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**TRAVELING PULSES FOR THE NONLOCAL AND
LATTICE KLEIN-GORDON EQUATIONS**

By

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A DISSERTATION

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ABSTRACT

TRAVELING PULSES FOR THE NONLOCAL AND LATTICE KLEIN-GORDON EQUATIONS

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The thesis includes two parts. In the first part, we study a nonlinear nonlocal Klein-Gordon equation on the whole real line. We first prove the existence of traveling pulses and then study the spatial asymptotic properties of these pulses and their instability. In the second part, we study a nonlinear lattice Klein-Gordon equation, which is actually a system of infinitely many ordinary differential equations. We show the existence and determine the spatial asymptotic properties of the traveling pulses using a similar method to that in the first part and then study the instability by numerical simulations. For both the nonlinear nonlocal and nonlinear lattice Klein-Gordon equations we illustrate that they share some similar properties with the limit equation, the classical Klein-Gordon equation, as a parameter ε approaches 0.

To my family

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CHAPTER 1

Introduction

Crystals are solids in which the constituent atoms are arranged regularly in a space lattice with specific geometrical symmetry elements. However, a perfect crystal, with every atom of the same type in the correct position, does not exist in nature. Actually, all crystals have defects. Crystal defects are very important in materials science and engineering because they govern many physical properties, such as color, conductivity, transparency, hardness, *etc.*. There is also much work on the mathematical model studying crystal defects (see [22] for instance).

Kresse and Truskinovsky [23] studied a lattice wave equation with nearest neighbor interaction to study the motion of lattice defects (see [34] for an analysis with longer range interaction). They consider a chain of particles attached at equal distance ε to a rigid background by bistable springs with energy density $w(u_n)$ where u_n is the vertical displacement of the particle with index n (Figure 1.1), then $\varepsilon w(u_n)$ gives the on site potential energy for particle with index n . Assume that the neighboring particles interact through standard harmonic forces, characterized by the modulus of elasticity $\mathcal{E}$ (Young' modulus) of the spring, then the total potential energy of the chain is in the form

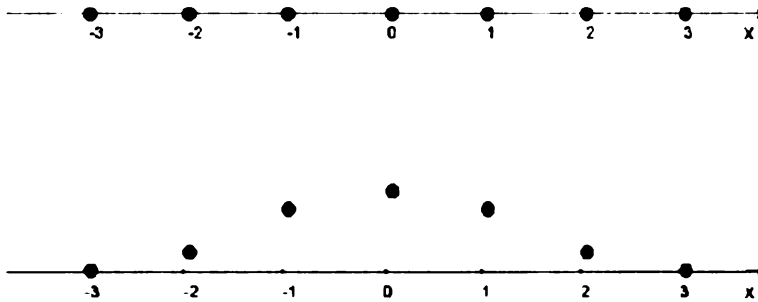


Figure 1.1. vertical displacement

$$V = \sum_n \frac{1}{2} \mathcal{E} \varepsilon \left(\frac{u_{n+1} - u_n}{\varepsilon} \right)^2 + \varepsilon w(u_n).$$

For simplicity assuming the spring has unit mass density, then the total kinetic energy is

$$K = \sum_n \frac{1}{2} \varepsilon \dot{u}^2.$$

Then the Euler-Lagrange equation gives

$$\ddot{u}_n - \frac{\mathcal{E}}{\varepsilon^2} [u_{n+1} - 2u_n + u_{n-1}] + w'(u_n) = 0.$$

The authors consider traveling waves, and so let $x = \varepsilon n$, write $u_n(t) = u(x, t)$ and seek a solution of the form $u(x, t) = u(\eta) = u(x - ct)$ satisfying the equation

$$c^2 \frac{d^2 u}{d\eta^2} - \frac{\mathcal{E}}{\varepsilon^2} [u(\eta + \varepsilon) - 2u(\eta) + u(\eta - \varepsilon)] + w'(u) = 0. \quad (1.1)$$

In [23] and [34], heteroclinic solutions were studied for a piecewise linear w' , in which case solutions can be found explicitly.

We consider a general lattice equation

$$\ddot{u}_n - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u_{n-k} + f(u_n) = 0, \quad n \in \mathbb{Z}, \quad (1.2)$$

where $0 < \varepsilon$, $\sum \alpha_k = 0$, $\alpha_0 < 0$, $\alpha_k = \alpha_{-k}$, and a certain ellipticity condition holds. Note that we allow infinite range and not just nearest neighbor or finite length interaction, although those are included as special cases. We want to study homoclinic waves for supersonic speeds c and rather general smooth nonlinearity. We assume that f is a C^2 function, satisfying $f(0) = 0$, and

$$\zeta_0 = \inf\{\zeta > 0 : F(\zeta) = 0\}$$

exists, where

$$F(\zeta) = \int_0^\zeta f(s) ds,$$

and $f(\zeta_0) > 0$. A typical example is the quadratic function $f(u) = u(u - a)$ with $a > 0$. Under such assumption the ODE

$$u'' + f(u) = 0$$

has a unique even, positive homoclinic solution (see [8]).

Equation (1.2) can be derived through the Hamiltonian. For instance, one could take the Hamiltonian on $\ell^2 \times \ell^2$ defined by

$$H(\mathbf{p}, \mathbf{u}) = \sum_n \left(\frac{1}{2} p_n^2 + \frac{1}{4\varepsilon^2} \sum_m \alpha_{n-m} (u_n - u_m)^2 + F(u_n) \right). \quad (1.3)$$

Here, $\mathbf{p} = (p_n) \in \ell^2$ represents the momentum and $\mathbf{u} = (u_n) \in \ell^2$, the position of particles relative to lattice sites. The pairwise interaction term, $\alpha_{n-m}(u_n - u_m)^2$, could be thought of as each pair of particles being connected by a linear spring. However, we will not require that these all produce restoring forces, i.e., some of the α'_k s could be negative. Such is the case with the Lennard-Jones potential, for instance.

Since the term with $n = m$ vanishes, we may take $\alpha_0 = -\sum_{k \neq 0} \alpha_k$. We will also assume interactions are symmetric, so that $\alpha_{-k} = \alpha_k$.

The on-site potential F is fairly general but has a local maximum or semi-maximum at 0 and a local minimum at some positive value, 1 say. Examples include $F(u) = (1 - u^2)^2$ or $u^2(u - 1)$, which cases have been well studied numerically, and analytically in the continuum differential equation setting.

The equations of motion produced by (1.3) are

$$\begin{aligned}\frac{du_n}{dt} &= \left(\frac{\partial H}{\partial \mathbf{p}} \right)_n = p_n, \\ \frac{dp_n}{dt} &= - \left(\frac{\partial H}{\partial \mathbf{u}} \right)_n = \frac{1}{\varepsilon^2} \sum_m \alpha_{n-m} u_m - f(u_n),\end{aligned}\tag{1.4}$$

where $f = F'$. This system is equivalent to (1.2).

Note that if we rescale η in (1.1) such that $z = \eta/\varepsilon$, the equation for $u(z) = u(\varepsilon\eta)$ becomes

$$c^2 \frac{d^2 u}{dz^2} - \mathcal{E}[u(z+1) - 2u(z) + u(z-1)] + \varepsilon^2 w'(u) = 0$$

and many authors have studied such lattice Klein-Gordon equations (see [9, 14, 16, 17, 30], and the references therein). Some computational studies on heteroclinic traveling waves for a cubic bistable w' have been reported in [9], and the existence of homoclinic traveling wave solutions for fixed $\varepsilon > 0$ and sufficiently small $\mathcal{E}$ has been proved in [20] by center manifold reduction (see also [14]). The existence of other solitary solutions, for example, breathers, was studied in [24, 31]. However, for equation (1.1), when w' is nonlinear, ε is small, and $\mathcal{E}$ is of moderate size, little is known about the existence of traveling waves or pulses. A major difficulty arises as one must deal with a nonlinear differential equation with significant advanced and retarded terms. That case is considered here.

Many other nonlinear infinite dimensional lattice systems have been studied during the last couple of decades. For example, the famous FPU lattice,

$$\ddot{u}_n = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1})$$

has received considerable attention, in particular having been studied by numerical simulations [13] (see also [12, 15]). More recently there have also appeared some rigorous results concerning traveling waves using a variety of methods. For instance, there are variational approaches used in [19, 26, 33]. In [18], Friesecke and Pego studied the qualitative properties of traveling pulses in FPU lattices from another point of view: they regard the traveling pulse as a perturbation of the solitary solution of a KdV equation. Their method is to look for the fixed point of an operator defined in terms of Fourier multipliers, but which is a reformulation of the problem at hand. The Fourier transform has also been used by other authors to study solitary waves (see [1, 12] etc, for both numerical and theoretical treatments). In this work, we will prove our main result following the idea of [18].

There is also some literature considering traveling waves for the damped discrete sine-Gordon equation with periodic boundary conditions (so it is a finite-dimensional system rather than an infinite-dimensional one), which models many physical systems such as circular arrays of Josephson junctions. The existence of traveling waves was proved in [21] by a fixed-point argument. At present, our methods do not apply to such systems, however.

Note that in (1.2) we can regard the term

$$\Delta_\epsilon u_n \equiv \frac{1}{\epsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u_{n-k}$$

as a discrete Laplacian with possibly infinite range interaction. Similar long range interactions have been introduced in a DNA model with long-range dipole-dipole interactions. The typical Hamiltonian is of the form

$$H = \sum_{n=1}^N \left(\frac{1}{2} \dot{u}_n^2 + V(u_n) \right) + \sum_{m=1}^N J_{mn} u_n u_m$$

and traveling breathers were numerically studied in [2, 3, 11].

While we do not intend for the foregoing to be a complete survey of previous results on lattice equations and traveling waves, we hope that it provides a context for our results.

We also consider the nonlocal wave equation for $u(x, t)$:

$$u_{tt} - \frac{1}{\varepsilon^2}(j_\varepsilon * u - u) + f(u) = 0, \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \quad (1.5)$$

where ε is a positive parameter and the kernel j_ε of the convolution is defined by

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right),$$

where $j(\cdot)$ is an even function with unit integral.

We can regard equation (1.5) (or (1.2)) as the wave analog of the nonlocal parabolic equation studied in [4], (or [5] for the lattice version)

$$u_t - \frac{1}{\varepsilon^2}(j_\varepsilon * u - u) + f(u) = 0, \quad (1.6)$$

(see also [6, 7, 10]). Equation (1.6) was introduced as an L^2 gradient flow for the energy functional

$$E(u) = \frac{1}{4} \int \int_{\mathbb{R}^2} j_\varepsilon(x - y)(u(x) - u(y))^2 dx dy + \int_{\mathbb{R}} F(u(x)) dx.$$

Similarly, when we apply Newton's second law

$$u_{tt} = -\text{grad } E(u),$$

a nonlocal wave equation, (1.5), is derived.

Note that, as $\varepsilon \rightarrow 0$, $\frac{1}{\varepsilon^2}(j_\varepsilon * u - u) \rightarrow du_{xx}$, formally and in some weak sense described in [4], where d is a constant determined by j . So we can also regard (1.5) as a nonlocal version of the classical nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + f(u) = 0. \quad (1.7)$$

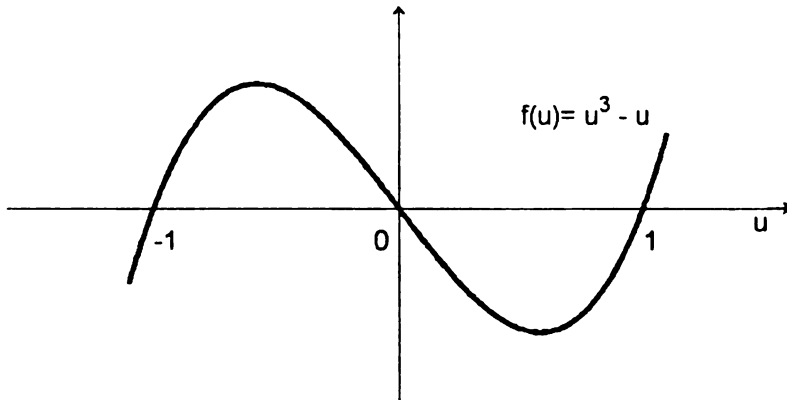


Figure 1.2. $f(u)$

Clearly, a traveling wave solution $u(\eta) = u(x - ct)$ satisfies the ordinary differential equation

$$(c^2 - 1)u'' + f(u) = 0$$

which has homoclinic solutions for speed $c^2 > 1$ and suitable nonlinear function f . For example, with $f = u^3 - u$ (Figure 1.2), the phase portrait is shown in Figure 1.3 and the solution is a single pulse (Figure 1.4). Actually, the ODE is integrable and the solution is given by a sech function

$$u(\eta) = \sqrt{2} \operatorname{sech} \left(\frac{\eta}{\sqrt{c^2 - 1}} \right).$$

Figure 1.5 shows how the pulse travels when the speed c is positive.

