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DIFFUSION FLAME STABILITY

By

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ABSTRACT

DIFFUSION FLAME STABILITY

By

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We analyze the solutions of a boundary value problem arising from a one-dimensional diffusion flame. We state properties satisfied by the linear operator of the system and prove existence of the steady state solutions for certain parameter values. It has been well established that the steady state solutions form an S-curve when no radiative heat losses are included. We show that this S-curve transforms into an island shaped curve and an ignition branch for large activation temperatures when radiation is included, but islands never form for low activation temperatures. We also analyze the stability of the steady state solutions by analyzing the eigenvalues of the linearized system. The evolution of stable oscillations is seen for certain parameter values by perturbing steady state solutions. Hopf bifurcation points are identified and classified as subcritical or supercritical and the regions in which small perturbations lead to stable oscillations are analyzed. A method for easily determining whether a Hopf bifurcation point of a general system of differential equations is subcritical or supercritical is developed. The method is used to more accurately identify the regions in which stable periodic solutions exist in diffusion flames.

For Germ

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1 Introduction

Solutions of a boundary value problem arising from a one-dimensional, non-premixed, film diffusion flame (in which the physics include only diffusion, chemical reaction, and possibly convection or volumetric heat losses due to radiation) have been investigated, both theoretically and numerically. When no radiating heat loss is included, Fendell [4] showed that plotting the maximum temperature of the steady state solution versus the Damköhler number yields an S-curve. Vance et al. [14] examined the response of a diffusion flame to small perturbations by studying the eigenvalues of the linearized system. In particular, Vance and colleagues discussed the impact of the Lewis number on the stability of the upper branch of the S-curve. Sohn et al. [11] investigated when the Lewis number is larger than unity by integrating the conservation equations numerically. Kukuck and Matalon [8], Cheatham and Matalon [2], and Kim et al. [6, 7] also explored the stability problem, but all used asymptotics instead of analyzing eigenvalues. Kim and colleagues predicted pulsating instability when the Lewis number is slightly greater than unity. Vance et al. [14] used eigenvalues not only to determine stability, but also to predict when flame oscillations would occur. Oscillatory behavior in flames was observed in micro-gravity candle experiments aboard the Mir space station, as well as during droplet burning experiments. Sohn and colleagues [11] also observed both decaying oscillations and oscillations leading to flame extinction.

In the later 1990s, the effect of heat loss on diffusion flames was studied. Cheatham and Matalon [1] included a linearized volumetric heat loss term, $h(T - T_0)$, in the energy equation. Kukuck and Matalon [8], using the same model, examined what effect this heat loss had near the upper turn of the S-curve. Sohn and colleagues [12] included heat loss due to radiation, which introduced a term of the form $RD(T^4 - T_0^4)$ to the temperature equation. This study showed that plotting the maximum temperature of the steady state solution versus the Damköhler number in this model yields an island curve, or isola, instead of the usual S-curve. In addition, they found a region of unstable steady flames which, when perturbed, evolved into stable oscillations. This was the first time such behavior was reported. However, the behavior seemed reasonable due to remarkably sustainable flame oscillations in a droplet flame experiment [12]. Christiansen and colleagues [3] included heat loss due to radiation in a way similar to Sohn et al. [12], but also included variable properties, full species multicomponent diffusion, and detailed multistep reaction chemistry. They, however, were not able to find stable oscillations. They explain that this is due either to the different geometry or to the fact that the stable oscillations appeared only in "an extremely narrow regime."

We will examine a system which includes a radiating heat loss term. We prove the existence of the solutions of such equations for certain parameter values and establish many important properties of the linear operator. We will also show how an isola emerges from the usual S-curve at large activation temperatures, but never develops for smaller activation temperatures. We demonstrate that when the ratio of characteristic chemical and radiation time scales is large, only a lower ignition branch remains. We then identify the parameter values affecting the stability of the steady state solutions. In addition, we will show when stable periodic solutions exist and describe a method for easily determining the existence of such solutions.

2 Mathematical Model

We will consider a one-dimensional diffusion flame which lies between two porous walls. Fuel issues from a large reservoir behind the wall at $x = -1$ and an oxidizer diffuses from a free stream through the wall at $x = 1$, see Figure 1.

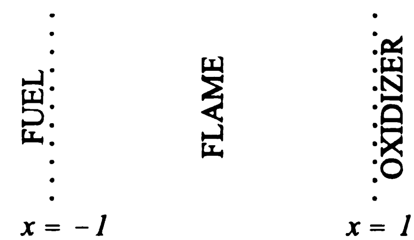

Figure 1: The one-dimensional diffusion flame between two porous walls

The equations governing the flame evolution over time $t > 0$ can be written as

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + W - RD(T^4 - T_0^4) \tag{1}$$

$$Le\frac{\partial Y_O}{\partial t} = \frac{\partial^2 Y_O}{\partial x^2} - W \tag{2}$$

$$Le\frac{\partial Y_f}{\partial t} = \frac{\partial^2 Y_f}{\partial x^2} - W. \tag{3}$$

Here T, the temperature, is a function of both the spatial and time coordinates, x and t, respectively. Y_O and Y_f, the oxidizer and fuel mass fractions, respectively, are also functions of x and t. Le is the Lewis number, taken to be the same for the fuel and oxidizer, and R is the ratio of characteristic chemical and radiation time scales. The reactivity term, W, which is a result of the chemistry, is given by

$$W = DY_O Y_f e^{-Ta/T}, \tag{4}$$

where D is the Damköhler number and T_a is the activation temperature. Note that all of these coordinates and functions have been nondimensionalized exactly as in Sohn et al. [12]. We assume the following boundary conditions at the porous walls:

$$T = T_0 \qquad Y_O = 0 \qquad Y_f = 1 \qquad \text{at} \qquad \mathrm{x} = -1 \tag{5}$$

$$T = T_0 \qquad Y_O = 1 \qquad Y_f = 0 \qquad \text{at} \qquad \mathrm{x} = +1. \tag{6}$$

This model is physically idealized. However, understanding this simple model can aid in understanding more complicated situations. For this reason, this model has been studied extensively before. In particular, the equations used by Vance et al. [14] are identical to the above equations 1 - 6 when $R = 0$, however Vance's equations include a convection term. In our experience, the introduction of this convection term does not change the resulting stability behavior substantially. The equations used by Sohn et al. [12] are also equivalent to the above equations when $T_a = 5$, $T_0 = 0.1$, and $Le = 1$. A slightly different configuration is preferred by Kukuck and Matalon [8]. Christiansen et al. [3] added many complicated real-world influences. For a discussion of radiative loss, see T'ien [13]. They all provide excellent physical descriptions of diffusion flames and cite many references.

Throughout the analysis, we shall take $T_0 = 0.1$, as Sohn et al. [12]. Except for a brief discussion in Sections 6 and 7, we will take $Le = 1$, also as Sohn and colleagues. In our experience, the introduction of other parameters, like the Peclet number of convection, different starting fuel fractions Y_f at $x = -1$, and different wall temperatures T_0 do not change the resulting stability behavior substantially.

3 Existence of the Steady State Solutions

We will begin by showing the existence of the steady state solutions for small values of the Damköhler number, D. In Section 5 we will find the steady state solutions numerically. We therefore know that these solutions exist, however, we will prove existence for only small D values.

For $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \epsilon C[-1,1]^3$, define the function $F(u)$ to be

$$F(u) = \begin{pmatrix} -Du_2u_3e^{-Ta/u_1} + RD(u_1^4 - T_0^4) \\ Du_2u_3e^{-Ta/u_1} \\ Du_2u_3e^{-Ta/u_1} \end{pmatrix}$$

and the operator Nu to be

$$\begin{aligned} Nu(x) &= \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix} \frac{1-x}{2} - \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix} \frac{-1-x}{2} \\ &+ \frac{1-x}{2} \int_{-1}^{x} (-1-s)F(u(s))ds + \frac{-1-x}{2} \int_{x}^{1} (1-s)F(u(s))ds. \end{aligned}$$

Claim 3.1 *u is a fixed point of N if and only if u solves the boundary value problem*

$$u''(x) = F(u(x))$$
$$u(-1) = \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix}, u(1) = \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix}. \tag{7}$$

Proof 3.1 *Let u be a fixed point of N. Hence,*

$$\begin{aligned} u(x) &= \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix} \frac{1-x}{2} - \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix} \frac{-1-x}{2} \\ &+ \frac{1-x}{2} \int_{-1}^{x} (-1-s) F(u(s)) ds + \frac{-1-x}{2} \int_{x}^{1} (1-s) F(u(s)) ds. \end{aligned}$$

So, $u(-1) = \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix}$ *and* $u(1) = \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix}$. *In addition,*

$$\begin{aligned} u'(x) &= \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix} \frac{-1}{2} - \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix} \frac{-1}{2} \\ &- \frac{1}{2} \int_{-1}^{x} (-1-s) F(u(s)) ds + \frac{1-x}{2} (-1-x) F(u(x)) \\ &- \frac{1}{2} \int_{x}^{1} (1-s) F(u(s)) ds - \frac{-1-x}{2} (1-x) F(u(x)) \end{aligned}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - \frac{1}{2}\int_{-1}^{x}(-1-s)F(u(s))ds - \frac{1}{2}\int_{x}^{1}(1-s)F(u(s))ds$$

and

$$\begin{aligned} u''(x) &= -\frac{1}{2}(-1-x)F(u(x)) + \frac{1}{2}(1-x)F(u(x)) \\ &= F(u(x)). \end{aligned}$$

Thus, u is a solution of the boundary value problem 7. To prove the converse, let u be a solution of the boundary value problem 7. Then

$$\begin{aligned} Nu(x) &= \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix} \frac{1-x}{2} - \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix} \frac{-1-x}{2} \\ &+ \frac{1-x}{2}\int_{-1}^{x}(-1-s)u''(s)ds + \frac{-1-x}{2}\int_{x}^{1}(1-s)u''(s)ds \\ &= \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix} \frac{1-x}{2} - \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix} \frac{-1-x}{2} \\ &+ \frac{1-x}{2}[(-1-s)u'(s)\Big|_{-1}^{x} + \int_{-1}^{x} u'(s)ds] \\ &+ \frac{-1-x}{2}[(1-s)u'(s)\Big|_{x}^{1} + \int_{x}^{1} u'(s)ds] \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} T_0 \\ 0 \\ 1 \end{pmatrix} \frac{1-x}{2} - \begin{pmatrix} T_0 \\ 1 \\ 0 \end{pmatrix} \frac{-1-x}{2} \\
&\quad + \frac{1-x}{2}[(-1-x)u'(x) - 0 + u(x) - u(-1)] \\
&\quad + \frac{-1-x}{2}[0 - (1-x)u'(x) + u(1) - u(x)] \\
&= u(x)
\end{aligned}
$$

and u is a fixed point of N. Hence u is a fixed point of N if and only if u is a solution of 7.

Fix $M > max\{T_0, 1\}$ and $0 < \epsilon < T_0$. Define

$$U = \{u \epsilon C[-1,1]^3 : \epsilon \leq u_1 \leq M, |u_2| \leq M, |u_3| \leq M\}.$$

For $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \epsilon C[-1,1]^3$, define the norm

$$||u|| = sup_{x\epsilon[-1,1]}(|u_1(x)| + |u_2(x)| + |u_3(x)|).$$

Claim 3.2 *N has a unique fixed point in U if*

$$D < min\left\{\frac{M-1}{8M^2e^{-Ta/M}}, \frac{M-T_0}{\frac{RT_0^4}{2}+M^2e^{-Ta/M}},\right.$$

$$\left.\frac{\epsilon - T_0}{\frac{RT_0^4}{2} - M^2e^{-Ta/M} - RM^4}, \frac{1}{48Me^{-Ta/M} + \frac{24M^2e^{-Ta/M}Ta}{\epsilon^2} + 32RM^3}\right\}.$$

Proof 3.2 *To prove that N has a unique fixed point, we will show that U is a complete, nonempty metric space, $N : U \to U$, and N is a contraction mapping. First, note that U is a closed, nonempty subset of the Banach space $C[-1,1]^3$. Hence, U is a complete, nonempty metric space. If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \epsilon U$, then clearly $N\mathbf{u}\epsilon C[-1,1]^3$. To show that $N\mathbf{u}\epsilon U$, we must show that $N\mathbf{u}$ satisfies the necessary bounds. Let $x\epsilon[-1,1]$, then*

$$\begin{aligned}
|(Nu(x))_2| &= \Big|\frac{1+x}{2} + \frac{1-x}{2}\int_{-1}^{x}(-1-s)Du_2u_3e^{-Ta/u_1}ds \\
&\quad + \frac{-1-x}{2}\int_{x}^{1}(1-s)Du_2u_3e^{-Ta/u_1}ds\Big| \\
&\le 1 + 2\int_{-1}^{x}|Du_2u_3e^{-Ta/u_1}|ds + 2\int_{x}^{1}|Du_2u_3e^{-Ta/u_1}|ds \\
&\le 1 + 4\int_{-1}^{1}|Du_2u_3e^{-Ta/u_1}|ds \\
&\le 1 + 4\cdot 2\cdot DM^2e^{-Ta/M} \\
&= 1 + 8DM^2e^{-Ta/M}.
\end{aligned}$$

So, $|(Nu(x))_2| \leq M$ *if* $D \leq \dfrac{M-1}{8M^2 e^{-Ta/M}}$. *Similarly,*

$$\begin{aligned}
|(Nu(x))_3| &= \Big|\frac{1-x}{2} + \frac{1-x}{2}\int_{-1}^{x}(-1-s)Du_2u_3e^{-Ta/u_1}ds \\
&\quad + \frac{-1-x}{2}\int_{x}^{1}(1-s)Du_2u_3e^{-Ta/u_1}ds\Big| \\
&\leq 1 + 8DM^2e^{-Ta/M}.
\end{aligned}$$

So, $|(Nu(x))_3| \leq M$ *if* $D \leq \dfrac{M-1}{8M^2 e^{-Ta/M}}$. *Now, note that*

$$\begin{aligned}
(Nu(x))_1 &= T_0 + \frac{1-x}{2}\int_{-1}^{x}(-1-s)(-Du_2u_3e^{-Ta/u_1} \\
&\quad + RDu_1^4 - RDT_0^4)ds \\
&\quad + \frac{-1-x}{2}\int_{x}^{1}(1-s)(-Du_2u_3e^{-Ta/u_1} + RDu_1^4 - RDT_0^4)ds \\
&= T_0 + \frac{x-1}{2}(\frac{x^2}{2} - \frac{1}{2} + (x+1))(-RDT_0^4) \\
&\quad + \frac{x-1}{2}\int_{-1}^{x}(s+1)(-Du_2u_3e^{-Ta/u_1} + RDu_1^4)ds \\
&\quad + \frac{x+1}{2}(\frac{1}{2} - \frac{x^2}{2} - (1-x))(-RDT_0^4) \\
&\quad + \frac{x+1}{2}\int_{x}^{1}(s-1)(-Du_2u_3e^{-Ta/u_1} + RDu_1^4)ds \\
&= T_0 + \frac{RDT_0^4}{2}(1-x^2) + \frac{D(1-x)}{2}\int_{-1}^{x}(s+1)u_2u_3e^{-Ta/u_1}ds \\
&\quad + \frac{D(x+1)}{2}\int_{x}^{1}(1-s)u_2u_3e^{-Ta/u_1}ds \\
&\quad + \frac{RD(x-1)}{2}\int_{-1}^{x}(s+1)u_1^4ds + \frac{RD(x+1)}{2}\int_{x}^{1}(s-1)u_1^4ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
(Nu(x))_1 &\leq T_0 + \frac{RDT_0^4}{2}(1-x^2) + \frac{D(1-x)}{2}(x+1)(x+1)M^2 e^{-Ta/M} \\
&\quad + \frac{D(x+1)}{2}(1-x)(1-x)M^2 e^{-Ta/M} \\
&= T_0 + \frac{RDT_0^4}{2}(1-x^2) + D(1-x^2)M^2 e^{-Ta/M} \\
&\leq T_0 + \frac{RDT_0^4}{2} + DM^2 e^{-Ta/M}.
\end{aligned}$$

So, $(Nu(x))_1 \leq M$ *if* $D \leq \dfrac{M - T_0}{\frac{RT_0^4}{2} + M^2 e^{-Ta/M}}$. *And,*

$$\begin{aligned}
(Nu(x))_1 &\geq T_0 + \frac{RDT_0^4}{2}(1-x^2) - \frac{D(1-x)}{2}(x+1)(x+1)M^2 e^{-Ta/M} \\
&\quad - \frac{D(x+1)}{2}(1-x)(1-x)M^2 e^{-Ta/M} \\
&\quad + \frac{RD(x-1)}{2}(x+1)(x+1)M^4 + \frac{RD(x+1)}{2}(1-x)(x-1)M^4 \\
&= T_0 + D(1-x^2)\left(\frac{RT_0^4}{2} - M^2 e^{-Ta/M} - RM^4\right) \\
&\geq T_0 + D\left(\frac{RT_0^4}{2} - M^2 e^{-Ta/M} - RM^4\right).
\end{aligned}$$

So, $(Nu(x))_1 \geq \epsilon$ *if* $D \leq \dfrac{\epsilon - T_0}{\frac{RT_0^4}{2} - M^2 e^{-Ta/M} - RM^4}$. *Therefore, N maps U into U.*

We will now show that N is a contraction mapping. Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, z =$

$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \epsilon U$ *and* $x \epsilon [-1, 1]$. *So,* $\epsilon \leq u_1, z_1 \leq M$ *and hence, by the Mean Value Theorem,*

$$|e^{-Ta/u_1} - e^{-Ta/z_1}| \leq e^{-Ta/M} \frac{Ta}{\epsilon^2} |u_1 - z_1|.$$

