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Local Regularization Methods for Nonlinear Volterra Integral
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ABSTRACT

Local Regularization Methods for Nonlinear Volterra Integral Equations of Hammerstein Type

By

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We develop a local regularization theory for the nonlinear Volterra problem of Hammerstein type. Our method retains the causal structure of the original Volterra problem and allows for fast sequential numerical solution. The fundamental difference between our method and the previous existing local regularization method for Hammerstein equations (Lamm and Dai, 2005) is that for our method we do not need to solve a nonlinear equation at every step of a numerical implementation. We only have to solve a nonlinear equation for the first step. We prove the convergence of the regularized solutions to the true solution as noise level in the data shrinks to zero with a certain convergence rate.

To my parents

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CHAPTER 1

Introduction

Volterra integral equations arise in a great many applications. For example, in population dynamics [23] [24], epidemic diffusion, reaction-diffusion in small cells [25], in nuclear reactor kinetics [2] and in general in evolutionary phenomena incorporating memory.

Of special interest are Volterra integral equations of Hammerstein type. In many applications, the problem can be written in terms of a Volterra integral equation of Hammerstein type, as for example in chemical absorption kinetics, in epidemic models, and also in situations when Laplace transform techniques are used to reduce systems of ordinary or partial differential equations to Volterra integral equations.

In this paper, we will study the solution of a nonlinear Volterra problem of Hammerstein type of the following form

$$Fu = f, \tag{1.1}$$

where F is the nonlinear Volterra operator given by

$$Fu(t) = \int_0^t k(t,s)g(u(s))\,ds \tag{1.2}$$

for suitable kernel k, nonlinear function g and f in the Range of F which will be clarified later. Before we get into details of this nonlinear problem, we will first give some brief introduction for the linear counterpart to this problem. Let us consider a *linear* first-kind Volterra integral equation for (1.1), where F is defined by

$$Fu(t) = \int_0^t k(t,s)u(s)\,ds \tag{1.3}$$

with the kernel $k \in L^2\left((0,T) \times (0,T)\right)$, where f is in the range of F and our goal is to find $\bar{u} \in L^2(0,T)$ or $C[0,T]$ which solves equation (1.1).

However, such problems are generally ill-posed due to the fact that the solutions u^δ which are obtained by solving (1.3) using imprecise measurement data f^δ do not depend continuously on data, i.e., very small errors in the measurement data f^δ could lead to large deviations in the solution u^δ as compared to the true solution $\bar{u}$. What we usually see for these kinds of ill-posed problems are highly oscillatory solutions using measurement data. This is very troublesome because in practice we never have exact data in hand. Since the available data always contain uncertainty, regularization methods have to be employed to stabilize the problem.

A classic and well-known example is the Inverse Heat Conduction Problem (IHCP). The problem can be stated as follows: applying heat on one end of a semi-infinite bar which we call location $x = 0$, we measure the temperature $f(t)$ as a function of time t at some location away from the heat source, which for simplicity we call location $x = 1$. The problem is to recover the temperature $u(t)$ at the heat source $x = 0$, and this problem can be formulated as solving equations (1.1) and (1.3) for u with the kernel given by $k(t,s) = \kappa(t-s)$, where

$$\kappa(t) = \frac{1}{2\sqrt{\pi}t^{3/2}}e^{-1/4t}.$$

This problem is a severely ill-posed linear Volterra problem.

One well-known regularization theory is that of Tikhonov regularization. The idea of Tikhonov regularization is that, instead of solving for u satisfying $Fu = f^{\delta}$, we solve a constrained minimization problem for u_{α}^{δ},

$$\min_{u} \|Fu - f^{\delta}\|^2 + \alpha\|Lu\|^2, \tag{1.4}$$

where f^{δ} is noisy data, α is the regularization parameter and L is a suitable (usually identity or differential) operator. The Tikhonov theory gives conditions under which there is choice of α such that as the noise level $\delta \to 0$, $\alpha(\delta) \to 0$, and the corresponding Tikhonov solution $u_{\alpha(\delta)}^{\delta}$ to (1.4) converges to the true solution $\bar{u}$.

However, there is a drawback associated with using Tikhonov regularization in solving Volterra problems. Volterra problems have a nice physical structure called *causal* structure. That means the solution u at any given time t does not affect the data f on the interval $[0, t)$. Therefore in finding $u(t)$, it makes sense to use future data f on the interval [t,T] and it does not make much sense to use all data f on the whole interval $[0, T]$. Tikhonov regularization however converts a causal problem to a non-causal problem, and this leads to nontrivial increases in costs of implementation.

In the mid-1990's, P. K. Lamm established the local regularization theory which is a generalization of a regularization scheme for the discretized IHCP developed by J. V. Beck in the late 1960's. While Beck's method was an approach developed to handle a finite dimensional problem, the local regularization theory can be placed in both finite and infinite dimensional settings. The theory can be applied to a wide class of linear first-kind Volterra problems [4] [5] [6]. Local regularization methods preserve the causal structure of the Volterra problems and therefore they have computational advantages over the classical regularization methods. For example, while Tikhonov regularization requires $O(N^3)$ flops for a discretized problem of dimension

N (or $O(N^2)$ if special structure is accounted for), local regularization requires $O(N^2)$ flops (or $O(NlogN)$ flops in the case of special structure). See Section 2.1 for some background of local regularization methods for linear first kind Volterra problems.

We now turn to some background on the regularization of nonlinear problems. Consider solving for u that satisfies equation (1.1), where $F : D(F) \subseteq X \to Y$ is a nonlinear operator between Hilbert spaces X and Y. We assume that

(1) F is continuous and

(2) F is weakly (sequentially) closed, i.e. for any sequence $\{u_n\} \subset D(F)$ such that $u_n \rightharpoonup u$ in X and $Fu_n \rightharpoonup f$ in Y, then $u \in D(F)$ and $Fu = f$.

Also assume equation (1.1) has a solution. Then there exists a u^*-minimum-norm solution u^+ for the data $f \in Y$, i.e.,

$$Fu^+ = f \quad \text{and} \quad \|u^+ - u^*\| = \min\{\|u - u^*\| : \quad Fu = f\}.$$

(This is by the weak closedness of F, and follows from the attainability assumption that equation (1.1) has an exact solution [17].)

If the nonlinear operator F is compact, one can give a sufficient condition for ill-posedness of (1.1) which is similar to its compact linear counterpart.

Proposition 1.0.1. *[17] Let F be a nonlinear compact and weakly closed operator between two Hilbert spaces X and Y, and let $D(F)$ be weakly closed. Moreover, assume that $Fu^+ = y$ and that there exists an ϵ such that $Fu = \bar{y}$ has a unique solution for all $\bar{y} \in R(F) \cap B_\epsilon(y)$. If there exists a sequence $\{u_n\} \subseteq D(F)$ such that*

$$u_n \rightharpoonup u^+ \quad \textit{but} \quad u_n \nrightarrow u^+,$$

then F^{-1} (defined on $R(F) \cap B_\epsilon(y)$) is not continuous in y.

Tikhonov regularization

As in the linear case, we can replace problem (1.1) by minimization problem:

$$\|Fu - f^{\delta}\|^2 + \alpha\|u - u^*\|^2 \to \min, \quad u \in D(F), \tag{1.5}$$

where $\alpha > 0$, $f^{\delta} \in Y$ is an approximation of the exact right-hand side f of (1.1) and $u^* \in X$, $\|f^{\delta} - f\| \leq \delta$. As in the linear case, any solution to (1.5) will be denoted by u_{α}^{δ}.

Tikhonov regularization gives the following convergence rate analysis

Theorem 1.0.1. *[17] Let $D(F)$ be convex, F continuous and weakly closed. Let $f^{\delta} \in Y$ with $\|f - f^{\delta}\| \leq \delta$ and let u^+ be an u^*-minimum-norm solution. Moreover, let the following conditions hold:*

(i) F is Fréchet-differentiable,

(ii) there exists $\gamma \geq 0$ such that $\|F'u^+ - F'u\| \leq \gamma\|u^+ - u\|$ for all $u \in D(F)$ in a sufficiently large ball around u^+,

(iii) there exists $w \in Y$ satisfying $u^+ - u^* = (F'u^+)^*w$ and

(iv) $\gamma\|w\| < 1$.

Then for the choice of $\alpha \sim \delta$, we obtain $\|u_{\alpha}^{\delta} - u^+\| = O(\sqrt{\delta})$.

An example of the application of Tikhonov regularization to a particular 1-smoothing convolution nonlinear Volterra Hammerstein problem is given in [17]. The problem is to consider the Hammerstein integral equation

$$F : H^1[0,1] \to L^2[0,1]$$

$$Fu(s) := \int_0^t (t-s)u^3(s)\,ds.$$

Since F is continuous, weakly closed, compact and injective [17], Proposition 1.0.1 implies that the problem of solving $Fu = f$ is ill-posed.

Consider the application of Tikhonov regularization method to this problem. In order to satisfy assumption (iii) about the source condition, u^+ and u^* have to satisfy quite strict smoothness conditions and particular boundary conditions. For example, u^+ and $u^* \in H^4$, $u_S^+(0) = u_S^*(0)$, $u_S^+(1) = u_S^*(1)$, $u^+(1) - u_{SS}^+(1) = u^*(1) - u_{SS}^*(1)$ and $u_{SSS}^+(1) = u_{SSS}^*(1)$ [17].

From the above example, we see that in order to use Tikhonov regularization theory on nonlinear Volterra problems of Hammerstein type, strict assumptions on the smoothness of the source conditions and particular boundary conditions are needed in order to achieve the desired convergence rate. Also, as for the linear Volterra problems, another disadvantage of Tikhonov regularization methods are that they destroy the causal nature of the Volterra problems and lead to nontrivial computational costs.

Lavrentiev's regularization

We now turn to the second common form of regularization for inverse Volterra problems, i.e. Lavrentiev Regularization. The idea of Lavrentiev Regularization is to solve an equation of the form

$$\alpha u + Fu = f. \tag{1.6}$$

Definition 1.0.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$. We say that f is monotonic if*

$$< x - y, f(x) - f(y) > \geq 0, \quad \forall x, y.$$

Consider the problem of solving for u that satisfies

$$\int_0^t k(t,s)u(s)\,ds + \int_0^t \hat{F}(t,s,u(s))\,ds = f(t), \quad t \in [0,T]. \tag{1.7}$$

It is proved in [18] that one can adapt the Lavrentiev method to identify u by solving the following equation

$$\alpha u(t) + \int_0^t k(t,s)u(s)\,ds + \int_0^t \hat{F}(t,s,u(s))\,ds = \frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-s)} f^{\delta}(s)\,ds. \quad (1.8)$$

Under suitable assumptions given by the next theorem, this equation is solvable on the interval $[0,T]$ and the solution u_{α}^{δ} to (1.8) approaches the true solution $\bar{u}$ as noise $\delta \to 0$ in an appropriate sense. See [22] for an introduction of the Lavrentiev method.

Theorem 1.0.2. *[22] Assume: 1. The vectors y, u belong to $\mathbb{R}^n$, $k(t,s) : \Delta \to \mathbb{R}^n$, $\hat{F}(t,s,u) : \Delta \times \mathbb{R}^n \to \mathbb{R}^n$ where*

$$\Delta := \{(t,s) : 0 \le s \le t \le T\}.$$

2. $\hat{F}$ is continuous and the partial derivative $\hat{F}_t(t,s,u)$ exists for a.e. $(t,s) \in \Delta$ and for all $u \in \mathbb{R}^n$.

3. For each u, $v \in \mathbb{R}^n$ and a.e. $(t,s) \in \Delta$ we have

$$\|\hat{F}(t,s,v) - \hat{F}(t,s,u)\| \le N(t,s)\|v-u\|, \quad \|\hat{F}_t(t,s,u) - \hat{F}_t(t,s,v)\| \le L(t,s)\|v-u\|$$

and

$$\sup_{t\in[0,T]} \int_0^t L^2(t,s)\,ds \le L, \qquad \sup_{t\in[0,T]} \int_0^t N^2(t,s)\,ds = N.$$

4. For every $t \in [0,T]$, the function $u \to \hat{F}(t,t,u) : \mathbb{R}^n \to \mathbb{R}^n$ is monotonic.

5. The kernel $k(t,s)$ is continuous for $0 \le s \le t \le T$ and $k(t,t) = I$ for $t \in [0,T]$.

6. The derivative $D_1k(t,s)$ exists a.e. and $\sup_{t\in[0,T]} \int_0^t \|D_1k(t,s)\|^2\,ds \le C$.

7. The true solution $\bar{u}$ is piecewise $W^{1,2}(0,T)$.

If $\|f^{\delta} - f\| \le \delta$, where $\delta > 0$ is a known tolerance, then for the choice of $\alpha = \alpha(\delta)$

such that

$$\lim_{\delta \to 0^+} \frac{\delta}{\alpha^2} = 0,$$

equation (1.8) *has a unique solution* u_α^δ *and* $u_\alpha^\delta \to \bar{u}$ *in* $L^2((0,T),\mathbb{R}^n)$.

Notice that by assumption 5, this theorem can only be applied to 1-smoothing (both convolution and nonconvolution type) Volterra problems of Hammerstein type. In this case, the nonlinear function g in (1.2) has to be monotonic. The advantage of this method over Tikhonov regularization is that it still preserves the causal structure of the Volterra problem. However because the added penalty term αu does not take the given operator F into consideration, the approximation is not as good as local regularization theory at least for the nonlinear Hammerstein problem from our numerical results. One reason this is the case is that convergence of u_α^δ to $u \in C[0,T]$ in the uniform norm is impossible if $u(0) \neq 0$, unless information about $u(0)$ (which is rarely known accurately) is built into the approximate equation (1.8). This fact tends to lead to bad approximations near $t = 0$. If (1.8) is solved sequentially (the usual case), this can lead to large errors on the entire interval. Please see Example 1 in Chapter 4 of the thesis and we refer to Figure 2 in [18] for comparison.

Since the results in [18] do not give a convergence rate in the case of noisy data, we briefly mention the work in [33] where the Lavrentiev method (1.6) is applied to equation (1.1), in the case of the operator F satisfying assumptions similar to those in Theorem 1.0.1 (for Tikhonov regularization) along with additional monotonicity and hemicontinuity assumptions on F. In this case the rate $\|u_\alpha^\delta - u^+\| = O(\delta^{1/2})$ is achieved, so that the rate for Lavrentiev regularization can be seen to be the same as that for Tikhonov regularization under similar smoothness hypotheses on u^+.

