqueness of equilibrium states for certain dense sets of functions [19]. Young studied the large deviation estimates for continuous maps of compact metric spaces and applied these results in differentiable maps and shift spaces [43]. In [37], the authors obtained the rate functions for certain maps based on the theory of Young Towers with exponential return time functions. Melbourne investigated the large deviation principles for a class of nonuniformly hyperbolic dynamical systems with polynomial decay of correlations and some moderate deviations [26]. In [27], Melbourne and Nicol studied the large deviation estimates for a large class of nonuniformly hyperbolic systems, which are defined on Young towers with summable decay of correlations. In [31], Pollicott and Sharp studied the large deviation behavior of the orbits of interval maps with indifferent fixed points, and obtained the polynomial and the exponential level I estimation results for functions, as well as the polynomial and the exponential level II estimation results for measures. We will study the large deviation estimates for two-dimensional almost Anosov diffeomorphims and apply these results to the study of the decay of correlations for Hölder observable functions.

The rest is organized as follows. In Chapter 1, some useful concepts and results are introduced. In Chapter 2, we study the polynomial lower and upper bounds for decay of correlations of a class of two-dimensional almost Anosov diffeomorphisms with respect to their SRB measures. It is discovered that the degrees of the bounds could be described by the expansion and contraction rates as the orbits approach the indifferent fixed point, and can be expressed by using coefficients of the third order terms in the Taylor expansions of the diffeomorphisms at the indifferent fixed points. In Chapter 3, it is to investigate the relationship between the existence of SRB measures and the differentiability of some almost Anosov diffeomorphisms near the indifferent fixed points in dimensions bigger than one, where the almost Anosov diffeomorphism is a system which is hyperbolic everywhere except for one point. As a consequence, there are twice-differentiable almost Anosov diffeomorphisms that admit infinite SRB measures in two or three-dimensional spaces; there exist twice-differentiable almost Anosov diffeomorphisms with SRB measures in dimensions bigger than three. Further, we obtain the polynomial lower and upper bounds for the correlation functions of some almost Anosov maps that admit SRB measures.

1.1 Preliminary

In this section, we introduce some basic definitions and useful properties.

Consider a non-singular measurable map $T : X \to X$, where X is measurable space, $\mathcal{B}$ is the σ algebra, μ is a σ-finite measure. The measure μ is called non-singular if $\mu(T^{-1}(E)) = 0$ is equivalent to $\mu(E) = 0$ for any $E \in \mathcal{B}$.

Definition 1.1.1. [41] The transfer operator of a non-singular map $(X, \mathcal{B}, \mu, T)$ is the operator $\mathcal{T} : L^1(\mu) \to L^1(\mu)$, which is defined by

$$\mathcal{T} f = \frac{d\mu_f \circ T^{-1}}{d\mu},$$

where μ_f is the measure $\mu_f(E) = \int_E f d\mu$.

Proposition 1.1.1. [41] There is a unique solution $\psi \in L^1(\mu)$ to the equation $\int \phi \cdot \psi d\mu = \int (\phi \circ T) \cdot f d\mu$ for any function $\phi \in L^\infty$. The solution is $\psi = \mathcal{T} f$.

Proposition 1.1.2. [41] The transfer operator is a positive bounded linear operator with norm one, and satisfying that

(1) for any $\phi \in L^1$ and $\psi \in L^\infty$, we have $\mathcal{T}[(\psi \circ T) \cdot \phi] = \psi \cdot (\mathcal{T}\phi)$, μ-almost everywhere;

(2) if T is a measure-preserving map, then for any $\phi \in L^1(\mu)$, we have $(\mathcal{T}\phi) \circ T = \mathbb{E}_\mu(\phi | T^{-1}\mathcal{B})$, μ-almost everywhere.

For any given map f and its invariant probability measure μ, the correlation function for two observable functions Φ and Ψ is defined by

$$\mathrm{Cor}_n(\Phi, \Psi; f, \mu) := \int (\Psi \circ (f^n))\Phi d\mu - \int \Phi d\mu \int \Psi d\mu,$$

where n is a positive integer.

Definition 1.1.2. Let μ be an f-invariant Borel probability measure and let $\mathcal{H}$ be a class of functions on M. We say that (f, μ) has exponential decay of correlations for functions in $\mathcal{H}$ if there is $0 < \tau < 1$ such that for any $\Phi, \Psi \in \mathcal{H}$, there exists $C = C(\Psi, \Phi)$ such that

$$|\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| \le C\tau^n.$$

Definition 1.1.3. Let μ be an f-invariant Borel probability measure and let $\mathcal{H}$ be a class of functions on M. We say that (f, μ) has polynomial decay of correlations for functions in $\mathcal{H}$ if there is $\tau > 0$ such that for any $\Phi, \Psi \in \mathcal{H}$, there exists $C = C(\Psi, \Phi)$ such that

$$|\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| \leq C n^{-\tau}.$$

Definition 1.1.4. Given a measurable space X with a probability measure ν and a measurable partition ξ, there exists a family of probability measures $\{\nu_x^\xi : \ x \in X\}$, which is called a canonical system of conditional measures for ν and ξ [39], satisfying that

(i) $\nu_x^\xi(\xi(x)) = 1$, where $\xi(x) \in \xi$ containing x;

(ii) for any measurable set $B \subset X$, the map $x \to \nu_x^\xi(B)$ is measurable;

(iii) $\nu(B) = \int_X \nu_x^\xi(B) d\nu(x)$.

Let M be a C^∞ compact Riemannian manifold without boundary. Let ν be the Lebesgue measure on M. Let μ be an invariant measure with respect to a map f on M, where $f : (M, \mu) \to (M, \mu)$ is a $C^{1+\alpha}$ measurable map with positive Lyapunov exponents almost everywhere, and $\alpha > 0$. It follows from Pesin theory [29] that the unstable manifold $W^u(x)$ exists almost everywhere and it is an immersed submanifold of M. Denote by ν_x^u the Riemannian measure induced on $W^u(x)$. Given a measurable partition ξ, if $\xi(x) \subset W^u(x)$ and $\xi(x)$ contains an open neighborhood of x in $W^u(x)$ for almost every x with respect to the measure μ, then ξ is said to be subordinate to unstable manifolds; further, if μ_x^ξ is absolutely continuous with respect to ν_x^u for μ almost everywhere $x \in M$, then the measure μ is said to have absolutely continuous conditional measures on unstable manifolds, where μ_x^ξ is a canonical system of conditional measures for μ and ξ [22].

Definition 1.1.5. An invariant Borel probability measure μ for the map f on M is said to be an SRB measure if

(a) f has positive Lyapunov exponents almost everywhere with respect to the measure μ;

(b) μ has absolutely continuous conditional measures on unstable manifolds.

Definition 1.1.6. An infinite invariant Borel probability measure μ for the map f on M is said to be an infinite SRB measure if

(i) there is a set E, for any open neighborhood V of the set E, one has $\mu(M \setminus V) < \infty$;

(ii) the first return map (see Definition 3.3.4) defined on the set $M \setminus V$ has positive Lyapunov exponents almost everywhere with respect to μ;

(iii) the measure μ has absolutely continuous conditional measures on unstable manifolds.

1.2 Renewal Theory

In this section, we talk about the application of the renewal theory in dynamical systems, which could be applied to study the systems with the polynomial return time. This method was introduced by Sarig [40], and was extended by Gouëzel [8].

Given a measurable dynamical system $(X, \mathcal{B}, \mu, T)$, a subset $A \in \mathcal{B}$, the induced transformation on A is $(A, \mathcal{B} \cap A, \mu_A, T_A)$, where $\mathcal{B} \cap A = \{B \cap A : \ B \in \mathcal{B}\}$, $\mu_A(E) = \frac{\mu(A \cap E)}{\mu(A)}$, and $T_A(x) = T^{R_A(x)}(x)$, where $R_A(x) := 1_A(x) \inf\{n \geq 1 : \ T^n(x) \in A\}$.

Proposition 1.2.1. [40, Proposition 1] For a conservative non-singular transformation $(X, \mathcal{B}, \mu, T)$, $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$. Set $T_n\phi := 1_A \mathcal{T}^n(\phi 1_A)$ and $R_n\phi = 1_A \mathcal{T}^n(\phi 1_{R_A = n})$. Then, for any $z \in \mathbb{D}$,

$$T(z) = (I - R(z))^{-1},$$

where

$$R(z)=\sum_{n=1}^{\infty} z^n R_n, \quad T(z)=\sum_{n=0}^{\infty} z^n T_n, \quad T_0=I, \quad z\in\overline{\mathbb{D}}.$$

And,

$$T_n=\sum_{k=1}^{n} R_k T_{n-k}=\sum_{k=0}^{n-1} T_k R_{n-k}.$$

Theorem 1.2.1. Let T_n be bounded linear operators on a Banach space $\mathcal{L}$ such that $T(z)=I+\sum_{n\geq 1} z^n T_n$ converges in $\mathrm{Hom}(\mathcal{L},\mathcal{L})$ for evry $z\in\mathbb{D}$,

(1) Rnewal Equation: for every $z\in\mathbb{D}$, $T(z)=(I-R(z))^{-1}$, where $R(z)=\sum_{n\geq 1} z^n R_n\in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ and $\sum\|R_n\|<\infty$.

(2) Spectral Gap: the spectrum of $R(1)$ consists of an isolated simple eigenvalue at 1 and a compact subset of $\mathbb{D}$.

(3) Aperiodicity: the spectral radius of $R(z)$ is strictly less than one for all $z\in\overline{\mathbb{D}}\setminus\{1\}$.

Let P be the eigenprojection of $R(1)$ at 1. If $\sum_{k>n}\|R_k\|=O(1/n^{\beta}$ for some $\beta>2$ and $PR'(1)P\neq 0$, then for all n

$$T_n=\frac{1}{\mu}P+\frac{1}{\mu^2}\sum_{k=n+1}^{\infty} P_k+E_n,$$

where μ is given by $PR'(1)P=\mu P$, $P_n=\sum_{l>n} PR_l P$, and $E_n\in\mathrm{Hom}(\mathcal{L},\mathcal{L})$ satisfy $\|E_n\|=O(1/n^{\lfloor\beta\rfloor})$.

Lemma 1.2.1 (Sarig, 2002; Gouëzel, 2004)**.** Let $(X,\mathcal{B},m,T,\mathcal{F})$ be a topologically mixing probability preserving Markov map, and $\log g_{m_F}$ has a $(T_F,\mathcal{F}_F)$ locally Hölder continuous version for some F, where $g_{m_F}=\frac{dm}{dm\circ T_F}$. Assume that T_F has the big image property, i.e., the measure of the images of the elements of the partition are bounded away from 0 (which is always true when the number of element in F is finite). If $m[R_F > n] = O(1/n^\gamma)$ with $\gamma > 1$, then there are $\kappa \in (0,1)$ and $C > 0$ such that for any $\Phi \in \mathcal{L}$ and $\Psi \in L^\infty$ supported inside F, one has

$$\left|\mathrm{Cor}_n(\Phi, \Psi; T, m) - \left(\sum_{k=n+1}^{\infty} m[R_F > k]\right) \int \Phi \int \Psi\right|$$
$$\leq C F_\gamma(n) \|\Psi\|_\infty \|\Phi\|_{\mathcal{L}},$$

where $F_\gamma(n) = 1/n^\gamma$, $if \gamma > 2$; $F_\gamma(n) = (\log n)/n^2$, if $\gamma = 2$; $F_\gamma(n) = 1/n^{2\gamma-2}$, if $2 > \gamma > 1$.

Chapter 2

Polynomial decay of correlations for almost Anosov diffeomorphisms

2.1 Introduction

The theory of dynamical systems plays an important role in the understanding of physical phenomena, and many systems in physics provide good models of dynamical systems such as the pendulum equation, Billiard systems, Lorentz gas, etc. ([17]). Some interesting physical systems are thought of as dynamical systems, like anomalous transport, fractional kinetics, [20, 48]. Many physical systems exhibit a variety of mixing properties. It is well known that hyperbolicity gives rise to exponential mixing with respect to the physical measures. For systems with slower decay rates, some different physical phenomena could be observed, e.g. sticky domain, intermittency, and so on ([35, 48]). In this work we present a simple model in the categogy of *invertible smooth* dynamical systems in which the systems have intermittent behavior ([34, 35]) and therefore the rates of mixing can be regarded as polynomial.

The systems we consider are C^r, $r \geq 4$, almost Anosov diffeomorphisms f of a two-dimensional manifold M with an indifferent fixed point p at which $Df_p = \text{id}$. We show that under some nondegeneracy conditions, if the coefficients of the third order terms in the Taylor expansions of f at p satisfy certain conditions then f has polynomial decay of correlations, and the degrees of the decay rates are given by the coefficients of the xy^2 and y^3 terms. [1]

Polynomial decay for one-dimensional expanding maps with an indifferent fixed point has been studied extensively (see e.g. [23, 33, 45, 13]). There are some systematic ways developed to obtain polynomial decay rates. The tower structures introduced in [44, 45] are widely used that can apply for both exponential and subexponential decay rates. The renew methods proposed in [40] provide a way to obtain upper and lower bound estimates. For higher-dimensional expanding maps with an indifferent periodic points, upper bounds estimates were made in [33]. Recently both upper and lower bound estimates were obtained in [15] for some non-Morkov maps. Though the methods in both [44] and [40] can be applied to invertible case, there are fewer results in this direction. Liverani and Martens investigated a class of area preserving maps on torus and obtained the upper bounds for the correlation functions [24]. In this work we obtain both upper and lower bound estimates of polynomial decay rates for diffeomorphisms.

Our strategy to prove the results is more or less standard. We first induce two-dimensional almost hyperbolic systems to one-dimensional almost expanding systems by collapsing the stable leaves in a Markov partitions, following the scheme described in [44] in particular. Then we use a corresponding theorem, stated in [40] (and [8] as well), for the induced systems to obtain polynomial decay rates, in which first return maps are used. The last step is to pass the rates we obtained for the induced systems to the original ones.

The most challenging part of the work is to estimate the size of the level sets $[\tau > n]$,

[1] We mention here that in the Taylor expansion, the conditions $Df_p = \mathrm{id}$ means that the linear terms are trivial, and hyperbolicity implies that the second order terms must vanish. So under the nondegeneracy conditions the third order terms determine the ergodic properties of the systems.

where τ is the first return time with respect to the set $M \setminus P$, where P is a rectangle whose interior contains p. Note that restricted to the unstable manifold of the indifferent fixed point p, the map has the form $f(r) \approx r + a_0 r^3$. (See (2.2.2) and (2.2.3) with $x = r$ and $y = 0$.) So if we take any point z in the the local unstable manifold of p, then the backward orbit $f^{-n}(z)$ converges to p at a speed proportional to $n^{-1/2}$, that is unsummable. Fortunately, the size of the level sets $[\tau > n]$ is of order between $n^{-1/\alpha}$ and $n^{-1/\beta}$, where $1/\beta > 1/\alpha > 2$, because the stable foliation is *not* Lipschitz continuous near the indifferent fixed point p! (See (2.2.4) for the value of α and β, and Proposition 2.5.1 for the estimates.) We obtain such estimates by controlling the slopes of the stable leaves at the points close to the local stable manifold of p.

Another problem comes from the last step, when we use the decay rates of the induced systems to obtain the decay rates of the original ones. In this step we need to estimate of the sizes of the rectangles after nth iteration. We use large deviation estimation to get that most rectangles shrink exponentially fast, and prove directly that other rectangles shrink fast enough, and the measure of the union of such rectangles is small.

It is well known that for almost expanding maps of the interval with indifferent fixed point $p = 0$, if $f(x) \approx x + x^{1+s}$, $s \in (0, 1)$, then the rates of decay of correlations are of the order $n^{-(1/s-1)}$. So faster decay rates are given by stronger expansion near the indifferent fixed point (smaller s). In our case, near the fixed point $f(x, y) \approx \big(x(1 + a_2 y^2), y(1 - b_2 y^2)\big)$, and $a_2/2b_2$ plays the role as $1/s$ in one-dimensional systems. The rates of decay are roughly of the order $n^{-(a_2/2b_2-1)}$. This means that the rates of decay for two-dimensional almost hyperbolic systems are determined by the effect of both contraction and expansion when orbits *approach* the indifferent fixed point, and faster decay rates are given by either stronger expansion (larger a_2) or weaker contraction (smaller b_2) or both. [2]

We would like to mention that besides [24], there are also some upper bound estimates for billiards (see [49] and the references therein). Also, the lower bound estimates are announced in [7].

The rest of the chapter is organized as follows. In Section 2.2, we introduce some related definitions and state the Main Theorem. In Section 2.3, we give the proof of the theorem. The proof consists of three major steps, which are carried out in three subsections. In Subsection 2.3.1, we introduce a quotient map by collapsing the map along the stable manifolds. In Subsection 2.3.2, we obtain both the lower and upper polynomial bounds for the induced systems. In Subsection 2.3.3, we obtain the polynomial bounds for Hölder continuous observables for the original systems. Section 2.4 is for distortion estimates, mainly used in Subsection 2.3.1. The size of the level sets are estimated in Section 2.5, where quantitative analysis is performed. And the decay rates of the size of rectangles are estimated in Sections 2.6 and 2.7.

2.2 Statement of results

In this section, some basic concepts and the main results are introduced.

Consider a C^∞ two-dimensional compact Riemannian manifold M without boundary, and the Riemannian measure on M is m. Let $\mathrm{Diff}^4(M)$ be the set of four times differentiable diffeomorphisms.

Definition 2.2.1. [[12] Definition 1] A map $f \in \mathrm{Diff}^4(M)$ is called an almost Anosov diffeomorphism, if there exist two continuous families of cones $x \to \mathcal{C}^u_x, \mathcal{C}^s_x$ such that, except

[2]We refer Remark 2.2.6 for the reasons that a_0 and b_0 are not involved here.

for a finite set S,

(i) $Df_x\mathcal{C}^u_x \subseteq \mathcal{C}^u_{f(x)}$ and $Df_x\mathcal{C}^s_x \supseteq \mathcal{C}^s_{f(x)}$;

(ii) $|Df_xv| > |v|$ for any $v \in \mathcal{C}^u_x$ and $|Df_xv| < |v|$ for any $v \in \mathcal{C}^s_x$.

Since S is a finite set, we only need to consider that S is an invariant set by studying f^n instead of f for some nonnegative integer n. Assume that S consists of a single fixed point p. A fixed point p is called *indifferent* if Df_p has an eigenvalue of modulus 1.

Remark 2.2.1. (i) By Proposition 4.2 in [12], there is an invariant decomposition of the tangent bundle into $TM = E^u \oplus E^s$, the decomposition is continuous except at the indifferent fixed point. By Definition 2.2.1, away from the fixed point angle between E^s and E^u is bounded away from zero.

(ii) It follows from Proposition 4.4 in [12] local unstable manifolds exist for all $x \in M$. Existence of local stable manifolds follows similarly.

Definition 2.2.2. [[12] Definition 2] An almost Anosov diffeomorphism f is said to be non-degenerate (up to the third order), if there exist constants $r_0 > 0$ and $\kappa^u, \kappa^s > 0$ such that for any $x \in B(S, r_0)$,

$$\begin{aligned} |Df_xv| &\geq (1 + \kappa^u d(x,S)^2)|v|, \ \forall v \in \mathcal{C}^u_x; \\ |Df_xv| &\leq (1 - \kappa^s d(x,S)^2)|v|, \ \forall v \in \mathcal{C}^s_x. \end{aligned} \tag{2.2.1}$$

By choosing a suitable coordinate system, there is a neighborhood $B(p, r^*)$ of p such that $p = (0,0)$ and f can be expressed as

$$f(x,y) = \Big(x(1+\phi(x,y)), y(1-\psi(x,y))\Big), \tag{2.2.2}$$

where $(x, y) \in \mathbb{R}^2$ and

$$\begin{aligned}\phi(x,y) &= a_0x^2 + a_1xy + a_2y^2 + O(|(x,y)|^3),\\ \psi(x,y) &= b_0x^2 + b_1xy + b_2y^2 + O(|(x,y)|^3).\end{aligned} \tag{2.2.3}$$

Remark 2.2.2. By (2.2.1), we know that $\phi(x,y), \psi(x,y) > 0$ for any $(x,y) \in B(p,r^*) \setminus \{p\}$. Hence, we have $a_0, a_2, b_0, b_2 > 0$. In this paper, we will consider the case $a_1 = b_1 = 0$.

In Lemma 7.1 of [12], it is in fact proved that if f is an almost Anosov diffeomorphism of a torus $M = \mathbb{T}^2$, then for any neighborhood U of p, there exists $\theta^* \in (0,1)$, such that the unstable subspaces are Hölder continuous with Hölder exponent θ^*.

By applying the renewal theory developed by [40] and [8], we could obtain the following results:

Main Theorem. Let $f \in \mathrm{Diff}^4(M)$ be a topologically mixing almost Anosov diffeomorphism that has an indifferent fixed point p at which (2.2.1)–(2.2.3) are satisfied. Suppose $a_0b_2 - a_2b_0 > 0$, $4b_2 < a_2$, and $a_1 = b_1 = 0$. Fix any $\alpha, \beta \in (0, 1/2)$ with

$$\frac{\alpha}{1+\alpha} < \beta < \frac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \frac{2b_2}{a_2} < \alpha. \tag{2.2.4}$$

Then for any neighborhood U of p, and any Hölder continuous functions Φ, Ψ with the exponent θ, $\operatorname{supp}\Phi, \operatorname{supp}\Psi \subset M \setminus U$, and $\int \Phi d\mu \int \Psi d\mu \neq 0$, we have

$$\frac{A'}{n^{\frac{1}{\beta}-1}} \leq \left|\mathrm{Cor}_n(\Phi, \Psi; f, \mu)\right| \leq \frac{A}{n^{\frac{1}{\alpha}-1}}, \tag{2.2.5}$$

where μ is an SRB measure, $\theta \in (\max\{(1/\beta - 1/\alpha)(3/2 + b_0/(2a_0))^{-1}, \theta^*\}, 1]$, and A' and A are positive constants dependent on Φ and Ψ.

Remark 2.2.3. The condition on topological mixing seems unnecessary. It can be proved that f is topologically conjugate to an Anosov diffeomorphism on the two-dimensional torus. Hence, f is topologically transitive, and M is the only basic set of f. By the spectral decomposition theorem, f is topologically mixing on M. However, since there is no suitable reference, we put this condition in the theorem.

Remark 2.2.4. (i) Since $\alpha < \frac{1}{2}$, the decay rates are faster than n^{-1}.

(ii) In inequalities (2.2.4), we can take $\alpha \gtrsim \frac{2b_2}{a_2}$ and $\beta \lesssim \frac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2}$. Hence $\frac{1}{\beta} - \frac{1}{\alpha} \gtrsim \frac{1}{2} + \frac{\alpha}{4}$, while the first inequalities in (2.2.4) is equivalent to $\frac{1}{\beta} - \frac{1}{\alpha} < 1$. So if $4b_2 < a_2$, then we can always choose α and β satisfying (2.2.4).

Remark 2.2.5. As we see in the above remark, $1/\beta - 1/\alpha \gtrsim 1/2 + \alpha/4$, and $1/2 + \alpha/4 < 1$. Hence, we can take $\theta \leq 1$. In particular, if $2b_2/a_2$ is sufficiently small, then θ can be close to $1/2$.