So,

$$\begin{aligned}
|u_2 u_3 e^{-Ta/u_1} - z_2 z_3 e^{-Ta/z_1}| &= |u_2 u_3 e^{-Ta/u_1} - z_2 u_3 e^{-Ta/u_1} \\
&\quad + z_2 u_3 e^{-Ta/u_1} - z_2 z_3 e^{-Ta/u_1} \\
&\quad + z_2 z_3 e^{-Ta/u_1} - z_2 z_3 e^{-Ta/z_1}| \\
&\leq |u_3 e^{-Ta/u_1}||u_2 - z_2| \\
&\quad + |z_2 e^{-Ta/u_1}||u_3 - z_3| \\
&\quad + |z_2 z_3||e^{-Ta/u_1} - e^{-Ta/z_1}| \\
&\leq M e^{-Ta/M} |u_2 - z_2| \\
&\quad + M e^{-Ta/M} |u_3 - z_3| \\
&\quad + M^2 e^{-Ta/M} \frac{Ta}{\epsilon^2} |u_1 - z_1| \\
&\leq K_1 ||u - z||,
\end{aligned}$$

where $K_1 = 2Me^{-Ta/M} + \dfrac{M^2 e^{-Ta/M} Ta}{\epsilon^2}$. *In addition,*

$$|u_1^4 - z_1^4| = |(u_1 - z_1)(u_1^3 + u_1^2 z_1 + u_1 z_1^2 + z_1^3)|$$

$$\begin{aligned}
&\leq 4M^3|u_1 - z_1| \\
&\leq K_2||u - z||,
\end{aligned}$$

where $K_2 = 4M^3$. *Hence,*

$$\begin{aligned}
||Nu - Nz|| &= \Bigg|\Bigg|\frac{1-x}{2}\int_{-1}^{x}(-1-s)(F(u(s)) - F(z(s)))ds \\
&\quad + \frac{-1-x}{2}\int_{x}^{1}(1-s)(F(u(s)) - F(z(s)))ds\Bigg|\Bigg| \\
&= \Bigg|\Bigg|\frac{1-x}{2}\int_{-1}^{x}(-1-s)\left(\begin{pmatrix} -Du_2u_3e^{-Ta/u_1} + RD(u_1^4 - T_0^4) \\ Du_2u_3e^{-Ta/u_1} \\ Du_2u_3e^{-Ta/u_1} \end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} -Dz_2z_3e^{-Ta/z_1} + RD(z_1^4 - T_0^4) \\ Dz_2z_3e^{-Ta/z_1} \\ Dz_2z_3e^{-Ta/z_1} \end{pmatrix}\right)ds \\
&\quad + \frac{-1-x}{2}\int_{x}^{1}(1-s)\left(\begin{pmatrix} -Du_2u_3e^{-Ta/u_1} + RD(u_1^4 - T_0^4) \\ Du_2u_3e^{-Ta/u_1} \\ Du_2u_3e^{-Ta/u_1} \end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} -Dz_2z_3e^{-Ta/z_1} + RD(z_1^4 - T_0^4) \\ Dz_2z_3e^{-Ta/z_1} \\ Dz_2z_3e^{-Ta/z_1} \end{pmatrix}\right)ds\Bigg|\Bigg| \\
&= sup_{x\epsilon[-1,1]}\Bigg[\Bigg|\frac{1-x}{2}\int_{-1}^{x}(-1-s)(-Du_2u_3e^{-Ta/u_1} \\
&\quad + RD(u_1^4 - T_0^4) - (-Dz_2z_3e^{-Ta/z_1} + RD(z_1^4 - T_0^4)))ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{-1-x}{2}\int_x^1 (1-s)(-Du_2u_3e^{-Ta/u_1} + RD(u_1^4 - T_0^4) \\
&-(-Dz_2z_3e^{-Ta/z_1} + RD(z_1^4 - T_0^4)))ds\Big| \\
&+\Big|\frac{1-x}{2}\int_{-1}^x (-1-s)(Du_2u_3e^{-Ta/u_1} - Dz_2z_3e^{-Ta/z_1})ds \\
&+\frac{-1-x}{2}\int_x^1 (1-s)(Du_2u_3e^{-Ta/u_1} - Dz_2z_3e^{-Ta/z_1})ds\Big| \\
&+\Big|\frac{1-x}{2}\int_{-1}^x (-1-s)(Du_2u_3e^{-Ta/u_1} - Dz_2z_3e^{-Ta/z_1})ds \\
&+\frac{-1-x}{2}\int_x^1 (1-s)(Du_2u_3e^{-Ta/u_1} - Dz_2z_3e^{-Ta/z_1})ds\Big|\Big] \\
\leq\ & 4\int_{-1}^1 \Big| - Du_2u_3e^{-Ta/u_1} \\
&+RD(u_1^4 - T_0^4) - (-Dz_2z_3e^{-Ta/z_1} + RD(z_1^4 - T_0^4))\Big|ds \\
&+4\int_{-1}^1 \Big|Du_2u_3e^{-Ta/u_1} - Dz_2z_3e^{-Ta/z_1}\Big|ds \\
&+4\int_{-1}^1 \Big|Du_2u_3e^{-Ta/u_1} - Dz_2z_3e^{-Ta/z_1}\Big|ds \\
\leq\ & 8(3D|u_2u_3e^{-Ta/u_1} - z_2z_3e^{-Ta/z_1}| + RD|u_1^4 - z_1^4|) \\
\leq\ & 8(3DK_1||u-z|| + RDK_2||u-z||) \\
=\ & c||u-z||,
\end{aligned}
$$

where $c = D(24K_1 + 8RK_2)$. *Thus,* N *is a contraction mapping if*

$$
\begin{aligned}
D &< \frac{1}{24K_1 + 8RK_2} \\
&= \frac{1}{48Me^{-Ta/M} + \frac{24M^2e^{-Ta/M}T_a}{\epsilon^2} + 32RM^3}.
\end{aligned}
$$

Hence, by the Contraction Mapping Theorem, N has a unique fixed point in U if

$$
\begin{aligned}
D \;&<\; min\Big\{\frac{M-1}{8M^2e^{-Ta/M}}, \frac{M-T_0}{\frac{RT_0^4}{2}+M^2e^{-Ta/M}},\\
&\frac{\epsilon-T_0}{\frac{RT_0^4}{2}-M^2e^{-Ta/M}-RM^4}, \frac{1}{48Me^{-Ta/M}+\frac{24M^2e^{-Ta/M}T_a}{\epsilon^2}+32RM^3}\Big\}.
\end{aligned}
$$

4 Properties of the Linear Operator

In this section, we will prove that the linear operator has important properties which we will use in later sections.

We begin by transforming the system 1 - 6 into a system for a perturbation of a steady state solution. Let the steady state solutions be denoted by $\tilde{T}, \tilde{Y_O}$, and $\tilde{Y_f}$. Also, assume that $\tilde{T}(x) > 0$ for all $x \epsilon [-1, 1]$. By expressing T, Y_O, and Y_f as the sum of the steady state solution and a perturbation, 1 - 3 can be written as

$$
\begin{aligned}
\frac{\partial}{\partial t}(\tilde{T(x)} + u_1(x,t)) &= \frac{\partial^2}{\partial x^2}(\tilde{T(x)} + u_1(x,t)) \\
&\quad + D(\tilde{Y_O(x)} + u_2(x,t)) \\
&\quad (\tilde{Y_f(x)} + u_3(x,t))e^{-Ta/(\tilde{T(x)} + u_1(x,t))} \\
&\quad - RD((\tilde{T(x)} + u_1(x,t))^4 - T_0^4) \qquad (8) \\
\frac{\partial}{\partial t}(\tilde{Y_O(x)} + u_2(x,t)) &= \frac{1}{Le}\frac{\partial^2}{\partial x^2}(\tilde{Y_O(x)} + u_2(x,t)) \\
&\quad - \frac{1}{Le}D(\tilde{Y_O(x)} + u_2(x,t)) \\
&\quad (\tilde{Y_f(x)} + u_3(x,t))e^{-Ta/(\tilde{T(x)} + u_1(x,t))} \qquad (9) \\
\frac{\partial}{\partial t}(\tilde{Y_f(x)} + u_3(x,t)) &= \frac{1}{Le}\frac{\partial^2}{\partial x^2}(\tilde{Y_f(x)} + u_3(x,t)) \\
&\quad - \frac{1}{Le}D(\tilde{Y_O(x)} + u_2(x,t)) \\
&\quad (\tilde{Y_f(x)} + u_3(x,t))e^{-Ta/(\tilde{T(x)} + u_1(x,t))}. \qquad (10)
\end{aligned}
$$

In order to linearize our nonlinear terms, we will define the following functions:

$$W(c) = D(\tilde{Y_O(x)} + cu_2(x,t))(\tilde{Y_f(x)} + cu_3(x,t))e^{-Ta/(\tilde{T(x)} + cu_1(x,t))}$$

$$Z(c) = RD((\tilde{T(x)} + cu_1(x,t))^4 - T_0^4).$$

Then the nonlinear term of 8 is simply $W(1) - Z(1)$ and the nonlinear term of 9 and 10 is $\frac{-1}{Le}W(1)$. In order to linearize these terms, we will first express $W(1)$ and $Z(1)$ as the Taylor expansions of $W(c)$ and $Z(c)$, respectively, about $c = 0$ evaluated at $c = 1$. Note that

$$W(1) = W(0) + W'(0) + \frac{1}{2}W''(0) + \frac{1}{6}W'''(0) + ... \tag{11}$$

and

$$Z(1) = Z(0) + Z'(0) + \frac{1}{2}Z''(0) + \frac{1}{6}Z'''(0) + ..., \tag{12}$$

where

$$\begin{aligned}
W(0) &= D\tilde{Y_o}\tilde{Y_f}e^{-Ta/\tilde{T}} \\
W'(0) &= D(u_2)\tilde{Y_f}e^{-Ta/\tilde{T}} + D\tilde{Y_o}(u_3)e^{-Ta/\tilde{T}} + D\tilde{Y_o}\tilde{Y_f}\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
W''(0) &= 2D(u_2)(u_3)e^{-Ta/\tilde{T}} + 2D(u_2)\tilde{Y_f}\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
&\quad +2D(u_3)\tilde{Y_o}\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
&\quad +D\tilde{Y_o}\tilde{Y_f}\Big(\frac{-2Ta(u_1)^2}{\tilde{T}^3} + \frac{T_a^2(u_1)^2}{\tilde{T}^4}\Big)e^{-Ta/\tilde{T}} \\
W'''(0) &= 6D(u_2)(u_3)\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
&\quad +3D(u_2)\tilde{Y_f}\Big(\frac{-2Ta(u_1)^2}{\tilde{T}^3} + \frac{T_a^2(u_1)^2}{\tilde{T}^4}\Big)e^{-Ta/\tilde{T}}
\end{aligned}$$

$$
\begin{aligned}
&+3D(u_3)\tilde{Y_O}\Big(\frac{-2T_a(u_1)^2}{\tilde{T}^3}+\frac{T_a^2(u_1)^2}{\tilde{T}^4}\Big)e^{-T_a/\tilde{T}}\\
&+D\tilde{Y_O}\tilde{Y_f}\Big(\frac{6T_a(u_1)^3}{\tilde{T}^4}-\frac{6T_a^2(u_1)^3}{\tilde{T}^5}+\frac{T_a^3(u_1)^3}{\tilde{T}^6}\Big)e^{-T_a/\tilde{T}}\\
Z(0) &= RD(\tilde{T}^4-T_0^4)\\
Z'(0) &= 4RD(u_1)\tilde{T}^3\\
Z''(0) &= 12RD(u_1)^2\tilde{T}^2\\
Z'''(0) &= 24RD(u_1)^3\tilde{T}.
\end{aligned}
$$

Using these expressions in 11 and 12 and using the fact that $\tilde{T}$, $\tilde{Y_O}$ and $\tilde{Y_f}$ are exact steady state solutions, equations 8 - 10 can be written as

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= Lu+f(u)\\
&= Lu+N_2(u,u)+N_3(u,u,u)+O(u^4),
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
u(x,t) &= \begin{pmatrix} u_1(x,t)\\ u_2(x,t)\\ u_3(x,t)\end{pmatrix},\\
N_2(u,u) &= \frac{1}{2}\begin{pmatrix} W''(0)-Z''(0)\\ -\frac{1}{Le}W''(0)\\ -\frac{1}{Le}W''(0)\end{pmatrix},
\end{aligned}
$$

$$
N_3(u,u,u) = \frac{1}{6}\begin{pmatrix} W'''(0) - Z'''(0) \\ -\frac{1}{Le}W'''(0) \\ -\frac{1}{Le}W'''(0) \end{pmatrix},
$$

and the linear operator L is given by

$$
\begin{pmatrix}
\frac{\partial^2}{\partial x^2} + D\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} - 4RD\tilde{T}^3 & D\tilde{Y}_f e^{-Ta/\tilde{T}} & D\tilde{Y}_o e^{-Ta/\tilde{T}} \\
-\frac{D}{Le}\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} & \frac{1}{Le}\frac{\partial^2}{\partial x^2} - \frac{D}{Le}\tilde{Y}_f e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_o e^{-Ta/\tilde{T}} \\
-\frac{D}{Le}\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_f e^{-Ta/\tilde{T}} & \frac{1}{Le}\frac{\partial^2}{\partial x^2} - \frac{D}{Le}\tilde{Y}_o e^{-Ta/\tilde{T}}
\end{pmatrix}.
$$

In addition, since the steady state solution satisfies the boundary conditions, the perturbation must satisfy the zero boundary conditions,

$$
u_1 = u_2 = u_3 = 0 \qquad \text{at} \qquad x = \pm 1. \tag{14}
$$

To establish the properties of L needed for out analysis, define the operators

$$
L_0 = \begin{pmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{1}{Le}\frac{\partial^2}{\partial x^2} \\ \frac{1}{Le}\frac{\partial^2}{\partial x^2} \end{pmatrix}
$$

and

$$
L_1 = \begin{pmatrix}
D\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} - 4RD\tilde{T}^3 & D\tilde{Y}_f e^{-Ta/\tilde{T}} & D\tilde{Y}_o e^{-Ta/\tilde{T}} \\
-\frac{D}{Le}\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_f e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_o e^{-Ta/\tilde{T}} \\
-\frac{D}{Le}\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_f e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_o e^{-Ta/\tilde{T}}
\end{pmatrix}.
$$

Then $L = L_0 + L_1$ and L_1 is a bounded operator. Let the domain of L be $D(L) = \{u' \epsilon AC[-1,1] : u'' \epsilon L^2[-1,1], u(-1) = u(1) = 0\}^3$, a subset of the Hilbert space $L^2[-1,1]^3$.

Before examining L, we will first analyze another linear operator. Define S to be

$$Su = u''$$

with the domain of S, $D(S) = \{u' \epsilon AC[-1,1] : u'' \epsilon L^2[-1,1], u(-1) = u(1) = 0\}$. It is shown in [9, page 81] that $-S$ is the operator associated with a sectorial sesquilinear form. In addition, S is self-adjoint and has compact resolvent [9, page 83]. It is also easy to see that for $u \epsilon L^2[-1,1]$,

$$S^{-1}u = \frac{x-1}{2}\int_{-1}^{x}(y+1)u(y)dy + \frac{x+1}{2}\int_{x}^{1}(y-1)u(y)dy$$

and the point spectrum of S is given by

$$\sigma_P(S) = \{\frac{-\pi^2 k^2}{4} : k = 1, 2, 3, \ldots\}.$$

Note that

$$L_0 = \begin{pmatrix} S & 0 & 0 \\ 0 & \frac{1}{Le}S & 0 \\ 0 & 0 & \frac{1}{Le}S \end{pmatrix}$$

and $D(L_0) = D(S)^3$. Clearly,

$$L_0^{-1} = \begin{pmatrix} S^{-1} & 0 & 0 \\ 0 & LeS^{-1} & 0 \\ 0 & 0 & LeS^{-1} \end{pmatrix}.$$

The compactness of S^{-1} implies the compactness of L_0^{-1} and hence L_0 has compact resolvent. Therefore, according to [9, page 38], the spectrum of L_0 is equal to its point spectrum and

$$\sigma(L_0) = \{\frac{-\pi^2 k^2}{4}, \frac{-\pi^2 k^2}{4Le} : k = 1, 2, 3, \ldots\}.$$

It is easy to see that L_0 is symmetric. Thus, L_0 is a symmetric, linear operator in the Hilbert space $L^2[-1,1]^3$ and zero, a real number, is in the resolvent set of L_0. Therefore, according to [9, page 71], L_0 is self-adjoint.

Since L_0 is self-adjoint, [9, page 73] states that $||(L_0 - \lambda)^{-1}|| = dist(\lambda, \sigma(L_0))^{-1}$ for all λ in the resolvent set of L_0. Hence, since L_1 is bounded, we see that there is a λ_0 in the resolvent set of L_0 such that $||L_1(L_0 - \lambda_0)^{-1}|| < 1$. Therefore, according to [9, page 25], $(1 + L_1(L_0 - \lambda_0)^{-1})^{-1}$ exists. It is also easy to see that $(L - \lambda_0)^{-1} = (L_0 - \lambda_0)^{-1}(1 + L_1(L_0 - \lambda_0)^{-1})^{-1}$. Since both $(L_0 - \lambda_0)^{-1}$ and $(1 + L_1(L_0 - \lambda_0)^{-1})^{-1}$ exist, λ_0 is in the resolvent set of L. Also, the compactness of $(L_0 - \lambda_0)^{-1}$ implies that $(L - \lambda_0)^{-1}$ is compact [9, page 31]. Thus, L has compact resolvent.