There are other regularization methods for nonlinear problems in the literature, for example, Landweber methods [26], where one may seek a solution u_k^δ such that

$$u_{k+1}^\delta = u_k^\delta + (F' u_k^\delta)^* (f^\delta - F u_k^\delta) \tag{1.9}$$

for $k = 0, 1, \ldots,$ where $u_0^\delta = u_0$ is the initial guess. A modified Landweber-method [29] based on the idea that for the numerical realization of (1.9), the use of a rough approximation F_m to F within the first iteration steps has no influence on the quality of the iterates, as long as the iteration is continued with a sufficiently good approximation to F. This leads to the iteration formula

$$u_{k+1}^\delta = u_k^\delta + (F'_{r(k)} u_k^\delta)^* (f^\delta - F_{r(k)} u_k^\delta).$$

Other regularization methods include Levenberg-Marquardt methods [28] where the author studied a Levenberg-Marquardt scheme for nonlinear inverse problems where the corresponding Lagrange (or regularization) parameter is chosen from an inexact Newton strategy; conjugate gradient methods [27], where the basic idea is to compute an approximate solution for the linearized problem in each Newton step with the conjugate gradient method as an inner iteration; iteratively regularized Gauss-Newton methods [31] [32]; and other Newton-like methods [30], etc. We refer [21] for extensive discussions of such methods.

In recent years, local regularization methods have been extended to some nonlinear Volterra problems, for example, the autoconvolution problem [12]. In 2005, Lamm and Dai studied the nonlinear Volterra Hammerstein problems and their idea is that if one treats $g(u(t))$ as the solution of (1.1) where F is given by (1.2), then solving for g is nothing more than solving a linear Volterra problem. However, this local regularization method requires one to solve a nonlinear equation at each step of numerical iteration which can be difficult in practice. See Section 2.2 for details.

Driven by applications and the need to have a regularization scheme that is easy to implement, we develop a local regularization method that not only preserves the causal structure of the Volterra problems but also gives accurate approximations and is easy to implement in practice.

We organize the paper in the following way: in Chapter 2, we first give some background on the local regularization methods for linear problems and nonlinear Hammerstein problems. Then we give our main results on the new local regularization theory for nonlinear Volterra integral equations of Hammerstein type with ν-smoothing *convolution* kernels. In Chapter 3, we extend our results to nonlinear Volterra integral equations of Hammerstein type with 1-smoothing *nonconvolution* kernels. In Chapter 4, we present some numerical results using our regularization theory.

CHAPTER 2

Hammerstein problem with ν-smoothing convolution kernel

We first motivate our work on the local regularization method for the Hammerstein problem by giving some background on the existing theory for the linear Volterra problem.

2.1 Linear Problems

We consider the problem of finding $\bar{u} \in C[0,T]$ solving

$$Fu = f \tag{2.1}$$

where F is the Volterra operator of convolution type given by

$$Fu(t) = \int_0^t k(t-s)u(s)\,ds, \quad t \in [0,T], \tag{2.2}$$

and f is in range of F.

A discussion of the existence and uniqueness of solutions of (2.1) may be found in [13] in the linear case. We call k the kernel of the operator F. Throughout we will assume that F satisfies a ν-smoothing condition for some $\nu = 1, 2, \dots,$ that is the kernel k satisfies

$$k \in C^{\nu}[0,T], \quad k^{(j)}(0) = 0, \quad j = 0, 1, \dots, \nu - 2, \quad k^{(\nu - 1)}(0) \neq 0, \tag{2.3}$$

where without loss of generality, we will take $k^{(\nu - 1)}(0) = 1$. It is well-known that the degree of ill-posedness of problem (2.1) is characterized by the degree of smoothness of the kernel k and the behavior of k at 0, the larger the value of ν, the worse the ill-posedness is. We will assume the desired $\bar{u}$ of (2.1) satisfies the Hölder condition

$$|\bar{u}(t) - \bar{u}(s)| \leq \bar{N}|t - s|^{\mu}, \tag{2.4}$$

for $0 < \mu \leq 1$, $\bar{N} := \bar{N}(\bar{u}) > 0$, and t, s in the interval of interest.

To motivate the sequential local regularization method for linear Volterra problems, we let $R > 0$ be a small fixed number and $r \in (0, R]$ a small parameter. Assume that equation (2.1) holds on an extended interval $[0, T + R]$. If data is not available past the original interval, then this can always be accomplished by decreasing the size of T slightly. Then $\bar{u}$ solves

$$\int_0^{t + \rho} k(t + \rho - s)u(s)\, ds = f(t + \rho), \quad t \in [0, T],\ \rho \in [0, r].$$

Split the integral at t, then change the variable of integration, we have

$$\int_0^t k(t + \rho - s)u(s)\, ds + \int_0^{\rho} k(\rho - s)u(t + s)\, ds = f(t + \rho), \quad t \in [0, T],\ \rho \in [0, r].$$

Now we integrate both sides of the equation with respect to a suitable Borel measure

$\eta_r(\rho)$ (which will be clarified later) on [0,r], so we have

$$\int_0^t \int_0^r k(t+\rho-s)\, d\eta_r(\rho) u(s)\, ds + \int_0^r \int_0^\rho k(\rho-s) u(t+s)\, ds\, d\eta_r(\rho)$$
$$= \int_0^r f(t+\rho)\, d\eta_r(\rho), \quad t \in [0,T]. \tag{2.5}$$

For simplicity, we define the following notations which we will use throughout this paper:

$$\|\cdot\| := \|\cdot\|_{L^\infty(0,T)}, \quad \|\cdot\|_\infty := \|\cdot\|_{L^\infty(0,T+R)}, \quad \|\cdot\|_r := \|\cdot\|_{L^\infty(0,r)}$$

and $\|q\|_I := \sup_{x \in I} |q(x)|$.

Note that $\bar{u}$ still satisfies (2.5) exactly. However, in practice, we only have in hand imprecise measurement data or perturbed data $f^\delta \in C[0,T+R]$, instead of the true data $f \in C[0,T+R]$, where f^δ satisfies

$$\|f^\delta - f\|_\infty \leq \hat{\delta} \quad \text{for some} \quad \hat{\delta} > 0. \tag{2.6}$$

Since solving for u from equation (2.5) when f^δ is in place of f is an ill-posed problem due to lack of continuous dependence on data, some regularization method needs to be employed.

The idea is if we momentarily hold u constant on a small interval $[t, t+r]$, then we can replace $u(t+s)$ by $u(t)$ in the second term of equation (2.5). And r serves as the regularization parameter. Then we obtain the regularization equation

$$a(r)u(t) + \int_0^t \tilde{k}_r(t-s)u(s)\, ds = \tilde{f}_r^\delta(t), \quad t \in [0,T], \tag{2.7}$$

where

$$\tilde{k}_r(t) = \int_0^r k(t+\rho)\, d\eta_r(\rho), \tag{2.8}$$

$$\tilde{f}_r^{\delta}(t) = \int_0^r f^{\delta}(t+\rho)\, d\eta_r(\rho), \tag{2.9}$$

$$a(r) = \int_0^r \int_0^{\rho} k(\rho - s)\, ds\, d\eta_r(\rho). \tag{2.10}$$

Notice that equation (2.7) is a well-posed second kind integral equation in u provided that $a(r) \neq 0$. Sufficient conditions for stability and convergence of solutions u to $\bar{u}$ include the hypotheses on the measures η_r given below:

The signed Borel measures $\eta_r(\rho)$ on $[0, r]$ satisfy the following conditions:

- (H_1) For $i = 0, 1, \ldots, \nu$, there is some $\sigma \in \mathbb{R}$ and $c_i = c_i(\nu) \in \mathbb{R}, c_\nu > 0$ independent of r, such that

$$\int_0^r \rho^i\, d\eta_r(\rho) = r^{i+\sigma}(c_i + \mathcal{O}(r)), \quad \text{as} \quad r \to 0.$$

- (H_2) The parameters c_i, $i = 0, 1, \ldots, \nu$, satisfy the condition that all roots of the polynomial $p_\nu(\lambda)$ defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + \cdots + \frac{c_1}{1!}\lambda + \frac{c_0}{0!}$$

have negative real part.

- (H_3) There exists a $\tilde{C} \geq 0$ independent of r such that

$$\left| \int_0^r h(\rho)\, d\eta_r(\rho) \right| \leq \tilde{C} \|h\|_r r^\sigma,$$

for all $h \in C[0, r]$ and all $r > 0$ sufficiently small.

It is worth noting that there are an infinite number of continuous and discrete families $\{\eta_r\}_{r > 0}$ of measures which are easily constructed and which satisfy the above assumptions. In what follows we provide two classes of measures satisfying $(H_1) - (H_3)$ and we refer to [10] for the proof. The first measure is a continuous measure.

Lemma 2.1.1. *[10] Let $\nu = 1, 2, \ldots$ be arbitrary and let $\psi \in L^1(0, 1)$ be given such that*

$$\int_0^1 \rho^\nu \psi(\rho)\, d\rho > 0.$$

Then the 'density' η_r for $r \in (0, R]$, $0 < R \leq 1$, defined by

$$\int_0^r g(\rho)\, d\eta_r(\rho) = \int_0^r g(\rho)\psi_r(\rho)\, d\rho, \quad g \in C[0, r],$$

where $\psi_r \in L^1(0, r)$ is given by

$$\psi_r(\rho) = \psi(\rho/r), \quad a.e. \quad \rho \in [0, r],$$

satisfies condition (H_1) (with $c_\nu = \int_0^1 \rho^\nu \psi(\rho)\, d\rho$ and $\sigma = 1$) and condition (H_3). Further, for all $\nu = 1,\ 2,\ \ldots$ and given arbitrary positive $\bar{c}$, m_1, $m_2, \ldots$ and m_ν, there is a unique polynomial ψ of degree ν so that the resulting family $\{\eta_r\}$ satisfies (H_1) with $c_\nu = \bar{c}$ and $\sigma = 1$, (H_2) with the roots of the polynomial p_ν in (H_2) given by $(-m_i)$, $i = 1, \ldots, \nu$ and (H_3).

The second measure is a discrete measure.

Lemma 2.1.2. *[10] Let $\nu = 1, 2, \ldots,$ be arbitrary and let $\beta_l, \tau_l \in \mathbb{R}$, $l = 0, 1, \ldots,$ L, be fixed so that*

$$0 \leq \tau_0 < \tau_1 < \cdots < \tau_{L-1} < \tau_L \leq 1, \tag{2.11}$$

and

$$\sum_{l=0}^{L} \beta_l \tau_l^{\nu} > 0. \tag{2.12}$$

Then the discrete measure η_r defined via

$$\int_0^r g(\rho)\, d\eta_r(\rho) = \sum_{l=0}^{L} \beta_l g(\tau_l r), \quad g \in C[0,r],$$

satisfies condition (H_1) (with $c_\nu = \sum_{l=0}^{L} \beta_l \tau_l^{\nu}$ and $\sigma = 0$) and condition (H_3). Further, for all $\nu = 1, 2, \ldots$ and given arbitrary positive $\bar{c}$, m_1, $m_2 \ldots$, and m_ν and for $L = \nu$, there is a unique choice of $\beta_0, \beta_1 \ldots, \beta_\nu$ satisfying (2.12) (for each given collection of $\{\tau_l\}$ satisfying (4.4)) and such that the resulting discrete measure η_r satisfies (H_1) with $c_\nu = \bar{c}$ and $\sigma = 0$, $(H2)$ with the roots of the polynomial p_ν in (H_2) given by $(-m_i)$, $i = 1, 2, \ldots, \nu$ and (H_3).

Under the conditions on the measure η_r the following lemma shows that $a(r) \neq 0$ for all $r > 0$ sufficiently small and all ν-smoothing k. Therefore the regularization equation (2.7) is always well-posed in these cases so that the solution to (2.7) depends continuously on data f^{δ}.

Lemma 2.1.3. *[10] Assume η_r satisfies (H_1) and (H_3). Then*

$$a(r) = \frac{c_\nu}{\nu!} r^{\sigma + \nu}(1 + O(r)),$$

so that $a(r) > 0$ for all $r > 0$ sufficiently small.

Using the above lemma, it is easy to see that

$$a(r) \geq \frac{c_\nu}{2\nu!} r^{\sigma + \nu} > 0 \quad \text{for } r > 0 \text{ sufficiently small.} \tag{2.13}$$

Further, under this construction we have from [10] the following theorem.

Theorem 2.1.1. *[10] Let $\bar{u}$ denote the solution of* (2.1) *given "true" data $f \in C[0, T + R]$ and assume $\bar{u}$ satisfies the Hölder condition* (2.4) *on $[0, T + R]$ with Hölder exponent $\mu \in (0, 1]$ and $R > 0$ small. Assume k is ν-smoothing and that $\{\eta_r\}$ is a family of signed Borel measures satisfying hypotheses $(H_1) - (H_3)$ for all $r \in (0, R]$. Then there is a constant $C > 0$ (depending only on the c_i defined in (H_1) and independent of r) such that if*

$$\|k^{(\nu)}\|_\infty < C,$$

and if $f^\delta \in C[0, T + R]$ satisfies (2.6)*, then*

$$|u_r^\delta(t) - \bar{u}(t)| \leq C_1 \frac{\hat{\delta}}{r^\nu} + C_2 r^\mu, \quad t \in [0, T],$$

for some C_1, $C_2 \geq 0$, so that the choice

$$r = r(\hat{\delta}) \sim \hat{\delta}^{\frac{1}{\mu + \nu}}$$

gives

$$|u_r^\delta(t) - \bar{u}(t)| = O(\hat{\delta}^{\frac{\mu}{\mu + \nu}}) \quad as \quad \delta \to 0,$$

uniform in $t \in [0, T]$.

We would like to point out that the above convergence result can be obtained using not only signed Borel measures but also *positive* Borel measures for ν-smoothing

Volterra problems with $\nu = 1, 2, 3, 4$. There is to date no convergence theory for positive Borel measures with $\nu > 4$ and in fact a sufficient condition for convergence is known to fail in these cases. For details, see [3], [4], [6] and [10].