Remark 2.2.6. To get decay rates of such a system we need to consider a first return map with respect to $M \setminus P$, where P is a rectangle with p in its interior. The decay rates are determined by the size of the level sets $[\tau = n]$, where τ is the first return time. For all large n, the sets are in regions close to the local stable manifold of p. More precisely, if f has the form given by (2.2.2) and (2.2.3) under some coordinate system, then the level sets $[\tau = n]$ are in regions of the form $\{(x, y) : \ 0 < |x| \ll r_1 \leq |y| \leq r_2\}$ for some $0 < r_1 < r_2$. In the regions a_0x^2 and b_0x^2 are much smaller than a_2y^2 and b_2y^2, and hence we have $f(x, y) \approx (x(1 + a_2y^2), y(1 - b_2y^2))$. So the degree of the rates of decay only depends on a_2 and b_2.

2.3 Proof of the main theorem

In this section, we prove the Main Theorem. The proof consists of three steps, and is carried out in three subsections. In the first step, we induce the system (f, M) to one-dimensional expanding system $(\bar{f}, \overline{M})$ with an indifferent fixed point $\bar{p}$ by taking a Markov partition $\mathcal{P}$ and then collapsing the stable manifolds in each element of the partition. In the second step, we apply a result of Sarig [40] to obtain the lower and upper bounds for the decay of correlations for observable functions on the reduced manifold $\overline{M}$, where the key step is to estimate the measure of the level sets $[\tau = n]$ for the first return time function τ. In the last step, we obtain the decay rates for (f, M) by using the estimates for $(\bar{f}, \overline{M})$, where the main ingredient is to estimate the size of the elements of the partition $\vee_{i=-n}^{n} f^i \mathcal{P}$.

2.3.1 Induce to one-dimensional map

Take a finite Markov partition $\mathcal{P} = \{P_0, P_1, \cdots, P_r\}$ such that $p \in \mathrm{int} P_0 \subset U$, where U is given in the Main Theorem. For any P_i and $x \in P_i$, denote by $\gamma^u(x)$ the connected component of unstable leaf containing x in P_i, and by $W^u(P_i)$ the set of all such leaves. And, $\gamma^s(x)$ and $W^s(P_i)$ are understood in a similar way.

Define an equivalent relation on M by $x \sim y$ if x and y are in the same stable leave $\gamma^s \in W^s(P_i)$ for some P_i. Denote by $\bar{x} = \gamma^s(x)$ the equivalent class that contains x. Denote $\overline{M} = M/\sim$. Let $\pi : M \to \overline{M}$ be the natural projection.

Denote by $\overline{\mathcal{B}}$ the completion of the Borel algebra of $\overline{M}$.

Since $\mathcal{P}$ is a Markov partition, $f(\gamma^s(x)) \subset \gamma^s(f(x))$ for any $x \in P_i$ with $f(x) \in P_j$. Hence, the quotient map $\overline{f} : \overline{M} \to \overline{M}$ given by $\overline{f}(\bar{x}) = \overline{f(x)}$ is well defined. Denote $\overline{P}_i = P_i/\sim$ and $\overline{\mathcal{P}} = \{\overline{P}_0, ..., \overline{P}_r\}$. Since $f(\gamma^u(x)) \supset \gamma^u(f(x))$ for any $x \in P_i$ with $f(x) \in P_j$, $\overline{\mathcal{P}}$ is a Markov partition for $\overline{f}$.

Fix an arbitrary $\hat{\gamma}_i^u \in W^u(P_i)$, $0 \le i \le r$. By abuse of notation we also let $\pi : P_i \to \hat{\gamma}_i^u$ be the sliding map along stable leaves such that for any $x \in P_i$, $\pi(x) = \gamma^s(x) \cap \hat{\gamma}_i^u = \hat{x}$, where $\gamma^s(x) \in W^s(P_i)$.

Now, we define a reference measure $\overline{\nu}$ on $\overline{M}$. For each $\gamma \in W^u(P_i)$, denote by m_γ the Lebesgue measure restricted to γ. We introduce the following function

$$u_n(x) := \sum_{i=0}^{n-1} \left(\log |Df_{x_i}|_{E^u_{x_i}}| - \log |Df_{\widehat{x}_i}|_{E^u_{\widehat{x}_i}}|\right),$$

where $x_i = f^i(x)$. By Lemma 2.4.1 in the next section, one has that u_n converges uniformly to some function u. We define ν by $d\nu_\gamma(x) := e^{u(x)} dm_\gamma(x)$. By (1) of Lemma 2.4.3 in the next section, we can define a measure $\overline{\nu}$ on $\overline{M}$ satisfying $\overline{\nu}|_{\overline{P}_i} = \nu_{\hat{\gamma}_i^u}$.

Note that the Jacobian of f with respect to ν is given by

$$J(f)(x) = |D(f)|_{E^u_x}| \cdot e^{u(f(x))} \cdot e^{-u(x)}$$

for ν_γ almost every $x \in M$. By (2) of Lemma 2.4.3, we have that $J(\overline{f})(\bar{x})$ can be defined as $J(f)(y)$ for any $y \in \gamma^s(x)$.

By Theorem B in [12], f has an SRB measure μ under our assumption. And, μ induces an invariant measure $\overline{\mu}$ on $\overline{M}$ in an obvious way. The estimates for bounded distortion given by Proposition 7.5 in [12] imply that the conditional measure is equivalent to the Lebesgue measure, when the measure is restricted to any unstable curve γ^u away from the indifferent fixed point p. Hence, $\overline{\mu}$ is an absolutely continuous invariant measure with respect to $\overline{\nu}$, and is equivalent to $\overline{\nu}$ away from $\bar{p}$.

Now, we obtain a Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ in the following sense (see [1, 40]):

(i) (Generator property) $\overline{\mathcal{B}}$ is complete and is the smallest σ-algebra containing $\cup_{n\geq 0}\overline{f}^{-n}(\overline{\mathcal{P}})$;

(ii) (Markov property) $\overline{\mathcal{P}}$ is a Markov partition, that is, for any $\overline{P}_i, \overline{P}_j \in \overline{\mathcal{P}}$, if $\bar{\mu}(\overline{f}(\overline{P}_i) \cap \overline{P}_j) > 0$, then $\overline{f}(\overline{P}_i) \supset \overline{P}_j$ (mod $\bar{\mu}$);

(iii) (Local invertibility) for any $\overline{P}_i \in \overline{\mathcal{P}}$ with $\bar{\mu}(\overline{P}_i) > 0$, $\overline{f} : \overline{P}_i \to \overline{f}(\overline{P}_i)$ is invertible with measurable inverse.

By the assumption that f is topologically mixing, the Markov map is irreducible.

2.3.2 Polynomial decay rates

Recall that the indifferent fixed point $p \in \operatorname{int} P_0$, and hence, $\bar{p} \in \operatorname{int} \overline{P}_0$. Denote $\widetilde{M} = \overline{M} \setminus \overline{P}_0$.

Take the first return map $\widetilde{f} = f^\tau$ of f with respect to $M \setminus P_0$, that is, $\widetilde{f}(x) = f^{\tau(x)}(x)$, where τ is the first return time, $\tau(x) = \min\{n > 0 : f^n(x) \in M \setminus P_0\}$. Clearly $\widetilde{f} : M \setminus P_0 \to M \setminus P_0$ induces a first return map from $\widetilde{M}$ to itself. For the sake of simplicity of notation we also denote it by $\widetilde{f}$.

Let $\mathfrak{T}' = \{[\tau = n] : n = 1, 2, \dots\}$ be a partition into the level sets. Then let $\mathfrak{T} = \mathfrak{T}' \vee \overline{\mathcal{P}}_0$, where $\overline{\mathcal{P}}_0 = \overline{\mathcal{P}} \setminus \{\overline{P_0}\}$ is the Markov partition of $\widetilde{M}$. It is clear that $\mathfrak{T}$ is a Markov partition of $\widetilde{M}$.

For any point $\bar{x}, \bar{y} \in \widetilde{M}$, the separation time is defined by

$$s(\bar{x}, \bar{y}) := \sup\{n \geq 0 : \ \widetilde{f}^i(\bar{y}) \in \mathfrak{T}(\widetilde{f}^i(\bar{x})), \ 0 \leq i \leq n\}.$$

We may also regard $s(x, y) = s(\bar{x}, \bar{y})$ if $x \in \bar{x}$ and $y \in \bar{y}$.

Let

$$\lambda = \sup\{\|Df_x|_{E^u_x}\|^{-1},\ \|Df_x|_{E^s_x}\| :\ x \in M \setminus P_0\}. \tag{2.3.1}$$

Clearly $\lambda \in (0,1)$. Let $\theta^* \in (0,1)$ as in Lemma 2.4.1, and then take $\theta \in [\theta^*, 1)$.

For any function Φ defined on $\overline{M}$, take a semi-norm by

$$D\Phi := \sup_{\bar{x},\bar{y}\in\widetilde{M}} \frac{|\Phi(\bar{x}) - \Phi(\bar{y})|}{\sqrt{\lambda}^{\,\theta s(\bar{x},\bar{y})}}.$$

Then we consider the Banach space

$$\mathcal{L} := \{\Phi : \operatorname{supp}\Phi \subset \widetilde{M},\ \|\Phi\|_\infty + D\Phi < \infty\}. \tag{2.3.2}$$

and take the norm in $\mathcal{L}$ by $\|\Phi\|_{\mathcal{L}} = \|\Phi\|_\infty + D\Phi$.

It is clear that $\mathcal{L}$ contains Hölder functions with Hölder exponent θ supported on $\widetilde{M}$. If $\Phi \in \mathcal{L}$, then for any $\bar{x}, \bar{y}$ with $s(\bar{x},\bar{y}) \geq n$, we have

$$|\Phi(\bar{x}) - \Phi(\bar{y})| \leq (D\Phi)\lambda^{\theta s(\bar{x},\bar{y})} \leq (D\Phi)(\lambda^\theta)^n \leq (D\Phi)(\sqrt{\lambda}^{\theta})^n.$$

That is, Φ is *locally Hölder continuous* in the sense given in [40] (see also [1]).

By Lemma 3.3.1, we know that $\log J(\widetilde{f}) \in \mathcal{L}$. By standard arguments, it is easy to know (e.g. see Lemma 2 in Subsection 3.1 in [44]) that $\widetilde{f}$ admits an absolutely continuous invariant measure $\widetilde{\mu}$ on $\widetilde{M}$ with the density function $\widetilde{h}$ with respect to $\widetilde{\nu}$, and the density function satisfies $\log \widetilde{h} \in \mathcal{L}$ and is bounded away from 0 and infinity. By uniqueness we know that $\widetilde{\mu}$ is the conditional measure mentioned in the last subsection with respect to $\widetilde{M}$.

The Jacobian of $\widetilde{f}$ with respect to $\widetilde{\mu}$ is given by

$$J_{\widetilde{\mu}}(\widetilde{f}) = J(\widetilde{f})\frac{\widetilde{h}\circ\widetilde{f}}{\widetilde{h}}.$$

Since both $\log J(\widetilde{f})$ and $\log\widetilde{h}$ are in $\mathcal{L}$, so is $-\log J_{\widetilde{\mu}}(\widetilde{f})$. Hence, $-\log J_{\widetilde{\mu}}(\widetilde{f})$ is locally Hölder continuous.

Now we are ready to apply the following theorem that is directly derived from Theorem 2 in [40].

Theorem. Let $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ be an irreducible measure preserving Markov map with $\bar{\mu}(\overline{M}) = 1$, and assume that $-\log|J_{\widetilde{\mu}}(\widetilde{f})|$ has a $(\widetilde{f}, \mathfrak{T})$-locally Hölder continuous version for $\overline{M}$. If g.c.d. $\{\tau(\bar{x}) - \tau(\bar{y}) : \bar{x}, \bar{y} \in \overline{M}\} = 1$, and $\bar{\mu}[\tau > n] = O(1/n^{\varrho})$ with $\varrho > 2$, then there exists $C > 0$ such that for any $\Phi \in \mathcal{L}$ and $\Psi \in L^{\infty}$ with $\operatorname{supp}\Phi, \operatorname{supp}\Psi \subset \widetilde{M}$, one has

$$\left|\mathrm{Cor}_n(\Phi, \Psi; \overline{f}, \bar{\mu}) - \left(\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]\right)\int\Phi\int\Psi\right| \leq CF_{\varrho}(n)\|\Psi\|_{\infty}\|\Phi\|_{\mathcal{L}},$$

where $F_{\varrho}(n) = O(1/n^{\varrho})$.

We have an irreducible measure preserving Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ by the previous subsection. By above arguments we know that $-\log|J_{\widetilde{\mu}}(\widetilde{f})|$ has a $(\widetilde{f}, \mathfrak{T})$-locally Hölder continuous version. It is clear that $\{\tau(\bar{x}) - \tau(\bar{y}) : \bar{x}, \bar{y} \in \overline{M}\} = 1$ by our construction. So, what we need to do is to estimate $\bar{\mu}[\tau > n]$, that is, to estimate the exponent ϱ.

Recall that $P = P_0$ is the element of the Markov Partition $\mathcal{P}$ with $p \in \operatorname{int} P$. Denote $Q = f^{-1}P \setminus P$. Clearly Q is a rectangle and the set of points $x \in M$ with $\tau(x) > 1$, where τ is the first return time given at the beginning of this subsection. Denote $Q_k = [\tau \geq k]$. Clearly $Q = Q_2$ and $Q_{k+1} \subset Q_k$ for any $k \geq 2$. Moreover, Q_k are rectangles such that for any $x \in Q_k$, $W^s(x, Q_k) = W^s(x, Q)$ and $W^u(x, Q_k) \subset W^u(x, Q)$.

For any unstable curve $\gamma^u \in W^u(Q)$, let $\gamma^u_k = \gamma^u \cap Q_k$. By Proposition 2.5.1, we know that there exist $D_\alpha > 0$ and $D_\beta > 0$ such that

$$\frac{D_\beta}{k^{\frac{1}{\beta}}} \le m^u_\gamma(\gamma^u_k) \le \frac{D_\alpha}{k^{\frac{1}{\alpha}}},$$

where α and β are given in the Main Theorem, and m^u_γ is the Lebesgue measure restricted to γ^u.

Denote by μ^u_γ the conditional measure of the SRB measure μ on γ^u. Since the distortion of f along any unstable curve is uniformly bounded above and below away from p (see Lemma 2.4.1, also Proposition 7.5 in [12]), so is the density function $\dfrac{d\mu^u_\gamma}{dm^u_\gamma}$. Hence, there exist $C_\alpha, C_\beta > 0$ such that

$$\frac{C_\beta}{k^{\frac{1}{\beta}}} \le \mu^u_\gamma(\gamma^u_k) \le \frac{C_\alpha}{k^{\frac{1}{\alpha}}}.$$

By integration, we get that similar inequalities are true for $\mu Q_k = \mu[\tau > k]$ with different constant coefficients, that is, there exist two positive constants $B_\alpha, B_\beta > 0$ such that

$$\frac{B_\beta}{k^{\frac{1}{\beta}}} \le \mu(Q_k) \le \frac{B_\alpha}{k^{\frac{1}{\alpha}}}. \tag{2.3.3}$$

It gives that $\displaystyle\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]$ has the order between $n^{-(\frac{1}{\alpha}-1)}$ and $n^{-(\frac{1}{\beta}-1)}$.

By (3.4.5), we can take $\varrho = 1/\alpha$. Since $F_\varrho(n)$ is of order of $n^{-\varrho}$ and $\varrho > \dfrac{1}{\beta} - 1$, we get that there exist $A_\alpha, A_\beta > 0$ such that

$$\frac{A_\beta}{n^{\frac{1}{\beta}-1}} \le \mathrm{Cor}_n(\Phi, \Psi; \overline{f}, \bar{\mu}) \le \frac{A_\alpha}{n^{\frac{1}{\alpha}-1}}. \tag{2.3.4}$$

2.3.3 Polynomial decay rates for diffeomorphisms

In this subsection, we establish polynomial decay of correlations for almost Anosov diffeomorphisms using the results we obtained in the reduced systems.

Recall that $\mathcal{P}$ is a Markov partition, and $P = P_0$ is the element of $\mathcal{P}$ containing p, and $M_0 = M \setminus P_0$.

We introduce a type of Hölder functions:

$$\mathcal{H}_\theta := \big\{\Phi : \ \exists H_\Phi > 0 \ \text{ s.t. } \ |\Phi(x) - \Phi(y)| \le H_\Phi |x-y|^\theta \text{ and } \operatorname{supp}(\Phi) \subset M_0\big\},$$

where $\theta \in (\max\{(1/\beta - 1/\alpha)(3/2 + b_0/(2a_0))^{-1}, \theta^*\}, 1]$, and $\theta^* \in (0,1)$ is specified in Lemma 7.1 of [12], which is dependent on the map f and the element P_0.

Set $\mathcal{P}_0 := \mathcal{P}$ and $\mathcal{P}_{k,n} := \bigvee_{i=k}^{n} f^{-i}(\mathcal{P}_0)$, and $\mathcal{P}_n = \mathcal{P}_{0,n}$.

For any $\Phi, \Psi \in \mathcal{H}_\theta$ and for any $k > 0$, we define $\overline{\Phi}_k$ by $\overline{\Phi}_k | B := \inf\{\Phi(x) : \ x \in f^k(B)\}$ for any $B \in \mathcal{P}_{0,2k}$, and define $\overline{\Psi}_k$ in the same way.

By Lemma 2.3.1 below, the direct calculation gives

$$\begin{aligned} &|\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu)| \\ \le& \Big| \int (\Psi \circ f^k - \overline{\Psi}_k) \circ (f^{n-k}) \cdot \Phi d\mu \Big| + \Big| \int (\Psi \circ f^k - \overline{\Psi}_k) d\mu \cdot \int \Phi d\mu \Big| \\ \le& (2 \max |\Phi|) \int |\Psi \circ f^k - \overline{\Psi}_k| d\mu \le (2 \max |\Phi|) \cdot \frac{C_A H_\Psi}{k^{\beta^*}}, \end{aligned} \tag{2.3.5}$$

where β^* is specified in Lemma 2.3.1.

For $\overline{\Phi}_k$ defined as above, let $\overline{\Phi}_k \mu$ be the signed measure whose density with respect to μ is $\overline{\Phi}_k$, and set $\Phi_k := \dfrac{d((f^k)_*(\overline{\Phi}_k \mu))}{d\mu}$.

Let $|\cdot|$ be the total variation of a signed measure, and note that $(f^k)_*(\Phi \circ (f^k)\mu) = \Phi\mu$, where $|\mu|(A) = \int_A d|\mu|$ for any Borel set $A \subset M$. Applying Lemma 2.3.1 for Φ we can get

$$\begin{aligned}
&\int |\Phi - \Phi_k| d\mu = |\Phi\mu - \Phi_k\mu|(M) = |(f^k)_*((\Phi \circ (f^k)\mu) - (f^k)_*(\overline{\Phi}_k\mu)|(M) \\
\leq &|(\Phi \circ f^k - \overline{\Phi}_k)\mu|(M) = \int |\Phi \circ f^k - \overline{\Phi}_k| d\mu \leq \frac{C_A H_\Phi}{k^{\beta^*}}.
\end{aligned}$$

Hence, by similar computation as previously, we have

$$\begin{aligned}
&\Big|\mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)\Big| \\
\leq &\Big|\int (\overline{\Psi}_k \circ (f^{n-k}))(\Phi - \Phi_k) d\mu\Big| + \Big|\int \overline{\Psi}_k d\mu \cdot \int (\Phi - \Phi_k) d\mu\Big| \\
\leq &(2 \max|\Psi|) \int |\Phi - \Phi_k| d\mu \leq (2 \max|\Psi|) \frac{C_A H_\Phi}{k^{\beta^*}}.
\end{aligned} \tag{2.3.6}$$

Now we show that $\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)$ can be expressed as functions only dependent on the unstable manifolds, which means that these functions are constant along stable manifolds on each element of P_i. Since $\overline{\Psi}_k$ is constant along stable manifolds on each rectangle $P_i \in \mathcal{P}$, we can regard it as a function on $\overline{M}$ as well. Also we have $\pi_*(\overline{\Phi}_k\mu) = \overline{\Phi}_k(\pi_*\mu) = \overline{\Phi}_k(\bar{\mu})$, and $\overline{f} \circ \pi = \pi \circ f$. So,

$$\begin{aligned}
&\int (\overline{\Psi}_k \circ (f^{n-k}))\Phi_k d\mu = \int (\overline{\Psi}_k \circ (f^{n-k})) d((f^k)_*(\overline{\Phi}_k\mu)) \\
= &\int \overline{\Psi}_k d((f^{n-k})_*(f^k)_*(\overline{\Phi}_k\mu)) = \int \overline{\Psi}_k d((f^n)_*(\overline{\Phi}_k\mu)) \\
= &\int \overline{\Psi}_k d(\pi_*(f^n)_*(\overline{\Phi}_k\mu)) = \int \overline{\Psi}_k d((\overline{f}^n)_*(\overline{\Phi}_k\bar{\mu})) = \int \overline{\Psi}_k \circ \overline{f}^n \cdot \overline{\Phi}_k d\bar{\mu},
\end{aligned}$$

and,

$$\int \Phi_k d\mu \int \overline{\Psi}_k d\mu = \int d((f^k)_*(\overline{\Phi}_k\mu)) \cdot \int \overline{\Psi}_k d\bar{\mu} = \int \overline{\Phi}_k d\bar{\mu} \cdot \int \overline{\Psi}_k d\bar{\mu}.$$

It means $|\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| = |\mathrm{Cor}_{n-k}(\overline{\Phi}_k, \overline{\Psi}_k; \overline{f}, \bar{\mu})|$. Hence, by (3.4.9) and (3.4.10), we have

$$
\begin{aligned}
&|\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| = |\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu)| \\
\leq &|\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu)| \\
+ &|\mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| + |\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| \\
= &(2 \max |\Phi|) \cdot \frac{C_A H_\Psi}{k^{\beta^*}} + (2 \max |\Psi|) \cdot \frac{C_A H_\Phi}{k^{\beta^*}} + |\mathrm{Cor}_{n-k}(\overline{\Phi}_k, \overline{\Psi}_k; \overline{f}, \bar{\mu})|.
\end{aligned}
$$

Take $k = [n/2]$. Since $\beta^* > \frac{1}{\beta} - 1$, by (2.3.4), we obtain that there exist $A > 2^{1/\alpha - 1} A_\alpha$ and $A' < 2^{1/\beta - 1} A_\beta$ such that

$$
\frac{A'}{n^{\frac{1}{\beta} - 1}} \leq |\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| \leq \frac{A}{n^{\frac{1}{\alpha} - 1}}.
$$

This completes the whole proof of the Main Theorem.

Lemma 2.3.1. Given any $\theta \in (\max\{(1/\beta - 1/\alpha)(3/2 + b_0/(2a_0))^{-1}, \theta^*\}, 1]$, there exist $C_A > 0$, $K > 0$ and $\beta^* = \beta^*(\theta) > \frac{1}{\beta} - 1$ such that for any $\Psi \in \mathcal{H}^\theta$ and $k \geq K$,

$$
\int \left| \Psi \circ f^k - \overline{\Psi}_k \right| d\mu \leq \frac{C_A H_\Psi}{k^{\beta^*}}.
$$

Proof. Recall that by the definition, $\overline{\Psi}_k | B := \inf\{\Psi(x) : \ x \in f^k(B)\}$, where $B \in \mathcal{P}_{0,2k}$. So for any x, there is $y \in \mathcal{P}_{0,2k}(x)$ such that $\Psi \circ f^k(x) - \overline{\Psi}_k(x) = \Psi \circ f^k(x) - \Psi \circ f^k(y)$. Since $\Psi \in \mathcal{H}_\theta$ and $f^k(\mathcal{P}_{0,2k}(x)) = \mathcal{P}_{-k,k}(f^k(x))$, we have that for $x \in B$ with $B \in \mathcal{P}_{0,2k}$,

$$
\begin{aligned}
&|\Psi \circ f^k(x) - \overline{\Psi}_k(x)| = |\Psi \circ f^k(x) - \Psi \circ f^k(y)| \\
\leq &H_\Psi |f^k(x) - f^k(y)|^\theta \leq H_\Psi \mathrm{diam}(f^k(B))^\theta = H_\Psi \mathrm{diam}(\mathcal{P}_{-k,k}(f^k(x)))^\theta.
\end{aligned}
$$

It means

$$|\Psi(x) - \overline{\Psi}_k(f^{-k}(x))| \le H_\Psi \mathrm{diam}(\mathcal{P}_{-k,k}(x))^\theta. \tag{2.3.7}$$

Hence, we need to estimate the diameter of the sets in $\mathcal{P}_{-k,k}$.