Since $-L_0$ is self-adjoint and hence $||(-L_0 - \lambda)^{-1}|| = dist(\lambda, \sigma(-L_0))^{-1}$ for all

λ in the resolvent set of $-L_0$, it is easy to see that, by definition, $-L_0$ is sectorial. Therefore, since $-L_0$ is sectorial, $-L_1$ is bounded and linear, and $D(-L_0)$ is a subset of $D(-B)$, [9, page 194] states that $-L$ is sectorial. We therefore may apply what we know about semilinear parabolic equations to equations of the form

$$\frac{\partial u}{\partial t} = Lu + f(t, u).$$

5 Steady State Solutions

We find steady state solutions of 1 - 6 numerically by using Mathematica's NDSolve. We transform 1 - 6 into an initial value problem by specifying $T'(-1)$, $Y_O'(-1)$, and $Y_f'(-1)$ instead of $T(1)$, $Y_O(1)$, and $Y_f(1)$. This is done in one of two ways. For a fixed value of the Damköhler number, D, we use Newton's Method to find $T'(-1)$, $Y_O'(-1)$, and $Y_f'(-1)$ such that $T(1) = T_0$, $Y_O(1) = 1$, and $Y_f(1) = 0$. Alternatively, for a fixed value of $T'(-1)$, we use Newton's Method to find D, $Y_O'(-1)$, and $Y_f'(-1)$ such that $T(1) = T_0$, $Y_O(1) = 1$, and $Y_f(1) = 0$. After finding a steady state solution, we continue to use this shooting method to find others. This can be done fast if a suitable continuation method is employed. We use either continuation in D or $T'(-1)$ and we must switch, often several times, to draw a curve of steady state solutions. When radiation is neglected ($R = 0$), plotting the maximum temperature (T_{max}) versus D gives the classical S-curve, see Figure 2. Including a convection term, as in

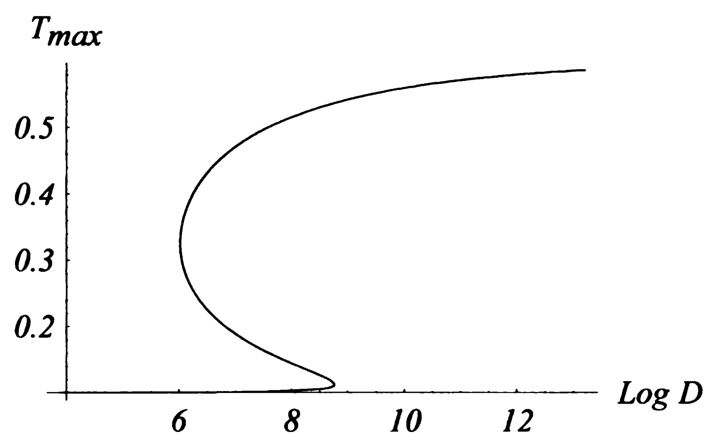

Figure 2: S-curve ($R = 0$) of steady states when $Le = 1$ and $T_a = 1.2$

Vance et al. [14], also yields this S-curve. When radiating heat loss is included, this

S-curve transforms. As R increases from 0, the back of the S-curve begins to push down. For small activation temperatures ($T_a \leq 1$), the back of the S-curve pushes down and eventually flattens down into the ignition branch, see Figure 3.

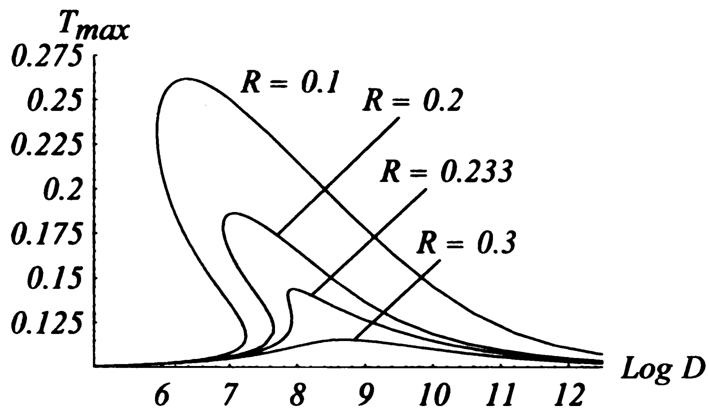

Figure 3: Transformation of S-curve when $Le = 1$ and $T_a = 1$

For large activation temperatures ($T_a > 1$), the transformation of the S-curve is more complicated. When R is small, the transformation of the S-curve is similar to the transformation seen for small activation temperatures. The curve looks like the S-curve with the back of the curve pushed down, see Figure 4. However, as R continues

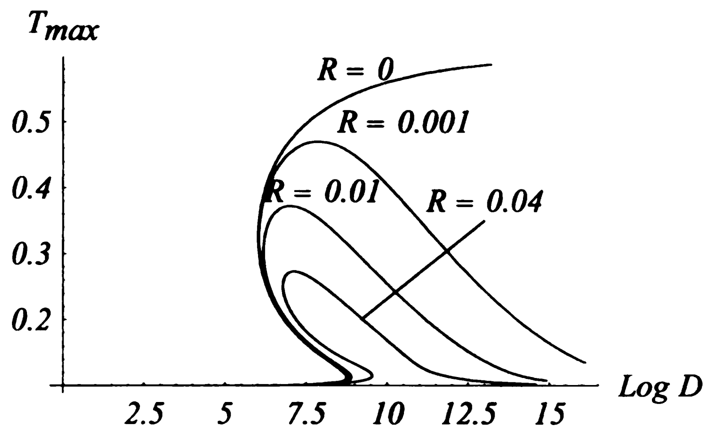

Figure 4: Transformation of S-curve for small R when $Le = 1$ and $T_a = 1.2$

to increase, the plot of the steady states looks quite different than those seen for small activation temperatures. Sohn et al. [12] showed that with certain radiation values, this plot gives an island curve, or isola. We see this behavior as well. As radiation increases and the back of the S-curve pushes down, the curve eventually breaks into an island and an ignition branch, see Figure 5. As R continues to increase, the island

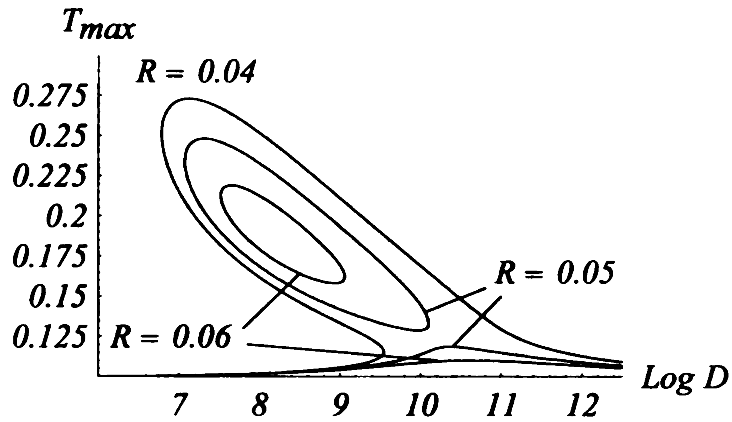

Figure 5: Formation and shrinking of islands when $Le = 1$ and $T_a = 1.2$

shrinks and eventually disappears leaving only the ignition branch, see Figure 6. With further increases in R, the ignition branch remains and approaches closer and closer to T_0. This transformation occurs when $T_a > 1$. The R values at which an island first appears and then disappears are summarized in Table 1 for various activation temperatures.

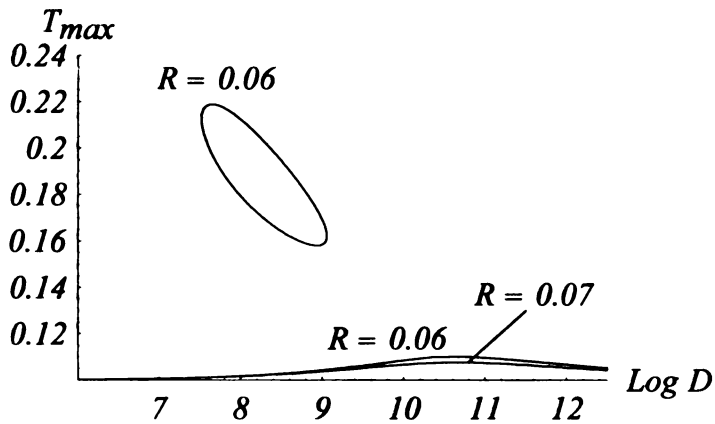

Figure 6: Disappearance of islands when $Le = 1$ and $T_a = 1.2$

T_a	R_{Appear}	$R_{Disappear}$
1.1	0.1111	0.117
1.2	0.05	0.07
2	3.9×10^{-5}	1.9×10^{-3}
3	3.1×10^{-9}	7×10^{-5}
5	1.2×10^{-17}	3.1×10^{-7}
6	6.7×10^{-22}	2.9×10^{-8}

Table 1: T_a vs. values of R where the islands appear and disappear

6 Stability of the Steady State Solutions

In this section we analyze the stability of the steady state solutions. A steady solution is called stable if all solutions of 1 - 6 that start as small perturbations of the steady state solution decay to the steady state. The steady state is unstable if there exists a number $r > 0$ such that for every number $\epsilon > 0$ one can find a perturbation of the steady solution which is initially closer to the steady solution than ϵ yet eventually differs from it by more than r.

We saw in Section 4 that every perturbation, u, of a steady state solution satisfies the differential equation

$$\frac{\partial u}{\partial t} = Lu + f(u),$$

where $-L$ is sectorial. Hence, according to [9, page 265], the stability of a steady state solution is determined by the spectrum of L. We also saw in Section 4 that L has compact resolvent. Therefore, by [9, page 38], the eigenvalues of L have no finite accumulation point and the spectrum of L is equal to its point spectrum. Hence, the steady state solution is stable if all the eigenvalues of L have negative real parts. On the other hand, if L has an eigenvalue with positive real part, the steady state solution is unstable. Therefore, to determine the stability of the steady states, we first solve the eigenvalue problem

$$
\begin{aligned}
Lu &= \sigma u \\
u(\pm 1, t) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

To solve this eigenvalue problem, we discretized the equations using a second-order central difference scheme and Mathematica solved the resulting matrix eigenvalue problem. Increasing mesh size allowed for the discretization error to be kept small.

On the graphs we will use thick curves to denote stable steady states and thin curves to denote unstable steady states. When the leading eigenvalue is complex, the curve will be dashed. For example, in Figure 8, the leading eigenvalue is real and positive on the middle branch. At point 1, the leading eigenvalue changes to a pair of complex values with positive real part. At point 2, the real part of the leading eigenvalue pair changes from positive to negative and the leading eigenvalue becomes real and negative at point 3. The stability of the S-curve varies little with varying parameter values. The lower branch seems to be always stable and the middle branch is always unstable. However, Vance et al. [14] showed that the stability of the beginning of the upper branch depends on the Lewis number. There exists an $Le_{crit} > 1$ such that if $Le < Le_{crit}$ then the entire upper branch consists of stable steady state solutions, see Figure 7. When $Le > Le_{crit}$ the beginning of the upper branch is unstable, see Figure 8. We see in Figure 9 how the interactions of the three leading eigenvalues determine the stability of the S-curve when $Le > Le_{crit}$. At point 1, the two positive leading eigenvalues join and form a complex conjugate pair

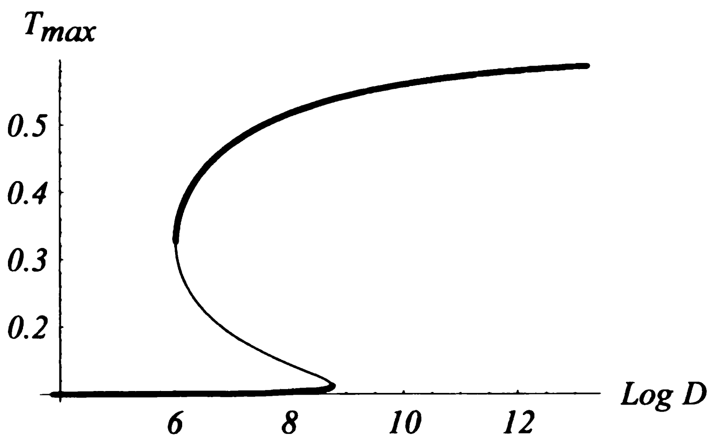

Figure 7: Stability of the S-curve when $Le = 1 < Le_{crit}$

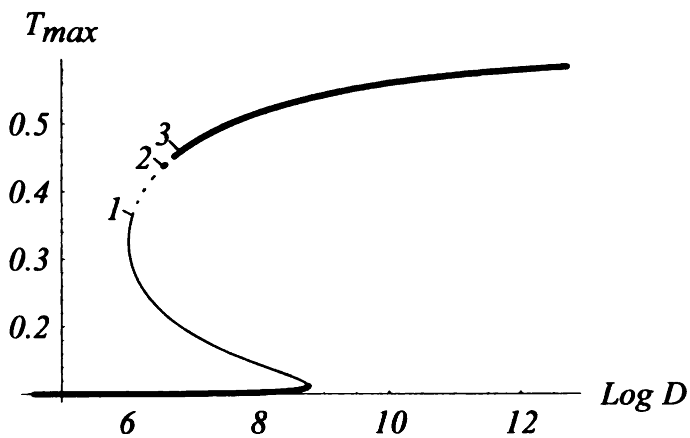

Figure 8: Stability of the S-curve when $Le = 5 > Le_{crit}$

of leading eigenvalues with positive real part. At point 2, the real part of this pair becomes negative, implying the stability of the steady state solution. At point 3, the real part of the complex conjugate pair of eigenvalues decreases below the negative, constant eigenvalue.

To examine the effect of R on the stability, we set $Le = 1$. When R is nonzero, the lower and middle branches seem to remain stable and unstable, respectively, however the stability of the upper branch is affected by the value of R. Since $1 < Le_{crit}$,

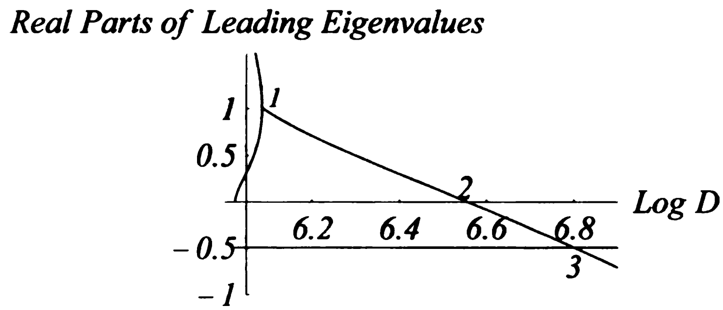

Figure 9: Real parts of the three leading eigenvalues at the beginning of the upper branch when $Le = 5$

the entire upper branch is stable when $R = 0$. When R is nonzero, the beginning of the upper branch is sometimes unstable, like the $R = 0$, $Le > Le_{crit}$ case. We will discuss when the beginning of the upper branch is unstable only when the behavior here proves to be interesting when examining the Hopf bifurcation points. The more interesting result of radiation is the appearance and disappearance of an interval of unstable steady solutions on the back of the S-curve, which we discuss in detail.

When $T_a < .95$, the upper branch remains stable for all radiation values. As R increases and the S-curve flattens out, the unstable middle branch shrinks and eventually disappears, creating a completely stable curve. But, when $T_a \geq .95$, we see the effect of radiation on the stability of the upper branch. When R is small and the back of the S-curve begins to push down, the stability is similar to the stability of the S-curve. Then, at a certain R value, an interval of unstable steady solutions appears on the back of the transformed S-curve. This interval appears for $T_a \geq .95$, however the curves depend on the T_a values. Let us first consider when $.95 \leq T_a \leq 1$.

In this case, we also see that the beginning of the upper branch becomes unstable at a certain R value. We saw in Section 5 that the curve flattens as R increases. As the curve flattens, the regions of unstable steady solutions shrink and eventually the entire curve becomes stable. An example of this stability behavior is seen in Figure 10 for $T_a = 1$.

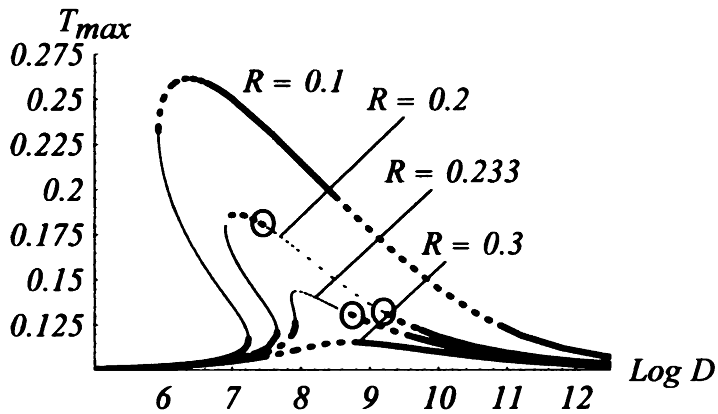

Figure 10: Stability of the steady states when $Le = 1$ and $T_a = 1$

When $T_a > 1$, the interval of unstable steady solutions appears before the transformed S-curve splits into the island and ignition branch, see Figure 11. For example, when $T_a = 1.2$, the interval of unstable steady states first appears when R is approximately 0.0064 and the island does not form until approximately $R = 0.05$. The interval persists on the island after the split, see Figure 12. As R increases further, the island becomes a collection of completely unstable steady states and then eventually disappears. The stable lower branch persists. We see this transformation for all $T_a > 1$. The first columns of Table 2 give the R values at which this interval of unstable steady solutions appears and disappears for varying T_a values.

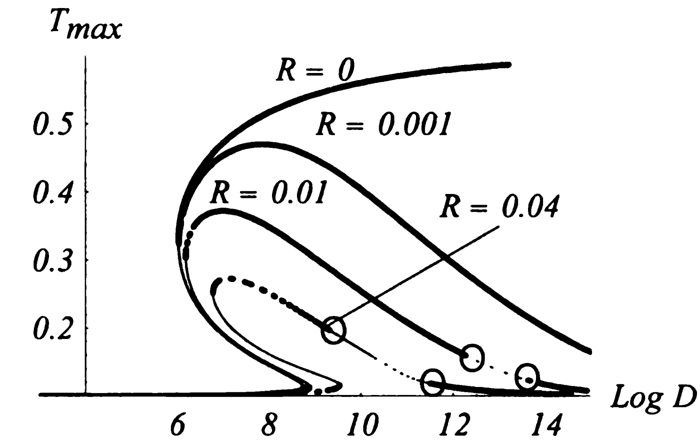

Figure 11: Stability of the steady states when $Le = 1$, $T_a = 1.2$, and R is low

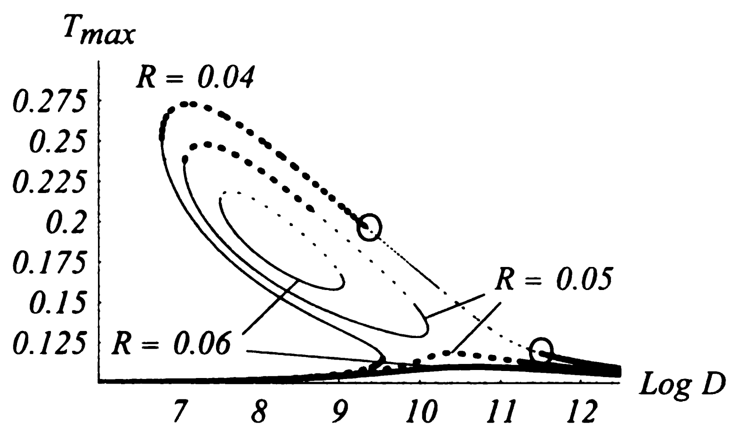

Figure 12: Stability of the steady states when $Le = 1$, $T_a = 1.2$, and R is high

T_a	R_{Appear}	$R_{Disappear}$	$R_{LeftSubcritical}$	$R_{RightSubcritical}$
.95	.28	.34	—	—
.97	.19	.30	—	—
1	.12	.25	.202	—
1.1	.026	.12	.096	.11
1.2	.0064	.058	.048	.034

Table 2: T_a vs. values of R where the interval of unstable steady solutions appears and disappears and values of R at which the bifurcation points at the ends of the interval become subcritical

7 Hopf Bifurcations

A Hopf bifurcation occurs when the leading eigenvalue is complex and its real part changes sign. At a Hopf bifurcation point, according to E. Hopf [5], a branch of time periodic solutions splits from the steady state solutions. These time periodic solutions may be stable or unstable. If the time periodic solutions exist on the side of the bifurcation point where the steady state solutions are stable, then the time periodic solutions are unstable and the bifurcation is called subcritical. If the branch of time periodic solutions exists where the steady state solutions are unstable, then the time periodic solutions are stable and the bifurcation is called supercritical. In this case, slightly perturbing an unstable steady solution near the bifurcation point will lead to a solution which will approach the time periodic solution as t grows. Sohn et al. [12] found stable periodic solutions on the island curves, implying the existence of supercritical Hopf bifurcation points of the system 1 - 6 for certain parameter values. We vary these parameter values to develop a better understanding of when these stable periodic solutions exist.