2.2 Existing results for the local regularization of nonlinear Hammerstein problems

While the theory for the local regularization methods of linear Volterra problems is rather complete, the same can not be said for the nonlinear theory. In recent years the local regularization theory has been extended to the nonlinear autoconvolution problem [12] and to the nonlinear Hammerstein problem [11]:

$$\int_0^t k(t-s)g(s,u(s))\,ds = f(t) \quad \text{for} \quad t \in [0,T], \tag{2.14}$$

where g is a nonlinear function on $\mathbb{R}$. A discussion of the existence and uniqueness of solutions of (2.14) can be found in [16]-[17]. Based on the idea for the linear problem, we let $R > 0$ be a small fixed number and assume that equation (2.14) still holds on an extended interval $[0, T+R]$. We may define the following nonlinear regularized equation

$$a(r)g(t,u(t)) + \int_0^t \tilde{k}_r(t-s)g(s,u(s))\,ds = \tilde{f}_r^\delta(t), \quad t \in [0,T], \tag{2.15}$$

where $\tilde{k}_r$, $\tilde{f}_r^\delta$ and $a(r)$ are given by (2.8) – (3.4) using a signed measure η_r satisfying (H_1) – (H_3). In a note in 2005, Lamm and Dai observed that if one lets $v(t) = g(t,u(t))$, then equation (2.15) is nothing more than equation (2.7) in the new variable $v(t)$, that is

$$a(r)v(t) + \int_0^t \tilde{k}_r(t-s)v(s)\,ds = \tilde{f}_r^\delta(t), \quad t \in [0,T]. \tag{2.16}$$

By the linear theory, if $f^\delta \in C[0, T+R]$, then there exists a unique solution $v_r^\delta \in C[0,T]$ of (2.16). But the goal is to find $u \in C[0,T]$ which solves (2.15). So the question is how to stably recover u from inverting the function g. For $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ continuous with

$$(g1) \quad \lim_{x \to +\infty} g(t,x) = +\infty, \qquad \lim_{x \to -\infty} g(t,x) = -\infty, \quad t \in [0,T],$$

$$(g2) \quad (g(t,x) - g(t,y))(x-y) > 0, \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad x, y \in \mathbb{R} \quad \text{with} \quad x \neq y,$$

then there exists a unique $u_r^\delta \in C[0,T]$ such that $g(t, u_r^\delta(t)) = v_r^\delta(t)$, $t \in [0,T]$. The convergence of u_r^δ to $\bar{u}$ is given by the following theorem:

Theorem 2.2.1. *[11] Let $\bar{u}$ denote the solution of* (2.14) *given "true" data $f \in C[0, T+R]$ and assume $\bar{u}$ satisfies* (2.4) *on $[0, T+R]$. Assume k is ν-smoothing and that $\{\eta_r\}$ is a family of signed Borel measures satisfying hypotheses $(H_1)-(H_3)$ for all $r \in (0,R]$. Assume further that g, g_t, $g_x : [0, T+R] \times \mathbb{R} \to \mathbb{R}$ are continuous with g_t, g_x bounded on set $[0, T+R] \times I$, where I is a bounded open interval in $\mathbb{R}$, such that $\bar{u}(t) \in I$ for $t \in [0, T+R]$. Assume also that g satisfies $(g1)-(g2)$ for $t \in [0, T+R]$ and*

$$(g3) \quad \textit{there exists} \quad \bar{c}_1 := \bar{c}_1(I) > 0 \quad \textit{such that} \quad (g(t,x)-g(t,y))(x-y) \geq \bar{c}_1|x-y|^2,$$

for all $t \in [0, T+R]$ and $x, y \in I$.
If $\|k^{(\nu)}\|_\infty < C$, for the constant C given in Theorem 2.1.1 *above, then if $f^\delta \in C[0, T+R]$ satisfies* (2.6) *then the choice*

$$r = r(\hat{\delta}) \sim \hat{\delta}^{\frac{1}{\mu+\nu}} \tag{2.17}$$

gives

$$|u_r^\delta(t) - \bar{u}(t)| = O(\hat{\delta}^{\frac{\mu}{\mu+\nu}}) \quad as \quad \hat{\delta} \to 0, \tag{2.18}$$

for $t \in [0, T]$.

Remark 2.2.1. *Notice that from the above theorem, we derive the same convergence rate as in the linear case. The assumptions* $(g1) - g(2)$ *guarantee a unique solution* $u_r^\delta \in C[0, T]$ *for the regularized equation* (2.15). *Assumptions on* g, g_t, g_x *on* I *make sure that* u_r^δ *converges to the true solution* $\bar{u}$.

However, notice that this theory given by (2.15) requires inverting the nonlinear function g in order to find the solution u_r^δ. In terms of numerical implementation this means the method requires solving a large-scale nonlinear system or numerous nonlinear equations which can be difficult in practice. Therefore our goal is to design a local regularization theory which avoids solving large number of nonlinear equations. That is, we want to derive a regularization equation such that the solution to this equation depends continuously on data and it converges to the true solution $\bar{u}$ when noise level shrinks to zero. At the same time, we want to be able to solve our regularization equation without solving a nonlinear equation at each step of a numerical iteration. Keeping this goal in mind, we present our local regularization theory in the next section.

2.3 New local regularization theory for Hammerstein equations

To motivate the sequential local regularization method for nonlinear Hammerstein problems, we let $R > 0$ be a small fixed number and assume that

$$\int_0^t k(t-s)g(u(s))\,ds = f(t) \quad \text{a.e. } t \in [0,T] \tag{2.19}$$

holds on an extended interval $[0, T+R]$. Assume $g \in C^1(I)$, where $I \subset \mathbb{R}$ is a bounded open interval, with g' bounded on I. The true solution $\bar{u}$ to (2.19) satisfies (2.4) and $\bar{u}(t) \in I$ for $t \in [0, T+R]$. Note that these are the same assumptions that required in [11]. We will let $r \in (0, R]$ be a small parameter. Then the "true" solution $\bar{u}$ of (2.19) satisfies

$$\int_0^{t+\rho} k(t-s+\rho)g(u(s))\,ds = f(t+\rho), \quad \text{a.e.} \quad t \in [0,T], \quad \rho \in [0,r].$$

Proceeding as in the linear problem, we obtain an approximate equation in u valid for a.e. $t \in [0,T]$, such that

$$\int_0^t \tilde{k}_r(t-s)g(u(s))\,ds + a(r)g(u(t)) = \tilde{f}_r^\delta(t), \tag{2.20}$$

where $\tilde{k}_r$, $\tilde{f}_r^\delta$ and $a(r)$ are given by (2.8)-(3.4) for $f^\delta \in C[0, T+R]$. The true solution $\bar{u}$ still satisfies the following equation

$$\int_0^t \int_0^r k(t-s+\rho)\,d\eta_r(\rho)g(\bar{u}(s))\,ds + \int_0^r \int_0^\rho k(\rho-s)g(\bar{u}(s+t))\,ds\,d\eta_r(\rho)$$
$$= \int_0^r f(t+\rho)\,d\eta_r(\rho), \quad \text{a.e.} \quad t \in [0,T]. \tag{2.21}$$

Since our goal is to solve for u numerically and avoid solving nonlinear function

g, we need to linearize the function g. Notice that for suitable functions $u(t) \in I$, $g(u(t)) \approx g(u(t-\tau))+g'(u(t-\tau))(u(t)-u(t-\tau))$, where $\tau = \tau(t)$ is assumed to satisfy $0 < \tau(t) < min\{t, r\}$, if $0 < t \leq T$ and $\tau(0) = 0$. Thus in (2.20), we replace $g(u(t))$ outside the integral by its approximation and we obtain our regularized equation in u valid for a.e. $t \in [0, T]$,

$$\begin{aligned}
&\int_0^t \tilde{k}_r(t-s)g(u(s))\,ds + a(r)g'(u(t-\tau))u(t) \\
&= -a(r)g(u(t-\tau)) + a(r)g'(u(t-\tau))u(t-\tau) + \tilde{f}_r^\delta(t), \quad \text{a.e.} \quad t \in [0, T]. \qquad (2.22)
\end{aligned}$$

So we seek a solution $u(t) = u_r^\delta(t)$, $u(t) \in I$ for $t \in [0, T]$ satisfying the regularization equation (2.22).

We hope to show that our regularization equation (2.22) is well-posed and the solution u to (2.22) approximates the true solution $\bar{u}$ in some appropriate norm (which will be clarified later) for suitable choices of the parameter r.

Subtracting (2.21) from (2.22) gives

$$\begin{aligned}
&\int_0^t \tilde{k}_r(t-s)\,[g(u(s)) - g(\bar{u}(s))]\,ds \\
&= \tilde{\delta}_r(t) + \int_0^r \int_0^\rho k(\rho - s)g(\bar{u}(s+t))\,ds\,d\eta_r(\rho) - a(r)g(u(t-\tau)) \\
&\quad - a(r)g'(u(t-\tau))u(t) + a(r)g'(u(t-\tau))u(t-\tau), \quad \text{a.e.} \quad t \in [0, T], \qquad (2.23)
\end{aligned}$$

where

$$\tilde{\delta}_r(t) = \int_0^r \delta(t+\rho)d\eta_r(\rho), \quad \delta(t) = f^\delta(t) - f(t), \quad \|\delta\|_\infty \leq \hat{\delta}, \quad \text{some} \quad \hat{\delta} > 0. \ (2.24)$$

Assume g' satisfies

$(g3')$ there exists a constant $\bar{c}_1 := \bar{c}_1(I) > 0$ such that $|g'(x)| \geq \bar{c}_1 > 0$ for $x \in I$.

Remark 2.3.1. *Theorem* 2.2.1 *is still true under a weaker hypothesis than* $(g3)$, *namely,*

$(g3a)$ $\exists c_1 > 0$ *such that* $|g(t,x) - g(t,y)| \geq c_1|x-y|$,

for $t \in [0, T+R]$ *and all* $x,\ y \in I$. *This latter hypothesis implies* $(g3')$ *if* g' *exists on* I *since* $\left|\frac{g(x+h)-g(x)}{h}\right| \geq \bar{c}_1 > 0$ *for* $|h|$ *sufficiently small. Because our local regularization method has to utilize* g' *term, it makes sense to assume* $(g3')$ *in our problem.*

By $(g3')$ and Inverse Function Theorem, we derive $g^{-1} \in C^1(D)$, where $D := g(I)$. Let $\bar{v}(t) = g(\bar{u}(t))$, for $t \in [0, T+R]$. Motivated by equation (2.23), we will seek a solution v, $v(t) \in D$ a.e. $t \in [0,T]$, of the following equation:

$$
\begin{aligned}
&\int_0^t \tilde{k}_r(t-s)\,[v(s) - \bar{v}(s)]\,ds \\
&= \tilde{\delta}_r(t) + \int_0^r \int_0^\rho k(\rho - s)\bar{v}(s+t)\,ds\,d\eta_r(\rho) - a(r)v(t-\tau) \\
&\quad - a(r)g'\left(g^{-1}(v(t-\tau))\right)g^{-1}(v(t)) + a(r)g'\left(g^{-1}(v(t-\tau))\right)g^{-1}(v(t-\tau)),
\end{aligned}
$$

or

$$
\int_0^t \tilde{k}_r(t-s)\,[v(s)-\bar{v}(s)]\,ds + a(r)\,[v(t)-\bar{v}(t)] = G_r(v)(t), \quad \text{a.e.} \quad t \in [0,T], \tag{2.25}
$$

where for $w \in L^{\infty}((0,T),D)$,

$$G_r(w)(t) = a(r)\left[w(t) - \bar{v}(t)\right] + \tilde{\delta}_r(t) + \int_0^r \int_0^\rho k(\rho - s)\bar{v}(s+t)\,ds\,d\eta_r(\rho)$$
$$- a(r)\bar{v}(t-\tau) - a(r)\left[w(t-\tau) - \bar{v}(t-\tau)\right] - a(r)g'\left(g^{-1}(w(t-\tau))\right)g^{-1}(w(t))$$
$$+ a(r)g'\left(g^{-1}(w(t-\tau))\right)g^{-1}(w(t-\tau)), \quad \text{a.e.} \quad t \in [0,T]. \tag{2.26}$$

Since $\delta \in C[0, T+R]$, then

$$G_r : L^{\infty}((0,T),D) \to L^{\infty}(0,T).$$

Define

$$B_r : L^{\infty}(0,T) \to L^{\infty}(0,T),$$

where

$$B_r(w)(t) := \int_0^t \tilde{k}_r(t-s)w(s)ds, \quad \text{a.e.} \quad t \in [0,T],$$

so that we can write (2.25) as:

$$\left(a(r)I + B_r\right)(v - \bar{v})(t) = G_r(v)(t), \quad \text{a.e.} \quad t \in [0,T]. \tag{2.27}$$

The following lemma is obtained using Theorem 3.1 of [10].

Lemma 2.3.1. *[10] The operator $(a(r)I + B_r) : L^{\infty}(0,T) \to L^{\infty}(0,T)$ is invertible with $(a(r)I + B_r)^{-1} \in \mathcal{L}\left(L^{\infty}(0,T), L^{\infty}(0,T)\right)$ and, if $\|k^{(\nu)}\|_{\infty} \leq C$, for the C given in Theorem* 2.1.1 *above, then*

$$\left\|\left(a(r)I + B_r\right)^{-1}\right\|_{\mathcal{L}\left(L^{\infty}(0,T), L^{\infty}(0,T)\right)} \leq \frac{1+m}{a(r)},$$

for $r > 0$ sufficiently small, where m is independent of r.

Now we are ready to prove the main results.

Theorem 2.3.1. *Let $\bar{u}$ denote the solution of* (2.19) *given "true" data $f \in C[0, T+R]$ and let the same assumptions hold as in Theorem* 2.2.1 *for $\bar{u}$, kernel k and signed Borel measures $\{\eta_r\}$. Let $g \in C^1(I)$ with g' bounded on I, where $I \subset \mathbb{R}$ is an open bounded interval, assume g satisfies* ($g3'$). *Assume further that*

($g4$) *there exists a constant $N > 0$ such that $|g'(x) - g'(y)| \leq N|x - y|$ for $x,\ y \in I$.*

Let $R > 0$ be sufficiently small and let $r \in (0, R]$ be arbitrary. Then there exists a θ independent of r such that, if $f^\delta \in C[0, T + R]$ satisfies (2.24) *with $\hat{\delta} \leq k_1 r^{\mu+\nu}$, then there is a unique solution v of* (2.27) *satisfying $\|v - \bar{v}\| \leq \theta r^\mu$. Further, the mapping $f^\delta \in \{w \in C[0, T + R], \|w - f\|_\infty \leq \hat{\delta}\} \mapsto v \in L^\infty((0, T), D)$ is continuous for all $r \in (0, R]$.*

Before proving Theorem 2.3.1, we need some lemmas.

Lemma 2.3.2. *If $\bar{v}(x) = g(\bar{u}(x))$, $x \in [0, T + R]$, then*

$$|\bar{v}(x) - \bar{v}(y)| \leq \|g'\|_I \bar{N} |x - y|^\mu$$

for a.e. $x, y \in [0, T + R]$ and μ defined in (2.4).

Proof. We have

$$\begin{aligned} \bar{v}(x) - \bar{v}(y) &= g(\bar{u}(x)) - g(\bar{u}(y)) \\ &= g'(\xi(\bar{u}, x, y))(\bar{u}(x) - \bar{u}(y)), \end{aligned}$$

where $\xi(\bar{u}, x, y) \in I$ since I is an open interval. Thus $|\bar{v}(x) - \bar{v}(y)| \leq \|g'\|_I \bar{N} |x - y|^\mu$, for a.e. $x,\ y \in [0, T + R]$. ☐

Lemma 2.3.3. *Assume that g satisfies assumption $(g3')$. Then*

$$|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{\bar{c}_1}|x-y| \quad \textit{for} \quad x, y \in D.$$

Proof. As stated earlier, our assumptions on g give $g^{-1} \in C^1(D)$; further, D can be seen to be an open interval due to the continuity of g and g^{-1}. For any x, $y \in D$, $|g^{-1}(x) - g^{-1}(y)| = (g^{-1})'(\xi(x,y))(x-y)$, where $\xi(x,y) \in D$. But

$$|(g^{-1})'(\xi(x,y))| = \left|\frac{1}{g'(g^{-1}(\xi(x,y)))}\right| \leq \frac{1}{\bar{c}_1}$$

since $g^{-1}(\xi(x,y)) \in I$. So

$$|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{\bar{c}_1}|x-y|.$$

□

Now we are ready to prove the above theorem.