Let $\delta \in (0, \delta_0)$, where δ_0 is given in Proposition 2.7.1. Let

$$S_k = \{x \in M \setminus P : \mathrm{diam}(\mathcal{P}_{-k,k}(x)) \ge e^{-k\delta}\}.$$

By Remark 2.2.1, there is a uniform lower bound for the angle between E^u_x and E^s_x for all $x \in M \backslash P$. Hence, there exist $C_\ell > 0$ such that for any $x \in S_k$, either there exists an unstable manifold $\gamma^u_k(x^u) \subset \mathcal{P}_{-k,k}(x)$ with the length larger than $C_\ell e^{-k\delta}$, where $x^u \in \mathcal{P}_{-k,k}(x)$, or there exists a stable manifold $\gamma^s_k(x^s) \subset \mathcal{P}_{-k,k}(x)$ with the length larger than $C_\ell e^{-k\delta}$, where $x^s \in \mathcal{P}_{-k,k}(x)$.

In the former case, by the fact $f^k(\gamma^u_k(x^u)) = \gamma^s_0(f^k(x^u))$, there is $C_d > 0$ and $y^u \in \gamma^u_k(x^u)$ such that $|Df^k_{y^u}|_{E^u_{y^u}}| < C_d e^{k\delta}$. Hence, by distortion given in Lemma 2.4.1, for any $y \in \gamma^u_k(x^u)$, $|Df^k_y|_{E^u_y}| < C_d J_u e^{k\delta}$, and then for any $z \in \gamma^s_k(y)$, $y \in \gamma^u_k(x^u)$, $|Df^k_z|_{E^s_z}| < C_d J_u J_s e^{k\delta}$, that is, the inequality holds for all $z \in \mathcal{P}_{-k,k}(x)$. In particular, we have $|Df^k_x|_{E^s_x}| < C_d J_u J_s e^{k\delta}$. Similarly, in the latter case, we can get that $|Df^{-k}_x|_{E^s_x}| < C'_d J'_s J'_u e^{k\delta}$ for some $C'_D > 0$, where J'_s and J'_u are given in Lemma 2.4.2. So we can get

$$S_k \subset \Big\{x \in M : \ |Df^k_x|_{E^u_x}| < Ee^{k\delta}\Big\} \bigcup \Big\{x \in M : \ |Df^{-k}_x|_{E^s_x}| < E'e^{k\delta}\Big\},$$

where $E = C_d J_u J_s$ and $E' = C'_d J'_u J'_s$. By applying Proposition 2.7.1, we get that there exist $C_D, C'_D > 0$ such that

$$\mu(S_k) \le \frac{C^*_D (\log k)^{2(1/\alpha - 1)}}{k^{1/\alpha - 1}},$$

where $C_D^* = C_D + C_D'$,

Let T_k be given in Proposition 2.6.1. By this proposition, $\mu(T_k) \le \dfrac{C_s \log k}{k^{1/\alpha}}$. For any $x \in T_k$, by Propositions 2.6.1 and 2.6.2, $\operatorname{diam}(\mathcal{P}_{-k,k}(x)) \le \dfrac{C_h}{k^{1/2+\alpha'}}$, where $\alpha' = \dfrac{b_0}{2a_0}$, and C_h is a constant larger than the constants C_s and C_u given by Proposition 2.6.1 and 2.6.2.

For any $x \notin T_k$, $\operatorname{diam}(\mathcal{P}_{-k,k}(x)) \le \dfrac{C_s}{k^{3/2+\alpha'}}$ by Proposition 2.6.1.

Hence, by invariance of μ and (2.3.7), the above estimates give

$$\begin{aligned}
&\int \left|\Psi \circ f^k - \overline{\Psi}_k\right| d\mu = \int \left|\Psi - \overline{\Psi}_k \circ f^{-k}\right| d\mu \\
=&\int_{T_k^c \cap S_k^c} |\Psi \circ f^k - \overline{\Psi}_k| d\mu + \int_{T_k^c \cap S_k} |\Psi \circ f^k - \overline{\Psi}_k| d\mu + \int_{T_k} |\Psi \circ f^k - \overline{\Psi}_k| d\mu \\
\le& H_\Psi e^{-k\delta\theta} + \frac{H_\Psi C_s^\theta}{k^{(3/2+\alpha')\theta}} \cdot \frac{C_D^* (\log k)^{2(1/\alpha - 1)}}{k^{1/\alpha - 1}} + \frac{H_\Psi C_s^\theta}{k^{(1/2+\alpha')\theta}} \cdot \frac{C_s \log k}{k^{1/\alpha}} \le \frac{C_A H_\Psi}{k^{\beta^*}}
\end{aligned}$$

for some $C_A > 0$ independent of Ψ, where $\beta^* > \left(\frac{3}{2} + \alpha'\right)\theta + \frac{1}{\alpha} - 1$. By the choice of θ, we have that $\beta^* > \dfrac{1}{\beta} - 1$. $\square$

2.4 Some distortion estimates

In this section we provide some distortion estimates which were used in Subsection 2.3.1 and will be used in Section 2.6 as well.

Lemma 2.4.1. There are positive constants $J_s, J_u > 0$, and $\theta^* \in (0, 1]$ such that for any $\gamma^s \in W^s(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^s$ and $n \ge 0$,

$$\log \frac{|Df_y^n|_{E_y^u}|}{|Df_x^n|_{E_x^u}|} \le J_s d^s(x, y)^{\theta^*}; \tag{2.4.1}$$

and for any $\gamma^u \in W^u(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^u$ and $n \geq 0$,

$$\log \frac{|Df_y^{-n}|_{E_y^u}|}{|Df_x^{-n}|_{E_x^u}|} \leq J_u d^u(x, y)^{\theta^*}. \tag{2.4.2}$$

Proof. Denote $P = P_0$. By the same method as in the proof of Lemma 7.4 in [12], we can get that there exists constant $I_s > 0$ such that if $\gamma^s \subset f^{-1}P \setminus P$ is a W^s-segment with $f^i \gamma^s \subset P$, $i = 1, \cdots n-1$, then for any $x, y \in \gamma^s$,

$$\log \frac{|Df_y^n|_{E_y^u}|}{|Df_x^n|_{E_x^u}|} \leq I_s d^u(x, y)^{\theta^*},$$

where $\theta^* = \theta$ is given in Lemma 7.1 of [12].

With this result we can get a proof of (2.4.1) using the same idea as in the proof of Proposition 7.5 in [12], whose details can be found in Proposition 3.1 in [16].

The second inequality (2.4.2) can be obtained similarly. $\square$

Similarly, we have the following result:

Lemma 2.4.2. There are two positive constants J'_s and J'_u, and $\theta^* \in (0, 1]$ such that for any $\gamma^s \in W^s(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^s$ and $n \geq 0$,

$$\log \frac{|Df_y^n|_{E_y^s}|}{|Df_x^n|_{E_x^s}|} \leq J'_s d^s(x, y)^{\theta^*};$$

and for any $\gamma^u \in W^u(P_i)$, $i = 1, \cdots, r$, $x, y \in \gamma^u$ and $n \geq 0$,

$$\log \frac{|Df_y^{-n}|_{E_y^s}|}{|Df_x^{-n}|_{E_x^s}|} \leq J'_u d^u(x, y)^{\theta^*}.$$

Lemma 2.4.3. (1) Let $\gamma^u, \hat{\gamma}_i^u \in W^u(P_i)$. For the sliding map $\pi : \gamma^u \to \hat{\gamma}_i^u$, one has that

$\pi_* \mu_\gamma = \mu_{\hat{\gamma}_i^u}$.

(2) $J(\overline{f})(x) = J(\overline{f})(y)$ for any $y \in \gamma^s(x)$.

Proof. The statements and proof are the same as (1) and (2) of Lemma 1 in Subsection 3.1 in [44]. □

Lemma 2.4.4. There are $C > 0$, $\lambda \in (0,1)$, and $\theta^* \in (0,1)$ such that for any $\gamma^u \in W^u(P_i)$, $i = 1, ..., r$, $x, y \in \gamma^u$,

$$\log \left| \frac{J(\widetilde{f})(x)}{J(\widetilde{f})(y)} \right| \leq C\sqrt{\lambda}^{\theta^* s(x,y)},$$

where $s(x,y)$ is given in Subsection 2.3.2.

Proof. For any $x \in \gamma^u \cap P_i$, $i \neq 0$, one has

$$J(\widetilde{f})(x) = |D\widetilde{f}_x|_{E_x^u}| \cdot e^{u(\widetilde{f}(x))} \cdot e^{-u(x)}.$$

Denote $\phi(x) = \log |D\widetilde{f}_x|_{E_x^u}|$. We can write

$$\begin{aligned}|u(x) - u(y)| \leq & \left| \sum_{i=0}^{k} [\phi(\widetilde{f}^i(x)) - \phi(\widetilde{f}^i(y))] \right| + \left| \sum_{i=0}^{k} [\phi(\widetilde{f}^i(\widehat{x})) - \phi(\widetilde{f}^i(\widehat{y}))] \right| \\ & + \left| \sum_{i=k+1}^{\infty} [\phi(\widetilde{f}^i(x)) - \phi(\widetilde{f}^i(\widehat{x}))] \right| + \left| \sum_{i=k+1}^{\infty} [\phi(\widetilde{f}^i(y)) - \phi(\widetilde{f}^i(\widehat{y}))] \right|.\end{aligned}$$

We take $k > 0$ such that $f^k = \widetilde{f}^{s^*(x,y)/2}$, where $s^*(x,y) = s(x,y)$ if $s(x,y)$ is even and $s^*(x,y) = s(x,y) + 1$, otherwise. Hence, $f^k(x), f^k(\hat{x}), f^k(y), f^k(\hat{y}) \notin P$, and (2.4.1) and

(2.4.2) can be applied to the sums of the right hand side. So, we can get

$$
\begin{aligned}
|u(x)-u(y)| \leq & J_u d^u(f^k(x), f^k(y))^{\theta^*} + J_u d^u(f^k(\hat{x}), f^k(\hat{y}))^{\theta^*} \\
& + J_s d^s(f^k(x), f^k(\hat{x}))^{\theta^*} + J_s d^s(f^k(y), f^k(\hat{y}))^{\theta^*}.
\end{aligned}
$$

Recall that λ is defined in (3.4.2). We can get that

$$
\begin{aligned}
& d^u(f^k(x), f^k(y))^{\theta^*} \\
= & d^u(\widetilde{f}^{s(x,y)}(x), \widetilde{f}^{s(x,y)}(y))^{\theta^*} \cdot \frac{d^u(\widetilde{f}^{s(x,y)^*/2}(x), \widetilde{f}^{s(x,y)^*/2}(y))^{\theta^*}}{d^u(\widetilde{f}^{s(x,y)}(x), \widetilde{f}^{s(x,y)}(y))^{\theta^*}} \leq C_d \lambda^{\theta^* s(x,y)/2},
\end{aligned}
$$

where C_d is determined by the maximum radius of each element in the Markov partition, we use the fact that $\widetilde{f}^{s(x,y)}(x)$ and $\widetilde{f}^{s(x,y)}(y)$ are in the same element of the Markov partition $\mathcal{P}$, and hence, $d^u(\widetilde{f}^{s(x,y)}(x), \widetilde{f}^{s(x,y)}(y))^{\theta^*}$ is uniformly bounded. Similarly, we have $d^u(f^k(\hat{x}), f^k(\hat{y}))^{\theta^*}$, $J_u d^s(f^k(x), f^k(\hat{x}))^{\theta^*}$, $J_u d^s(f^k(y), f^k(\hat{y}))^{\theta^*} \leq C' \lambda^{\theta^* s(x,y)/2}$, where C' is a positive constant. Hence,

$$
|u(x)-u(y)| \leq 4C\lambda^{\theta^* s(x,y)/2},
$$

where C is a positive constant.

Since $\log|D\widetilde{f}_x|_{E_x^u}| - \log|D\widetilde{f}_y|_{E_y^u}|$ and $u(\widetilde{f}(x)) - u(\widetilde{f}(y))$ can be estimated in a similar way, we get the inequality we need.

This competes the proof. $\square$

2.5 Rates of convergence of the level sets

In this section, we prove Proposition 2.5.1 that is the key step to estimate the term $\mu[\tau > n]$.

Recall that $Q = Q_2 = f^{-1}P \setminus P$, and $Q_i = [\tau \geq i]$ for $i \geq 2$.

Note that the map f has a local product structure, that is, there exist positive constants ϵ and δ such that for any $x, y \in M$ with $d(x, y) \leq \delta$, $[x, y] := W^u_\epsilon(x) \cap W^s_\epsilon(y)$ contains exactly one point.

Take a coordinate system in a neighborhood U^* of p such that the map has the form given in (2.2.2) and (2.2.3). Hence, the y-axis and x-axis are the stable and unstable manifold of p, respectively. Recall that we assume $a_1 = 0 = b_1$.

Let $r > 0$ be small such that the ball centered at p of radius r is contained in U^*. We also assume that $P = P_0$ is small enough such that P, $f(P)$, and $f^{-1}(P)$ are contained in the ball.

Proposition 2.5.1. Suppose $\alpha, \beta \in (0, 1)$ satisfies $\beta < \dfrac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \dfrac{2b_2}{a_2} < \alpha$. Then there exist D_α, $D_\beta > 0$ such that for any unstable curve $\gamma^u \in W^u(Q)$, for any $k > 0$, we have

$$\frac{D_\beta}{k^{\frac{1}{\beta}}} \leq m^u_\gamma(\gamma^u_k) \leq \frac{D_\alpha}{k^{\frac{1}{\alpha}}}.$$

where $\gamma^u_k = \gamma^u \cap Q_k$ and m^u_γ is the Lebesgue measure restricted to γ^u.

Proof. Let $\gamma^u \in W^u(Q)$ be an unstable curve in Q. Denote $q = \gamma^u \cap W^s_\varepsilon(p)$.

For any $z = (x, y) \in \gamma^u$, denote $z_1 = (x_1, y_1) = f(z)$, and $\bar{z} = (\bar{x}, \bar{y}) = [z, fz] = W^u(z) \cap W^s(fz)$. Since both z_1 and $\bar{z}$ are in the same stable curve, $z \in Q_k$ if and only if $\bar{z} \in Q_{k-1}$. So if z is an endpoint of γ^u_k, then $\bar{z}$ is an endpoint of γ^u_{k-1}. In order to estimate the length of γ^u_k, we estimate the ratio $m^u_\gamma(\gamma^u_{k-1})/m^u_\gamma(\gamma^u_k)$ firstly. This is equivalent to estimate $\bar{x}/x$.

Denote by v_z^s a real number or ∞ such that $(v_z^s, 1)$ is a tangent vector of $W_r^s(z)$. Take the function $\hat{\rho}$ on $[0, r]$ as in Proposition 2.5.2. By Lemmas 2.5.2 and 2.5.4 below, we know that if $z = z_0$ is sufficiently close to q, then

$$-\Big(\frac{a_2}{b_2} + \hat{\rho}(y_0)\Big)(1 - x_0^{\alpha})\frac{x_0}{y_0} \leq v_{z_0}^s \leq -\Big(\frac{a_2}{b_2} + \hat{\rho}(y_0)\Big)(1 - x_0^{\beta})\frac{x_0}{y_0}.$$

With the estimates for v_z^s, we can get by Lemmas 2.5.3 and 2.5.5 that there exist $E_\alpha, E_\beta > 0$ such that

$$x_0 + E_\alpha x_0^{1+\alpha} \leq \bar{x}_0 \leq x_0 + E_\beta x_0^{1+\alpha}.$$

If we denote $s_k = m^u(\gamma_k^u)$, the inequalities mean

$$s_k + E_\alpha s_k^{1+\alpha} \leq s_{k-1} \leq s_k + E_\beta s_k^{1+\alpha}.$$

for all k sufficiently large. Hence, it follows (e.g. see Lemma 3.1 in [14]) that there exist D_α, $D_\beta > 0$ such that for all $k > 0$,

$$\frac{D_\beta}{k^{\frac{1}{\beta}}} \leq s_k \leq \frac{D_\alpha}{k^{\frac{1}{\alpha}}}.$$

This is what we need. $\square$

To obtain Lemmas 2.5.2 and 2.5.4, we consider v_z^s, where z is near the y-axis. Assume that v_z^s has the form

$$v_z^s = -\rho\frac{x}{y},$$

where $\rho = \rho(x, y)$.

Since $(v_z^s, 1)$ is in the stable cone at z, without loss of generality, assume that

$$-1 \le v_z^s \le 1, \quad \forall z \in B(p, r). \tag{2.5.1}$$

Let ρ ba a function defined on U^*. Set $z_1 := f(z)$ and $\rho_1 := \rho(z_1)$. Define

$$\begin{aligned}\Delta_\rho(x, y) :&= (\rho - \rho_1)(1 + \phi)(1 - \psi) + \rho_1 y(1 + \phi)\psi_y - y(1 - \psi)\phi_y \\ &- \rho_1 \rho x(1 + \phi)\psi_x + \rho x(1 - \psi)\phi_x,\end{aligned}$$

where $\phi = \phi(x, y)$ and $\psi = \psi(x, y)$. We need the following facts.

Lemma 2.5.1 ([12] Lemma 8.3)**.** If $v_z^s \le -\rho(z)\frac{x}{y}$ and $0 \le \Delta_\rho(x, y)$, then $v_{z_1}^s \le -\rho(z_1)\frac{x_1}{y_1}$. The result also holds if all " $\leqslant$ " are replaced by " $\geqslant$ ".

To get more precise form of ρ, we need the following results.

Proposition 2.5.2 ([12] Proposition 8.4)**.** There exists a Lipschitz function $\hat\rho$ on $[0, r]$ with $\hat\rho(0) = 0$ satisfying the following two equations:

$$\begin{aligned}\Delta_{\frac{a_2}{b_2}+\hat\rho}(0, y) &= (\hat\rho(y) - \hat\rho(y_1^{(0)}))(1 + \phi)(1 - \psi) \\ &+ \left(\frac{a_2}{b_2} + \hat\rho(y_1^{(0)})\right) y(1 + \phi)\psi_y - y(1 - \psi)\phi_y = 0,\end{aligned}$$

and

$$b_2 \log(1 + \phi) + a_2 \log(1 - \psi) - b_2 \int_{y_1^{(0)}}^{y} \frac{\hat\rho(t)}{t} dt = 0, \tag{2.5.2}$$

where $\phi = \phi(0, y)$, $\psi = \psi(0, y)$, and $y_1^{(0)} = y(1 - \psi(0, y))$.

The upper bound estimates have been proved in [12]. We state the corresponding lemmas here for completion, which are Lemmas 9.1 and 9.2 in [12]

Lemma 2.5.2. Suppose $\alpha a_2 > 2b_2$, $0 < \alpha < 1$, and $a_0 b_2 - a_2 b_0 > 0$. Then for any point $q = (0, y_q)$ with $y_q > 0$ small, there exists $\epsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W^u_\epsilon(q)$ with $x_0 > 0$,

$$v^s_{z_0} \geq -\left(\frac{a_2}{b_2} + \hat{\rho}(y_0)\right)(1 - x_0^\alpha)\frac{x_0}{y_0}.$$

Lemma 2.5.3. Let $z_0 = (x_0, y_0)$ with $x_0 > 0$. If for all $z = (x, y)$ in the stable curve that joins $\bar{z}_0$ and z_1,

$$v^s_z \geq -\left(\frac{a_2}{b_2} + \hat{\rho}(y)\right)(1 - x^\alpha)\frac{x}{y},$$

then

$$\bar{x}_0 \geq x_0 + E_\alpha x_0^{1+\alpha},$$

where E_α is a positive constant dependent on y_0.

The following lemma is the key step to get the lower bound estimates for $\bar{x}_0/x_0$.

Lemma 2.5.4. Given any $\alpha, \beta \in (0, 1)$ with $\beta < \dfrac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \dfrac{2b_2}{a_2} < \alpha$. Then for any point $q = (0, y_q)$ with $y_q > 0$ small, there exists $\varepsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W^u_\varepsilon(q)$ with $x_0 > 0$ small,

$$v^s_{z_0} \leq -\left(\frac{a_2}{b_2} + \hat{\rho}(y_0)\right)(1 - x_0^\beta)\frac{x_0}{y_0}. \tag{2.5.3}$$

Proof. For each $z_0 = (x_0, y_0) \in W^u_r(q)$, $z_i = (x_i, y_i) = f^i(z_0)$, define

$$c_0 := 0, \quad c_i := \frac{A_1 x_0^\beta y_0^2}{\prod_{j=0}^{i-1}\Big(1 - \theta_0 y_j \psi_y(0, y_j)\Big)} \qquad \forall i \geq 1,$$

where $A_1 = \dfrac{a_2}{2b_2}(2b_2 - \beta a_2)$ and θ_0 is specified in Lemma 2.5.6. It is evident that

$$c_{i+1} - c_i = c_{i+1}\theta_0 y_i \psi_y(0, y_i), \qquad \forall i > 0. \tag{2.5.4}$$

Set

$$\rho_i := \rho(z_i) = \left(\frac{a_2}{b_2} + \hat{\rho}(y_i)\right)(1 - x_i^{\beta}),\ i \geq 0, \tag{2.5.5}$$

and

$$\tilde{\rho}_i := \rho_i - c_i,\ i \geq 0. \tag{2.5.6}$$

For any $z_i = (x_i, y_i)$, set

$$\begin{aligned}\Delta_{\tilde{\rho}_i}(x_i, y_i) :=& (\tilde{\rho}_i - \tilde{\rho}_{i+1})(1 + \phi_i)(1 - \psi_i)\\ &+ \tilde{\rho}_{i+1} y_i (1 + \phi_i)\psi_y(x_i, y_i) - y_i(1 - \psi_i)\phi_y(x_i, y_i)\\ &- \tilde{\rho}_i \tilde{\rho}_{i+1} x_i (1 + \phi_i)\psi_x(x_i, y_i) + \tilde{\rho}_i x_i (1 - \psi_i)\phi_x(x_i, y_i),\end{aligned}$$

where $\phi_i = \phi(z_i) = \phi(x_i, y_i)$, $\psi_i = \psi(z_i) = \psi(x_i, y_i)$.