One way of determining the type of Hopf bifurcation is to make a small initial perturbation of an unstable steady state solution near the bifurcation point and solve 1 - 6 directly using a higher order finite difference scheme. Two different types of flame behaviors may occur on the unstable side of a bifurcation point. Either the perturbation diverges to extinction, see Figure 13, or the solution approaches a periodic solution, see Figure 14. If the solution stabilizes and clearly approaches a periodic solution, as in Figure 14, then the bifurcation is supercritical. However, we

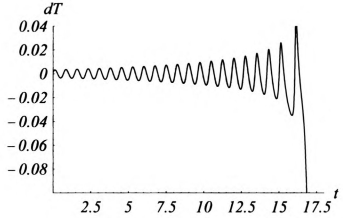

Figure 13: Plot of the difference between the solution of the perturbed problem and steady solution in which the perturbation leads to extinction

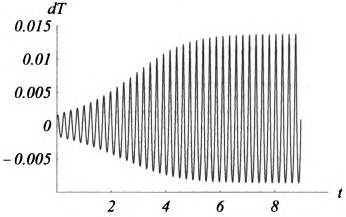

Figure 14: Plot of the difference between the solution of the perturbed problem and steady solution in which the perturbation approaches a stable periodic solution

may also see the solution diverge to extinction when the bifurcation is supercritical. If the perturbation is too large or the unstable steady state solution is too far from the bifurcation point, the solution may not approach an existing stable periodic solution. For example, there is a supercritical Hopf bifurcation point at approximately $D = 1.59 \times 10^7$ when $Le = 1$, $T_a = 3$, and $R = 10^{-5}$. By making a small perturbation near the bifurcation point, say at $D = 1.62 \times 10^7$, we see the solution approach a stable periodic orbit and hence this stable periodic solution exists. However, if we make a larger perturbation at this point or move farther from the bifurcation point, say to $D = 1.622 \times 10^7$, the solution diverges to extinction. On the other hand, if the bifurcation is subcritical, there is no stable periodic solution to approach and every perturbation of every unstable steady state near the bifurcation point will eventually lead to extinction. When looking for stable periodic solutions, it is important to search near the bifurcation point and make small perturbations to prevent extinction when a stable periodic solution exists. Unfortunately, this can be time consuming. If one is too close to the bifurcation point or the perturbation is too small, then it takes too long to determine whether a periodic solution evolves. It may appear as if the solution is periodic when in fact it is just growing very slowly, see Figure 15. For

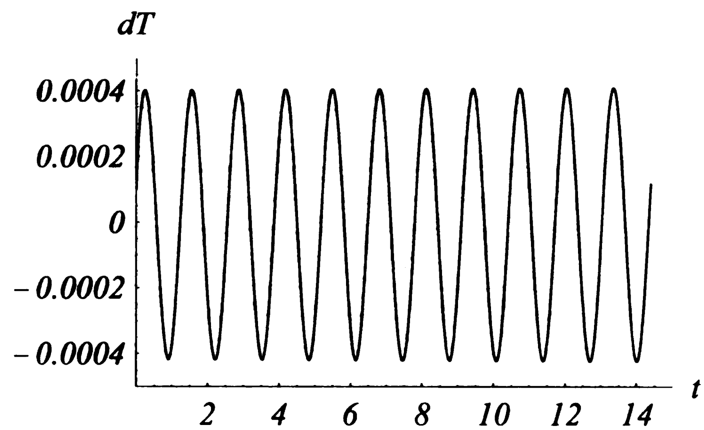

Figure 15: Plot of the difference between the solution of the perturbed problem and steady solution in which the perturbation grows slowly, but does not approach a stable periodic solution

example, there is a subcritical Hopf bifurcation point at approximately $D = 703.64$ when $Le = 5$, $T_a = 1.2$, and $R = 0$. An initial perturbation of size 5×10^{-5} leads to a growth rate of only about 1% in amplitude after the first cycle. A first noticeable increase in the growth rate in this case happens if we start with a 300 times larger initial perturbation (or continue for about 5000 periods), and it leads to extinction after about an 800 times larger initial perturbation. These small growth rates near the bifurcation point make it difficult to determine the type of Hopf bifurcation. One way of verifying the behavior of the solution is to examine a few oscillations and use the frequency and growth rate to compute the apparent eigenvalue. If the solution seems to be behaving exactly as the eigenvalues predict, then the solution will likely diverge to extinction. However, if the solution seems to grow more slowly than expected, then the nonlinear terms may be affecting the solution enough to stabilize it. Such analyses aid in finding possible supercritical Hopf bifurcation points.

Our findings of oscillations persisting for hundreds or thousands of periods contrast sharply with statements made by Sohn et al. in [12] and in the previous nonlinear analysis (Sohn et al. in [11]), concluding that the oscillations should be terminated after a few cycles. However, it is unlikely that an actual experiment would produce these kinds of oscillations which would last virtually unchanged for hundreds of cycles. The key reason for being able to produce graphs like Figure 15 is the ability to choose D very close to the bifurcation value and make small perturbations of the unstable steady state. For example, in the situation described earlier when $Le = 5$, $T_a = 1.2$, and $R = 0$, if the Damköhler number is dropped to $D = 700$ from $D = 703.5$, the growth rate jumps from 0.0004 to 0.0100, i.e. a 0.5% drop in the Damköhler number causes a 25 fold increase in growth rate, which shortens the period of persistence of oscillations by roughly a factor of 25.

When $R = 0$ we have the S-curve, as we saw in Section 5. Vance et al. [14]

showed that when $Le < Le_{crit}$, the middle branch is unstable and the upper branch is stable. In addition, although they found flame oscillations when $Le < Le_{crit}$, they found no Hopf bifurcation points. On the other hand, when $Le > Le_{crit}$, Vance and colleagues showed that the beginning of the upper branch is unstable and there is a Hopf bifurcation point where the stability changes on the upper branch. However, in this case, all perturbations of unstable steady solutions near Hopf bifurcation points led to extinction. We made many calculations for many different values of physical parameters, yet we always found the bifurcation to be subcritical when $R = 0$. When $Le = 1$ and R is nonzero, we saw in Section 6 that there are sometimes Hopf bifurcation points at the beginning of the upper branch. However, we found almost all Hopf bifurcation points here to be subcritical. There is a very small range of parameter values for which the Hopf bifurcation points here are supercritical. This will be discussed below.

As we did in Section 6, we will set $Le = 1$ to discuss the effect of R on the bifurcation points. For small R, the stability analysis is similar to the stability of the S-curve and therefore no supercritical Hopf bifurcation points were found. As R increases, though, and the interval of unstable steady solutions on the back of the S-curve appears, supercritical Hopf bifurcations appear at one or both ends of the interval, see Figures 10, 11, and 12. Supercritical Hopf bifurcation points are circled in the figures. Subcritical Hopf bifurcation points are not circled. We discuss the Hopf bifurcations for various values of T_a below.

When $T_a < .95$, we saw no Hopf bifurcation points. When $.95 \leq T_a \leq 1$, there are three Hopf bifurcation points which appear. There is a Hopf bifurcation point at the beginning of the upper branch and one at each end of the interval of unstable steady solutions on the back of the transformed S-curve. When these points first appear, the Hopf bifurcation point at the beginning of the upper branch is subcritical and the Hopf bifurcation points at the ends of the interval of unstable steady states are supercritical. The Hopf bifurcation point at the right endpoint of the interval of unstable steady states remains supercritical as long as it appears for all $.95 \leq T_a \leq 1$. For $.95 \leq T_a < .98$, the Hopf bifurcation point at the left endpoint of the interval of unstable steady states remains supercritical as long as it appears. In this case, the Hopf bifurcation point at the beginning of the upper branch becomes supercritical right before it disappears. This is the only set of parameter values for which we see a supercritical Hopf bifurcation point in this location and the bifurcation point is supercritical for only a very small range of R. For example, when $T_a = .97$, this bifurcation point is still subcritical at $R = .255$, however it is supercritical when it disappears by $R = .256$. When $.98 \leq T_a \leq 1$, the Hopf bifurcation at the beginning of the upper branch is subcritical as long as it appears. The left endpoint of the interval of unstable steady states remains supercritical until right before it disappears when it changes to subcritical. This bifurcation point is subcritical for only a small range of R. For example, when $T_a = 1$, it is still supercritical at $R = .201$, however it is subcritical when it disappears at around $R = .204$.

When $T_a > 1$, the bifurcations are supercritical when they first appear. However, as R increases, the Hopf bifurcations first become subcritical and then they disappear. Also when $T_a > 1$, at a certain R value, the S-curve splits into an island and an ignition branch. Whether the bifurcations become subcritical before or after the split depends on the T_a value. For lower T_a values ($T_a \leq 1.2$), the bifurcation points change from supercritical to subcritical before the curve splits into an island and ignition branch. For example, when $T_a = 1.2$ and $R = 0.01$, supercritical Hopf bifurcations occur at $D = 268973$ and $D = 843180$ on the back of the S-curve. However, when the island forms at approximately $R = 0.05$, these bifurcation points are subcritical. The bifurcation points disappear altogether by $R = 0.06$. At higher values of T_a ($T_a > 1.2$), the bifurcation is supercritical when the island forms. We saw in Table 1 that when $Le = 1$ and $T_a = 3$, an island forms at approximately $R = 3.1 \times 10^{-9}$ and there are no longer islands by $R = 7 \times 10^{-5}$. In Figures 16 and 17, we see that there are stable periodic solutions on the islands when $R = 10^{-7}$ and $R = 10^{-5}$. However, when $R = 5 \times 10^{-5}$, although there is still a Hopf bifurcation point on the island, it is subcritical. When $R = 6.4 \times 10^{-5}$, the entire island is unstable and hence there are no Hopf bifurcation points on the island. We will further investigate the R value at which the bifurcation changes from supercritical to subcritical later. The existence of stable oscillations on islands was first observed by Sohn et al. [12]. Sohn and colleagues always found stable oscillations on the islands since they considered only $T_a = 5$, a relatively large value of T_a.

Slightly perturbing an unstable steady solution near a supercritical Hopf bifurcation point leads to stable oscillations. When far from the bifurcation point, perturbations may not lead to stable oscillations. How near the unstable steady state must be to the bifurcation point in order for a small perturbation to lead to stable oscillations depends on the bifurcation point. For smaller T_a values, perturbing any of the unstable steady states in between two supercritical Hopf bifurcation points on

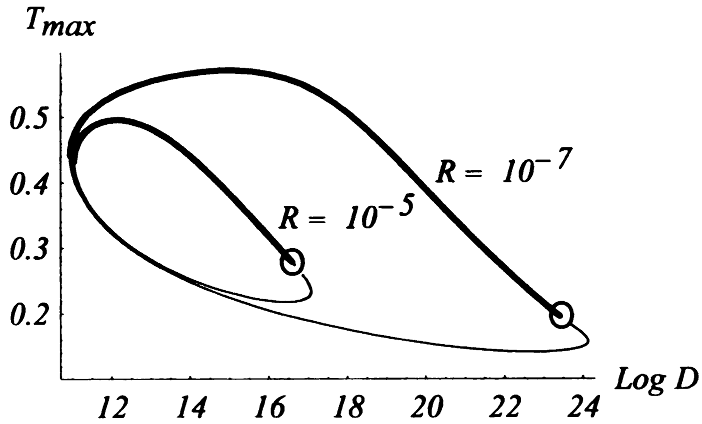

Figure 16: Stability of the islands when $Le = 1$, $T_a = 3$, and R is low

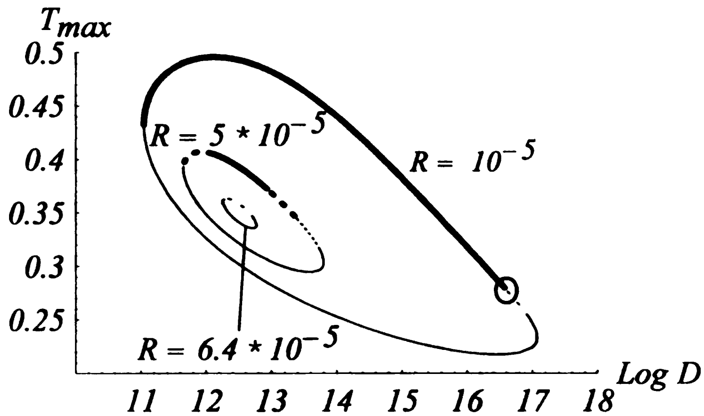

Figure 17: Stability of the islands when $Le = 1$, $T_a = 3$, and R is high

the back of the curve leads to stable oscillations. For larger T_a values, perturbing the unstable steady states near the bifurcation points leads to stable oscillations; but perturbing an unstable steady state in the middle of this interval leads to extinction. In this case, the size of the interval in which perturbations approach stable periodic solutions varies. We showed that when $T_a > 1$, the Hopf bifurcation points on the back of the S-curve (or island) change from supercritical to subcritical as R increases. We see that as this R approaches the point at which the type of Hopf bifurcation changes, the size of the interval in which small perturbations lead to stable oscillations approaches zero. We can examine these perturbation interval sizes to approximate the points at which the bifurcation points change from supercritical to subcritical. For example, when $T_a = 3$, $Le = 1$, and $R = 1 \times 10^{-5}$, there is a supercritical Hopf bifurcation point when $D = 1.59 \times 10^7$. Perturbing steady solutions between this point and $D = 1.62 \times 10^7$ leads to stable oscillations, but perturbing steady states at larger values of D leads to extinction. Therefore, the perturbation interval size when $R = 1 \times 10^{-5}$ is approximately 3.0×10^5. However, when R increases to 2.1×10^{-5} the size of this interval decreases to around 3.3×10^3. We estimated the perturbation interval sizes at several R values between 1×10^{-5} and 2.1×10^{-5} and used those values to approximate a function, $T(R)$, which gives the perturbation interval size at R. We first expressed this function as its Taylor series centered at R_0, the point at which the bifurcation changes from supercritical to subcritical. We then used the perturbation interval sizes computed at the largest four values of R : $1.6 \times 10^{-5}, 1.8 \times 10^{-5}, 2 \times 10^{-5}$, and 2.1×10^{-5} to estimate the third order Taylor polynomial. Using this approximation of T, we see that the perturbation interval size becomes zero, and hence the bifurcation point becomes subcritical, at approximately $R = 2.3 \times 10^{-5}$. This order three Taylor polynomial approximation of T is plotted in Figure 18, along with the seven perturbation interval sizes computed. The last two columns of Table 2 give the R values at which the two bifurcation points at the ends of the interval of unstable steady states become subcritical for various activation temperatures. However, these values were found by using the method developed in Section 8, instead of by analyzing perturbation interval sizes.

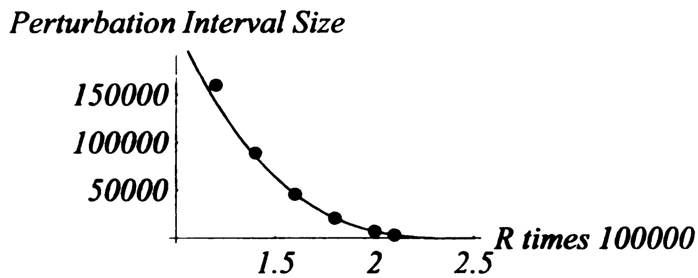

Figure 18: Plot of R vs. the perturbation interval size when $Le = 1$ and $T_a = 3$

8 Analytically Identifying Supercritical Hopf Bifurcation Points of a General System

In the previous section we discussed classifying Hopf bifurcation points as either subcritical or supercritical by perturbing unstable steady solutions near a bifurcation point and examining the behavior of the solution of the perturbed problem. We addressed some of the difficulties in using this method and how supercritical Hopf bifurcation points may appear to be subcritical. In this section, we will develop a method for determining whether a Hopf bifurcation point of a general system is subcritical or supercritical. Instead of examining the apparent behavior of solutions, this method will use the structure of the equations and the eigenvalues to explicitly state whether stable periodic solutions exist. This method was partially developed by Renardy in [10], however, his argument depended on the existence of inverse functions which do not exist at the bifurcation point.

