PATTERNS IN SET PARTITIONS AND RESTRICTED GROWTH FUNCTIONS

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ABSTRACT

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In this thesis we study two related notions of pattern avoidance. One is in set partitions σ of $[n] = \{1, 2 \ldots, n\}$ which are families of nonempty subsets $B_1, \ldots, B_k$ whose disjoint union is $[n]$, written $\sigma = B_1/\ldots/B_k \vdash S$. The other is in restricted growth functions (RGFs) which are words $w = a_1a_2\ldots a_n$ of positive integers such that $a_1 = 1$ and $a_i \leq 1+\max\{a_1, \ldots, a_{i-1}\}$ for $i > 1$. The concept of pattern avoidance is built on a standardization map st on an object O, be it a set partition or RGF, where $\mathrm{st}(O)$ is obtained by replacing the ith smallest integer with i. A set partition σ will contain a pattern π if σ has a subpartition which standardizes to π, and when σ does not contain π we say σ avoids π. Pattern avoidance in RGFs is defined similarly. This work is the study of the generating functions for Wachs and White's statistics on RGFs over the avoidance classes of set partitions and RGFs.

The first half of the thesis concentrates on set partitions. We characterize most of these generating functions for avoiding single and multiple set partitions of length three, and we highlight the longer pattern 14/2/3, a partition of [4], as its avoidance class has a particularly nice characterization. The second half of this thesis will present our results about the generating functions for RGF patterns, starting with those of length three. We find many equidistribution properties which we prove using integer partitions and the hook decomposition of Young diagrams. For certain patterns of any length we provide a recursive formula for their generating functions including the pattern $12\ldots k$. We finish this presentation by discussing the patterns 1212 and 1221 which have connections to noncrossing and nonnesting partitions, respectfully. We find connections to two-colored Motzkin paths and define explicit bijections between these combinatorial objects.

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1 Introduction

The work in this thesis finds itself at the intersection of two lines of research: one on pattern avoidance and the other on generating functions. At this intersection, mathematicians have found many interesting results and unexpected connections between previously unrelated objects. Below, we write about the early results which inspired research in these fields, summarize our own research, and present some of the unexpected connections we found between combinatorial objects.

Pattern avoidance started not in the field of mathematics but with its founder, Donald Knuth [Knu73], in the field of computer science. He investigated stack-sortable permutations and found that the pattern 231 was the only obstruction. For example, the permutation 416325 is not stack-sortable because it contains the subword 462 whose elements are in the same relative order as 231 and so counts as an occurrence of that pattern. Whereas 216354 is stack-sortable since it avoids 231 in that no three-element subsequence has its elements in this relative order. Additionally, he found that the number of 231-avoiding permutations length n is $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number named after the Belgian mathematician Eugène Charles Catalan in the 1800s. These are numbers of great interest and have over two hundred combinatorial interpretations which can be found in Richard Stanley's book [Sta99] and his addendum [Sta]. Among the list is another of Knuth's findings which is that C_n also counts the number of length n permutations which avoid any single pattern of length three.

Mathematicians then began to count and characterize permutations which avoid longer patterns and multiple patterns. In the 1980s Richard Stanley and Herbert Wilf independently formulated what came to be known as the Stanley-Wilf conjecture, which proposed that if $\mathfrak{S}_n(\pi)$ is the collection of length n permutations which avoid a permutation pattern π then $\lim_{n\to\infty} \sqrt[n]{\#\mathfrak{S}_n(\pi)}$ is a real number depending on π, where $\#\mathfrak{S}_n(\pi)$ is the cardinality of the set. This is in stark contrast to the full symmetric group which has a much higher growth rate. The conjecture has since been proved in 2004 by Adam Marcus and Gábor Tardos in [MT04]. The subject grew as mathematicians considered subclasses of permutations and defined various kinds of pattern avoidance for other combinatorial objects including, but not restricted to, set partitions, restricted growth functions, even/odd permutations, and involutions ([Sag10],[Kla96],[JM08],[SS85]). In our preliminary chapter 2, we define all relevant terms for set partitions and restricted growth functions.

The second line of work on which this thesis builds is the study of generating functions. Though not the beginning of the subject, the early 1900s saw British Major Percy MacMahon prove a result about the equality of two generating functions using two statistics on permutations [Mac78]. One of these statistics is the major index, or maj named after his title, which is defined on a permutation $\pi = \pi(1)\ldots\pi(n)$ to be

$$\operatorname{maj}(\pi) = \sum_{\pi(i)>\pi(i+1)} i$$

and the other is the number of inversions, or inv, defined as

$$\operatorname{inv}(\pi) = \#\{(i,j) :\ i<j,\ \pi(i)>\pi(j)\}.$$

He found that the generating functions over all permutations length n using inv and maj were the same,

$$\sum_{\pi\in\mathfrak{S}_n} q^{\operatorname{maj}(\pi)} = \sum_{\pi\in\mathfrak{S}_n} q^{\operatorname{inv}(\pi)},$$

and any other statistic with an equivalent distribution (that is, an equal generating function) is called Mahonian. This result was later proven bijectivly by Dominique Foata [Foa68]. These functions are equal to the Gaussian q-analogue for $n!$, equation (10). See Stanley's book [Sta97] for details.

One can combine these two lines of work by considering generating functions for various statistics over an avoidance class of permutations rather than over the full symmetric group. Dokos et. al. [DDJ$^+$12] were the first to make a comprehensive study of the statistics maj and inv over avoidance classes of length three patterns and found connections to lattice paths, integer partitions, and Foata's second fundamental bijection.

This thesis considers pattern avoidance in set partitions and restricted growth functions which will be defined shortly. On these two sets of objects, which are in bijection with each other, we define two notions of pattern avoidance. The generating functions we consider use Wachs and White's [WW91] four fundamental statistics on restricted growth functions. Throughout the paper we introduce Gaussian polynomials, Young diagrams, integer partitions, and two-colored Motzkin paths since these objects will be essential for some proofs.

The rest of this thesis is structured as follows. We start by presenting our results about set partitions in Chapter 3. Our study fully characterizes the generating functions for all four of Wachs and White's statistics [WW91] over the avoidance classes for single and multiple patterns of length three except for the single pattern 123 which we only partially characterize. The longer pattern 14/2/3, a partition of [4], has its own section 3.3 due to its particularly nice characterization.

We then proceed to presenting our results about RGFs in Chapter 4. We note at the end of the preliminary section 2.2.2 that pattern avoidance for set partitions and RGFs is the same for some patterns. In Section 4.1 we characterize generating function for the remaining length three patterns $v = 112, 122$. We find many equidistribution properties which we prove using integer partitions and the hook decomposition of Young diagrams. In Section 4.2 we provide a recursive formula which can generate functions for certain RGF patterns of any length including the pattern $12 \ldots k$. We finish this presentation by discussing the patterns 1212 and 1221 in Sections 4.3 and 4.4, which have connections to noncrossing and nonnesting partitions, respectfully. We further present their connection to two-colored Motzkin paths by defining explicit bijections between these combinatorial objects.

2 Preliminaries

2.1 Set partitions and restricted growth functions

Let us begin by defining our terms. Consider a finite set S. A *set partition* σ of S is a family of nonempty subsets $B_1, \ldots, B_k$ whose disjoint union is S, written $\sigma = B_1/\ldots/B_k \vdash S$. The B_i are called *blocks* and we will usually suppress the set braces and commas in each block for readability. We will be particularly interested in set partitions of $[n] := \{1, 2, \ldots, n\}$ and will use the notation

$$\Pi_n = \{\sigma \ : \ \sigma \vdash [n]\}.$$

To illustrate $\sigma = 145/2/3 \vdash [5]$. If $T \subseteq S$ and $\sigma = B_1/\ldots/B_k \vdash S$ then there is a corresponding *subpartition* $\sigma' \vdash T$ whose blocks are the nonempty intersections $B_i \cap T$. To continue our example, if $T = \{2, 4, 5\}$ then we get the subpartition $\sigma' = 2/45 \vdash T$.

Our other objects of interest are restricted growth functions. A sequence $w = a_1 a_2 \ldots a_n$ of positive integers is a *restricted growth function* (RGF) if it satisfies the conditions

1. $a_1 = 1$, and
2. for $i \geq 2$ we have
$$a_i \leq 1 + \max\{a_1, \ldots, a_{i-1}\}. \tag{1}$$

For example, $w = 11213224$ is an RGF, but $w = 11214322$ is not since $4 > 1+\max\{1, 1, 2, 1\}$. The number of elements of w is called its *length* and denoted $|w|$. Define

$$R_n = \{w \ : \ w \text{ is an RGF of length } n\}.$$

There is a bijection between RGFs and set partitions. To describe it, we will henceforth write all $\sigma = B_1/B_2/\ldots/B_k \vdash [n]$ in *standard form* which means that

$$\min B_1 < \min B_2 < \cdots < \min B_k.$$

Note that this implies $\min B_1 = 1$. Given $\sigma = B_1/\ldots/B_k \vdash [n]$ in standard form, we construct an associated word $w(\sigma) = a_1 a_2 \ldots a_n$ where

$$a_i = j \text{ if and only if } i \in B_j.$$

Returning to our running example, we have $w(145/2/3) = 12311$. More generally, for any set P of set partitions, we let $w(P)$ denote the set of $w(\sigma)$ for $\sigma \in P$.

Proposition 2.1. *If a partition $\sigma \vdash [n]$ is written in standard form, then $w(\sigma)$ is an RGF and the map $\sigma \mapsto w(\sigma)$ is a bijection $\Pi_n \to R_n$.* □

2.2 Pattern avoidance

2.2.1 Pattern avoidance in set partitions and RGFs

The concept of pattern is built on the standardization map. Let O be an object with labels which are positive integers. The *standardization* of O, $\mathrm{st}(O)$, is obtained by replacing all occurrences of the smallest label in O by 1, all occurrences of the next smallest by 2, and so on. Say that $\sigma \vdash [n]$ *contains* π *as a pattern* if it contains a subpartition σ' such that $\mathrm{st}(\sigma') = \pi$. In this case σ' is called an *occurrence* or *copy* of π in σ. Otherwise, we say that σ *avoids* π and let

$$\Pi_n(\pi) = \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids } \pi\}.$$

In our running example, $\sigma = 145/2/3$ contains $\pi = 1/23$ since $\mathrm{st}(2/45) = 1/23$. But σ avoids $12/3$ because if one takes any two elements from the first block of σ then it is impossible to find an element from another block bigger than both of them. Klazar [Kla96, Kla00a, Kla00b] was the first to study this approach to set partition patterns.

We can now define patterns in terms of RGFs. Given RGFs v, w we call v a *pattern* in w if there is a subword w' of w with $\mathrm{st}(w') = v$. By subword we mean any subsequence $a_{i_1} a_{i_2} \ldots a_{i_k}$ of $w = a_1 \ldots a_n$ where the i_j's are increasing and not necessarily consecutive.

The use of the terms "occurrence," "copy," and "avoids" in this setting are the same as for set partitions. Given and RGF v we let

$$R_n(v) = \{w \in R_n \ : \ w \text{ avoids } v\}.$$

As before, consider $w = w(145/2/3) = 12311$. Then w contains $v = 121$ because either of the subwords 121 or 131 of w standardize to v. However, w avoids $v = 122$ since the only repeated elements of w are ones. Note that this is in contrast to the fact that 145/2/3 contains 1/23 where $w(1/23) = 122$. Given a set S of set partitions we write

$$w(S) = \{w(s) : \ s \in S\}.$$

The next result connects these two notions of pattern avoidance.

Proposition 2.2. *Suppose that partitions π and σ have RGFs $v = w(\pi)$ and $w = w(\sigma)$. If w contains v then σ contains π, but not necessarily conversely. Equivalently, we have $R_n(v) \supseteq w(\Pi_n(\pi))$.* □

2.2.2 Avoidance classes and cardinalities for patterns with three elements

Sagan [Sag10] described the set partitions in $\Pi_n(\pi)$ for each $\pi \in \Pi_3$. We include his result translated into the language of RGFs. For a proof please see [Sag10]. To state his result, we will need some definitions. The *initial run* of an RGF w is the longest prefix of the form $12\ldots m$. Write a^l to indicate a string of l copies of the integer a. Call the word w *layered* if it has the form $w = 1^{n_1}2^{n_2}\ldots m^{n_m}$, equivalently, if it is weakly increasing. For a partition pattern π we will sometimes write $R_n(\pi)$ for $w(\Pi_n(\pi))$.

Theorem 2.3 ([Sag10])**.** *For $n \geq 1$, we have the following characterizations.*

1. $R_n(1/2/3) = \{w \in R_n \ : \ w$ *consists of only* 1*s and* 2*s*$\}$.

2. $R_n(1/23) = \{w \in R_n \ : \ w$ *is obtained by inserting a single* 1 *into a word*

of the form $1^l 23\ldots m$ *for some* $l, m \geq 1\big\}$.

3. $R_n(13/2) = \{w \in R_n \ :\ w$ *is layered*$\}$.

4. $R_n(12/3) = \{w \in R_n \ :\ w$ *has initial run* $1\ldots m$ *and* $a_{m+1} = \cdots = a_n \leq m\}$.

5. $R_n(123) = \{w \in R_n \ :\ w$ *has no element repeated more than twice*$\}$. □

Using these characterizations, it is a simple matter to find the cardinalities of the avoidance classes.

Corollary 2.4 ([Sag10])**.** *We have the following cardinalities.*

$$\#\Pi_n(1/2/3) = \#\Pi_n(13/2) = 2^{n-1},$$

$$\#\Pi_n(1/23) = \#\Pi_n(12/3) = 1 + \binom{n}{2},$$

$$\#\Pi_n(123) = \sum_{k\geq 0} \binom{n}{2k} (2k)!!$$

where $(2k)!! = (1)(3)(5)\ldots(2k-1)$. □

Sagan also described the sets $R_n(v)$ for all $v \in R_3$, and the proof can be found in his paper [Sag10].

Theorem 2.5 ([Sag10])**.** *We have the following characterizations.*

1. $R_n(111) = \{w \in R_n \ :\$ *every element of* w *appears at most twice*$\}$.

2. $R_n(112) = \{w \in R_n \ :\ w$ *has initial run* $12\ldots m$ *and* $m \geq a_{m+1} \geq a_{m+2} \geq \cdots \geq a_n\}$.

3. $R_n(121) = \{w \in R_n \ :\ w$ *is layered*$\}$.

4. $R_n(122) = \{w \in R_n \ :\$ *every element* $j \geq 2$ *of* w *appears only once*$\}$.

5. $R_n(123) = \{w \in R_n \ :\ w$ *contains only* $1s$ *and* $2s\}$. □

Using this result, it is not hard to compute the cardinalities of the classes.

Corollary 2.6 ([Sag10]). *We have*

$$\#R_n(112) = \#R_n(121) = \#R_n(122) = \#R_n(123) = 2^{n-1}$$

and

$$\#R_n(111) = \sum_{i \geq 0} \binom{n}{2i} (2i)!!$$

where $(2i)!! = 1 \cdot 3 \cdot 5 \cdot \ \ldots \ \cdot (2i-1)$. □

As noted in Proposition 2.2, if $v = w(\pi)$ then we always have $R_n(v) \supseteq w(\Pi_n(\pi))$. But for certain π we have equality. In particular, as shown in [Sag10], this is true for $\pi = 123, 13/2, 1/2/3$ and the corresponding $v = 111, 121, 123$.

2.3 Statistics and generating functions

Our object, in part, is to prove generalizations of the formulae in corollaries 2.4 and 2.6 using the statistics of Wachs and White and their generating functions. They defined four statistics on RGFs denoted lb, ls, rb, and rs where the letters l, r, b, and s stand for left, right, bigger, and smaller, respectively. We will explicitly define the lb statistic and the others are defined analogously. Given a word $w = a_1 a_2 \ldots a_n$, let

$$\mathrm{lb}(a_j) = \#\{a_i \ : \ i < j \text{ and } a_i > a_j\}.$$

Otherwise put, $\mathrm{lb}(a_j)$ counts the number of integers which are to the left of a_j in w and bigger than a_j. Note that multiple copies of the same integer which is left of and bigger than a_j are only counted once. Note, also, that $\mathrm{lb}(a_j)$ also depends on w and not just the value of a_j. But context will ensure that there is no confusion. For an example, if $w = 12332412$ then for $a_5 = 2$ we have $\mathrm{lb}(a_5) = 1$ since three is the only larger integer which occurs before the two. For w itself, define

$$\mathrm{lb}(w) = \mathrm{lb}(a_1) + \mathrm{lb}(a_2) + \cdots + \mathrm{lb}(a_n).$$

Continuing our example,

$$\mathrm{lb}(12332412) = 0 + 0 + 0 + 0 + 1 + 0 + 3 + 2 = 6.$$

Finally, given an RGF, v, we consider the generating function

$$\mathrm{LB}_n(v) = \mathrm{LB}_n(v; q) = \sum_{w \in R_n(w)} q^{\mathrm{lb}(w)}$$

and similarly for the other three statistics. Sometimes we will be able to prove things about multivariate generating functions such as

$$F_n(v) = F_n(v; q, r, s, t) = \sum_{w \in R_n(v)} q^{\mathrm{lb}(w)} r^{\mathrm{ls}(w)} s^{\mathrm{rb}(w)} t^{\mathrm{rs}(w)}.$$

Similarly, we can define Wachs and White's statistics for set partitions where for a set partition $\pi \vdash [n]$ we define lb of π to be equal to $\mathrm{lb}(w(\pi))$. To simplify notation, we will write $\mathrm{lb}(\pi)$ for the more cumbersome $\mathrm{lb}(w(\pi))$. We similarly define generating functions

$$\mathrm{LB}_n(\pi) = \mathrm{LB}_n(\pi; q) = \sum_{\sigma \in \Pi_n(\pi)} q^{\mathrm{lb}(\sigma)}$$

and analogously for the other statistics. Again, often, we will even be able to compute the multivariate generating function

$$F_n(\pi) = F_n(\pi; q, r, s, t) = \sum_{\sigma \in \Pi_n(\pi)} q^{\mathrm{lb}(\sigma)} r^{\mathrm{ls}(\sigma)} s^{\mathrm{rb}(\sigma)} t^{\mathrm{rs}(\sigma)}.$$

Though we use similar notation for both set partitions and RGFS there should be no confusion since either π or v will be inside the parentheses which will indicate the intended function.

3 Set partitions

In this chapter we present our results about set partitions. We characterize the generating functions for all four Wachs and White statistics on all length three patterns except 123 where we have only partial results. We find many equidistribution properties of the form $\mathrm{LB}_n(\pi) = \mathrm{RS}_n(\pi)$ for $\pi \in \{1/2/3, 1/23, 13/2, 14/2/3\}$ and $\mathrm{LS}_n(\pi) = \mathrm{RB}_n(\pi)$ for $\pi \in \{1/2/3, 13/2\}$. Also, we find equidistribution between avoidance classes using different patterns such as $\mathrm{LB}_n(1/23) = \mathrm{RS}_n(12/3)$ and $\mathrm{LS}_n(1/23) = \mathrm{RB}_n(12/3)$. This is a theme we also find when studying RGFs in Chapter 4. The characterization for partitions in the avoidance class of 14/2/3 have a particularly nice form which aids us in showing that $\mathrm{LB}_n(14/2/3, \pi) = \mathrm{RS}_n(14/2/3, \pi)$ for $\pi \in \{13/2/4, 1/2/\ldots/t\}$.

3.1 Single patterns of length three

3.1.1 The pattern 1/2/3

We first consider the set partition 1/2/3. We begin by presenting the four-variable generating function from which we derive the generating functions associated with each individual statistic.

Theorem 3.1. *We have*

$$F_n(1/2/3) = 1 + \sum_{l=1}^{n-1} r^{n-l}s^l + \sum_{l=2}^{n-1}\sum_{k=0}^{n-l-1}\sum_{i,j\geq 1}\binom{n-i-j-k-2}{l-i-j} q^{l-i}r^{n-l}s^{l-\delta_{k,0}j}t^{n-l-k}$$

where $\delta_{k,0}$ is the Kronecker delta function.

Proof. By Theorem 2.3, any word $w \in R_n(1/2/3)$ is composed solely of ones and twos. Let l denote the number of ones in w. If such a word is weakly increasing, it is easy to see that these words contribute

$$1 + \sum_{l=1}^{n-1} r^{n-l}s^l$$

to the generating function.

Otherwise, let w have at least one descent and l ones. We can see that the word w has the form $1^i w' 1^j 2^k$, where $i, j \geq 1$, the subword w' begins and ends with a two, and $0 \leq k \leq n - l - 1$.

For such w the lb statistic is given by the number of ones after the first two, that is, by the number of ones not in 1^i. Thus, $\mathrm{lb}(w) = l - i$. The ls statistic is given by the total number of twos in w, namely $n - l$. For the rb statistic, if k is non-zero, then each one in w contributes to the statistic. Otherwise, only the ones that are not in 1^j contribute. Combining the two cases gives $\mathrm{rb}(w) = l - \delta_{k,0} j$. Finally, the rs statistic is given by the number of twos in w', namely $n - l - k$. Putting all four statistics together produces

$$q^{\mathrm{lb}(w)} r^{\mathrm{ls}(w)} s^{\mathrm{rb}(w)} t^{\mathrm{rs}(w)} = q^{l-i} r^{n-l} s^{l-\delta_{k,0} j} t^{n-l-k}.$$

Choosing the number of ways of arranging the ones in w' gives a coefficient of

$$\binom{n-i-j-k-2}{l-i-j}.$$

Summing over i, j, k, l and combining the cases gives our desired polynomial. □

The equations in the following corollary can be derived either by specialization of the four-variable generating function in Theorem 3.1 and standard hypergeometric series techniques or by using the ideas in the proof of the previous result and ignoring the other three statistics.