In this work we will study homoclinic traveling wave solutions (traveling pulses) of (1.5) (and of (1.2)), i.e., solutions of the form $u(x, t) = u(x - ct)$ (or $u_n(t) = u(\varepsilon n - ct)$) which decay to zero at infinity.

It will become apparent that for both the lattice and continuum equations, our method does not seem to apply to the subsonic case $c^2 < d$. Also, the Lorentz invariance, which produces, for stationary solutions, subsonic traveling waves for the

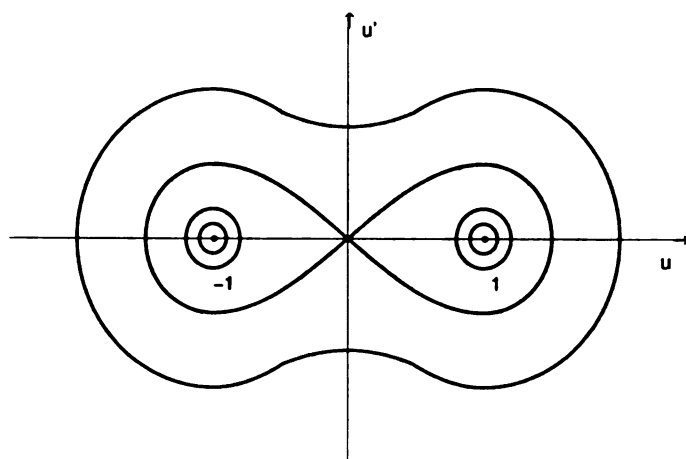


Figure 1.3. phase portrait

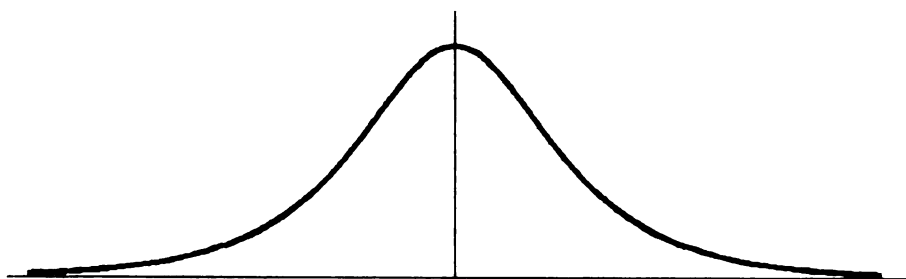


Figure 1.4. single pulse

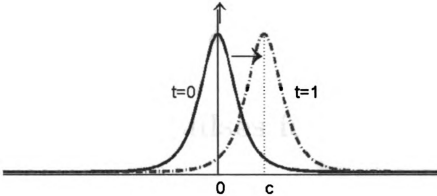


Figure 1.5. traveling pulse

usual nonlinear wave equation, does not hold for the nonlocal version. Since the corresponding limit ODEs have heteroclinic solutions for bistable nonlinearity f , one may conjecture that there exist heteroclinic traveling waves for our models, at least when $\varepsilon > 0$ is small. On the other hand, even though the limit equation does not have homoclinic solutions, since the convolution operator is bounded, we suspect that our models still possess traveling pulse solutions with subsonic speed provided ε is sufficiently large.

The dissertation is organized as follows: In chapter 2 we study the traveling pulses for the nonlocal Klein-Gordon equation (1.5). We first show the existence and spatial asymptotic properties of the traveling pulses, then we prove that the traveling pulses are unstable. In chapter 3 we study the lattice Klein-Gordon equation (1.2). We show the existence of pulses, and determine their spatial asymptotic properties using similar methods to those in Chapter 2, then illustrate the instability thorough numerical simulations.

CHAPTER 2

Traveling pulses for the nonlocal Klein-Gordon equation

2.1 Existence of traveling pulses

In this section we study the existence of traveling pulses for the nonlocal Klein-Gordon equation (1.5).

Throughout this dissertation we assume the following conditions:

(A1) $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(0) = -a < 0$; $f(\zeta_0) > 0$, where

$$\zeta_0 \equiv \inf\{\zeta > 0 : F(\zeta) = 0\} \text{ and } F(\zeta) = \int_0^\zeta f(s)ds,$$

and

(A2) $j(x) \in L^1(\mathbb{R})$ is even, has unit integral,

$$\lim_{z \rightarrow 0} \frac{\hat{j}(z) - 1}{z^2} = -d \quad \text{and} \quad \hat{j}(z) \geq 1 - d_1 z^2,$$

where $0 < d \leq d_1$ are constants and the Fourier transform is given by $\hat{j}(z) \equiv \int_{-\infty}^{\infty} e^{-izx} j(x) dx$.

Remark A typical example for f is the quadratic function $f(u) = u(u - a)$ with

$a > 0$. The assumption **(A1)** guarantees that the ODE

$$u'' + f(u) = 0$$

has a unique even, positive homoclinic solution ([8]).

Remark If $\widehat{j} \in C^2$ then $d = \frac{1}{2} \int_{-\infty}^{\infty} j(x)x^2 dx$. It is easy to see that $\frac{1}{2}e^{-|x|}$, $\frac{1}{\sqrt{\pi}}e^{-x^2}$ and $\frac{2}{\pi} \frac{1}{(1+x^2)^2}$ satisfy **(A2)**, with $d(=d_1)$ given by 1, $\frac{1}{4}$ and $\frac{1}{2}$ respectively.

The idea is to change to traveling wave coordinates, formally take the Fourier transform, and reformulate the problem of existence of a traveling pulse as a fixed point problem for an operator defined in terms of the nonlinearity and a certain Fourier multiplier.

In (1.5), let $u(x, t) = u(x - ct) = u(\eta)$, so that $u(\eta)$ should satisfy the equation

$$c^2 u'' - \frac{1}{\varepsilon^2} (j_\varepsilon * u - u) = -g(u) + au, \quad (2.1)$$

where $g(u) = f(u) + au$. Applying the Fourier transform, equation (2.1) becomes

$$-c^2 \xi^2 \hat{u} - \frac{1}{\varepsilon^2} (\widehat{j_\varepsilon} \cdot \hat{u} - \hat{u}) = -\widehat{g(u)} + a\hat{u}$$

or

$$(c^2 \xi^2 + l_\varepsilon(\xi) + a)\hat{u} = \widehat{g(u)},$$

where $l_\varepsilon = \frac{1}{\varepsilon^2} (\widehat{j_\varepsilon} - 1)$. Thus, an equivalent formulation is

$$\hat{u} = p_\varepsilon(\xi) \widehat{g(u)},$$

where

$$p_\varepsilon(\xi) = \frac{1}{c^2 \xi^2 + l_\varepsilon(\xi) + a}. \quad (2.2)$$

The inverse Fourier transform gives

$$u = \check{p}_\varepsilon * g(u), \quad (2.3)$$

where $\check{p}_\varepsilon$ is the inverse transform of p_ε .

Define the operator

$$P_\varepsilon(u) \equiv \check{p}_\varepsilon * g(u),$$

and write (2.3) as

$$u = P_\varepsilon(u). \quad (2.4)$$

Note that, due to **(A2)**,

$$l_\varepsilon(\xi) = \frac{1}{\varepsilon^2}(\widehat{j}_\varepsilon(\xi) - 1) = \frac{\widehat{j}(\varepsilon\xi) - 1}{(\varepsilon\xi)^2} \cdot \xi^2 \rightarrow -d\xi^2$$

as $\varepsilon \rightarrow 0$, so $p_\varepsilon \approx \frac{1}{(c^2 - d)\xi^2 + a}$ for ε small. Thus, when $\varepsilon \rightarrow 0$, (2.4) formally becomes

$$u = P_0(u), \quad (2.5)$$

where $P_0(u) \equiv \check{p}_0 * g(u)$ and

$$p_0(\xi) = \frac{1}{(c^2 - d)\xi^2 + a}.$$

Clearly, (2.5) is equivalent to

$$u = ((d - c^2)\partial^2 + a)^{-1}(g(u)),$$

that is,

$$(c^2 - d)u'' = au - g(u),$$

or

$$(c^2 - d)u'' + f(u) = 0. \quad (2.6)$$

By the results of Berestycki and Lions in [8], under the assumption **(A1)**, (2.6) has a unique even, positive homoclinic solution for each $c^2 > d$, which we denote by u_0 .

Thus, u_0 is a fixed point of operator P_0 . We can write equation (2.5) in the form

$$u = P_0(u) + (P_\varepsilon - P_0)(u),$$

and look for a fixed point near u_0 . The basic idea to solve this is to consider $\bar{u} = u - u_0$.

Since $\bar{u}$ satisfies

$$\bar{u} = P_\varepsilon(\bar{u} - u_0) - P_0(u),$$

which is equivalent to

$$\bar{u} - DP_0(u_0)\bar{u} = P_\varepsilon(\bar{u} - u_0) - P_0(u) - DP_0(u_0)\bar{u},$$

we want to solve the equation

$$\bar{u} = (I - DP_0(u_0))^{-1}(P_\varepsilon(\bar{u} + u_0) - P_0(u_0) - DP_0(u_0)\bar{u}). \quad (2.7)$$

The operator on the right hand side turns out to be a contraction map on a small ball about 0 when $\varepsilon > 0$ is sufficiently small. Under these conditions the operators $F = P_0$ and $\tilde{F} = P_\varepsilon - P_0$ satisfy the assumptions in the following lemma from [18] :

Lemma 2.1.1 *Suppose F and $\tilde{F}$ are C^1 maps from a ball in a Banach space E into E . Suppose F has a fixed point ϕ_0 and that $A \equiv I - DF(\phi_0)$ is invertible with $\|A^{-1}\| \leq C_0 < \infty$. Suppose that there exist positive constants C_1, C_2, θ , and δ such that*

(i)

$$\|DF(\phi) - DF(\phi_0)\| \leq C_1 < C_0^{-1} \quad \text{and} \quad \|D\tilde{F}(\phi)\| \leq C_2$$

whenever $\|\phi - \phi_0\|_E \leq \delta$, where $\|\cdot\|$ represents the operator norm ;

(ii)

$$C_0(C_1 + C_2) \leq \theta < 1;$$

(iii)

$$\|\tilde{F}(\phi_0)\|_E < \delta(1 - \theta)/C_0.$$

Then the equation $\phi = F(\phi) + \tilde{F}(\phi)$ has a unique solution satisfying $\|\phi - \phi_0\|_E \leq \delta$.

This will be used to prove our main theorem:

Theorem 2.1.2 *Under the assumptions (A1) and (A2), for any speed c satisfying $c^2 > d_1$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, equation (1.5) has a unique nonzero solution u_ε in the set*

$$\{u \in H^1(\mathbb{R}) : u \text{ is even, } \|u - u_0\|_{H^1} < \delta\},$$

where u_0 is the even, positive homoclinic solution of (2.6) and $\delta > 0$ depends on j, f and c , and satisfies $\delta < \|u_0\|_{H^1}$.

Remark The condition $\delta < \|u_0\|_{H^1}$ excludes the trivial solution $u \equiv 0$.

2.2 Proof of the Theorem

Note that u_0 is the fixed point of P_0 . To prove the theorem we apply Lemma 2.1.1 with the operators

$$\tilde{F} = P_\varepsilon - P_0, \quad \text{and} \quad F = P_0.$$

First, we show $\tilde{F}$ and F are C^1 maps in $E = H_\varepsilon^1 = \{u \in H^1(\mathbb{R}) : u \text{ is even}\}$.

Step 1 P_ε is an operator from H^1 to H^1 .

Denote $\|\cdot\| = \|\cdot\|_{L^2}$ and $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$. We also use $\|\cdot\|$ for the operator norm, where the range and domain spaces are understood. For any $u \in H^1$, by Plancherel's theorem,

$$\begin{aligned} \|P_\varepsilon(u)\| &= \|\check{p}_\varepsilon * g(u)\| \\ &= \frac{1}{\sqrt{2\pi}} \|p_\varepsilon \widehat{g(u)}\| \\ &\leq \|p_\varepsilon\|_\infty \|g(u)\| \\ &\leq \|p_\varepsilon\|_\infty \|g(u)/u\|_\infty \|u\|, \end{aligned} \tag{2.8}$$

where $g(u)/u \in L^\infty$ because $g'(0) = 0$ and $u \in L^\infty$.

We also have,

$$\begin{aligned}
\|(P_\varepsilon(u))'\| &= \|\check{p}_\varepsilon * (g'(u)u')\| \\
&= \frac{1}{\sqrt{2\pi}} \|p_\varepsilon(\widehat{g'(u)u'})\| \\
&\leq \|p_\varepsilon\|_\infty \|g'(u)u'\| \\
&\leq \|p_\varepsilon\|_\infty \|g'(u)\|_\infty \|u'\|.
\end{aligned} \tag{2.9}$$

Thus, P_ε is an operator from H^1 to H^1 .