Let H be a Hilbert space of complex, vector-valued function whose inner product has the property that the inner product of real functions is real. Then, for all $u, v \epsilon H$,

$$\begin{aligned}
(\overline{u}, \overline{v}) &= (Re(u) - iIm(u), Re(v) - iIm(v)) \\
&= (Re(u), Re(v)) + (Im(u), Im(v)) + i((Re(u), Im(v)) - (Im(u), Re(v))) \\
&= \overline{(Re(u) + iIm(u), Re(v) + iIm(v))} \\
&= \overline{(u, v)}.
\end{aligned}$$

Consider a general system

$$\dot{u} = L(\lambda)u + f(u;\lambda), \tag{15}$$

where, for each t, $u(t)\epsilon H$, λ is a parameter, L is a linear operator, and f is nonlinear. Assume that $u = 0$ is a steady solution of 15. We will examine real solutions of this differential equation. Let the domain of f be denoted by $D(f)$. Assume $f(u;\lambda)$ and $L(\lambda)u$ are both real when u is real and f can be written as

$$f(u;\lambda) \quad = \quad N_2(u,u;\lambda) + N_3(u,u,u;\lambda) + O(|u|^4),$$

where N_2 and N_3 are the quadratic and cubic parts of f, respectively. Note that since $f(u;\lambda)$ is real when u is real, both $N_2(u,u;\lambda)$ and $N_3(u,u,u;\lambda)$ are real when u is real. Assume that f, N_2, and N_3 have the following properties for all $u_1, u_2, u_3 \epsilon D(f)$:

1. $\overline{f(u_1)} = f(\overline{u_1})$

2. $N_2(u_1,u_2) = N_2(u_2,u_1)$

3. $N_3(u_1,u_2,u_3) = N_3(u_1,u_3,u_2) = N_3(u_2,u_1,u_3)$

4. N_2 and N_3 are linear in each coordinate.

Note that these properties imply that for $u, v, w \epsilon H$,

$$\begin{aligned} N_2(u,v) &= \frac{1}{4}(N_2(u+v,u+v) - N_2(u-v,u-v)) \qquad (16) \\ N_3(u,v,w) &= \frac{1}{24}(N_3(u+v+w,u+v+w,u+v+w) \end{aligned}$$

$$+N_3(u-v-w,u-v-w,u-v-w)$$

$$+N_3(-u+v-w,-u+v-w,-u+v-w)$$

$$+N_3(-u-v+w,-u-v+w,-u-v+w)). \tag{17}$$

Hence, we see that $N_2(u,v)$ and $N_3(u,v,w)$ are real if u, v, and w are real. This fact, along with the linearity of N_2 and N_3, imply that for $u, v, w \epsilon H$,

$$\begin{aligned}
N_2(\overline{u},\overline{v}) &= N_2(Re(u) - iIm(u), Re(v) - iIm(v)) \\
&= N_2(Re(u), Re(v)) - N_2(Im(u), Im(v)) \\
&\quad -i(N_2(Im(u), Re(v)) + N_2(Re(u), Im(v))) \\
&= \overline{N_2(u,v)}
\end{aligned}$$

and

$$\begin{aligned}
N_3(\overline{u},\overline{v},\overline{w}) &= N_3(Re(u) - iIm(u), Re(v) - iIm(v), Re(w) - iIm(w)) \\
&= N_3(Re(u), Re(v), Re(w)) - N_3(Im(u), Im(v), Re(w)) \\
&\quad -N_3(Im(u), Re(v), Im(w)) - N_3(Re(u), Im(v), Im(w)) \\
&\quad +i(N_3(Im(u), Im(v), Im(w)) - N_3(Im(u), Re(v), Re(w)) \\
&\quad -N_3(Re(u), Im(v), Re(w)) - N_3(Re(u), Re(v), Im(w))) \\
&= \overline{N_3(u,v,w)}.
\end{aligned}$$

Now, let us examine 15 near a Hopf bifurcation point. Assume that the system has a Hopf bifurcation point at $\lambda = 0$. That is, assume $L(\lambda)$ has a complex conjugate pair of leading eigenvalues $\sigma(\lambda)$ and $\overline{\sigma(\lambda)}$,

$$\begin{aligned} Re(\sigma(0)) &= 0, \\ Im(\sigma(0)) &\neq 0, \\ \frac{d(Re\sigma(\lambda))}{d\lambda}(0) &\neq 0, \end{aligned}$$

and the rest of the spectrum of $L(\lambda)$ is in the left half plane. Let ωi, where $\omega > 0$ denote $\sigma(0)$. Therefore, $\pm\omega i$ is the complex conjugate pair of leading eigenvalues of $L(0)$ at the bifurcation point. Let $a(\lambda)$ and $a^*(\lambda)$ denote the eigenvector and adjoint eigenvector, respectively, corresponding to $\sigma(\lambda)$. So,

$$L(\lambda)a(\lambda) = \sigma(\lambda)a(\lambda) \qquad \text{and} \qquad L^*(\lambda)a^*(\lambda) = \overline{\sigma(\lambda)}a^*(\lambda).$$

Without loss, assume $(a(\lambda), a^*(\lambda)) = 1$. Also, note that

$$\begin{aligned} (a^*(\lambda), \overline{a(\lambda)}) &= (a^*(\lambda), \frac{\overline{\sigma(\lambda)} - \sigma(\lambda)}{\overline{\sigma(\lambda)} - \sigma(\lambda)}\overline{a(\lambda)}) \\ &= (a^*(\lambda), \frac{L(\lambda) - \sigma(\lambda)}{\overline{\sigma(\lambda)} - \sigma(\lambda)}\overline{a(\lambda)}) \\ &= \frac{1}{\sigma(\lambda) - \overline{\overline{\sigma(\lambda)}}}\left[(L^*(\lambda)a^*(\lambda), \overline{a(\lambda)}) - (\overline{\sigma(\lambda)}a^*(\lambda), \overline{a(\lambda)})\right] \\ &= \frac{1}{\sigma(\lambda) - \overline{\overline{\sigma(\lambda)}}}\left[(\overline{\sigma(\lambda)}a^*(\lambda), \overline{a(\lambda)}) - (\overline{\sigma(\lambda)}a^*(\lambda), \overline{a(\lambda)})\right] \\ &= 0. \end{aligned} \tag{18}$$

Consider the projection $P(\lambda)$ given by

$$P(\lambda)u = u - (u,a^*(\lambda))a(\lambda) - (u,\overline{a^*(\lambda)})\overline{a(\lambda)}$$

with the domain of P equal to H. The normalization of the eigenvector and adjoint eigenvector and 18 imply that

$$P(\lambda)a(\lambda) = P(\lambda)\overline{a(\lambda)} = 0. \tag{19}$$

Hence,

$$\begin{aligned} P(\lambda)^2 u &= P(\lambda)(u - (u,a^*(\lambda))a(\lambda) - (u,\overline{a^*(\lambda)})\overline{a(\lambda)}) \\ &= P(\lambda)u - (u,a^*(\lambda))P(\lambda)a(\lambda) - (u,\overline{a^*(\lambda)})P(\lambda)\overline{a(\lambda)} \\ &= P(\lambda)u \end{aligned}$$

and $P(\lambda)$ is indeed a projection. We will now find $P^*(\lambda)$.

Claim 8.1 $P^*(\lambda)v = v - (v,a(\lambda))a^*(\lambda) - (v,\overline{a(\lambda)})\overline{a^*(\lambda)}$.

Proof 8.1 *If* $u, v \epsilon H$, *then*

$$\begin{aligned} (P(\lambda)u,v) &= (u - (u,a^*(\lambda))a(\lambda) - (u,\overline{a^*(\lambda)})\overline{a(\lambda)}, v) \\ &= (u,v) - (u,a^*(\lambda))(a(\lambda),v) - (u,\overline{a^*(\lambda)})(\overline{a(\lambda)},v) \\ &= (u,v) - (u,\overline{(a(\lambda),v)}a^*(\lambda)) - (u,\overline{(\overline{a(\lambda)},v)}\,\overline{a^*(\lambda)}) \\ &= (u, v - \overline{(a(\lambda),v)}a^*(\lambda) - \overline{(\overline{a(\lambda)},v)}\,\overline{a^*(\lambda)}). \end{aligned}$$

Later we will need to know the null space of $P^*(\lambda)$.

Claim 8.2 *The null space of $P^*(\lambda)$ is Span($a^*(\lambda), \overline{a^*(\lambda)}$).*

Proof 8.2 *We will first show that $P^*(\lambda)a^*(\lambda) = P^*(\lambda)\overline{a^*(\lambda)} = 0$.*

$$\begin{aligned}
P^*(\lambda)a^*(\lambda) &= a^*(\lambda) - (a^*(\lambda), a(\lambda))a^*(\lambda) - (a^*(\lambda), \overline{a(\lambda)})\overline{a^*(\lambda)} \\
&= a^*(\lambda) - 1 * a^*(\lambda) - 0 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
P^*(\lambda)\overline{a^*(\lambda)} &= \overline{a^*(\lambda)} - (\overline{a^*(\lambda)}, a(\lambda))a^*(\lambda) - (\overline{a^*(\lambda)}, \overline{a(\lambda)})\overline{a^*(\lambda)} \\
&= \overline{a^*(\lambda)} - 0 - 1 * \overline{a^*(\lambda)} \\
&= 0.
\end{aligned}$$

So, Span($a^*(\lambda), \overline{a^*(\lambda)}$) is a subset of the null space of $P^*(\lambda)$. Now, let v be in the null space of $P^*(\lambda)$. Then

$$\begin{aligned}
P^*(\lambda)v &= 0 \\
v - (v, a(\lambda))a^*(\lambda) - (v, \overline{a(\lambda)})\overline{a^*(\lambda)} &= 0 \\
v &= (v, a(\lambda))a^*(\lambda) + (v, \overline{a(\lambda)})\overline{a^*(\lambda)} \\
v &= c_1 a^*(\lambda) + c_2 \overline{a^*(\lambda)}
\end{aligned}$$

for constants c_1 and c_2. So, the null space of $P^*(\lambda)$ is a subset of Span($a^*(\lambda), \overline{a^*(\lambda)}$)

and the null space of $P^*(\lambda)$ is equal to $Span(a^*(\lambda), \overline{a^*(\lambda)})$.

Throughout the rest of this section, assume that u is a real solution of 15. Now, define

$$z = (u, a^*(\lambda)) \qquad \text{and} \qquad y = P(\lambda)u.$$

Then z is a scalar and

$$\begin{aligned}
za(\lambda) + \overline{za(\lambda)} + y &= (u, a^*(\lambda))a(\lambda) + (u, \overline{a^*(\lambda)})\overline{a(\lambda)} + u \\
&\quad -(u, a^*(\lambda))a(\lambda) - (u, \overline{a^*(\lambda)})\overline{a(\lambda)} \\
&= u.
\end{aligned}$$

So, we may write u as the decomposition

$$u = za(\lambda) + \overline{za(\lambda)} + y \qquad \text{where} \qquad z = (u, a^*(\lambda)) \text{ and } y = P(\lambda)u.$$

Note that since u is real and $za(\lambda) + \overline{za(\lambda)} = 2Re(za(\lambda))$, y is real as well. Now,

$$\begin{aligned}
\dot{z} &= (u, \dot{a^*(\lambda)}) \\
&= (\dot{u}, a^*(\lambda)) \\
&= (L(\lambda)u + f(za(\lambda) + \overline{za(\lambda)} + y; \lambda), a^*(\lambda)) \\
&= (L(\lambda)u, a^*(\lambda)) + (f(za(\lambda) + \overline{za(\lambda)} + y; \lambda), a^*(\lambda)) \\
&= (u, L^*(\lambda)a^*(\lambda)) + (f(za(\lambda) + \overline{za(\lambda)} + y; \lambda), a^*(\lambda)) \\
&= (u, \overline{\sigma(\lambda)}a^*(\lambda)) + (f(za(\lambda) + \overline{za(\lambda)} + y; \lambda), a^*(\lambda)) \\
&= \sigma(\lambda)(u, a^*(\lambda)) + (f(za(\lambda) + \overline{za(\lambda)} + y; \lambda), a^*(\lambda))
\end{aligned}$$

$$
\begin{aligned}
&= \sigma(\lambda)z + (f(za(\lambda) + \overline{za(\lambda)} + y;\lambda), a^*(\lambda)) \\
&= \sigma(\lambda)z + g(z,y;\lambda), \qquad (20)
\end{aligned}
$$

where

$$
g(z,y,\lambda) \;=\; (f(za(\lambda) + \overline{za(\lambda)} + y;\lambda), a^*(\lambda)).
$$

Similarly,

$$
\dot{\overline{z}} \;=\; \overline{\sigma(\lambda)}\overline{z} + \overline{g}(z,y;\lambda), \qquad (21)
$$

where

$$
\overline{g}(z,y,\lambda) \;=\; (f(za(\lambda) + \overline{za(\lambda)} + y;\lambda), \overline{a^*(\lambda)}).
$$

And,

$$
\begin{aligned}
\dot{y} &= P(\lambda)\dot{u} \\
&= P(\lambda)(L(\lambda)u + f(u;\lambda)) \\
&= P(\lambda)L(\lambda)(za(\lambda) + \overline{za(\lambda)} + y) + P(\lambda)f(za(\lambda) + \overline{za(\lambda)} + y;\lambda) \\
&= z\sigma(\lambda)P(\lambda)a(\lambda) + \overline{z\sigma(\lambda)}P(\lambda)\overline{a(\lambda)} + P(\lambda)L(\lambda)y + P(\lambda)f(za(\lambda) + \overline{za(\lambda)} + y;\lambda) \\
&= P(\lambda)L(\lambda)y + P(\lambda)f(za(\lambda) + \overline{za(\lambda)} + y;\lambda)
\end{aligned}
$$

by 19. Hence,

$$\dot{y} = B(\lambda)y + h(z, y; \lambda), \tag{22}$$

where

$$B(\lambda) = P(\lambda)L(\lambda)$$

and

$$h(z, y; \lambda) = P(\lambda)f(za(\lambda) + \overline{za(\lambda)} + y; \lambda).$$

Claim 8.3 *$B(\lambda)$ is singular.*

Proof 8.3 *Assume that $B(\lambda)$ has an inverse, $B(\lambda)^{-1}$. Then, noting that $P(\lambda)^2 = P(\lambda)$ since $P(\lambda)$ is a projection,*

$$\begin{aligned}
B(\lambda)B(\lambda)^{-1} &= I \\
P(\lambda)L(\lambda)B(\lambda)^{-1} &= I \\
P(\lambda)^2L(\lambda)B(\lambda)^{-1} &= P(\lambda) \\
P(\lambda)L(\lambda)B(\lambda)^{-1} &= P(\lambda) \\
B(\lambda)B(\lambda)^{-1} &= P(\lambda),
\end{aligned}$$

a contradiction since $P(\lambda)$ is not the identity. Hence, $B(\lambda)$ is singular.

Claim 8.4 *$(B(\lambda)-\alpha)^{-1}$ exists for all α in the resolvent set of $L(\lambda)$, except zero.*

Proof 8.4 *We will denote the resolvent set of $B(\lambda)$ by $\rho(B(\lambda))$ and the resolvent set of $L(\lambda)$ minus zero by $\rho(L(\lambda))-\{0\}$. We will show that $\rho(L(\lambda))-\{0\}$ is a subset of $\rho(B(\lambda))$. Let α be an element of $\rho(L(\lambda))-\{0\}$. Let $g\epsilon H$ and define*

$$h = (L(\lambda)-\alpha)^{-1}P(\lambda)g - \frac{(g,a^*(\lambda))}{\alpha}a(\lambda) - \frac{(g,\overline{a^*(\lambda)})}{\alpha}\overline{a(\lambda)}.$$

Then

$$\begin{aligned}
(B(\lambda)-\alpha)h &= P(\lambda)L(\lambda)(L(\lambda)-\alpha)^{-1}P(\lambda)g - \alpha(L(\lambda)-\alpha)^{-1}P(\lambda)g \\
&\quad -\frac{(g,a^*(\lambda))}{\alpha}P(\lambda)L(\lambda)a(\lambda) + (g,a^*(\lambda))a(\lambda) \\
&\quad -\frac{(g,\overline{a^*(\lambda)})}{\alpha}P(\lambda)L(\lambda)\overline{a(\lambda)} + (g,\overline{a^*(\lambda)})\overline{a(\lambda)} \\
&= P(\lambda)(L(\lambda)-\alpha)(L(\lambda)-\alpha)^{-1}P(\lambda)g + \alpha P(\lambda)(L(\lambda)-\alpha)^{-1}P(\lambda)g \\
&\quad -\alpha(L(\lambda)-\alpha)^{-1}P(\lambda)g - \frac{(g,a^*(\lambda))}{\alpha}\sigma(\lambda)P(\lambda)a(\lambda) \\
&\quad +(g,a^*(\lambda))a(\lambda) - \frac{(g,\overline{a^*(\lambda)})}{\alpha}\overline{\sigma(\lambda)}P(\lambda)\overline{a(\lambda)} \\
&\quad +(g,\overline{a^*(\lambda)})\overline{a(\lambda)} \\
&= P(\lambda)P(\lambda)g + \alpha P(\lambda)(L(\lambda)-\alpha)^{-1}P(\lambda)g - \alpha(L(\lambda)-\alpha)^{-1}P(\lambda)g \\
&\quad +(g,a^*(\lambda))a(\lambda) + (g,\overline{a^*(\lambda)})\overline{a(\lambda)} \\
&= P(\lambda)g - \alpha((L(\lambda)-\alpha)^{-1}P(\lambda)g, a^*(\lambda))a(\lambda) \\
&\quad -\alpha((L(\lambda)-\alpha)^{-1}P(\lambda)g, \overline{a^*(\lambda)})\overline{a(\lambda)} \\
&\quad +(g,a^*(\lambda))a(\lambda) + (g,\overline{a^*(\lambda)})\overline{a(\lambda)} \\
&= P(\lambda)g - \alpha(P(\lambda)g, ((L(\lambda)-\alpha)^{-1})^*a^*(\lambda))a(\lambda)
\end{aligned}$$

$$-\alpha(P(\lambda)g,((L(\lambda)-\alpha)^{-1})^*\overline{a^*(\lambda)})\overline{a(\lambda)}$$

$$+(g,a^*(\lambda))a(\lambda)+(g,\overline{a^*(\lambda)})\overline{a(\lambda)}. \tag{23}$$

According to [9, page 57], $((L(\lambda)-\alpha)^{-1})^* = (L^*(\lambda)-\overline{\alpha})^{-1}$. *In addition, since* $L^*(\lambda)a^*(\lambda)=\overline{\sigma(\lambda)}a^*(\lambda)$,

$$\begin{aligned}(L^*(\lambda)-\overline{\alpha})a^*(\lambda) &= (\overline{\sigma(\lambda)}-\overline{\alpha})a^*(\lambda)\\ \frac{1}{\overline{\sigma(\lambda)}-\overline{\alpha}}a^*(\lambda) &= (L^*(\lambda)-\overline{\alpha})^{-1}a^*(\lambda).\end{aligned}$$

Similarly, $(L^*(\lambda)-\overline{\alpha})^{-1}\overline{a^*(\lambda)} = \frac{1}{\sigma(\lambda)-\overline{\alpha}}\overline{a^*(\lambda)}$. *Hence, by 23,*

$$\begin{aligned}(B(\lambda)-\alpha)h &= P(\lambda)g-\frac{\alpha}{\sigma(\lambda)-\alpha}(P(\lambda)g,a^*(\lambda))a(\lambda)\\ &\quad -\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(P(\lambda)g,\overline{a^*(\lambda)})\overline{a(\lambda)}\\ &\quad +(g,a^*(\lambda))a(\lambda)+(g,\overline{a^*(\lambda)})\overline{a(\lambda)}\\ &= g-(g,a^*(\lambda))a(\lambda)-(g,\overline{a^*(\lambda)})\overline{a(\lambda)}-\frac{\alpha}{\sigma(\lambda)-\alpha}(g-(g,a^*(\lambda))a(\lambda)\\ &\quad -(g,\overline{a^*(\lambda)})\overline{a(\lambda)},a^*(\lambda))a(\lambda)-\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(g-(g,a^*(\lambda))a(\lambda)\\ &\quad -(g,\overline{a^*(\lambda)})\overline{a(\lambda)},\overline{a^*(\lambda)})\overline{a(\lambda)}+(g,a^*(\lambda))a(\lambda)+(g,\overline{a^*(\lambda)})\overline{a(\lambda)}\\ &= g-\frac{\alpha}{\sigma(\lambda)-\alpha}(g,a^*(\lambda))a(\lambda)+\frac{\alpha}{\sigma(\lambda)-\alpha}(g,a^*(\lambda))(a(\lambda),a^*(\lambda))a(\lambda)\\ &\quad +\frac{\alpha}{\sigma(\lambda)-\alpha}(g,\overline{a^*(\lambda)})(\overline{a(\lambda)},a^*(\lambda))a(\lambda)-\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(g,\overline{a^*(\lambda)})\overline{a(\lambda)}\\ &\quad +\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(g,a^*(\lambda))(a(\lambda),\overline{a^*(\lambda)})\overline{a(\lambda)}\\ &\quad +\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(g,\overline{a^*(\lambda)})(\overline{a(\lambda)},\overline{a^*(\lambda)})\overline{a(\lambda)}\\ &= g-\frac{\alpha}{\sigma(\lambda)-\alpha}(g,a^*(\lambda))a(\lambda)+\frac{\alpha}{\sigma(\lambda)-\alpha}(g,a^*(\lambda))a(\lambda)\end{aligned}$$

$$
-\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(g,\overline{a^*(\lambda)})\overline{a(\lambda)}+\frac{\alpha}{\overline{\sigma(\lambda)}-\alpha}(g,\overline{a^*(\lambda)})\overline{a(\lambda)}
$$

$$
= g.
$$

Thus, $\alpha\epsilon\rho(B(\lambda))$ and $\rho(L(\lambda))-\{0\}$ is a subset of $\rho(B(\lambda))$. Therefore, $(B(\lambda)-\alpha)^{-1}$ exists for all α in the resolvent set of $L(\lambda)$, except zero.