Corollary 3.2. *We have*

$$\mathrm{LB}_n(1/2/3) = \mathrm{RS}_n(1/2/3) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k,$$

and

$$\mathrm{LS}_n(1/2/3) = \mathrm{RB}_n(1/2/3) = (r+1)^{n-1}.$$ □

In view of the preceding corollary, it would be nice to find explicit bijections $\phi : R_n(1/2/3) \to R_n(1/2/3)$ and $\psi : R_n(1/2/3) \to R_n(1/2/3)$ such that ϕ takes lb to rs and ψ takes ls to rb. In the next two propositions, we present such bijections.

Proposition 3.3. *There exists an explicit bijection $\phi : R_n(1/2/3) \to R_n(1/2/3)$ such that for $v \in R_n(1/2/3)$,*

$$\mathrm{lb}(v) = \mathrm{rs}(\phi(v)).$$

Proof. Let $v = a_1 a_2 \ldots a_n \in R_n(1/2/3)$. Define

$$\phi(v) = a_1(3 - a_n)(3 - a_{n-1}) \ldots (3 - a_3)(3 - a_2).$$

Because $v \in R_n(1/2/3)$, by Theorem 2.3, it must be composed of only ones and twos and begin with a one. It is clear that $\phi(v)$ has the same form, so ϕ is well defined. Also, ϕ is its own inverse and is therefore a bijection.

If $\mathrm{lb}(v) = k$, then v must contain a subword $v' = 21^k$ and no subword of the form 21^l, with $l > k$. In fact, this condition is clearly equivalent to $\mathrm{lb}(v) = k$. It follows that $\phi(v') = 2^k1$ is a subword of $\phi(v)$ and $\phi(v)$ has no subword 2^l1 with $l > k$. Therefore, $\mathrm{rs}(\phi(v)) = k = \mathrm{lb}(v)$, as desired. □

Proposition 3.4. *There exists an explicit bijection $\psi : R_n(1/2/3) \to R_n(1/2/3)$ such that for $v \in R_n(1/2/3)$,*

$$\mathrm{ls}(v) = \mathrm{rb}(\psi(v)).$$

Proof. Let $v \in R_n(1/2/3)$. If $v = 1^n$, then define $\psi(v) = v$. Clearly in this case $\mathrm{ls}(v) = 0 = \mathrm{rb}(v)$.

Otherwise, let $v = a_1 a_2 \ldots a_{i-1} a_i 1^{n-i}$ where $a_i = 2$ and $n - i \geq 0$. Define

$$\psi(v) = (3 - a_i)(3 - a_{i-1}) \ldots (3 - a_2)(3 - a_1)1^{n-i}.$$

The proof is now similar to that of Proposition 3.3, using the fact that the 1^{n-i} at the end of v contributes to neither ls or rb. $\square$

3.1.2 The pattern $1/23$

In this section we will determine $F_n(1/23)$, and thus the generating functions for all four statistics. We will find that lb and rs are equal for any $w \in R_n(1/23)$.

Theorem 3.5. *We have*

$$F_n(1/23) = (rs)^{\binom{n}{2}} + \sum_{m=1}^{n-1} \sum_{j=1}^{m} (qt)^{j-1} r^{\binom{m}{2}} s^{(n-m)(m-1)+m-j+\binom{m-1}{2}}.$$

Proof. If σ avoids $1/23$ we know from Theorem 2.3 that the associated RGF is obtained by inserting a single 1 into a word of the form $1^l 23 \dots m$ for some $l \geq 0$ and $m \geq 1$. If $l = 0$ then the inserted 1 must be at the beginning of the word in order for w to be a RGF, so $w = 12 \dots n$. If $l > 0$ then the inserted 1 can be inserted after j for any $1 \leq j \leq m$, and the maximal letter m satisfies $1 \leq m \leq n-1$. If w has maximal letter m and we insert the 1 after j then w is completely determined to be $1^{n-m} 23 \dots j 1 \dots m$.

In summary, either $w = 12 \dots n$ or w is determined by the choice of $1 \leq j \leq m$ and $1 \leq m \leq n-1$. If $w = 12 \dots n$ then $\mathrm{rb}(w) = \mathrm{ls}(w) = \binom{n}{2}$ and $\mathrm{lb}(w) = \mathrm{rs}(w) = 0$. For all other w we have the following:

1. $\mathrm{lb}(w) = j - 1$,

2. $\mathrm{ls}(w) = \binom{m}{2}$,

3. $\mathrm{rb}(w) = (n-m)(m-1) + m - j + \binom{m-1}{2}$, and

4. $\mathrm{rs}(w) = j - 1$.

1. Only the inserted 1 has elements which are left and bigger which are the numbers 2 through j. So $\mathrm{lb}(w) = j - 1$.

2. Since w is an RGF every letter i contributes $i-1$ to the ls giving a total of $\mathrm{ls}(w) = 1+\cdots+(m-1) = \binom{m}{2}$.

3. The first $n-m$ ones of w each have $m-1$ elements which are right and bigger, so they contribute $(n-m)(m-1)$ to the rb. The inserted 1 has $m-j$ letters which are right and bigger. Any element i such that $2 \le i \le m$ appears only once and contributes $m-i$ to the rb. This means we have an additional $(m-2)+\cdots+0 = \binom{m-1}{2}$. Hence $\mathrm{rb}(w) = (n-m)(m-1)+m-j+\binom{m-1}{2}$.

4. The only elements which have a number right and smaller are the elements 2 through j, and the only number which is right and smaller of these elements is the inserted 1. Hence $\mathrm{rs}(w) = j-1$.

Summing over all the valid values for m and j gives us our equality. $\square$

The following result can be quickly seen by specializing Theorem 3.5 or its demonstration, so we have omitted the proofs.

Corollary 3.6. *We have* $\mathrm{lb}(w) = \mathrm{rs}(w)$ *for all words* $w \in R_n(1/23)$ *and*

$$\mathrm{LB}_n(1/23) = \mathrm{RS}_n(1/23) = 1 + \sum_{j=1}^{n-1}(n-j)q^{j-1}.$$

Also

$$\mathrm{LS}_n(1/23) = r^{\binom{n}{2}} + \sum_{m=1}^{n-1} m r^{\binom{m}{2}},$$

and

$$\mathrm{RB}_n(1/23) = s^{\binom{n}{2}} + \sum_{m=1}^{n-1}\sum_{j=1}^{m} s^{(n-m)(m-1)+m-j+\binom{m-1}{2}}. \qquad \square$$

3.1.3 The pattern 13/2

In this section, we begin by evaluating the four-variable generating function $F_n(13/2)$. Goyt and Sagan [GS09] have previously proven a theorem regarding the single-variable generating functions for the ls and rb statistics, and we will adapt their map and proof to obtain the multi-variate generating function for 13/2. This generating function is closely related to integer partitions. A *reverse partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *of an integer* t is a weakly increasing sequence of positive integers such that $\sum_{i=1}^{k} \lambda_i = t$. The λ_i are called *parts.* Additionally, we will define an integer partition $n - \lambda = (n - \lambda_k, \ldots n - \lambda_2, n - \lambda_1)$. Let $|\lambda| = \sum_{i=1}^{k} \lambda_i$. We will denote by D_{n-1} the set of reverse integer partitions with distinct parts of size at most $n - 1$.

Theorem 3.7. *We have*

$$F_n(13/2) = \prod_{i=1}^{n-1} (1 + r^{n-i} s^i).$$

Proof. Suppose $w \in R_n(13/2)$. By Theorem 2.3, w is layered and so lb and rs are zero, resulting in no contribution to the generating function. For the other two statistics, since w is layered it has the form $w = 1^{n_1} 2^{n_2} \ldots m^{n_m}$ where m is the maximum element of w. Define $\phi : R_n(13/2) \to D_{n-1}$ by

$$\phi(w) = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$$

where $\lambda_j = \sum_{i=1}^{j} n_i$ for $1 \leq j \leq m - 1$. Note that since the n_j are positive, the λ_j are distinct, increasing, and less than n since the sum never includes n_m. Thus the map is well defined.

We now show that ϕ is a bijection by constructing its inverse. Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$, consider for $1 \leq j \leq m$, the differences $n_j = \lambda_j - \lambda_{j-1}$, where we define $\lambda_0 = 0$ and $\lambda_m = n$. It is easy to see that sending λ to $w = 1^{n_1} 2^{n_2} \ldots m^{n_m}$ is a well-defined inverse for ϕ.

We next claim that if $\phi(w) = \lambda$ then $\mathrm{rb}(w) = |\lambda|$. Indeed, from the form of w and λ we see that

$$\mathrm{rb}(w) = \sum_{i=1}^{m-1} n_i(m - i) = \sum_{j=1}^{m-1} \sum_{i=1}^{j} n_i = |\lambda|.$$

Similarly we obtain $\mathrm{ls}(w) = |n - \lambda|$. It follows that

$$F_n(13/2) = \sum_{\lambda \in D_{n-1}} r^{|n-\lambda|} s^{|\lambda|} = \prod_{i=1}^{n-1} (1 + r^{n-i} s^i)$$

as desired. $\square$

The generating function of each individual statistic is easy to obtain by specialization of Theorem 3.7 so we have omitted the proofs.

Corollary 3.8 ([GS09]). *We have*

$$\mathrm{LB}_n(13/2) = 2^{n-1} = \mathrm{RS}_n(13/2)$$

and

$$\mathrm{LS}_n(13/2) = \prod_{i=1}^{n-1}(1+q^i) = \mathrm{RB}_n(13/2). \qquad \square$$

3.1.4 The pattern 12/3

In this section, we determine $F_n(12/3)$. The other polynomials associated with 12/3 are obtained as corollaries. We find this avoidance class interesting because it leads to a connection with number theory.

Theorem 3.9. *We have*

$$F_n(12/3) = (rs)^{\binom{n}{2}} + \sum_{m=1}^{n-1}\sum_{i=1}^{m} q^{(n-m)(m-i)} r^{\binom{m}{2}+(n-m)(i-1)} s^{\binom{m}{2}} t^{m-i}. \tag{2}$$

Proof. By Theorem 2.3, the elements of $R_n(12/3)$ are the words of the form

$$w = 123\ldots m i^{n-m}$$

where $i \le m$. If $w = 123\ldots n$ then $\mathrm{ls}(w) = \mathrm{rb}(w) = \binom{n}{2}$ and $\mathrm{lb}(w) = \mathrm{rs}(w) = 0$. Otherwise $m < n$. In this case, we will show the following:

1. $\mathrm{lb}(w) = (n-m)(m-i)$,

2. $\mathrm{ls}(w) = \binom{m}{2} + (n-m)(i-1)$,

3. $\mathrm{rb}(w) = \binom{m}{2}$,

4. $\mathrm{rs}(w) = m - i$.

1. There are $n - m$ copies of i in w after the initial run and these are the only elements contributing to lb. Each of these i's has the elements $(i+1)(i+2)\ldots m$ to its left that are bigger than it. So $\mathrm{lb}(i) = m - i$ for all such i and $\mathrm{lb}(w) = (n-m)(m-i)$.

2. Each element w_j of w has $\mathrm{ls}(w_j) = w_j - 1$ by condition (1). Using this and the form of w easily yields the desired equality.

3. This is similar to the previous case, noting that only the initial run of w contributes to rb.

4. We can see that the only elements w_j with $\mathrm{rs}(w_j) > 0$ will be those in the initial run such that $w_j > i$. These are precisely the elements $(i+1)(i+2)\ldots m$ and each element has exactly one element to its right that is smaller than it. So $\mathrm{rs}(w) = m - i$.

Summing over the valid values of m and i, we have (2). □

The next corollary follows easily by specialization of (3.9).

Corollary 3.10. *We have*

$$\mathrm{LS}_n(12/3) = r^{\binom{n}{2}} + \sum_{m=1}^{n-1} \sum_{i=1}^{m} r^{\binom{m}{2}+(n-m)(i-1)},$$

and

$$\mathrm{RB}_n(12/3) = s^{\binom{n}{2}} + \sum_{m=1}^{n-1} m s^{\binom{m}{2}},$$

as well as

$$\mathrm{RS}_n(12/3) = 1 + \sum_{k=0}^{n-2} (n-k-1)t^k.$$ □

The coefficients of $\mathrm{LB}_n(12/3)$ have an interesting interpretation.

Proposition 3.11. *We have*

$$\mathrm{LB}_n(12/3) = \sum_{k=0}^{\lfloor (n-1)^2/4 \rfloor} D_k q^k, \tag{3}$$

where $D_k = \#\{d \geq 1 : d \mid k \text{ and } d + \frac{k}{d} + 1 \leq n\}$.

Proof. Set $r = s = t = 1$ in (2). We begin by showing the degree of $\mathrm{LB}_n(12/3)$ is $\lfloor (n-1)^2/4 \rfloor$. If $w \in R_n(12/3)$ then, by Theorem 2.3, we have $w = 12\ldots mi^{n-m}$ for some m and $i \leq m$.

In order to maximize the $\mathrm{lb}(w)$, we can assume $i = 1$. So, using the formula for $\mathrm{lb}(w)$ derived in the proof of Theorem 3.9, we must maximize $(n-m)(m-1)$. We take the derivative with respect to m and set the equation equal to zero to obtain $n - 2m + 1 = 0$ and $m = \frac{n+1}{2}$. To get integer values of m, we obtain

$$\begin{cases} m = \frac{n+1}{2} \text{ if } n \text{ is odd,} \\ m = \left\lceil \frac{n+1}{2} \right\rceil \text{ or } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ if } n \text{ is even.} \end{cases} \tag{4}$$

In either case, the maximum value of lb is $\lfloor (n-1)^2/4 \rfloor$.

We now show the coefficient of q^k is D_k. As before, let $w = 123\ldots mi^{n-m}$ be a word associated with a set partition that avoids 12/3 and let $\mathrm{lb}(w) = k$. If we let $d = n - m$ be the number of i's, it is clear that $\mathrm{lb}(w) = d(m-i) = k$ and therefore, $m - i = \frac{k}{d}$. Because w must be of length n, we now must determine which divisors d of k are valid. Each of the d trailing i's has $\frac{k}{d}$ elements to its left and bigger. Because $i \geq 1$, the leading one cannot be such an element. Thus in order for w to be of length n we must have $d + \frac{k}{d} + 1 \leq n$. $\square$

The above formulation of $\mathrm{LB}_n(12/3)$ leads to the following corollary, showing a connection to number theory.

Corollary 3.12. *When $k \leq n-2$, we have $D_k = \tau(k)$, the number-theoretic function which counts the divisors of k.*

Proof. We show that if $k \leq n-2$ then all positive divisors d of k are valid. We know that $d + \frac{k}{d} \leq k+1$ because $d = 1$ and $d = k$ are the divisors of k which maximize $d + \frac{k}{d}$. Thus, we have $d + \frac{k}{d} + 1 \leq k + 2 \leq n$. Therefore every positive divisor of k satisfies the inequality in the definition of D_k, and this implies $D_k = \tau(k)$. □

For our final result of this section, we provide two interesting relationships between the avoidance classes $\Pi(1/23)$ and $\Pi(12/3)$.

Proposition 3.13. *For $n \geq 0$, we have the following equalities:*

$$\mathrm{LB}_n(1/23) = \mathrm{RS}_n(12/3),$$
$$\mathrm{LS}_n(1/23) = \mathrm{RB}_n(12/3).$$

Proof. We will prove this theorem by providing a bijection that maps from $R_n(1/23)$ to $R_n(12/3)$. This bijection will interchange the lb and rs statistics, as well as the ls and rb statistics. Let w be an element of $R_n(1/23)$. By Theorem 2.3, we know that w is of the form $1^l 23 \ldots m$, with possibly a single one inserted. Let j be the number of ones in w, and let i be the index of the rightmost one in w. We define $\phi : R_n(1/23) \mapsto R_n(12/3)$ as

$$\phi(w) = 123 \ldots (n-j+1)(n-i+1)^{j-1}.$$

From the characterization of $R_n(12/3)$ provided in Theorem 2.3, it follows that $\phi(w)$ is indeed contained in $R_n(12/3)$. Furthermore, by Corollary 2.4 we know that $\#R_n(1/23) = \#R_n(12/3)$. It is also immediate that ϕ is injective, which then gives that ϕ is a bijection.

Now we show that ϕ takes the lb statistic to the rs statistic. First, note that if w is a member of $R_n(1/23)$ with $\mathrm{lb}(w) = 0$, then w must be of the form

$$w = 1^l 23 \ldots (n-l+1),$$

for some l with $1 \leq l \leq n$. In this case $i = j = l$. Therefore when we apply ϕ, we are left with

$$\phi(w) = 123\dots(n-l+1)(n-l+1)^{l-1},$$

and it follows that $\mathrm{rs}(\phi(w)) = 0$. Now consider the case where $\mathrm{lb}(w) = k$, for $k > 0$. In this instance, w must be of the form

$$w = 1^l 23\dots(k+1)1(k+2)\dots(n-l).$$

It follows that the rightmost one in w has index $l+k+1$, and that there are $l+1$ ones in w. Thus when we apply ϕ, we get

$$\phi(w) = 123\dots(n-l)(n-l-k)^l,$$

which satisfies $\mathrm{rs}(\phi(w)) = k$.

Finally, we show that ϕ takes the ls statistic to the rb statistic. From the proof of Theorem 3.5, we know that if $w \in R_n(1/23)$ with maximum value m, then $\mathrm{ls}(w) = \binom{m}{2}$. Similarly, from the proof of Theorem 3.9, if $w' \in R_n(12/3)$ with maximum value m', then $\mathrm{rb}(w) = \binom{m'}{2}$. Since ϕ preserves maximum values, it follows that $\mathrm{ls}(w) = \mathrm{rb}(\phi(w))$. $\square$

3.1.5 The pattern 123

The reader will have noticed that for the other four set partitions of [3], we provided a 4-variable generating function describing all four statistics on the avoidance class of those partitions. The pattern 123, however, is much more difficult to deal with and so we will content ourselves with results about the individual statistics. Note that Theorem 4.18 gives us an alternative method for computing $\mathrm{LS}_n(123)$ and $\mathrm{RS}_n(123)$ (using the corresponding RGF 111) via recursion. We will start with the left-smaller statistic.

Theorem 3.14. *We have*

$$\mathrm{LS}_n(123) = \sum_{m=\lceil n/2 \rceil}^{n} \left[\sum_{L} \left(\prod_{g=1}^{n-m} (m - \ell_g + g) \right) q^{\binom{m}{2} + \sum\limits_{\ell \in L} (\ell - 1)} \right] \tag{5}$$

where the inner sum is over all subsets $L = \{\ell_1, \ell_2, \ldots, \ell_{n-m}\}$ *of* $[m]$ *with* $\ell_1 > \cdots > \ell_{n-m}$.

Proof. We start by noting that if the maximum value of an element of a word is m, then there must be $n - m$ repeated elements in the word, i.e., elements i that appear after the initial occurrence of i. The bounds on our outer sum are given by the largest possible value of m being n, and the smallest possible value of m being $\lceil n/2 \rceil$, since we can repeat each element a maximum of two times. We will now build our word w by starting with a base sequence $12 \ldots m$ and adding in repeated elements. The base sequence will contribute $1 + 2 + \cdots + (m-1) = \binom{m}{2}$ to $\mathrm{ls}(w)$. Let L be the set of repeated elements we want to add to w. Then L must contain $n - m$ elements from $[m]$, and since w can have no element appear more than twice, L can have no element appear more than once. For each element $\ell \in L$ that we add to our base sequence, we will increase $\mathrm{ls}(w)$ by $\ell - 1$. So for any word w with maximum m formed in this way, we have $\mathrm{ls}(w) = \binom{m}{2} + \sum_{\ell \in L} (\ell - 1)$.

To find how many possible words can be so created, we start with our base sequence $12 \ldots m$, and build up our word by placing in the repeated elements from L one at a time. There are $m - (\ell_1 - 1)$ spots where we can place the largest repeated element, ℓ_1: anywhere after the original occurrence of ℓ_1. Then when we place our second repeated element, ℓ_2, we will have $m - (\ell_2 - 1) + 1$ spots, where the plus one comes from the extra space the first repeated element added in front of ℓ_2. In general, when we place ℓ_g we will have $m - (\ell_g - 1) + (g - 1) = m - \ell_g + g$ places to put it. The condition $\ell_1 > \cdots > \ell_{n-m}$ is used since it implies that regardless of where ℓ_i is placed, one will have the same number of choices for the placement of ℓ_{i+1}. Multiplying all these terms together and then summing over all possible subsets L of $[m]$ gives us the coefficient of q. Finally, summing over all possible maximums of the words in the avoidance class gives us equation (5). $\square$

For comparison, we include here the recursion obtained by specializing Theorem 4.18.

Corollary 3.15. *We have* $\mathrm{LS}_0(123) = \mathrm{LS}_1(123) = 1$ *and for* $n > 1$

$$\mathrm{LS}_n(123) = q^{n-1}\,\mathrm{LS}_{n-1}(123) + (n-1)q^{n-2}\,\mathrm{LS}_{n-2}(123) \qquad \square$$

We were only able to find explicit expressions for certain coefficients of the polynomials generated from other statistics. We will now look at the left-bigger statistic.