By contradiction, suppose that (2.5.3) is incorrect. It is to show that for $y_q > 0$ small enough, there is $\varepsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W^u_\varepsilon(q)$ with $q = (0, y_q)$, $x_0, y_0 > 0$,

$$v^s_{z_i} \geq -\tilde{\rho}_i \frac{x_i}{y_i} \qquad \text{and} \qquad 0 \geq \Delta_{\tilde{\rho}_i}(x_i, y_i),$$

this, together with Lemma 2.5.1, yields that

$$v^s_{z_{i+1}} \geq -\tilde{\rho}_{i+1} \frac{x_{i+1}}{y_{i+1}}.$$

By Lemma 2.5.6 below, we can take $\varepsilon > 0$ small enough such that $c_{n_0} > 1 + \max\{a_2/b_2 + \hat{\rho}(y_i) : y \in [0, r]\}$ and hence, $\tilde{\rho}_{n_0} < -1$ for some $n_0 = n(z_0)$. Since c_i increases with i, it follows that $\tilde{\rho}_i < -1$ for any $i \geq n_0$. Note that x_i is increasing and y_i is decreasing when the orbit under the iteration of f is in the neighborhood of the origin. Then there exists

$n_1 \geq n_0$ such that $v^s_{z_{n_1}} > -\tilde{\rho}_{n_1} \frac{x_{n_1}}{y_{n_1}} > 1$. This contradicts (2.5.1).

Now, we will show that for all $i \geq 0$ with $x_i < y_i$,

$$\Delta_{\tilde{\rho}_i}(x_i, y_i) \leq 0.$$

Note that by (2.2.2) and (2.2.3)

$$\phi(x,y) = \phi(0,y) + O(x^2 + xy^2), \quad \psi(x,y) = \psi(0,y) + O(x^2 + xy^2), \tag{2.5.7}$$

$$\phi_y(x,y) = \phi_y(0,y) + O(x^2 + xy), \quad \psi_y(x,y) = \psi_y(0,y) + O(x^2 + xy). \tag{2.5.8}$$

Also,

$$\phi(x,y) = a_2 y^2 + O(x^2 + xy^2 + y^3) = a_2 y^2 + O(x^2 + y^3), \tag{2.5.9}$$

$$y\psi_y(x,y) = 2b_2 y^2 + O(x^2 y + xy^2 + y^3) = 2b_2 y^2 + O(x^2 y + y^3), \tag{2.5.10}$$

$$x\phi_x(x,y), x\psi_x(x,y) = O(x^2 + x^2 y + xy^2) = O(x^2 + xy^2). \tag{2.5.11}$$

Since $y_i - y_{i+1} = y_i - y_i(1 - \psi(x_i, y_i)) = y_i \psi(x_i, y_i)$, and $\hat{\rho}$ is Lipschitz continuous,

$$\hat{\rho}(y_i) - \hat{\rho}(y_{i+1}) = O(y_i - y_{i+1}) = O(y_i \psi(x_i, y_i)) = O(y_i x_i^2 + y_i^3). \tag{2.5.12}$$

Denote $y^{(0)}_{i+1} := y_i(1 - \psi(0, y_i))$. Then $y_{i+1} - y^{(0)}_{i+1} = O(y_i(\psi(0, y_i) - \psi(x_i, y_i)))$ and hence, by (2.5.8),

$$\hat{\rho}(y_{i+1}) - \hat{\rho}(y^{(0)}_{i+1}) = O(y_{i+1} - y^{(0)}_{i+1}) = O(x_i^2 y_i + x_i y_i^2). \tag{2.5.13}$$

Note $(1+a)^\beta - 1 = \beta a + O(a^2)$. By (2.5.9), we have

$$x_{i+1}^\beta - x_i^\beta = x_i^\beta((1+\phi(x_i,y_i))^\beta - 1) = \beta a_2 x_i^\beta y_i^2 + x_i^\beta O(x_i^2 + y_i^3). \tag{2.5.14}$$

First, using (2.5.4), (2.5.7), (2.5.12), (2.5.13), and (2.5.14), we get

$$\begin{aligned}
&(\tilde\rho_i - \tilde\rho_{i+1})(1+\phi_i)(1-\psi_i)\\
=&\Big(\hat\rho(y_i) - \hat\rho(y_{i+1})\Big)(1+\phi_i)(1-\psi_i)\\
&+\Big(\frac{a_2}{b_2}(x_{i+1}^\beta - x_i^\beta) + \big(\hat\rho(y_{i+1})x_{i+1}^\beta - \hat\rho(y_i)x_i^\beta\big)\Big)(1+\phi_i)(1-\psi_i)\\
&+(c_{i+1}-c_i)(1+\phi_i)(1-\psi_i)\\
=&\Big(\hat\rho(y_i) - \hat\rho(y_{i+1}^{(0)})\Big)(1+\phi(0,y_i))(1-\psi(0,y_i))\\
&+\frac{a_2}{b_2}\beta a_2 x_i^\beta y_i^2 + (c_{i+1}-c_i)(1+\phi_i)(1-\psi_i)\\
&+O(x_i^2 y_i + x_i y_i^2) + x_i^\beta O(x_i^2 + y_i^3).
\end{aligned} \tag{2.5.15}$$

Next, using (2.5.5) and (2.5.6), and then using (2.5.7), (2.5.8), (2.5.10), and (2.5.13), we get

$$\begin{aligned}
&\tilde\rho_{i+1} y_i \psi_y(x_i,y_i)(1+\phi_i) - y_i(1-\psi_i)\phi_y(x_i,y_i)\\
=&\Big(\frac{a_2}{b_2} + \hat\rho(y_{i+1})\Big) y_i\psi_y(x_i,y_i)(1+\phi_i) - y_i(1-\psi_i)\phi_y(x_i,y_i)\\
&-\Big(\frac{a_2}{b_2} + \hat\rho(y_{i+1})\Big) x_{i+1}^\beta y_i\psi_y(x_i,y_i)(1+\phi_i) - c_{i+1}y_i\psi_y(x_i,y_i)(1+\phi_i)\\
=&\Big(\frac{a_2}{b_2} + \hat\rho(y_{i+1}^{(0)})\Big) y_i\psi_y(0,y_i)(1+\phi(0,y_i)) - y_i(1-\psi(0,y_i))\phi_y(0,y_i)\\
&-\frac{a_2}{b_2} 2b_2 x_i^\beta y_i^2 - c_{i+1}y_i\psi_y(0,y_i) + y_i O(x_i^2 + x_i y_i) + x_i^\beta y_i O(x_i^2 + y_i^2).
\end{aligned} \tag{2.5.16}$$

Also, denote

$$R_{\tilde{\rho}}(x_i, y_i) := -\tilde{\rho}_i\tilde{\rho}_{i+1}x_i(1+\phi_i)\psi_x(x_i, y_i) + \tilde{\rho}_i x_i(1-\psi_i)\phi_x(x_i, y_i). \tag{2.5.17}$$

The equations (2.5.15)-(2.5.17), (2.5.11), and Proposition 2.5.2 give

$$\begin{aligned}\Delta_{\tilde{\rho}_i}(x_i, y_i) &= -\frac{a_2}{b_2}(2b_2 - \beta a_2)x_i^\beta y_i^2 + (c_{i+1} - c_i)(1+\phi_i)(1-\psi_i) \\ &\quad - c_{i+1}y_i\psi_y(0, y_i) + R_{\tilde{\rho}}(x_i, y_i) + O(x_i^2 y_i + x_i y_i^2) + x_i^\beta O(x_i^2 + y_i^3).\end{aligned} \tag{2.5.18}$$

Note that the choice of β implies $2b_2 - \beta a_2 > 0$. By (2.5.11), we have

$$R_{\tilde{\rho}}(x_i, y_i) \begin{cases} = O(x_i^2 + x_i y_i^2) & \text{if } \tilde{\rho}_i \geq -1; \\ < 0 & \text{if } \tilde{\rho}_i < 0. \end{cases}$$

For $i = 0$, $c_0 = 0$ and $c_1 = \dfrac{a_2}{2b_2}(2b_2 - \beta a_2)x_0^\beta y_0^2$ by the definition of c_i. Hence,

$$\Delta_{\tilde{\rho}_0}(x_0, y_0) = -\frac{a_2}{2b_2}(2b_2 - \beta a_2)x_0^\beta y_0^2 - c_1 y_i\psi_y(0, y_i) + O(x_0^2 + x_0 y_0^3 + x_0^\beta y_0^3) < 0,$$

since we assume x_0 is small compared with y_0.

For $0 < i \leq n_0(z_0)$, where n_0 is given in Lemma 2.5.6, by (2.5.4), we have

$$(c_{i+1} - c_i)(1+\phi_i)(1-\psi_i) - c_{i+1}y_i\psi_y(0, y_i) = -c_{i+1}(1-\theta_0)y_i\psi_y(0, y_i) + y_i^2 O(x_i^2 + y_i^2).$$

So, we have

$$\Delta_{\tilde{\rho}_i}(x_i, y_i) = -\frac{a_2}{2b_2}(2b_2 - \beta a_2)x_i^\beta y_i^2 - c_{i+1}(1-\theta_0)y_i\psi_y(0, y_i) + O(x_i^2 + x_i y_i^3 + x_0^\beta y_i^3) < 0,$$

since we have $Kx_i < y^{1+\beta/(2-\beta)}$, or $x_i^2 \le K^{-(2-\beta)} x_i^\beta y_i^2$, for some $K > 0$ sufficiently large.

If $i \ge n_0$, then $\tilde{\rho}_i < 0$. Hence, $R_{\tilde{\rho}}(x_i, y_i) < 0$. Then by (2.5.18),

$$\begin{aligned}\Delta_{\tilde{\rho}_i}(x_i, y_i) = &-\frac{a_2}{2b_2}(2b_2 - \beta a_2)x_i^\beta y_i^2 - c_{i+1}(1-\theta_0)y_i\psi_y(0, y_i)\\ &-|R_{\tilde{\rho}}(x_i, y_i)| + O(x_i^2 y_i + x_i y_i^2) + x_i^\beta O(x_i^2 + y_i^3)) < 0.\end{aligned}$$

This completes the proof. □

Lemma 2.5.5. Let $z_0 = (x_0, y_0)$ with $x_0, y_0 > 0$. If for all $z = (x, y)$ in the stable curve that joins $\bar{z}_0$ and z_1,

$$v_z^s \le -\left(\frac{a_2}{b_2} + \hat{\rho}(y_0)\right)(1 - x_0^\beta)\frac{x_0}{y_0}, \tag{2.5.19}$$

then

$$\bar{x}_0 \le x_0 + E_\beta x_0^{1+\beta},$$

where E_β is a positive constant dependent on y_0.

Proof. Since $(v_z^s, 1)$ forms a tangent line of the stable manifold $W_r^s(z)$, (2.5.19) gives

$$\frac{dx}{dy} \le -\Big(\frac{a_2}{b_2} + \hat{\rho}(y)\Big)(1 - x^\beta)\frac{x}{y},$$

which implies that

$$\frac{dx}{x(1-x^\beta)} + \Big(\frac{a_2}{b_2} + \hat{\rho}(y)\Big)\frac{dy}{y} \le 0.$$

Integrating the function from $z_1 = (x_1, y_1)$ to $\bar{z}_0 = (\bar{x}_0, \bar{y}_0)$, we have

$$\log\frac{\bar{x}_0}{x_1} - \frac{1}{\beta}\log\frac{1-\bar{x}_0^\beta}{1-x_1^\beta} + \frac{a_2}{b_2}\log\frac{\bar{y}_0}{y_1} + \int_{y_1}^{\bar{y}_0}\frac{\hat{\rho}(y)}{y}dy \le 0.$$

In the following discussions, we omit the subscript 0. The above inequality gives

$$\frac{\bar{x}}{x_1} \le \Big(\frac{1-\bar{x}^\beta}{1-x_1^\beta}\Big)^{\frac{1}{\beta}} \Big(\frac{y_1}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \exp\Big(-\int_{y_1}^{\bar{y}} \frac{\hat{\rho}(y)}{y} dy\Big).$$

This, together with $x_1 = x(1+\phi(x,y))$ and $y_1 = y(1-\psi(x,y))$, yields that

$$\frac{\bar{x}}{x} \le (1+\phi(x,y))(1-\psi(x,y))^{\frac{a_2}{b_2}} \Big(\frac{1-\bar{x}^\beta}{1-x_1^\beta}\Big)^{\frac{1}{\beta}} \Big(\frac{y}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \exp\Big(-\int_{y(1-\psi(x,y))}^{\bar{y}} \frac{\hat{\rho}(y)}{y} dy\Big).$$

By (2.5.7), $\phi(x,y) = \phi(0,y) + O(x^2+xy^2)$ and $\psi(x,y) = \psi(0,y) + O(x^2+xy^2)$. Hence, $\displaystyle\int_{y(1-\psi(x,y))}^{y(1-\bar{\psi}(0,y))} \frac{\hat{\rho}(y)}{y} dy = O(x)$, where we treat y as a constant. By (2.5.2), one has

$$\begin{aligned}
&(1+\phi(x,y))(1-\psi(x,y))^{\frac{a_2}{b_2}} \exp\Big(-\int_{y(1-\psi(x,y))}^{\bar{y}} \frac{\hat{\rho}(y)}{y} dy\Big) \\
=&(1+O(x^2)) \exp\Big(\int_{\bar{y}}^{y} \frac{\hat{\rho}(y)}{y} dy\Big).
\end{aligned}$$

Since $\bar{z} = (\bar{x},\bar{y})$ and $z = (x,y)$ are in the same local unstable manifold, one has that

$$|\bar{y} - y| \le N(\bar{x} - x) \le N(x_1 - x) = Nx\phi,$$

where N is a positive constant. So,

$$\Big(\frac{y}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \le \Big(1 + \frac{Nx\phi}{\bar{y}}\Big)^{\frac{a_2}{b_2}} = 1 + O(x) \quad \text{and} \quad \exp\Big(\int_{\bar{y}}^{y} \frac{\hat{\rho}(y)}{y} dy\Big) = 1 + O(x).$$

Now we get

$$\frac{\bar{x}}{x} \le \Big(\frac{1-\bar{x}^\beta}{1-x_1^\beta}\Big)^{\frac{1}{\beta}} \big(1 + O(x)\big).$$

Using the facts $x_1^\beta = x^\beta(1+\phi)^\beta = x^\beta + \beta x^\beta\phi + x^\beta O(\phi^2)$ and $x < \bar{x}$, we have

$$\frac{1-\bar{x}^\beta}{1-x_1^\beta} = 1 + \frac{x_1^\beta - \bar{x}^\beta}{1-x_1^\beta} = 1 + \frac{x^\beta + \beta x^\beta\phi - \bar{x}^\beta + x^\beta O(\phi^2)}{1-x_1^\beta} \le 1 + \frac{\beta x^\beta\phi + x^\beta O(\phi^2)}{1-x_1^\beta}.$$

Therefore,

$$\frac{\bar{x}}{x} \le 1 + E_\beta x^\beta,$$

where E_β is a positive constant dependent on y_0.

This completes the proof. $\square$

Lemma 2.5.6. Suppose $\alpha, \beta \in (0,1)$ satisfies $\beta < \dfrac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \dfrac{2b_2}{a_2} < \alpha$. Then there exist $\theta_0 \in (0,1)$ and $\eta > \dfrac{a_2b_2}{a_2^2+b_2^2}$ such that for any positive constants K and N, a point $q = (0, y_q)$ with $y_q > 0$ small, there is $\varepsilon > 0$ such that for any $z_0 = (x_0, y_0) \in W_\varepsilon^u(q)$ with $x_0 > 0$, the following inequalities hold simultaneously for some positive integer $n = n(z_0)$:

$$x_0^\beta y_0^2 \prod_{j=0}^{n} \Big(1 - \theta_0 y_j \psi_y(0, y_j)\Big)^{-1} \ge N, \qquad Kx_n < y^{1+\eta}.$$

Proof. Since $\beta < \dfrac{2a_2b_2}{a_2^2+a_2b_2+b_2^2} = \dfrac{2a_2b_2}{\left(a_2^2+b_2^2\right)\left(1 + \dfrac{a_2b_2}{a_2^2+b_2^2}\right)}$, there is $\gamma > 1 + \dfrac{a_2b_2}{a_2^2+b_2^2}$ such that $\beta = \dfrac{2a_2b_2}{\gamma(a_2^2+b_2^2)}$. Take $\dfrac{a_2b_2}{a_2^2+b_2^2} < \eta < \gamma - 1$ and then take $\theta_0 > 0$ such that

$$1 > \theta_0 > \max\Big\{\frac{\gamma\beta}{2},\ \frac{2-(\gamma-1)\beta+\eta\beta}{2}\Big\}.$$

Clearly we have $\dfrac{\beta}{2-\beta} < \dfrac{2a_2b_2}{2(a_2^2+a_2b_2+b_2^2) - 2a_2b_2} = \dfrac{a_2b_2}{a_2^2+b_2^2} < \eta$.

By the choices of θ_0 and γ, we could assume that K is large enough such that if $Kx \le y$, then

$$1 - \theta_0 y \psi_y(0, y) \leq 1 - \theta_1 2 b_2 y^2 \leq (1-\psi)^{2\theta_2} \tag{2.5.20}$$

and

$$(1+\phi)^{\beta}(1-\psi)^{2-\gamma\beta} \leq 1, \tag{2.5.21}$$

where θ_1 and θ_2 satisfy

$$\max\left\{\frac{\gamma\beta}{2},\ \frac{2-(\gamma-1)\beta+\beta\eta}{2}\right\} < \theta_2 < \theta_1 < \theta_0. \tag{2.5.22}$$

Hence, for any $z_0 = (x_0, y_0)$ with $Kx_0 < y_0$, by (2.5.21), we have

$$x_1^{\beta} y_1^{2-\gamma\beta} \leq x_0^{\beta}(1+\phi_0)^{\beta} y_0^{2-\gamma\beta}(1-\psi_0)^{2-\gamma\beta} \leq x_0^{\beta} y_0^{2-\gamma\beta}. \tag{2.5.23}$$

Set $n := n(z_0)$ as the largest positive integer such that $Kx_n \leq y_n^{1+\eta}$ and $Kx_{n+1} > y_{n+1}^{1+\eta}$. Since $0 < y < 1$, we have that if $Kx < y^{1+\eta}$, then $Kx < y$. So,

$$x_0^{\beta} y_0^{2-\gamma\beta} \geq x_{n+1}^{\beta} y_{n+1}^{2-\gamma\beta} \geq K^{-\beta} y_{n+1}^{\beta(1+\eta)} y_{n+1}^{2-\gamma\beta} = K^{-\beta} y_{n+1}^{2+(1-\gamma)\beta+\eta\beta}.$$

By (2.5.20) and (2.5.23), we get

$$\begin{aligned}
&\frac{x_0^{\beta} y_0^2}{\prod_{j=0}^{n}\Big(1-\theta_0 y_j \psi_y(0, y_j)\Big)} \geq \frac{y_0^{2\theta_2} x_0^{\beta} y_0^2}{y_0^{2\theta_2} \prod_{j=0}^{n}(1-\psi_j)^{2\theta_2}} \\
\geq &\frac{x_0^{\beta} y_0^{2+2\theta_2}}{y_{n+1}^{2\theta_2}} \geq \frac{y_0^{2\theta_2-\gamma\beta}}{K^{\beta} y_{n+1}^{2\theta_2-(2-(\gamma-1)\beta+\eta\beta)}}.
\end{aligned}$$

By (2.5.22), $2\theta_2 - (2-(\gamma-1)\beta+\eta\beta) > 0$. Hence, if z_0 is sufficiently close to q, then y_{n+1}

can be arbitrarily small and the right hand side of the inequality can be arbitrarily large. This lemma is thus proved. $\square$

2.6 Estimates of the size of elements of $\mathcal{P}_{-k,k}$

Recall that $\mathcal{P}$ is a Markov partition. Denote $\mathcal{P}_{k,n} = \vee_{i=k}^{n} f^i(\mathcal{P})$ and $\mathcal{P}_n = \mathcal{P}_{0,n}$. Denote by $\mathcal{P}_{k,n}(x)$ the element of $\mathcal{P}_{k,n}$ that contains x.

Also, denote by $\gamma_n^s(x)$ the connected stable curves that contains x and is contained in $\mathcal{P}_n(x)$, and by $\gamma_n^u(x)$ the connected unstable curves that contains x and is contained in $\mathcal{P}_{-n,0}(x)$.

Recall that m^s is the Lebesgue measure restricted to stable curves. Recall also that $Q = Q_2 = f^{-1}P \setminus P$, and $Q_k = [\tau \geq k]$, $k \geq 2$, are introduced in Subsection 2.3.2. Denote $R_k = [\tau = k] = Q_k \setminus Q_{k+1}$ for $k \geq 2$. Then we denote $Q_k^+ = f^\tau(Q_k)$ and $R_k^+ = f^\tau(R_k) = f^k(R_k)$, where f^τ is the first return map of f with respect to $M_0 = M \setminus P_0$. Clearly $Q_k = \cup_{i=k}^{\infty} R_i$ and $Q_k^+ = \cup_{i=k}^{\infty} R_i^+$.

Proposition 2.6.1. There exist $K_s > 0$ and $C_s > 0$ such that for any $k \geq K_s$, we can find a set T_k with the following properties:

(i) $\mu(T_k) \leq \dfrac{C_s \log k}{k^{1/\alpha}}$;

(ii) $m^s(\gamma_k^s(x)) \leq \dfrac{C_s}{k^{1/2+\alpha'}}$ for any $x \in T_k$;

(iii) $m^s(\gamma_k^s(x)) \leq \dfrac{C_s}{k^{3/2+\alpha'}}$ for any $x \notin T_k \cup P$,

where $\alpha' = b_0/2a_0$.

Proof. Take $K_s \geq 2K_1$, where K_1 is given in Corollary 2.6.1.

Recall that λ is defined in (3.4.2). For each $k > 0$, take $\ell = \ell_k = -\left\lfloor \frac{\log k}{\log \lambda} \right\rfloor$. Then for any $j \geq \ell_k$, $\lambda^j < \frac{1}{k}$.

Define

$$T_k = \bigcup_{i=0}^{\ell} (f^\tau)^i (Q^+_{\lfloor k/2 \rfloor}),$$

where τ is the first return time with respect to $M \setminus P$. By (3.4.5), $\mu(Q_{\lfloor k/2 \rfloor}) \leq \frac{2^{1/\alpha} B_\alpha}{k^{1/\alpha}}$ for some $B_\alpha > 0$. Since μ is preserved under the map f^τ, we can get

$$\mu(T_k) \leq \frac{2^{1/\alpha} B_\alpha}{k^{1/\alpha}} \cdot \ell \leq \frac{C' \log k}{k^{1/\alpha}}$$

for some $C' > 0$. Hence, we get part (i) if $C_s \geq C'$.

For any $x \in M$, denote $x_k := f^{-k}(x)$. If $x_k \in P$, we define $\tau(x_k) = \min\{i > 0 : f^i(x_k) \in M \setminus P\}$, the first time the orbit of x_k enter $M \setminus P$.

We now prove a claim stronger than the requirements in (ii) and (iii): For any $x \notin P$, the inequality in (ii) holds for any $x \in T_k$ with $x_k \in P$ and $\tau(x_k) > k/2$; and that in (iii) holds otherwise.

If $x_k \notin P$, then by Corollary 2.6.2(i), $m^s(\gamma^s_k(x)) \leq \frac{C_2}{k^{3/2+\alpha'}}$.