According to The Center Manifold Theorem, u approaches an invariant manifold. On the manifold, there are constants $\{d_i(\lambda)\}_{i=1}^4$ such that

$$
\begin{aligned}
y &= d_1(\lambda)z^2+d_2(\lambda)z\overline{z}+\overline{d_1(\lambda)}\overline{z}^2+d_3(\lambda)z^3+d_4(\lambda)z^2\overline{z}+\overline{d_4(\lambda)}z\overline{z}^2 \\
&\quad +\overline{d_3(\lambda)}\overline{z}^3+O(|z|^4).
\end{aligned}
$$

Note that some of the coefficients are conjugates due to the fact that y is real. Now,

$$
\begin{aligned}
\dot{y} &= 2d_1(\lambda)z\dot{z}+d_2(\lambda)\overline{z}\dot{z}+d_2(\lambda)z\dot{\overline{z}}+2\overline{d_1(\lambda)}\overline{z}\dot{\overline{z}}+3d_3(\lambda)z^2\dot{z}+2d_4(\lambda)z\overline{z}\dot{z} \\
&\quad +d_4(\lambda)z^2\dot{\overline{z}}+\overline{d_4(\lambda)}\overline{z}^2\dot{z}+2\overline{d_4(\lambda)}z\overline{z}\dot{\overline{z}}+3\overline{d_3(\lambda)}\overline{z}^2\dot{\overline{z}}+O(|z|^4).
\end{aligned}
$$

We can solve for these coefficients $\{d_i(\lambda)\}_{i=1}^4$ by setting this expression for $\dot{y}$ equal to the expression given in 22. Note that $g(z,y,\lambda)=O(|z|^2)$ and $\overline{g}(z,y,\lambda)=O(|z|^2)$. So, by 20 and 21,

$$
\begin{aligned}
B(\lambda)d_1(\lambda)z^2+B(\lambda)d_2(\lambda)z\overline{z} &= 2d_1(\lambda)\sigma(\lambda)z^2+2d_1(\lambda)zg(z,y;\lambda) \\
+B(\lambda)\overline{d_1(\lambda)}\overline{z}^2+B(\lambda)d_3(\lambda)z^3 &\quad +d_2(\lambda)\sigma(\lambda)\overline{z}z+d_2(\lambda)\overline{z}g(z,y;\lambda) \\
+B(\lambda)d_4(\lambda)z^2\overline{z}+B(\lambda)\overline{d_4(\lambda)}z\overline{z}^2 &\quad +d_2(\lambda)\overline{\sigma(\lambda)}z\overline{z}+d_2(\lambda)z\overline{g}(z,y;\lambda)
\end{aligned}
$$

$$+B(\lambda)\overline{d_3(\lambda)}\overline{z}^3 + h(z,y;\lambda) \qquad +2\overline{d_1(\lambda)\sigma(\lambda)}\overline{z}^2 + 2\overline{d_1(\lambda)}\overline{z}\overline{g}(z,y;\lambda)$$

$$+3d_3(\lambda)\sigma(\lambda)z^3 + 2d_4(\lambda)\sigma(\lambda)z^2\overline{z}$$

$$+d_4(\lambda)\overline{\sigma(\lambda)}z^2\overline{z} + \overline{d_4(\lambda)}\sigma(\lambda)z\overline{z}^2$$

$$+2\overline{d_4(\lambda)\sigma(\lambda)}z\overline{z}^2 + 3\overline{d_3(\lambda)\sigma(\lambda)}\overline{z}^3$$

$$+O(|z|^4). \tag{24}$$

By analyzing f more closely and remembering that N_2 and N_3 are linear in each coordinate,

$$\begin{aligned}
f(za(\lambda) + \overline{za(\lambda)} &= N_2(za(\lambda) + \overline{za(\lambda)} + d_1(\lambda)z^2 + d_2(\lambda)z\overline{z} + \overline{d_1(\lambda)}\overline{z}^2 \\
+d_1(\lambda)z^2 + d_2(\lambda)z\overline{z} &\quad +d_3(\lambda)z^3 + d_4(\lambda)z^2\overline{z} + \overline{d_4(\lambda)}z\overline{z}^2 + \overline{d_3(\lambda)}\overline{z}^3 + O(|z|^4), \\
+\overline{d_1(\lambda)}\overline{z}^2 + d_3(\lambda)z^3 &\quad za(\lambda) + \overline{za(\lambda)} + d_1(\lambda)z^2 + d_2(\lambda)z\overline{z} + \overline{d_1(\lambda)}\overline{z}^2 \\
+d_4(\lambda)z^2\overline{z} + \overline{d_4(\lambda)}z\overline{z}^2 &\quad +d_3(\lambda)z^3 + d_4(\lambda)z^2\overline{z} + \overline{d_4(\lambda)}z\overline{z}^2 + \overline{d_3(\lambda)}\overline{z}^3 \\
+\overline{d_3(\lambda)}\overline{z}^3 + O(|z|^4);\lambda) &\quad +O(|z|^4);\lambda) + N_3(za(\lambda) + \overline{za(\lambda)} + d_1(\lambda)z^2 + d_2(\lambda)z\overline{z} \\
&\quad +\overline{d_1(\lambda)}\overline{z}^2 + d_3(\lambda)z^3 + d_4(\lambda)z^2\overline{z} + \overline{d_4(\lambda)}z\overline{z}^2 + \overline{d_3(\lambda)}\overline{z}^3 \\
&\quad +O(|z|^4), za(\lambda) + \overline{za(\lambda)} + d_1(\lambda)z^2 + d_2(\lambda)z\overline{z} + \overline{d_1(\lambda)}\overline{z}^2 \\
&\quad +d_3(\lambda)z^3 + d_4(\lambda)z^2\overline{z} + \overline{d_4(\lambda)}z\overline{z}^2 + \overline{d_3(\lambda)}\overline{z}^3 + O(|z|^4), \\
&\quad za(\lambda) + \overline{za(\lambda)} + d_1(\lambda)z^2 + d_2(\lambda)z\overline{z} + \overline{d_1(\lambda)}\overline{z}^2 + d_3(\lambda)z^3 \\
&\quad +d_4(\lambda)z^2\overline{z} + \overline{d_4(\lambda)}z\overline{z}^2 + \overline{d_3(\lambda)}\overline{z}^3 + O(|z|^4);\lambda) + O(|z|)^4 \\
&= N_2(a(\lambda), a(\lambda);\lambda)z^2 + 2N_2(a(\lambda), \overline{a(\lambda)};\lambda)z\overline{z} \\
&\quad +N_2(\overline{a(\lambda)}, \overline{a(\lambda)};\lambda)\overline{z}^2 + (2N_2(a(\lambda), d_1(\lambda);\lambda)
\end{aligned}$$

$$
\begin{aligned}
&+N_3(a(\lambda), a(\lambda), a(\lambda); \lambda))z^3 + (2N_2(a(\lambda), d_2(\lambda); \lambda) \\
&+2N_2(\overline{a(\lambda)}, d_1(\lambda); \lambda) + 3N_3(a(\lambda), a(\lambda), \overline{a(\lambda)}; \lambda))z^2\overline{z} \\
&+(2N_2(a(\lambda), \overline{d_1(\lambda)}; \lambda) + 2N_2(\overline{a(\lambda)}, d_2(\lambda); \lambda) \\
&+3N_3(a(\lambda), \overline{a(\lambda)}, \overline{a(\lambda)}; \lambda))z\overline{z}^2 + (2N_2(\overline{a(\lambda)}, \overline{d_1(\lambda)}; \lambda) \\
&+N_3(\overline{a(\lambda)}, \overline{a(\lambda)}, \overline{a(\lambda)}; \lambda))\overline{z}^3 + O(|z|^4). \qquad (25)
\end{aligned}
$$

By rewriting h and g with this representation of f and setting terms of equal degree in 24 equal to each other, we can solve for the coefficients $\{d_i(\lambda)\}_{i=1}^4$. We use Claims 8.3 and 8.4 to determine which operators are invertible and which are not, noting that, except for $\sigma(\lambda)$ and $\overline{\sigma(\lambda)}$, the spectrum of $L(\lambda)$ is in the left half plane. The first coefficient $d_1(\lambda)$ must satisfy

$$
\begin{aligned}
B(\lambda)d_1(\lambda) &= 2\sigma(\lambda)d_1(\lambda) \\
+P(\lambda)N_2(a(\lambda), a(\lambda); \lambda) & \\
d_1(\lambda) &= -(B(\lambda) - 2\sigma(\lambda))^{-1}P(\lambda)N_2(a(\lambda), a(\lambda); \lambda) \qquad (26)
\end{aligned}
$$

and $\overline{d_1(\lambda)}$ must satisfy

$$
\begin{aligned}
B(\lambda)\overline{d_1(\lambda)} + P(\lambda)N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda) &= 2\overline{\sigma(\lambda)d_1(\lambda)} \\
\overline{d_1(\lambda)} &= -(B(\lambda) - 2\overline{\sigma(\lambda)})^{-1}P(\lambda)N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda).
\end{aligned}
$$

The coefficient $d_2(\lambda)$ must satisfy

$$B(\lambda)d_2(\lambda) + 2P(\lambda)N_2(a(\lambda), \overline{a(\lambda)}; \lambda) = \sigma(\lambda)d_2(\lambda) + \overline{\sigma(\lambda)}d_2(\lambda).$$

This equation cannot be solved as the $d_1(\lambda)$ and $\overline{d_1(\lambda)}$ equations since at the bifurcation point $\sigma(\lambda) + \overline{\sigma(\lambda)} = 0$ and $B(\lambda)$ is singular. So, we must find $d_2(\lambda)$ in another way. Since $B(\lambda) = P(\lambda)L(\lambda)$, this equation becomes

$$\begin{aligned} P(\lambda)L(\lambda)d_2(\lambda) + 2P(\lambda)N_2(a(\lambda), \overline{a(\lambda)}; \lambda) &= \sigma(\lambda)d_2(\lambda) + \overline{\sigma(\lambda)}d_2(\lambda) \\ P(\lambda)(L(\lambda)d_2(\lambda) + 2N_2(a(\lambda), \overline{a(\lambda)}; \lambda)) &= (\sigma(\lambda) + \overline{\sigma(\lambda)})d_2(\lambda). \qquad (27) \end{aligned}$$

In order for such an equation to have a solution, the righthand side must be in the range of $P(\lambda)$. Thus, this equation will have a solution if and only if $((\sigma(\lambda) + \overline{\sigma(\lambda)})d_2(\lambda), v_i) = 0$ for all v_i in the null space of $P^*(\lambda)$. At the bifurcation point, this will clearly hold since $\sigma(0) + \overline{\sigma(0)} = 0$, but if $(d_2(\lambda), v_i) = 0$ for all v_i in the null space of $P^*(\lambda)$, then the equation will have a solution at each λ, instead of just at the bifurcation point. According to Claim 8.2, a basis for the null space of $P^*(\lambda)$ is $\{a^*(\lambda), \overline{a^*(\lambda)}\}$. If $(d_2(\lambda), a^*(\lambda)) = 0$ and $(d_2(\lambda), \overline{a^*(\lambda)}) = 0$, then

$$\begin{aligned} P(\lambda)L(\lambda)d_2(\lambda) &= L(\lambda)d_2(\lambda) - (L(\lambda)d_2(\lambda), a^*(\lambda))a(\lambda) - (L(\lambda)d_2(\lambda), \overline{a^*(\lambda)})\overline{a(\lambda)} \\ &= L(\lambda)d_2(\lambda) - (d_2(\lambda), L^*(\lambda)a^*(\lambda))a(\lambda) - (d_2(\lambda), L^*(\lambda)\overline{a^*(\lambda)})\overline{a(\lambda)} \\ &= L(\lambda)d_2(\lambda) - (d_2(\lambda), \overline{\sigma(\lambda)}a^*(\lambda))a(\lambda) - (d_2(\lambda), \sigma(\lambda)\overline{a^*(\lambda)})\overline{a(\lambda)} \\ &= L(\lambda)d_2(\lambda) - \sigma(\lambda)(d_2(\lambda), a^*(\lambda))a(\lambda) - \overline{\sigma(\lambda)}(d_2(\lambda), \overline{a^*(\lambda)})\overline{a(\lambda)} \end{aligned}$$

$$= \quad L(\lambda)d_2(\lambda).$$

Hence, 27 becomes

$$\begin{aligned}
L(\lambda)d_2(\lambda)+ &= (\sigma(\lambda)+\overline{\sigma(\lambda)})d_2(\lambda) \\
2P(\lambda)N_2(a(\lambda),\overline{a(\lambda)};\lambda) & \\
(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))d_2(\lambda) &= -2P(\lambda)N_2(a(\lambda),\overline{a(\lambda)};\lambda) \\
d_2(\lambda) &= -2(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1} \\
& \quad P(\lambda)N_2(a(\lambda),\overline{a(\lambda)};\lambda), \qquad (28)
\end{aligned}$$

a quantity which exists both at and away from the bifurcation point.

By setting the coefficients on z^3 equal and substituting in the values of $d_1(\lambda)$ and $d_2(\lambda)$,

$$\begin{aligned}
B(\lambda)d_3(\lambda)+P(\lambda)(2N_2(a(\lambda),d_1(\lambda);\lambda) &= 2d_1(\lambda)(N_2(a(\lambda),a(\lambda);\lambda),a^*(\lambda)) \\
+N_3(a(\lambda),a(\lambda),a(\lambda);\lambda)) \qquad & \quad +d_2(\lambda)(N_2(a(\lambda),a(\lambda);\lambda),\overline{a^*(\lambda)}) \\
& \quad +3\sigma(\lambda)d_3(\lambda) \\
d_3(\lambda) &= (B(\lambda)-3\sigma(\lambda))^{-1}[-P(\lambda)(2N_2(a(\lambda), \\
& \quad -(B(\lambda)-2\sigma(\lambda))^{-1} \\
& \quad P(\lambda)N_2(a(\lambda),a(\lambda);\lambda);\lambda) \\
& \quad +N_3(a(\lambda),a(\lambda),a(\lambda);\lambda)) \\
& \quad -2(B(\lambda)-2\sigma(\lambda))^{-1}P(\lambda)N_2(a(\lambda),
\end{aligned}$$

$$a(\lambda);\lambda)(N_2(a(\lambda),a(\lambda);\lambda),a^*(\lambda))$$
$$-2(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1}P(\lambda)$$
$$N_2(a(\lambda),\overline{a(\lambda)};\lambda)$$
$$(N_2(a(\lambda),a(\lambda);\lambda),\overline{a^*(\lambda)})].$$