Theorem 3.16. *We have the following.*

1. *The degree of* $\mathrm{LB}_n(123)$ *is*
$$\left\lfloor \frac{n(n-1)}{6} \right\rfloor.$$

2. *The leading coefficient of* $\mathrm{LB}_n(123)$ *is*
$$\begin{cases} k! & \text{if } n = 3k \text{ or } 3k+1, \\ (k+2)k! & \text{if } n = 3k+2, \end{cases}$$
for some nonnegative integer k.

Proof. We will show that a word of the form $w = 12\ldots iw_{i+1}\ldots w_n$ with $w_{i+1},\ldots,w_n$ being a permutation of the interval $[1, n-i]$ will provide a maximum lb which is $\lfloor (n(n-1))/6 \rfloor$.

First we will prove that the elements after the initial run $12\ldots i$ must be less than or equal to i. Note that, by definition of the initial run, $w_{i+1} \le i$. Now suppose, towards a contradiction, that for some $j \in [i+2, n]$, there was some element $w_j > i$. Then, since w is an RGF, we must have $w_k = i+1$ for some $k \in [i+2, j]$. But by switching w_k and w_{i+1}, we would increase lb by at least one since $w_{i+1} \le i$. So if any element after the initial run is greater than i, lb is not maximum.

Next we will show that the elements after the initial run have to be exactly those in the interval $[1, n-i]$, up to reordering. Suppose towards contradiction there was some element $t \in [1, n-i]$ that did not appear in the sequence after the initial run, and instead there appeared some element $s \in [n-i+1, i]$. Then $\mathrm{lb}(s) = i - s$. But $\mathrm{lb}(t) = i - t$, and since $s > t$, it follows that $\mathrm{lb}(t) > \mathrm{lb}(s)$. Therefore, if we want to maximize lb, we must have the sequence after the initial run being exactly the interval $[1, n-i]$, up to reordering.

Now that we've established that our word is of the form $w = 12\ldots i w_{i+1} \ldots w_n$ with $w_{i+1}, \ldots, w_n$ being exactly those elements in the interval $[1, n-i]$, we simply need to maximize lb using some elementary calculus as follows

$$\begin{aligned}
\mathrm{lb}(w) &= (i - w_{i+1}) + (i - w_{i+2}) + \cdots + (i - w_n) \\
&= (i-1) + (i-2) + \cdots + (2i - n) \\
&= \frac{(4n+1)i - 3i^2 - n^2 - n}{2}. \qquad (6)
\end{aligned}$$

Considering i as a real variable and differentiating gives us the maximum value of $\mathrm{lb}(w)$ when $i = (4n+1)/6$. We must modify this slightly since we want i to be integral. Rounding i to the nearest integer gives

$$i = \begin{cases} \left\lfloor \frac{4n+1}{6} \right\rfloor & \text{if } n = 3k, \\ \left\lceil \frac{4n+1}{6} \right\rceil & \text{if } n = 3k+1, \\ \left\lfloor \frac{4n+1}{6} \right\rfloor \text{ or } \left\lceil \frac{4n+1}{6} \right\rceil & \text{if } n = 3k+2, \end{cases}$$

for some nonnegative integer k.

Plugging each value of n and i back into equation (6) gives us an lb of $\lfloor (n(n-1))/6 \rfloor$ in all cases. As we've mentioned before, the elements $w_{i+1}, \ldots, w_n$ must be exactly those in the interval $[1, n-i]$, but the ordering doesn't matter. This means the leading coefficient of $\mathrm{LB}_n(123)$ will be precisely the number of ways to permute the $n-i$ elements after the initial run. This gives us our second result. $\square$

Some of the following theorems will involve Fibonacci numbers. Recall that the nth

Fibonacci number F_n is defined recursively as

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 1$ and $F_1 = 1$.

Theorem 3.17. *We have the following coefficients.*

1. *The constant term of* $\mathrm{LB}_n(123)$ *is* F_n.

2. *The coefficient of* q *in* $\mathrm{LB}_n(123)$ *is* $(n-2)F_{n-2}$.

Proof. If $\mathrm{lb}(\sigma) = 0$, then $w = w(\sigma)$ must be layered. Let $L(n)$ be the set of layered words $w(\sigma)$ with $\sigma \in \Pi_n(123)$. It follows that the constant term of $\mathrm{LB}_n(123)$ is $\#L(n)$. Define $L_i(n) = \{w \in L(n) \mid w \text{ starts with } i \text{ ones}\}$. Then $\#L(n) = \#L_1(n) + \#L_2(n)$. But $\#L_i(n) = \#L(n-i)$ for $i = 1, 2$, since if w begins with i ones then the rest of the word is essentially a layered word with $n-i$ elements. Therefore, $\#L(n) = \#L(n-1) + \#L(n-2)$. Since $\#L(0) = 1$ and $\#L(1) = 1$, we have $\#L(n) = F_n$.

To prove the second claim, let $w \in R_n(123)$ with $\mathrm{lb}(w) = 1$. Then there must be exactly one descent in w and it must be of the form $w_{j+1} = w_j - 1$ for some $2 \leq j \leq n-1$. Removing w_j and w_{j+1} from w and then subtracting one from all w_k with $k > j+1$ gives an element $w' \in R_{n-2}$ which is layered. So, from the previous paragraph, there are F_{n-2} choices for w'. Further, there were $n-2$ choices for j and so the total number of w is $(n-2)F_{n-2}$. □

We will now look at the right-smaller statistic.

Theorem 3.18. *We have the following.*

1. *The degree of* $\mathrm{RS}_n(123)$ *is*
$$\left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

2. *The leading coefficient of* $\mathrm{RS}_n(123)$ *is* 1 *when* n *is odd, and* 2 *when* n *is even.*

3. *The constant term of* $\mathrm{RS}_n(123)$ *is* F_n.

Proof. The proof of the first result is very similar to the proof of the degree of $\mathrm{LB}_n(123)$. When looking at the right-smaller statistic, the word that maximizes rs is of the form $w = 12\ldots i(n-i)\ldots 21$, where $12\ldots i$ is the initial run. Calculating $\mathrm{rs}(w)$ gives

$$\mathrm{rs}(w) = (n-i)(i-1), \tag{7}$$

and differentiating with respect to the real variable i and maximizing gives $i = (n+1)/2$. Since we want i to be integral, we have

$$i = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{n+1}{2} \right\rfloor \text{ or } \left\lceil \frac{n+1}{2} \right\rceil & \text{if } n \text{ is even.} \end{cases}$$

Plugging each value of i and n into (7) gives $\lfloor (n-1)^2/4 \rfloor$ in both cases. Also, the number of choices for i gives the leading coefficient of $\mathrm{RS}_n(123)$.

The proof for the constant term of $\mathrm{RS}_n(123)$ is the same as for $\mathrm{LB}_n(123)$ since for any w we have $\mathrm{rs}(w) = 0$ if and only if $\mathrm{lb}(w) = 0$. $\square$

Again, since $w(\Pi_n(123)) = R_n(111)$ we have the following corollary for Theorem 4.18 where the Gaussian q-analogue $[n]_q$ is defined in equation (9).

Corollary 3.19. *We have* $\mathrm{RS}_0(123) = 1$ *and for* $n \geq 1$

$$\mathrm{RS}_n(123) = \mathrm{RS}_{n-1}(123) + [n-1]_q \, \mathrm{RS}_{n-2}(123). \qquad \square$$

Our final result of this section gives the degree of $\mathrm{RB}_n(123)$. It follows immediately from the easily proved fact that the word which maximizes rb is $w = 12\ldots n$.

Theorem 3.20. $\mathrm{RB}_n(123)$ *is monic and has degree* $\binom{n}{2}$. $\square$

Avoidance Class	Associated RGFs
$\Pi_n(1/2/3, 1/23)$	$1^n,\ 1^{n-1}2,\ 1^{n-2}21$
$\Pi_n(1/2/3, 13/2)$	1^m2^{n-m} for all $1 \le m \le n$
$\Pi_n(1/2/3, 12/3)$	$1^n,\ 12^{n-1},\ 121^{n-2}$
$\Pi_n(1/23, 13/2)$	$1^{n-m+1}23\ldots m$ for all $1 \le m \le n$
$\Pi_n(1/23, 12/3)$	$1^n,\ 12\ldots(n-1)1,\ 12\ldots n$
$\Pi_n(1/23, 123)$	$12\ldots n,\ 12\ldots(n-1)$ with an additional 1 inserted
$\Pi_n(13/2, 12/3)$	$12\ldots m^{n-m+1}$ for all $1 \le m \le n$
$\Pi_n(13/2, 123)$	layered RGFs with at most two elements in each layer
$\Pi_n(12/3, 123)$	$12\ldots(n-1)m$ for all $1 \le m \le n$

Table 1 Avoidance classes avoiding two partitions of [3] and associated RGFs

3.2 Multiple pattern avoidance

Rather than avoiding a single pattern, one can avoid multiple patterns. Define, for any set P of set partitions

$$\Pi_n(P) = \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids every } \pi \in P\}.$$

Similarly adapt the other notations we have been using. Goyt [Goy08] characterized that cardinalities of $\Pi_n(P)$ for any $P \subseteq \mathfrak{S}_3$. Our goal in this section is to do the same for $F_n(P)$. We will not include those P containing both $1/2/3$ and 123 since it is easy to see from Theorem 2.3 that there are no such partitions for $n \geq 5$.

Table 1 shows the avoidance classes and the resulting restricted growth functions that arise from avoiding two patterns of length 3. These as well as the entries in Table 2 also appear in Goyt's work, but we include them here for completeness. For ease of references, we give a total order to Π_3 as follows

$$1/2/3,\ 1/23,\ 13/2,\ 12/3,\ 123 \tag{8}$$

and list the elements of any set P in lexicographic order with respect to (8). Finally, for any $P \subseteq \Pi_3$ we have $\Pi_n(P) = \Pi_n$ for $n < 3$. So we assume for the rest of this section that $n \geq 3$.

The next result translates this table into generating functions. This is routine and only uses techniques we have seen in earlier sections so the proof is omitted. The function $F_n(13/2, 123)$ is due to Goyt and Sagan [GS09] where the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ is an extension of the one defined in equation (11). The multivariate version defines

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + q^{n-1}$$

so the binomial analogue is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

We can recover the one-variable version by letting $p = 1$.

Theorem 3.21. *For $n \geq 3$ we have*

1. $F_n(1/2/3, 1/23) = 1 + rs^{n-1} + qrs^{n-2}t$,

2. $F_n(1/2/3, 13/2) = 1 + \sum_{i=1}^{n-1} r^i s^{n-i}$,

3. $F_n(1/2/3, 12/3) = 1 + rs^{n-1} + q^{n-2}rst$,

4. $F_n(1/23, 13/2) = 1 + \sum_{i=1}^{n-1} r^{\binom{n-i+1}{2}} s^{\binom{n}{2}-\binom{i}{2}}$,

5. $F_n(1/23, 12/3) = 1 + (qt)^{n-2}(rs)^{\binom{n-1}{2}} + (rs)^{\binom{n}{2}}$,

6. $F_n(1/23, 123) = (rs)^{\binom{n}{2}} + r^{\binom{n-1}{2}} \sum_{i=0}^{n-2} (qt)^i s^{\binom{n}{2}-i-1}$,

7. $F_n(13/2, 12/3) = 1 + \sum_{i=1}^{n-1} r^{\binom{n}{2}-\binom{i}{2}} s^{\binom{n-i+1}{2}}$,

8. $F_n(13/2, 123) = \sum_{k \geq 0} (rs)^{\binom{n}{2}-k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}_{r,s}$, *and*

9. $F_n(12/3, 123) = (rs)^{\binom{n}{2}} + s^{\binom{n-1}{2}} \sum_{i=0}^{n-2} (qt)^i r^{\binom{n}{2}-i-1}$. $\square$

Note that from this theorem we immediately get the following nice equidistribution results.

Corollary 3.22. *Consider the generating function $F_n(P)$ where $P \subseteq \Pi_3$.*

1. *We have $F_n(P)$ invariant under switching q and t if $13/2 \in P$ or P is one of*

$$\{1/2/3, 1/23\};\ \{1/23, 12/3\};\ \{1/23, 123\};\ \{12/3, 123\}.$$

2. *We have $F_n(P)$ invariant under switching r and s if P is one of*

$$\{1/2/3, 13/2\};\ \{1/23, 12/3\}.$$

3. We have the following equalities between generating functions for different P:

$$F_n(1/23, 13/2; q, r, s, t) = F_n(13/2, 12/3; q, s, r, t)$$

and

$$F_n(1/23, 123; q, r, s, t) = F_n(12/3, 123; q, s, r, t). \qquad \square$$

Next, we will examine the outcome of avoiding three and four partitions of [3]. We can see the avoidance classes and the resulting restricted growth functions in Table 2. The entries in this table can easily be turned into a polynomial by the reader if desired. Avoiding all five partitions of [3] is not included because it would contain both 1/2/3 and 123.

Avoidance Class	Associated RGFs
$\Pi_n(1/2/3, 1/23, 13/2)$	1^n, $1^{n-1}2$
$\Pi_n(1/2/3, 1/23, 12/3)$	1^n, 121 when $n = 3$
$\Pi_n(1/2/3, 13/2, 12/3)$	1^n, 12^{n-1}
$\Pi_n(1/23, 13/2, 12/3)$	1^n, $12\dots n$
$\Pi_n(1/23, 13/2, 123)$	$1^2 2\dots(n-1)$, $12\dots n$
$\Pi_n(1/23, 12/3, 123)$	$12\dots(n-1)1$, $12\dots n$
$\Pi_n(13/2, 12/3, 123)$	$12\dots(n-2)(n-1)^2$, $12\dots n$
$\Pi_n(1/2/3, 1/23, 13/2, 12/3)$	1^n
$\Pi_n(1/23, 13/2, 12/3, 123)$	$12\dots n$

Table 2 Avoidance classes and associated RGFs avoiding three and four partitions of [3]

3.3 The pattern 14/2/3

In this section we study the pattern 14/2/3. Its avoidance class has a very nice characterization, Lemma 3.23 below, which facilitates proving enumerative results.

Our first theorem concerns applying the lb statistic, from which a connection arises between 14/2/3-avoiding set partitions and integer compositions. First, we characterize $R_n(14/2/3)$. We define the index i to be a *dale of height* a in w if $a_i = a$ and

$$a_i = \max\{a_1, \ldots, a_{i-1}\} - 1.$$

Lemma 3.23. *For an RGF w, w is contained in $R_n(14/2/3)$ if and only if w meets the following restrictions:*

- *for $i \geq 2$ we have $a_i \geq \max\{a_1, \ldots, a_{i-1}\} - 1$, and*
- *if w has a dale of height a, then w does not have a dale of height $a+1$.*

Proof. Let σ avoid 14/2/3. Assume, towards contradiction, that there existed an a_i in $w = w(\sigma)$ with $a_i < \max\{a_1, \ldots, a_{i-1}\} - 1$ and let $a = a_i$. By the structure of restricted growth functions, this implies that $a(a+1)(a+2)a$ exists as a subword in w. But then these four elements give rise to an occurrence of 14/2/3 in σ, which is a contradiction. This shows the first inequality. Now assume that there existed dales of height a and height $a+1$ in w. This would require w to contain $(a+1)a(a+2)(a+1)$ as a subword, which again implies an occurrence of 14/2/3 in σ. This shows the height requirement for dales.

Now assume that σ is a partition with $w = w(\sigma)$ meeting the listed requirements. If σ contained 14/2/3 as a pattern, then $abca$ must occur as a subword in w, with $a \neq b \neq c$. If a was the minimum value in this subword, then either $a < b - 1$ or $a < c - 1$, which contradicts the first restriction put on w in view of the second a in the subword. Further, if a was the maximum value in this subword, then either $b < a - 1$ or $c < a - 1$, raising the same contradiction in view of the first a. Similarly, we can rule out $c < a < b$. Thus the only remaining possibility is that $b < a < c$. By the first condition in the lemma, it then must be that the subword is exactly $a(a-1)(a+1)a$, which contradicts the restriction on dales. Thus σ avoids 14/2/3, showing the reverse implication. $\square$

Note that a dale in a word w contributes exactly one to $\mathrm{lb}(w)$. And by the previous lemma, dales are the only source of lb for words in $R_n(14/2/3)$. For the proof of our theorem about LB(14/2/3) we will also need the following notion: call i a *left-right maximum of value* a in w if $a_i = a$ and

$$a_i > \max\{a_1, \ldots, a_{i-1}\}.$$

Being an RGF is equivalent to having left-right maxima of values $1, 2, \ldots, m$ for some m.

Theorem 3.24. *For $n \geq 1$, we have*

$$\mathrm{LB}_n(14/2/3) = 2^{n-1} + \sum_{k=1}^{n-2} \left[\sum_{m \geq 2} \binom{n-1}{k+m-1} \sum_{j \geq 1} \binom{k-1}{j-1} \binom{m-j}{j} \right] q^k.$$

Proof. It is easy to see that the constant term in this polynomial comes from the layered partitions of $[n]$, all of which avoid 14/2/3. Now consider the coefficient of q^k for $k \geq 1$. From the discussion before the statement of the theorem, for a word in $R_n(14/2/3)$ to have an lb of k, it must have k dales. Further, we know that $i = 1$ is always a left-right maximum of value 1 in any RGF, and that $i = 1$ is never a dale. It follows by Lemma 3.23 that, to completely characterize an RGF of lb equal to k and maximum value m in $R_n(14/2/3)$, it suffices to specify the remaining $m-1$ left-right maxima and the k dale indices. As such, there are $\binom{n-1}{m+k-1}$ ways to choose a set I which is the union of these two index sets.

Let $I = \{i_1 < i_2 < \cdots < i_{m+k-1}\}$ be such a set. We will indicate indices chosen for dales by coloring them blue, and left-right maxima by coloring them red. We define a *run* to be a maximal sequence of indices $i_c, i_{c+1}, \ldots, i_d$ which is monochromatic. Let j be the number of blue runs, and let b_s be the number of indices in the sth blue run, for $1 \leq s \leq j$. As these numbers count the dales in w, we must have

$$b_1 + b_2 + \cdots + b_j = k,$$

or equivalently that $b_1, \ldots, b_j$ form an integer composition of k. Thus there are $\binom{k-1}{j-1}$ ways of choosing j blue runs.

Now note that I must start with a red run, and can end with either a red or blue run. Thus there are j or $j+1$ red runs. Let r_t be the length of the tth red run, for $1 \le t \le j+1$, where we set $r_{j+1} = 0$ if there are j red runs. Furthermore, by the dale height restriction in Lemma 3.23, we have $r_t \ge 2$ for $2 \le t \le j$. Now as before, we have

$$r_1 + r_2 + \cdots + r_{j+1} = m - 1,$$

subject to $r_1 \ge 1$, $r_2, \ldots, r_j \ge 2$, and $r_{j+1} \ge 0$. Using a standard composition manipulation, we can put this sum in correspondence with a composition of $m - j + 1$ into $j + 1$ parts, which gives $\binom{m-j}{j}$ ways to choose the red runs. Putting everything together and summing over the possible values of m and j gives the coefficient of q^k as

$$\sum_{m \ge 2} \binom{n-1}{k+m-1} \sum_{j \ge 1} \binom{k-1}{j-1} \binom{m-j}{j}.$$

All that is left is to give appropriate bounds for k. It follows by Lemma 3.23 that $w = 121^{n-2}$ is in $R_n(14/2/3)$ and that w gives a maximizing lb of $n-2$. This gives $1 \le k \le n-2$, and provides the correct parameters for the polynomial. □

From the previous theorem, and from the characterization of $R_n(14/2/3)$, several corollaries follow.

Corollary 3.25. *We have*

$$\mathrm{LB}_n(14/2/3) = \mathrm{RS}_n(14/2/3).$$

Proof. We proceed by finding a bijection ϕ that takes $R_n(14/2/3)$ to itself, and that takes the lb statistic to the rs statistic. Let w be a member of $R_n(14/2/3)$. From Lemma 3.23, we can partition w into sections based on the dales of w. Specifically, let a_i be a letter in w, and let $a = a_i$. If there is no dale of height a or $a-1$ in w, then it follows that every copy of a is adjacent in w. That is to say, we can break w into

$$w = w_1 a^l w_2,$$

with $a_j < a$ for all a_j in w_1, and $a_k > a$ for all a_k in w_2. Call such a string a *plateau* of w. It follows that plateaus in w contribute nothing to $\mathrm{lb}(w)$ or $\mathrm{rs}(w)$. We will let ϕ act trivially on the plateaus of w.