Step 2 P_ε maps even functions to even functions.

Let u be even, then $g(u)$ is even and it is easy to check that the convolution of two even functions is even.

Step 3 For $u \in H^1$, the linearized operator $DP_\varepsilon(u) : H^1 \rightarrow H^1$ is bounded.

For any $v \in H^1$,

$$DP_\varepsilon(u)v = \check{p}_\varepsilon * (g'(u)v).$$

Thus,

$$\begin{aligned}
\|DP_\varepsilon(u)(v)\| &= \|\check{p}_\varepsilon * (g'(u)v)\| \\
&\leq \|p_\varepsilon\|_\infty \|g'(u)\|_\infty \|v\|,
\end{aligned}$$

and

$$\begin{aligned}
\|(DP_\varepsilon(u)(v))'\| &= \|(\check{p}_\varepsilon * (g'(u)v))'\| \\
&\leq \|p_\varepsilon\|_\infty \|(g'(u)v)'\| \\
&\leq \|p_\varepsilon\|_\infty (\|g''(u)\|_\infty \|u'\| \|v\|_\infty + \|g'(u)\|_\infty \|v'\|) \\
&\leq C\|v\|_{H^1},
\end{aligned}$$

where we used the embedding inequality, $\|v\|_{L^\infty} \leq C\|v\|_{H^1}$. Therefore, $DP_\varepsilon(u)$ is bounded.

Step 4 The operator $DP_\epsilon(u)$ is continuous with respect to u .

For any $u, u_1 \in H^1$,

$$\begin{aligned} \|DP_\epsilon(u)(v) - DP_\epsilon(u_1)(v)\| &= \|\check{p}_\epsilon * [(g'(u) - g'(u_1))v]\| \\ &\leq \|p_\epsilon\|_\infty \|g'(u) - g'(u_1)\|_\infty \|v\| \\ &\leq \|p_\epsilon\|_\infty \|g''\|_\infty \|u - u_1\|_\infty \|v\|, \end{aligned}$$

and

$$\begin{aligned} &\|(DP_\epsilon(u)(v) - DP_\epsilon(u_1)(v))'\| \\ &= \|\check{p}_\epsilon * [(g'(u) - g'(u_1))v]'\| \\ &= \|\check{p}_\epsilon * [(g''(u)u' - g''(u)u'_1 + g''(u)u'_1 - g''(u_1)u'_1)v] + \check{p}_\epsilon * [(g'(u) - g'(u_1))v']\| \\ &\leq \|p_\epsilon\|_\infty \|g''(u)\|_\infty \|u' - u'_1\| \|v\|_\infty + \|p_\epsilon\|_\infty \|g''(u) - g''(u_1)\|_\infty \|u'_1\|_\infty \|v\| \\ &\quad + \|p_\epsilon\|_\infty \|g'(u) - g'(u_1)\|_\infty \|v'\|. \end{aligned}$$

Hence

$$\|DP_\epsilon(u) - DP_\epsilon(u_1)\| \rightarrow 0$$

as $\|u - u_1\|_{H^1} \rightarrow 0$. Therefore DP_ϵ is continuous and P_ϵ is C^1 . Similarly, P_0 is a C^1 map from E to E .

We will finish the proof using the following lemmas.

Lemma 2.2.1 *The operator $T = DF(u_0) : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ is compact.*

Proof: Note that

$$\begin{aligned} DF(u_0)v &= \check{p}_0 * (g'(u_0)v) \\ &= \left(\frac{1}{(c^2 - d)\xi^2 + a} \right) * (g'(u_0)v) \\ &= \frac{1}{2(c^2 - d)\beta} e^{-\beta|\cdot|} * (g'(u_0)v), \end{aligned}$$

where $\beta = \sqrt{\frac{a}{c^2 - d}}$. Since u_0 decays at $\pm\infty$, we can prove that $DF(u_0)$ is compact using the Fréchet-Kolmogorov Theorem (see [35]):

A subset K in L^p ($1 \leq p < \infty$) is precompact if and only if it satisfies

1. $\sup_{u \in K} \|u\|_p < \infty$;
2. $\|u(\cdot + h) - u(\cdot)\|_p \rightarrow 0$ uniformly in $u \in K$ as $h \rightarrow 0$;
3. $\lim_{\alpha \rightarrow \infty} \int_{|x| > \alpha} |u(x)|^p dx = 0$ uniformly in $u \in K$.

First we show $T : L^2 \rightarrow L^2$ is compact. Let A be a bounded subset in L^2 and let $K = TA$. We need to show K is precompact in L^2 .

(1) As in (2.8) we have

$$\|Tv\|_2 \leq C(u_0)\|v\|_2 \quad \text{for } v \in A.$$

(2) For $v \in A$,

$$\begin{aligned} & \int_{-\infty}^{\infty} |Tv(x+h) - Tv(x)|^2 dx \\ &= \|(\check{p}_0(\cdot + h) - \check{p}_0(\cdot)) * (g'(u_0)v)\|_2^2 \\ &\leq \|(\check{p}_0(\cdot + h) - \check{p}_0(\cdot))\|_{\infty}^2 \|g'(u_0)\|_{\infty}^2 \|v\|_2^2 \rightarrow 0 \end{aligned}$$

uniformly in $v \in A$ as $h \rightarrow 0$ because

$$\begin{aligned} |(\check{p}_0(\cdot + h))(\xi) - p_0(\xi)| &= |e^{ih\xi} p_0(\xi) - p_0(\xi)| \\ &= |e^{ih\xi} - 1| p_0(\xi) \rightarrow 0 \end{aligned}$$

uniformly in ξ as $h \rightarrow 0$, since $p_0(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

3) Since u_0 decays at $\pm\infty$, for any $\gamma > 0$, there exists $M > 0$ such that

$$|g'(u_0(y))| < \gamma \quad \text{for } |y| > M.$$

Choose $\alpha > M$ such that $2Me^{2\beta M} \int_{\alpha}^{\infty} e^{-2\beta x} dx < \gamma$. Then

$$\begin{aligned}
& \int_{\alpha}^{\infty} \left| \int_{-\infty}^{\infty} \check{p}_0(x-y) g'(u_0(y)) v(y) dy \right|^2 dx \\
&= \int_{\alpha}^{\infty} \left| \left(\int_{-M}^M + \int_M^{\infty} + \int_{-\infty}^{-M} \right) \check{p}_0(x-y) g'(u_0(y)) v(y) dy \right|^2 dx \\
&\leq C \int_{\alpha}^{\infty} \left| \int_{-M}^M e^{-\beta|x-y|} g'(u_0(y)) v(y) dy \right|^2 dx + \\
&\quad + 2\gamma^2 \int_{\alpha}^{\infty} \left(\int_M^{\infty} |\check{p}_0(x-y) v(y)| dy \right)^2 dx + 2\gamma^2 \int_{\alpha}^{\infty} \left(\int_{-\infty}^{-M} |\check{p}_0(x-y) v(y)| dy \right)^2 dx \\
&\leq C \int_{\alpha}^{\infty} \left| \int_{-M}^M e^{-\beta(x-y)} g'(u_0(y)) v(y) dy \right|^2 dx + 4\gamma^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\check{p}_0(x-y) v(y)| dy \right)^2 dx \\
&\leq C \int_{\alpha}^{\infty} e^{-2\beta x} dx \left| \int_{-M}^M e^{-\beta y} g'(u_0(y)) v(y) dy \right|^2 + 4\gamma^2 \|\check{p}_0 * |v|\|_2^2 \\
&\leq C \int_{\alpha}^{\infty} e^{-2\beta x} dx e^{2\beta M} \|g'(u_0(\cdot))\|_{\infty}^2 (2M) \|v\|_2^2 + 4\gamma^2 \|p_0\|_{\infty}^2 \|v\|_2^2 \\
&\leq C\gamma \|g'(u_0(\cdot))\|_{\infty}^2 \|v\|_2^2 + 4\gamma^2 \|p_0\|_{\infty}^2 \|v\|_2^2.
\end{aligned}$$

Thus, this integral converges to 0 uniformly for $v \in A$ as $\alpha \rightarrow \infty$, and so $T : L^2 \rightarrow L^2$ is compact.

Next we show $T : H^1 \rightarrow H^1$ is compact. Let u_n be a bounded sequence in H^1 . We want to show Tu_n has a convergent subsequence in H^1 . Since T is compact in L^2 , there exists a subsequence, still denoted by u_n , and $u, v \in L^2$ such that

$$Tu_n \rightarrow u,$$

and

$$Tu'_n \rightarrow v, \text{ as } n \rightarrow \infty.$$

Define $S : L^2 \rightarrow L^2$ by

$$Su \equiv \check{p}_0 * (g''(u_0) u'_0 u).$$

Then $S : L^2 \rightarrow L^2$ is compact by the same discussion above because we only need $\|g''(u_0)\|_\infty < \infty$ and $g''(u_0)u_0(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. So there exists a subsequence, still denoted by u_n , and $w \in L^2$ such that

$$Su_n \rightarrow w \quad \text{in } L^2, \quad \text{as } n \rightarrow \infty.$$

Therefore for $\phi \in C_0^\infty$,

$$\begin{aligned} \int u \phi' dx &= \lim_{n \rightarrow \infty} \int Tu_n \phi' dx \\ &= - \lim_{n \rightarrow \infty} \int (Tu_n)' \phi dx \\ &= - \lim_{n \rightarrow \infty} \int (\check{p}_0 * (g'(u_0)u_n))' \phi dx \\ &= - \left[\lim_{n \rightarrow \infty} \int (\check{p}_0 * (g'(u_0)u_n')) \phi dx + \int (\check{p}_0 * (g''(u_0)u_0' u_n)) \phi dx \right] \\ &= - \lim_{n \rightarrow \infty} \langle Tu_n', \phi \rangle - \lim_{n \rightarrow \infty} \langle Su_n, \phi \rangle \\ &= - \langle v + w, \phi \rangle \\ &= - \int (v + w) \phi dx, \end{aligned}$$

so $u' = v + w \in L^2$, and $(Tu_n)' \rightarrow u'$ in L^2 . Thus, $Tu_n \rightarrow u$ in H^1 . The claim is proved. $\square$

Lemma 2.2.2 *The operator $A = I - T$ is invertible in E .*

Proof: We only need to show that the equation $u - Tu = 0$ has no nonzero solutions in E . Note that equation $u - Tu = 0$ is equivalent to

$$u = \check{p}_0 * (g'(u_0)u),$$

that is

$$-(c^2 - d)u'' + au = g'(u_0)u, \quad (2.10)$$

to which u'_0 is a solution. We find the general solution by reduction of order: let $u = u'_0 y$, then y satisfies

$$2u''_0 y' + u'_0 y'' = 0;$$

let $y' = z$, then

$$\begin{aligned} \frac{z'}{z} &= -2 \frac{u''_0}{u'_0}, \\ \ln z &= -2 \ln(u'_0) + C, \end{aligned}$$

so

$$z = C(u'_0)^{-2},$$

and

$$y = C_1 + C_2 \int_1^x (u'_0)^{-2}(s) ds.$$

Thus we get the general solution

$$u = u'_0(C_1 + C_2 \int_1^x (u'_0)^{-2}(s) ds).$$

where C_1 and C_2 are arbitrary constants. Since

$$\lim_{x \rightarrow \infty} u'_0 \int_1^x (u'_0)^{-2} ds = \lim_{x \rightarrow \infty} \frac{\int_1^x (u'_0)^{-2} ds}{1/u'_0} = - \lim_{x \rightarrow \infty} \frac{1}{u''_0},$$

which is infinite and we are considering solutions in H^1 , C_2 has to be zero and any H^1 solution of (2.10) must be of the form $u = Cu'_0$, which is odd. Therefore (2.10) has no nonzero solution in E . By the Fredholm alternative, $A = I - T$ is invertible and A^{-1} is bounded. $\square$

Proof of Theorem 2.2

Fix $0 < \theta < 1$ and let C_0 be positive such that $\|A^{-1}\| \leq C_0 < \infty$. Since $F \in C^1$ we can choose $\delta > 0$ small such that for a constant $C_1 > 0$ satisfying $C_1 < \frac{\theta}{2C_0}$,

$$\|DF(u) - DF(u_0)\| \leq C_1, \quad \text{for } \|u - u_0\|_{H^1} \leq \delta.$$

Since

$$\|D\tilde{F}(u)v\|_{H^1} = \|(\tilde{p}_\varepsilon - \tilde{p}_0) * (g'(u)v)\|_{H^1} \leq C\|p_\varepsilon - p_0\|_\infty \|v\|_{H^1},$$

where C depends on $\|u\|_\infty$, we have

$$\|D\tilde{F}(u)\| \leq C\|p_\varepsilon - p_0\|_\infty.$$

Note that, because of **(A2)** (which implies that $(\hat{j}(z) - 1)/z^2$ is bounded) and

$$|p_\varepsilon - p_0| = \left| \frac{-d\xi^2 - l_\varepsilon(\xi)}{[c^2\xi^2 + l_\varepsilon(\xi) + a][(c^2 - d)\xi^2 + a]} \right| \leq \frac{\left| d + \frac{\hat{j}(\varepsilon\xi) - 1}{(\varepsilon\xi)^2} \right| \xi^2}{[(c^2 - d_1)\xi^2 + a][(c^2 - d)\xi^2 + a]},$$

we have $|p_\varepsilon - p_0| \rightarrow 0$ uniformly in ξ as $\varepsilon \rightarrow 0$. Thus there exists a positive ε_0 , which depends on δ , such that for any $\varepsilon < \varepsilon_0$ one has

$$\|D\tilde{F}(u)\| \leq C_2, \quad \text{for } \|u - u_0\|_{H^1} < \delta,$$

where $C_2 < \frac{\theta}{2C_0}$. Similar to the discussion in **Step 1**, $\tilde{F} : H^1 \rightarrow H^1$ maps bounded sets into bounded sets and

$$\|\tilde{F}(u_0)\| \leq \|p_\varepsilon - p_0\|_\infty \left(\left\| \frac{g(u_0)}{u_0} \right\|_\infty + \|g'(u_0)\|_\infty \right).$$

Hence we can choose small ε_0 again, such that

$$\|\tilde{F}(u_0)\| \leq \delta(1 - \theta)/C_0 \quad \text{for } \varepsilon < \varepsilon_0.$$

By Lemma 2.1.1, $u = F(u) + \tilde{F}(u)$ has a unique solution in the set

$\{u \in E : \|u - u_0\|_{H^1} < \delta\}$. This completes the proof.