The $z^2\bar{z}$ terms give

$$\begin{aligned}
B(\lambda)d_4(\lambda)+P(\lambda)(2N_2(a(\lambda),d_2(\lambda);\lambda) &= 2\sigma(\lambda)d_4(\lambda)+\overline{\sigma(\lambda)}d_4(\lambda)\\
+2N_2(\overline{a(\lambda)},d_1(\lambda);\lambda) &\quad +2d_1(\lambda)(2N_2(a(\lambda),\overline{a(\lambda)};\lambda),a^*(\lambda))\\
+3N_3(a(\lambda),a(\lambda),\overline{a(\lambda)};\lambda)) &\quad +d_2(\lambda)(N_2(a(\lambda),a(\lambda);\lambda),a^*(\lambda))\\
&\quad +d_2(\lambda)(2N_2(a(\lambda),\overline{a(\lambda)};\lambda),\overline{a^*(\lambda)})\\
&\quad +2\overline{d_1(\lambda)}(N_2(a(\lambda),a(\lambda);\lambda),\overline{a^*(\lambda)})\\
d_4(\lambda) &= (B(\lambda)-(2\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1}[-P(\lambda)\\
&\quad (2N_2(a(\lambda)-2(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1}\\
&\quad P(\lambda)N_2(a(\lambda),\overline{a(\lambda)};\lambda);\lambda)+2N_2(\overline{a(\lambda)},\\
&\quad -(B(\lambda)-2\sigma(\lambda))^{-1}P(\lambda)N_2(a(\lambda),\\
&\quad a(\lambda);\lambda);\lambda)+3N_3(a(\lambda),a(\lambda),\overline{a(\lambda)};\lambda))\\
&\quad -2(B(\lambda)-2\sigma(\lambda))^{-1}P(\lambda)N_2(a(\lambda),\\
&\quad a(\lambda);\lambda)(2N_2(a(\lambda),\overline{a(\lambda)};\lambda),a^*(\lambda))\\
&\quad -2(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1}P(\lambda)\\
&\quad N_2(a(\lambda),\overline{a(\lambda)};\lambda)(N_2(a(\lambda),a(\lambda);\lambda),
\end{aligned}$$

$$
\begin{aligned}
& a^*(\lambda)) - 2(L(\lambda) - (\sigma(\lambda) + \overline{\sigma(\lambda)}))^{-1} \\
& P(\lambda)N_2(a(\lambda), \overline{a(\lambda)}; \lambda)(2N_2(a(\lambda), \\
& \overline{a(\lambda)}; \lambda), \overline{a^*(\lambda)}) \\
& -2(B(\lambda) - 2\overline{\sigma(\lambda)})^{-1}P(\lambda) \\
& N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda)(N_2(a(\lambda), a(\lambda); \lambda), \\
& \overline{a^*(\lambda)})]
\end{aligned}
$$

and the $z\bar{z}^2$ terms give

$$
\begin{aligned}
B(\lambda)\overline{d_4(\lambda)} &= \sigma(\lambda)\overline{d_4(\lambda)} + 2\overline{\sigma(\lambda)d_4(\lambda)} \\
+P(\lambda)(2N_2(a(\lambda), \overline{d_1(\lambda)}; \lambda) &\quad +2d_1(\lambda)(N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda), a^*(\lambda)) \\
+2N_2(\overline{a(\lambda)}, d_2(\lambda); \lambda) &\quad +d_2(\lambda)(2N_2(a(\lambda), \overline{a(\lambda)}; \lambda), a^*(\lambda)) \\
+3N_3(a(\lambda), \overline{a(\lambda)}, \overline{a(\lambda)}; \lambda)) &\quad +d_2(\lambda)(N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda), \overline{a^*(\lambda)}) \\
&\quad +2\overline{d_1(\lambda)}(2N_2(a(\lambda), \overline{a(\lambda)}; \lambda), \overline{a^*(\lambda)}) \\
\overline{d_4(\lambda)} &= (B(\lambda) - (\sigma(\lambda) + 2\overline{\sigma(\lambda)}))^{-1}[-P(\lambda)(2N_2(a(\lambda), \\
&\quad -(B(\lambda) - 2\overline{\sigma(\lambda)})^{-1}P(\lambda)N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda); \lambda) \\
&\quad +2N_2(\overline{a(\lambda)}, -2(L(\lambda) - (\sigma(\lambda) + \overline{\sigma(\lambda)}))^{-1} \\
&\quad P(\lambda)N_2(a(\lambda), \overline{a(\lambda)}; \lambda); \lambda) + 3N_3(a(\lambda), \overline{a(\lambda)}, \overline{a(\lambda)}; \lambda)) \\
&\quad -2(B(\lambda) - 2\sigma(\lambda))^{-1}P(\lambda)N_2(a(\lambda), a(\lambda); \lambda)(N_2(\overline{a(\lambda)}, \\
&\quad \overline{a(\lambda)}; \lambda), a^*(\lambda)) - 2(L(\lambda) - (\sigma(\lambda) + \overline{\sigma(\lambda)}))^{-1}P(\lambda) \\
&\quad N_2(a(\lambda), \overline{a(\lambda)}; \lambda)(2N_2(a(\lambda), \overline{a(\lambda)}; \lambda), a^*(\lambda))
\end{aligned}
$$

$$-2(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1}P(\lambda)N_2(a(\lambda),$$

$$\overline{a(\lambda)};\lambda)(N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda),\overline{a^*(\lambda)})$$

$$-2(B(\lambda)-2\overline{\sigma(\lambda)})^{-1}P(\lambda)N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda)$$

$$(2N_2(a(\lambda),\overline{a(\lambda)};\lambda),\overline{a^*(\lambda)})].$$

Lastly, the $\overline{z}^3$ terms give

$$\begin{aligned}
B(\lambda)\overline{d_3(\lambda)} &= 3\overline{\sigma(\lambda)d_3(\lambda)} \\
+P(\lambda)(2N_2(\overline{a(\lambda)},\overline{d_1(\lambda)};\lambda) &\quad +d_2(\lambda)(N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda),a^*(\lambda)) \\
+N_3(\overline{a(\lambda)},\overline{a(\lambda)},\overline{a(\lambda)};\lambda)) &\quad +2\overline{d_1(\lambda)}(N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda),\overline{a^*(\lambda)}) \\
\overline{d_3(\lambda)} &= (B(\lambda)-3\overline{\sigma(\lambda)})^{-1}[-P(\lambda)(2N_2(\overline{a(\lambda)}, \\
&\quad -(B(\lambda)-2\overline{\sigma(\lambda)})^{-1}P(\lambda)N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda);\lambda) \\
&\quad +N_3(\overline{a(\lambda)},\overline{a(\lambda)},\overline{a(\lambda)};\lambda)) \\
&\quad -2(L(\lambda)-(\sigma(\lambda)+\overline{\sigma(\lambda)}))^{-1}P(\lambda)N_2(a(\lambda),\overline{a(\lambda)};\lambda) \\
&\quad (N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda),a^*(\lambda)) \\
&\quad -2(B(\lambda)-2\overline{\sigma(\lambda)})^{-1}P(\lambda)N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda) \\
&\quad (N_2(\overline{a(\lambda)},\overline{a(\lambda)};\lambda),\overline{a^*(\lambda)})].
\end{aligned}$$

Now, by equation 20,

$$\begin{aligned}
\dot{z} &= \sigma(\lambda)z+(f(za(\lambda)+\overline{z}\overline{a(\lambda)}+d_1(\lambda)z^2+\ldots+\overline{d_3(\lambda)}\overline{z}^3+O(|z|^4);\lambda),a^*(\lambda)) \\
&= \sigma(\lambda)z+A_1(\lambda)z^2+A_2(\lambda)z\overline{z}+A_3(\lambda)\overline{z}^2+B_1(\lambda)z^3+B_2(\lambda)z^2\overline{z}+B_3(\lambda)z\overline{z}^2
\end{aligned}$$

$$+B_4(\lambda)\bar{z}^3 + O(|z|^4), \tag{29}$$

where, by 25,

$$\begin{aligned}
A_1(\lambda) &= (N_2(a(\lambda), a(\lambda); \lambda), a^*(\lambda)) && (30)\\
A_2(\lambda) &= (2N_2(a(\lambda), \overline{a(\lambda)}; \lambda), a^*(\lambda)) && (31)\\
A_3(\lambda) &= (N_2(\overline{a(\lambda)}, \overline{a(\lambda)}; \lambda), a^*(\lambda))\\
B_1(\lambda) &= (2N_2(a(\lambda), d_1(\lambda); \lambda) + N_3(a(\lambda), a(\lambda), a(\lambda); \lambda), a^*(\lambda))\\
B_2(\lambda) &= (2N_2(a(\lambda), d_2(\lambda); \lambda) + 2N_2(\overline{a(\lambda)}, d_1(\lambda); \lambda)\\
&\quad +3N_3(a(\lambda), a(\lambda), \overline{a(\lambda)}; \lambda), a^*(\lambda)) && (32)\\
B_3(\lambda) &= (2N_2(a(\lambda), \overline{d_1(\lambda)}; \lambda) + 2N_2(\overline{a(\lambda)}, d_2(\lambda); \lambda)\\
&\quad +3N_3(a(\lambda), \overline{a(\lambda)}, \overline{a(\lambda)}; \lambda), a^*(\lambda))\\
B_4(\lambda) &= (2N_2(\overline{a(\lambda)}, \overline{d_1(\lambda)}; \lambda) + N_3(\overline{a(\lambda)}, \overline{a(\lambda)}, \overline{a(\lambda)}; \lambda), a^*(\lambda)).
\end{aligned}$$

To understand the behavior of the solution z of 29, we will find a related equation which is in Poincaré normal form. Thus, we set, for some values $\{a_i(\lambda)\}_{i=1}^{7}$ which we will choose later,

$$\begin{aligned}
w &= z + a_1(\lambda)z^2 + a_2(\lambda)z\bar{z} + a_3(\lambda)\bar{z}^2 + a_4(\lambda)z^3 + a_5(\lambda)z^2\bar{z} + a_6(\lambda)z\bar{z}^2\\
&\quad +a_7(\lambda)\bar{z}^3 + O(|z|^4). && (33)
\end{aligned}$$

We would like to write

$$
\begin{aligned}
z &= w + b_1(\lambda)w^2 + b_2(\lambda)w\overline{w} + b_3(\lambda)\overline{w}^2 + b_4(\lambda)w^3 + b_5(\lambda)w^2\overline{w} + b_6(\lambda)w\overline{w}^2 \\
&\quad + b_7(\lambda)\overline{w}^3 + O(|w|^4).
\end{aligned} \tag{34}
$$

Note that

$$
\begin{aligned}
z^2 &= w^2 + 2b_1(\lambda)w^3 + 2b_2(\lambda)w^2\overline{w} + 2b_3(\lambda)w\overline{w}^2 + O(|w|^4) \\
z\overline{z} &= w\overline{w} + \overline{b_1(\lambda)}w\overline{w}^2 + \overline{b_2(\lambda)}w^2\overline{w} + \overline{b_3(\lambda)}w^3 + b_1(\lambda)w^2\bar{w} \\
&\quad + b_2(\lambda)w\overline{w}^2 + b_3(\lambda)\overline{w}^3 + O(|w|^4) \\
\overline{z}^2 &= \overline{w}^2 + 2\overline{b_1(\lambda)}\overline{w}^3 + 2\overline{b_2(\lambda)}w\overline{w}^2 + 2\overline{b_3(\lambda)}w^2\overline{w} + O(|w|^4) \\
z^3 &= w^3 + O(|w|^4) \\
z^2\overline{z} &= w^2\overline{w} + O(|w|^4) \\
z\overline{z}^2 &= w\overline{w}^2 + O(|w|^4) \\
\overline{z}^3 &= \overline{w}^3 + O(|w|^4).
\end{aligned}
$$

By substituting these expressions into equation 33, we see that

$$
\begin{aligned}
w &= w + w^2(b_1(\lambda) + a_1(\lambda)) + w\overline{w}(b_2(\lambda) + a_2(\lambda)) + \overline{w}^2(b_3(\lambda) + a_3(\lambda)) \\
&\quad + w^3(b_4(\lambda) + 2a_1(\lambda)b_1(\lambda) + a_2(\lambda)\overline{b_3(\lambda)} + a_4(\lambda)) + w^2\overline{w}(b_5(\lambda) \\
&\quad + 2a_1(\lambda)b_2(\lambda) + a_2(\lambda)\overline{b_2(\lambda)} + a_2(\lambda)b_1(\lambda) + 2a_3(\lambda)\overline{b_3(\lambda)} \\
&\quad + a_5(\lambda)) + w\overline{w}^2(b_6(\lambda) + 2a_1(\lambda)b_3(\lambda) + a_2(\lambda)\overline{b_1(\lambda)}
\end{aligned}
$$

$$+a_2(\lambda)b_2(\lambda)+2a_3(\lambda)\overline{b_2(\lambda)}+a_6(\lambda))+\overline{w}^3(b_7(\lambda)$$
$$+a_2(\lambda)b_3(\lambda)+2a_3(\lambda)\overline{b_1(\lambda)}+a_7(\lambda))+O(|w|^4).$$

In order for this equation to hold, we must have that the coefficients on the terms of degree higher than one are zero. We can write our $\{b_i(\lambda)\}_{i=1}^7$ in terms of the $\{a_i(\lambda)\}_{i=1}^7$ to ensure that the necessary terms are zero, although we will not write those values of $\{b_i(\lambda)\}_{i=1}^7$ here. So, we see that we can write z as 34 by having our $\{b_i(\lambda)\}_{i=1}^7$ defined in this way. Now,

$$\begin{aligned}
\dot{w} &= \dot{z}+2a_1(\lambda)z\dot{z}+a_2(\lambda)z\dot{\overline{z}}+a_2(\lambda)\overline{z}\dot{z}+2a_3(\lambda)\overline{z}\dot{\overline{z}}+3a_4(\lambda)z^2\dot{z}+2a_5(\lambda)z\overline{z}\dot{z}\\
&\quad+a_5(\lambda)z^2\dot{\overline{z}}+a_6(\lambda)\overline{z}^2\dot{z}+2a_6(\lambda)z\overline{z}\dot{\overline{z}}+3a_7(\lambda)\overline{z}^2\dot{\overline{z}}+O(|z|^4)\\
&= \sigma(\lambda)z+c_1(\lambda)z^2+c_2(\lambda)z\overline{z}+c_3(\lambda)\overline{z}^2+c_4(\lambda)z^3+c_5(\lambda)z^2\overline{z}+c_6(\lambda)z\overline{z}^2\\
&\quad+c_7(\lambda)\overline{z}^3+O(|z|)^4, \qquad (35)
\end{aligned}$$

where, by 29,

$$\begin{aligned}
c_1(\lambda) &= A_1(\lambda)+2\sigma(\lambda)a_1(\lambda)\\
c_2(\lambda) &= A_2(\lambda)+\overline{\sigma(\lambda)}a_2(\lambda)+\sigma(\lambda)a_2(\lambda)\\
c_3(\lambda) &= A_3(\lambda)+2\overline{\sigma(\lambda)}a_3(\lambda)\\
c_4(\lambda) &= B_1(\lambda)+2a_1(\lambda)A_1(\lambda)+a_2(\lambda)\overline{A_3(\lambda)}+3\sigma(\lambda)a_4(\lambda)\\
c_5(\lambda) &= B_2(\lambda)+2a_1(\lambda)A_2(\lambda)+a_2(\lambda)\overline{A_2(\lambda)}+\\
&\quad a_2(\lambda)A_1(\lambda)+2a_3(\lambda)\overline{A_3(\lambda)}+2\sigma(\lambda)a_5(\lambda)+\overline{\sigma(\lambda)}a_5(\lambda)
\end{aligned}$$

$$
\begin{aligned}
c_6(\lambda) &= B_3(\lambda) + 2a_1(\lambda)A_3(\lambda) + a_2(\lambda)\overline{A_1(\lambda)} + a_2(\lambda)A_2(\lambda) \\
&\quad +2a_3(\lambda)\overline{A_2(\lambda)} + \sigma(\lambda)a_6(\lambda) + 2\overline{\sigma(\lambda)}a_6(\lambda) \\
c_7(\lambda) &= B_4(\lambda) + a_2(\lambda)A_3(\lambda) + 2a_3(\lambda)\overline{A_1(\lambda)} + 3\overline{\sigma(\lambda)}a_7(\lambda).
\end{aligned}
$$

By using 34 to express each z in terms of w, 35 becomes

$$
\begin{aligned}
\dot{w} &= \sigma(\lambda)w + (\sigma(\lambda)b_1(\lambda) + c_1(\lambda))w^2 + (\sigma(\lambda)b_2(\lambda) + c_2(\lambda))w\overline{w} + (\sigma(\lambda)b_3(\lambda) \\
&\quad +c_3(\lambda))\overline{w}^2 + (\sigma(\lambda)b_4(\lambda) + 2b_1(\lambda)c_1(\lambda) + \overline{b_3(\lambda)}c_2(\lambda) + c_4(\lambda))w^3 \\
&\quad +(\sigma(\lambda)b_5(\lambda) + 2b_2(\lambda)c_1(\lambda) + \overline{b_2(\lambda)}c_2(\lambda) + b_1(\lambda)c_2(\lambda) + 2\overline{b_3(\lambda)}c_3(\lambda) \\
&\quad +c_5(\lambda))w^2\overline{w} + (\sigma(\lambda)b_6(\lambda) + 2b_3(\lambda)c_1(\lambda) + \overline{b_1(\lambda)}c_2(\lambda) + b_2(\lambda)c_2(\lambda) \\
&\quad +2\overline{b_2(\lambda)}c_3(\lambda) + c_6(\lambda))w\overline{w}^2 + (\sigma(\lambda)b_7(\lambda) + b_3(\lambda)c_2(\lambda) + 2\overline{b_1(\lambda)}c_3(\lambda) \\
&\quad +c_7(\lambda))\overline{w}^3 + O(|w|^4).
\end{aligned} \tag{36}
$$

Since both the $\{b_i(\lambda)\}_{i=1}^7$ and $\{c_i(\lambda)\}_{i=1}^7$ are expressions of $\{a_i(\lambda)\}_{i=1}^7$, we can choose our $\{a_i(\lambda)\}_{i=1}^7$ to make all of the above coefficients on terms of degree higher than one, except the coefficient on the $w^2\overline{w}$ term, to be zero. At the bifurcation point, $\sigma(0) + \overline{\sigma}(0) = 0$, which causes the $a_5(0)$ terms in the coefficient of $w^2\overline{w}$ to vanish. Therefore, we cannot eliminate this term in the differential equation at the bifurcation point. We will not include the values of all the coefficients here, but instead will give only those values which prove useful. By substituting in the chosen values of $\{a_i(\lambda)\}_{i=1}^{7}$, the above equation 36 becomes

$$\begin{aligned}\dot{w} &= \sigma(\lambda)w + (\sigma(\lambda)b_5(\lambda) + 2b_2(\lambda)c_1(\lambda) + \overline{b_2(\lambda)}c_2(\lambda) + b_1(\lambda)c_2(\lambda) \\ &\quad +2\overline{b_3(\lambda)}c_3(\lambda) + c_5(\lambda))w^2\overline{w} + O(|w|^4).\end{aligned}$$