If this is not the case, then there is a dale of height a or $a-1$ in w. By Lemma 3.23 again, both a and $a-1$ can not be dale heights. So suppose $a-1$ is a dale height. It follows that the occurrences of a and $a-1$ in w are adjacent and we have

$$w = w_1 (a-1)^{l_0} a^{j_1} (a-1)^{l_1} \ldots a^{j_t} (a-1)^{l_t} w_2,$$

with $l_0, \ldots, l_{t-1} > 0$, $l_t \geq 0$, and $j_1, \ldots, j_t > 0$. Further, we have $a_j < a-1$ for all a_j in w_1, and $a_k > a$ for all a_k in w_2. Such a string will be called a *dale section* of w. Breaking up w in this manner shows that such a dale section contributes $l_1 + \cdots + l_t$ to $\mathrm{lb}(w)$, and either $j_1 + \cdots + j_{t-1}$ or $j_1 + \cdots + j_t$ to $\mathrm{rs}(w)$, depending on whether or not $l_t = 0$. As such, if

$$d = (a-1)^{l_0} a^{j_1} (a-1)^{l_1} \ldots a^{j_t} (a-1)^{l_t}$$

is a dale section in w, we let

$$\phi(d) = \begin{cases} (a-1)^{l_0} a^{l_1} (a-1)^{j_1} \ldots a^{l_t} (a-1)^{j_t} & \text{if } l_t > 0, \\ (a-1)^{l_0} a^{l_1} (a-1)^{j_1} \ldots a^{l_{t-1}} (a-1)^{j_{t-1}} a^{j_t} & \text{if } l_t = 0. \end{cases}$$

It follows that ϕ exchanges lb and rs for a dale section.

Now by the nature of $R_n(14/2/3)$, we know that w is merely a concatenation of plateaus and dale sections. Having defined ϕ on these parts of w, we define $\phi(w)$ by applying ϕ to the plateaus and dale sections of w in a piecewise manner. It follows that ϕ is a bijection, since it is an involution. Finally, since $\mathrm{lb}(w)$ and $\mathrm{rs}(\phi(w))$ are sums over the dale sections of w and $\phi(w)$, and since ϕ exchanges the two statistics on each dale section, it follows that we have $\mathrm{lb}(w) = \mathrm{rs}(\phi(w))$. □

Corollary 3.26. *For $t \geq 2$, we have*

$$\mathrm{LB}_n(14/2/3, 1/2/\ldots/t) = \sum_{i=0}^{t-2} \binom{n}{i} + \sum_{k=1}^{n-2} \left[\sum_{m=2}^{t-1} \binom{n-1}{k+m-1} \sum_{j\geq 1} \binom{k-1}{j-1} \binom{m-j}{j} \right] q^k$$

and the equality

$$\mathrm{LB}_n(14/2/3, 1/2/\ldots/t) = \mathrm{RS}_n(14/2/3, 1/2/\ldots/t).$$

Proof. Avoiding $1/2/\ldots/t$ as well as $14/2/3$ adds the restriction that words must have maximum value less than or equal to $t-1$. Following the proof of Theorem 3.24 with this additional restriction gives the generating function $\mathrm{LB}_n(14/2/3, 1/2/\ldots/t)$.

Next, we note that the same bijection from Corollary 3.25 also provides a bijection from $R_n(14/2/3, 1/2\ldots/t)$ to itself, since ϕ preserves maximum values. The same map then ensures the second equality. □

Corollary 3.27. *The polynomial* $\mathrm{LB}_n(14/2/3, 123)$ *has degree* $\lfloor n/3 \rfloor$ *and leading coefficient equal to*

$$\begin{cases} 1 & \textit{if } n = 3k, \\ n & \textit{if } n = 3k+1, \\ \frac{3n^2-7n+14}{6} & \textit{if } n = 3k+2, \end{cases}$$

for some integer k.

Proof. Avoiding the pattern 123 as well as 14/2/3 adds the restriction that letters can be repeated at most twice in a word. Adapting the notation used in the proof of Corollary 3.25, this implies that, for $w \in R_n(14/2/3, 123)$, the dale sections of w must have length equal to 3 or 4. Further, these dale sections can only contribute 1 to $\mathrm{lb}(w)$. Thus to maximize $\mathrm{lb}(w)$, we maximize the number of dale sections contained in w. It follows from the restrictions on w that this leads to a maximum of $\lfloor n/3 \rfloor$.

We now move to the leading coefficient. If $n = 3k$ for some integer k, then it is clear that the only RGF w in $R_n(14/2/3, 123)$ that achieves this maximum is

$$w = 121343\ldots(2k-1)2k(2k-1),$$

giving a leading coefficient of 1.

Now let $w \in R_n(14/2/3, 123)$ for $n = 3k+1$. It follows that w either has one dale section of length 4, or one plateau of length 1. In the first case, we note that a dale section of length 4 has the form $a(a+1)(a+1)a$ or $a(a+1)a(a+1)$. As there will be k total dales in w, we have k choices for which dale section to extend, and 2 choices for how to extend it. This gives $2k$ possible words of the first form. Now assume w has a plateau of length 1. Note that, once the index of this plateau has been chosen, the rest of the word is uniquely determined. As such, we can choose to place the plateau directly in front of any of the k dale sections, or after the last dale section in w. This gives $k+1$ possible words of the second form. Summing over both possibilities now gives a leading coefficient of $n = 3k+1$.

Finally, we have $w \in R_n(14/2/3, 123)$ for $n = 3k+2$. There are four distinct possibilities for w in this case. First, w could contain one plateau of length 2. This gives $k+1$ possibilities as in the previous paragraph. The second possibility is that w contains two plateaus of length 1. If these plateaus are adjacent, then as in the previous case we have $k+1$ possibilities. Otherwise, we choose 2 distinct places from these options, giving $\binom{k+1}{2}$ more words. In the third case, w contains one plateau of length 1 and one dale section of length 4. We have $k+1$ choices for the plateau, and $2k$ possibilities for the dale section, giving $2k(k+1)$ words of this form. Finally, w could contain two dale sections of length 4. In this case, we choose two dale sections to extend. As there are two distinct ways to extend each dale section, this gives $4\binom{k}{2}$ such words. Summing over these four cases and using the substitution $n = 3k+2$ gives the final result. $\square$

Our last corollary regarding the pattern 14/2/3 involves multiple pattern avoidance with two partitions of [4]. First, we need a lemma.

Lemma 3.28. *For an RGF w, w is contained in $R_n(14/2/3, 13/2/4)$ if and only if w meets the following restrictions:*

- *For $i \geq 2$ we have $a_i \geq \max\{a_1, \ldots, a_{i-1}\} - 1$, and*
- *If i is a dale of height a, then $a_j = a$ or $a_j = a+1$ for all $j > i$.*

Proof. First, let σ avoid 14/2/3 and 13/2/4, and let $w = w(\sigma)$. Since $R_n(14/2/3, 13/2/4)$ is a subset of $R_n(14/2/3)$, the first inequality follows from Lemma 3.23. Now assume that i is a dale of height a in w, and assume towards a contradiction that there exists a_j in w with $j > i$, $a_j \neq a$, and $a_j \neq a+1$. From the first inequality, it must be that $a_j > a+1$. Because w is an RGF, it follows that $a(a+1)a(a+2)$ exists as a subword in w. But now these four elements will cause an occurrence of 13/2/4 in σ, which is a contradiction.

For the reverse implication, let σ be a partition with $w = w(\sigma)$ satisfying the above restrictions. From Lemma 3.23, it follows that σ will avoid 14/2/3. To see that σ will also avoid 13/2/4, note that if σ contained 13/2/4, then the subword $abac$ would exist in w, with $a \neq b \neq c$. Using the first inequality, we can rule out all cases except $b < a < c$. But, as this implies a dale of height b in w, this would lead to a contradiction with respect to the second restriction put on w by the lemma. Thus σ must also avoid 13/2/4. $\square$

Corollary 3.29. *We have*

$$\mathrm{LB}_n(14/2/3, 13/2/4) = 2^{n-1} + \sum_{k=1}^{n-2} \left[\sum_{m \geq 2} \binom{n-1}{k+m-1} \right] q^k$$

and

$$\mathrm{LB}_n(14/2/3, 13/2/4) = \mathrm{RS}_n(14/2/3, 13/2/4).$$

Proof. Following the proof of Theorem 3.24, we note that the constant term in this polynomial comes from the layered partitions of $[n]$. Now consider a word w in $R_n(14/2/3, 13/2/4)$ with lb equal to k and maximum value m, for $k \geq 1$. From the previous lemma, it follows that the k dales in w must come to the right of the m left to right maxima in w. As the leading one in w provides the first left to right maximum, it suffices to choose $k+m-1$ other indices where we place the remaining left to right maxima in the left-most $m-1$ indices, and the k dales afterwards. This gives $\binom{n-1}{k+m-1}$ such words, and summing over all possible values of m gives the coefficient of q^k for $k \geq 1$.

Finally, we note that the bijection from Corollary 3.25 also takes $R_n(14/2/3, 13/2/4)$ to itself. This gives the second equality. □

4 Restricted growth functions

In this chapter we present our results about RGFs. In the preliminary chapter we noted that the two notions for pattern avoidance were the same for all RGF patterns of length three except 112 and 122. In Section 4.1 we determine all four generating functions for these patterns, and show many equalities between their generating functions including $\mathrm{LB}_n(112) = \mathrm{RS}_n(112) = \mathrm{LB}_n(122)$. This result uses Gaussian polynomials, integer partitions, and the hook decomposition of Young diagrams. The following section presents recursive formulae for generating the functions for certain patterns of any length including $12\ldots k$ and 1^k. The final sections discuss the patterns 1212 and 1221 which are related to noncrossing and nonnesting partition, respectfully. We define two bijections from these avoidance classes to two-colored Motzkin paths and prove $\mathrm{RS}_n(1212) = \mathrm{LB}_n(1212) = \mathrm{LB}_n(1221)$. We also determine the four-variate generating functions for these patterns paired with any length three pattern.

4.1 Single patterns of length 3

4.1.1 Patterns related to integer partitions in a rectangle

In this subsection, we show bijectively that three of the generating functions under consideration are the same. Moreover, the common value can be expressed in terms of the q-binomial coefficients which count integer partitions in a rectangle. First we need some definitions about q-analogues and integer partitions.

We let

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}. \tag{9}$$

We can now define a q-analogue of the factorial, letting

$$[n]_q! = [1]_q[2]_q \cdots [n]_q. \tag{10}$$

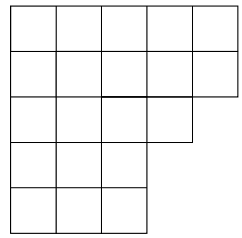

Figure 1 The Young diagram for $\lambda = (5, 5, 4, 3, 3)$

Finally, we define the *q-binomial coefficients* or *Gaussian polynomials* as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{11}$$

By convention, $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$.

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *of an integer* t is a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{k} \lambda_i = t$. We call the λ_i *parts* and let $|\lambda| = \sum_{i=1}^{k} \lambda_i$. The *Young diagram* of a partition λ is an array of boxes with k left-justified rows, where the ith row has λ_i boxes. For example the partition $\lambda = (5, 5, 4, 3, 3)$ would correspond to the Young diagram in Figure 1. Sometimes we will need to refer to particular boxes in the Young diagram. We let (i, j) denote the box in the ith row and jth column of the Young diagram.

If β is an $r \times \ell$ rectangle and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, we say that the Young diagram of λ *fits inside* β if $k \leq r$ and $\lambda_1 \leq \ell$ and we will denote this by $\lambda \subseteq \beta$. When $\lambda \subseteq \beta$, we will draw λ and β so together so that their $(1, 1)$ boxes coincide, as can be seen in Figure 2. For β an $r \times \ell$ rectangle it is well-known that

$$\begin{bmatrix} r+\ell \\ \ell \end{bmatrix}_q = \sum_{\lambda \subseteq \beta} q^{|\lambda|}.$$

Now that we have the proper terminology, we can prove our first equidistribution theorem of this section.

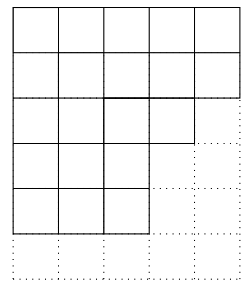

Figure 2 The Young diagram for $\lambda = (5, 5, 4, 3, 3)$ in the 6×5 rectangle β

Theorem 4.1. *We have*

$$\mathrm{LB}_n(112) = \mathrm{RS}_n(112) = \mathrm{LB}_n(122) = \sum_{t \geq 0} \begin{bmatrix} n-1 \\ t \end{bmatrix}_q .$$

In other words, each of the above polynomials is the generating function for integer partitions counted with multiplicity given by the number of rectangles into which they fit. We will establish Theorem 4.1 through four propositions.

First, we need a few definitions. A sequence of integers $u_1 \dots u_n$ is called *unimodal* if there exists an index i with

$$u_1 \leq u_2 \leq \cdots \leq u_i \geq u_{i+1} \geq \cdots \geq u_n.$$

We define a *rooted unimodal composition* $u = u_1 \dots \boldsymbol{u_m} \dots u_n$ to be a sequence of nonnegative integers, together with a distinguished element called the *root* and displayed in boldface type, having the following properties:

1. u is unimodal.

2. $u_1 = u_n = 0$.

3. If u is rooted at $\boldsymbol{u_m}$, then $\boldsymbol{u_m} = \max(u)$.

4. We have $|u_j - u_{j+1}| \leq 1$ for all j.

We define $|u| = u_1 + \cdots + u_n$ and let

$$A_n = \{u : u = u_1 \ldots u_n \text{ is a rooted unimodal composition}\}.$$

For example $u = 00012\mathbf{2}22110000$ is a rooted unimodal composition with root $\boldsymbol{u_6} = \mathbf{2}$ and $|u| = 11$. The rooted unimodal compositions are useful to show that $\mathrm{RS}_n(112)$ is a sum of Gaussian polynomials.

Proposition 4.2. *We have*

$$\mathrm{RS}_n(112) = \sum_{u \in A_n} q^{|u|}.$$

Proof. We will construct a bijection $\psi : R_n(112) \to A_n$ such that $\mathrm{rs}(w) = |\psi(w)|$. Let $w = a_1 \ldots a_n$, where m is the index of the first maximum of w. We will construct $\psi(w) = u = u_1 \ldots \boldsymbol{u_m} \ldots u_n$ by letting $u_i = \mathrm{rs}(a_i)$. For example if $w = 1234553221$ then $\psi(w) = 0123\mathbf{3}32110$.

We begin by showing ψ is well defined. Let $w \in R_n(112)$ and $u = \psi(w)$. By Theorem 2.5, w has some initial run $12 \ldots m$ which is followed by a weakly decreasing sequence with terms at most m. We will show u satisfies properties 1–4 above.

First, properties 1 and 3 follow from the fact that w is unimodal with maximum m. Because $a_1 = 1$ and a_n is the right-most element, we have $\mathrm{rs}(a_1) = \mathrm{rs}(a_n) = 0$ and thus, property 2. The fourth property holds because before the maximum index m, adjacent elements increase by one, and after that index the sequence is weakly decreasing.

We now define ψ^{-1}. Let $u = u_1 \ldots \boldsymbol{u_m} \ldots u_n \in A_n$. Let $\ell(j)$ be the index of the last occurrence of j in $u_1 \ldots \boldsymbol{u_m}$. We construct $w = \psi^{-1}(u)$ so that

$$w = 123 \ldots m a_{m+1} \ldots a_n$$

where for $m + 1 \leq i \leq n$ we have $a_i = \ell(u_i)$. For example if $u = 001122\mathbf{2}21000$ then $w = 123456774222$. To show ψ^{-1} is well defined, it suffices to show $a_i \geq a_{i+1}$ for $i \geq m$.

But this follows since $u_1 \dots \boldsymbol{u_m}$ is a weakly increasing sequence and so $u_i \geq u_{i+1}$ for $i \geq m$ implies $\ell(u_i) \geq \ell(u_{i+1})$.

Next, we show that the two maps are indeed inverses. First, assume $\psi(w) = u$ and $\psi^{-1}(u) = v = v_1 \dots v_n$. Let m be the index of the first maximum in w so that $\boldsymbol{u_m}$ is the root in u. We will show $a_i = v_i$ for all $1 \leq i \leq n$. We know $a_i = i$ for all $i \leq m$. Since $\boldsymbol{u_m}$ is rooted in u then, by definition of ψ^{-1}, we have that v also begins with $12 \dots m$. For $i > m$, there must be an index $k \leq m$ with $k = a_k = a_i$ If follows that $u_k = \mathrm{rs}(a_k) = \mathrm{rs}(a_i) = u_i$. Furthermore, k must be the largest index less than or equal to m which satisfies the last equality since $a_{k+1} = a_k + 1$ so that $u_{k+1} > u_i$. It follows, by definition of ℓ, that $v_i = \ell(u_i) = k = a_i$. The proof that $\psi(\psi^{-1}(u)) = u$ is similar.

If $\psi(w) = u$ then $u_i = \mathrm{rs}(a_i)$ and so $\mathrm{rs}(w) = |u|$. Therefore

$$\mathrm{RS}_n(112) = \sum_{u \in A_n} q^{|u|}$$

as desired. □

Let

$$B_n = \bigcup_{m \geq 1} \{(\lambda, \beta) : \lambda \text{ an integer partition and } \lambda \subseteq \beta, \text{ for } \beta \text{ an } (m-1) \times (n-m) \text{ rectangle}\}.$$

As discussed above,

$$\sum_{(\lambda,\beta) \in B_n} q^{|\lambda|} = \sum_{m \geq 1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q = \sum_{t \geq 0} \begin{bmatrix} n-1 \\ t \end{bmatrix}_q.$$

Proposition 4.3. *We have*

$$\sum_{u \in A_n} q^{|u|} = \sum_{t \geq 0} \begin{bmatrix} n-1 \\ t \end{bmatrix}_q.$$

Proof. From the discussion just before this proposition, it suffices to construct a bijection $\varphi : A_n \to B_n$ such that if $u \in A_n$ and $\varphi(u) = (\lambda, \beta)$ then $|u| = |\lambda|$. Let $u = u_1 \dots \boldsymbol{u_m} \dots u_n \in$

A_n. Then we construct $\varphi(u) = (\lambda, \beta)$ as follows. First, we use the index of the root of u to determine that β will be a $(m-1) \times (n-m)$ rectangle. Consider the diagonal in β formed by coordinates $(1,1), (2,2), \ldots$ and diagonals above and below this one. Then, going from southwest to northeast, take the first u_i squares along each diagonal as i varies from 2 to $n-1$ to form the diagram for λ. For example the rooted unimodal composition $u = 001233\mathbf{3}32210$ will give $\lambda = (5,5,4,3,3)$ in the 6×5 rectangle β shown in Figure 2.

Properties 1, 2, and 4 ensure that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a well-defined Young diagram corresponding to an integer partition. Now we must check that $\lambda \subseteq \beta$. Using the ordering in which we constructed the diagonals of λ, observe that the number of diagonals up to and including the main diagonal is k. As the main diagonal of λ corresponds to element $\boldsymbol{u_m}$ and we begin in our construction with element u_2, we have $k \leq m-1$. Similarly, we can see that λ_1 is equal to the number of diagonals after and including the main diagonal. Thus $\lambda_1 \leq n-m$, as we end with element u_{n-1}. Therefore $\lambda \subseteq \beta$.

If $(\lambda, \beta) \in B_n$, where β is an $(m-1) \times (n-m)$ rectangle, then the root of $\varphi^{-1}(\lambda, \beta)$ will be at index m. The entries of $\varphi^{-1}(\lambda, \beta)$ are obtained using the diagonals of λ so as to reverse the above construction. As φ^{-1} is very similar to φ, we leave the process of checking that φ^{-1} is well defined and the inverse of φ to the reader. In addition, it is clear from the definitions that $|u| = |\lambda|$. $\square$

It should be mentioned that we originally proved Proposition 4.3 using a bijection involving hook decompositions, similar to Section 3 of the paper of Barnabei et al. [BBES14]. Although the above proof was found to be simpler, it may be interesting to further explore connections between patterns in RGFs and hook decompositions.

Proposition 4.4. *We have*

$$\mathrm{LB}_n(112) = \sum_{t \geq 1} \begin{bmatrix} n-1 \\ t \end{bmatrix}_q .$$

Proof. We will construct a bijection $\rho : R_n(112) \to B_n$ such that $\mathrm{lb}(w) = |\lambda|$, where $w = a_1 \ldots a_n \in R_n(112)$ and $\rho(w) = (\lambda, \beta)$. Let the initial run of w be $a_1 a_2 \ldots a_m = 12 \ldots m$ so that $m = \max(w)$.

First, we let β be an $(n-m) \times (m-1)$ rectangle. We then let

$$\lambda = (m - a_n, m - a_{n-1}, \dots, m - a_{m+1}),$$

permitting parts equal to zero. For example, $w = 123456633211$ would map to (λ, β) shown in Figure 2.

As m is the maximum of w, we have $0 \leq m - a_i \leq m - 1$ for $m + 1 \leq i \leq n$. In addition, $a_{m+1} \geq a_{m+2} \geq \cdots \geq a_n$ and thus the parts of λ are weakly decreasing. Therefore λ is well defined and fits inside β. Constructing ρ^{-1} is a simple matter which we leave to the reader.

Now notice that in w, $\mathrm{lb}(a_i) = 0$ for all $1 \leq i \leq m$. For $m < i \leq n$, we have that $\mathrm{lb}(a_i) = m - a_i = \lambda_{n-i+1}$. Thus

$$\mathrm{lb}(w) = \mathrm{lb}(a_{m+1}) + \mathrm{lb}(a_{m+2}) + \cdots + \mathrm{lb}(a_n) = \lambda_{n-m} + \lambda_{n-m-1} + \cdots + \lambda_1 = |\lambda|$$

as desired. □

Proposition 4.5. *We have*

$$\mathrm{LB}_n(112) = \mathrm{LB}_n(122).$$

Proof. We will construct a bijection $\eta : R_n(112) \to R_n(122)$ such that $\mathrm{lb}(w) = \mathrm{lb}(\eta(w))$. Let $w = a_1 \dots a_n \in R_n(112)$ with maximum m. To construct $\eta(w)$ we start with the sequence $12 \dots m$. For every a_i, where a_i is not in the initial run of w, we will insert a 1 just to the right of element $m - a_i + 1$ in $\eta(w)$. Note that $1 \leq a_i \leq m$ ensures that this element always exists and thus η is well defined. For example if $w = 12345664331$ then $\eta(w) = 11231411561$. Clearly η is invertible.