Proof of Theorem 2.3.1. Since $\bar{u}(t) \in I$, so $\bar{v}(t) \in D$. Consider the ball $M := \{v \in L^\infty(0,T) : \|v - \bar{v}\| \leq \theta r^\mu\}$ for some number θ (independent of r) to be determined and μ defined in (2.4). We claim that any $v \in M$, we have $v(t) \in D$ for a.e. $t \in [0,T]$ when $r > 0$ is sufficiently small. Indeed, since $\bar{v}$ is continuous, the set $R(\bar{v}) = \{\bar{v}(t), \quad t \in [0,T]\}$ is a closed bounded interval $[a,b]$ in D. Since D is open, the interval $[a - \theta r^\mu, b + \theta r^\mu] \subset D$ for $r > 0$ sufficiently small. Therefore the claim is true.

For $v \in M$, $G_r(v) \in L^\infty(0,T)$, so it makes sense to apply the operator

$(a(r)I + B_r)^{-1}$ on $G_r(v)$. Thus

$$v = (a(r)I + B_r)^{-1} G_r(v) + \bar{v} = H_r(v),$$

where

$$H_r : L^{\infty}((0,T), D) \to L^{\infty}(0,T),$$

is given by

$$H_r(v) := (a(r)I + B_r)^{-1} G_r(v) + \bar{v}.$$

Our goal now is to show that there is a unique solution $v_r^{\delta} \in L^{\infty}((0,T), D)$ solving the equation:

$$v = H_r(v),$$

so that such a v will uniquely solve (2.27).

We will prove by the contraction mapping theorem: so we want to show that $H_r : M \to M$ and is a contraction. First we show that H_r maps M to M for $r > 0$ sufficiently small.

By Lemma 2.3.1 and for $v \in M$,

$$\begin{aligned}
\|H_r(v) - \bar{v}\| &= \| (a(r)I + B_r)^{-1} G_r(v)\| \\
&\leq \| (a(r)I + B_r)^{-1} \|_{\mathcal{L}(L^{\infty}(0,T), L^{\infty}(0,T))} \|G_r(v)\| \\
&\leq \frac{1+m}{a(r)} \|G_r(v)\|.
\end{aligned}$$

We will add and subtract $a(r)\bar{v}(t-\tau)$, $a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(\bar{v}(t))$ and $a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(\bar{v}(t-\tau))$, then regroup on the right hand side of (2.26), we obtain

$$
\begin{aligned}
G_T(v)(t) &= \tilde{\delta}_T(t) + \int_0^r \int_0^\rho k(\rho - s)\bar{v}(s+t)\, ds\, d\eta_T(\rho) - a(r)\bar{v}(t-\tau) \\
&\quad + a(r)\,[v(t) - \bar{v}(t)] - a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(v(t)) + a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(\bar{v}(t)) \\
&\quad - a(r)[v(t-\tau) - \bar{v}(t-\tau)] + a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(v(t-\tau)) \\
&\quad - a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(\bar{v}(t-\tau)) \\
&\quad - a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(\bar{v}(t)) + a(r)g'(g^{-1}(v(t-\tau)))g^{-1}(\bar{v}(t-\tau)) \\
&= \tilde{\delta}_T(t) + \int_0^r \int_0^\rho k(\rho - s)(\bar{v}(s+t) - \bar{v}(t-\tau))\, ds\, d\eta_T(\rho) \\
&\quad + a(r)\left[v(t) - \bar{v}(t) - g'(g^{-1}(v(t-\tau)))\left(g^{-1}(v(t)) - g^{-1}(\bar{v}(t))\right)\right] \\
&\quad - a(r)\left[v(t-\tau) - \bar{v}(t-\tau) - g'(g^{-1}(v(t-\tau)))\left(g^{-1}(v(t-\tau)) - g^{-1}(\bar{v}(t-\tau))\right)\right] \\
&\quad - a(r)g'(g^{-1}(v(t-\tau)))\left(g^{-1}(\bar{v}(t)) - g^{-1}(\bar{v}(t-\tau))\right), \quad \text{a.e.} \quad t \in [0,T]. \qquad (2.28)
\end{aligned}
$$

Therefore, for a.e. $t \in [0,T]$,

$$
|G_T(v)(t)| \le \sum_{i=1}^{5} T_T^{(i)}(t),
$$

where

$$
\begin{aligned}
T_T^{(1)}(t) &= |\tilde{\delta}_T(t)|, \\
T_T^{(2)}(t) &= \left|\int_0^r \int_0^\rho k(\rho - s)(\bar{v}(s+t) - \bar{v}(t-\tau))\, ds\, d\eta_T(\rho)\right|, \\
T_T^{(3)}(t) &= a(r)\left|v(t) - \bar{v}(t) - g'(g^{-1}(v(t-\tau)))\left(g^{-1}(v(t)) - g^{-1}(\bar{v}(t))\right)\right|, \qquad (2.29) \\
T_T^{(4)}(t) &= a(r)\left|v(t-\tau) - \bar{v}(t-\tau) - g'(g^{-1}(v(t-\tau)))\left(g^{-1}(v(t-\tau)) - g^{-1}(\bar{v}(t-\tau))\right)\right|, \\
&\qquad (2.30) \\
T_T^{(5)}(t) &= a(r)\left|g'(g^{-1}(v(t-\tau)))\left(g^{-1}(\bar{v}(t)) - g^{-1}(\bar{v}(t-\tau))\right)\right|, \qquad (2.31)
\end{aligned}
$$

and where we have used the fact that $a(r) > 0$ for $r > 0$ sufficiently small from Lemma 2.1.3. By (H_3), we can show that

$$|T_r^{(1)}(t)| = \left| \int_0^r \delta(t+\rho)\, d\eta_r(\rho) \right| \leq \tilde{C}\hat{\delta} r^\sigma.$$

Now consider the second term on the right hand side of (2.28):

let $p(\rho, t) := \int_0^\rho k(\rho - s)(\bar{v}(s+t) - \bar{v}(t-\tau))\, ds$. By Lemma 2.3.2

$$\begin{aligned} |p(\rho, t)| &\leq \int_0^\rho |k(\rho - s)| \|g'\|_I \bar{N}(s+\tau)^\mu \, ds \\ &\leq \|k\|_r \|g'\|_I \bar{N} \int_0^\rho (s+\tau)^\mu \, ds \\ &\leq \|k\|_r \|g'\|_I \bar{N} 2^\mu r^{\mu+1} \quad \text{a.e.} \quad t \in [0, T] \end{aligned}$$

and by assumption of the above theorem, $p(\cdot, t) \in C[0, r]$ for any $t \in [0, T]$. Further, for any $s \in [0, r]$

$$k(s) = \frac{1}{(\nu - 1)!} s^{\nu - 1} + R_{\nu - 1}(s),$$

where

$$R_{\nu - 1}(s) = \frac{k^{(\nu)}(\xi)}{\nu!} s^\nu,$$

$0 < \xi < s$. Therefore

$$\|k\|_r \leq \frac{r^{\nu - 1}}{(\nu - 1)!} + \frac{\|k^{(\nu)}\|_r}{\nu!} r^\nu.$$

So for a.e. $t \in [0, T]$,

$$|p(\rho, t)| \leq \|g'\|_I \bar{N} 2^\mu r^{\mu+1} \frac{r^{\nu - 1}}{(\nu - 1)!} \left(1 + \frac{\|k^{(\nu)}\|_r}{\nu} r \right) \leq \|g'\|_I \bar{N} 2^{\mu+1} \frac{r^{\nu + \mu}}{(\nu - 1)!},$$

for $r > 0$ sufficiently small. So by assumption (H_3) on the measure, we obtain

$$T_r^{(2)}(t) \leq \tilde{C}\|p\|_r r^{\sigma} \leq \tilde{C}\|g'\|_I \bar{N} 2^{\mu+1} \frac{r^{\nu+\mu+\sigma}}{(\nu-1)!},$$

for $r > 0$ sufficiently small.

For a.e. $t \in [0, T]$, we use the fact that $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$ for suitable x, to write

$$\begin{aligned}
T_r^{(3)}(t) &= a(r)\left|(v(t) - \bar{v}(t))[1 - g'(g^{-1}(v(t-\tau)))(g^{-1})'(\xi(v, \bar{v}, t))]\right| \\
&\leq a(r)\theta r^{\mu}\left|\frac{g'(g^{-1}(\xi(v, \bar{v}, t))) - g'(g^{-1}(v(t-\tau)))}{g'(g^{-1}(\xi(v, \bar{v}, t)))}\right| \\
&\leq a(r)\frac{\theta r^{\mu}}{\bar{c}_1} N |g^{-1}(\xi(v, \bar{v}, t)) - g^{-1}(v(t-\tau))| \\
&= a(r)\frac{\theta r^{\mu}}{\bar{c}_1} N \frac{1}{\bar{c}_1} |\xi(v, \bar{v}, t) - v(t-\tau)|
\end{aligned}$$

where we have used Lemma 2.3.2 and Lemma 2.3.3. Further, for a.e. $t \in [0, T]$,

$$min\{v(t), \bar{v}(t)\} < \xi(v, \bar{v}, t) < max\{v(t), \bar{v}(t)\},$$

so $|\xi(v, \bar{v}, t) - \bar{v}(t)| \leq \theta r^{\mu}$ and $\xi(v, \bar{v}, t) \in D$ for $r > 0$ sufficiently small. Therefore

$$\begin{aligned}
&|\xi(v, \bar{v}, t) - v(t-\tau)| \leq |\xi(v, \bar{v}, t) - \bar{v}(t)| + |\bar{v}(t) - \bar{v}(t-\tau)| + |\bar{v}(t-\tau) - v(t-\tau)| \\
&\leq \theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu} + \theta r^{\mu} \\
&= 2\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu}, \quad \text{a.e.} \quad t \in [0, T],
\end{aligned}$$

and thus

$$T_r^{(3)}(t) \le a(r)\frac{\theta r^{\mu}}{\bar{c}_1^2}N(2\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu})$$
$$= a(r)\frac{\theta}{\bar{c}_1^2}N(2\theta + \|g'\|_I \bar{N})r^{2\mu}, \quad \text{a.e.} \quad t \in [0,T]. \tag{2.32}$$

Similarly, for $r > 0$ sufficiently small,

$$T_r^{(4)}(t) = a(r)\left|(v(t-\tau) - \bar{v}(t-\tau))\,\frac{g'(g^{-1}(\xi(v,\bar{v},t-\tau))) - g'(g^{-1}(v(t-\tau)))}{g'(g^{-1}(\xi(v,\bar{v},t-\tau)))}\right|$$
$$\le a(r)\frac{\theta r^{\mu} N}{\bar{c}_1^2}|\xi(v,\bar{v},t-\tau) - v(t-\tau)|$$
$$\le a(r)\frac{\theta^2 N}{\bar{c}_1^2}r^{2\mu}, \quad \text{a.e.} \quad t \in [0,T], \tag{2.33}$$

because $|\xi(v,\bar{v},t-\tau) - v(t-\tau)| \le |\bar{v}(t-\tau) - v(t-\tau)| \le \theta r$, a.e. $t \in [0,T]$.

Finally the last term on the right hand side of (2.28) is

$$T_r^{(5)}(t) = a(r)\left|g'(g^{-1}(v(t-\tau)))\right| |\bar{u}(t) - \bar{u}(t-\tau)|$$
$$\le a(r)\|g'\|_I \bar{N} r^{\mu} \quad \text{a.e.} \quad t \in [0,T] \quad \text{and} \quad r \quad \text{sufficiently small.} \tag{2.34}$$

It follows that

$$\|G_r(v)\| \le \tilde{C}\hat{\delta} r^{\sigma} + \tilde{C}\|g'\|_I \bar{N} 2^{\mu+1} \frac{r^{\nu+\mu+\sigma}}{(\nu-1)!}$$
$$+ a(r)\frac{\theta}{\bar{c}_1^2}N(2\theta + \|g'\|_I \bar{N})r^{2\mu} + a(r)\frac{\theta^2 N}{\bar{c}_1^2}r^{2\mu}$$
$$+ a(r)\|g'\|_I \bar{N} r^{\mu}, \quad \text{for } r > 0 \text{ sufficiently small.}$$

Thus, we have from Lemma 2.1.3,

$$
\begin{aligned}
&\|H_r(v) - \bar{v}\| \le \frac{1+m}{a(r)}\|G_r(v)\| \\
&\le \frac{(1+m)}{(c_\nu/2\nu!)r^\nu}\tilde{C}\hat{\delta} \\
&\quad + 2^{(\mu+2)}\frac{(1+m)\nu}{c_\nu}\tilde{C}\|g'\|_I\bar{N}r^\mu \\
&\quad + (1+m)\frac{\theta}{\bar{c}_1^2}N(2\theta + \|g'\|_I\bar{N})r^{2\mu} + (1+m)\frac{\theta^2 N}{\bar{c}_1^2}r^{2\mu} \\
&\quad + (1+m)\|g'\|_I\bar{N}r^\mu,
\end{aligned}
$$

for all $r > 0$ sufficiently small.

Let $\hat{\delta} = \hat{\delta}(r)$ satisfy $\hat{\delta} \le k_1 r^{\mu+\nu}$, for some $k_1 > 0$. Then

$$
\frac{(1+m)}{(c_\nu/2\nu!)r^\nu}\tilde{C}\hat{\delta} \le 2(1+m)\tilde{C}\frac{k_1\nu!}{c_\nu}r^\mu,
$$

for all $r > 0$ sufficiently small.