If $x_k \in P$ and $\tau(x_k) \leq k/2$, then we have $f^\tau(x_k) \notin P$ and $k - \tau(x_k) \geq \max\{K_1, k/2\}$. Using Corollary 2.6.2(i) with $f^{\tau(x_k)}(x_k)$ and $x = f^{k-\tau(x_k)}(f^{\tau(x_k)}(x_k))$ we get

$$m^s(\gamma^s_k(x)) \leq \frac{C_2}{(k - \tau(x_k))^{3/2+\alpha'}} \leq \frac{2^{3/2+\alpha'} C_2}{k^{3/2+\alpha'}}.$$

If $x_k \in P$, $\tau(x_k) > k/2$ and $x \notin T_k$, then we have $\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k)) \subset Q^+_{\lfloor k/2 \rfloor}$. By

Corollary 2.6.2(ii) we have $m^s(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))) \le \dfrac{C_2}{\lfloor k/2 \rfloor^{1/2+\alpha'}} \le \dfrac{2^{1/2+\alpha'}C_2}{k^{1/2+\alpha'}}$. On the other hand, $x \notin T_k$ implies $k - \tau(x_k) \ge \tau(f^\tau(x_k)) + \tau((f^\tau)^2(x_k)) + \cdots + \tau((f^\tau)^\ell(x_k))$. Hence $\|Df_y^{k-\tau(x_k)}|_{E^s_y}\| \le \lambda^\ell \le \dfrac{1}{k}$ for any $y \in \gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))$ by the choice of ℓ. Note that $f^{k-\tau(x_k)}\big(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))\big) = \tau^s_k(x)$. We get

$$m^s(\gamma^s_k(x)) \le \frac{1}{k} \cdot m^s\big(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))\big) \le \frac{1}{k} \cdot \frac{2^{1/2+\alpha'}C_2}{k^{1/2+\alpha'}} = \frac{2^{1/2+\alpha'}C_2}{k^{3/2+\alpha'}}.$$

On the other hand, if $x_k \in P$, $\tau(x_k) > k/2$ and $x \in T_k$, then we can only get

$$m^s(\gamma^s_k(x)) \le m^s\big(\gamma^s_{\tau(x_k)}(f^{\tau(x_k)}(x_k))\big) \le \frac{C_2}{\lfloor k/2 \rfloor^{1/2+\alpha'}} \le \frac{2^{1/2+\alpha'}C_2}{k^{1/2+\alpha'}}.$$

Now we get what we claimed if we take $C_s = 2^{1/2+\alpha'}C_2$. $\square$

Proposition 2.6.2. There exist $K_u > 0$ and $C_u > 0$ such that for any $k \ge K_u$, $m^u(\gamma^u_k(x)) \le \dfrac{C_u}{k^{1/\alpha}}$ for any $x \notin P$.

Proof. The proof is similar to that for Proposition 2.6.1 by using the estimates given in Proposition 2.5.1 for $\gamma^u_k \in W^u(Q_k)$, instead of Corollary 2.6.2 for $\gamma^s_k \in W^s(Q^+_k)$. $\square$

To prove Lemma 2.6.3 below, we need the following facts.

Lemma 2.6.1 ([14] Lemmas 3.1 and 3.2)**.** If

$$t_{n-1} \ge t_n + Ct_n^{1+\varrho} + O(t_n^{1+\varrho'}) \qquad \forall n > 0, \tag{2.6.1}$$

where $\varrho' > \varrho$, then for all large n,

$$t_n \leq \frac{1}{(\varrho C(n+k))^{1/\varrho}} + O\left(\frac{1}{(n+k)^{\delta'}}\right), \tag{2.6.2}$$

for some $\delta' > 1/\varrho$ and $k \in \mathbb{Z}$.

Moreover, if (2.6.2) holds and for all $n > 0$,

$$r(t_n) \leq 1 - C' t_n^{\varrho} + O(t_n^{1+\varrho'}),$$

where $C' > 0$, then there exists $D > 0$ such that for all $k_0 > k$,

$$\prod_{i=k_0-k}^{n+k_0-k} r(t_i) \leq D\left(\frac{k}{n+k}\right)^{C'/\varrho C}.$$

The results remain true if we interchange "$\leq$" and "$\geq$". Therefore, if (2.6.1) becomes an equality, then so does (2.6.2).

Lemma 2.6.2 ([12] Propositions 2.6 and 2.8)**.** For any $\varepsilon > 0$, there exists a constant $0 < r_* \leq r_0$ such that for any $r \in (0, r_*)$ and $x \in B(p, r)$, $t \in (0, 1]$, $j = 1, \cdots, \left\lfloor \frac{2}{t^2} \right\rfloor$, we have

$$(1-\epsilon)|tx| \leq \left|f^j(tx)\right| \leq (1+\epsilon)|tx|;$$

and for any $x, y \in B(p, r)$ with $|\Theta(x, y)| \leq |\Theta(x, f(x))|$ and $|y| = t|x|$, $t \in (0, 1]$, we have

$$\begin{aligned} |\Theta(y, f^j(y))| &\leq |\Theta(x, f(x))| + \varepsilon|x|^2 \qquad \forall\ 0 \leq j \leq \left\lfloor \frac{1}{t^2} \right\rfloor; \\ |\Theta(y, f^j(y))| &\geq |\Theta(x, f(x))| - \varepsilon|x|^2 \qquad \forall \left\lfloor \frac{1}{t^2} \right\rfloor \leq j \leq \left\lfloor \frac{2}{t^2} \right\rfloor, \end{aligned}$$

where r_0 is specified in Definition 2.2.2, and $\Theta(x, y)$ denotes the angle from x to y counter-clockwise in $\mathbb{R}^2$.

Lemma 2.6.3. There exists $C_1 > 0$ such that for any $x \in Q$ with $n = \tau(x)$, $\|Df_x^n|_{E_x^s}\| \leq \dfrac{C_1}{n^{3/2+\alpha'}}$, where $\alpha' = b_0/2a_0$.

Proof. Choose $\theta^u, \theta^s > 0$ small. Then take sectors $\mathcal{S}^u = \{z \in U : |\angle(z, E_p^u)| \leq \theta^u\}$ and $\mathcal{S}^s = \{z \in U : |\angle(z, E_p^s)| \leq \theta^s\}$, where $\angle(z, E_p^u)$ is the angle between the vector from p to z and the line E_p^u. Then let $\mathcal{S}^c = P \setminus (\mathcal{S}^s \cup \mathcal{S}^s)$.

If $N_0 > 0$ is large enough, then for any $x \in Q_{N_0}$, the orbit of x passes through $\mathcal{S}^s$, $\mathcal{S}^c$, and $\mathcal{S}^u$ consecutively before it leaves P. Note that if $x \in R_n \subset Q_{N_0}$, then $n = n_x = \tau(x) \geq N_0$. We take n^s,n^c, $n^u > 0$ such that $n^s = \max\{j > 0 : f^i(x) \in \mathcal{S}^s,\ \forall 1 \leq i \leq j\}$, $n^c = \max\{j > 0 : f^{n^s+i}(x) \in \mathcal{S}^c,\ \forall 1 \leq i \leq j\}$, and $n^u = n_x - n^s - n^c$. That is, $x, f(x), \ldots, f^{n^s}(x) \in \mathcal{S}^s$, $f^{n^s+1}(x), \ldots, f^{n^s+n^c}(x) \in \mathcal{S}^c$, and $f^{n^s+n^c+1}(x), \ldots, f^{n_x}(x) \in \mathcal{S}^u$.

Note that (2.2.3) implies that f has the form $f(r) = r(1 - b_2r^2 + O(r^3))$ restricted to $W_\varepsilon^s(p)$, and Df has the form $Df|_{E^s} = 1 - 3b_2r^2 + O(r^3)$ restricted to E_x^s for $x = (0, r) \in W_\varepsilon^s(p)$. Hence, by Lemma 2.6.1, for any point $\hat{x} \in W_\varepsilon^s(p) \cap Q$, $|f^n(\hat{x})| \approx \dfrac{1}{\sqrt{2b_2n}}$ and $\|Df_{\hat{x}}^n|_{E_{\hat{x}}^s}\| \sim \dfrac{\hat{d}_s}{\sqrt{n^3}}$ for some constant $\hat{d}_s > 0$, where $a_k \approx b_k$ means $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = 1$, and $a_k \sim b_k$ means a_k/b_k is bounded away from 0 and infinity. Since the points in $\mathcal{S}^s$ are close to $W_\varepsilon^s(p)$, we can get that there exist $c_s > c_s' > 0$ and $d_s > d_s' > 0$ such that

$$\frac{c_s'}{\sqrt{n^s}} \leq |f^{n^s}(x)| \leq \frac{c_s}{\sqrt{n^s}} \qquad \text{and} \qquad \frac{d_s'}{\sqrt{(n^s)^3}} \leq \|Df_x^{n^s}|_{E_x^s}\| \leq \frac{d_s}{\sqrt{(n^s)^3}}. \tag{2.6.3}$$

Now we consider the part of the orbit in $\mathcal{S}^c$. Take $z \in \mathcal{S}^s$ such that $f^k(z) \in \mathcal{S}^u \cap Q_{N_0}^+$ with some $k > 0$. Define k^s and k^c in a way similar with that of n^s and n^c as above, that is, k^s is the largest positive integer such that $f^1(z), \ldots, f^{k^s}(z) \in \mathcal{S}^s$, and k^c is the largest positive integer such that $f^{k^s+1}(z), \ldots, f^{k^s+k^c}(z) \in \mathcal{S}^c$. Consider Lemma 2.6.2 with ε small. If N_0 is sufficiently large, then for $x \in Q_{N_0}$, $|f^{n^s}(x)| = t|f^{k^s}(z)|$ is small. Hence, by Lemma 2.6.2, for $i = 0, 1, \ldots, n^c$,

$$(1-\varepsilon)^{k^c}|f^{n^s}(x)| \leq |f^{n^s+i}(x)| \leq (1+\varepsilon)^{k^c}|f^{n^s}(x)| \quad \text{and} \quad n^c \sim \frac{k^c}{t^2} = \frac{k^c|f^{k^s}(z)|^2}{|f^{n^s}(x)|^2}.$$

So, there exist $c_n > c_n' > 0$ and $c_c > c_c' > 0$ such that for $i = 0, 1, \ldots, n^c$,

$$\frac{c_n'}{|f^{n^s}(x)|^2} \leq n^c \leq \frac{c_n}{|f^{n^s}(x)|^2}, \quad \text{and} \quad c_c'|f^{n^s}(x)| \leq |f^{n^s+i}(x)| \leq c_c|f^{n^s}(x)|. \tag{2.6.4}$$

Note that (2.2.2) and (2.2.3) imply that there exist $c > c' > 0$ such that $1 - c|y|^2 \leq \|Df_y|_{E^s_y}\| \leq 1 - c'|y|^2$ for any y with $|y|$ small. Hence, by taking $y = f^{n^s+i}(x)$, $i = 0, 1, \ldots, n^c$, we obtain that there exist $0 < d_c' \leq d_c < 1$ such that

$$d_c' \leq \|Df^{n^c}_{f^{n^s}(x)}|_{E^s_{f^{n^s}(x)}}\| \leq d_c. \tag{2.6.5}$$

For the part of the orbit in $\mathcal{S}^u$, we note that (2.2.3) implies that f has the form $f(r) = r(1 + a_0r^2 + O(r^3))$ restricted to $W^u_\varepsilon(p)$, and Df has the form $Df|_{E^s} = 1 - b_0r^2 + O(r^3)$ restricted to E^s_x for $x = (r, 0) \in W^u_\varepsilon(p)$. Hence, by Lemma 2.6.1, for any point $\hat{x} \in W^u_\varepsilon(p)$, $|f^{-n}(\hat{x})| \approx \dfrac{1}{\sqrt{2a_0n}}$ and $\|Df^n_{\hat{x}}|_{E^s_{\hat{x}}}\| \sim \dfrac{1}{n^{b_0/2a_0}}$. Since points in $\mathcal{S}^u$ are close to $W^u_\varepsilon(p)$, we can get that there exist $c_u > c_u' > 0$ and $d_u > d_u' > 0$ such that

$$\begin{aligned} \frac{c_u'}{\sqrt{n^u}} &\leq |f^{n^s+n^c}(x)| \leq \frac{c_u}{\sqrt{n^u}}, \\ \frac{d_u'}{(n^u)^{b_0/2a_0}} &\leq \|Df^{n^u}_{f^{n^s+n^c}(x)}|_{E^s_{f^{n^s+n^c}(x)}}\| \leq \frac{d_u}{(n^u)^{b_0/2a_0}}. \end{aligned} \tag{2.6.6}$$

By the second inequality of (2.6.4), $|f^{n^s+n^c}(x)| \sim |f^{n^s}(x)|$. Hence, by (2.6.3), (2.6.4), and (2.6.6), all n^s, n^c and n^u are roughly proportional. Since $n^s + n^c + n^u = n = n_x$, we know that there exist $\rho^s, \rho^u \in (0,1)$ such that $n^s \geq \rho^s n$ and $n^u \geq \rho^u n$. So by (2.6.3), (2.6.5) and (2.6.6), we get

$$\|Df_x^n|_{E_x^s}\| \leq \frac{C_1}{n^{3/2+b_0/2a_0}}$$

for some $C_1 > 0$.

The proof is completed. $\square$

Corollary 2.6.1. There exists $K_1 > 0$ such that for any $n > K_1$, if $x, f^n(x) \notin P$, then $\|Df_x^n|_{E_x^s}\| \leq \dfrac{C_1}{n^{3/2+\alpha'}}$, where C_1 and α' are as in Lemma 2.6.3.

Proof. Take $K_1' > 0$ such that $\dfrac{C_1}{k^{3/2+\alpha'}} \cdot \dfrac{C_1}{n^{3/2+\alpha'}} \leq \dfrac{C_1}{\big(2(k+n)\big)^{3/2+\alpha'}}$, whenever $k, n \geq K_1'$.

Let $S = S_{K_1'} = \{f^i(x) \in P : x \in Q_{K_1'}, i = 1, \ldots, n_x - 1\}$, where $n_x = \tau(x)$. Since f is uniformly hyperbolic on $M \setminus \overline{S}$, there exists $\rho = \rho_S \in (0,1)$ such that $\|Df_z|_{E_x^s}\| \leq \rho$ for any $x \in M \setminus \overline{S}$. Take $K_1'' > 0$ such that for any $n \geq K_1''$, $\rho^n \leq \dfrac{C_1}{(2n)^{3/2+\alpha'}}$.

Take $K_1 = \max\{2K_1', 2K_1''\}$. For $x, f^n x \notin P$ with $n \geq K_1$, we denote $I = \{i \in (1,n) : f^i(x) \notin S\}$, and let k_x be the cardinality of I. If $k_x \geq n/2 > K_1''$, then

$$\|Df_x^n|_{E_x^s}\| \leq \prod_{i \in I} \|Df_{f_i(x)}|_{E^s_{f_i(x)}}\| \leq \rho^{k_x} \leq \frac{C_1}{(2k_x)^{3/2+\alpha'}} \leq \frac{C_1}{n^{3/2+\alpha'}}.$$

If $k_x \leq n/2$, then we may assume that the orbit $\{x, \ldots, f^{n-1}(x)\}$ passes through $Q_{K_1'}$ ℓ times. Let $k_1 < k_2 < \cdots < k_\ell < n$ such that $f^{k_j}(x) \in Q_{K_1'}$, $j = 1, \ldots, \ell$. Denote

$n_j = \tau(f^{k_j}(x))$. So, we have $n_j \geq K_1'$ for all j. Now we get

$$\|Df_x^n|_{E_x^s}\| \leq \prod_{1\leq j\leq \ell} \|Df^{n_j}_{f^{k_j}(x)}|_{E^s_{f^{k_j}(x)}}\| \leq \prod_{1\leq j\leq \ell} \frac{C_1}{n_j^{3/2+\alpha'}}$$
$$\leq \frac{C_1}{\big(2(n_1+\cdots+n_\ell)\big)^{3/2+\alpha'}} = \frac{C_1}{\big(2(n-k_x)\big)^{3/2+\alpha'}} \leq \frac{C_1}{n^{3/2+\alpha'}},$$

where we use the fact $n_1 + \cdots + n_\ell = n - k_x > n/2$.

This completes the proof. □

Recall that Q_n, R_n, Q_n^+, R_n^+ and $\gamma_n^s(x)$ are given at the beginning of this section. Also, we have $Q_n^+ \in \mathcal{P}_n$.

Corollary 2.6.2. There exists $C_2 > 0$ such that for any $k > 0$,

(i) $m^s\big(\gamma_k^s(f^k(x))\big) \leq \dfrac{C_2}{k^{3/2+\alpha'}}$ if $x, f^k(x) \notin P$;

(ii) $m^s\big(\gamma_k^s(x)\big) \leq \dfrac{C_2}{k^{1/2+\alpha'}}$ if $x \in Q_k^+$.

Proof. (i) Note that $f^n(\gamma_0^s(x)) = \gamma_k^s(f^k(x))$. By Corollary 2.6.1, and distortion estimates given in Lemma 2.4.2, we can get that $m^s\big(\gamma_k^s(f^k(x))\big) \leq \dfrac{C_1'}{k^{3/2+\alpha'}} \cdot m^s\big(\gamma_0^s(x)\big)$ for some $C_1' > 0$. Then we use the fact that $m^s\big(\gamma_0^s(x)\big)$ are bounded above for all $x \in M$.

(ii) Note that for $y \in R_i$, $f^i(y) \in R_i^+$ and $f^i(\gamma_0^s(y_i)) = \gamma_i^s(f^i(y_i))$. By using the same arguments as above, and using Lemma 2.6.3 to replace Corollary 2.6.1, we can get $m^s\big(\gamma_i^s(f^i(y))\big) \leq \dfrac{C_2}{i^{3/2+\alpha'}}$ for all $y \in R_i$. Since for any $x \in Q_k^+$, $\gamma_k^s(x)$ is the union of the stable curves $\gamma_i^s(z_i)$, $z_i \in R_i^+ \cap \gamma_k^s(x)$, $i = k, k+1, \ldots$, we get that $m^s(\gamma_k^s(x)) \leq \displaystyle\sum_{i=k}^{\infty} \frac{C_2}{i^{3/2+\alpha'}}$. Now we can increase C_2 to get the result of part (ii). □

2.7 Some large deviation estimation

In this section, we study the large deviation estimates for the observable function $\Psi \in \mathcal{L}$ with respect to the quotient map $\bar{f}$. We adopt the discussions used in [32].

Recall that $(\overline{f}, \overline{M})$ is the one-dimensional system induced from (f, M), and $(\tilde{f}, \widetilde{M})$ is the first return maps of $\overline{f}$ with respect to $\widetilde{M} = \overline{M} \setminus \overline{P}_0$.

Lemma 2.7.1. Let $0 < \alpha < \frac{1}{2}$. Given any $\epsilon > 0$, for any function $\Psi \in \mathcal{L}$ satisfying $|\int \Psi d\bar{\mu}| \geq \epsilon$, one has that

$$\bar{\mu}\Big\{\overline{x} \in \overline{M} : \ \Big|\sum_{i=0}^{n-1} \Big(\Psi(\bar{f}^i(\overline{x})) - \int \Psi d\bar{\mu}\Big)\Big| > n\epsilon\Big\} = O((\log n)^{2(\frac{1}{\alpha}-1)} n^{-(\frac{1}{\alpha}-1)}). \tag{2.7.1}$$

The transfer operator of the Markov map $\bar{f}$ is defined as follows:

$$\mathcal{T}\Psi(\overline{x}) = \sum_{\bar{f}\overline{y}=\overline{x}} g_{\bar{\mu}}(\overline{y})\Psi(\overline{y}),$$

where $g_{\bar{\mu}} = d\bar{\mu}/d\bar{\mu} \circ \bar{f}$ and $\Psi \in L^1(\overline{M})$. Since $\bar{\mu}$ is invariant with respect to the quotient map $\bar{f}$, $g_{\bar{\mu}}$ is said to be the g-function of $\bar{\mu}$.

Define the following operators:

$$T_n\Psi := 1_{\overline{Q}}\mathcal{T}^n(\Psi \cdot 1_{\overline{Q}}), \quad R_n\Psi := 1_{\overline{Q}}\mathcal{T}(\Psi \cdot 1_{[R_{\overline{Q}}=n]}).$$

By Proposition 1 of [40], one has the renewal equation:

$$T(z) = (I - R(z))^{-1}, \quad z \in \mathbb{D},$$

where $\mathbb{D}$ is the unit disk in the complex plane, and

$$R(z) = \sum_{n=1}^{\infty} z^n R_n, \quad T(z) = I + \sum_{n=1}^{\infty} z^n T_n, \quad z \in \mathbb{D}.$$

Proof of Lemma 2.7.1. For convenience, set $\Phi := \Psi - \int \Psi d\bar{\mu}$.

It follows from (2.3.4) and the fact that $\bar{\mu}$ is an invariant measure of $\bar{f}$ that

$$\begin{aligned} &\Big|\int \Phi \circ \bar{f}^k \cdot \Phi d\bar{\mu}\Big| = \Big|\int \Big(\Psi \circ \bar{f}^k - \int \Psi d\bar{\mu}\Big)\Big(\Psi - \int \Psi d\bar{\mu}\Big) d\bar{\mu}\Big| \\ =&\Big|\int \Psi \circ \bar{f}^k \cdot \Psi d\bar{\mu} - \int \Psi \circ \bar{f}^k d\bar{\mu} \int \Psi d\bar{\mu}\Big| = |\mathrm{Cor}_n(\Psi, \Psi; \bar{f}, \bar{\mu})| \leq \frac{C(\Psi)}{k^{\frac{1}{\alpha}-1}}. \end{aligned}$$

By the renewal theory, Theorem 1 in [40] or Theorem 1.1 in [8],

$$T_n = \frac{1}{r}\mathsf{Pr} + \frac{1}{r^2}\sum_{k=n+1}^{\infty} \mathsf{P}_k + E_n,$$

where Pr is the eigenprojection of $R(1)$ at 1, r is given by $\mathsf{Pr}R'(1)\mathsf{Pr} = r\mathsf{Pr}$, $\mathsf{P}_n = \sum_{l>n} \mathsf{Pr}R_l\mathsf{Pr}$, $E_n \in \mathrm{Hom}(\mathcal{L}, \mathcal{L})$. By using Lemma 6.5 in [8] and (3.4.5), we have that $\|R_n\| = O(\frac{1}{n^{\alpha}})$. So, we have $\|E_n\| = o(1/n^{\frac{1}{\alpha}-1})$.

By the fact that $\mathsf{Pr}\Phi = \int_{\overline{Q}} \Phi d\bar{\mu}$ (see the proof of Theorem 2 in [40]), $\int \Phi d\bar{\mu} = 0$, and Theorem 1.2 in [8], one has

$$\int \|\mathcal{T}^n \Phi\| d\bar{\mu} = \int \|T_n \Phi\| d\bar{\mu} = O\Big(\frac{1}{n^{\frac{1}{\alpha}-1}}\Big).$$

Next, it is to apply the method of the proof of Proposition 2.3 in [32] to prove (2.7.1).