2.3 Asymptotic behavior of traveling pulses

Here we provide an example and show that in this case the traveling pulse decays exponentially at infinity, something that is not obvious for this class of equations and may not be true in general (see [6] section 5).

Consider the case $j(x) = \frac{1}{2}e^{-|x|}$. Since $\hat{j}(\xi) = \frac{1}{1 + \xi^2}$, then

$$p_\varepsilon(\xi) = \frac{1}{c^2\xi^2 - \frac{\xi^2}{1 + \varepsilon^2\xi^2} + a};$$

and

$$p_0(\xi) = \frac{1}{(c^2 - 1)\xi^2 + a}.$$

Note that (A2) holds with $d = d_1 = 1$.

We have the following estimate for the Fourier multipliers:

Lemma 2.3.1 *Let $p_1(\xi) = p_\varepsilon - p_0$, then $\check{p}_1$ decays exponentially and there exist positive constants C_ε and γ , where $C_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that*

$$|\check{p}_1| \leq C_\varepsilon e^{-\gamma|x|}.$$

Proof:

$$\begin{aligned} p_1(\xi) &= \frac{1}{c^2\xi^2 - \frac{\xi^2}{1 + \varepsilon^2\xi^2} + a} - \frac{1}{(c^2 - 1)\xi^2 + a} \\ &= \frac{(-i\xi)(-i\xi)}{c^2(\xi^2 + \alpha_+)(\xi^2 + \alpha_-)} \cdot \frac{1}{c^2 - 1} \left(1 - \frac{a}{(c^2 - 1)\xi^2 + a}\right), \end{aligned}$$

where

$$\alpha_\pm = \frac{\eta_1 \pm \sqrt{\eta_1^2 - 4ac^2\varepsilon^2}}{2c^2\varepsilon^2} \quad \text{and} \quad \eta_1 = c^2 - 1 + a\varepsilon^2.$$

Since $c^2 > 1$ we have that $\alpha_\pm > 0$. Note that for $\alpha > 0$

$$(e^{-\alpha|x|\operatorname{sgn}x})^\wedge = \frac{-2i\alpha\xi}{\xi^2 + \alpha^2},$$

and so

$$\left(\frac{-i\xi}{\xi^2 + \alpha_{\pm}}\right)^{\gamma} = \frac{1}{2} \frac{1}{\sqrt{\alpha_{\pm}}} e^{-\sqrt{\alpha_{\pm}}|x|} \operatorname{sgn} x.$$

Notice that $\frac{1}{\sqrt{\alpha_+}} = o(1)$ and $\frac{1}{\sqrt{\alpha_-}} = O(1)$ when $\varepsilon \rightarrow 0$, and therefore

$$\begin{aligned} \check{p}_1 = \\ C'_\varepsilon e^{-\sqrt{\alpha_+}|x|} \operatorname{sgn} x * \left[e^{-\sqrt{\alpha_-}|x|} \operatorname{sgn} x - (e^{-\sqrt{\alpha_-}|x|} \operatorname{sgn} x) * \left(\sqrt{\frac{c^2-1}{a}} e^{-\sqrt{\frac{a}{c^2-1}}|x|} \right) \right], \end{aligned}$$

where $C'_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, there exists $\gamma > 0$ and $C_\varepsilon \rightarrow 0$ such that

$$|\check{p}_1| \leq C_\varepsilon e^{-\gamma|x|}.$$

□

Using Lemma 2.3.1 we conclude

Theorem 2.3.2 *When $j(x) = \frac{1}{2}e^{-|x|}$, the solution u_ε obtained in Theorem 2.1.2 decays exponentially at $\pm\infty$.*

Proof: From Lemma 2.3.1, since $\check{p}_\varepsilon = \check{p}_1 + \check{p}_0$,

$$\check{p}_\varepsilon \leq C'_\varepsilon e^{-\gamma'|x|},$$

where $C'_\varepsilon > 0$ and $\gamma' = \min\{\gamma, \sqrt{\frac{a}{c^2-1}}\}$. Also, since $g'(0) = 0$ we have

$$\lim_{x \rightarrow \pm\infty} \frac{g(u_\varepsilon(x))}{u_\varepsilon(x)} = \lim_{s \rightarrow 0} \frac{g(s)}{s} = 0.$$

Note that for any $x \in \mathbb{R}$

$$|u_\varepsilon(x)| = |(\check{p}_\varepsilon * g(u_\varepsilon))(x)| \leq C'_\varepsilon \int_{-\infty}^{\infty} e^{-\gamma'|x-y|} \left| \frac{g(u_\varepsilon(y))}{u_\varepsilon(y)} \right| |u_\varepsilon(y)| dy,$$

and so we draw the conclusion using Lemma 5.3 in [18]:

Lemma Suppose that h and k are bounded non-negative functions defined on $\mathbb{R}$ with $\lim_{|x| \rightarrow \infty} k(x) = 0$. Let $\beta > 0$ and suppose that

$$h(x) \leq \int_{-\infty}^{\infty} e^{\beta|x-y|} k(y) h(y) dy$$

for all real x . Then $e^{\alpha|\cdot|} h \in L^\infty$ whenever $\alpha < \beta$. Moreover, if K_1, K_2 and M are constants such that $\|h\|_\infty \leq K_1$, $\|k\|_\infty \leq K_2$, and $k(x) \leq \min\{\beta/4, (e^{-\alpha} - e^{-\beta})/2\}$ for $|x| \geq M$, then

$$\|e^{\alpha|\cdot|} h\|_\infty \leq C(\alpha, \beta, K_1, K_2, M).$$

□

2.4 Instability of traveling pulses

In this section we study the instability of those traveling pulses for the nonlocal Klein-Gordon equation discussed earlier. Recall that the equation has the form

$$u_{tt} - \frac{1}{\varepsilon^2} (j_\varepsilon * u - u) + f(u) = 0, \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \quad (2.11)$$

where the kernel j_ε of the convolution is defined by

$$j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right),$$

but for simplicity, throughout this section we only consider a particular $j(\cdot) = \frac{1}{2} e^{-|\cdot|}$.

We also restrict our attention to the case $f(u) = u^3 - u$. The traveling pulse $\phi(\xi) = \phi(x - ct)$ satisfies the second order integro-differential equation

$$c^2 \phi_{\xi\xi} - \frac{1}{\varepsilon^2} (j_\varepsilon * \phi - \phi) + f(\phi) = 0. \quad (2.12)$$

Since the traveling pulse $\phi(x - ct)$ is just one member of a continuum of traveling pulses obtained by translation, the question of stability or asymptotic stability is broader than it would be for an isolated stationary state. It will be shown that there are solutions with initial data arbitrarily close to ϕ in L^2 which not only leave an L^2 neighborhood of $\phi(\cdot - ct)$ for t sufficiently large, but also leave an L^2 neighborhood of any fixed translate. That is, there exists $\varepsilon_1 > 0$, such that for all $\delta_1 > 0$ and $k \in \mathbb{R}$, there exists initial data $(u_0, u_1) \in L^2 \times L^2$ with $\|u_0 - \phi\|_{L^2} < \delta_1$ and $\|u_1 + c\phi'\|_{L^2} < \delta_1$, while

$$\|u(t, x) - \phi(x - ct - k)\|_{L^2} > \varepsilon_1 \quad \text{for arbitrarily large } t > 0,$$

where $u(t, x)$ is the solution to (2.11) with initial values

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x). \quad (2.13)$$

As we point out in Section 2.1, equation (2.11) can be compared with the classical nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + f(u) = 0, \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \quad (2.14)$$

and for (2.14) many authors have studied the stability and instability of standing waves, see for example [25, 29, 33].

To show the instability of our traveling pulses for (2.11) we first study the continuity of traveling pulses with respect to the wave speed. To indicate the dependence of P_ε on c , we use the notation $P_{\varepsilon, c}$.

Lemma 2.4.1 *u_c is continuous with respect to wave speed c .*

Proof: Let c_1, c_2 satisfy $c_1^2 > 1, c_2^2 > 1$, and let u_{c_1}, u_{c_2} be the traveling pulses for (2.11) derived from Theorem 2.1.2, with wave speeds c_1 and c_2 , respectively. Then they are fixed points of operators P_{ε, c_1} and P_{ε, c_2} , respectively. Denote

$$\bar{u}_1 = u_{c_1} - u_{0, c_1}, \quad \bar{u}_2 = u_{c_2} - u_{0, c_2},$$

where $u_{0,c}$ is the positive, even, homoclinic solution to (2.6), then by (2.7), $\bar{u}_1$ and $\bar{u}_2$ are the fixed points of the contraction map

$$R_c(\bar{u}) \equiv (I - DP_{0,c}(u_{0,c}))^{-1}(P_{\varepsilon,c}(\bar{u} + u_{0,c}) - P_{0,c}(u_{0,c}) - DP_{0,c}(u_{0,c})\bar{u}), \quad (2.15)$$

for $c = c_1, c_2$ respectively, in the set

$$\{u \in H^1(\mathbb{R}) : u \text{ is even, } \|u\|_{H^1} < \delta\}.$$

Then, the contraction property of R_{c_1} implies that

$$\begin{aligned} \|u_{c_1} - u_{c_2}\|_{H^1} &= \|u_{c_1} - u_{0,c_1} - u_{c_2} + u_{0,c_2} + u_{0,c_1} - u_{0,c_2}\|_{H^1} \\ &\leq \|\bar{u}_1 - \bar{u}_2\|_{H^1} + \|u_{0,c_1} - u_{0,c_2}\|_{H^1} \\ &= \|R_{c_1}(\bar{u}_1) - R_{c_2}(\bar{u}_2)\|_{H^1} + \|u_{0,c_1} - u_{0,c_2}\|_{H^1} \\ &\leq \|R_{c_1}(\bar{u}_1) - R_{c_1}(\bar{u}_2)\|_{H^1} + \|R_{c_1}(\bar{u}_2) - R_{c_2}(\bar{u}_2)\|_{H^1} + \|u_{0,c_1} - u_{0,c_2}\|_{H^1} \\ &\leq \theta \|\bar{u}_1 - \bar{u}_2\|_{H^1} + \|R_{c_1}(\bar{u}_2) - R_{c_2}(\bar{u}_2)\|_{H^1} + \|u_{0,c_1} - u_{0,c_2}\|_{H^1}, \end{aligned}$$

where $\theta < 1$. Therefore,

$$\|u_{c_1} - u_{c_2}\|_{H^1} \leq \frac{1}{1-\theta} (\|R_{c_1}(\bar{u}_2) - R_{c_2}(\bar{u}_2)\|_{H^1} + (1+\theta)\|u_{0,c_1} - u_{0,c_2}\|_{H^1}).$$

Note that $u_{0,c} = \psi(\frac{1}{\sqrt{c^2-1}}x)$ where ψ is the positive nondegenerate homoclinic solution of $u'' + f(u) = 0$, thus

$$\begin{aligned} \|u_{0,c_1} - u_{0,c_2}\|_{H^1} &= \|\psi(\frac{1}{\sqrt{c_1^2-1}}x) - \psi(\frac{1}{\sqrt{c_2^2-1}}x)\|_{H^1} \\ &\rightarrow 0 \end{aligned}$$

as $c_2 \rightarrow c_1$.