So

$$
\begin{aligned}
\|H_r(v) - \bar{v}\| \le &\left[2(1+m)\tilde{C}\frac{k_1\nu!}{c_\nu} + 2^{(\mu+2)}\frac{(1+m)\nu}{c_\nu}\tilde{C}\|g'\|_I\bar{N} + (1+m)\|g'\|_I\bar{N}\right]r^\mu \\
+ &\left[(1+m)\frac{\theta}{\bar{c}_1^2}N(2\theta + \|g'\|_I\bar{N}) + (1+m)\frac{\theta^2 N}{\bar{c}_1^2}\right]r^{2\mu}.
\end{aligned}
$$

To have $\|H_r(v) - \bar{v}\| \le \theta r^\mu$ for some $\theta > 0$ and all $r > 0$ sufficiently small, a sufficient condition is

$$
2(1+m)\tilde{C}\frac{k_1\nu!}{c_\nu} + 2^{\mu+2}\frac{(1+m)\nu}{c_\nu}\tilde{C}\|g'\|_I\bar{N} + (1+m)\|g'\|_I\bar{N} < \frac{\theta}{2},
$$

for $r > 0$ sufficiently small. So let

$$0 < \theta < 2(2(1+m)\tilde{C}\frac{k_1\nu!}{c_\nu} + 2^\mu + 3\frac{(1+m)\nu}{c_\nu}\tilde{C}\|g'\|_I\bar{N} + (1+m)\|g'\|_I\bar{N}). \quad (2.35)$$

Then $\|H_r(v) - \bar{v}\| \leq \theta r^\mu$ for $r > 0$ sufficiently small where θ is defined by (2.35). Therefore

$$H_r : M \longrightarrow M$$

for $r > 0$ sufficiently small provided $\hat{\delta} = \hat{\delta}(r)$ satisfies $\hat{\delta} \leq k_1 r^{\mu+\nu}$ for all such r. Now we want to show that for any $v_1, v_2 \in M = \{v \in L^\infty(0,T) : \|v - \bar{v}\| \leq \theta r^\mu\}$, we have

$$\|H_r(v_1) - H_r(v_2)\| \leq \alpha\|v_1 - v_2\|,$$

for $0 \leq \alpha < 1$. Since

$$\|H_r(v_1) - H_r(v_2)\| \leq \frac{1+m}{a(r)}\|G_r(v_1) - G_r(v_2)\|,$$

we note that for $r > 0$ sufficiently small, we have for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\frac{1}{a(r)}[G_r(v_1)(t) - G_r(v_2)(t)] \\
&= [v_1(t) - v_2(t)] - [v_1(t-\tau) - v_2(t-\tau)] \\
&\quad - g'(g^{-1}(v_1(t-\tau)))[g^{-1}(v_1(t)) - g^{-1}(v_2(t))] \\
&\quad + g'(g^{-1}(v_2(t-\tau)))[g^{-1}(v_1(t-\tau)) - g^{-1}(v_2(t-\tau))] \\
&\quad - g^{-1}(v_2(t))[g'(g^{-1}(v_1(t-\tau))) - g'(g^{-1}(v_2(t-\tau)))] \\
&\quad - g^{-1}(v_1(t-\tau))[g'(g^{-1}(v_2(t-\tau))) - g'(g^{-1}(v_1(t-\tau)))] \\
&= \sum_{i=1}^{3} S_r^{(i)}(t), \qquad (2.36)
\end{aligned}$$

where

$$S_r^{(1)}(t) = \left[v_1(t) - v_2(t) - g'(g^{-1}(v_1(t-\tau)))\left(g^{-1}(v_1(t)) - g^{-1}(v_2(t))\right)\right] \tag{2.37}$$

$$\begin{aligned} S_r^{(2)}(t) = & -\left[v_1(t-\tau) - v_2(t-\tau)\right. \\ & \left. -g'(g^{-1}(v_2(t-\tau)))\left(g^{-1}(v_1(t-\tau)) - g^{-1}(v_2(t-\tau))\right)\right] \end{aligned} \tag{2.38}$$

$$S_r^{(2)}(t) = \left[g^{-1}(v_1(t-\tau)) - g^{-1}(v_2(t))\right]\left[g'(g^{-1}(v_1(t-\tau))) - g'(g^{-1}(v_2(t-\tau)))\right]. \tag{2.39}$$

Use similar arguments to those used in obtaining (2.32), for $r > 0$ sufficiently small we have

$$\begin{aligned} S_r^{(1)}(t) &= |(v_1(t) - v_2(t))[1 - g'(g^{-1}(v_1(t-\tau)))(g^{-1})'(\xi(v_1, v_2, t))]| \\ &\leq \frac{\|v_1 - v_2\|}{\bar{c}_1}|g'(g^{-1}(\xi(v_1, v_2, t))) - g'(g^{-1}(v_1(t-\tau)))| \\ &\leq \frac{\|v_1 - v_2\|}{\bar{c}_1}N|g^{-1}(\xi(v_1, v_2, t)) - g^{-1}(v_1(t-\tau))| \\ &\leq \frac{\|v_1 - v_2\|}{\bar{c}_1^2}N|\xi(v_1, v_2, t) - v_1(t-\tau)| \\ &\leq \frac{\|v_1 - v_2\|}{\bar{c}_1^2}N[3\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu}], \quad \text{a.e.} \quad t \in [0, T], \end{aligned}$$

where here we have used the fact that for a.e. $t \in [0, T]$,

$$min\{v_1(t), v_2(t)\} < \xi(v_1, v_2, t) < max\{v_1(t), v_2(t)\}$$

and

$$\xi(v_1, v_2, t) \in D \quad \text{for a.e.} \quad t \in [0, T],$$

so that

$$
\begin{aligned}
&|\xi(v_1, v_2, t) - v_1(t-\tau)| \\
&\leq |\xi(v_1, v_2, t) - \bar{v}(t)| + |\bar{v}(t) - \bar{v}(t-\tau)| + |\bar{v}(t-\tau) - v_1(t-\tau)| \\
&\leq 2\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu} + \theta r^{\mu} \\
&\leq 3\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu}, \quad \text{a.e.} \quad t \in [0, T].
\end{aligned}
$$

Similarly, using arguments like those used to obtain (2.33),

$$
\begin{aligned}
S_r^{(2)}(t) &= \left| (v_1(t-\tau) - v_2(t-\tau)) \frac{g'(g^{-1}(\xi(v_1, v_2, t-\tau))) - g'(g^{-1}(v_2(t-\tau)))}{g'(g^{-1}(\xi(v_1, v_2, t-\tau)))} \right| \\
&\leq \|v_1 - v_2\| \frac{N}{\bar{c}_1^2} |\xi(v_1, v_2, t-\tau) - v_2(t-\tau)| \\
&\leq \|v_1 - v_2\| \frac{N}{\bar{c}_1^2} 2\theta r^{\mu}, \quad \text{a.e.} \quad t \in [0, T],
\end{aligned}
$$

because $|\xi(v_1, v_2, t-\tau) - v_2(t-\tau)| \leq |v_1(t-\tau) - v_2(t-\tau)| \leq 2\theta r^{\mu}$, a.e. $t \in [0, T]$. Finally,

$$
\begin{aligned}
S_r^{(3)}(t) &= \left| \left(g^{-1}(v_1(t-\tau)) - g^{-1}(v_2(t))\right) \cdot \left[g'(g^{-1}(v_1(t-\tau))) - g'(g^{-1}(v_2(t-\tau)))\right] \right| \\
&\leq \frac{1}{\bar{c}_1}(2\theta + \|g'\|_I \bar{N}) r^{\mu} N \frac{1}{\bar{c}_1} \|v_1 - v_2\| \\
&= \frac{1}{\bar{c}_1^2}(2\theta + \|g'\|_I \bar{N}) N r^{\mu} \|v_1 - v_2\|, \quad \text{a.e.} \quad t \in [0, T],
\end{aligned}
$$

where we have used

$$
\begin{aligned}
&|g^{-1}(v_1(t-\tau)) - g^{-1}(v_2(t))| \\
&\leq \frac{1}{\bar{c}_1} \left[|v_1(t-\tau) - \bar{v}(t-\tau)| + |\bar{v}(t-\tau) - \bar{v}(t)| + |\bar{v}(t) - v_2(t)| \right] \\
&\leq \frac{1}{\bar{c}_1}(2\theta + \|g'\|_I \bar{N}) r^{\mu}, \quad \text{a.e.} \quad t \in [0, T].
\end{aligned}
$$

Thus

$$\|H_r(v_1) - H_r(v_2)\| \le \frac{1+m}{a(r)}\|G_r(v_1) - G_r(v_2)\|$$
$$\le (1+m)\left\{\frac{N}{\bar{c}_1^2}(3\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu}) + \frac{N}{\bar{c}_1^2}2\theta r^{\mu} + \frac{1}{\bar{c}_1^2}(2\theta + \|g'\|_I \bar{N})N r^{\mu}\right\}\|v_1 - v_2\|$$
$$= \alpha(r)\|v_1 - v_2\|,$$

where

$$\alpha(r) := \frac{(1+m)Nr^{\mu}}{\bar{c}_1^2}(7\theta + 2\|g'\|_I \bar{N}).$$

Therefore

$$\|H_r(v_1) - H_r(v_2)\| \le \alpha(r)\|v_1 - v_2\|,$$

where $\alpha(r) \in [0,1)$ provided $r > 0$ is sufficiently small.

Thus H_r is a contraction in the ball M for all $r > 0$ sufficiently small, provided $\hat{\delta} = \hat{\delta}(r)$ satisfies $\hat{\delta} \le k_1 r^{\mu + \nu}$ for all such r. Therefore equation (2.27) has a unique solution $v_r^{\delta} \in L^{\infty}((0,T), D)$ and $\|v_r^{\delta} - \bar{v}\| \le \theta r^{\mu}$ where θ is defined by (2.35).

Now we show that this solution v_r^{δ} depends continuously on the data f^{δ}. Fix $r > 0$ sufficiently small and let $\hat{\delta} = \hat{\delta}(r)$ satisfy $\hat{\delta} \le k_1 r^{\mu + \nu}$. Let $f_1^{\delta}, f_2^{\delta} \in C[0, T+R]$ satisfy

$$\|f_i^{\delta} - f\|_{\infty} \le \hat{\delta}, \quad i = 1, 2.$$

Replace $\tilde{\delta}_r(t)$ in equation (2.28) by $\tilde{\delta}_{r,i}(t)$ where $\tilde{\delta}_{r,i}(t)$ is defined as in (2.24) using f_i^{δ} instead of f^{δ} respectively for $i = 1, 2$. Then there exists a unique solution $v_{r,i}^{\delta}$ in ball M which is defined in Theorem 2.3.1 solving $v = H_{r,i}(v)$ respectively for $i = 1, 2$.

Further, using arguments similar to those used to prove H_r is a contraction,

$$\begin{aligned}\|v^\delta_{r,1} - v^\delta_{r,2}\| &= \|H_{r,1}(v^\delta_{r,1}) - H_{r,2}(v^\delta_{r,2})\| \\ &\le \frac{1+m}{a(r)}\|G_{r,1}(v^\delta_{r,1}) - G_{r,2}(v^\delta_{r,2})\| \\ &\le \alpha(r)\|v^\delta_{r,1} - v^\delta_{r,2}\| + \frac{1+m}{a(r)}\left|\int_0^r \left(f_1^\delta(t+\rho) - f_2^\delta(t+\rho)\right) d\eta_r(\rho)\right| \\ &\le \alpha(r)\|v^\delta_{r,1} - v^\delta_{r,2}\| + \frac{2(1+m)\tilde{C}\nu!\|f_1^\delta - f_{p2}^\delta\|_\infty}{c_\nu r^\nu},\end{aligned}$$

so

$$\|v^\delta_{r,1} - v^\delta_{r,2}\| \le \frac{1}{1-\alpha(r)} \cdot \frac{2(1+m)\tilde{C}\nu!\|f_1^\delta - f_2^\delta\|_\infty}{c_\nu r^\nu},$$

where $\alpha(r) \in (0,1)$ for this fixed r. Thus continuous dependence of solutions on data is obtained for equation (2.25). This completes the proof. □

Remark 2.3.2. *The only new assumption we need for our theorem is assumption* ($g4$) *on g' which is not surprising since our theory use g' explicitly so we expect to have some assumptions on g'. Also our assumption* ($g3'$) *is in fact weaker than assumption* ($g3a$) *(which could have been used in place of* ($g3$) *in [11]) in the case when g' exists. Using* ($g3'$) *alone guarantees existence of a unique solution $u_r^\delta \in L^\infty(0,T)$ which solves $u_r^\delta(t) = g^{-1}(v_r^\delta(t))$ a.e. $t \in [0,T]$ where $v_r^\delta(t) \in D$ for a.e. $t \in [0,T]$.*

Corollary 2.3.1. *Assume $\bar{u}$, f, and g still satisfy the assumptions given in Theorem* 2.3.1. *For $k = 1, 2, \ldots$, let $f^{\delta_k} \in C[0, T+R]$ satisfy* (2.24) *with $\hat{\delta}_k > 0$ where $\hat{\delta}_k \to 0$ as $k \to \infty$ and let $r_k = r_k(\hat{\delta}_k) > 0$ be selected satisfying $d_1\hat{\delta}_k^{\frac{1}{\nu+\mu}} \le r_k \le d_2\hat{\delta}_k^{\frac{1}{\nu+\mu}}$ for some constants d_1, $d_2 > 0$ and $\hat{\delta}_k \to 0$ as $k \to \infty$. Then for k sufficiently large, equation* (2.22) *has a unique solution $u_{r_k}^{\delta_k} = u_{r_k(\hat{\delta}_k)}^{\delta_k} \in L^\infty((0,T), I)$ satisfying*

$$\|u_{r_k}^{\delta_k} - \bar{u}\| \le \hat{C}\hat{\delta}_k^{\frac{\mu}{\mu+\nu}} \text{ as } k \longrightarrow \infty \text{ for some } \hat{C} \text{ independent of } k \text{ and } \hat{\delta}_k. \quad (2.40)$$

Further, the mapping $f^{\delta_k} \in \{w_k \in C[0, T+R], \|w_k - f\|_\infty \le \hat{\delta}_k\} \mapsto u_{r_k}^{\delta_k} \in L^\infty((0,T), I)$ *is continuous for all* k *sufficiently large.*

Remark 2.3.3. *The rate of convergence in* (2.40) *is in fact the optimal rate for local regularization of linear* ν*-smoothing problems under the assumption of* $\bar{u}$ *Hölder continuous with Hölder exponent* $\mu \in (0, 1]$.

Proof of Corollary 2.3.1. By Theorem 2.3.1, for each fixed k sufficiently large, equation (2.27) has a unique solution $v_{r_k}^{\delta_k} \in L^\infty((0,T), D)$ satisfying $\|v_{r_k}^{\delta_k} - \bar{v}\| \le \theta r_k^\mu$. Therefore, we can define

$$u_{r_k}^{\delta_k}(t) := g^{-1}(v_{r_k}^{\delta_k}(t)),$$

and we obtain that $u_{r_k}^{\delta_k}(t) \in I$ for a.e. $t \in [0,T]$.

Therefore,

$$\begin{aligned} \left|v_{r_k}^{\delta_k}(t) - \bar{v}(t)\right| &= \left|g(u_{r_k}^{\delta_k}(t)) - g(\bar{u}(t))\right| \\ &= \left|g'(\xi(u_{r_k}^{\delta_k}, \bar{u}, t))(u_{r_k}^{\delta_k}(t) - \bar{u}(t))\right| \\ &\ge \bar{c}_1 \left|u_{r_k}^{\delta_k}(t) - \bar{u}(t)\right| \qquad \text{a.e.} \quad t \in [0,T]. \end{aligned} \tag{2.41}$$

So

$$\left|u_{r_k}^{\delta_k}(t) - \bar{u}(t)\right| \le \frac{1}{\bar{c}_1}\left|v_{r_k}^{\delta_k}(t) - \bar{v}(t)\right| \le \frac{\theta}{\bar{c}_1}(r_k)^\mu := \hat{C}(r_k)^\mu, \quad \text{a.e.} \quad t \in [0,T],$$

where $\hat{C} = \dfrac{\theta}{\bar{c}_1}$. The above is true for any k sufficiently large. Therefore the approximate solution $u_{r_k}^{\delta_k}$ converges to the true solution $\bar{u}$ with order $(r_k)^\mu$ in L^∞-norm as $k \to \infty$.