By Proposition 1.2 in [41] and the fact that $\bar{f}$ is measure preserving with respect to the measure $\bar{\mu}$, $\mathbb{E}_{\bar{\mu}}(\Phi|\bar{f}^{-k}\overline{\mathcal{B}}) = (\mathcal{T}^k\Phi) \circ \bar{f}^k$ for any positive integer k and $\Phi \in L^1(\overline{M})$. By direct computation,

$$
\begin{aligned}
&\bar{\mu}\Big\{\overline{x}\in\overline{M}:\ \Big|\sum_{i=0}^{n-1}\Phi(\bar{f}^i(\overline{x}))\cdot\Big|>n\epsilon\Big\}\leq\frac{1}{(n\epsilon)^{2\vartheta}}\int\Big|\sum_{i=0}^{n-1}\Phi(\bar{f}^i(\overline{x}))\Big|^{2\vartheta}d\bar{\mu}(\overline{x})\\
\leq&\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}}\Big(\|\Phi\|_{2\vartheta}+240\sum_{k=1}^{n}k^{-1/2}\|\mathbb{E}_{\bar{\mu}}(\Phi\circ\bar{f}^k|\overline{\mathcal{B}})\|_{2\vartheta}\Big)^{2\vartheta}\\
=&\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}}\Big(\|\Phi\|_{2\vartheta}+240\sum_{k=1}^{n}k^{-1/2}\|\mathbb{E}_{\bar{\mu}}(\Phi|\bar{f}^{-k}\overline{\mathcal{B}})\|_{2\vartheta}\Big)^{2\vartheta}\\
=&\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}}\Big(\|\Phi\|_{2\vartheta}+240\sum_{k=1}^{n}k^{-1/2}\|\mathcal{T}^k\Phi\|_{2\vartheta}\Big)^{2\vartheta}\\
\leq&\frac{Cn^{\vartheta}}{(n\epsilon)^{2\vartheta}}\Big(\|\Phi\|_{2\vartheta}+240\|\Phi\|_{\infty}^{(2\vartheta-1)/(2\vartheta)}\sum_{k=1}^{n}k^{-1/2}\Big(\int|\mathcal{T}^k\Phi|d\bar{\mu}\Big)^{\frac{1}{2\vartheta}}\Big)^{2\vartheta}\\
\leq&\frac{C}{n^{\vartheta}\epsilon^{2\vartheta}}\Big(\|\Phi\|_{2\vartheta}+240\|\Phi\|_{\infty}^{(2\vartheta-1)/(2\vartheta)}\sum_{k=1}^{n}\frac{1}{k}\Big)^{2\vartheta},
\end{aligned}
$$

where $\vartheta=\frac{1}{\alpha}-1>1$ and Corollary 1 from [28] is used in the second inequality. This shows (2.7.1). □

Finally we show a proposition which is used in Subsection 2.3.3.

Proposition 2.7.1. There exists $\delta_0>0$ such that for any $0<\delta<\delta_0$, $E,E'>0$, we can find C_D, $C_D'>0$ respectively and $N_d>0$ satisfying

$$\mu\Big\{x\in M:\ |Df_x^n|_{E_x^u}|<Ee^{n\delta}\Big\}\leq\frac{C_D(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}};\tag{2.7.2}$$

$$\mu\Big\{x\in M:\ |Df_x^{-n}|_{E_x^s}|<E'e^{n\delta}\Big\}\leq\frac{C_D'(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}}\tag{2.7.3}$$

for all $n\geq N_d$.

Proof. Without loss of generality, we can assume that $E=E'=1$. This is because we can always take N_d sufficiently large and incease δ to some $\delta^* > \delta$ such that $Ee^{n\delta} \le e^{n\delta^*}$ for all $n > N_d$.

Now let us prove (2.7.2).

For the finite Markov partition $\mathcal{P} = \{P_0, P_1, \cdots, P_r\}$ and fixed $\hat{\gamma}_i^u \in W^u(P_i)$, $0 \le i \le r$, consider the following function

$$\psi(x) = \begin{cases} 0 & \text{if } x \in P_0; \\ \log |Df_{\pi(x)}|_{E^u_{\pi(x)}}| & \text{if } x \notin P_0, \end{cases}$$

where π is the sliding map defined in Subsection 2.3.1. Clearly ψ is constant along the stable manifolds in P_i, $0 \le i \le r$. It can be regarded as an element in $\mathcal{L}$ as well. It is evident that $\int \psi d\bar{\mu} > 0$.

Since f is uniformly hyperbolic on $M \setminus P$, there exist two positive constants C_u and C'_u such that

$$C_u \le \log |Df_x|_{E^u_x}| \le C'_u \quad \forall x \in M \setminus P.$$

Hence, if we let $C_L = \dfrac{C_u}{C'_u}$ and $C'_L = \dfrac{C'_u}{C_u}$, then

$$C_L \le \frac{\log |Df_x|_{E^u_x}|}{\log |Df_{\pi(x)}|_{E^u_{\pi(x)}}|} \le C'_L \quad \forall x \in P_i,\ i \ne 0.$$

So,

$$\begin{aligned} \log |Df^n_x|_{E^u_x}| &= \sum_{i=0}^{n-1} \log |Df_{f^i(x)}|_{E^u_{f^i(x)}}| \\ &\ge \sum_{i=0}^{n-1} \mathbb{1}_{M\setminus P_0} \log |Df_{f^i(x)}|_{E^u_{f^i(x)}}| \ge C_L \sum_{i=0}^{n-1} \psi(f^i(x)), \end{aligned}$$

where $\mathbb{1}_{M\setminus P_0}$ is the indicator function. Hence,

$$\Big\{x \in M: \ \frac{1}{n}\log|Df_x^n|_{E_x^u}| < \delta\Big\} \subset \Big\{x \in M: \ \frac{1}{n}\sum_{i=0}^{n-1}\psi(f^i(x)) < \frac{\delta}{C_L}\Big\} \tag{2.7.4}$$

for any $\delta > 0$.

Take $\delta_0 = C_L \int \psi d\mu$, and let $0 < \delta < \delta_0$. Set $\epsilon := \int \psi d\mu - \delta/C_L$. Clearly $\epsilon > 0$. Recall that we mentioned that ψ can be regarded as functions in $\mathcal{L}$. So by Lemma 2.7.1, one has that

$$\bar{\mu}\Big\{\overline{x} \in \overline{M}: \ \Big|\sum_{i=0}^{n-1}\Big(\psi(\bar{f}^i(\overline{x})) - \int \psi d\bar{\mu}\Big)\Big| > n\epsilon\Big\} = O((\log n)^{2(\frac{1}{\alpha}-1)} n^{-(\frac{1}{\alpha}-1)}),$$

and therefore,

$$\bar{\mu}\Big\{\overline{x} \in \overline{M}: \ \frac{1}{n}\sum_{i=0}^{n-1}\psi(\bar{f}^i(\overline{x})) < \int \psi d\bar{\mu} - \epsilon\Big\} = O((\log n)^{2(\frac{1}{\alpha}-1)} n^{-(\frac{1}{\alpha}-1)}). \tag{2.7.5}$$

By (2.7.4) and (2.7.5), and the fact that $\bar{\mu}$ is the quotient measure of μ, we have that

$$\begin{aligned}
&\mu\Big\{x \in M: \ |Df_x^n|_{E_x^u}| < e^{n\delta}\Big\} \le \mu\Big\{x \in M: \ \frac{1}{n}\sum_{i=0}^{n-1}\psi(f^i(x)) < \frac{\delta}{C_L}\Big\} \\
=&\bar{\mu}\Big\{x \in M: \ \frac{1}{n}\sum_{i=0}^{n-1}\psi(\bar{f}^i(x)) < \int \psi d\bar{\mu} - \epsilon\Big\} \le \frac{C_D(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}}
\end{aligned}$$

for some $C_D > 0$. This is (2.7.2).

To get (2.7.3), we introduce the following function

$$\psi(x) = \begin{cases} 0 & \text{if } x \in P_0; \\ -\log |Df_{\pi(x)}|_{E^s_{\pi(x)}}| & \text{if } x \notin P_0. \end{cases}$$

Hence ψ is constant along the stable manifolds and can be regarded as a function in $\mathcal{L}$. It is also obvious that $\int \psi d\bar{\mu} > 0$. By using similar methods as above, we can obtain

$$\mu\Big\{x \in M : \ |Df^n_x|_{E^s_x}| > e^{-n\delta}\Big\} \leq \frac{C'_D(\log n)^{2(\frac{1}{\alpha}-1)}}{n^{\frac{1}{\alpha}-1}}$$

for some $C'_D > 0$. Note that E^s is one-dimensional. So $|Df^{-n}_{f^n(x)}|_{E^s_{f^n(x)}}| < e^{n\delta}$ if and only if $|Df^n_x|_{E^s_x}| > e^{-n\delta}$. Since μ is an invariant measure, we get (2.7.3). $\square$

Chapter 3

Some statistical properties of almost Anosov diffeomorphisms with spectral gap

3.1 Introduction

The existence of an invariant measure of a map is a basic problem in ergodic theory [25]. In smooth ergodic theory, one important result of Sinai is that a twice-differentiable Anosov diffeomorphism on a compact connected Riemannian manifold has an invariant measure, which has absolutely continuous conditional measures on unstable manifolds [42]. This was generalized to Axiom-A systems by Bowen and Ruelle [4]. Based on Sinai, Ruelle, and Bowen's work, a kind of invariant measures with absolutely continuous conditional measures on unstable manifolds is called SRB measures. The maps with SRB measures have some good dynamical properties in physics [5]. For more information on SRB measures, please refer to the survey [47]. Lots of results about the existence of SRB measures have been obtained for many systems, for example, non-uniformly hyperbolic systems by Pesin [2], singular systems by Katok et al. [18], the billiard systems and so on [6, 21].

In smooth ergodic theory and physics, some interesting systems are generated by functions of high differentiability. For example, the Lorenz system, Logistic map, Hénon map, and so on [38]. And, the differentiability of the maps affects the dynamics. For example, there exists a one-dimensional map $T : [0, 1] \to [0, 1]$, which is piecewise twice-differentiable expanding and the derivative at one fixed point is equal to one, but T can not admit a finite absolutely continuous invariant measure [30]; Hu and Young obtained that some twice-differentiable almost Anosov diffeomorphisms defined on two-dimensional spaces admit infinite SRB measures [16]; Hu obtained some results on the existence of SRB measures and infinite SRB measures for almost Anosov systems [12].

For a mixing dynamical system, the correlation function provides us with the quantitative description about how fast the state of the system becomes uncorrelated with its future status. The SRB measures play an important role in the study of the correlation functions. The transfer operator with some function spaces is a powerful tool in the study of the decay rate of correlation functions. For instance, the idea of the construction of "Young Tower" has been successfully applied to the study of many systems with exponential decay rates, like Hénon map, piecewise hyperbolic systems, scattering billiards and so on [45]. The estimation of the polynomial upper bounds for the correlation functions of some systems is obtained by the "coupling method" [46]. Later, the estimation of the polynomial lower bounds for the correlation functions of some maps is studied by the "renewal theory" [40], which is sharped by Gouëzel's results [8].

There exist many interesting results about the estimation of the correlation functions of the maps on two-dimensional manifolds. For instance, the work of Benedicks and Young on Hénon map proved the existence of SRB measures, exponential decay of correlations and so on [3], the study of Liverani and Martens on a class of area preserving maps on torus gave the upper bounds for the correlation functions [24]. In [9], the upper bounds for the correlation functions have been obtained for some systems with one center unstable direction Manneville-Pomeau-like map by Hatomoto.

In this chapter, we provide some almost Anosov maps defined on spaces with dimensions no less than two, since there are few examples in higher dimensions, these maps could be regarded as the generalization of Manneville-Pomeau-like maps in higher dimensions. And, we study the existence of SRB or infinite SRB measures for this type of maps. We obtain that the differentiability of the maps near the indifferent fixed points and the dimension of the spaces affect the existence of SRB measures (See Theorem 3.2.1). As a consequence, there are twice-differentiable almost Anosov diffeomorphisms that admit infinite SRB measures in spaces with dimensions equal to two or three, which is a generalization of the results of [16]; there exist twice-differentiable almost Anosov diffeomorphisms that admit SRB measures in spaces with dimensions bigger than three. Further, we apply the renewal theory to investigate the polynomial lower and upper bounds for maps that admit SRB measures (see Theorem 3.2.2).

The rest is organized as follows. In Section 3.2, some basic definitions and the main results are introduced. In Section 3.3, it is to study the existence of SRB or infinite SRB measures in spaces with dimensions bigger than or equal to two. In Section 3.4, the polynomial lower and upper bounds are obtained by using the renewal theory. This section consists of three parts. In Subsection 3.4.1, a quotient map by collapsing the map along the stable manifolds is introduced. In Subsection 3.4.2, both the lower and upper polynomial bounds for the decay rate of the correlation functions are obtained by using the renewal theory, where the observable functions are defined on the quotient manifold. In Subsection 3.4.3, the polynomial bounds for Hölder observable functions for the original diffeomorphisms are obtained.

3.2 Main results

In this section, the main results are introduced.

Assume M is a C^∞ compact Riemannian manifold without boundary, and the dimension of M is $m \geq 2$. Let f be a diffeomorphism defined on M satisfying the following properties:

(1) if $m \geq 3$, the map f is twice-differentiable on M, and has a fixed point p;

(1)′ if $m = 2$, the map f is $C^{1+\eta}$ on M, and has a fixed point p, where $\eta > 0$;

(2) the map f is topologically mixing, and topologically conjugate with an Anosov diffeomorphism;

(3) there exist a constant $0 < \kappa^s < 1$ and a continuous function κ^u with

$$\kappa^u(x) \begin{cases} = 1 & \text{at } x = p \\ > 1 & \text{elsewhere} \end{cases},$$

and there is a decomposition of the tangent space T_xM:

$$T_xM = E_x^u \oplus E_x^s,$$

such that

$$|Df_x|_{E_x^s}| \leq \kappa^s, \; m(Df_x|_{E_x^u}) \geq \kappa^u(x), \; Df_p|_{E_p^u} = id, \; \kappa^s L_0 < 1,$$

where

$$m(Df_x|_{E_x^u}) = \inf_{v \in E_x^u, v \neq 0} \frac{|Df_x v|}{|v|}, \; L_0 = \sup_{x \in M, v \in E_x^u, v \neq 0} \frac{|Df_x v|}{|v|};$$

(4) the dimension of E^u_x and E^s_x is $m-1$ and 1, respectively;

(5) there is a coordinate system on a small neighborhood U of p such that the map f can be written as follows:

$$\begin{aligned} &f(x_1, x_2, ..., x_{m-1}, x_m) \\ =&((1 + |(x_1, ..., x_{m-1})|^\eta + \rho x_m^2)x_1, ..., (1 + |(x_1, ..., x_{m-1})|^\eta + \rho x_m^2)x_{m-1}, \kappa_s x_m) \end{aligned} \tag{3.2.1}$$

where ρ is a nonzero constant and $|(x_1, ..., x_{m-1})| = \sqrt{\sum_{i=0}^{m-1} x_i^2}$.

Remark 3.2.1. The ρx_m^2 term could be replaced by some general differentiable function $\psi(x_m)$ with $\psi(0) = 0$.

Theorem 3.2.1. For the diffeomorphism f satisfying the above assumptions and $m \geq 2$, if $\eta > 0$ and $\min\{2, m-2\} \leq \eta < m-1$, then there exists an SRB measure; if $\eta \geq m-1$, then there is an infinite SRB measure.

Corollary 3.2.1. For the diffeomorphism f satisfying the assumptions of Theorem 3.2.1, if f admits an SRB measure, then for any continuous function $\phi : M \to \mathbb{R}$, for μ-a.e. $x \in M$, one has

$$\frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) \to \int \phi d\mu \text{ as } n \to \infty,$$

where μ is the SRB measure introduced in Theorem 3.2.1; if f has an infinite SRB measure, then for ν-a.e. $x \in M$, one has

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \delta_p \text{ as } n \to \infty,$$

where δ_z is the Dirac measure at z and ν is the Lebesgue measure on M, and the above convergence is in the weak star topology.

By applying the renewal theory developed by [40] and [8], we could obtain the following results:

Theorem 3.2.2. Let f be an almost Anosov diffeomorphism satisfying the above assumptions and m be an integer no less than 2. For $\eta > 0$, $\min\{m-2, 2\} \leq \eta < m-1$, and $m-1-\eta < \theta \leq 1$, any neighborhood V of p, and any Hölder functions Φ, Ψ with exponent θ, $\mathrm{supp}\Phi, \mathrm{supp}\Psi \subset M \setminus V$, and $\int \Phi d\mu \int \Psi d\mu \neq 0$, we have

$$\frac{A'(\Phi,\Psi)}{n^{\frac{m-1}{\eta}-1}} \leq |\mathrm{Cor}_n(\Phi,\Psi; f,\mu)| \leq \frac{A(\Phi,\Psi)}{n^{\frac{m-1}{\eta}-1}}, \tag{3.2.2}$$

where μ is an SRB measure specified in Theorem 3.2.1, and $A'(\Phi,\Psi)$ and $A(\Phi,\Psi)$ are positive constants dependent on Φ and Ψ.

3.3 The existence of SRB measures

In this section, it is to show Theorem 3.2.1. In the following discussions, assume that f satisfies the assumptions of Theorem 3.2.1.

For any $x \in M$, let E^u_x and E^s_x be the unstable and stable tangent spaces at x, respectively. For any positive constant β, set

$$E^u_x(\beta) := \{v \in E^u_x : \ |v| \leq \beta\}, \ E^s_x(\beta) := \{v \in E^s_x : \ |v| \leq \beta\},$$

and

$$E_x(\beta) := E_x^u(\beta) \times E_x^s(\beta).$$

Proposition 3.3.1. (a) For any $x \in M$, the maps $x \to E_x^u$ and $x \to E_x^s$ are continuous.

(b) There exist two continuous foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ on M tangential to E^u and E^s, respectively, such that

(1) the stable leaf $\mathcal{F}^s(x)$ is the stable manifold at x,

$$\mathcal{F}^s(x) = \{y \in M : \ d(f^k(x), f^k(y)) \leq C_s(\kappa^s)^k d(x,y), \ \forall k \geq 0\};$$

(2) the unstable leaf $\mathcal{F}^u(x)$ is the unstable manifold at x, that is,

$$\mathcal{F}^u(x) = \{y \in M : \ \lim_{k\to\infty} d(f^{-k}(x), f^{-k}(y)) = 0\};$$

(3) there exist positive constants β and C_u such that $\mathcal{F}_\beta^u(x)$ is the component of $\mathcal{F}^u(x) \cap \exp_x(E_x(\beta))$ containing x, then $\exp_x^{-1}(\mathcal{F}_\beta^u(x))$ is the graph of a function $\phi_x^u : E_x^u(\beta) \to E_x^s(\beta)$ with $\phi_x^u(0) = 0$ and $\|\phi_x^u\|_{C^1} \leq C_u$, where exp is the exponential map. Similar results also hold for $\mathcal{F}_\beta^s(x)$.

Proof. Part (a) can be derived from the gap assumption $\kappa^s < 1 = \inf\{\kappa^u(x) : \ x \in M\}$.

Part (b) can be obtained by Theorems 5.5 and 5A.1 in [11]. □

For convenience, denote by $W^u(x) := \mathcal{F}^u(x)$ and $W_\beta^u(x) := \mathcal{F}_\beta^u(x)$ the unstable manifold and local unstable manifold at x, respectively; one could define the stable manifold $W^s(x)$ and the local stable manifold $W_\beta^s(x)$, similarly. For $y \in W^s(x)$, denote by $d^s(x,y)$ the minimal distance from x to y along the stable manifold; for $y \in W^u(x)$, let $d^u(x,y)$

be the minimal distance between x and y on the unstable manifold, where the metric on $W^s(x)/W^u(x)$ is induced by the Riemannian metric restricted to $W^s(x)/W^u(x)$.

By Proposition 3.3.1, one has the following result immediately.

Corollary 3.3.1. The map f has a local product structure, that is, there are constants $0 < \epsilon < \beta$ such that for any $y, z \in M$ with $d(y, z) < \epsilon$, one has that $[y, z] = W^u_\beta(y) \cap W^s_\beta(z)$ and $[z, y] = W^u_\beta(z) \cap W^s_\beta(y)$ contain exactly one point, respectively.

Definition 3.3.1. [38] Given a set X in M, if for any $x, y \in X$, one has that $[x, y], [y, x] \in X$, then X is said to be a rectangle. A rectangle P is called proper if $\overline{\mathrm{int}P} = P$.

By Corollary 3.3.1, it is reasonable to define the following rectangle

$$[\gamma^s, \gamma^u] = \{W^u_\beta(x) \cap W^s_\beta(y) : \ x \in \gamma^s, \ y \in \gamma^u\},$$

where γ^s and γ^u are stable and unstable leaves, respectively, and the diameter of these two leaves are sufficiently small. In the following discussions, assume that the rectangles are defined by $[\gamma^s, \gamma^u]$ with sufficiently small diameter. For any rectangle X and $x \in X$, set $W^u(x, X) := W^u_\beta(x) \cap X$ and $W^s(x, X) := W^s_\beta(x) \cap X$. Given two rectangles X_1 and X_2, if for any $x \in X_1$ with $f^k(x) \in X_2$ for some $k \geq 0$, one has that $f^k(W^u(x, X_1)) \cap X_2 = W^u(f^k(x), X_2)$, then it is said that $f^k(X_1)$ u-crosses X_2; if for any $x \in X_1$ with $f^{-k}(x) \in X_2$ for some $k \geq 0$, one has that $f^{-k}(W^s(x, X_1)) \cap X_2 = W^s(f^{-k}(x), X_2)$, then it is said that $f^{-k}(X_1)$ s-crosses X_2.

Definition 3.3.2. [38] A Markov partition of M is a finite covering $\mathcal{P} = \{P_0, P_1, ..., P_l\}$ of M by proper rectangles satisfying that

(i) $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$ for $i \neq j$;

(ii) if $x \in \mathrm{int}P_i$ and $f(x) \in \mathrm{int}P_j$, then $f(W^u(x, P_i)) \supset W^u(f(x), P_j)$ and $f(W^s(x, P_i)) \subset W^s(f(x), P_j)$.

By Assumption (2), there exists a Markov partition for the map f. Suppose the Markov partition is $\mathcal{P} = \{P_0, ..., P_l\}$ with $p \in \mathrm{int}(P_0)$. Suppose that $P = P_0$ and the radius of any element in this Markov partition is sufficiently small.

Definition 3.3.3. [16] Suppose that W_1 and W_2 are two W^u-leaves, and let the holonomy map $H : W_1 \to W_2$ be continuous map defined by the sliding map along the stable manifolds. The stable manifold W^s is Lipschitz if H is Lipschitz for every (W_1, W_2, H).

Proposition 3.3.2. The stable manifold W^s is Lipschitz. Further, for any $\delta > 0$, there is $C_L > 0$ such that for any (W_1, W_2, H) with $d^s(x, H(x)) < \delta$ for any $x \in W_1$, the Lipschitz constant is less than C_L. Further, if $m \geq 3$, then the holonomy map is differentiable.

Proof. First, it is to study the case $m \geq 3$. By Assumptions (1) and (3), the map f is twice-differentiable and $\kappa^s L_0 < 1$. One could apply the method used in the proof of Theorem 6.3 in [10] to obtain that the stable foliation is C^1. Hence, the stable manifold W^s is Lipschitz.

Second, it is to consider the case $m = 2$. By Assumption $(1)'$, f is $C^{1+\eta}$. So, the arguments for $m \geq 3$ do not work. One could apply the arguments used in the proof of Proposition 2.5 in [16] to obtain the Lipschitz property of the stable manifold W^s.

This completes the proof. □

Lemma 3.3.1. [12, Lemma 8.1] Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers, C and α be two positive constants.

(i) If $a_{k-1} \geq a_k + C a_k^{1+\alpha}$ for any $k \geq 1$, then there exist $D > 0$ and $k_0 \geq 1$ such that $a_k \leq D(k - k_0)^{-\frac{1}{\alpha}}$ for sufficiently large k.

(ii) If $a_{k-1} \leq a_k + Ca_k^{1+\alpha}$ for any $k \geq 1$, then there exist $D' > 0$ and $k_0' \geq 1$ such that $a_k \geq D'(k+k_0')^{-\frac{1}{\alpha}}$ for sufficiently large k.