To check that lb is preserved, note that in w the initial run does not contribute to lb and in $\eta(w)$, none of the terms greater than 1 contribute to lb. Let a_i such that $i > m$. Then $\mathrm{lb}(a_i) = m - a_i$. If we examine the 1 placed into $\eta(w)$ because of a_i, we notice that it has $m - a_i$ terms greater than 1 to its left. Therefore the lb of this 1 is $m - a_i$. Thus, $\mathrm{lb}(w) = \mathrm{lb}(\eta(w))$. $\square$

Combining the above propositions yields Theorem 4.1.

4.1.2 Patterns related to integer partitions with distinct parts

Next, we will explore a connection to integer partitions with distinct parts. It is well-known that the generating function for partitions with distinct parts of size at most $n-1$ is

$$\prod_{i=1}^{n-1}(1+q^i).$$

As noted in the introduction, for the pattern 121 we have $R_n(121) = w(\Pi_n(13/2))$. So we can use the following result of Goyt and Sagan who studied the ls statistic on $\Pi_n(13/2)$.

Proposition 4.6 ([GS09]). *We have*

$$\mathrm{LS}_n(121) = \prod_{i=1}^{n-1}(1+q^i).$$ $\square$

The following result establishes that, once again, three of our generating functions are the same.

Theorem 4.7. *We have the equalities*

$$\mathrm{LS}_n(112) = \mathrm{LS}_n(121) = \mathrm{RB}_n(122) = \prod_{i=1}^{n-1}(1+q^i).$$

As before, we break the proof of this result into pieces.

Proposition 4.8. *We have*

$$\mathrm{LS}_n(112) = \mathrm{LS}_n(121).$$

Proof. We will construct a bijection $\xi : R_n(112) \to R_n(121)$ such that $\mathrm{ls}(w) = \mathrm{ls}(\xi(w))$. Given $w \in R_n(112)$ we will construct $\xi(w)$ by rearranging the elements of w in weakly increasing order. For the inverse, if we are given a layered RGF, v, then we use the first element of each layer to form an initial run and rearrange the remaining elements in weakly decreasing order.

For any RGF $w = a_1 \ldots a_n$ we have $\mathrm{ls}(a_i) = a_i - 1$. Since w and $\xi(w)$ are rearrangements of each other, ls is preserved. □

Proposition 4.9. *We have*

$$\mathrm{LS}_n(112) = \mathrm{RB}_n(122).$$

Proof. Let $\eta : R_n(112) \to R_n(122)$ be as in Proposition 4.5. To see that $\mathrm{ls}(w) = \mathrm{rb}(\eta(w))$, first note that $\mathrm{ls}(a_i) = a_i - 1$. By construction the initial run of w has ls that is equal to the total rb of the leading 1 and elements greater than 1 in $\eta(w)$. In addition, for each a_i not in the initial run of w, we place a 1 to the right of $m - a_i + 1$ in $\eta(w)$, and therefore there are $a_i - 1$ elements to its right that are larger than it. Thus $\mathrm{ls}(w) = \mathrm{rb}(\eta(w))$. □

Combining the above propositions, we obtain Theorem 4.7.

4.1.3 Patterns not related to integer partitions

In this section, we present two more connections between the generating functions of patterns of length 3. The first is as follows.

Theorem 4.10. *We have*

$$\mathrm{RS}_n(122) = \mathrm{LB}_n(123) = \mathrm{RS}_n(123) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k.$$

Proof. It was shown in [DDG$^+$16] that

$$\mathrm{LB}_n(123) = \mathrm{RS}_n(123) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k.$$

So it suffices to construct a bijection $f : R_n(122) \to R_n(123)$ that preserves the rs statistic. First, recall that by Theorem 2.5, words in $R_n(123)$ contain only 1s and 2s and that for

$w \in R_n(122)$, every element $j \geq 2$ of w appears only once. Given $w = a_1 \ldots a_n \in R_n(122)$, we will construct $f(w) = b_1 \ldots b_n$ by replacing each element $j \geq 2$ in w with a 2. This is a bijection, as any word in $R_n(122)$ is uniquely determined by the placement of its ones. In addition, $\mathrm{rs}(a_i) = \mathrm{rs}(b_i)$ by construction so that $\mathrm{rs}(w) = \mathrm{rs}(f(w))$. $\square$

The second establishes yet another connection between statistics on $R_n(112)$ and $R_n(122)$.

Theorem 4.11. *We have*

$$\mathrm{RB}_n(112) = \mathrm{LS}_n(122) = \sum_{m=0}^{n} \binom{n-1}{n-m} q^{\binom{m}{2}}.$$

Proof. For the first equality, let $\eta : R_n(112) \to R_n(122)$ be as in Propositions 4.5 and 4.9. We will show that for $w \in R_n(112)$, we have $\mathrm{rb}(w) = \mathrm{ls}(\eta(w))$. Because w is unimodal, only the initial run contributes to rb. If m is the largest element in the initial run of w, then $\mathrm{rb}(w) = 1 + 2 + \cdots + (m-1) = \binom{m}{2}$. Similarly, only the elements greater than 1 in $\eta(w)$ contribute to ls. By construction, the largest element in $\eta(w)$ is m as well. Thus, $\mathrm{ls}(\eta(w)) = 1 + 2 + \cdots + (m-1) = \binom{m}{2}$.

To show that $\mathrm{RB}_n(112) = \sum_m \binom{n-1}{n-m} q^{\binom{m}{2}}$ it suffices, from what we did in the previous paragraph, to count the number of $w \in R_n(112)$ with initial run $12 \ldots m$. Notice that once the elements in the weakly decreasing sequence following the initial run have been selected, there is only one way to order them. For that sequence we must choose $n-m$ elements from the set $[m]$, allowing repetition, yielding a total of $\binom{n-1}{n-m}$ as desired. $\square$

It is remarkable that the map η connects so many of the statistics on $R_n(112)$ and $R_n(122)$; see the proofs of Propositions 4.5, 4.9, and 4.11. The four-variable generating functions $F_n(v; q, r, s, t)$ can be used to succinctly summarize these demonstrations as follows.

Theorem 4.12. *We have*

$$F_n(112; q, r, s, 1) = F_n(122; q, s, r, 1). \qquad \square$$

4.2 Recursive Formulae and Longer Words

In this section we will investigate generating functions for avoidance classes of various RGFs of length greater than three. This includes a recursive formula for computing the generating functions for longer words in terms of shorter ones.

Recall that $w+k$ denotes the word obtained by adding the nonnegative integer k to every element of w. Note that if w is an RGF and k is nonzero, then $w+k$ will not be an RGF. However, the word $\bar{w} = 12\ldots k(w+k)$ obtained by concatenating the increasing sequence $12\ldots k$ with $w+k$, will be an RGF. In fact, there is a relationship between the generating functions for w and $\bar{w}$ for certain statistics. In the following theorem, we show that this relationship holds for the ls and rs statistic. We note that in [MS12, Propositions 2.1 and 2.2], Mansour and Shattuck use the same method to find the cardinality of the avoidance class of the pairs of patterns $\{1222, 12332\}$ and $\{1222, 12323\}$.

Theorem 4.13. *Let v be an RGF and $\bar{v} = 1(v+1)$. Then*

$$\mathrm{LS}_n(\bar{v}) = \sum_{j=0}^{n-1} \binom{n-1}{j} q^j \,\mathrm{LS}_j(v)$$

and

$$\mathrm{RS}_n(\bar{v}) = \sum_{j=0}^{n-1} \sum_{k=0}^{j} \binom{n+k-j-2}{k} q^k \,\mathrm{RS}_j(v).$$

Proof. We start by building the avoidance class of $\bar{v}$ out of the avoidance class of v. We do so by taking a word w in the avoidance class of v, forming $1(w+1)$, and then adding a sufficient number of ones to $1(w+1)$ to obtain a word $\bar{w}$ of length n which avoids $\bar{v}$. We then count how adding these ones affects the respective statistics.

We first establish that avoidance is preserved in this process. Let $w \in R_j(v)$. Since w avoids v, we know $1(w+1)$ avoids $1(v+1) = \bar{v}$. Now we need to show that forming $\bar{w}$ by adding $n-j-1$ ones to $1(w+1)$ in any manner will result in $\bar{w}$ avoiding $\bar{v}$. If $\bar{w} \notin R_n(\bar{v})$, then there is a subword w' of w such that $\mathrm{st}(w') = \bar{v}$. Since $\bar{v} = 1(v+1)$, the smallest element of w' must appear only at the beginning of the subword, and must be a 1 since $1(w+1)$ avoided $\bar{v}$. But removing the unique 1 and standardizing the remaining elements shows that there is a subword of w that standardizes to v. This is a contradiction. Therefore, we must have $\bar{w} \in R_n(\bar{v})$. Similarly, every word in $R_n(\bar{v})$ with $n-j$ ones can be turned into a word in $R_j(v)$ by removing all ones and standardizing. If this word wasn't in $R_j(v)$, then it would contain a subword that standardized to v. As before, this would mean the original word contained $1(v+1) = \bar{v}$, which is a contradiction. Therefore, we can construct every word in $R_n(\bar{v})$ from the words in $R_j(v)$ for $j \in [0, n-1]$.

We now translate this process into the generating function identities. First we will focus on the LS formula. We can choose any $w \in R_j(v)$, and place the elements of $w+1$ in our word $\bar{w}$ in $\binom{n-1}{j}$ different ways since we must leave the first position free to be a one. Then we fill in the rest of the positions with ones. Since we added 1 to each element of $w \in R_j(v)$ and added a one to the beginning of the word, we have $\mathrm{ls}(\bar{w}) = \mathrm{ls}(w) + j$. So

$$\mathrm{LS}_n(\bar{v}) = \sum_{\bar{w} \in R_n(\bar{v})} q^{\mathrm{ls}(\bar{w})} = \sum_{j=0}^{n-1} \sum_{w \in R_j(v)} \binom{n-1}{j} q^j q^{\mathrm{ls}(w)} = \sum_{j=0}^{n-1} \binom{n-1}{j} q^j \, \mathrm{LS}_j(v).$$

For the RS formula, instead of all j elements of $w+1$ increasing the statistic, only the k elements of $w+1$ that are to the left of the rightmost one in $\bar{w}$ will contribute. If we choose where to place these elements, then everything else is forced. We start with $n-1$ positions available, and disregard $j-k+1$ for the rightmost one and the elements of $w+1$ that appear after it. Thus we have $(n-1)-(j-k+1) = n+k-j-2$ positions to choose from. Summing over all values of j and k gives the RS formula. $\square$

In the paper of Dokos et al. [DDJ+12], the authors introduced the notion of statistical Wilf equivalence. We will consider how this idea can be applied to the four statistics we have been studying. We define two RGFs v and w to be ls-*Wilf-equivalent* if $\mathrm{LS}_n(v) = \mathrm{LS}_n(w)$ for all n, and denote this by

$$v \overset{\mathrm{ls}}{\equiv} w.$$

Similarly define an equivalence relation for the other three statistics. Let st denote any of our four statistics. Given any equivalence $v \overset{\mathrm{st}}{\equiv} w$, we can use Theorem 4.13 to generate an infinite number of related equivalences.

Corollary 4.14. *Suppose $v \overset{\mathrm{st}}{\equiv} w$. Then for any $k \geq 1$ we have*

$$12\dots k(v+k) \overset{\mathrm{st}}{\equiv} 12\dots k(w+k).$$

Proof. For $\mathrm{st} = \mathrm{ls}, \mathrm{rs}$ this follows immediately from Theorem 4.13 and induction on k. For the other two statistics, note that the same ideas as in the proof of Theorem 4.13 can be used to show that one can write down the generating function for st over $R_n(12\dots k(v+k))$ in terms of the generating functions for st over $R_j(v)$ for $j \leq n$ although the expressions are more complicated. Thus induction can also be used in these cases as well. □

Applying this corollary to the equivalences in Theorem 4.5, Proposition 4.8, and Theorem 4.10 yields the following result.

Corollary 4.15. *For all $k \geq 1$, we have*

$$\begin{aligned} 12\dots kk(k+1) &\overset{\mathrm{lb}}{\equiv} 12\dots k(k+1)(k+1), \\ 12\dots kk(k+1) &\overset{\mathrm{ls}}{\equiv} 12\dots k(k+1)k, \\ 12\dots k(k+1)(k+1) &\overset{\mathrm{rs}}{\equiv} 12\dots k(k+1)(k+2). \end{aligned}$$ □

We will now demonstrate how these formulae can be used to find the generating functions for a family of RGFs by finding $\mathrm{LS}_n(12\dots k)$ for a general k. We begin by finding the degree of $\mathrm{LS}_n(12\dots k)$ through a purely combinatorial approach before using Theorem 4.13 to give a formula for the generating function itself.

Proposition 4.16. *The generating function $\mathrm{LS}_n(12\dots k)$ is monic and*

$$\deg \mathrm{LS}_n(12\dots k) = \binom{k-2}{2} + (k-2)(n-k+2).$$

Proof. Consider $w = w_1 \dots w_n \in R_n(12\dots k)$ with an initial run of length ℓ. That is, for all $i \in [1,\ell], w_i = i$. Then $\mathrm{ls}(w_{\ell+1}) = w_{\ell+1} - 1$. So we see that $\mathrm{ls}(w_{\ell+1})$ is largest when $w_{\ell+1} = \ell + 1$. However, we cannot have $w_k = k$ without containing $12\dots k$. Therefore, we must have $\ell = k-1$, and $w_i \le k-1$ for all $i \in [k,n]$.

Further, since our initial run contains all elements between 1 and $k-1$, we have $\mathrm{ls}(w_i) = w_i - 1$ for all $i \in [k,n]$. Thus, to maximize ls, we must have $w_i = k-1$ for all $i \in [k,n]$. This gives us the unique word $w = 12\dots(k-2)(k-1)\dots(k-1)$, and a small computation shows $\mathrm{ls}(w) = 0+1+2+\cdots+(k-2)+(n-k+1)(k-2) = \binom{k-1}{2} + (k-2)(n-k+1)$. $\square$

To obtain a formula for $L_n(12\dots k)$ we will use the q-analogues introduced earlier, often suppressing the subscript q for readability. Let

$$K_{m,n} = \frac{[m+1]^{n-1} - 1}{[m]}.$$

We will need the following facts about $K_{m,n}$. Writing $[m+1]^{n-1} = (1+q[m])^{n-1}$ and expanding by the binomial theorem gives

$$K_{m,n} = \sum_{j=1}^{n-1} \binom{n-1}{j} q^j [m]^{j-1}. \tag{12}$$

We also have

$$\frac{1}{[m]}(K_{m+1,n} - K_{1,n}) = \sum_{j=1}^{n-1} \binom{n-1}{j} q^j K_{m,j} \tag{13}$$

which can be obtained by substituting the definition of $K_{m,j}$ into the sum and then applying the previous equation.

Finally we define, for $k \ge 3$,

$$c_k = 1 - \sum_{j=1}^{k-3} \frac{1}{[j]!} c_{k-j}.$$

Note that when $k=3$ the sum is empty and so $c_3 = 1$. While the following expression for $\mathrm{LS}_n(12\ldots k)$ is a sum, note that the number of terms depends only on k and not on n making it efficient for computation.

Theorem 4.17. *For $k \geq 3$, we have*

$$\mathrm{LS}_n(12\ldots k) = 1 + \sum_{i=1}^{k-2} \frac{1}{[i-1]!} c_{k-i+1} K_{i,n}.$$

Proof. We proceed with a proof by induction. In [DDG+16], the authors show that $\mathrm{LS}_n(1/2/3) = [2]^{n-1}$ for the set partition $1/2/3$. Recall that a set partition avoids $1/2/3$ if and only if its corresponding RGF avoids 123. Therefore $\mathrm{LS}_n(1/2/3) = \mathrm{LS}_n(123) = [2]^{n-1}$ for $n \geq 1$. Rewriting this as $\mathrm{LS}_n(123) = 1 + K_{1,n}$ gives our base case for $k = 3$.

Suppose the equality held for $k \geq 3$. Then, using Theorem 4.13 as well as equations (12) and (13),

$$\begin{aligned}
\mathrm{LS}_n(12\ldots k+1) &= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} q^j \, \mathrm{LS}_j(12\ldots k) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} q^j \left(1 + \sum_{i=1}^{k-2} \frac{1}{[i-1]!} c_{k-i+1} K_{i,j} \right) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} q^j + \sum_{i=1}^{k-2} \frac{1}{[i-1]!} c_{k-i+1} \left(\sum_{j=1}^{n-1} \binom{n-1}{j} q^j K_{i,j} \right) \\
&= 1 + K_{1,n} + \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} (K_{i+1,n} - K_{1,n}) \\
&= 1 + K_{1,n} \left(1 - \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} \right) + \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} K_{i+1,n} \\
&= 1 + c_{k+1} K_{1,n} + \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} K_{i+1,n} \\
&= 1 + \sum_{i=1}^{k-1} \frac{1}{[i-1]!} c_{k-i+2} K_{i,n}
\end{aligned}$$

which completes the induction. □

Let 1^m denote the RGF consisting of m copies of one. The ideas in the proof of Theorem 4.13 can be used to give recursive formulae for this pattern. It would be interesting to find other patterns where this reasoning could be applied.

Theorem 4.18. *For $m \geq 0$, we have*

$$\mathrm{LS}_n(1^m) = \sum_{j=1}^{m-1} \binom{n-1}{j-1} q^{n-j}\, \mathrm{LS}_{n-j}(1^m)$$

and

$$\mathrm{RS}_n(1^m) = \mathrm{RS}_{n-1}(1^m) + \sum_{j=2}^{m-1} \sum_{k=0}^{n-j} \binom{j+k-2}{k} q^k\, \mathrm{RS}_{n-j}(1^m).$$

Proof. Let w avoid 1^m. Then w can be uniquely obtained by taking a w' avoiding 1^m and inserting j ones in $w'+1$, where $1 \leq j \leq m-1$ and a one must be inserted at the beginning of the word. The formula for ls now follows since the binomial coefficient counts the number of choices for the non-initial ones, $\mathrm{LS}_{n-j}(1^m)$ is the contribution from $w'+1$, and q^{n-j} is the obtained from the interaction between the initial one and $w'+1$. The reader should now have no problem modifying the proof of the rs formula in Theorem 4.13 to apply to this case. □

4.3 The pattern 1212

4.3.1 Noncrossing partitions

The set partitions which avoid the pattern 13/24 are called *non-crossing* and are of interest, in part, because of their connection with Coxeter groups and Catalan numbers. See the memoir of Armstrong [Arm09] for more information. In this case the set containment in Proposition 2.2 can be turned into an equality as we will show next. Note that $w(13/24) = 1212$.

Proposition 4.19. *We have*

$$R_n(1212) = w(\Pi_n(13/24)).$$

Proof. As just noted, it suffices to show that if π contains 13/24, then $w(\pi)$ contains 1212. By definition, if π contains 13/24, then $w(\pi)$ contains a subword $abab$ for some $a \neq b$. If $a < b$, then this will standardize to 1212 as desired. If $a > b$ then, because $w(\pi)$ is a restricted growth function, there must be some occurrence of b before the leftmost occurrence of a in $w(\pi)$. Thus $w(\pi)$ also contains a subword $baba$ which is a copy of 1212 in $w(\pi)$. $\square$

With this proposition in hand, we now focus on gaining information about these partitions by studying $R_n(1212)$. We begin by applying the rs statistic to $R_n(1212)$, and in doing so obtain a q-analogue of the standard Catalan recursion. We first need the following lemma regarding 1212-avoiding restricted growth functions.

Lemma 4.20. *For an RGF $w = a_1 \ldots a_n$, the following are equivalent:*

(1) The RGF w avoids 1212.

(2) There are no $abab$ subwords in w.

(3) If $a_i = a_{i'}$ for some $i < i'$ then, for all $j' > i'$, either $a_{j'} \leq a_{i'}$ or $a_{j'} > \max\{a_1, \ldots, a_{i'}\}$

.

Proof. The equivalence of the first two statements follows from the proof of Proposition 4.19. It thus suffices to show that (2) and (3) are equivalent. First, let $w = a_1 \ldots a_n$ be an RGF with no $abab$ subword, and let $a_i = a_{i'}$ for some $i < i'$. Assume, towards contradiction, that there exists a j' with $j' > i'$ and $a_{i'} < a_{j'} \leq \max\{a_1, \ldots a_{i'}\}$. This implies that there exists a j with $j < i'$ and $a_j = a_{j'}$. If $i < j < i'$, then $a_i a_j a_{i'} a_{j'}$ forms an $abab$ subword in w, a contradiction. If this is not the case, then since $a_j > a_i$ and w is an RGF, there must exist another occurrence of the letter a_i preceding a_j. This letter, combined with a_j, $a_{i'}$, and $a_{j'}$

still creates an $abab$ subword, which is again a contradiction. This shows that (2) implies (3).