Continuous dependence of $u_{r_k}^{\delta_k}$ on f^{δ_k} follows from continuous dependence of $v_{r_k}^{\delta_k}$ on f^{δ_k} and estimates like (2.41). □

CHAPTER 3

Hammerstein Problem with nonconvolution kernel

3.1 The regularized Hammerstein equation

We study the following nonlinear Volterra problem:

$$Fu = f, \tag{3.1}$$

where F is the nonlinear operator given by

$$Fu(t) = \int_0^t k(t,s)g(u(s))\,ds \quad \text{a.e.} \quad t \in [0,T], \tag{3.2}$$

where $f \in Range(F) \subseteq L^\infty(0,T)$ for $u \in L^\infty(0,T)$ suitably defined.

Here $k(t,s)$ is called the nonconvolution kernel. We will assume the kernel k is a 1-smoothing kernel, that is

$$k \in C^1\left([0,T+R] \times [0,T+R]\right), \quad \text{and} \quad k(t,t) \neq 0 \quad \text{for} \quad t \in [0,T+R].$$

Without loss of generality, we assume $k(t,t) = 1$. For a large class of kernels k, the solution of (3.1) is a ill-posed problem due to the fact that the solution of (3.1) does not depend on data in a continuous way. It is for this reason that some kind of regularization of (3.1) must occur. In order to motivate our method, we will make the same type of assumptions that we have for the convolution problems in Chapter 2 for the nonlinear function g and the true solutions $\bar{u}$. That is, we will let $g \in C^1(I)$. Assume the true solution $\bar{u} \in C([0, T+R], I)$ of (3.1) satisfies Hölder inequality (2.4) and $\bar{u}(t) \in I$ for $t \in [0, T+R]$. We will let $r \in (0, R]$ be a small parameter. Then using the same idea as in Chapter 2 for the nonlinear problem: we extend the integral slightly into the future, split the integral and do a change of variable to the second integral, we obtain

$$\int_0^t k(t+\rho, s) g(u(s))\, ds + \int_0^\rho k(t+\rho, t+s) g(u(t+s))\, ds = f(t+\rho) \quad \text{a.e.} \quad t \in [0,T].$$

Then integrate with respect to a signed Borel measure $\{\eta_r\}$ which satisfies $(H_1)-(H_3)$, and change the order of integration to the first integral, we then obtain for a.e. $t \in [0, T]$,

$$\int_0^t \int_0^r k(t+\rho, s)\, d\eta_r(\rho) g(u(s))\, ds + \int_0^r \int_0^\rho k(t+\rho, t+s) g(u(t+s))\, ds\, d\eta_r(\rho) = \tilde{f}_r(t),$$

where $\tilde{f}_r(t) = \int_0^r f(t+\rho)\, d\eta_r(\rho)$.

If we approximate $k(t+\rho, t+s)g(u(t+s))$ by $k(t,t)g(u(t)) = g(u(t))$ in the second integral above, we then have an approximating equation

$$\int_0^t \int_0^r k(t+\rho, s)\, d\eta_r(\rho) g(u(s))\, ds + a(r) g(u(t)) = \tilde{f}_r^\delta(t), \quad \text{a.e.} \quad t \in [0,T]$$

where $a(r) = \int_0^r \rho\, d\eta_r(\rho)$ and $\tilde{f}_r^\delta$ is defined by (2.9). If we linearize $g(u(t))$ at $u(t-\tau)$, we then obtain our regularization equation

$$\int_0^t \int_0^\tau k(t+\rho,s)\,d\eta_r(\rho)g(u(s))\,ds + a(r)[g(u(t-\tau)) + g'(u(t-\tau))(u(t)-u(t-\tau))] = \tilde{f}_r^\delta(t), \quad \text{a.e.} \quad t\in[0,T]. \tag{3.3}$$

From (H_1) we have

$$a(r) = \int_0^\tau \rho\,d\eta_r(\rho) = r^{1+\sigma}(c_1 + O(r)) \geq \frac{1}{2}c_1 r^{1+\sigma} > 0, \tag{3.4}$$

for $r>0$ sufficiently small. The true solution $\bar{u}$ satisfies for a.e. $t\in[0,T]$,

$$\int_0^t \int_0^\tau k(t+\rho,s)\,d\eta_r(\rho)g(\bar{u}(s))\,ds + \int_0^\tau \int_0^\rho k(t+\rho,t+s)g(\bar{u}(t+s))\,ds\,d\eta_r(\rho) = \tilde{f}_r(t). \tag{3.5}$$

Because the assumption on k limits the convergence theory to mildly ill-posed problems of solving for (3.1), we make the following remark.

Remark 3.1.1. *We note that this* 1*-smoothing assumption on* k *is standardly found in the theoretical convergence arguments for methods which preserve the Volterra nature of the original problem. The hypotheses of several well-known methods which preserve causality are discussed in [7] and [8]. Local regularization theory has been extended to the linear nonconvolution problem using the assumption of* k 1*-smoothing [7]. In 2000, Lamm and Scofield observed that the theoretical assumption* $k(t,t)\neq 0$ *does not appear to be needed in numerical method for the local regularization method they present for the linear problem. Numerical examples for* k *not satisfying the assumption* $k(t,t)=0$ *may be found in [4]. Thus the* 1*-smoothing assumption in nonconvolution problems is more a theoretical limitation than a practical one. Here we extend the existing* *theory for* 1*-smoothing nonconvolution linear problems to* 1*-smoothing nonconvolution Hammerstein problems.*

Subtract (3.5) from (3.3) and regroup the terms by adding and subtracting, we obtain for a.e. $t \in [0, T]$,

$$\begin{aligned}
&\int_0^t \int_0^r k(t+\rho, s)\, d\eta_r(\rho)[g(u(s)) - g(\bar{u}(s))]\, ds + a(r)[g(u(t)) - g(\bar{u}(t))] \\
&= a(r)[g(u(t)) - g(\bar{u}(t))] + \tilde{\delta}_r(t) + \int_0^r \int_0^\rho k(t+\rho, t+s) g(\bar{u}(t+s))\, ds\, d\eta_r(\rho) \\
&\quad - a(r) g(\bar{u}(t-\tau)) - a(r)[g(u(t-\tau)) - g(\bar{u}(t-\tau))] \\
&\quad - a(r) g'(u(t-\tau)) u(t) + a(r) g'(u(t-\tau)) u(t-\tau), \qquad (3.6)
\end{aligned}$$

where $\tilde{\delta}_r(t)$ is defined by (2.24)

By (3.4), we know that $a(r) > 0$ for $r > 0$ sufficiently small. So for fixed $r > 0$ sufficiently small, we can divide $a(r)$ on both sides of equation (3.6) to obtain for a.e. $t \in [0, T]$,

$$\begin{aligned}
&\frac{1}{a(r)} \int_0^t \int_0^r k(t+\rho, s)\, d\eta_r(\rho)[g(u(s)) - g(\bar{u}(s))]\, ds + [g(u(t)) - g(\bar{u}(t))] \\
&= [g(u(t)) - g(\bar{u}(t))] + \frac{1}{a(r)} \tilde{\delta}_r(t) \\
&\quad + \frac{1}{a(r)} \int_0^r \int_0^\rho k(t+\rho, t+s) g(\bar{u}(t+s))\, ds\, d\eta_r(\rho)] - g(\bar{u}(t-\tau)) \\
&\quad - (g(u(t-\tau)) - g(\bar{u}(t-\tau))) - g'(u(t-\tau)) u(t) + g'(u(t-\tau)) u(t-\tau). \qquad (3.7)
\end{aligned}$$

Assume g satisfies $(g3')$ and let $\bar{v}(t) := g(\bar{u}(t))$ for $t \in [0, T+R]$. By $(g3')$ and Inverse Function Theorem, we derive $g^{-1} \in C^1(D)$, where $D := g(I)$. Motivated by (3.7), for fixed $r > 0$ sufficiently small, we will seek a solution v, $v(t) \in D$ a.e.

$t \in [0,T]$ of the following equation:

$$(B_r + I)(v - \bar{v}) = F_r(v), \tag{3.8}$$

where

$$B_r : L^\infty(0,T) \to L^\infty(0,T), \tag{3.9}$$

defined by

$$B_r(v)(t) := \frac{1}{a(r)} \int_0^t \int_0^r k(t+\rho, s)\, d\eta_r(\rho) v(s)\, ds. \tag{3.10}$$

If $\delta \in L^\infty(0, T+R)$, then $F_r : L^\infty((0,T), D) \to L^\infty(0,T)$ is defined by

$$\begin{aligned}
F_r(v)(t) := {} & (v(t) - \bar{v}(t)) + \frac{1}{a(r)}\tilde{\delta}_r(t) \\
& + \frac{1}{a(r)} \int_0^r \int_0^\rho k(t+\rho, t+s)\bar{v}(t+s)\, ds\, d\eta_r(\rho) \\
& - \bar{v}(t-\tau) - (v(t-\tau) - \bar{v}(t-\tau)) \\
& - g'\left(g^{-1}(v(t-\tau))\right) g^{-1}(v(t)) + g'\left(g^{-1}(v(t-\tau))\right) g^{-1}(v(t-\tau)) \\
= {} & \frac{\tilde{\delta}_r(t)}{a(r)} + \frac{\int_0^r \int_0^\rho (k(t+\rho, t+s)\bar{v}(t+s) - \bar{v}(t-\tau))\, ds\, d\eta_r(\rho)}{a(r)} \\
& + \left[v(t) - \bar{v}(t) - g'\left(g^{-1}(v(t-\tau))\right)\left(g^{-1}(v(t)) - g^{-1}(\bar{v}(t))\right)\right] \\
& - \left[v(t-\tau) - \bar{v}(t-\tau) - g'\left(g^{-1}(v(t-\tau))\right)\left(g^{-1}(v(t-\tau)) - g^{-1}(\bar{v}(t-\tau))\right)\right] \\
& - g'\left(g^{-1}(v(t-\tau))\right)\left(g^{-1}(\bar{v}(t)) - g^{-1}(\bar{v}(t-\tau))\right),
\end{aligned} \tag{3.11}$$

for a.e. $t \in (0,T)$.

3.2 Convergence and well-posedness results

Before we present our main results, we will study the properties of the operator $B_r + I$ first.

Lemma 3.2.1. *For any $r > 0$ sufficiently small, let B_r be given by* (3.9)-(3.10). *If $k \in C^1([0,T+R]\times[0,T+R])$, then the operator B_r+I is invertible with $(B_r+I)^{-1} \in \mathscr{L}(L^\infty(0,T), L^\infty(0,T))$ and there exists a constant $\hat{C}$ independent of r, such that $\|(B_r+I)^{-1}\|_{\mathscr{L}(L^\infty(0,T),L^\infty(0,T))} \leq \hat{C}$ for all $r > 0$ sufficiently small.*

Proof. For any $r > 0$ sufficiently small, by Taylor expansion

$$\begin{aligned}
&\frac{\int_0^r k(t+\rho,s)\,d\eta_r(\rho)}{a(r)}\\
&= \frac{\int_0^r [k(t,s) + D_1 k(\xi(t,\rho),s)\rho]\,d\eta_r(\rho)}{a(r)}\\
&= \frac{\int_0^r d\eta_r(\rho)}{a(r)}\cdot k(t,s) + \frac{\int_0^r D_1 k(\xi(t,\rho),s)\rho\,d\eta_r(\rho)}{a(r)}\\
&= \frac{k(t,s)}{\epsilon_r} + \bar{K}_r(t,s),
\end{aligned}$$

where

$$\epsilon_r := \frac{a(r)}{\int_0^r d\eta_r(\rho)} = \frac{\int_0^r \rho\,d\eta_r(\rho)}{\int_0^r d\eta_r(\rho)}$$

and

$$\bar{K}_r(t,s) := \frac{\int_0^r D_1 k(\xi(t,\rho),s)\rho\,d\eta_r(\rho)}{a(r)}.$$

Consider the equation

$$(B_r + I)(w)(t) = \bar{f}(t), \quad \text{a.e.} \quad t \in [0,T]. \tag{3.12}$$

This is a second kind integral equation in w. If $\bar{f} \in L^\infty(0,T)$, then there exists a unique $w \in L^\infty(0,T)$ which solves equation (3.12), i.e. $(B_r + I)^{-1} : L^\infty(0,T) \to L^\infty(0,T)$ [13]. This is true for any $r > 0$ sufficiently small. By definition

$$\begin{aligned}|\bar{K}_r(t,s)| &= \frac{\left|\int_0^r D_1 k(\xi(t,\rho),s)\rho\, d\eta_r(\rho)\right|}{\left|\int_0^r \rho\, d\eta_r(\rho)\right|} \\ &\le \frac{\tilde{C}\|D_1 k\|_1 r^{1+\sigma}}{r^{1+\sigma}(c_1 + O(r))} \le \frac{2\tilde{C}\|D_1 k\|_1}{c_1}\end{aligned}$$

for $r > 0$ sufficiently small and $t \in [0, T+R]$. Therefore,

$$\|\bar{K}_r\|_\infty \le \frac{2\tilde{C}\|D_1 k\|_\infty}{c_1}.$$

From the proof of Lemma 4.1 of [9] we have

$$\begin{aligned}\|w\| &\le 2\|\bar{f}\| \exp\left(\|D_1\bar{k}\|_\infty + 2\frac{2\tilde{C}\|D_1 k\|_\infty}{c_1}\right) \\ &= \hat{C}\|\bar{f}\|,\end{aligned}$$

where $\hat{C} := 2\exp\left(\|D_1\bar{k}\|_\infty + 2\frac{2\tilde{C}\|D_1 k\|_\infty}{c_1}\right)$ independent of r. Since

$$\|w\| = \|(B_r + I)^{-1}\bar{f}\| \le \hat{C}\|\bar{f}\|,$$

we obtain

$$\|(B_r + I)^{-1}\|_{\mathscr{L}(L^\infty(0,T), L^\infty(0,T))} \le \hat{C}.$$

□

If $\delta \in L^\infty(0, T+R)$, then by Lemma 3.2.1, for $r > 0$ sufficiently small, equation

(3.8) is equivalent to

$$(v - \bar{v}) = (B_r + I)^{-1} F_r(v), \tag{3.13}$$

or

$$v(t) = H_r(v)(t), \quad \text{a.e. } t \in [0, T], \tag{3.14}$$

where

$$H_r : L^\infty((0,T), D) \to L^\infty(0,T),$$

is defined by

$$H_r(v) := (B_r + I)^{-1} F_r(v) + \bar{v}. \tag{3.15}$$

Now we present our main results.