To show $\|R_{c_1}(\bar{u}_2) - R_{c_2}(\bar{u}_2)\|_{H^1} \rightarrow 0$ as $c_2 \rightarrow c_1$, we need the following results:

1) $\|P_{\varepsilon,c_1}(\bar{u}_2) - P_{\varepsilon,c_2}(\bar{u}_2)\|_{H^1} \rightarrow 0$ as $c_2 \rightarrow c_1$.

This is true because

$$\begin{aligned} \|P_{\varepsilon,c_1}(\bar{u}_2) - P_{\varepsilon,c_2}(\bar{u}_2)\|_{L^2} &= \|\check{p}_{\varepsilon,c_1} * \bar{u}_2^3 - \check{p}_{\varepsilon,c_2} * \bar{u}_2^3\|_{L^2} \\ &= \|p_{\varepsilon,c_1} - p_{\varepsilon,c_2}\|_{L^2} \|\bar{u}_2^3\|_{L^2} \\ &\rightarrow 0 \end{aligned}$$

as $c_2 \rightarrow c_1$, and a similar calculation for the H_1 norm.

2) For any $v \in H^1$, $\|DP_{0,c_1}(u_{0,c_1})v - DP_{0,c_2}(u_{0,c_2})v\|_{H^1} \rightarrow 0$ as $c_2 \rightarrow c_1$.

Notice that $DP_{0,c}(u_{0,c})v = \check{p}_{0,c} * (g'(u_{0,c})v)$, we can first see in the L^2 norm

$$\begin{aligned} \|DP_{0,c_1}(u_{0,c_1})v - DP_{0,c_2}(u_{0,c_2})v\|_{L^2} &= \|\check{p}_{0,c_1} * (g'(u_{0,c_1})v) - \check{p}_{0,c_2} * (g'(u_{0,c_2})v)\|_{L^2} \\ &\leq \|\check{p}_{0,c_1} * (g'(u_{0,c_1})v) - \check{p}_{0,c_1} * (g'(u_{0,c_2})v)\|_{L^2} \\ &\quad + \|\check{p}_{0,c_1} * (g'(u_{0,c_2})v) - \check{p}_{0,c_2} * (g'(u_{0,c_2})v)\|_{L^2} \\ &\leq \|\check{p}_{0,c_1} * [(g'(u_{0,c_1}) - g'(u_{0,c_2}))v]\|_{L^2} \\ &\quad + \|(\check{p}_{0,c_1} - \check{p}_{0,c_2}) * (g'(u_{0,c_2})v)\|_{L^2} \\ &\rightarrow 0 \end{aligned}$$

as $c_2 \rightarrow c_1$, and similarly we can show the convergence in H^1 .

3) $A_c \equiv (I - DP_{0,c}(u_{0,c}))^{-1}$ is continuous with respect to c .

To show this, denote $T_c = DP_{0,c}(u_{0,c})$. Then

$$\begin{aligned} (I - T_{c_1})^{-1} - (I - T_{c_2})^{-1} &= (I - T_{c_1})^{-1} - (I - T_{c_1} + T_{c_1} - T_{c_2})^{-1} \\ &= (I - T_{c_1})^{-1} - [(I - T_{c_1})(I + (I - T_{c_1})^{-1}(T_{c_1} - T_{c_2}))]^{-1}, \end{aligned}$$

and $B \equiv -(I - T_{c_1})^{-1}(T_{c_1} - T_{c_2})$ satisfies $\|B\| \rightarrow 0$ as $c_2 \rightarrow c_1$. Thus for c_2 close to c_1 ,

$$\begin{aligned} (I - T_{c_1})^{-1} - (I - T_{c_2})^{-1} &= (I - T_{c_1})^{-1} - (I - B)^{-1}(I - T_{c_1})^{-1} \\ &= (I - T_{c_1})^{-1} - (I + B + B^2 + \cdots)(I - T_{c_1})^{-1}, \\ &= -B(I - B)^{-1}(I - T_{c_1})^{-1} \rightarrow 0 \end{aligned}$$

as $c_2 \rightarrow c_1$.

By denoting $S_c(\bar{u}) \equiv P_{\epsilon,c}(\bar{u} + u_{0,c}) - P_{0,c}(u_{0,c}) - DP_{0,c}(u_{0,c})\bar{u}$, we get

$$\begin{aligned} \|R_{c_1}(\bar{u}_2) - R_{c_2}(\bar{u}_2)\|_{H^1} &= \|A_{c_1}S_{c_1}(\bar{u}_2) - A_{c_2}S_{c_2}(\bar{u}_2)\|_{H^1} \\ &= \|A_{c_1}S_{c_1}(\bar{u}_2) - A_{c_1}S_{c_2}(\bar{u}_2) + A_{c_1}S_{c_2}(\bar{u}_2) - A_{c_2}S_{c_2}(\bar{u}_2)\|_{H^1} \\ &\leq \|A_{c_1}(S_{c_1} - S_{c_2})(\bar{u}_2)\|_{H^1} + \|(A_{c_1} - A_{c_2})S_{c_2}(\bar{u}_2)\|_{H^1} \\ &\rightarrow 0 \end{aligned}$$

as $c_2 \rightarrow c_1$ because both operator R_c and function $u_{0,c}$ are continuous with respect to c and $\|S_{c_2}(\bar{u}_2)\|_{H^1}$ is bounded as $c_2 \rightarrow c_1$.

Therefore

$$\begin{aligned} \|u_{c_1} - u_{c_2}\|_{H^1} &\leq \frac{1}{1 - \theta}(\|R_{c_1}(\bar{u}_2) - R_{c_2}(\bar{u}_2)\|_{H^1} + (1 + \theta)\|u_{0,c_1} - u_{0,c_2}\|_{H^1}) \\ &\rightarrow 0 \end{aligned}$$

as $c_2 \rightarrow c_1$, which proves the lemma. □

Theorem 2.4.2 *The traveling pulse of (2.11) derived in Theorem 2.1.2 is unstable.*

Proof: Let $\phi = u_{c_1}$ be a traveling pulse corresponding to wave speed c_1 . Then for $\varepsilon_1 \equiv \frac{1}{4}\|u_{c_1}\|_{L^2}$, and any $\delta_1 > 0$, by Lemma 2.4.1, there exist a speed c_2 which is close to c_1 and a traveling pulse u_{c_2} for (2.11) such that

$$\|u_{c_1} - u_{c_2}\|_{H^1} < \delta_1 \quad \text{and} \quad \|c_1 u'_{c_1} - c_2 u'_{c_2}\|_{L^2} < \delta_1.$$

Without loss of generality, we can assume $c_2 > c_1 > 0$. Therefore, $u_{c_2}(x - ct)$ is a solution of (2.11) and it is initially close to the traveling pulse u_{c_1} . However, for any fixed k , there exist $M_2 > M_1 > 0$ such that

$$\left(\int_{-M_1}^{M_1} (u_{c_1}(x - k))^2 dx\right)^{\frac{1}{2}} > \frac{\sqrt{2}}{4}\|u_{c_1}\|_{L^2} = \sqrt{2}\varepsilon_1,$$

and

$$\left(\int_{-\infty}^{-M_2} (u_{c_2}(x - k))^2 dx\right)^{\frac{1}{2}} < \varepsilon_1,$$

thus

$$\begin{aligned} & \|u_{c_1}(x - c_1 t - k) - u_{c_2}(x - c_2 t)\|_{L^2}^2 \\ &= \int_{-\infty}^{\infty} (u_{c_1}(x - c_1 t - k) - u_{c_2}(x - c_2 t))^2 dx \\ &\geq \int_{-M_1 + c_1 t}^{M_1 + c_1 t} (u_{c_1}(x - c_1 t - k) - u_{c_2}(x - c_2 t))^2 dx \\ &\geq \int_{-M_1 + c_1 t}^{M_1 + c_1 t} (u_{c_1}(x - c_1 t - k))^2 dx - \int_{-M_1 + c_1 t}^{M_1 + c_1 t} (u_{c_2}(x - c_2 t))^2 dx \\ &\geq \int_{-M_1}^{M_1} (u_{c_1}(x - k))^2 dx - \int_{-M_1 + c_1 t - c_2 t}^{M_1 + c_1 t - c_2 t} (u_{c_2}(x))^2 dx \\ &\geq 2\varepsilon_1^2 - \int_{-\infty}^{-M_2} (u_{c_2}(x))^2 dx \\ &\geq 2\varepsilon_1^2 - \varepsilon_1^2 = \varepsilon_1^2, \end{aligned}$$

which shows the instability. □

CHAPTER 3

Traveling pulses for the lattice Klein-Gordon equation

3.1 Existence of traveling pulses

In this section we study the homoclinic traveling pulse solutions of the lattice system

$$\ddot{u}_n - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u_{n-k} + f(u_n) = 0, \quad n \in \mathbb{Z} \quad (3.1)$$

where $\varepsilon > 0$ and α_k satisfies

$$\begin{aligned} \text{(A3)} \quad \sum_{k=-\infty}^{\infty} \alpha_k &= 0, \quad \alpha_0 < 0, \quad \alpha_k = \alpha_{-k}, \quad \sum_{k \geq 1} \alpha_k k^2 = d > 0 \\ \text{and} \quad \sum_{k \geq 1} |\alpha_k| k^2 &= \bar{d} < \infty. \end{aligned}$$

With the ansatz $u_n(t) = u(\varepsilon n - ct) = u(\eta)$, we get the following differential equation with infinitely many advanced and delayed terms:

$$c^2 \frac{d^2 u}{d\eta^2} - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u(\eta - k\varepsilon) + f(u) = 0. \quad (3.2)$$

We proceed as before, taking Fourier transforms and reformulating as a fixed point problem.

Applying the Fourier transform, equation (3.2) becomes

$$-c^2\xi^2\hat{u} - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k e^{i\varepsilon k\xi} \hat{u} + \widehat{f(u)} = 0.$$

Using **(A3)** we may write

$$\begin{aligned} \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k e^{i\varepsilon k\xi} \hat{u} &= \frac{1}{\varepsilon^2} \sum_{k \geq 1} \alpha_k (e^{i\varepsilon k\xi} - 2 + e^{-i\varepsilon k\xi}) \hat{u} \\ &= -\frac{4}{\varepsilon^2} \sum_{k \geq 1} \alpha_k \sin^2\left(\frac{\varepsilon k\xi}{2}\right) \hat{u}. \end{aligned}$$

Therefore, we may write our equation as

$$[(c^2 - \sum_{k \geq 1} \alpha_k k^2 \text{sinc}^2(\frac{\varepsilon k\xi}{2}))\xi^2 + a]\hat{u} = \widehat{g(u)},$$

where $\text{sinc}(z) = \frac{\sin z}{z}$.

Let

$$q_\varepsilon(\xi) = \frac{1}{(c^2 - \sum_{k \geq 1} \alpha_k k^2 \text{sinc}^2(\frac{\varepsilon k\xi}{2}))\xi^2 + a},$$

then we can write the equation in the form

$$u = Q_\varepsilon(u),$$

where Q_ε is the operator defined by $Q_\varepsilon(u) \equiv \tilde{q}_\varepsilon * g(u)$. Since

$$\text{sinc}^2\left(\frac{\varepsilon k\xi}{2}\right) = 1 - \frac{1}{8}\varepsilon^2\xi^2 + o(\varepsilon^4\xi^4),$$

$q_\varepsilon(\xi)$ has the limit

$$p_0(\xi) = \frac{1}{(c^2 - d)\xi^2 + a}$$

as $\varepsilon \rightarrow 0$. Therefore, formally when $\varepsilon \rightarrow 0$, we have the limit equation

$$u = P_0(u),$$

where, as before, $P_0(u) \equiv \check{p}_0 * g(u)$; and the integral equation is equivalent to

$$(c^2 - d)u'' = -f(u), \quad (3.3)$$

which has the nondegenerate homoclinic solution, u_0 , discussed previously. By a method similar to that used to study the nonlocal wave equation with a continuous convolution operator, we can again prove the existence of a traveling pulse u_ε near u_0 for $c^2 > D_1$, where

$$D_1 \equiv \sup_z \sum_{k \geq 1} \alpha_k k^2 \text{sinc}^2(kz)$$

satisfies $D_1 \geq d$.

Theorem 3.1.1 *If conditions (A1) and (A3) are satisfied, then for any speed c satisfying $c^2 > D_1$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, equation (3.3) has a unique solution u_ε in the set*

$$\{u \in H^1(\mathbb{R}) : u \text{ is even, } \|u - u_0\|_{H^1} < \delta\},$$

where u_0 is the even, positive homoclinic solution of (3.3), and $\delta > 0$ depends on f , c , and the coefficients α_k , and can be chosen so that $\delta < \|u_0\|_{H^1}$.