Now we show that if w contains an $abab$ subword, then w cannot satisfy (3). Indeed, by the discussion in the proof of Proposition 4.19, if w contains an $abab$ subword then, without loss of generality, we may assume $a < b$. Thus the second occurrence of b in the subword will violate condition (3). This completes the proof of the equivalence of the statements. $\square$

We now move to a recursive way of producing words in $R_n(1212)$. Given two words u and v, we write uv to denote the concatenation of u and v. Furthermore, if $u = a_1 \dots a_k$ then let $u+1 = (a_1+1)\dots(a_k+1)$. With this notation we can state the following corollary of Lemma 4.20.

Corollary 4.21. *If u is in $R_{n-1}(1212)$ then both $1u$ and $1(u+1)$ are in $R_n(1212)$.*

Proof. Let u be an element of $R_{n-1}(1212)$. By the previous lemma, we know that u does not contain any $abab$ subwords. Prepending a 1 to u will not create any such subword, as otherwise this would imply an $abab$ subword in u using its leading 1. Therefore $1u$ is contained in $R_n(1212)$. Furthermore, adding one to each element in u to create $u+1$ will not introduce an $abab$ subword, and prepending a 1 to create $1(u+1)$ will not create an $abab$ subword as there is only one copy of 1 in $1(u+1)$. Thus $1(u+1)$ is also contained in $R_n(1212)$. $\square$

With these results in hand, we move to one of the main results of this section. For two words w and u, we will use the set notation $w \cap u = \emptyset$ to denote that w and u have no elements in common. The next theorem gives a q-analogue of the usual recursion for the Catalan numbers. It will also be used to establish a connection between $R_n(1212)$ and lattice paths.

Theorem 4.22. *We have*

$$\mathrm{RS}_0(1212) = 1,$$
$$\mathrm{RS}_1(1212) = 1,$$

and for $n \geq 2$,

$$\mathrm{RS}_n(1212) = 2\,\mathrm{RS}_{n-1}(1212) + \sum_{k=1}^{n-2} q^k\,\mathrm{RS}_k(1212)\,\mathrm{RS}_{n-k-1}(1212).$$

Proof. To prove the recursion, we partition $R_n(1212)$ into three disjoint subsets S, T, and U as follows:

$$S = \{w \in R_n(1212) \;:\; a_1 = 1 \text{ and there are no other 1s in } w\},$$
$$T = \{w \in R_n(1212) \;:\; a_1a_2 = 11\},$$
$$U = \{w \in R_n(1212) \;:\; a_1a_2 = 12 \text{ and there is at least one other 1 in } w\}.$$

We claim that we can also describe S as the set of words defined by

$$S = \{w = 1(u+1) \;:\; u \in R_{n-1}(1212)\}. \tag{14}$$

To see this, let u be a word in $R_{n-1}(1212)$. From Corollary 4.21, we know $w = 1(u+1)$ is an element of $R_n(1212)$, and by definition of $u+1$, the only 1 in w will be a_1. This gives one containment. Now let w be an element of S as originally defined. Since the leading one in w is unique, let $u+1$ denote the last $n-1$ letters in w. By Lemma 4.20, w contains no *abab* subword; in particular, $u+1$ contains no *abab* subword. Standardizing $u+1$ to the RGF u will not create any *abab* subwords, and thus u will be contained in $R_{n-1}(1212)$. This gives the reverse containment, from which we conclude that the two sets are equal. A similar proof, without standardization of the subword, allows us to describe T as the set

$$T = \{w = 1u \ : \ u \in R_{n-1}(1212)\}. \tag{15}$$

Now note that for any RGF u, we have $\mathrm{rs}(u) = \mathrm{rs}(1(u+1))$ and $\mathrm{rs}(u) = \mathrm{rs}(1u)$. Using this fact, and the above characterization of the sets, we can see that S and T must contribute $\mathrm{RS}_{n-1}(1212)$ each to the total $\mathrm{RS}_n(1212)$ polynomial.

Finally, we claim that we can characterize U as

$$\begin{aligned} U = \{w = 1(u+1)1v \ : u \in R_k(1212) \text{ for } 1 \le k \le n-2, \\ \mathrm{st}(1v) \in R_{n-k-1}(1212), v \cap (u+1) = \emptyset\}. \end{aligned} \tag{16}$$

First, let w be contained in U as defined at the beginning of the proof. By definition of U, w has a nonempty subword of the form $u+1$ consisting of all entries between the first and second 1 in w. Let the length of u be k. As with the set S, $u+1$ will standardize to u, an RGF in $R_k(1212)$. Now let v be the last $n-k-2$ letters in w, so that our word is of the form

$$w = 1(u+1)1v.$$

Since a 1 is repeated before v, we must have $v_i = 1$ or $v_i > \max(u+1)$ for all i by Lemma 4.20, where v_i is the ith letter of v. This gives $v \cap (u+1) = \emptyset$. Furthermore, there is no $abab$ subword contained in $1v$, and standardizing the subword will not create an $abab$ pattern. Thus $\mathrm{st}(1v)$ is contained in $R_{n-k-1}(1212)$. This shows one inclusion between the two versions of U. Now let u be an element of $R_k(1212)$, and let $1v'$ be an element of $R_{n-k-1}(1212)$. Corollary 4.21 gives that $1(u+1)$ avoids 1212 as well. Now from the RGF $1v'$, we create the word $1v$ by setting

$$(1v)_i = \begin{cases} (1v')_i & \text{if } (1v')_i = 1 \\ (1v')_i + \max(u) & \text{if } (1v')_i \neq 1. \end{cases}$$

We claim that $w = 1(u+1)1v$ is a member of $R_n(1212)$. To see this, note that $u+1$ contains no *abab* subwords, and further $u+1$ shares no integers in common with the rest of w. Therefore $u+1$ cannot contribute to an *abab* subword in w. Thus if such a subword existed in w, it must also exist in $11v$. This is impossible as it would imply an *abab* subword in $1v'$, contradicting our choice of $1v'$. We have now shown the reverse set containment, which implies the desired equality of the two sets.

With this characterization of U, we can now decompose $\mathrm{rs}(w)$ for w in U as

$$\mathrm{rs}(w) = \mathrm{rs}(u+1) + k + \mathrm{rs}(1v),$$

where the middle term comes from the contribution to rs caused by comparing the elements of $u+1$ with the second 1 in w. Summing over all possibilities of k, u, and v, and noting that the rs of a word is not affected by standardization, we can see that U will contribute

$$\sum_{k=1}^{n-2} q^k \, \mathrm{RS}_k(1212) \, \mathrm{RS}_{n-k-1}(1212).$$

Adding the results obtained from S, T, and U now gives the desired total. □

For the next result, we first recall the definition of a Motzkin path. A Motzkin path P of length n is a lattice path in the plane which starts at $(0,0)$, ends at $(n,0)$, stays weakly above the x-axis, and which uses vector steps in the form of up steps (1,1), horizontal steps (1,0), and down steps (1,-1). Let $\mathcal{M}_n$ denote the set of all Motzkin paths of length n. We write $P = s_1 \ldots s_n$ for such a path, where

$$s_i = \begin{cases} U \text{ if the } i\text{th step is an up step,} \\ L \text{ if the } i\text{th step is a horizontal step,} \\ D \text{ if the } i\text{th step is a down step.} \end{cases}$$

Given a step s_i in P, we can realize s_i geometrically as a line segment in the plane connecting two lattice points in the obvious way. With this in mind, define the *level* of s_i, $l(s_i)$, to be the lowest y-coordinate in s_i. Note that the level statistic provides a natural pairing of up steps with down steps in a Motzkin path. Namely, we associate an up step s_i with the first down step s_j, $j > i$, which is at the same level as s_i, i.e. $l(s_i) = l(s_j)$. We will call such steps *paired*.

We now define a *two-colored Motzkin path* R of length n to be a Motzkin path of length n whose level steps are individually colored using one of the colors a or b. We will call an a-colored level step an *a-step* and a b-colored level step a *b-step*. For a two-colored Motzkin path $R = s_1 \dots s_n$ we will still use s_i equal to U or D for up steps and down steps, but will use a or b instead of L to show the color of the level steps. Let $\mathcal{M}_n^2$ denote the set of all two-colored Motzkin paths of length n. For two paths $P = s_1 \dots s_n$ and $Q = t_1 \dots t_m$ we write $PQ = s_1 \dots s_n t_1 \dots t_m$ to indicate their concatenation.

Let the *area* of a path R, $\mathrm{area}(R)$, denote the area enclosed between R and the x-axis. Defining

$$M_n(q) = \sum_{R \in \mathcal{M}_n^2} q^{\mathrm{area}(R)}, \tag{17}$$

Drake [Dra09] proved the following recursion.

Theorem 4.23 ([Dra09])**.** *We have $M_0(q) = 1$ and, for $n \geq 1$,*

$$M_n(q) = 2M_{n-1}(q) + \textstyle\sum_{k=1}^{n-2} q^k M_k(q) M_{n-k-1}(q). \qquad \square$$

Using Theorems 4.22 and 4.23 as well as induction on n immediately gives the following equality.

Corollary 4.24. *We have*

$$\mathrm{RS}_n(1212) = M_{n-1}(q)$$

for all $n \geq 1$. $\square$

Interestingly, it turns out that we also have $\mathrm{LB}_n(1212) = \mathrm{LB}_n(1221) = M_{n-1}(q)$ which will be proved in Section 4.4. In our next result, we prove the previous corollary directly via a bijection between $\mathcal{M}^2_{n-1}$ and $R_n(1212)$.

Theorem 4.25. *There is an explicit bijection* $\psi : \mathcal{M}^2_{n-1} \to R_n(1212)$ *such that* $\operatorname{area}(R) = \operatorname{rs}(\psi(R))$ *for all* $R \in \mathcal{M}^2_{n-1}$.

Proof. Given $R = s_1 \dots s_{n-1} \in \mathcal{M}^2_{n-1}$ we define $\psi(R) = w = a_1 a_2 \dots a_n$ as follows. Let $a_1 = 1$ and

$$a_{i+1} = \begin{cases} \max(a_1 \dots a_i) + 1 & \text{if } s_i \text{ equals } U \text{ or } b, \\ a_i & \text{if } s_i = a, \\ a_j & \text{if } s_i = D \text{ is paired with the up step } s_j. \end{cases}$$

By way of example, we have $\psi(UaUDbDUaD) = 1223241551$. We first show that ψ is well defined. By definition we have $a_1 = 1$ and, for $i > 1$, a_i is either equal to a_j for some $j < i$ or $\max(a_1 \dots a_{i-1})+1$. This implies that that a_i is a positive integer and $a_i \leq \max(a_1 \dots a_{i-1})+1$ for all i, so w is an RGF.

For the avoidance condition, note that a number in w will appear more than once for only two reasons. The first is because of a-steps which will give us a consecutive string of this number. The second is because of a paired up step and down step. Suppose, towards a contradiction, that w has the pattern 1212 and so will have a subsequence $ijij$ with $i < j$. Since we have repeated i's which are not part of a consecutive string we must have a paired up and down step which give us two i's. Similarly because of the repeated j's we have a paired up and down step which give us two j's. However, this means that $s_1 \dots s_{n-1}$ has a subword $UUDD$, but the first U and the first D are paired and the second U and D are paired which is not possible. So w avoids 1212.

To motivate the definition of the inverse note that, in the definition of ψ, if $s_i = U$ then we have an increase $a_i < a_{i+1}$. Since the up step must have a paired down step s_j there must be some $j > i$ with $a_j = a_i$. If instead $s_i = b$ we have an increase $a_i < a_{i+1}$, but our map will not further repeat a_i. If $s_i = a$ then $a_i = a_{i+1}$. Finally consider if $s_i = D$. We claim that in this case $a_i > a_{i+1}$. Indeed, this down step has a paired up step s_k with $k < i$ and, since s_k is an up step, we have $a_{k+1} = \max(a_1 \dots a_k) + 1$. Since s_i is s_k's paired down step every $j \in [k+1, i]$ will have a_j equal to $\max(a_1 \dots a_{j-1}) + 1$ or equal to a number whose index is earlier in the interval $[k+1, i]$. So for all $j \in [k+1, i]$ we have $a_j > a_k = a_{i+1}$. As result $a_i > a_{i+1}$ as claimed. This discussion leads us to define, for $w = a_1 a_2 \dots a_n \in R_n(1212)$, the lattice path $\psi^{-1}(w) = R = s_1 \dots s_{n-1}$ where

$$s_i = \begin{cases} a \text{ if } a_i = a_{i+1}, \\ b \text{ if } a_i < a_{i+1} \text{ and } \nexists\ j > i+1 \text{ such that } a_j = a_i, \\ U \text{ if } a_i < a_{i+1} \text{ and } \exists\ j > i+1 \text{ such that } a_j = a_i, \\ D \text{ if } a_i > a_{i+1}, \end{cases}$$

By our previous discussion, this map is an inverse on the image of ψ. Since it is known that $|\mathcal{M}_{n-1}^2| = C_n = |R_n(1212)|$, where C_n is the nth Catalan number, ψ must be a bijection.

Lastly we will show that $\text{area}(R) = \text{rs}(g(R))$. Consider a letter a_i. We want to count the number of distinct elements to the right and smaller than a_i. We will consider which steps s_k with $k \geq i$ make a_{k+1} smaller than a_i. If $s_k = a$ then $a_{k+1} = a_k$ which is either equal to a_i or doesn't bring about a new distinct number so these steps need not be considered. If s_k equals b or U then a_{k+1} is larger than all previous numbers, so is not smaller than a_i. So the only steps which could result in something to the right and smaller than a_i are down steps $s_k = D$. Let $s_\ell = U$ be its paired up step. First we will consider the case when $\ell = i$. In this case, $a_{k+1} = a_i$ so a_{k+1} is not smaller than a_i. If instead $\ell > i$, we have $a_{k+1} = a_\ell$ and since a_ℓ is right of a_i the number a_{k+1} is not a distinct number right of a_i. Our last case is that $\ell < i$. We showed earlier that if s_ℓ is an up step paired with the down step s_k, then for all $j \in [\ell+1, k]$ we have $a_j > a_{k+1} = a_\ell$. Since $i \in [\ell+1, k]$ it follows that $a_i > a_{k+1}$ which shows that a_{k+1} is to the right and smaller than a_i. Finally, we also have that for all j in $[i, k]$, $a_j > a_{k+1}$. Thus a_{k+1} is the first occurrence of this letter that appears to the right of a_i, and so a_{k+1} is counted by rs.

This means that $\mathrm{rs}(a_i)$ is equal to the number of down steps weakly to the right of s_i such that its paired up step is strictly to the left of s_i. In the case of s_i equal to a, b, or U this calculation is equal to the level of the step. In the case of $s_i = D$ this calculation is equal to level of the step plus one. All together this gives the total area under the path R. Since this also counts $\mathrm{rs}(w)$ we have that $\mathrm{area}(R) = \mathrm{rs}(w)$. □

4.3.2 Combinations with other patterns

Next we examine RGFs that avoid 1212 and another pattern of length 3. As the patterns 121, 122, and 112 are all subpatterns of 1212, the only interesting cases to look at are $R_n(111, 1212)$ and $R_n(123, 1212)$. We start by calculating $\mathrm{RS}_n(111, 1212)$. It is easy to combine Theorem 2.5 and Lemma 4.20 to characterize $R_n(111, 1212)$.

Lemma 4.26. *We have*

$$R_n(111, 1212) = \{w \in R_n(1212) \ : \ \textit{every element of } w \textit{ appears at most twice}\}.$$

for all $n \geq 0$. □

The following proposition is similar to Theorem 4.22 in several respects. First, this proposition provides a q-analogue of the standard Motzkin recursion and is proved using very similar recursive techniques as before. Furthermore, it will also be used to connect $R_n(111, 1212)$ to lattice paths.

Proposition 4.27. *We have*

$$\mathrm{RS}_0(111, 1212) = 1,$$

$$\mathrm{RS}_1(111, 1212) = 1,$$

and for $n \geq 2$,

$$\mathrm{RS}_n(111, 1212) = \mathrm{RS}_{n-1}(111, 1212) + \sum_{k=0}^{n-2} q^k \,\mathrm{RS}_k(111, 1212)\,\mathrm{RS}_{n-k-2}(111, 1212).$$

Proof. We follow the proof of Theorem 4.22 by partitioning $R_n(111, 1212)$ into the sets

$$\begin{aligned}
S &= \{w \in R_n(111, 1212) \ : \ a_1 = 1 \text{ and there are no other 1s in } w\},\\
T &= \{w \in R_n(111, 1212) \ : \ a_1 a_2 = 11\},\\
U &= \{w \in R_n(111, 1212) \ : \ a_1 a_2 = 12 \text{ and there is a single other 1 in } w\}.
\end{aligned}$$

Using the same reasoning as in Theorem 4.22 and adding the restrictions of avoiding 111 gives the equivalent characterizations

$$\begin{aligned}
S &= \{w = 1(u+1) \ : \ u \in R_{n-1}(111, 1212)\}\\
T &= \{w = 11(u+1) \ : \ u \in R_{n-2}(111, 1212)\}\\
U &= \{w = 1(u+1)1v \ : \ u \in R_k(111, 1212) \text{ for } 1 \leq k \leq n-2\\
&\qquad\qquad \mathrm{st}(v) \in R_{n-k-2}(111, 1212), v \cap 1(u+1) = \emptyset\}.
\end{aligned}$$

From this the desired recurrence easily follows. □

The next result provides an explicit bijection between $R_n(111, 1212)$ and $\mathcal{M}_n$. We first extend the level statistic defined in the previous section to paths. Given a Motzkin path $P = s_1 \ldots s_n$, we define the level of the path, $l(P)$, to be

$$l(P) = \sum_{i=1}^{n} l(s_i).$$

It should be noted that if we impose a rectangular grid of unit squares on the first quadrant of the plane, then $l(P)$ simply counts the total area of the unit squares contained below P

and above the x-axis. We will use our bijection to calculate the generating function for the level statistic taken over all Motzkin paths of length n.

Theorem 4.28. *For $n \geq 0$, we have*

$$\mathrm{RS}_n(111, 1212) = \sum_{P \in \mathcal{M}_n} q^{l(P)}.$$

Proof. We start by defining a bijection $\phi : R_n(111, 1212) \mapsto \mathcal{M}_n$. For any $w = a_1 \ldots a_n$, we let $\phi(w) = P$, where $P = s_1 \ldots s_n$ and

$$s_i = \begin{cases} U \text{ if } a_i = a_j \text{ for some } j > i, \\ L \text{ if } a_i \neq a_j \text{ for any } j \neq i, \\ D \text{ if } a_i = a_j \text{ for some } j < i. \end{cases}$$

To show that ϕ is well defined, first note that since w contains at most two copies of any integer, the three cases are disjoint and cover all possibilities. We also need to show that P is a Motzkin path. But this is true because the definition of ϕ induces a bijection between the up steps and down steps of $\phi(w)$ in which each up step precedes its corresponding down step.

We will need the fact that this bijection between up and down steps induced by the definition of ϕ is exactly the same as the pairing relationship in the path $\phi(w)$. Formally, we have that $i < j$ and $a_i = a_j$ if and only if s_i is the up step paired with the down step s_j. To see this, assume $i < j$ and $a_i = a_j$. Consider the subword $a_i \ldots a_j$. As w avoids 111 and 1212, we must have $a_k > a_i$ for each $i < k < j$. Furthermore, if $i < k < j$ and if $a_k = a_{k'}$ for some other k', we clearly must also have $i < k' < j$. Thus the subpath $s_{i+1} \ldots s_{j-1}$ is a Motzkin path translated to start at the level of s_{i+1}. It follows that s_i and s_j must be paired. This in fact proves the equivalence, as the pairing relationship on a Motzkin path is unique.

Before inverting ϕ, it will be useful to look again at its definition. Recall that any sequence $w = a_1 a_2 \dots a_n$ of integers has a *left to right maximum* at i if $a_i > \max(a_1 \dots a_{i-1})$. If w is an RGF then clearly the left to right maxima occur exactly when $a_i = \max(a_1 \dots a_{i-1}) + 1$. Another characterization for RGFs is that w has a left to right maximum at i if and only if a_i is the first occurrence of that value in w. So if $w \in R_n(111, 1212)$, the left to right maxima occur precisely at those i corresponding to the first two cases in the definition of ϕ.

Now to invert ϕ, let $P = s_1 \dots s_n$ be a path in $\mathcal{M}_n$. We define $\phi^{-1}(P) = a_1 \dots a_n$ by $a_1 = 1$ and, for $j \geq 2$,

$$a_j = \begin{cases} \max(a_1 \dots a_{j-1}) + 1 & \text{if } s_j = U \text{ or } s_j = L, \\ a_i & \text{if } s_j \text{ is a down step paired with } s_i. \end{cases}$$

The proof that this function is well defined is similar to the one given for ϕ and so omitted. And from the description of ϕ in terms of left to right maxima as well as our remarks about ϕ's relationship to the pairing bijection, it should be clear that this is the inverse function.

It now suffices to show that $\mathrm{rs}(w) = l(\phi(w))$ for any w in our avoidance class. Let $w = a_1 \dots a_n$ and $\phi(w) = s_1 \dots s_n$. We will prove the stronger statement that $\mathrm{rs}(a_i) = l(s_i)$ for $1 \leq i \leq n$. To do this, note that if $l(s_i) = k$, then there are precisely k down steps to the right of s_i whose paired up steps precede s_i in $\phi(w)$.