Theorem 3.2.1. *Let $\bar{u}$ denote the solution of* (3.2) *given "true" data $f \in C[0, T+R]$ and let the same assumptions hold as in Theorem 2.2.1 for $\bar{u}$ and signed Borel measure $\{\eta_r\}$. Let g satisfy the same assumptions as in Theorem 2.3.1. Assume k is 1-smoothing, i.e. $k \in C^1([0,T+R] \times [0,T+R])$ and $k(t,t) = 1$. Let $R > 0$ be sufficiently small and let $r \in (0, R]$ be arbitrary. Then there exists a θ independent of r such that, if $f^\delta \in C[0, T+R]$ satisfies* (2.24) *with $\hat{\delta} \leq k_1 r^{\mu+1}$, for μ the Hölder exponent on $\bar{u}$, then there is a unique solution v of* (3.8) *satisfying $\|v - \bar{v}\| \leq \theta r^\mu$. Further, the mapping $f^\delta \in \{w \in C[0,T+R], \|w - f\|_\infty \leq \hat{\delta}\} \mapsto v \in L^\infty((0,T), D)$ is continuous for all $r > 0$ sufficiently small.*

Proof. We will use the same type of arguments like in Theorem 2.3.1. That is, we will first define a ball

$$M := \left\{ v \in L^\infty(0,T) : \|v - \bar{v}\| \leq \theta r^\mu \right\},$$

for some θ independent of r and $\mu \in (0,1]$ defined by (2.4), then use the Contraction Mapping Theorem to prove our result.

Since $\bar{u}(t) \in I$, so $\bar{v}(t) \in D$. By previous discussion in Chapter 2, we know D is an open interval and for any $v \in M$, we have $v(t) \in D$ for a.e. $t \in [0, T]$ when $r > 0$ is sufficiently small. We will show that there exists a unique solution v solving the equation

$$v = H_r(v),$$

so that such a v will uniquely solve equation (3.8). First we will show that H_r maps M into M.

For $v \in M$, by Lemma 3.2.1,

$$\begin{aligned}
&\|H_r(v) - \bar{v}\| = \|(B_r + I)^{-1} F_r(v)\| \\
&\leq \|(B_r + I)^{-1}\|_{\mathscr{L}(L^\infty(0,T), L^\infty(0,T))} \|F_r(v)\| \\
&\leq \hat{C} \|F_r(v)\|.
\end{aligned}$$

For $r > 0$ sufficiently small, by equation (3.11), we have for a.e. $t \in [0, T]$,

$$|F_r(v)(t)| \leq \sum_{i=1}^{5} P_r^{(i)}(t),$$

where

$$\begin{aligned}
P_r^{(1)}(t) &:= \frac{\left|\tilde{\delta}_r(t)\right|}{a(r)}, \\
P_r^{(2)}(t) &:= \frac{\left|\int_0^r \int_0^\rho \left(k(t+\rho, t+s)\bar{v}(t+s) - \bar{v}(t-\tau)\right) ds\, d\eta_r(\rho)\right|}{a(r)}, \\
P_r^{(3)}(t) &:= \left|v(t) - \bar{v}(t) - g'\left(g^{-1}(v(t-\tau))\right)\left(g^{-1}(v(t)) - g^{-1}(\bar{v}(t))\right)\right|, \\
P_r^{(4)}(t) &:= \left|v(t-\tau) - \bar{v}(t-\tau) - g'\left(g^{-1}(v(t-\tau))\right)\left(g^{-1}(v(t-\tau)) - g^{-1}(\bar{v}(t-\tau))\right)\right|, \\
P_r^{(5)}(t) &:= \left|g'\left(g^{-1}(v(t-\tau))\right)\left(g^{-1}(\bar{v}(t)) - g^{-1}(\bar{v}(t-\tau))\right)\right|,
\end{aligned}$$

and where we have used the fact that $a(r) > 0$ for $r > 0$ sufficiently small.

By (H_3), we have

$$P_r^{(1)}(t) \leq \frac{\left|\int_0^r \delta(t+\rho)\, d\eta_r(\rho)\right|}{\frac{1}{2}c_1 r^{1+\sigma}} \leq \frac{\tilde{C}\hat{\delta} r^{\sigma}}{\frac{1}{2}c_1 r^{1+\sigma}} = \frac{2\tilde{C}\hat{\delta}}{c_1 r}$$

for $r > 0$ sufficiently small. If $\hat{\delta} \leq k_1 r^{\mu+1}$, then

$$P_r^{(1)}(t) \leq M_1 r^{\mu} \quad \text{where} \quad M_1 := \frac{2\tilde{C}k_1}{c_1}. \tag{3.16}$$

Now consider the integrand of $P_r^{(2)}(t)$. We have for $t \in [0,T]$, $s, \rho \in [0,r]$,

$$\begin{aligned} &|k(t+\rho, t+s)\bar{v}(t+s) - \bar{v}(t-\tau)| \\ &\leq |k(t+\rho, t+s)\bar{v}(t+s) - k(t+\rho, t+s)\bar{v}(t-\tau)| \\ &\quad + |k(t+\rho, t+s)\bar{v}(t-\tau) - k(t+\rho, t)\bar{v}(t-\tau)| \\ &\quad + |k(t+\rho, t)\bar{v}(t-\tau) - k(t,t)\bar{v}(t-\tau)| \\ &\leq \|k\|_\infty \|g'\|_I \bar{N} 2^{\mu} r^{\mu} + \|D_2 k\|_\infty r \|g\|_I \\ &\quad + \|D_1 k\|_\infty r \|g\|_I = \bar{M}_2 r^{\mu} + \bar{M}_3 r, \end{aligned} \tag{3.17}$$

where $\bar{M}_2 := 2^{\mu}\|k\|_\infty \|g'\|_I \bar{N}$, $\quad \bar{M}_3 := \|D_2 k\|_\infty \|g\|_I + \|D_1 k\|_\infty \|g\|_I$, and where we have used Lemma 2.3.2. Therefore by (H_3), for $r > 0$ sufficiently small,

$$P_r^{(2)}(t) \leq \frac{\tilde{C}(\bar{M}_2 r^{\mu} + \bar{M}_3 r) r \cdot r^{\sigma}}{\frac{1}{2}c_1 r^{1+\sigma}} = \frac{2\tilde{C}(\bar{M}_2 r^{\mu} + \bar{M}_3 r)}{c_1} = M_2 r^{\mu} + M_3 r, \tag{3.18}$$

where $M_2 := \dfrac{2\tilde{C}\bar{M}_2}{c_1}$, $M_3 := \dfrac{2\tilde{C}\bar{M}_3}{c_1}$.

Notice that $P_r^{(3)}(t)$, $P_r^{(4)}(t)$ and $P_r^{(5)}(t)$ are exactly the same as $T_r^{(3)}(t)$, $T_r^{(4)}(t)$

and $T_r^{(5)}(t)$ respectively defined by (2.29) – (2.31), under the same assumptions on g and $\bar{u}$, except for a factor of $a(r)$, i.e. $P_r^{(i)}(t) = \frac{1}{a(r)} T_r^{(i)}(t)$ for $i = 3, 4, 5$. Therefore by similar arguments, we derive for $r > 0$ sufficiently small and a.e. $t \in [0, T]$,

$$P_r^{(3)}(t) \leq \frac{\theta}{\bar{c}_1^2} N(2\theta + \|g'\|_I \bar{N}) r^{2\mu}, \tag{3.19}$$

$$P_r^{(4)}(t) \leq \frac{\theta^2 N}{\bar{c}_1^2} r^{2\mu}, \tag{3.20}$$

$$P_r^{(5)}(t) \leq \|g'\|_I \bar{N} r^{\mu} = M_4 r^{\mu}, \tag{3.21}$$

and where $M_4 := \|g'\|_I \bar{N}$. Therefore by (3.16)-(3.21), we have

$$\|H_r(v) - \bar{v}\| \leq \hat{C}[(M_1 + M_2 + M_4) r^{\mu} + M_3 r + o(r^{\mu})].$$

For $r > 0$ sufficiently small, to have $\|H_r(v) - \bar{v}\| \leq \theta r^{\mu}$ for some $\theta > 0$, a sufficient condition is

$$\hat{C}(M_1 + M_2 + M_3 + M_4) < \frac{\theta}{2}.$$

So let

$$\theta := 2\hat{C}(M_1 + M_2 + M_3 + M_4),$$

then we have $\|H_r(v) - \bar{v}\| \leq \theta r^{\mu}$ for $r > 0$ sufficiently small. Therefore $H_r : M \to M$.

Now we want to show for any $v_1, v_2 \in M = \{v \in L^{\infty}(0, T) : \|v - \bar{v}\| \leq \theta r^{\mu}\}$, we have $\|H_r(v_1) - H_r(v_2)\| \leq \alpha \|v_1 - v_2\|$ for $0 \leq \alpha < 1$ and $r > 0$ sufficiently small. Since $\|H_r(v_1) - H_r(v_2)\| \leq \hat{C}\|F_r(v_1) - F_r(v_2)\|$, using similar computations as in

Chapter 2 to derive (2.36), we have

$$F_r(v_1)(t) - F_r(v_2)(t) = \sum_{i=1}^{3} S_r^{(i)}(t), \tag{3.22}$$

where $S_r^{(i)}$ are defined by (2.37), (2.38) and (2.39) for $i = 1, 2, 3$ respectively. Therefore

$$\begin{aligned}\|F_r(v_1) - F_r(v_2)\| &\leq \frac{\|v_1 - v_2\|}{\bar{c}_1^2} N(3\theta r^{\mu} + \|g'\|_I \bar{N} r^{\mu}) \\ &\quad + \|v_1 - v_2\| \frac{N}{\bar{c}_1^2} 2\theta r^{\mu} + \frac{1}{\bar{c}_1^2}(2\theta + \|g'\|_I \bar{N}) N r^{\mu} \|v_1 - v_2\| \\ &= \|v_1 - v_2\| \frac{N r^{\mu}}{\bar{c}_1^2}(7\theta + 2\|g'\|_I \bar{N}) \\ &= \beta(r)\|v_1 - v_2\|,\end{aligned}$$

where $\beta(r) := \frac{N r^{\mu}}{\bar{c}_1^2}(7\theta + 2\|g'\|_I \bar{N})$.

So we have

$$\begin{aligned}\|H_r(v_1) - H_r(v_2))\| &\leq \hat{C}\|F_r(v_1) - F_r(v_2)\| \\ &= \bar{\alpha}(r)\|v_1 - v_2\|,\end{aligned}$$

where $\bar{\alpha}(r) := \hat{C}\beta(r)$.

Hence for $v_1, v_2 \in M$, we have $\|H_r(v_1) - H_r(v_2)\| \leq \alpha(r)\|v_1 - v_2\|$, and $\alpha(r) \in [0, 1)$ provided $r > 0$ is sufficiently small. Thus equation (3.8) has a unique solution $v_r^{\delta} \in L^{\infty}((0, T), D)$ in ball M for $r > 0$ sufficiently small.

For the proof of continuous dependence on the data, by the same type of arguments that we have used in the proof of Theorem 2.3.1. Let $v^{\delta}_{r,i}$ denote the solution of (3.8) associated with data f^{δ}_i and $\|f^{\delta}_i - f\|_{\infty} \leq \hat{\delta}$, $i = 1, 2$. For fixed $r > 0$ sufficiently small, we obtain

$$\begin{aligned}
&\|v^{\delta}_{r,1} - v^{\delta}_{r,2}\| = \|H_{r,1}(v^{\delta}_{r,1}) - H_{r,2}(v^{\delta}_{r,2})\| \\
&\leq \hat{C}\|F_{r,1}(v^{\delta}_{r,1}) - F_{r,2}(v^{\delta}_{r,2})\| \\
&\leq \bar{\alpha}(r)\|v^{\delta}_{r,1} - v^{\delta}_{r,2}\| + \hat{C}\left|\frac{\int_0^r \left(f^{\delta}_1(t+\rho) - f^{\delta}_2(t+\rho)\right) d\eta_r(\rho)}{a(r)}\right| \\
&\leq \bar{\alpha}(r)\|v^{\delta}_{r,1} - v^{\delta}_{r,2}\| + \frac{2\hat{C}\tilde{C}\|f^{\delta}_1 - f^{\delta}_2\|_{\infty}}{c_1 r}.
\end{aligned}$$

So

$$\|v^{\delta}_{r,1} - v^{\delta}_{r,2}\| \leq \frac{1}{1-\bar{\alpha}(r)} \cdot \frac{2\hat{C}\tilde{C}}{c_1 r} \cdot \|f^{\delta}_1 - f^{\delta}_2\|_{\infty},$$

where $\bar{\alpha}(r) \in (0,1)$ for this fixed r. The above arguments are true for any $r > 0$ sufficiently small. Therefore, continuous dependence of solutions on data is obtained for equation (3.8) for $r > 0$ sufficiently small. □

Corollary 3.2.1. *Assume all the assumptions hold as in Theorem 3.2.1. Then for* $r_k = r_k(\hat{\delta}_k) > 0$ *selected satisfying* $d_1\hat{\delta}_k^{\frac{1}{\nu+\mu}} \leq r_k \leq d_2\hat{\delta}_k^{\frac{1}{\nu+\mu}}$ *for some constants* d_1, $d_2 > 0$ *and for* $\hat{\delta}_k \to 0$ *as* $k \to \infty$, *equation* (3.3) *has a unique solution* $u^{\delta_k}_{r_k} = u^{\delta_k}_{r_k(\hat{\delta}_k)} \in L^{\infty}((0,T), I)$ *satisfying*

$$\|u^{\delta_k}_{r_k} - \bar{u}\| \leq \bar{c}\hat{\delta}_k^{\frac{\mu}{\mu+\nu}}$$

as $k \to 0$ *for some constant* $\bar{c}$ *independent of* k *and* $\hat{\delta}_k$. *Further, the mapping*

$$f^{\delta_k} \in \{w_k \in C[0, T+R], \|w_k - f\|_{\infty} \leq \hat{\delta}_k\} \mapsto u^{\delta_k}_{r_k} \in L^{\infty}((0,T), I)$$

is continuous for all k sufficiently large.

The proof of the above corollary is similar to the proof for Corollary 2.3.1.