Lemma 3.3.2. Let $h : [-1,1] \to \mathbb{R}$ be a map, which can be written as $h(x) = x(1 + x^\eta + o(x^\eta))$ for x in a small neighborhood I of 0, where $\eta > 0$. For any $a_0 \in (0,1] \cap I$, set $a_k := h^{-k}(a_0)$, $k \geq 1$. Given any positive integer $m \geq 2$, if $0 < \eta < m-1$, then $\sum_{k=1}^{\infty} a_k^{m-1} < \infty$; if $\eta \geq m-1$, then $\sum_{k=1}^{\infty} a_k^{m-1} = \infty$.

Proof. By Lemma 3.3.1, one has that for sufficiently large k, $a_k \approx k^{-\frac{1}{\eta}}$. So, if $0 < \eta < m-1$, then $\sum_{k=1}^{\infty} a_k^{m-1} < \infty$; if $\eta \geq m-1$, then $\sum_{k=1}^{\infty} a_k^{m-1} = \infty$. □

Proposition 3.3.3. Given any small rectangle P containing p in its interior, there are two positive constants δ and D such that if Δ is a disk contained in some unstable manifold with $\operatorname{diam}(\Delta) \leq \delta$ and $\Delta \cap P = \emptyset$, then for any $y, z \in \Delta$ and $k > 0$,

$$D^{-1} \leq \frac{|Df_y^{-k}|_{E_y^u}|}{|Df_z^{-k}|_{E_z^u}|} \leq D. \tag{3.3.1}$$

Proof. First of all, we will study the case $m = 3$, the arguments for $m = 3$ below also work for $m > 3$.

Assume that P is sufficiently small such that $P \cup f(P) \subset U$ and $\operatorname{diam}(P \cup f(P)) < \epsilon$, where U is introduced in Assumption (5) and ϵ is specified in Corollary 3.3.1. So, the map H introduced in Definition 3.3.3 is well defined with respect to the local unstable manifolds contained in U. By Assumption (5), one does not need to consider the curvature on the unstable manifold $\Delta \subset U$. Let $d^u(y,z)$ denote the metric on Δ with respect to the Riemannian metric restricted to the unstable manifold. Let $d(y,z)$ be the distance defined by the Euclidean metric. By Assumption (5), one could assume that $d(y,z) = d^u(y,z)$, where y and z are in a common unstable manifold contained in U.

One could assume that there is a positive constant δ such that $P = [W^s_\delta(p), W^u_\delta(p)]$, it is also reasonable to define

$$\tau = \inf_{w \in \partial^s P \cap W^u(p,P)} \{d^u(w, f(w)) : \text{ the minimal length curve contained in } W^u(p)$$

$$\text{joining } w \text{ and } f(w)\}, \tag{3.3.2}$$

where $\partial^s P = \{w \in P : \ w \notin \mathrm{int} W^u(w, P)\}$ and $\partial^s(f(P))$ is defined similarly.

Now, it is to investigate the distortion estimation along any unstable submanifold. Consider $\Delta \subset ((f(P) \setminus P) \cap W^u(x))$ for some $x \in f(P) \setminus P$, and $f^{-i}(\Delta) \subset P$ for $1 \le i \le k-1$, then for any $y, z \in \Delta$,

$$\log \frac{|Df_y^{-k}|_{E^u_y}|}{|Df_z^{-k}|_{E^u_z}|} \le E_1 \frac{d^u(y,z)}{\tau}, \tag{3.3.3}$$

where E_1 is a positive constant determined later.

By (3.2.1), the local unstable manifold for the point in U is contained in some horizon plane, where the horizon plane could be represented by

$$\{(x_1, x_2, x_3) : \ x_3 \text{ is equal to some constant}\}.$$

So, assume that $\Delta \subset \{(x_1, x_2, x_3) : \ x_3 = E_2\}$, where E_2 is a real number. Set $\Delta_i := f^{-i}(\Delta)$, $0 \le i \le k$. Hence,

$$\Delta_i \subset A_i := \{(x_1, x_2, x_3) : \ x_3 = \kappa_s^{-i} E_2\}, \ 0 \le i \le k.$$

Next, let us introduce a function $\phi(w) = |Df_w^{-1}|_{E^u_w}|$, where $|Df_w^{-1}|_{E^u_w}| = \det(Df_w^{-1}|_{E^u_w})$. Now, it is to study the analytic expression of $\phi(w)$ on U. By (3.2.1), set $r := x_1^2 + x_2^2$, the map f restricted to the unstable manifold can be written as

$$((1+r^{\frac{\eta}{2}})x_1+\psi_1(x_1,x_2,x_3),(1+r^{\frac{\eta}{2}})x_2+\psi_2(x_1,x_2,x_3)),$$

where $\psi_1(x_1,x_2,x_3)=\rho x_3^2x_1$, $\psi_2(x_1,x_2,x_3)=\rho x_3^2x_2$. By direct calculation, the Jacobian matrix of $\phi(w)$ with respect to x_1 and x_2 is

$$\begin{pmatrix} 1+r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}-1}x_1^2+\frac{\partial\psi_1}{\partial x_1} & \eta r^{\frac{\eta}{2}-1}x_1x_2+\frac{\partial\psi_1}{\partial x_2} \\ \eta r^{\frac{\eta}{2}-1}x_1x_2+\frac{\partial\psi_2}{\partial x_1} & 1+r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}-1}x_2^2+\frac{\partial\psi_2}{\partial x_2} \end{pmatrix}.$$

Hence, the determinant is

$$\begin{aligned}
&\left(1+r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}-1}x_1^2+\frac{\partial\psi_1}{\partial x_1}\right)\left(1+r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}-1}x_2^2+\frac{\partial\psi_2}{\partial x_2}\right)\\
&-\left(\eta r^{\frac{\eta}{2}-1}x_1x_2+\frac{\partial\psi_1}{\partial x_2}\right)\left(\eta r^{\frac{\eta}{2}-1}x_1x_2+\frac{\partial\psi_2}{\partial x_1}\right)\\
=&1+(2+\eta)r^{\frac{\eta}{2}}+(1+\eta)r^{\eta}+\frac{\partial\psi_1}{\partial x_1}(1+r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}-1}x_2^2)+\frac{\partial\psi_2}{\partial x_2}(1+r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}-1}x_1^2)\\
&+\frac{\partial\psi_1}{\partial x_1}\frac{\partial\psi_2}{\partial x_2}-\eta r^{\frac{\eta}{2}-1}x_1x_2\frac{\partial\psi_2}{\partial x_1}-\eta r^{\frac{\eta}{2}-1}x_1x_2\frac{\partial\psi_1}{\partial x_2}-\frac{\partial\psi_1}{\partial x_2}\frac{\partial\psi_2}{\partial x_1}\\
=&1+(2+\eta)r^{\frac{\eta}{2}}+(1+\eta)r^{\eta}+\rho x_3^2(2+2r^{\frac{\eta}{2}}+\eta r^{\frac{\eta}{2}})+\rho^2x_3^4. \qquad (3.3.4)
\end{aligned}$$

Hence, for fixed x_3, the level curves for the function $\phi(w)$ are circles contained in some horizon plane. This, together with (3.2.1) and (3.3.4), yields that the image of level curves under f are also level curves.

Denote by $y_i := f^{-i}(y)$ and $z_i := f^{-i}(z)$, $i\geq 0$. Let O_1 be the plane containing the x_3-axis and the point y, O_2 be the plane containing the x_3-axis and the point z. By (3.2.1), one has that $y_i\in O_1$ and $z_i\in O_2$, $0\leq i\leq k$. Denote by l_{z_i} the level curves contained in

A_i. The set $l_{z_i} \cap O_1$ has two points, take $z_i^* \in l_{z_i} \cap O_1$, which is closer to the point y_i. Let S_i be the line segment in the plane A_i joining the points z_i^* and y_i. By (3.2.1), one has that $S_{i+1} = f^{-1}(S_i)$, and there is a line segment Γ_0 such that $S_0 \subset \Gamma_0$ and the end points of Γ_0 are two points w and w', where $w \in \partial^s P$ and $w' \in \partial^s f(P)$, and $\partial^s P$ is specified in (3.3.2). Set $\Gamma_i := f^{-i}(\Gamma_0)$, $i \geq 0$. So, length$(\Gamma_0) \geq C_0\tau$ and $S_i \subset \Gamma_i$, where C_0 is a positive constant determined by Proposition 3.3.2.

By Theorem 9.2 in [36] and f is twice-differentiable in Assumption (1), one has that

$$\begin{aligned}\big||Df_{y_i}^{-1}|_{E^u_{y_i}}| - |Df_{z_i^*}^{-1}|_{E^u_{z_i^*}}|\big| &= \left|\int_0^1 D(|Df^{-1}_{(z_i^*+t(y_i-z_i^*))}|_{E^u_{(z_i^*+t(y_i-z_i^*))}}|)(y_i - z_i^*)dt\right| \\ \leq & C_1|D(|Df_{y_i}^{-1}|_{E^u_{y_i}}|)||y_i - z_i^*| \leq C_2 d(y_i, z_i^*), \end{aligned} \tag{3.3.5}$$

where C_1 and C_2 are two positive constants, these are derived by the fact that $|Df^{-1}|_{E^u}|$ is uniformly continuous and differentiable, since f is twice-differentiable on the compact manifold M, and the point z_i^* falls into a uniformly small neighborhood of the point y_i for sufficiently large i. So, for $j \leq k$, by Assumption (1), one has

$$\begin{aligned}\log\frac{|Df_y^{-j}|_{E^u_y}|}{|Df_z^{-j}|_{E^u_z}|} &\leq \log\prod_{i=0}^{j-1}\left(1 + \frac{\big||Df_{y_i}^{-1}|_{E^u_{y_i}}| - |Df_{z_i}^{-1}|_{E^u_{z_i}}|\big|}{|Df_{z_i}^{-1}|_{E^u_{z_i}}|}\right) \leq C_3\sum_{i=0}^{j-1}\big||Df_{y_i}^{-1}|_{E^u_{y_i}}| - |Df_{z_i}^{-1}|_{E^u_{z_i}}|\big| \\ =& C_3\sum_{i=0}^{j-1}\big||Df_{y_i}^{-1}|_{E^u_{y_i}}| - |Df_{z_i^*}^{-1}|_{E^u_{z_i^*}}|\big| \leq C_3C_2\sum_{i=0}^{j-1}|y_i - z_i^*| = C_3C_2\sum_{i=0}^{j-1}d^u(y_i, z_i^*),\end{aligned} \tag{3.3.6}$$

where $C_3 > 0$ is a constant dependent on f.

Since Γ_i is a line segment, for any $w \in \Gamma_i$, let $|Df^{-1}|_{\Gamma_i(w)}| = |Df^{-1}(w)\vec{v}_w|$, where $\vec{v}_w$ is a tangent vector of the curve Γ_i at the point w with unit length, and $|Df^{-1}(w)\vec{v}_w|$ is the length of the vector $Df^{-1}(w)\vec{v}_w$.

It follows from the twice-differentiability of f and Theorem 9.2 in [36] that

$$\begin{aligned}&\big||Df^{-1}|_{\Gamma_i(y_i)}| - |Df^{-1}|_{\Gamma_i(z_i^*)}|\big| = \left|\int_0^1 D(|Df^{-1}|_{\Gamma_i(z_i^*+t(y_i-z_i^*))}|)(y_i - z_i^*)dt\right| \\ \le& C_4|D(|Df^{-1}|_{\Gamma_i(y_i)}|)||y_i - z_i^*| \le C_5 d(y_i, z_i^*) \le C_5 \mathrm{length}(\Gamma_i), \end{aligned} \tag{3.3.7}$$

where C_4 and C_5 are two positive constants. Hence, for $j \le k$, one has

$$\begin{aligned}&\log \frac{|Df^{-j}|_{\Gamma_0(y)}|}{|Df^{-j}|_{\Gamma_0(z^*)}|} \le \log \prod_{i=0}^{j-1} \left(1 + \frac{\big||Df^{-1}|_{\Gamma_i(y_i)}| - |Df^{-1}|_{\Gamma_i(z_i^*)}|\big|}{|Df^{-1}|_{\Gamma_i(z_i^*)}|}\right) \\ \le& C_6 \sum_{i=0}^{j-1} \big||Df^{-1}|_{\Gamma_i(y_i)}| - |Df^{-1}|_{\Gamma_i(z_i^*)}|\big| \le C_6 C_5 \sum_{i=0}^{j-1} |y_i - z_i^*| = C_6 C_5 \sum_{i=0}^{j-1} d^u(y_i, z_i^*), \end{aligned} \tag{3.3.8}$$

where $C_6 > 0$ is a constant dependent on f. Thus, one has

$$\frac{d^u(y_j, z_j^*)}{\mathrm{length}(\Gamma_j)} \le C_7 \frac{d^u(y, z^*)}{\mathrm{length}(\Gamma_0)}, \ \forall j \le k, \tag{3.3.9}$$

where C_7 is a positive constant dependent on f.

Let $\hat{\Gamma}_i$ be the image of Γ_i under the map $H : \Delta_i \to W^u(p)$, $0 \le i \le k$. Since $\hat{\Gamma}_i$'s are pairwise disjoint and Proposition 3.3.2, one has that

$$\sum_{i=1}^{j-1} \mathrm{length}(\Gamma_i) \le \sum_{i=1}^{j-1} C_L \mathrm{length}(\hat{\Gamma}_i) \le C_L \mathrm{diam}(W^u(p, P)). \tag{3.3.10}$$

Hence, it follows from (3.3.6)–(3.3.10) that

$$\log \frac{|Df_y^{-j}|_{E_y^u}|}{|Df_z^{-j}|_{E_z^u}|} \le \sum_{i=0}^{j-1} C_3C_2 d^u(y_i, z_i^*)$$
$$\le C_3C_2C_7C_L \mathrm{diam}(W^u(p,P)) \frac{d^u(y,z^*)}{\mathrm{length}(\Gamma_0)} \le E_1 \frac{d^u(y,z^*)}{\tau} \le E_1 \frac{d^u(y,z)}{\tau},$$

where $E_1 = C_3C_2C_7C_L\mathrm{diam}(W^u(p,P))/C_0$. This verifies (3.3.3).

Now, it is to show (3.3.1).

By the properties of the Markov partition, there is a constant $\delta > 0$ such that the diameter of Δ is less than δ and if $\mathrm{int}(\Delta_i) \cap (f(P) \setminus P) \neq \emptyset$, then $\Delta_i \subset f(P) \setminus P$. Suppose that the number of the orbits of y and z comes back to P is s_0, and there exist positive integers k_i and l_i, $1 \le i \le s_0$, such that

$$P \cap \Delta_j \neq \emptyset, \ \forall j \in \bigcup_{1 \le i \le s_0} ((k_i, k_i + l_i) \cap \mathbb{Z}),$$

and

$$P \cap \Delta_j = \emptyset, \ \forall j \notin \bigcup_{1 \le i \le s_0} ((k_i, k_i + l_i) \cap \mathbb{Z}),$$

where $\Delta_j = f^{-j}(\Delta)$. So,

$$\log \frac{|Df_y^{-k}|_{E_y^u}|}{|Df_z^{-k}|_{E_z^u}|} = \sum_{i=1}^{s_0} \log \frac{|Df_{y_{k_i}}^{-l_i}|_{E_{y_{k_i}}^u}|}{|Df_{z_{k_i}}^{-l_i}|_{E_{z_{k_i}}^u}|} + \sum_{i=0}^{s_0} \sum_{j=k_i+l_i}^{k_{i+1}-1} \log \frac{|Df_{y_j}^{-1}|_{E_{y_j}^u}|}{|Df_{z_j}^{-1}|_{E_{z_j}^u}|}.$$

The first part can be estimated by (3.3.3), and the second part is outside of P, which is a geometric sequence by Assumption (3), where f is uniformly hyperbolic outside of P. So, (3.3.1) holds.

Finally, it is to study the case $m = 2$ and $0 < \eta < 1$. For the case that $m = 2$ and $\eta \geq 1$, one could apply similar arguments for $m = 3$ as above.

Suppose that $P \cup f(P) \subset U$ and $\mathrm{diam}(P \cup f(P)) < \epsilon$. Fix any $0 < \delta \leq \epsilon$, it is to verify that if Δ is homermorphic to an interval such that $\Delta \subset ((f(P) \setminus P) \cap W^u(x))$ for some $x \in f(P) \setminus P$, $\mathrm{diam}(\Delta) \leq \delta$, and $f^{-i}(\Delta) \subset P$ for $1 \leq i \leq k-1$, then for any $y, z \in \Delta$,

$$\log \frac{|Df_z^{-k}|_{E_z^u}|}{|Df_y^{-k}|_{E_y^u}|} \leq D'' d^u(y,z)^{\vartheta}, \tag{3.3.11}$$

where D'' is a positive constant and $\vartheta = \frac{\eta}{1+\eta}$.

By Proposition 3.3.2, it suffices to study the distortion estimates along the unstable manifold of the indifferent fixed point p, that is, it is enough to study $f : W^u(p,P) \to W^u(p)$. It is evident that f is injective when it is restricted to $W^u(p,P)$ and $f^{-1}(W^u(p,P)) \subset W^u(p,P)$. Suppose $f(x_1) = x_1 + x_1^{1+\eta} + \phi(x_1)$ for $x_1 > 0$, when f is restricted to the unstable manifold, where $\phi(x_1)$ is the higher order term. In Assumption (5), there is no higher order term, the reason we add this higher order term is that we find the arguments here also work for the map with this higher order term. In other words, if $m = 2$ and $0 < \eta < 1$, we could generalize Assumption (5). So, assume that $\Delta \subset f(W^u(p,P)) \setminus W^u(p,P)$. Hence, one has that for any $y, z \in \Delta$ with $d(y,z) \leq |y|/2$,

$$d(f(y), f(z)) \geq (1 + \bar{C}_1'|y|^{\eta}) d(y,z),$$

$$\log \left| \frac{\det Df(y)}{\det Df(z)} \right| \leq \bar{C}_2 |y|^{\eta-1} d(y,z),$$

where $\bar{C}_1'$ and $\bar{C}_2$ are two positive constants. For any $y, z \in \Delta$, set $y_i := f^{-i}(y)$ and

$z_i := f^{-i}(z)$. By direct calculation, one has that

$$d(y_i, z_i)^{1-\vartheta} \leq d(y_i, y_{i+1})^{1-\vartheta} \leq \bar{D}_1|y_i + y_i^{\eta+1} - y_i|^{1-\vartheta} = \bar{D}_1|y_i|^{(1+\eta)(1-\vartheta)} = \bar{D}_1|y_i|,$$

where $\bar{D}_1$ is a positive constant. It follows from Lemma 3.3 in [14] that (3.3.11) holds.

By using (3.3.11) and the same argument in the case $m \geq 3$ as above, one can show (3.3.1) holds for $m = 2$ and $0 < \eta < 1$.

This completes the proof. □

Proposition 3.3.4. The unstable manifold $W^u(p)$ and the stable manifold $W^s(p)$ are dense in M, respectively.

Proof. It is to show that $W^u(p)$ is dense. Similar arguments also work for $W^s(p)$.

Take any rectangle X with $\text{int}X \neq \emptyset$. Take a strictly smaller rectangle $\hat{X} \subset \text{int}X$. It follows Assumption (2) that there is $k > 0$ such that $f^{-k}(\hat{X}) \cap P \neq \emptyset$. So, if k is sufficiently large, then $f^{-k}(X)$ s-crosses P. Hence, $f^k(W^u(p, P)) \cap X \neq \emptyset$. This completes the proof. □

Definition 3.3.4. Given any subset $E \subset M$, the first return map is given by $g = f^{\tau(x)}(x) : M \setminus E \to M \setminus E$, where $\tau(x) = \min\{i > 0 : f^i(x) \in M \setminus E\}$ is the first return time function with respect to the set E.

Lemma 3.3.3. There exists an ergodic invariant Borel probability measure ν_g for the map g, which has absolutely continuous conditional measures on the unstable manifolds of f.

Proof. First of all, it is to show the existence of an invariant measure, which has absolutely continuous conditional measures on unstable manifolds.

Suppose that $P = [W^u_\delta(p), W^s_\delta(p)]$, where δ is a small positive constant. Denote $\hat{P} := f(P) \setminus P$. If the dimension of M is bigger than two, then $\hat{P}$ is connected. Set $Q := W^u(x, \hat{P})$.

Denote by ν_Q the Lebesgue measure on Q, and $(g_*^k \nu_Q)(E) = \nu_Q(g^{-k}(E))$. Take a limit of the sequence $\frac{1}{k}\sum_{i=0}^{k-1} g_*^i(\nu_Q)$ in the weak star topology, denoted by ν_g. It is evident that ν_g is invariant.

Now, it is to show that ν_g has absolutely continuous conditional measures on unstable manifolds.

For any small rectangle K in $M \setminus P$, each component of $g^i(Q)$ is a disjoint union of W^u leaves, which are contained in some element from the Markov partition, and if any component of $g^i(Q)$ intersects K, then it u-crosses K by Assumption (2) and the discussions used in the proof of Proposition 3.3.4. Let ρ_i be the density of $g_*^i(\nu_Q)$ with respect to the Lebesgue measure on $g^i(Q)$, where ν_Q is the Riemannian measure ν induced on Q. It follows from Proposition 3.3.3 that for any x, y in the same component of $g^i(Q) \cap K$,

$$D^{-1} \leq \frac{\rho_i(x)}{\rho_i(y)} \leq D,$$

where D is independent of i. It is evident that similar estimates on the limit densities could be obtained.

Finally, the ergodicity of g with respect to ν_g can be derived by the discussions in the proof of Lemma 5.3 in [16], where the application of Lemma 5.1 in the proof of Lemma 5.3 of [16] is replaced by Assumption (2).

This completes the proof.

$\square$

Denote $S := f^{-1}P \setminus P$, where $P = P_0$ is the element of the Markov partition $\mathcal{P}$ containing p. Without loss of generality, assume that $P = [W_\delta^u(p), W_\delta^s(p)]$. It is evident that S consists of points $x \in M$ with $\tau(x) > 1$, where τ is the first return time function defined in Definition

3.3.4. Set $S^{(k)} := [\tau \geq k]$. So, one has that $S = S^{(2)}$ and $S^{(k+1)} \subset S^{(k)}$ for any $k \geq 2$. Further, one has that $W^s(x, S^{(k)}) = W^s(x, S)$ and $W^u(x, S^{(k)}) \subset W^u(x, S)$ for any $x \in S^{(k)}$.

For any unstable leaf $\gamma^u \in W^u(S)$, denote by $\gamma_k^u = \gamma^u \cap S^{(k)}$. By Assumption (5) and Lemma 3.3.1, one has that there exist $D'_\eta > 0$ and $D_\eta > 0$ such that

$$\frac{D'_\eta}{k^{\frac{m-1}{\eta}}} \leq \nu_\gamma^u(\gamma_k^u) \leq \frac{D_\eta}{k^{\frac{m-1}{\eta}}}, \tag{3.3.12}$$

where ν_γ^u is the Lebesgue measure restricted to γ^u.

Finally, it is to show Theorem 3.2.1.

Proof. Set $R_i := \{x \in M \setminus P : \ R(x) = i\}$. Denote

$$\mu := \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} f_*^j(\nu_g | R_i),$$

where ν_g is the measure specified in Lemma 3.3.3. So, μ has absolutely continuous conditional measures on the unstable manifolds by Lemma 3.3.3.