Now assume $\mathrm{rs}(a_i) = k$. By definition, there are k integers $a_{j_1}, \dots, a_{j_k}$ to the right and smaller of a_i. As w is an RGF, each of these integers also appear to the left of a_i in w. By the definition of ϕ, the $s_{j_1}, \dots, s_{j_k}$ are down steps which follow s_i in $\phi(w)$ whose paired up steps precede s_i. This gives $l(s_i) \geq k$.

To see that we actually have equality, assume that there is another down step s_l which follows s_i in $\phi(w)$. We know that in w, $a_i \leq a_l$, as a_l does not contribute to $\mathrm{rs}(a_i)$. If $a_i = a_l$, then in fact s_i and s_l must be paired via level, and thus s_l does not change $l(s_i)$. Finally, we deal with the case $a_i < a_l$. As s_l is a down step, there must exist another letter $a_{l'}$ in w with $l' < l$ and $a_{l'} = a_l$. In order for w to be an RGF and to avoid 1212, one can see that we must also have $i < l'$. Hence s_l and its paired up step $s_{l'}$ both follow s_i in $\phi(w)$, and thus s_l will still not affect $l(s_i)$. Thus we have $\mathrm{rs}(a_i) = l(s_i) = k$ as desired. □

We now conclude the section with a simple proposition characterizing $R_n(123, 1212)$. As the result follows easily from Theorem 2.5, Proposition 4.20, and standard counting techniques, we leave the proof to the reader.

Proposition 4.29. *If w is contained in $R_n(123, 1212)$, then*

$$w = 1^l 2^i 1^{n-i-l}$$

for some $l \geq 1$, $i \geq 0$ satisfying $l + i \leq n$. As such, for $n \geq 0$ we have

$$\mathrm{LB}_n(123, 1212) = \mathrm{RS}_n(123, 1212) = 1 + \sum_{k=0}^{n-2} (n-k-1) q^k$$

and

$$\mathrm{LS}_n(123, 1212) = \mathrm{RB}_n(123, 1212) = 1 + \sum_{k=1}^{n-1} (n-k) q^k.$$ □

4.4 The pattern 1221

4.4.1 Nonnesting partitions

The term nonnesting has been defined in different ways in the literature. In some sources a nonnesting partition is a partition π where we can never find four elements $a < x < y < b$ such that $a, b \in A$ and $x, y \in B$ for two distinct blocks A, B. This is the sense used in Klazar's paper [Kla96] and is equivalent to a partition avoiding 14/23.

In other papers, including Klazar's article [Kla00b], a partition π is nonnesting if, whenever there are four elements $a < x < y < b$ such that $a, b \in A$ and $x, y \in B$ for two distinct blocks A, B, then there exists a $c \in A$ such that $x < c < y$. This definition is often given using arc diagrams. We draw the *arc diagram* of a partition of $[n]$ by writing 1 through n on a straight line and drawing arcs (a,b) if $a<b$ are in a block and consecutive when writing the block in increasing order, see Figure 3. A *nesting* is a pair of arcs (a,b) and (x,y) such that $a<x<y<b$, and we will say in this case that the pair of arcs *nest*. For completeness, we prove the equivalence of this condition with the second definition of a nonnesting partition. It is known that the number of partitions satisfying either of these two equivalent conditions is the Catalan numbers, C_n.

Proposition 4.30. *The following conditions are equivalent for a partition π.*

1. *If there are four elements $a<x<y<b$ such that $a,b\in A$ and $x,y\in B$ for two distinct blocks A,B then there exists a $c\in A$ such that $x<c<y$.*

2. *The arc diagram for π contains no nestings.*

Proof. We will first show that if a partition fails condition 1, then its arc diagram has a nesting. Say that there are four elements $a<x<y<b$ such that $a,b\in A$ and $x,y\in B$ for two distinct blocks A,B but there is no $c\in A$ such that $x<c<y$. Since there is an $a\in A$ with $a<x$ there is a largest element $\bar{a}\in A$ where $\bar{a}<x$. Similarly, there is a smallest $\bar{b}\in A$ with $y<\bar{b}$. Since there is no element of A between x and y, $(\bar{a},\bar{b})$ must be an arc. Also there is a smallest element $\bar{y}\in B$ such that $x<\bar{y}$ so that $(x,\bar{y})$ is an arc. Since $\bar{a}<x<\bar{y}<\bar{b}$ these arcs nest which is a contradiction.

Conversely, assume that the arc diagram has two arcs (a,b) and (x,y) which nest with $a<x<y<b$. By construction of the arcs, this implies that a and b are consecutive elements in their block A, so there does not exist a $c\in A$ such that $a<c<b$ and the first condition is false. $\square$

There is another notion of nonnesting which we will call left nonnesting and can be defined by a different collection of arcs. For each block B we will draw all arcs of the form $(\min B,b)$ with $b\in B\setminus\{\min B\}$, and call the diagram with these arcs the *left arc diagram*. An example is displayed in Figure 3. If a partition's left arc diagram has no pair of arcs

Figure 3 The arc diagram and left arc diagram for the partition 134/267/5.

which nest then we will call this partition *left nonnesting* to distinguish our term from the previous two definitions of nonnesting. Let this set be LNN_n.

Jelínek and Mansour in their paper [JM08] defined nonnesting by requiring that a partition's associated RGF avoid 1221, and they found that $|R_n(1221)| = C_n$, the nth Catalan number. For a partition π, it turns out that $w(\pi)$ avoids 1221 if and only if its left arc diagram contains no nestings. As result, the partitions in LNN_n are also counted by the nth Catalan number.

Proposition 4.31. *We have*

$$R_n(1221) = w(LNN_n).$$

Proof. First we will show that if a partition's left arc diagram contains a nesting then its associated RGF has the pattern 1221. Let $\pi = B_1/\ldots/B_k$ be a partition of $[n]$. Say that its left arc diagram has a nesting which means that we have arcs (a, b) and (x, y) such that $a < x < y < b$. Since these are arcs from the left arc diagram we know that $a = \min B_i$ and $x = \min B_j$ for some distinct blocks B_i and B_j, and since we order the blocks of π so that their minimum elements increase we know that $i < j$. As result $w(\pi)$ has the subword $ijji$ which is the pattern 1221.

Conversely, say that we have an RGF w with the pattern 1221, so it has a subword $ijji$ with $i < j$. Pick the subword so that the first i and j are the first occurrences of these letters in w. Thus they correspond to minima in their respective blocks of the corresponding partition π. It follows that the two i's and two j's give rise to nesting arcs in the left arc diagram of π. $\square$

The rest of this section will describe $R_n(1221)$, some of its generating functions, and some connections to other patterns. We will prove that $\mathrm{LB}_n(1221) = \mathrm{RS}_n(1212)$ by showing that there exists a bijection from two-colored Motzkin paths to $R_n(1221)$ which maps area to lb, and then the result will follow from Theorem 4.25. We further use this bijection and previous methods to determine the generating function for some pairs of RGFs which include 1221. We end the section by showing $\mathrm{LB}_n(1221) = \mathrm{LB}_n(1212)$ and summarizing all the equalities we have proved.

4.4.2 The pattern 1221 by itself

For an RGF $w = w_1w_2 \ldots w_n$ we will call a letter w_i *repeated* if there exists a $j < i$ such that $w_j = w_i$. If a letter is not a repeated letter, we will call it a *first occurrence*. Since w is an RGF, the first occurrences are exactly the left-right maxima.

Lemma 4.32. *A word $w \in R_n(1221)$ if and only if the subword of all repeated letters in w is weakly increasing.*

Proof. Say that w contains the pattern 1221 and so has a subword $abba$ for some $a < b$. The second ba are repeated letters in w. This implies that there is a decrease in the subword of all repeated letters.

Conversely, say that the subword of all repeated letters of w has an decrease ba with $a < b$. Since these are repeated letters in an RGF the first b of w appears earlier, and the first a in w appears earlier than the first b. Hence we have a subword $abba$ with $a < b$ and the pattern 1221. □

Using the previous lemma we can define a surjection $\mathrm{inc} : R_n \to R_n(1221)$. The map will take a $w \in R_n$ and will output $\mathrm{inc}(w) = v$ which is w with its subword of repeated letters put in weakly increasing order. For example if $w = 1112221331$ then $\mathrm{inc}(w) = 1112112323$.

To see this map is well defined we must first show that v is an RGF. But the subword of repeated letters is rearranged to be weakly increasing which forces the maximum of any prefix to weakly decrease. Since the left-right maxima of w do not move in this process, they do not change in passing to v so that the latter is still an RGF. Also, v avoids 1221 by Lemma 4.32, showing inc is well defined.

In the next lemma we show that inc preserves lb. Note that because w is an RGF, all the numbers in the interval $[w_i + 1, \max\{w_1, \ldots, w_{i-1}\}]$ appear to the left of w_i and are larger than w_i, so

$$\operatorname{lb}(w_i) = \max\{w_1, \ldots, w_{i-1}\} - w_i. \tag{18}$$

Lemma 4.33. *Let v be a rearrangement of w such that both have the same left-right maxima in the same places. Then* $\operatorname{lb}(v) = \operatorname{lb}(w)$*. In particular,* $\operatorname{lb}(w) = \operatorname{lb}(\operatorname{inc}(w))$.

Proof. Since w and v only have their repeated letters rearranged and their left-right maxima fixed, we know $\max\{w_1, \ldots, w_i\} = \max\{v_1, \ldots, v_i\}$ for all i and $\{v_1, \ldots, v_n\} = \{w_1, \ldots, w_n\}$ as multisets. Using equation (18)

$$\operatorname{lb}(w) = \sum_{i=1}^{n} (\max\{w_1, \ldots, w_{i-1}\} - w_i) = \sum_{i=1}^{n} (\max\{v_1, \ldots, v_{i-1}\} - v_i) = \operatorname{lb}(v).$$

The special case of $v = \operatorname{inc}(w)$ now follows from the definition of the function. □

We wish to show that the generating function $\mathrm{RS}_n(1212)$ discussed in Section 4.3 is equal to $\mathrm{LB}_n(1221)$. The proof will be similar to that of Theorem 4.25 in that we will construct a bijection β from two-colored Motzkin paths length $n-1$ to $R_n(1221)$ which maps area to lb. The map β will not be difficult to describe. However, proving that β is a bijection will require a detailed argument. We define a map $\alpha : R_k(1221) \to R_{k+2}(1221)$ and provide the following lemma to assist us. This map will be useful when discussing two-colored Motzkin paths which are obtained from a smaller path by prepending an up step and appending a down step. Given any $v \in R_k(1221)$ we define $\bar{v} = \bar{v}_1 \bar{v}_2 \ldots \bar{v}_k$ such that

$$\bar{v}_i = \begin{cases} v_i + 1 & \text{if } v_i \text{ is a first occurrence,} \\ v_i & \text{else.} \end{cases} \tag{19}$$

It is not hard to see that $u = 1\bar{v}1$ is an RGF, but it may not avoid 1221, so we define

$$\alpha(v) = \mathrm{inc}(u)$$

which is in $R_{k+2}(1221)$ by Lemma 4.32. For example, if $v = 1212344$ will have $u = 1\bar{v}1 = 123124541$ and $\alpha(v) = 123114524$.

Lemma 4.34. *For $k \geq 0$ the map $\alpha : R_k(1221) \to R_{k+2}(1221)$ is an injection. Furthermore, the image of α is precisely the $w \in R_{k+2}(1221)$ satisfying the following three properties.*

(i) The word w has more than one 1 and ends in a repeated letter.

(ii) If w_i is a repeated letter then $w_i < \max\{w_1, \ldots, w_{i-1}\}$.

(iii) If, for $i \leq j$, we have w_{i-1} and w_{j+1} are repeated letters with $w_i w_{i+1} \ldots w_j$ all first occurrences then $w_{j+1} < w_i - 1$.

Proof. We will start by showing that α is injective. Given a $v \in R_k(1221)$, consider $u = 1\bar{v}1$. We can easily recover $\bar{v}$ from u by removing the first and last 1, and can further recover v by decreasing all left-right maxima in $\bar{v}$ by one. We finish showing that α is injective by recovering u from $w = \mathrm{inc}(u)$. Note that since v avoids 1221, its subword r of all repeated letters is weakly increasing. The subword of all repeated letters in $u = 1\bar{v}1$ is then $r1$. Making this subword increasing results in the subword of all repeated letters in w being $1r$. We can thus recover u by replacing $1r$ in w by $r1$.

Next, we show that w satisfies all three properties. Since $u = 1\bar{v}1$ has more than one 1 and ends in a repeated letter, the RGF $w = \mathrm{inc}(u)$ does as well. Property (i) is thus satisfied. Next we show property (ii) by first showing that u satisfies property (ii). If v_i is a repeated letter then we always have $v_i \leq \max\{v_1, \ldots, v_{i-1}\}$. Since we increased all first occurrences to get $\bar{v}$ and left the repeated letters the same we have $\bar{v}_i < \max\{\bar{v}_1, \ldots, \bar{v}_{i-1}\}$. And clearly the two new ones in u do not change this inequality. As previously noted, the value in the place of a given repeated letter can only get weakly smaller in passing from u to $w = \mathrm{inc}(u)$.

And since left-right maxima don't change, w also satisfies property (ii). Lastly, we will show property (iii). Consider the situation where $w_i w_{i+1} \dots w_j$ are all first occurrences but w_{i-1} and w_{j+1} are repeated letters. But then w_{j+1} was in position $i-1$ in u which is also a position in $\bar{v}$. And the element in position i of u is w_i which is a left-right maximum. Since left-right maxima in v were increased by one in passing from v to $\bar{v}$ we have $w_{j+1} < w_i - 1$ as desired. $\square$

Our goal is to define a map $\beta : \mathcal{M}^2_{n-1} \to R_n(1221)$ which maps area to lb. Before we define β we will discuss a partition of the region under $R = s_1 \dots s_{n-1} \in \mathcal{M}^2_{n-1}$ which will aid us in showing that area is sent to lb. Figure 4 gives an example of this process where different shadings indicate parts of the partition. Recall that $l(s_i)$ is the level, or smallest y-value, of s_i. If $s_i = D$, we define $A(s_i)$ to be equal to the area in the same row between s_i and its paired up step but excluding the area under other down steps or a-steps. In Figure 4, $A(s_5) = 1$, $A(s_8) = A(s_{12}) = 2$ and $A(s_9) = 5$. The area under R can be partitioned as follows. The rectangle under an a-step s_i will be a part with area $l(s_i)$. For example, in the figure we have the area $l(s_4) = 2$. Our other parts will be associated to down steps. Given a down step s_i, its part will consist of all area counted by $A(s_i)$ together with the area of the rectangle under the down step which is given by $l(s_i)$ for a total of $A(s_i)+l(s_i)$. Returning to our example, steps s_5, s_8, s_9, and s_{12} contribute total areas $2, 3, 5$, and 2, respectively. Since this partitions all the area under R we have

$$\operatorname{area}(R) = \sum_{s_i=a} l(s_i) + \sum_{s_i=D} (A(s_i) + l(s_i)). \tag{20}$$

Next we will define a map $\beta : \mathcal{M}^2_{n-1} \to R_n(1221)$ such that $\operatorname{area}(R) = \operatorname{lb}(\beta(R))$. Before we define $\beta(R)$ we will define an RGF, $v(R) = v_1 \dots v_n$, by letting $v_1 = 1$ and

$$v_{i+1} = \begin{cases} \max\{v_1, \dots, v_i\} + 1 & \text{if } s_i \text{ equals } U \text{ or } b, \\ \max\{v_1, \dots, v_i\} - l(s_i) & \text{if } s_i = a, \\ \max\{v_1, \dots, v_i\} - A(s_i) - l(s_i) & \text{if } s_i = D, \end{cases}$$

for $i \geq 0$. For the two-colored Motzkin path R in Figure 4 we have $v(R) = 1234225631786$.

A comparison of the first case in the definition of v with the other two shows that the left-right maxima of v are consecutive integers starting at 1. So to show that v is an RGF we only have to prove that $v_{i+1} > 0$ for all $s_i \in \{a, D\}$. Note that for all $i \geq 1$ we have that $\max\{v_1, \dots, v_i\}$ is equal to one more than the number of b-steps plus the number of up steps in the first $i-1$ steps. The level $l(s_i)$ of any horizontal step is at most the number of previous up steps, so for $s_i = a$ we have $v_{i+1} = \max\{v_1, \dots, v_i\} - l(s_i) > 0$. Note that the area counted by $A(s_i)$ between $s_i = D$ and its corresponding up step excluding the area under other a-steps or down steps is at most the number of up steps plus b-steps between and including the paired up and down step. Also, the level of the down step is at most the number of up steps strictly before its paired up step. All together $A(s_i) + l(s_i)$ is at most the number of up steps and b-steps in the first $i-1$ steps. As result, for $s_i = D$ we have $v_{i+1} = \max\{v_1, \dots, v_i\} - A(s_i) - l(s_i) > 0$. Hence, v is an RGF. However, $v(R)$ may not avoid 1221, so we define

$$\beta(R) = \mathrm{inc}(v(R))$$

which avoids 1221 by Lemma 4.32. For the two-colored Motzkin path R in Figure 4 we have $\beta(R) = 1234125623786$.

Next we show that $\mathrm{area}(R) = \mathrm{lb}(v)$ which will imply that $\mathrm{area}(R) = \mathrm{lb}(\beta(R))$ by Lemma 4.33. It is easy to see that $\mathrm{lb}(v_1) = 0$ and if s_i is b or U then $\mathrm{lb}(v_{i+1}) = 0$. Next consider $s_i = a$ so $v_{i+1} = \max\{v_1, \dots, v_i\} - l(s_i)$. By equation (18) we have $\mathrm{lb}(v_{i+1}) = l(s_i)$. Lastly, if $s_i = D$ then $v_{i+1} = \max\{v_1, \dots, v_i\} - A(s_i) - l(s_i)$. By equation (18) again,

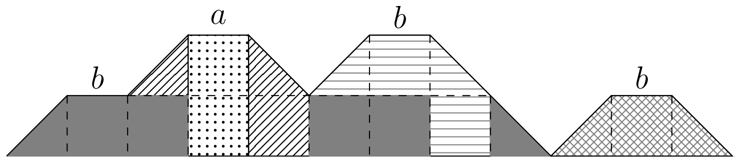

Figure 4 Two-colored Motzkin path

$\mathrm{lb}(v_{i+1}) = A(s_i) + l(s_i)$. As a result

$$\mathrm{lb}(v) = \sum_{s_i=a} l(s_i) + \sum_{s_i=D} (A(s_i) + l(s_i)) = \mathrm{area}(R)$$

by equation (20).

We now show that the β map behaves nicely with respect to two of the usual decompositions of Motzkin paths.

Lemma 4.35. *Let P and Q be two-colored Motzkin paths with $\beta(P) = x$ and $\beta(Q) = 1y$. The map β has the following properties.*

1. $\beta(PQ) = x(y + \max(x) - 1)$.
2. $\beta(UPD) = \alpha(x)$.

Proof. To prove statement 1, we first claim that

$$v(PQ) = v(P)(q + \max(v(P)) - 1)$$

where q is $v(Q)$ with its initial 1 deleted. It is clear from the definition of v that the first $|P|$ positions of $v(PQ)$ are $v(P)$. Also by definition of v, the first occurrences other than the initial 1 are in bijection with the union of the up steps and b-steps. It follows that the subword of first occurrences in the last $|Q|$ positions of $v(PQ)$ is the same as the corresponding subword in q with all elements increased by $\max(v(P)) - 1$. Thus the maximum value in any prefix of $v(PQ)$ ending in these positions is increased over the corresponding maximum in q by this amount. Furthermore, the areas and levels of down steps and a-steps in Q in that portion of PQ are the same since P ends on the x-axis. So, using the definition of v for these types of steps, the last $|Q|$ positions of $v(PQ)$ are exactly $q' = q + \max(v(P)) - 1$. To prove the equation for β, it suffices to show that the inc operator only permutes elements within $v(P)$ and within q'. But this is true because all elements of q' are greater than or equal to those of $v(P)$.

To prove the second statement, first consider $v := v(P) = v_1 \dots v_k$ and $u := v(UPD) = u_1 \dots u_{k+2}$. We claim that $u = 1\bar{v}1$ where $\bar{v}$ is v but with all its first occurrences increased by one. Clearly u begins with a 1. To see it must also end with 1, note that since the last step of $UPD = s_1 \dots s_{k+1}$ is down step and this path does not touch the axis between its initial and final points, we have $l(s_{k+1}) = 0$ and $A(s_{k+1})$ is the total number of up steps and b-steps in UPD. It now follows from the definition of the map v and our interpretation of the maximum of a prefix that $u_{k+2} = 1$. Let u' be u with its initial and final 1's removed. To see that $u' = \bar{v}$, first note that every step of UPD except the first is preceded by one more up step than in P. It follows every first occurrence of v is increased by one in passing to u'. But the area under each a-step and under each down step also increases by one during that passage. So the differences defining the v-map in such cases will stay the same for these repeated entries. It should now be clear that $u' = \bar{v}$. It follows immediately that $\beta(UPD) = \mathrm{inc}(1\bar{v}1) = \alpha(x)$. □

Before we show that β is a bijection we will need a method for determining from the image of a path where that path first returns to the x-axis. The following lemma will provide the key.