Proof: Notice that when $c^2 > D_1$, both $\check{q}_\varepsilon$ and $\check{p}_0$ are well defined. Therefore, similar to the proof of Theorem 2.2, we only need to show that for a given positive ρ , there exists a small $\varepsilon_0 > 0$ such that $\|q_\varepsilon - p_0\|_\infty < \rho$ for $\varepsilon < \varepsilon_0$. Since

$$q_\varepsilon(\xi) - p_0(\xi) = \frac{\sum_{k \geq 1} \alpha_k k^2 \text{sinc}^2\left(\frac{\varepsilon k \xi}{2}\right) \xi^2 - d \xi^2}{[(c^2 - \sum_{k \geq 1} \alpha_k k^2 \text{sinc}^2\left(\frac{\varepsilon k \xi}{2}\right)) \xi^2 + a][(c^2 - d) \xi^2 + a]},$$

and $\text{sinc}^2\left(\frac{\varepsilon k \xi}{2}\right) \leq 1$, there exists $\xi_0 > 0$ such that for $|\xi| > \xi_0$ we have

$$|q_\epsilon - p_0| < \frac{(D_1 + d)\xi^2}{((c^2 - D_1)\xi^2 + a)((c^2 - d)\xi^2 + a)} < \rho.$$

Notice that for $|\xi| \leq \xi_0$,

$$\sum_{k \geq 1} \alpha_k k^2 \text{sinc}^2\left(\frac{\epsilon k \xi}{2}\right) - d = \sum_{k \geq 1} \alpha_k k^2 [\text{sinc}^2\left(\frac{\epsilon k \xi}{2}\right) - 1] \rightarrow 0$$

uniformly as $\epsilon \rightarrow 0$ because $\sum_{k \geq 1} |\alpha_k| k^2 < \infty$. Therefore, $q_\epsilon \rightarrow p_0$ uniformly as $\epsilon \rightarrow 0$, allowing us to draw the desired conclusion. $\square$

3.2 Asymptotic behavior of traveling pulses

For a lattice with finite range interactions, we give an estimate for the tails of the traveling pulse:

Theorem 3.2.1 *Suppose that $\alpha_k = 0$ for $|k| > k_0$ and suppose $c^2 > \bar{d}$. There exists $\epsilon_1 > 0$ such that for $\epsilon < \epsilon_1$, the solution $u_\epsilon(z)$ obtained in Theorem 3.1.1 decays to zero exponentially as $z \rightarrow \pm\infty$.*

Proof: As in Section 2.3, we first show that $\tilde{q}_\epsilon$ decays exponentially at $\pm\infty$. To estimate the inverse Fourier transform of q_ϵ , we need to estimate $q_\epsilon(\xi)$ for complex $\xi = x + yi$.

Notice that

$$\begin{aligned}
|a + c^2\xi^2|^2 &= |a + c^2(x^2 - y^2) + 2c^2xyi|^2 \\
&= a^2 + 2ac^2(x^2 - y^2) + c^4(x^2 - y^2)^2 + 4c^4x^2y^2 \\
&= a^2 + 2ac^2(x^2 - y^2) + c^4(x^2 + y^2)^2 \\
&= \frac{3}{4}a^2 + ac^2x^2 - 3ac^2y^2 + [\frac{1}{4}a^2 + ac^2(x^2 + y^2) + c^4(x^2 + y^2)^2] \\
&\geq \frac{3}{4}a^2 - 3ac^2y^2 + (\frac{a}{2} + c^2(x^2 + y^2))^2 \\
&\geq (\frac{a}{2} + c^2|\xi|^2)^2
\end{aligned}$$

if we choose y such that $\frac{3}{4}a^2 - 3ac^2y^2 \geq 0$, i.e., $|y| \leq \gamma \equiv \frac{\sqrt{a}}{2|c|}$. A direct calculation shows that

$$|\sin^2(\frac{\varepsilon k \xi}{2})| = \frac{1}{4}[e^{-\varepsilon k y} + e^{\varepsilon k y} - 2] + \sin^2(\frac{\varepsilon k x}{2}).$$

Thus,

$$|\operatorname{sinc}^2(\frac{\varepsilon k \xi}{2})| = \frac{4}{\varepsilon^2 k^2 |\xi|^2} |\sin^2(\frac{\varepsilon k \xi}{2})| = \frac{e^{-\varepsilon k y} + e^{\varepsilon k y} - 2}{\varepsilon^2 k^2 (x^2 + y^2)} + \frac{4 \sin^2(\frac{\varepsilon k x}{2})}{\varepsilon^2 k^2 (x^2 + y^2)}.$$

Since

$$\frac{e^{-\varepsilon k y} + e^{\varepsilon k y} - 2}{\varepsilon^2 k^2 y^2} = \frac{e^{-\varepsilon k y} (e^{\varepsilon k y} - 1)^2}{\varepsilon^2 k^2 y^2} \rightarrow 1$$

uniformly for $|y| \leq \gamma$ and $|k| \leq k_0$ as $\varepsilon \rightarrow 0$, there exists a positive ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$

$$\frac{e^{-\varepsilon k y} + e^{\varepsilon k y} - 2}{\varepsilon^2 k^2 (x^2 + y^2)} = \frac{e^{-\varepsilon k y} + e^{\varepsilon k y} - 2}{\varepsilon^2 k^2 y^2} \frac{y^2}{x^2 + y^2} < (1 + \frac{c^2 - \bar{d}}{2\bar{d}}) \frac{y^2}{x^2 + y^2}.$$

Also, it is easily seen that

$$\frac{4 \sin^2(\frac{\varepsilon k x}{2})}{\varepsilon^2 k^2 (x^2 + y^2)} = \frac{4 \sin^2(\frac{\varepsilon k x}{2})}{\varepsilon^2 k^2 x^2} \frac{x^2}{x^2 + y^2} \leq \frac{x^2}{x^2 + y^2}.$$

Therefore we have for $|y| \leq \gamma$, $|k| \leq k_0$, and $\varepsilon \in (0, \varepsilon_1]$

$$|\operatorname{sinc}^2(\frac{\varepsilon k \xi}{2})| \leq (1 + \frac{c^2 - \bar{d}}{2\bar{d}}) \frac{y^2}{x^2 + y^2} + \frac{x^2}{x^2 + y^2} \leq 1 + \frac{c^2 - \bar{d}}{2\bar{d}}.$$

So, from the definition of q_ε , for $|y| \leq \gamma$ and $\varepsilon \in (0, \varepsilon_1]$

$$\begin{aligned} |q_\varepsilon| &= \frac{1}{|(c^2 - \sum_{k \geq 1} \alpha_k k^2 \operatorname{sinc}^2(\frac{\varepsilon k \xi}{2})) \xi^2 + a|} \\ &\leq \frac{1}{|c^2 \xi^2 + a| - |\sum_{k \geq 1} \alpha_k k^2 \operatorname{sinc}^2(\frac{\varepsilon k \xi}{2}) \xi^2|} \\ &\leq \frac{1}{\frac{a}{2} + c^2 |\xi|^2 - \sum_{k \geq 1} |\alpha_k| k^2 (1 + \frac{c^2 - \bar{d}}{2\bar{d}}) |\xi|^2} \\ &= \frac{1}{\frac{a}{2} + c^2 |\xi|^2 - \bar{d} (1 + \frac{c^2 - \bar{d}}{2\bar{d}}) |\xi|^2} \\ &= \frac{2}{a + (c^2 - \bar{d}) |\xi|^2}. \end{aligned}$$

Thus $q_\varepsilon(\xi)$ is analytic on the strip $\operatorname{Im} \xi < \gamma$. Therefore, for $x > 0$,

$$\begin{aligned} |e^{\gamma x} \check{q}_\varepsilon| &= |\int_{-\infty}^{\infty} e^{ixz + \gamma x} q_\varepsilon(z) dz| \\ &= |\int_{C_\gamma} e^{ix\xi} q_\varepsilon(\xi + \gamma i) d\xi| \end{aligned}$$

where $C_j = \{z - \gamma i : -\infty < z < \infty\}$, and by Cauchy's integral theorem, we have

$$\begin{aligned} |e^{\gamma x} \check{q}_\varepsilon| &= |\int_{-\infty}^{\infty} e^{ixz} q_\varepsilon(z + \gamma i) dz| \\ &\leq \int_{-\infty}^{\infty} \frac{2}{a + (c^2 - \bar{d}) |z + \gamma i|^2} dz \\ &\leq \int_{-\infty}^{\infty} \frac{2}{a + (c^2 - \bar{d}) |z|^2} dz \\ &= C < \infty. \end{aligned}$$

Thus,

$$|\tilde{q}_\varepsilon| \leq C e^{-\gamma|x|}.$$

The proof of the theorem is completed using the same idea as that in section 2.3. $\square$

Remark For both the lattice and continuum equations, the subsonic case when $c^2 < d$ is still an open problem. Since the corresponding limit ODEs have heteroclinic solutions for bistable nonlinearity f , one may conjecture that there exist heteroclinic traveling waves for our models, at least when $\varepsilon > 0$ is small. On the other hand, even though the limit equation does not have homoclinic solutions, since the convolution operator is bounded, we suspect that our models still possess traveling pulse solutions with subsonic speed provided ε is sufficiently large. Our analysis breaks down, however, for $c^2 < d_1$ (or D_1 in the lattice case) due to singularities in the transformed kernel p_ε (or q_ε , respectively).

3.3 Instability of traveling pulses

In this section we consider the lattice Klein-Gordon equations

$$\ddot{u}_n - \frac{1}{\varepsilon^2}(u_{n+1} - 2u_n + u_{n-1}) - u_n + u_n^3 = 0, \quad n \in \mathbb{Z} \quad (3.4)$$

for which the existence of the traveling pulse was proved in section 3.1. To illustrate the instability we use 4th order Runge-Kutta method solve the Cauchy problem for system (3.4) with the initial data

$$u_n(0) = \phi_0(\varepsilon n), \quad u'_n(0) = -c\phi_0(\varepsilon n), \quad (3.5)$$

where $\phi_0(x) = \sqrt{2} \operatorname{sech}(\frac{x}{\sqrt{c^2 - 1}})$ is the solution of the ODE

$$(c^2 - 1)u'' - u + u^3 = 0.$$

Since we proved that the traveling pulse of (3.4) is in a small neighborhood of ϕ_0 when ε is small, we can regard ϕ_0 as a perturbation of the traveling pulse. The numerical simulations show that even though the solution of (3.4) and (3.5) is close to the traveling pulse initially, it becomes significantly different rapidly. Figures 3.1–3.8 show the solution $u_n(t)$ of (3.4) at time $t = 1, 5, 6, 7, 8, 10, 15$, and $t = 20$. Figures 3.9–3.16 show the difference between $u_n(t)$ and $\phi_0(\varepsilon n - ct)$ at time $t = 1, 5, 8, 10, 15$, and $t = 20$. We use the following choice of parameters in our simulation: $N = 1001$, $n \in [\frac{-N+1}{2}, \frac{N-1}{2}]$, $\varepsilon = 0.1$, $c = 1.1$, and $\Delta t = 0.05$. For the boundary conditions, we choose

$$u_l = \frac{u_{l+1}^2}{u_{l+2}}$$

on the left boundary where $l = \frac{-N-1}{2}$ and

$$u_m = \frac{u_{m-1}^2}{u_{m-2}}$$

on the right boundary where $m = \frac{N+1}{2}$.