Let $\widetilde{S}^{(i)}$ be the projection of $S^{(i)}$ onto $W^u(p, P)$ along W^s. By Proposition 3.3.2, one has

$$\mu(S^{(i)}) \approx \nu(\widetilde{S}^{(i)}),$$

where $\nu(\widetilde{S}^{(i)})$ is the volume or the Lebesgue measure of $\widetilde{S}^{(i)}$ restricted to $W^u(p, P)$. This, together with the fact that $f^i(S^{(i)})$ are pairwise disjoint subsets of P, μ is invariant, (3.3.12), and Lemma 3.3.2, yields that

$$\mu(P) \approx \sum_{i=1}^{\infty} \mu(f^i(S^{(i)})) = \sum_{i=1}^{\infty} \mu(S^{(i)}) \approx \sum_{i=1}^{\infty} \nu(\widetilde{S}^{(i)}) \approx \sum_{i=1}^{\infty} i^{-\frac{m-1}{\eta}},$$

which is convergent whenever $0 < \eta < m-1$, divergent whenever $\eta \geq m-1$. Hence, if $0 < \eta < m-1$, then f has an SRB measure; if $\eta \geq m-1$, the map f admits an infinite SRB measure. Further, for any open neighborhood V of p, the set $M \setminus (\cap_{i=-k}^{k} f^i(P))$ contains the set $M \setminus V$ for large enough k, and it is a finite set with respect to the measure μ. This yields that μ is at most σ-finite.

This completes the proof of Theorem 3.2.1. $\square$

Now, it is to prove Corollary 3.2.1.

Proof. This could be derived by using the Birkhoff Ergodic Theorem and the arguments used in the proof of Theorem B in [16]. $\square$

3.4 Decay of correlations

In this section, it is to verify Theorem 3.2.2. The proof is split into three parts, which are studied in three subsections. In the first subsection, a quotient one-dimensional expanding system $(\bar{f}, \overline{M})$ with an indifferent fixed point $\bar{p}$ for the original map (f, M) is introduced by taking the Markov partition $\mathcal{P}$ and collapsing the stable manifolds in each element of the partition. In the second subsection, the lower and upper bounds for the decay of correlations for observable functions on the quotient manifold $\overline{M}$ is obtained by using the renewal theory. In the last subsection, the decay rates for the original system (f, M) is studied by using the estimates for the quotient map, where the main estimation is the size of the elements in the set $f^k(\mathcal{M}_{2k})$, where $\mathcal{M}_k$ is specified in (3.4.6).

3.4.1 Induce to one-dimensional map

In this subsection, a quotient map by collapsing the map along the stable manifolds is introduced.

For the given finite Markov partition $\mathcal{P} = \{P_0, P_1, \cdots, P_l\}$, assume that $p \in \mathrm{int}P_0 \subset V$, where V is specified in Theorem 3.2.2. Given any P_i and $x \in P_i$, let $\gamma^s(x)$ be the connected component of stable leaf containing x in P_i, and $W^s(P_i)$ be the set of all such leaves. And, $\gamma^u(x)$ and $W^u(P_i)$ are defined similarly.

Now, it is to introduce an equivalent relation on M by $x \sim y$, whenever x and y are belong to the same stable leave $\gamma^s \in W^s(P_i)$ for some P_i. Let $\bar{x} = \gamma^s(x)$ be the equivalent class containing x for $x \in P_i$. Set $\overline{M} := M/\sim$. There is a natural projection map $\pi : M \to \overline{M}$. Let $\overline{\mathcal{B}}$ be the completion of the Borel algebra of $\overline{M}$.

It follows from the fact that $\mathcal{P}$ is a Markov partition that $f(\gamma^s(x)) \subset \gamma^s(f(x))$ for any $x \in P_i$ with $f(x) \in P_j$. So, the quotient map $\overline{f} : \bar{x} \in \overline{M} \to \overline{f(x)} \in \overline{M}$ is well defined. Set $\overline{P}_i := P_i/\sim$ and $\overline{\mathcal{P}} := \{\overline{P}_0, ..., \overline{P}_l\}$. From the fact that $f(\gamma^u(x)) \supset \gamma^u(f(x))$ for any $x \in P_i$ and $f(x) \in P_j$, it follows that $\overline{\mathcal{P}}$ is a Markov partition for $\overline{f}$.

Take an arbitrary $\hat{\gamma}_i^u \in W^u(P_i)$, $0 \le i \le l$. By abuse of notation, let $\pi : P_i \to \hat{\gamma}_i^u$ be the sliding map along stable leaves such that for any $x \in P_i$, $\pi(x) = \hat{x} := \gamma^s(x) \cap \hat{\gamma}_i^u$, where $\gamma^s(x) \in W^s(P_i)$.

Now, it is to define a reference measure $\overline{\upsilon}$ on $\overline{M}$. For each $\gamma \in W^u(P_i)$, denote by ν_γ the Lebesgue measure restricted to γ. We introduce the following function

$$u_k(x) := \sum_{i=0}^{k-1} \big(\log |Df_{x_i}|_{E^u_{x_i}}| - \log |Df_{\widehat{x}_i}|_{E^u_{\widehat{x}_i}}|\big),$$

where $x_i = f^i(x)$ and $\widehat{x}_i = f^i(\widehat{x})$. By Proposition 3.3.3, one has that u_k converges uniformly to some function u. The measure υ is defined by $d\upsilon_\gamma(x) := e^{u(x)} d\nu_\gamma(x)$. By using the statement and proof of (1) of Lemma 1 in Subsection 3.1 in [45], it is reasonable to introduce a measure $\overline{\upsilon}$ on $\overline{M}$ satisfying $\overline{\upsilon}|_{\overline{P}_i} = \upsilon_{\hat{\gamma}_i^u}$.

From the definition above, the Jacobian of f with respect to υ is

$$J(f)(x) = |D(f)|_{E_x^u}| \cdot e^{u(f(x))} \cdot e^{-u(x)} \tag{3.4.1}$$

for υ_γ almost every $x \in M$. By applying the statement and proof of (2) of Lemma 1 in Subsection 3.1 in [45], one has that $J(\overline{f})(\bar{x})$ can be defined as $J(f)(x)$ for any $x \in \gamma_x^s$.

Let μ be the SRB measure of f obtained by Theorem 3.2.1. So, μ induces an invariant measure $\overline{\mu}$ on $\overline{M}$ naturally. By Proposition 3.3.3, one has the equivalence of the conditional measure and the Lebesgue measure, whenever the measure is restricted to any unstable leaf γ^u away from the indifferent fixed point p. This tells us that $\overline{\mu}$ is equivalent to $\overline{\upsilon}$ away from $\bar{p}$, and it has an absolutely continuous measure with respect to $\overline{\upsilon}$.

Now, one has the following Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ (see [1, 40]):

(i) (Generator property) The complete and smallest σ-algebra containing $\cup_{k\geq 0}\overline{f}^{-k}(\overline{\mathcal{P}})$ is $\overline{\mathcal{B}}$;

(ii) (Markov property) $\overline{f}(\overline{P}_i) \supset \overline{P}_j$ (mod $\bar{\nu}$), whenever $\bar{\mu}(\overline{f}(\overline{P}_i) \cap \overline{P}_j) > 0$ for any $\overline{P}_i, \overline{P}_j \in \overline{\mathcal{P}}$;

(iii) (Local invertibility) the map $\overline{f} : \overline{P}_i \to \overline{f}(\overline{P}_i)$ is invertible with measurable inverse for any $\overline{P}_i \in \overline{\mathcal{P}}$ with $\bar{\mu}(\overline{P}_i) > 0$.

It follows from Assumption (2) that this Markov map is irreducible.

3.4.2 Polynomial decay rates

In this subsection, the lower and upper bounds for the decay rates of the observable functions for the induced system $(\bar{f}, \overline{M})$ is investigated by applying the renewal theory.

Set $\widetilde{M} := \overline{M} \setminus \overline{P}$. Recall that $g = f^{\tau}$ is the first return map on $M \setminus P$ and $P = P_0$. It is evident that g on $M \setminus P$ induces a first return map from $\widetilde{M}$ to itself, denoted by $\widetilde{f}$. It follows from $p \in \operatorname{int} P$ that $\bar{p} \in \operatorname{int} \overline{P}$.

Let $\overline{\mathcal{P}}_0 = \overline{\mathcal{P}} \setminus \{\overline{P_0}\}$ be the Markov partition of $\widetilde{M}$. Set $\mathfrak{T} := \mathfrak{T}' \vee \overline{\mathcal{P}}_0$, where $\mathfrak{T}' = \{\mathfrak{T}_k = [\tau = k] : k = 1, 2, \cdots\}$ is a partition into sets with the same return time.

The separation time is given by

$$s(\bar{x}, \bar{y}) := \sup\{k \geq 0 : \ \widetilde{f}^i(\bar{y}) \in \mathfrak{T}(\widetilde{f}^i(\bar{x})), \ 0 \leq i \leq k\}, \ \forall \bar{x}, \bar{y} \in \widetilde{M}.$$

For any $x \in \bar{x}$ and $y \in \bar{y}$, it is also reasonable to set $s(x, y) := s(\bar{x}, \bar{y})$.

It follows from Assumption (3) that the map f is uniformly hyperbolic outside of any neighborhood of the fixed point p. One could define

$$\lambda := \sup\{\|Df_x|_{E_x^u}\|^{-1}, \ \|Df_x^{-1}|_{E_x^s}\|^{-1} : \ x \in M \setminus P\}, \tag{3.4.2}$$

where $\|Df_x|_{E_x^u}\| = \sup_{v \in E_x^u, v \neq 0} \frac{|Df_x v|}{|v|}$ and $\|Df_x^{-1}|_{E_x^s}\| = \sup_{v \in E_x^s, v \neq 0} \frac{|Df_x^{-1} v|}{|v|}$. It is evident that $\lambda \in (0, 1)$.

Next, it is to introduce a Banach space defined on $\overline{M}$:

$$\mathcal{L} := \{\Phi : \operatorname{supp}\Phi \subset \widetilde{M}, \ \|\Phi\|_{\mathcal{L}} := \|\Phi\|_{\infty} + D\Phi < \infty\}, \tag{3.4.3}$$

where $\|\cdot\|_{\mathcal{L}}$ is the norm, $D\Phi$ is a semi-norm given by

$$D\Phi := \sup_{\bar{x},\bar{y}\in\widetilde{M}} \frac{|\Phi(\bar{x})-\Phi(\bar{y})|}{\lambda^{\theta s(\bar{x},\bar{y})}},$$

$\eta > 0$, $\min\{2, m-2\} \leq \eta < m-1$, and $m-1-\eta \leq \theta \leq 1$.

Remark 3.4.1. It is evident that $\mathcal{L}$ contains Hölder functions with exponent θ and the support contained in $\widetilde{M}$.

Hence, one has the following result by using the renewal theory.

Lemma 3.4.1. Assume $\eta > 0$, $\min\{2, m-2\} \leq \eta < m-1$, and $m-1-\eta \leq \theta \leq 1$. The Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ is irreducible and measure preserving. There exists $C > 0$ such that for any $\Phi \in \mathcal{L}$ and $\Psi \in L^\infty$ with $\mathrm{supp}\Psi \subset \widetilde{M}$, one has

$$\left|\mathrm{Cor}_n(\Phi, \Psi; \overline{f}, \bar{\mu}) - \left(\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]\right) \int \Phi \int \Psi\right| \leq CF_\varrho(n)\|\Psi\|_\infty \|\Phi\|_{\mathcal{L}}, \tag{3.4.4}$$

where $\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]$ has order $n^{-(\varrho-1)}$, $F_\varrho(n) = O(1/n^{2\varrho-2})$, and $\varrho = \frac{m-1}{\eta}$.

Proof. It follows from the discussions in the previous subsection that the Markov map $(\overline{M}, \overline{\mathcal{B}}, \bar{\mu}, \overline{f}, \overline{\mathcal{P}})$ is irreducible measure preserving.

Next, it is to apply Theorem 6.3 in [8] to show (3.4.4).

First, it is to prove that $\widetilde{f}$ has big image property. This could be derived by the finiteness of the Markov partition $\overline{\mathcal{P}}$ and the discussions about the the big image property in Section 6.2 in [8].

Second, it is to verify that $\log J(\widetilde{f})$ is locally Hölder continuous, which is introduced in [40] (see also [1]).

It follows from Proposition 3.3.3 that $\log J(\widetilde{f}) \in \mathcal{L}$. By applying similar arguments used in Lemma 2 in Subsection 3.1 in [45], one has that $\widetilde{f}$ admits an absolutely continuous invariant measure $\widetilde{\mu}$ on $\widetilde{M}$ with the density function $\widetilde{h}$ with respect to $\widetilde{v}$, and the density function satisfies $\log \widetilde{h} \in \mathcal{L}$ and is bounded away from 0 and infinity. By uniqueness we know that $\widetilde{\mu}$ is the conditional measure mentioned in the last subsection with respect to $\widetilde{M}$.

The Jacobian of $\widetilde{f}$ with respect to $\widetilde{\mu}$ is defined as follows

$$J_{\widetilde{\mu}}(\widetilde{f}) = J(\widetilde{f}) \frac{\widetilde{h} \circ \widetilde{f}}{\widetilde{h}}.$$

By the fact that $\log J(\widetilde{f})$ and $\log \widetilde{h}$ are in $\mathcal{L}$, one has that $-\log J_{\widetilde{\mu}}(\widetilde{f})$ is also in $\mathcal{L}$, yielding that $-\log J_{\widetilde{\mu}}(\widetilde{f})$ is locally Hölder continuous.

Now, it is to prove that greatest common divisor of $\{\tau(\bar{x}) - \tau(\bar{y}) : \bar{x}, \bar{y} \in \overline{M}\}$ is one, and $\bar{\mu}[\tau > k] = O(1/k^{\varrho})$.

It follows from our construction that the greatest common divisor of $\{\tau(\bar{x}) - \tau(\bar{y}) : \bar{x}, \bar{y} \in \overline{M}\}$ is one. So, one only needs to estimate $\bar{\mu}[\tau > k]$. Let $\gamma^u \in W^u(S)$ be any unstable leaf. Denote by μ^u_γ the conditional measure of the SRB measure μ when it is restricted to γ^u. By Proposition 3.3.3, the distortion of f along any unstable leaf is uniformly bounded. Similar conclusions also work for the density function $\dfrac{d\mu^u_\gamma}{d\nu^u_\gamma}$.

Hence, by (3.3.12), there exist $C_1', C_1 > 0$ such that

$$\frac{C_1}{n^{\varrho}} \le \mu^u_\gamma(\gamma^u_n) \le \frac{C_1'}{n^{\varrho}}.$$

By direct integration and Proposition 3.3.2, one has that similar inequalities also hold for $\mu[\tau > n]$ with different positive constant coefficients, that is, there exist two positive constants B_1' and B_1 such that

$$\frac{B_1'}{n^\varrho} \leq \mu[\tau > n] \leq \frac{B_1}{n^\varrho}. \tag{3.4.5}$$

It gives that $\displaystyle\sum_{k=n+1}^{\infty} \bar{\mu}[\tau > k]$ has the order $n^{-(\varrho-1)}$.

It follows from Theorem 6.3 in [8] that the statement of this lemma is correct.

The proof is completed. □

3.4.3 Polynomial decay rates for diffeomorphisms

In this subsection, it is to establish the polynomial decay rates of correlation function for the almost Anosov diffeomorphisms by using the results in previous subsections.

First, it is to introduce a type of Hölder functions:

$$\mathcal{H}_\theta := \left\{\Phi : \ \exists H_\Phi > 0 \ \text{ s.t. } \ |\Phi(x) - \Phi(y)| \leq H_\Phi |x-y|^\theta \text{ and } \mathrm{supp}(\Phi) \subset M \setminus P\right\},$$

where $m \geq 2$, $\eta > 0$, $\min\{m-2, 2\} \leq \eta < m-1$, and $m - 1 - \eta < \theta \leq 1$.

Set

$$\mathcal{M}_0 := \mathcal{P} = \{P_0, P_1, \cdots, P_l\} \text{ and } \mathcal{M}_k := \bigvee_{i=0}^{k} f^{-i}(\mathcal{M}_0), \tag{3.4.6}$$

where k is any positive integer.

For any $n \in \mathbb{Z}$, let $k = k(n)$ be a number smaller than n, which will be given later. For any $\Phi, \Psi \in \mathcal{H}_\theta$, by direct calculation, one has

$$
\begin{aligned}
&|\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| = |\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu)| \\
\leq &|\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu)| \\
+&|\mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| + |\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)|,
\end{aligned}
\tag{3.4.7}
$$

where $\overline{\Psi}_k$ is a function defined by $\overline{\Psi}_k|A := \inf\{\Psi(x) : \ x \in f^k(A)\}$ for any $A \in \mathcal{M}_{2k}$, $\overline{\mathcal{P}}_k$ is defined analogously, and $\mathcal{P}_k := \frac{d((f^k)_*(\overline{\mathcal{P}}_k\mu))}{d\mu}$, where $\overline{\mathcal{P}}_k\mu$ is the signed measure satisfying that the density with respect to μ is $\overline{\mathcal{P}}_k$.

First, it is to estimate the diameter of the set $(f^k(\mathcal{M}_{2k}(x))$, which is useful in the study the each item in the inequality (3.4.7). Since f is uniformly hyperbolic outside of P, it suffices to estimate the diameter of the set $f^k(\cap_{i=0}^{2k} f^{-i}(P))$, and it is to show that

$$
f^k(\cap_{i=0}^{2k} f^{-i}(P)) \leq C_d k^{-\frac{1}{\eta}}, \tag{3.4.8}
$$

where C_d is a positive constant.

Now, it is to study the diameter of $f^k(\cap_{i=0}^{2k} f^{-i}(P))$ along the stable direction. By Assumption (3), one has

$$
\mathrm{diam}(f^k(\mathcal{M}_{2k}(x))) \leq C_s'(\kappa^s)^k;
$$

the diameter of $f^k(\cap_{i=0}^{2k} f^{-i}(P))$ along the unstable direction can be derived by using Assumption (5) and the discussions used in the proof of (3.3.12):

$$
\mathrm{diam}(f^k(\mathcal{M}_{2k}(x))) \leq C_u' k^{-\frac{1}{\eta}},
$$

where C_s' and C_u' are two positive constants. Hence, (3.4.8) holds.

Second, by direct calculation and (3.4.8), one has

$$\begin{aligned}
&|\mathrm{Cor}_{n-k}(\Phi, \Psi \circ f^k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu)| \\
\leq &\Big| \int (\Psi \circ f^k - \overline{\Psi}_k) \circ (f^{n-k}) \cdot \Phi d\mu \Big| + \Big| \int (\Psi \circ f^k - \overline{\Psi}_k) d\mu \cdot \int \Phi d\mu \Big| \\
\leq &(2 \max |\Phi|) \int |(\Psi \circ f^k - \overline{\Psi}_k)| d\mu \leq (2 \max |\Phi|) \cdot \frac{C_d^\theta}{k^{\frac{\theta}{\eta}}}.
\end{aligned} \tag{3.4.9}$$

Third, denote by $|\cdot|$ the total variation of a signed measure. By using the fact $((f^k)_*((\mathcal{P} \circ (f^k)\mu) = \mathcal{P}\mu$ and (3.4.8), one has

$$\begin{aligned}
&\Big|\mathrm{Cor}_{n-k}(\Phi, \overline{\Psi}_k; f, \mu) - \mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)\Big| \\
\leq &\Big| \int (\overline{\Psi}_k \circ (f^{n-k}))(\Phi - \Phi_k) d\mu \Big| + \Big| \int \overline{\Psi}_k d\mu \cdot \int (\Phi - \Phi_k) d\mu \Big| \\
\leq &(2 \max |\Psi|) \int |\Phi - \Phi_k| d\mu = (2 \max |\Psi|) |\mathcal{P}\mu - \mathcal{P}_k \mu|(M) \\
= &(2 \max |\Psi|) |(f^k)_*((\mathcal{P} \circ (f^k)\mu) - (f^k)_*(\overline{\mathcal{P}}_k \mu)|(M) \\
\leq &(2 \max |\Psi|) |(\mathcal{P} \circ f^k - \overline{\mathcal{P}}_k)\mu|(M) = (2 \max |\Psi|) \int |\mathcal{P} \circ f^k - \overline{\mathcal{P}}_k| d\mu \\
\leq &(2 \max |\Psi|) \cdot \frac{C_d^\theta}{k^{\frac{\theta}{\eta}}}.
\end{aligned} \tag{3.4.10}$$

Fourth, it is to show that

$$|\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)| = |\mathrm{Cor}_{n-k}(\overline{\Phi}_k, \overline{\Psi}_k; \overline{f}, \bar{\mu})|. \tag{3.4.11}$$

In other words, $\mathrm{Cor}_{n-k}(\Phi_k, \overline{\Psi}_k; f, \mu)$ can be determined by functions, which are constant along stable manifolds on each $P_i \in \mathcal{P}$. Since $\overline{\Phi}$ and $\overline{\Psi}$ are constant along stable manifolds contained in any P_i, one could treat $\overline{\Phi}$ and $\overline{\Psi}$ as functions defined on $\overline{M}$. This, together with $\overline{f} \circ \pi = \pi \circ f$ and $\pi_*(\overline{\Phi}_k \mu) = \overline{\Phi}_k(\pi_* \mu)$, implies that

$$\int (\overline{\Psi}_k \circ (f^{n-k}))\Phi_k d\mu = \int (\overline{\Psi}_k \circ (f^{n-k}))d((f^k)_*(\overline{\Phi}_k \mu)) = \int \overline{\Psi}_k d((f^{n-k})_*(f^k)_*(\overline{\Phi}_k \mu))$$
$$= \int \overline{\Psi}_k d((f^n)_*(\overline{\Phi}_k \mu)) = \int \overline{\Psi}_k d(\pi_*(f^n)_*(\overline{\Phi}_k \mu)) = \int \overline{\Psi}_k d((\overline{f}^n)_*(\overline{\Phi}_k \bar{\mu})) = \int \overline{\Psi}_k \circ \overline{f}^n \cdot \overline{\Phi}_k d\bar{\mu},$$

and,

$$\int \Phi_k d\mu \int \overline{\Psi}_k d\mu = \int d((f^k)_*(\overline{\Phi}_k \mu)) \cdot \int \overline{\Psi}_k d\bar{\mu} = \int \overline{\Phi}_k d\bar{\mu} \cdot \int \overline{\Psi}_k d\bar{\mu}.$$

This shows (3.4.11).

By Lemma 3.4.1, one has that

$$\frac{A'}{(n-k)^{\varrho-1}} \leq |\mathrm{Cor}_{n-k}(\overline{\Phi}_k, \overline{\Psi}_k; \overline{f}, \bar{\mu})| \leq \frac{A}{(n-k)^{\varrho-1}}, \tag{3.4.12}$$

where A and A' are two positive constants.

Furthermore, since $\eta > 0$, $\min\{m-2, 2\} \leq \eta < m-1$, and $m-1-\eta < \theta \leq 1$, one has $\frac{\theta}{\eta} > \frac{m-1}{\eta} - 1 = \varrho - 1$.

Therefore, take $k = [n/2]$, by (3.4.7), (3.4.9), (3.4.10), (3.4.11), and (3.4.12), one has

$$\frac{A'(\Phi, \Psi)}{n^{\varrho-1}} \leq |\mathrm{Cor}_n(\Phi, \Psi; f, \mu)| \leq \frac{A(\Phi, \Psi)}{n^{\varrho-1}},$$

where $A'(\Phi, \Psi)$ and $A(\Phi, \Psi)$ are two positive constants. This verifies (3.2.2).

This completes the whole proof of Theorem 3.2.2.

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