Lemma 4.36. *Given paths $P \in \mathcal{M}^2_{k-3}$ and Q with $k \geq 3$, the word $\beta(UPDQ) = w$ has w_k as the right-most repeated letter such that $w_1 w_2 \dots w_k$ satisfies all three properties in Lemma 4.34.*

Proof. Given a path $R = UPDQ \in \mathcal{M}^2_{n-1}$ as stated, by Lemma 4.35 we know that if we write $\beta(Q) = 1q$ then

$$w = \beta(R) = \alpha(\beta(P))(q + m - 1)$$

where $m = \max(\alpha(\beta(P)))$. Lemma 4.34 implies that the prefix $w_1 \ldots w_k = \alpha(\beta(P))$ satisfies all three properties. So it suffices to show that if there exists another repeated letter w_i after w_k then $w_1 \ldots w_i$ fails property (ii) or property (iii). In particular, it suffices to show such a failure for the prefix where w_i is the next repeated letter after w_k since any other prefix under consideration contains $w_1 \ldots w_i$.

If $i = k+1$ then, since every element of q is increased by $m-1$ and w_{k+1} is repeated, we must have $w_{k+1} = m = \max\{w_1, \ldots, w_k\}$, contradicting property (ii). If instead $i > k+1$ then w_{k+1} is a first occurrence and $w_{k+1} = \max\{w_1, \ldots, w_k\} + 1 = m + 1$. By definition of w_i, we have that $w_{k+1}, \ldots, w_{i-1}$ are all first occurrences with w_k and w_i repeated letters. Note that all elements in q were at least 1 and then increased by $m - 1$, so we must have $w_i \geq m = w_{k+1} - 1$ which contradicts property (iii). □

It will be helpful for us to be able to refer to the special repeated letter mentioned in the lemma above. So, given an RGF $w = w_1 \ldots w_n$, if there exists a right-most repeated letter w_k such that $w_1 w_2 \ldots w_k$ satisfies all three properties in Lemma 4.34 then we will say that w_k *breaks the word* w. Note that if such a repeated letter exists, its index k is unique.

Theorem 4.37. *The map* $\beta : \mathcal{M}^2_{n-1} \to R_n(1221)$ *is a bijection and* $\operatorname{area}(R) = \operatorname{lb}(\beta(R))$.

Proof. We have already shown that β is a well-defined map and that $\operatorname{area}(R) = \operatorname{lb}(\beta(R))$. Since $|\mathcal{M}^2_{n-1}| = C_n = |R_n(1221)|$, to show β is a bijection it suffices to show β is injective. We prove this by induction on n. It is easy to see that β is an injection for $n \leq 2$. We now assume that $n > 2$ and $\beta : \mathcal{M}^2_{k-1} \to R_k(1221)$ is injective for all $k < n$.

We will discuss three cases for paths $R \in \mathcal{M}^2_{n-1}$ and in each case we will show that R maps to a word distinct from the other words in that case and also from the words in previous cases.

First consider all paths R which start with an a-step so that $R = aQ$ for some path Q. By Lemma 4.35, we have $\beta(R) = 11y$ where $\beta(Q) = 1y$. Injectivity of the map now follows from the fact that, by induction, it is injective on paths Q of length $n-2$.

Our second case consists of paths R of the form $R = bQ$. Now $\beta(R) = 12(y+1)$ with y as above. Clearly these are distinct from the words in the previous paragraph and injectivity within this case follows by induction as before.

For the last case, consider all paths R which start with an up step so we can write $R = UPDQ$ for paths $P \in \mathcal{M}^2_{k-3}$ and Q where $k \geq 3$. By Lemma 4.35 we have $w := \beta(R) = \alpha(\beta(P))(y + \max(\alpha(\beta(P))) - 1)$, and by Lemma 4.36 the repeated letter w_k breaks the word w. Note that because $\alpha(\beta(P)) = w_1 \ldots w_k$ satisfies property (i) in Lemma 4.34, w has more than one 1 and so can not agree with a word from the second case above. But since R starts with an up step, w starts with the prefix 12 and so can not be a word from the first case. Finally, by uniqueness of the index of w_k, the injectivity of the map α, and induction the word w is uniquely determined among all words in this case. This finishes the proof that β is injective. □

Combining the previous result and the definition of $M_n(q)$ in equation (17) we have the following Corollary.

Corollary 4.38. *We have*

$$\mathrm{LB}_n(1221) = M_{n-1}(q).$$ □

4.4.3 Combinations with other patterns

Next we consider the RGFs which avoid 1221 and another length three pattern. Since 121 and 122 are subwords of 1221 these cases are not interesting, so we will focus on 111, 112, and 123.

Theorem 4.39. *We have for $L_n := \mathrm{LB}_n(111, 1221)$ that $L_0 = L_1 = 1$ and, for $n \geq 2$, and*

$$L_n = L_{n-1} + L_{n-2} + \sum_{k=1}^{n-2} q^k L_{k-1} L_{n-k-1}$$

Proof. Let $\mathcal{N}_n$ be the collection of two-colored Motzkin paths $R \in \mathcal{M}_n^2$ such that $\beta(R)$ avoids 111. Define $N_{-1}(q) = 1$ and, for $n \geq 0$,

$$N_n(q) = \sum_{R \in \mathcal{N}_n^2} q^{\mathrm{area}(R)}.$$

By Theorem 4.37 we only need to show that $N_n(q) = R_{n+1}(111, 1221)$ satisfies an equivalent recurrence and initial conditions. By definition $N_{-1}(q) = 1$ and $N_0(q) = 1$ because of the empty path. So we wish to show that for $n \geq 1$

$$N_n(q) = N_{n-1}(q) + N_{n-2}(q) + \sum_{k=0}^{n-2} q^{k+1} N_{k-1}(q) N_{n-k-2}(q). \tag{21}$$

We partition $\mathcal{M}_n^2$ as in the proof of Theorem 4.37:

$$S = \{R = aQ : \ Q \in \mathcal{M}_{n-1}^2\},$$
$$T = \{R = bQ : \ Q \in \mathcal{M}_{n-1}^2\},$$
$$U = \{R = UPDQ : \ P \in \mathcal{M}_k^2, \ Q \in \mathcal{M}_{n-k-2}^2 \text{ and } k \in [0, n-2]\},$$

We claim that when we restrict this partition to paths in $\mathcal{N}_n$ we have

$$S_{\mathcal{N}} = \{R = abQ : \ Q \in \mathcal{N}_{n-2}\},$$
$$T_{\mathcal{N}} = \{R = bQ : \ Q \in \mathcal{N}_{n-1}\},$$
$$U_1 = \{R = UDQ : \ Q \in \mathcal{N}_{n-2}\},$$
$$U_2 = \{R = UbPDQ : \ P \in \mathcal{N}_{k-1}, \ Q \in \mathcal{N}_{n-k-2} \text{ and } k \in [n-2]\},$$

where the set U breaks into two subsets. From the second partition we will be able to deduce the desired recursion.

Consider a path $R = aQ \in S$. We claim that $\beta(R)$ avoids 111 if and only if $Q = bQ'$ for $Q' \in \mathcal{N}_{n-2}$ which will show that S restricts to $S_{\mathcal{N}}$. If we write $\beta(Q) = 1y$ we have $\beta(R) = 11y$. The word $\beta(R)$ avoids 111 if and only if the word y has no 1's and at most two copies of every other number. Note that the second case considered in Theorem 4.37 contained all paths which started with a b-step and that these paths were mapped bijectively to words with exactly one 1. It is also clear that $y = \beta(Q') + 1$ has at most two copies of every number greater than one if and only if the same is true of $\beta(R)$. The claim now follows. Because $\text{area}(R) = \text{area}(Q')$ summing over all paths in this case gives us the term $N_{n-2}(q)$.

If instead $R = bQ \in T$ then, using that notation of Lemma 4.35 $\beta(R) = 12(y + 1) = 1(\beta(Q) + 1)$. So $\beta(R)$ avoids 111 if and only if $\beta(Q)$ does. It follows that T restricts to $T_{\mathcal{N}}$. Because $\text{area}(R) = \text{area}(Q)$ summing over all paths in this case gives us the term $N_{n-1}(q)$.

Next, we consider paths $R = UPDQ$ from the third part U. First consider the case where P has length 0 so $R = UDQ$. We want to prove that $\beta(R)$ avoids 111 if and only if $\beta(Q)$ avoids 111 since this will show that the collection of paths in U with $|P| = 0$ restricts to U_1. If we write $\beta(Q) = 1y$ we have $\beta(R) = 121(y + 1)$. Thus $\beta(Q)$ avoids 111 if and only if $\beta(R)$ does and the restriction is as claimed. Because $\text{area}(R) = 1 + \text{area}(Q)$ summing over all paths in this case gives us the term $qN_{n-2}(q)$ which is the $k = 0$ term in equation (21).

Lastly, consider a path $R = UPDQ$ with $|P| = k \in [n - 2]$ which are the remaining paths in U. We will show that $\beta(R)$ avoids 111 if and only if $P = bP'$ and both the words $\beta(P')$ and $\beta(Q)$ avoid 111. This will show that the remaining paths in U restrict to U_2 in the second partition. First we make an observation about $\alpha(\beta(P))$. Let $m = \max(\beta(P))$ and $\{1^{s_1}, \ldots, m^{s_m}\}$ be the multiset of all letters in $\beta(P)$. The map α increases all first occurrences by one and adds two 1's but otherwise doesn't affect the collection of letters. So the multiset of letters in $\alpha(\beta(P))$ is $\{1^{s_1+1}, \ldots, m^{s_m}, m + 1\}$. If we write $\beta(Q) = 1y$ we have $\beta(R) = \alpha(\beta(P))(y + m)$ since $m = \max(\alpha(\beta(P))) - 1$. If $\{1^{t_1}, \ldots, \bar{m}^{t_{\bar{m}}}\}$ is the multiset of letters in $\beta(Q)$ then the multiset of letters in $\beta(R)$ is $\{1^{s_1+1}, \ldots, m^{s_m}, (m+1)^{t_1}, \ldots, (m+\bar{m})^{t_{\bar{m}}}\}$. So $\beta(R)$ avoids 111 if and only if there are at most two of any element in this set which is equivalent to $s_1 = 1$, $s_i \leq 2$ for $i > 1$, and $t_i \leq 2$ for all $i \geq 1$. Further this implies that $\beta(R)$ avoids 111 if and only if $Q \in \mathcal{N}_{n-k-2}$ and $\beta(P)$ has exactly one 1 and avoids 111. Just as in our first case, $\beta(P)$ has exactly one 1 and avoids 111 if and only if $P = bP'$ for some $P' \in \mathcal{N}_{k-1}$. Because $\text{area}(R) = \text{area}(P') + \text{area}(Q) + k + 1$ summing over all paths in this case gives us the term $q^{k+1}N_{k-1}(q)N_{n-k-2}(q)$ for $k > 0$.

This completes the proof of the theorem. $\square$

The next two avoidance classes can be characterized by a combination of Theorem 2.5 and Lemma 4.32. The proofs are straightforward and so not included.

Proposition 4.40. *We have*

$$R_n(112, 1221) = \{12 \ldots mk^{n-m} : \ k \in [m]\}.$$

As such, for $n \geq 0$ we have

1. $F_n(112, 1221) = \sum_{m=1}^{n} \sum_{k=1}^{m} q^{(n-m)(m-k)} r^{\binom{m}{2}+(n-m)(k-1)} s^{\binom{m}{2}} t^{m-k}$,
2. $LB_n(112, 1221) = \sum_{m=1}^{n} \sum_{k=1}^{m} q^{(n-m)(m-k)}$,
3. $\mathrm{LS}_n(112, 1221) = \sum_{m=1}^{n} \sum_{k=1}^{m} q^{\binom{m}{2}+(n-m)(k-1)}$,
4. $\mathrm{RB}_n(112, 1221) = \sum_{m=1}^{n} mq^{\binom{m}{2}}$, *and*
5. $\mathrm{RS}_n(112, 1221) = \sum_{i=1}^{n} iq^{n-i}$. $\square$

Proposition 4.41. *We have*

$$R_n(123, 1221) = \{1^n, 11^i21^j2^k : \ i+j+k = n-2, \ \text{and } i, j, k \geq 0\}.$$

As such, for $n \geq 0$ we have, using the Kronecker delta function,

1. $F_n(123, 1221) = 1 + \sum_{\substack{i+j+k=n-2 \\ i,j,k \geq 0}} q^j r^{k+1} s^{i+1+j(1-\delta_{0,k})} t^{1-\delta_{0,j}}$,

2. $\mathrm{LB}_n(123, 1221) = 1 + \sum_{j=0}^{n-2} (n-j-1) q^j$,

3. $\mathrm{LS}_n(123, 1221) = 1 + \sum_{k=0}^{n-2} (n-k-1) q^{k+1}$,

4. $\mathrm{RB}_n(123, 1221) = 1 + q^{n-1} + \sum_{k=1}^{n-2} (k+1) q^k$, *and*

5. $\mathrm{RS}_n(123, 1221) = n + \binom{n-1}{2} q$. □

4.4.4 More about the pattern 1212

It turns out that the generating function $\mathrm{LB}_n(1212)$ is also equal to $M_{n-1}(q)$. Instead of showing this directly, we prove that $\mathrm{LB}_n(1212) = \mathrm{LB}_n(1221)$ and then Corollary 4.38 completes the proof. In the process we also show $\mathrm{LS}_n(1212) = \mathrm{LS}_n(1221)$.

Proposition 4.42. *The restriction* inc : $R_n(1212) \to R_n(1221)$ *is a bijection which preserves* lb *and* ls.

Proof. By Lemma 4.32 we have $\mathrm{lb}(w) = \mathrm{lb}(\mathrm{inc}(w))$. This map also preserves ls because w and $\mathrm{inc}(w)$ are rearrangements of each other and $\mathrm{ls}(w_i) = w_i - 1$ for any RGF w.

Now we only need to show that inc : $R_n(1212) \to R_n(1221)$ is bijective. Since $|R_n(1212)| = C_n = |R_n(1221)|$ it suffices to show the map is injective. Assume that $v = v_1 v_2 \ldots v_n$ and $w = w_1 w_2 \ldots w_n$ are two distinct words which avoid 1212, but $\mathrm{inc}(v) = \mathrm{inc}(w)$. This means that v and w share the same positions of first occurrences, and the same multiset of repeated letters. But since $v \neq w$ there is then a smallest index $i \geq 1$ such that $v_1 \ldots v_{i-1} = w_1 \ldots w_{i-1}$ but $v_i \neq w_i$. Without loss of generality let $v_i = a$, $w_i = b$, and $a < b$. We have noted that v and w have their first occurrences at the same indices, so v_i and w_i must be repeated letters.

Since w is an RGF, the first occurrence of a and b must occur before w_i, so v also has the subword ab before v_i. However, because v and w have the same collection of repeated letters the b which is w_i in w must occur some time after v_i in v. This means that v has the subword $abab$ contradicting Lemma 4.20. □

Corollary 4.43. *For $k \geq 0$ we have*

$$F_n(1212; q, r, 1, 1) = F_n(1221; q, r, 1, 1),$$

$$F_n(1^k, 1212; q, r, 1, 1) = F_n(1^k, 1221; q, r, 1, 1),$$

and

$$F_n(12\ldots k, 1212; q, r, 1, 1) = F_n(12\ldots k, 1221; q, r, 1, 1).$$

Proof. The bijection f in Proposition 4.42 preserves the number of times any letter appears and preserves the maximal letter. The equalities follow from this fact. □

Using Corollary 4.24, Propositions 4.29 and 4.41, and Corollaries 4.38 and 4.43 we have the following equalities which summarize many of the equations we have from results in this section.

Corollary 4.44. *We have, for all $n \geq 0$,*

$$\mathrm{LB}_n(1212) = \mathrm{RS}_n(1212) = \mathrm{LB}_n(1221) = M_n(q),$$

$$\mathrm{LB}_n(111, 1212) = \mathrm{LB}_n(111, 1221),$$

$$\mathrm{LS}_n(111, 1212) = \mathrm{LS}_n(111, 1221),$$

$$\mathrm{LB}_n(123, 1212) = \mathrm{RS}_n(123, 1212) = \mathrm{LB}_n(123, 1221),$$

and

$$\mathrm{LS}_n(123, 1212) = \mathrm{RB}_n(123, 1212) = \mathrm{LS}_n(123, 1221). \qquad \square$$

REFERENCES

REFERENCES

[Arm09] Drew Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.*, 202(949):x+159, 2009.

[BBES14] Marilena Barnabei, Flavio Bonetti, Sergi Elizalde, and Matteo Silimbani. Descent sets on 321-avoiding involutions and hook decompositions of integer partitions. *J. Combin. Theory Ser. A*, 128(1):132–148, November 2014.

[CDD$^+$13] William Y. C. Chen, Alvin Y. L. Dai, Theodore Dokos, Tim Dwyer, and Bruce E. Sagan. On 021-avoiding ascent sequences. *Electron. J. Combin.*, 20(1):Paper 76, 6, 2013.

[CEKS13] Szu-En Cheng, Sergi Elizalde, Anisse Kasraoui, and Bruce E. Sagan. Inversion polynomials for 321-avoiding permutations. *Discrete Math.*, 313(22):2552–2565, 2013.

[DDG$^+$] Samantha Dahlberg, Robert Dorward, Jonathan Gerhard, Thomas Grubb, Carlin Purcell, Lindsey Reppuhn, and Bruce E. Sagan. Restricted growth function patterns and statistics. In preparation.

[DDG$^+$16] Samantha Dahlberg, Robert Dorward, Jonathan Gerhard, Thomas Grubb, Carlin Purcell, Lindsey Reppuhn, and Bruce E. Sagan. Set partition patterns and statistics. *Discrete Math.*, 339, 2016.

[DDJ$^+$12] Theodore Dokos, Tim Dwyer, Bryan P. Johnson, Bruce E. Sagan, and Kimberly Selsor. Permutation patterns and statistics. *Discrete Math.*, 312(18):2760–2775, 2012.

[Dra09] Brian Drake. Limits of areas under lattice paths. *Discrete Math.*, 309(12):3936–3953, 2009.

[DS11] Paul Duncan and Einar Steingrímsson. Pattern avoidance in ascent sequences. *Electron. J. Combin.*, 18(1):Paper 226, 17, 2011.

[Foa68] Dominique Foata. On the Netto inversion number of a sequence. *Proc. Amer. Math. Soc.*, 19:236–240, 1968.

[GM09] Adam M. Goyt and David Mathisen. Permutation statistics and q-Fibonacci numbers. *Electron. J. Combin.*, 16(1):Research Paper 101, 15 pp., 2009.

[Goy08] Adam M. Goyt. Avoidance of partitions of a three-element set. *Adv. in Appl. Math.*, 41(1):95–114, 2008.

[GS09] Adam M. Goyt and Bruce E. Sagan. Set partition statistics and q-Fibonacci numbers. *European J. Combin.*, 30(1):230–245, 2009.

[HR01] J. Haglund and J. B. Remmel. Rook theory for perfect matchings. *Adv. in Appl. Math.*, 27(2-3):438–481, 2001. Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).

[JM08] Vít Jelínek and Toufik Mansour. On pattern-avoiding partitions. *Electron J. Combin.*, 15(R39):1–52, 2008.

[Kil] Kendra Killpatrick. Wilf equivalence for the charge statistic. Preprint `arXiv:1204.3121`.

[Kla96] Martin Klazar. On *abab*-free and *abba*-free set partitions. *European J. Combin.*, 17(1):53–68, 1996.

[Kla00a] Martin Klazar. Counting pattern-free set partitions. I. A generalization of Stirling numbers of the second kind. *European J. Combin.*, 21(3):367–378, 2000.

[Kla00b] Martin Klazar. Counting pattern-free set partitions. II. Noncrossing and other hypergraphs. *Electron. J. Combin.*, 7:Research Paper 34, 25 pp., 2000.

[Knu73] Donald E. Knuth. *The art of computer programming. Volume 3*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973. Sorting and searching, Addison-Wesley Series in Computer Science and Information Processing.

[Kre72] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.

[Mac78] Percy Alexander MacMahon. *Collected papers. Vol. I.* MIT Press, Cambridge, Mass., 1978. Combinatorics, Mathematicians of Our Time, Edited and with a preface by George E. Andrews, With an introduction by Gian-Carlo Rota.

[MS12] Toufik Mansour and Mark Shattuck. Pattern-avoiding set partitions and catalan numbers. *Electron. J. Combin.*, 18:1–18, 2012.

[MT04] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A*, 107(1):153–160, 2004.

[Sag10] Bruce E. Sagan. Pattern avoidance in set partitions. *Ars Combin.*, 94:79–96, 2010.

[SS85] Rodica Simion and Frank W. Schmidt. Restricted permutations. *European J. Combin.*, 6(4):383–406, 1985.

[SS12] Bruce E. Sagan and Carla D. Savage. Mahonian pairs. *J. Combin. Theory Ser. A*, 119(3):526–545, 2012.

[Sta] Richard Stanley. Catalan addendum. New problems for *Enumerative Combinatorics. Vol. 2*, available at `http://math.mit.edu/`~rstan/ec/catadd.pdf.

[Sta97] Richard P. Stanley. *Enumerative Combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[Sta99] Richard P. Stanley. *Enumerative Combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[WW91] Michelle Wachs and Dennis White. p, q-Stirling numbers and set partition statistics. *J. Combin. Theory Ser. A*, 56(1):27–46, 1991.