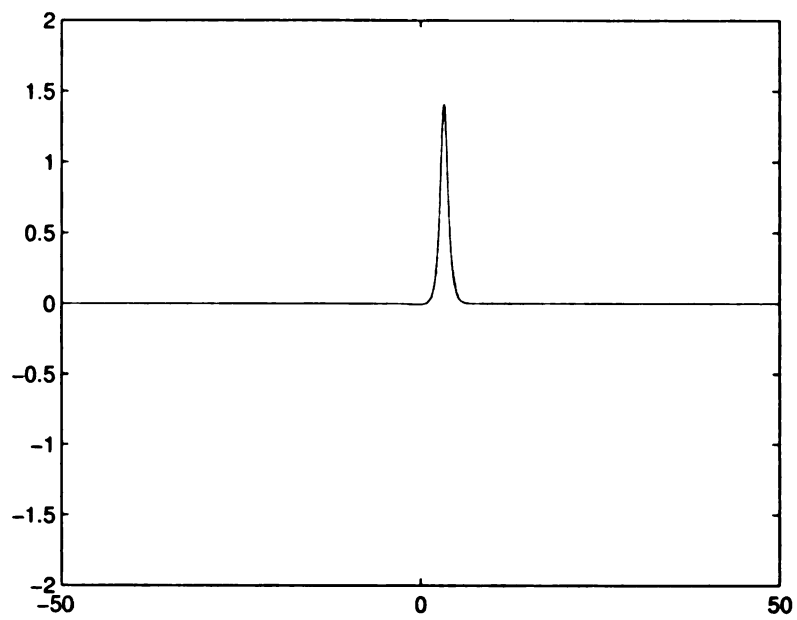


Figure 3.1. $u_n(t), t = 1$

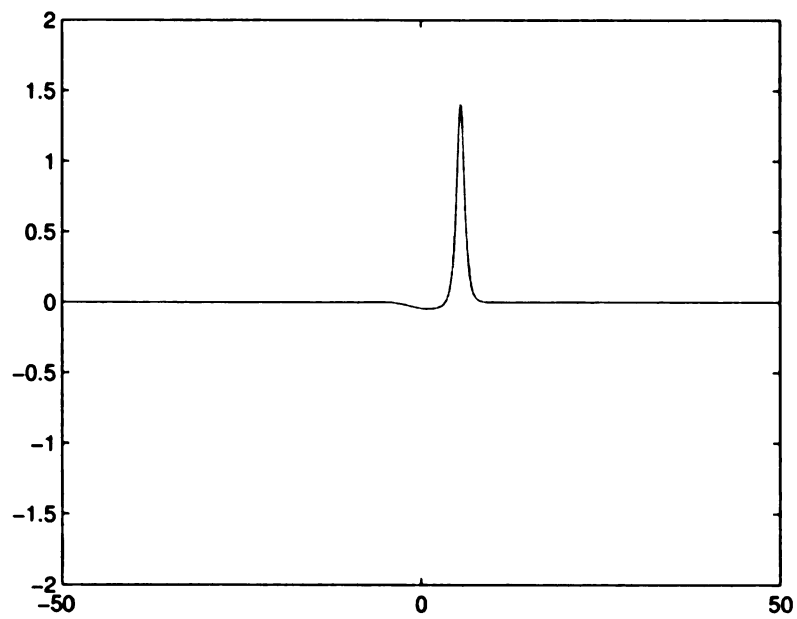


Figure 3.2. $u_n(t), t = 5$

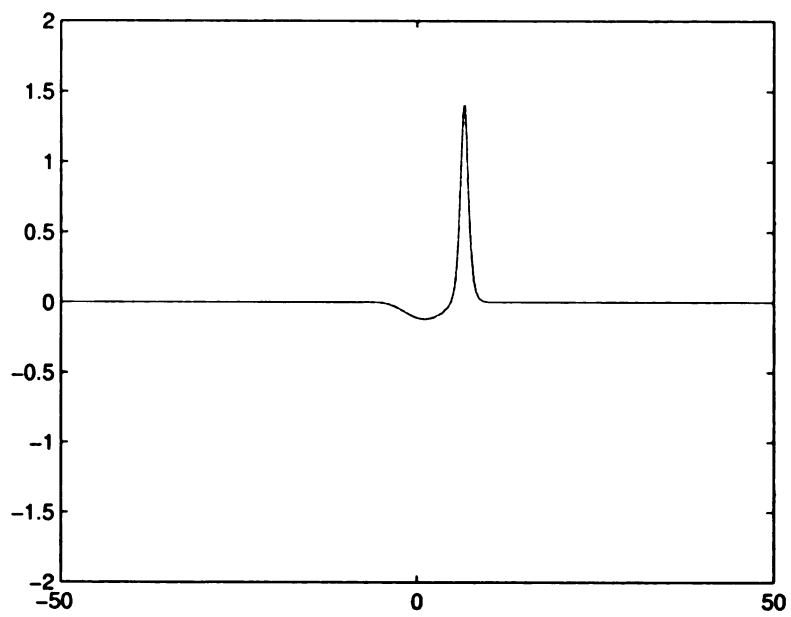


Figure 3.3. $u_n(t), t = 6$

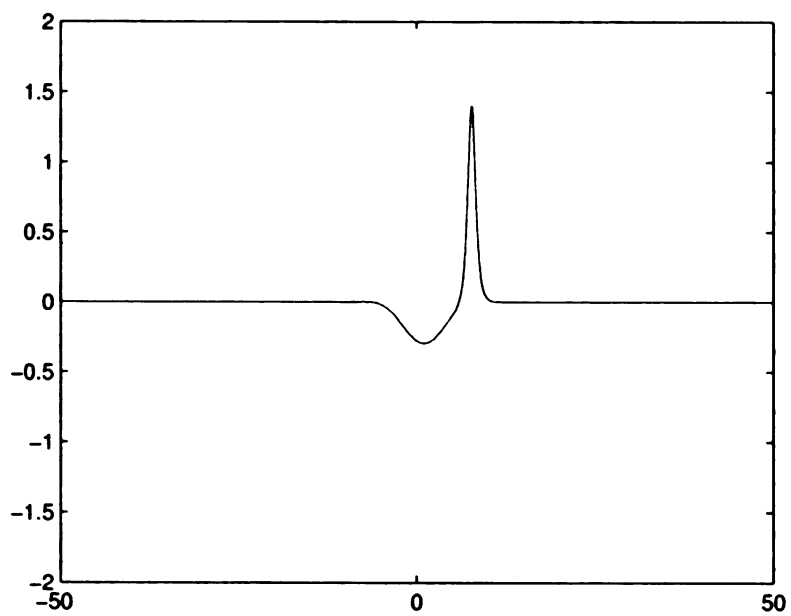


Figure 3.4. $u_n(t), t = 7$

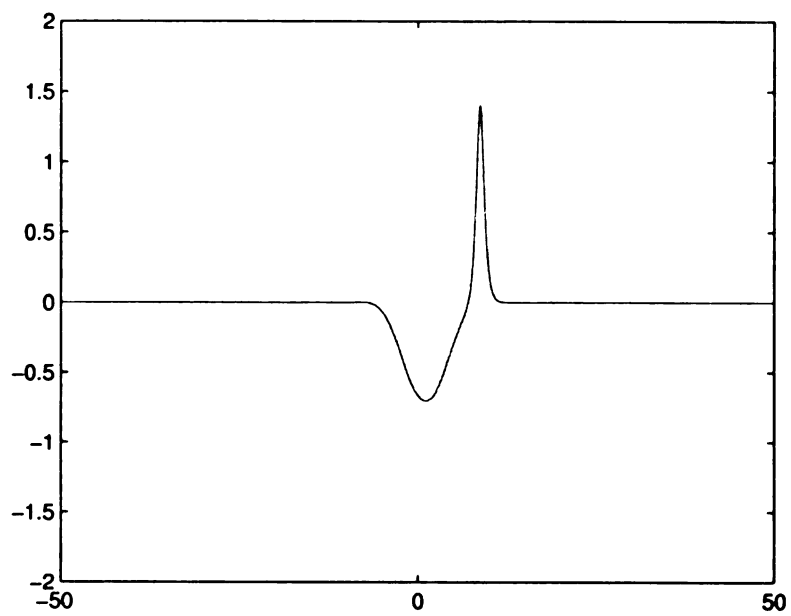


Figure 3.5. $u_n(t), t = 8$

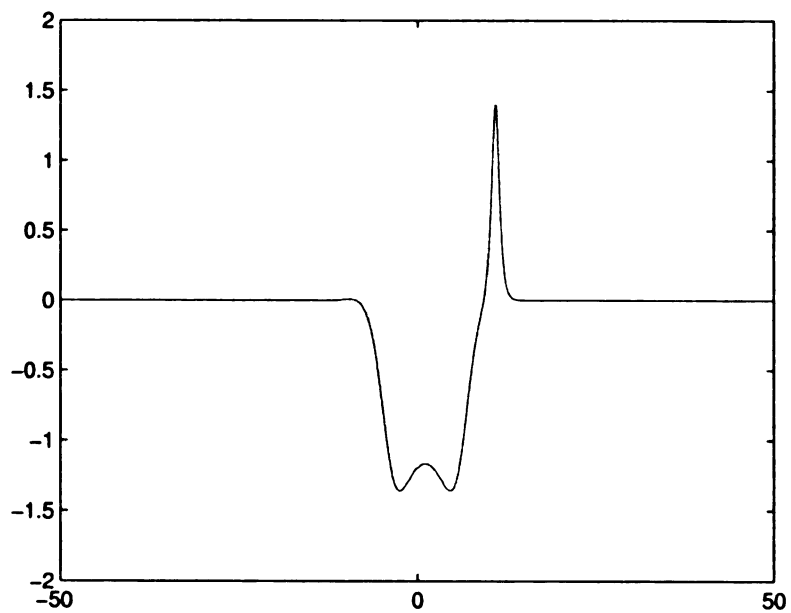


Figure 3.6. $u_n(t), t = 10$

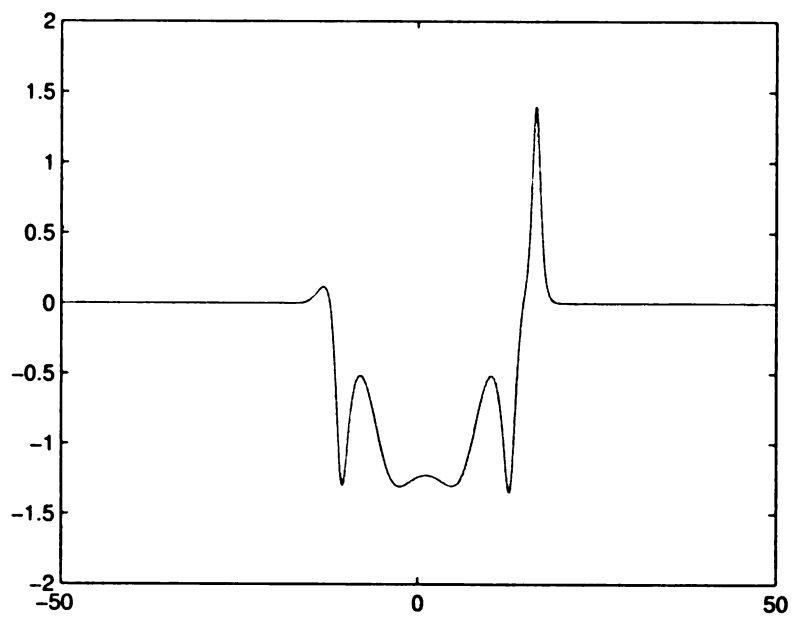


Figure 3.7. $u_n(t), t = 15$

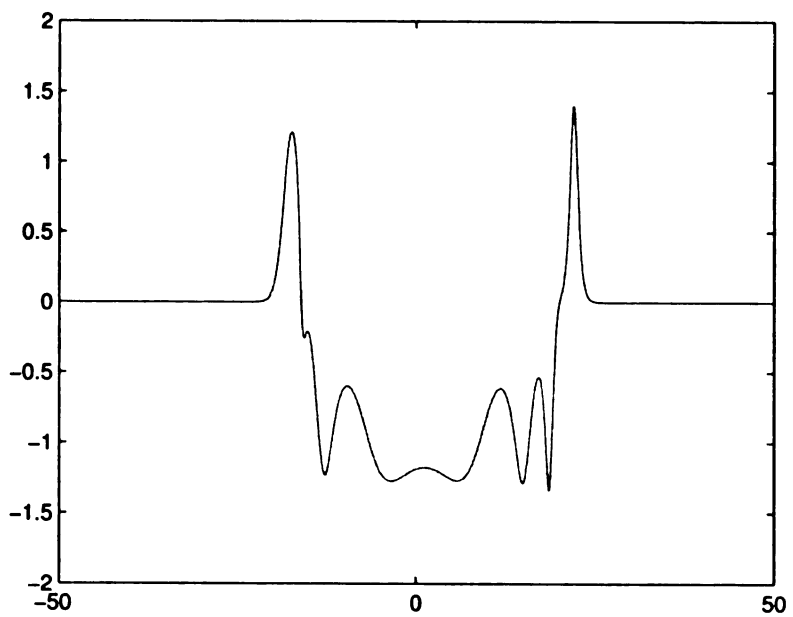


Figure 3.8. $u_n(t), t = 20$

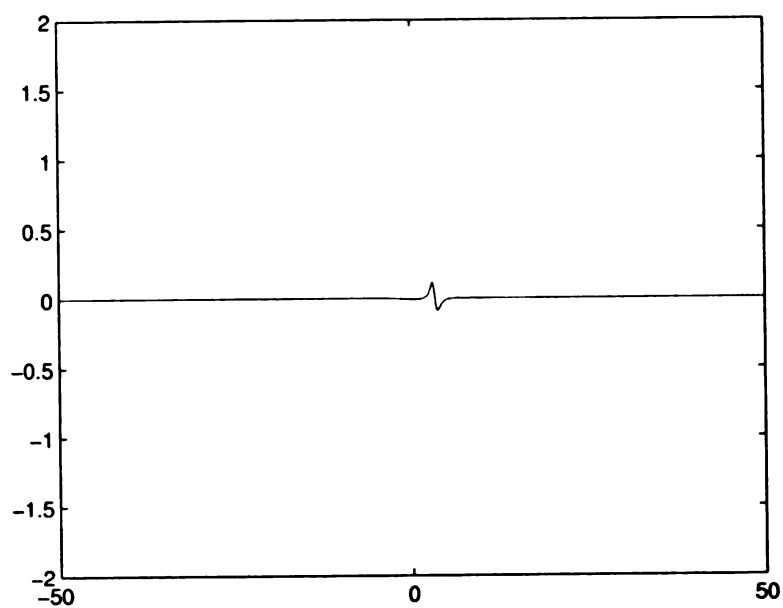


Figure 3.9. $u_n(t) - \phi_0(\epsilon n - ct), t = 1$

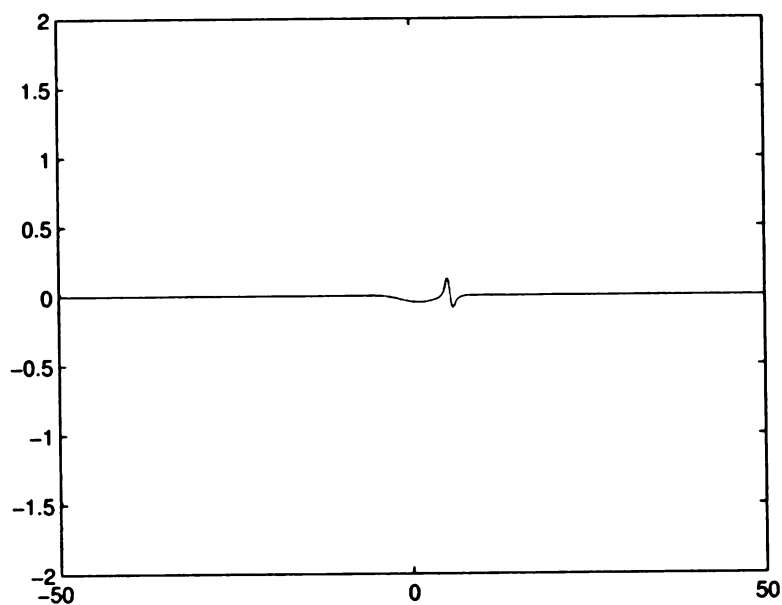