Now, the expressions for these $b_i(\lambda)$ and $c_i(\lambda)$ values are

$$\begin{aligned}
b_1(\lambda) &= \frac{A_1(\lambda)}{\sigma(\lambda)} \\
b_2(\lambda) &= \frac{A_2(\lambda)}{\overline{\sigma(\lambda)}} \\
b_3(\lambda) &= \frac{A_3(\lambda)}{-\sigma(\lambda) + 2\overline{\sigma(\lambda)}} \\
b_5(\lambda) &= -a_5(\lambda) + \frac{3A_1(\lambda)A_2(\lambda)}{\sigma(\lambda)\overline{\sigma(\lambda)}} + \frac{A_2(\lambda)\overline{A_2(\lambda)}}{\sigma(\lambda)\overline{\sigma(\lambda)}} \\
&\quad + \frac{2A_3(\lambda)\overline{A_3(\lambda)}}{(2\sigma(\lambda) - \overline{\sigma(\lambda)})(-\sigma(\lambda) + 2\overline{\sigma(\lambda)})} \\
c_1(\lambda) &= -A_1(\lambda) \\
c_2(\lambda) &= -\frac{A_2(\lambda)\sigma(\lambda)}{\overline{\sigma(\lambda)}} \\
c_3(\lambda) &= A_3(\lambda) - \frac{2A_3(\lambda)\overline{\sigma(\lambda)}}{-\sigma(\lambda) + 2\overline{\sigma(\lambda)}} \\
c_5(\lambda) &= B_2(\lambda) - \frac{2A_1(\lambda)A_2(\lambda)}{\sigma(\lambda)} + 2a_5(\lambda)\sigma(\lambda) - \frac{A_1(\lambda)A_2(\lambda)}{\overline{\sigma(\lambda)}} - \frac{A_2(\lambda)\overline{A_2(\lambda)}}{\overline{\sigma(\lambda)}} \\
&\quad + a_5(\lambda)\overline{\sigma(\lambda)} - \frac{2A_3(\lambda)\overline{A_3(\lambda)}}{-\sigma(\lambda) + 2\overline{\sigma(\lambda)}}.
\end{aligned}$$

Hence, by substituting in these values and using the fact that $w^2\overline{w} = |w|^2 w$,

$$\dot{w} = \sigma(\lambda)w + \beta|w|^2 w + O(|w|^4), \qquad (37)$$

where

$$\beta = \frac{-A_1(\lambda)A_2(\lambda)(2\overline{\sigma(\lambda)}+\sigma(\lambda))}{|\sigma(\lambda)|^2} - \frac{|A_2(\lambda)|^2}{\overline{\sigma(\lambda)}} + B_2(\lambda) + \frac{2|A_3(\lambda)|^2}{\sigma(\lambda)-2\overline{\sigma(\lambda)}}$$
$$+a_5(\lambda)(\sigma(\lambda)+\overline{\sigma(\lambda)}).$$

This is in Poincaré normal form. Thus, its stability is fully analyzed in [5]. We know that the periodic solutions which bifurcate from the steady solution $w = 0$ are stable if and only if $Re(\beta) < 0$ at the bifurcation point. Now, at the bifurcation point since $\sigma(0) = \omega i$,

$$\beta = \frac{A_1(0)A_2(0)i}{\omega} - \frac{|A_2(0)|^2 i}{\omega} + B_2(0) - \frac{2|A_3(0)|^2 i}{3\omega}.$$

So,

$$Re(\beta) = Re\Big(\frac{A_1(0)A_2(0)i}{\omega} + B_2(0)\Big)$$

and the bifurcating periodic solutions are stable if and only if $Re\Big(\frac{A_1(0)A_2(0)i}{\omega} + B_2(0)\Big) < 0$. Let us now find the values needed to compute this value. By 26, 28, 30, 31, and 32,

$$A_1(0) = (N_2(a(0), a(0); 0), a^*(0))$$

$$A_2(0) = (2N_2(a(0), \overline{a(0)}; 0), a^*(0))$$

$$B_2(0) = (2N_2(a(0), d_2(0); 0) + 2N_2(\overline{a(0)}, d_1(0); 0) + 3N_3(a(0), a(0), \overline{a(0)}; 0), a^*(0))$$

$$
\begin{aligned}
d_1(0) &= -(B(0) - 2i\omega)^{-1} P(0) N_2(a(0), a(0); 0) \\
d_2(0) &= -2L(0)^{-1} P(0) N_2(a(0), \overline{a(0)}; 0).
\end{aligned}
$$

Now, remember that

$$
\begin{aligned}
z &= w + b_1(\lambda) w^2 + b_2(\lambda) w\overline{w} + b_3(\lambda)\overline{w}^2 + b_4(\lambda) w^3 + b_5(\lambda) w^2\overline{w} + b_6(\lambda) w\overline{w}^2 \\
&\quad + b_7(\lambda)\overline{w}^3 + O(|w|^4), \\
y &= d_1(\lambda) z^2 + d_2(\lambda) z\overline{z} + \overline{d_1(\lambda)}\overline{z}^2 + d_3(\lambda) z^3 + d_4(\lambda) z^2\overline{z} + \overline{d_4(\lambda)} z\overline{z}^2 \\
&\quad + \overline{d_3(\lambda)}\overline{z}^3 + O(|z|^4),
\end{aligned}
$$

and

$$
u = z a(\lambda) + \overline{z}\overline{a(\lambda)} + y.
$$

Hence, the bifurcating periodic solutions of the system 37 are stable if and only the bifurcating periodic solutions of the original system 15 are stable. So, the periodic solutions of 15 which bifurcate from $u = 0$ are stable if only if $Re(\beta) < 0$. For this reason, we call $Re(\beta)$ the stability coefficient of 15.

9 Identifying Supercritical Hopf Bifurcation Points of Our System

In this section we will use the method established in Section 8 to develop another method for classifying the Hopf bifurcation points of the system 1 - 6. We saw in Section 4 that a perturbation of a steady solution must satisfy

$$\begin{aligned}\frac{\partial u}{\partial t} &= Lu + f(u) \\ &= Lu + N_2(u,u) + N_3(u,u,u) + O(u^4), \qquad (38)\end{aligned}$$

where

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix}$$

and the linear operator L is given by

$$\begin{pmatrix} \frac{\partial^2}{\partial x^2} + D\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} - 4RD\tilde{T}^3 & D\tilde{Y}_f e^{-Ta/\tilde{T}} & D\tilde{Y}_o e^{-Ta/\tilde{T}} \\ -\frac{D}{Le}\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} & \frac{1}{Le}\frac{\partial^2}{\partial x^2} - \frac{D}{Le}\tilde{Y}_f e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_o e^{-Ta/\tilde{T}} \\ -\frac{D}{Le}\tilde{Y}_o\tilde{Y}_f\frac{Ta}{\tilde{T}^2}e^{-Ta/\tilde{T}} & -\frac{D}{Le}\tilde{Y}_f e^{-Ta/\tilde{T}} & \frac{1}{Le}\frac{\partial^2}{\partial x^2} - \frac{D}{Le}\tilde{Y}_o e^{-Ta/\tilde{T}} \end{pmatrix}.$$

Also,

$$N_2(u,u) = \frac{1}{2}\begin{pmatrix} W''(0) - Z''(0) \\ -\frac{1}{Le}W''(0) \\ -\frac{1}{Le}W''(0) \end{pmatrix}$$

and

$$N_3(u,u,u) = \frac{1}{6}\begin{pmatrix} W'''(0) - Z'''(0) \\ -\frac{1}{Le}W'''(0) \\ -\frac{1}{Le}W'''(0) \end{pmatrix},$$

where

$$\begin{aligned}
W''(0) &= 2D(u_2)(u_3)e^{-Ta/\tilde{T}} + 2D(u_2)\tilde{Y_f}\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
&\quad + 2D(u_3)\tilde{Y_o}\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
&\quad + D\tilde{Y_o}\tilde{Y_f}\Big(\frac{-2Ta(u_1)^2}{\tilde{T}^3} + \frac{T_a^2(u_1)^2}{\tilde{T}^4}\Big)e^{-Ta/\tilde{T}} \\
W'''(0) &= 6D(u_2)(u_3)\frac{Ta(u_1)}{\tilde{T}^2}e^{-Ta/\tilde{T}} \\
&\quad + 3D(u_2)\tilde{Y_f}\Big(\frac{-2Ta(u_1)^2}{\tilde{T}^3} + \frac{T_a^2(u_1)^2}{\tilde{T}^4}\Big)e^{-Ta/\tilde{T}} \\
&\quad + 3D(u_3)\tilde{Y_o}\Big(\frac{-2Ta(u_1)^2}{\tilde{T}^3} + \frac{T_a^2(u_1)^2}{\tilde{T}^4}\Big)e^{-Ta/\tilde{T}} \\
&\quad + D\tilde{Y_o}\tilde{Y_f}\Big(\frac{6Ta(u_1)^3}{\tilde{T}^4} - \frac{6T_a^2(u_1)^3}{\tilde{T}^5} + \frac{T_a^3(u_1)^3}{\tilde{T}^6}\Big)e^{-Ta/\tilde{T}} \\
Z''(0) &= 12RD(u_1)^2\tilde{T}^2 \\
Z'''(0) &= 24RD(u_1)^3\tilde{T}.
\end{aligned}$$

And, the perturbation must satisfy the zero boundary conditions,

$$u_1 = u_2 = u_3 = 0 \qquad \text{at} \qquad x = \pm 1. \tag{39}$$

Define $H = \{u \epsilon AC[-1,1] : u' \epsilon L^2[-1,1], u(-1) = u(1) = 0\}^3$ as a subset of $L^2[-1,1]^3$. It follows from [9, page 81] that H is a Hilbert space of complex, vector-valued functions. Let $||\cdot||$ denote the norm in H. Note that $u = 0$ is a steady solution of 38 in H. By examining the forms of $W''(0)$ and $Z''(0)$, we see that we can write

$$N_2(u,u) = \begin{pmatrix} \Sigma_1 \leq i,j \leq 3 a_{i,j} u_i u_j \\ \Sigma_1 \leq i,j \leq 3 b_{i,j} u_i u_j \\ \Sigma_1 \leq i,j \leq 3 c_{i,j} u_i u_j \end{pmatrix}$$

and

$$N_3(u,u,u) = \begin{pmatrix} \Sigma_1 \leq i,j,k \leq 3 d_{i,j,k} u_i u_j u_k \\ \Sigma_1 \leq i,j,k \leq 3 e_{i,j,k} u_i u_j u_k \\ \Sigma_1 \leq i,j,k \leq 3 f_{i,j,k} u_i u_j u_k \end{pmatrix},$$

where $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, $1 \leq i,j \leq 3$ and $d_{i,j,k}$, $e_{i,j,k}$, $f_{i,j,k}$, $1 \leq i,j,k \leq 3$ are functions of x. Also note that these functions may be chosen so that $a_{i,j} = a_{j,i}$, $b_{i,j} = b_{j,i}$, $c_{i,j} = c_{j,i}$, $d_{i,j,k} = d_{i,k,j} = d_{j,i,k}$, $e_{i,j,k} = e_{i,k,j} = e_{j,i,k}$, and $f_{i,j,k} = f_{i,k,j} = f_{j,i,k}$ for all $1 \leq i,j,k \leq 3$. Therefore, it follows from 16 and 17 that,

$$N_2(u,v) = \begin{pmatrix} \Sigma_{1 \leq i,j \leq 3} a_{i,j} u_i v_j \\ \Sigma_{1 \leq i,j \leq 3} b_{i,j} u_i v_j \\ \Sigma_{1 \leq i,j \leq 3} c_{i,j} u_i v_j \end{pmatrix}$$

and

$$N_3(u,v,w) = \begin{pmatrix} \Sigma_{1 \leq i,j,k \leq 3} d_{i,j,k} u_i v_j w_k \\ \Sigma_{1 \leq i,j,k \leq 3} e_{i,j,k} u_i v_j w_k \\ \Sigma_{1 \leq i,j,k \leq 3} f_{i,j,k} u_i v_j w_k \end{pmatrix}.$$

It is easy to see in these forms that N_2 and N_3 are linear in each coordinate and that $N_2(u,v) = N_2(v,u)$ and $N_3(u,v,w) = N_3(u,w,v) = N_3(v,u,w)$ for all $u, v, w \epsilon H$. Also, note that $f(\overline{u}) = \overline{f(u)}$. Hence f, N_2, and N_3 have properties 1 - 4 in Section 8.

Suppose that 38 has a Hopf bifurcation point at the Damköhler number D_0. Define our bifurcation parameter, λ, to be $\lambda = D - D_0$. Thus, 38 has a Hopf bifurcation point at $\lambda = 0$. Note that the linearized system L depends on D and hence $L = L(\lambda)$. In addition, let us use the same notation for the leading eigenvalue, eigenvector, projection, and so on, as we did in Section 8. The following theorem is a result of Section 8.

Theorem 9.1 *Assume that 38 has a Hopf bifurcation point at D_0 and $Re(\beta) < 0$*

where

$$Re(\beta) = Re\Big(\frac{A_1(0)A_2(0)i}{\omega} + B_2(0)\Big)$$

and

$$\begin{aligned}
A_1(0) &= (N_2(a(0),a(0);0),a^*(0)) \\
A_2(0) &= (2N_2(a(0),\overline{a(0)};0),a^*(0)) \\
B_2(0) &= (2N_2(a(0),d_2(0);0) + 2N_2(\overline{a(0)},d_1(0);0) + 3N_3(a(0),a(0),\overline{a(0)};0),a^*(0)) \\
d_1(0) &= -(B(0)-2i\omega)^{-1}P(0)N_2(a(0),a(0);0) \\
d_2(0) &= -2L(0)^{-1}P(0)N_2(a(0),\overline{a(0)};0).
\end{aligned}$$

Then there is a time periodic solution $\theta(x,t)$ of 1 - 6, a critical Damköhler number D_{crit}, and an $\epsilon > 0$ such that if $\begin{pmatrix} \tilde{T} \\ \tilde{Y_o} \\ \tilde{Y_f} \end{pmatrix}$ is a steady state solution of 1 - 6 at D, where D is between D_0 and D_{crit} and $p(x)$ is an initial perturbation of $\begin{pmatrix} \tilde{T} \\ \tilde{Y_o} \\ \tilde{Y_f} \end{pmatrix}$ with $||p(x)|| < \epsilon$, then $\begin{pmatrix} \tilde{T} \\ \tilde{Y_o} \\ \tilde{Y_f} \end{pmatrix} + u(x,t)$, the solution of 1 - 6 with $u(x,0) = p(x)$, approaches $\theta(x,t)$ as $t \to \infty$.

Note that all the values needed for computing the stability coefficient, such as the eigenvalue and eigenvector, can be found (or very closely approximated) by Mathematica. As in the method for determining the stability of the steady states (see Section 6), this is done by discretizing and therefore changing the operator $L(\lambda)$ into a real matrix. Hence, finding the adjoint eigenvector, $a^*(\lambda)$, can even be done easily by finding an eigenvector of the transpose of $L(\lambda)$.

Using Mathematica to compute the stability coefficient, β, is much more straightforward than searching for stable periodic solutions as we did in Section 7. Determining where a Hopf bifurcation point changes from supercritical to subcritical is now done with ease. For example, in Section 7, we examined when $T_a = 3$ and $Le = 1$. We saw that for smaller R values, the size of the interval in which small perturbations lead to stable oscillations is relatively large. However, as R increases, the size of this interval decreases and approaches zero as R approaches the point at which the bifurcation point becomes subcritical. By analyzing the sizes of these intervals for various R values, we found that the bifurcation changed from supercritical to subcritical at approximately $R = 2.3 \times 10^{-5}$. Finding this value of R required much time and many computations. However, this result is easily verified in Figure 19. We see that the stability coefficient changes signs, and hence the bifurcation point changes from supercritical to subcritical, at approximately $R = 2.3 \times 10^{-5}$. It is in this way that we computed the R values at which the bifurcation points changed from supercritical to subcritical in Table 2.

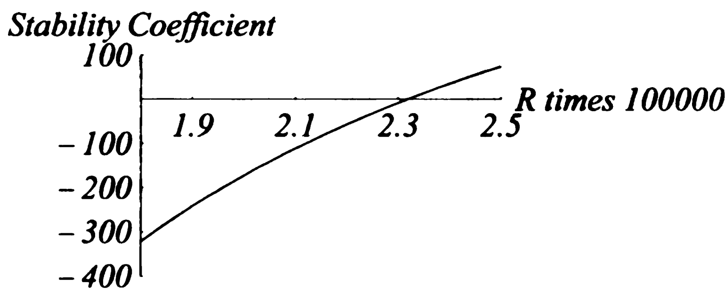

Figure 19: Plot of R vs. the stability coefficient when $Le = 1$ and $T_a = 3$

10 Conclusions

We organize our conclusions by examining different ranges of T_a and explaining the transformation of the steady state solutions, the stability, and the Hopf bifurcations as R increases.

When $T_a \leq 1$ and R increases from zero, the back of the S-curve pushes down to T_0 and no islands form. When $T_a < .95$, the upper branch remains stable for all values of R. When $.95 \leq T_a \leq 1$, the beginning of the upper branch becomes unstable and an interval of unstable steady solutions appears on the back of the transformed S-curve as R increases. The Hopf bifurcation point at the right endpoint of the interval of unstable steady solutions on the back of the curve is always supercritical. In addition, the left endpoint of the interval of unstable steady states on the back of the curve is a supercritical Hopf bifurcation point when it first appears and the Hopf bifurcation point at the beginning of the upper branch is subcritical when it first appears. When $.95 \leq T_a < .98$, the Hopf bifurcation point at the left endpoint of the interval of unstable steady states remains supercritical as long as it appears. The Hopf bifurcation at the beginning of the upper branch becomes supercritical right before it disappears. When $.98 \leq T_a \leq 1$, the Hopf bifurcation point at the beginning of the upper branch is always subcritical. The Hopf bifurcation point at the left endpoint of the interval of unstable steady solutions becomes subcritical right before it disappears.

When $T_a > 1$ and R is small, the transformation of the S-curve is similar to the $T_a \leq 1$ case. As the $T_a \leq 1$ case, an interval of unstable steady solutions appears on the back of the transformed S-curve and the Hopf bifurcation points at the ends of the interval are supercritical when they first appear. After the appearance of this interval, the transformed S-curve breaks, creating an island and a lower branch. The lower branch is completely stable. The island contains a Hopf bifurcation point. When $1 < T_a \leq 1.2$, the bifurcation points change from supercritical to subcritical before the island forms. When $T_a > 1.2$, the bifurcation point on the island is supercritical when the island forms, but changes to subcritical as R increases further. As R increases, the island shrinks and the lower branch pushes down closer to T_0. At a certain value of R, as the island shrinks, the island becomes a collection of completely unstable steady solutions and then disappears. The stable lower branch persists.

The method established in Section 8 made determining the type of Hopf bifurcation a straightforward process. Although we can often correctly identify whether a Hopf bifurcation point is subcritical or supercritical by perturbing unstable steady solutions near the Hopf bifurcation point, it is time consuming and at times it may appear as though a supercritical Hopf bifurcation point is subcritical. By finding the stability coefficient, we clearly establish when stable periodic solutions exist.

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