Figure 3.10. $u_n(t) - \phi_0(\epsilon n - ct), t = 5$

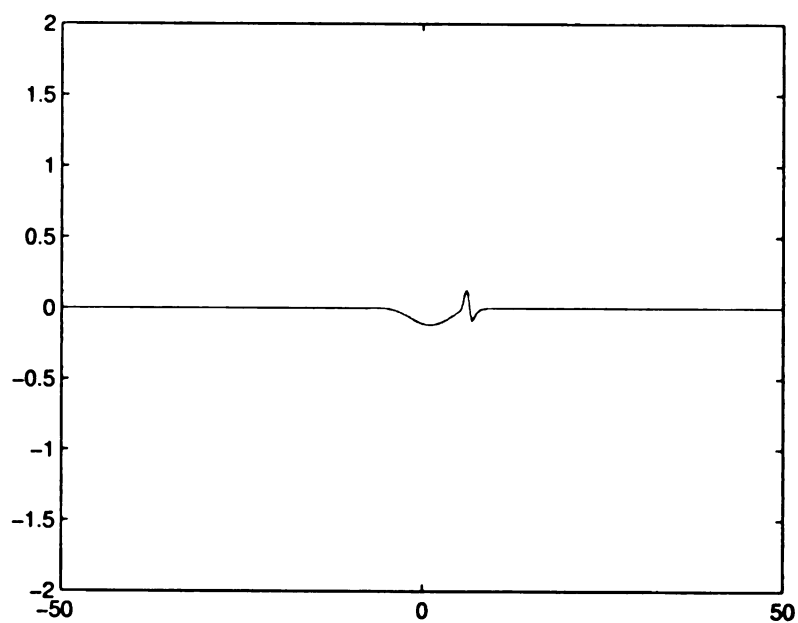


Figure 3.11. $u_n(t) - \phi_0(\epsilon n - ct), t = 6$

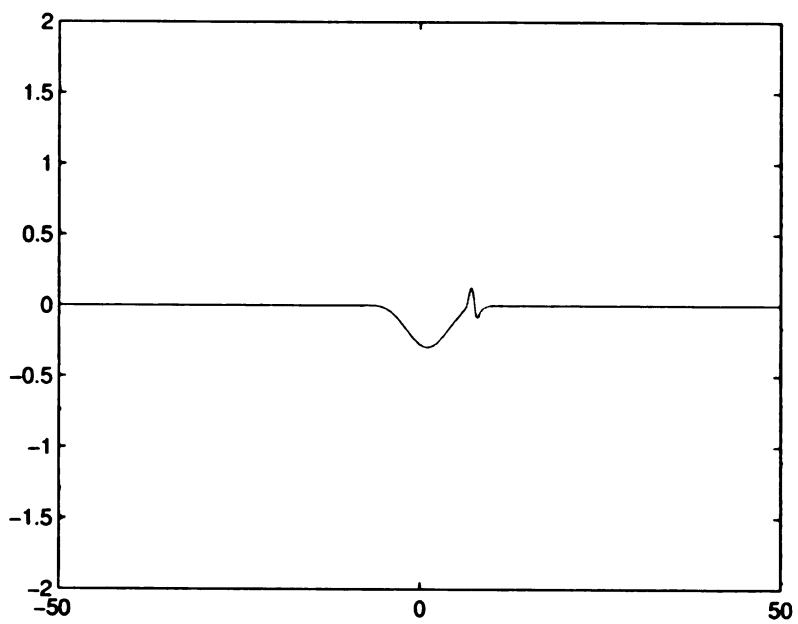


Figure 3.12. $u_n(t) - \phi_0(\epsilon n - ct), t = 7$

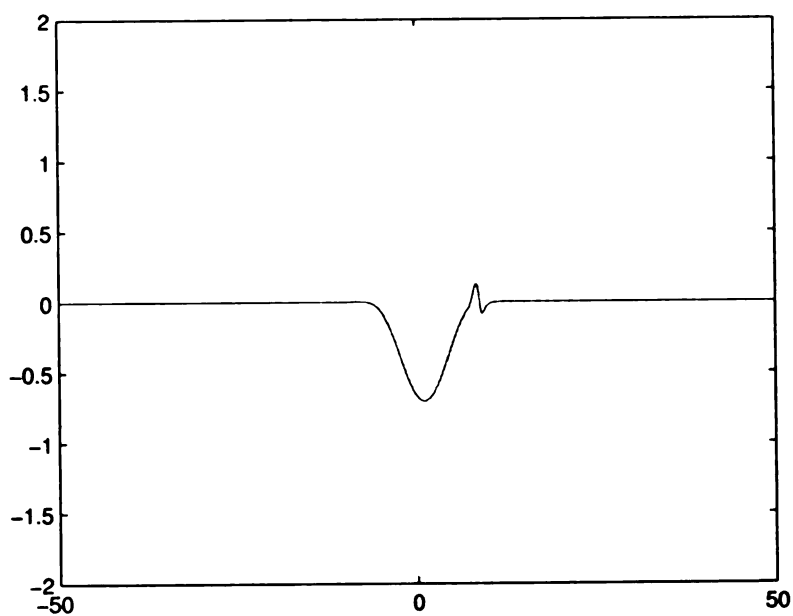


Figure 3.13. $u_n(t) - \phi_0(\epsilon n - ct), t = 8$

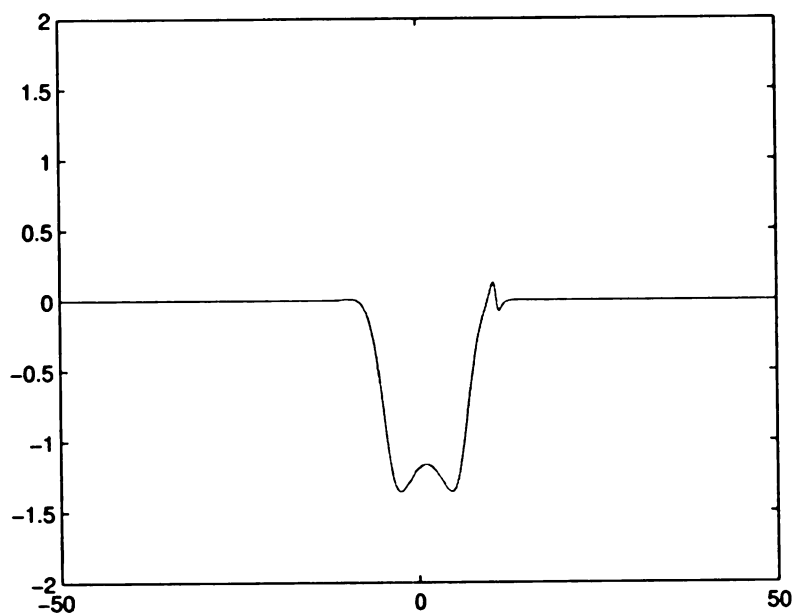


Figure 3.14. $u_n(t) - \phi_0(\epsilon n - ct), t = 10$

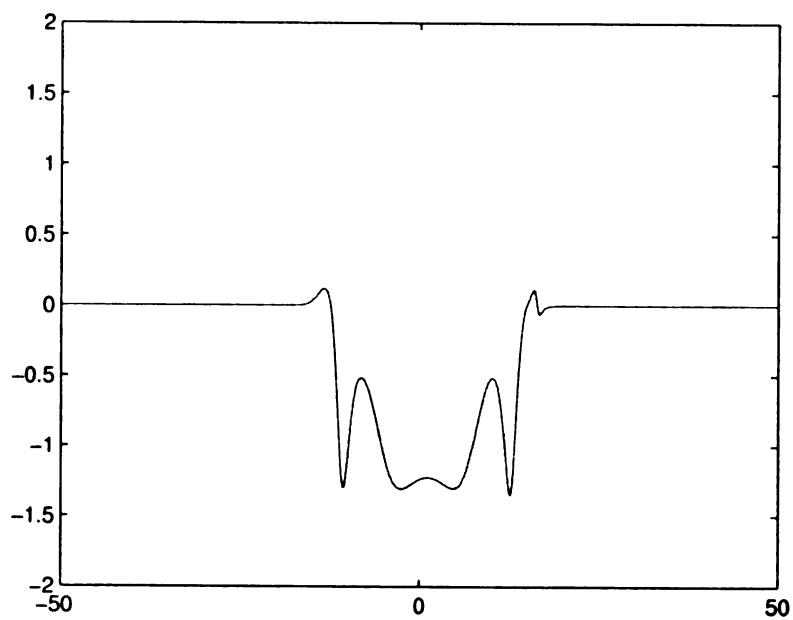


Figure 3.15. $u_n(t) - \phi_0(\epsilon n - ct), t = 15$

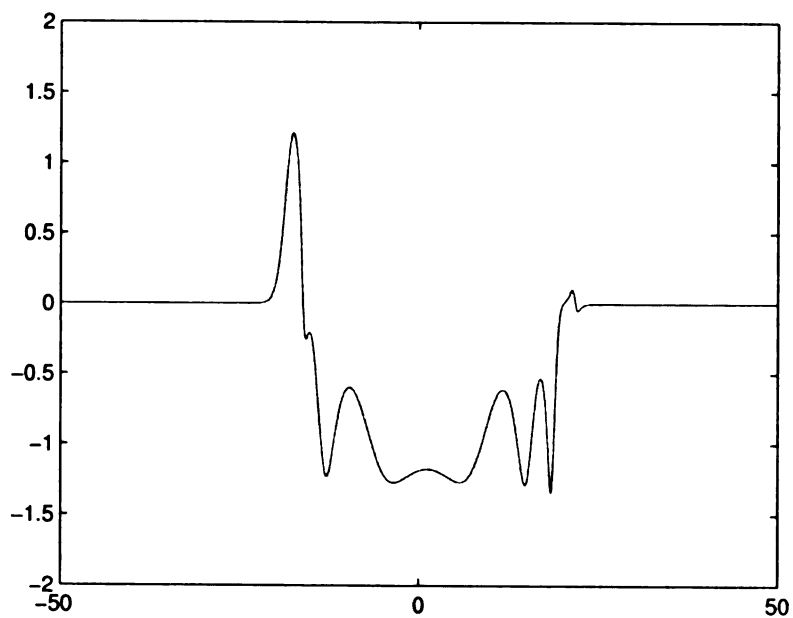


Figure 3.16. $u_n(t) - \phi_0(\epsilon n - ct), t = 20$

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