CHAPTER 4

Discretization and Numerical Implementation

4.1 ν-Smoothing Convolution Kernel

We first consider the implementation for our regularized equation with ν-smoothing convolution kernel. Recall the regularized equation is

$$\int_0^t \tilde{k}_r(t-s)g(u(s))\,ds + a(r)g'(u(t-\tau))u(t) + a(r)g(u(t-\tau)) \\ - a(r)g'(u(t-\tau))u(t-\tau) = \tilde{f}_r^{\delta}(t), \tag{4.1}$$

which came from a linearization of certain terms in the equation

$$\int_0^t \tilde{k}_r(t-s)g(u(s))\,ds + a(r)g(u(t)) = \tilde{f}_r^{\delta}(t), \tag{4.2}$$

where $\tilde{k}$, $\tilde{f}_r^\delta$ and $a(r)$ are defined by (2.8), (2.9) and (3.4). Let $N = 1, 2, 3, \ldots$, and partition our interval $[0, T]$ into N equally spaced subintervals. That is, we let

$$\Delta t = \frac{T}{N},$$

$$t_i = i\Delta t, \quad i = 0, 1, \ldots, N.$$

Let $\chi_j(t)$ be the usual characteristic functions on the interval $[t_{j-1}, t_j)$ for $j = 1, 2, \ldots, N$. We seek constants c_j, $j = 1, 2, \ldots, N$, so that the step function

$$u(t) := \sum_{j=1}^{N} c_j \chi_j(t), \quad t \in [0, T], \tag{4.3}$$

satisfies (4.2) at the collocation point t_1 (since $u(t)$ in (4.3) has no "past" information on the interval $[0, t_1)$, and satisfies (4.1) at $t = t_i$ for $i = 1, 2, \ldots, N-1$. Let $r := R\Delta t$, where R is the number of future subintervals that we will use. Note that it is not practical if the number of future intervals is more than the number of subintervals on $[0, T]$. Therefore we take $R \in \{1, 2, \ldots, N\}$. By our requirements on τ from Chapter 2, we will let

$$\tau = \Delta t, \quad \text{for solving } c_i, \text{ for } i = 2, 3, \ldots, N. \tag{4.4}$$

In order to solve for c_1, we solve the equation (4.2) at t as $t \to t_1^-$, that is

$$\lim_{t \to t_1^-} \left[\int_0^t \tilde{k}_r(t-s) g\left(\sum_{j=1}^{N} c_j \chi_j(s) \right) ds + a(r) g\left(\sum_{j=1}^{N} c_j \chi_j(t) \right) \right] = \lim_{t \to t_1^-} \tilde{f}_r^\delta(t),$$

or

$$g(c_1) \int_0^{t_1} \tilde{k}_r(t_1 - s)\, ds + a(r) g(c_1) = \tilde{f}_r^\delta(t_1),$$

since if $t \in [0, t_1)$, then $\chi_j(t) = 1$, for $j = 1$ and $\chi_j(t) = 0$ for $j = 2, 3, \ldots, N$. Or we can write the above equation as

$$\Delta_1 g(c_1) + a(r) g(c_1) = \tilde{f}_r^\delta(t_1), \tag{4.5}$$

where

$$\Delta_j := \int_0^{t_1} \tilde{k}_r(t_j - s)\, ds = \int_0^{t_1} \int_0^r k(t_j + \rho - s)\, d\eta_r(\rho)\, ds,$$

for $j = 1, \ldots, N-1$. So

$$g(c_1) = \frac{\tilde{f}_r^\delta(t_1)}{a(r) + \Delta_1}. \tag{4.6}$$

To find $c_2, c_3, \ldots, c_N$, we solve the following collocation equation:

$$\begin{aligned}
&\int_0^{t_i} \tilde{k}_r(t_i - s) g\left(\sum_{j=1}^N c_j \chi_j(s)\right) ds + a(r) g'\left(\sum_{j=1}^N c_j \chi_j(t_i - \tau)\right) \sum_{j=1}^N c_j \chi_j(t_i) \\
&\quad + a(r) g\left(\sum_{j=1}^N c_j \chi_j(t_i - \tau)\right) - a(r) g'\left(\sum_{j=1}^N c_j \chi_j(t_i - \tau)\right) \sum_{j=1}^N c_j \chi_j(t_i - \tau) \\
&= \tilde{f}_r^\delta(t_i), \quad \text{for } i = 1, 2, \ldots, N-1,
\end{aligned} \tag{4.7}$$

where $t_i - \tau = t_{i-1}$ for $i = 1, \ldots, N-1$, therefore

$$\chi_j(t_{i-1}) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi_j(t_i) = \begin{cases} 1, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

So the first term on the left hand side of (4.7) is

$$\int_0^{t_i} \tilde{k}_r(t_i - s) g\left(\sum_{j=1}^{N} c_j \chi_j(s)\right) ds = \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} \tilde{k}_r(t_i - s) g(c_k)\, ds$$

$$= \sum_{k=1}^{i} g(c_k) \int_0^{t_1} \tilde{k}_r(t_i - (s + t_{k-1}))\, ds,$$

$$= \sum_{k=1}^{i} g(c_k) \Delta_{i-k+1} \quad i = 1, \ldots, N-1.$$

The second term on the left hand side of (4.7) is

$$a(r) g'\left(\sum_{j=1}^{N} c_j \chi_j(t_i - \tau)\right) \sum_{j=1}^{N} c_j \chi_j(t_i)$$

$$= a(r) g'\left(\sum_{j=1}^{N} c_j \chi_j(t_{i-1})\right) \sum_{j=1}^{N} c_j \chi_j(t_i)$$

$$= a(r) g'(c_i) c_{i+1}, \quad i = 1, \cdots, N-1.$$

The third term on the left hand side of (4.7) is

$$a(r) g\left(\sum_{j=1}^{N} c_j \chi_j(t_i - \tau)\right) = a(r) g\left(\sum_{j=1}^{N} c_j \chi_j(t_{i-1})\right) = a(r) g(c_i),$$

for $i = 1, \cdots, N-1$.

And the last term on the left hand side of (4.7) is

$$a(r) g'\left(\sum_{j=1}^{N} c_j \chi_j(t_i - \tau)\right) \sum_{j=1}^{N} c_j \chi_j(t_i - \tau)$$

$$= a(r) g'\left(\sum_{j=1}^{N} c_j \chi_j(t_{i-1})\right) \sum_{j=1}^{N} c_j \chi_j(t_{i-1}) = a(r) g'(c_i) c_i, \quad i = 1, \cdots, N-1.$$

Therefore our collocation equation (4.7) can be written as

$$\sum_{k=1}^{i} g(c_k)\Delta_{i-k+1} + a(r)g'(c_i)c_{i+1} + a(r)g(c_i) - a(r)g'(c_i)c_i = \tilde{f}_r^{\delta}(t_i),$$

for $i = 1, \cdots, N-1$.

We note that only equation (4.6) is a nonlinear equation, which we must solve for c_1. Once $c_1, c_2, \ldots, c_i$ are found, we can solve for c_{i+1} by the following *linear* equation:

$$c_{i+1} = \frac{1}{a(r)g'(c_i)} \left\{ \tilde{f}_r^{\delta}(t_i) - a(r)g(c_i) + a(r)g'(c_i)c_i - \sum_{k=1}^{i} g(c_k)\Delta_{i-k+1} \right\} \tag{4.8}$$

for $i = 1, 2, \ldots, N-1$.

The computation of $a(r)$, $\tilde{f}_r^{\delta}(t_i)$, and Δ_i rely on the choice of the measure η. We will show their computation in Section 4.3 for one particular measure.

4.2 1-Smoothing Nonconvolution Kernel

In this section, we consider the discretized equation for 1-smoothing nonconvolution kernel (not necessarily satisfying $k(t,t) = 1$). Using the same method as in the above section for deriving equation (4.6) and (4.8) , we derive the following formulas to compute c_1 and c_j for $j = 2, \ldots, N$ respectively,

$$g(c1) = \frac{\tilde{f}_r^{\delta}(t_1)}{a(r)k(t_1,t_1) + \Delta_{1,0}},$$

where

$$a(r) = \int_0^r \rho \, d\eta_r(\rho)$$

and

$$\Delta_{i,j} := \int_0^{t_1} \int_0^{r} k(t_i + \rho, s + t_j)\, d\eta_r(\rho)\, ds,$$

for $i = 1, \ldots, N-1$; $j = 0, \ldots, N-2$. And

$$c_{i+1} = \frac{\tilde{f}_r^{\delta}(t_i) - a(r)k(t_i, t_i)g(c_i) + a(r)k(t_i, t_i)g'(c_i)c_i - \sum_{k=1}^{i} g(c_k)\Delta_{i,k-1}}{a(r)k(t_i, t_i)g'(c_i)}$$

for $i = 1, 2, \ldots, N-1$,

4.3 Numerical Results

For all numerical examples in this thesis, we used Matlab to evaluate the collocation-based discretization over the space of piecewise constant functions (defined on a uniform grid of N+1 points starting from 0 and ending at T). For simplicity, we used Lebesgue measure, i.e.

$$\int_0^{r} m(\rho)\, d\eta_r(\rho) := \int_0^{r} m(\rho)\, d\rho,$$

for $m \in C[0, r]$. For the ν-smoothing convolution examples, we defined the kernel to be

$$k(t) = \frac{t^{\nu-1}}{(\nu-1)!},$$

while for the 1-smoothing nonconvolution case, we used as an example

$$k(t, s) = ts + 1.$$

Remark 4.3.1. *Notice that Lebesgue measure is a positive Borel measure that satisfies assumptions* (H_1)–(H_3) *for* $\nu \leq 3$ *[6]. It fails assumption* (H_2) *for* $\nu = 4$ *[6] and it has been shown in [3] that there exists no family of positive Borel measures that satisfies*

(H_2) for the case $\nu \geq 5$. Therefore, by this choice of measure, we will only consider ν-smoothing problems for $\nu \leq 3$. However, it is worth noting that it does not mean that our method only applies to ν-smoothing problems for $\nu \leq 3$. In order to use our method for higher ν-smoothing problems, an appropriate signed Borel measure needs to be picked, for example, see Lemma 2.1.1 and Lemma 2.1.2 for the construction of such measures. Also, we want to point out that the above choices of kernel k are only for the purpose of simplicity. There are infinite number of choices for the kernel k available.

Using Lebesgue measure, we have

$$a(r) = \int_0^r \int_0^\rho \frac{(\rho - s)^{\nu - 1}}{(\nu - 1)!} \, ds \, d\rho = \frac{r^{\nu + 1}}{(\nu + 1)!},$$

for $\nu = 1,\ 2,\ 3$ for the convolution examples. And

$$a(r) = \int_0^r \rho \, d\rho = \frac{r^2}{2},$$

for the nonconvolution examples.

Further, we used the approximation

$$\int_0^r m(\rho) \, d\rho \approx \Delta t \sum_{j=1}^{R} m(t_{j-1}),$$

so

$$\tilde{f}_r^\delta(t_i) = \int_0^r f^\delta(t_i + \rho) \, d\rho = \Delta t \sum_{j=1}^{R} f^\delta(t_{i+j-1}).$$

For the convolution examples,

$$\Delta_j = \frac{1}{(\nu + 1)!} \left[(t_j + r)^{\nu + 1} - t_j^{\nu + 1} - (t_{j-1} + r)^{\nu + 1} + t_{j-1}^{\nu + 1} \right],$$

for $j = 1, 2, \ldots, N-1$, and for the nonconvolution examples,

$$\Delta_{i,j} = (t_i r + \frac{r^2}{2})\frac{t_1^2}{2} + \left[(t_i t_j + 1)r + t_j \frac{r^2}{2}\right] t_1,$$

for $i = 1, \ldots, N-1$; $j = 0, \ldots, N-2$.

We selected our true solution $\bar{u}$ ahead of time, then generated the data function f by $f(t) = \int_0^t k(t,s) g(\bar{u}(s))\, ds$ for $t \in [0, T]$. We then added random uniform noise to f at discrete values of $f(t)$ for $t = t_i$, where $i = 1, 2, \ldots, N$, to generate noisy data f^δ. We let δ to represent the relative error. In each of the examples, we show the recovered solution with regularization using our method against the true solution $\bar{u}$. In all pictures below, the dashed line represents the true solution $\bar{u}$ and the solid line expresses the approximate solution computed according to our method. Example 1– Example 6 are for convolution kernels, while Example 7– Example 8 are for nonconvolution kernels.

Example 1. In this example, we consider a 1-smoothing kernel $k(t) = 1$ with true solution

$$\bar{u}(t) = \begin{cases} 1 + cos2t & cos2t > 0, \\ 0 & cos2t = 0, \\ -1 + cos2t & cos2t < 0, \end{cases}$$

for $t \in [0, 10]$. And we choose our nonlinear function g to be $g(u) = u + u^3$. Below are three pictures corresponding to three relative errors. See Figures 4.1–4.3. See [18] for a comparison of local regularization to Lavrentiev regularization on the same example.

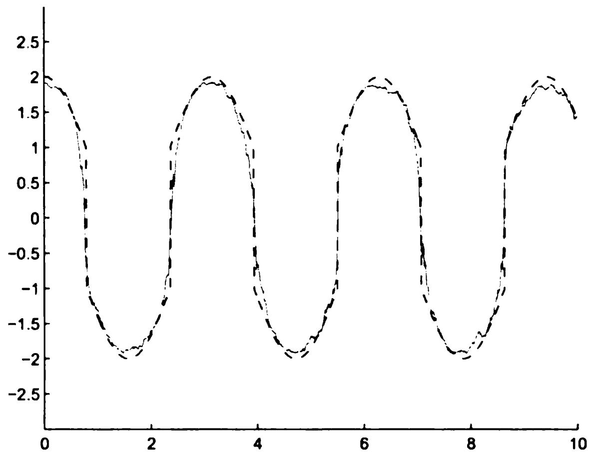

Figure 4.1. Example 1 (a 1-smoothing kernel): solution with regularization, $\delta = 10\%$, $N = 1000$, $R = 45$.

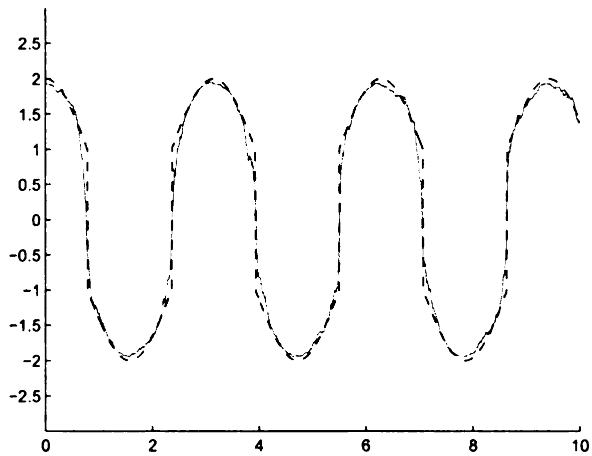

Figure 4.2. Example 1, continued: solution with regularization, $\delta = 5\%$, $N = 1000$, $R = 35$.

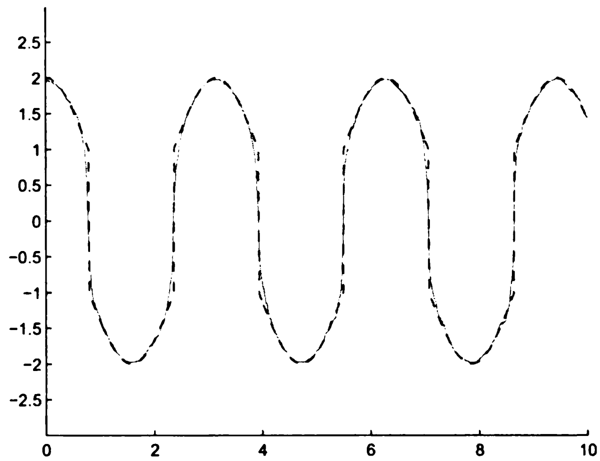

Figure 4.3. Example 1, continued: solution with regularization, $\delta = 1\%$, $N = 1000$, $R = 20$.

Example 2. In this example, we consider a 3-smoothing kernel $k(t) = 0.5t^2$, with the true solution $\bar{u} = 8(t - .4)^2 + 1$, $t \in [0, 1]$, and $g(u) = u^3$. Compare to the same example handled by solving a nonlinear equation for every i, $i = 1, \ldots, N$, in [11]. See Figures 4.4–4.7.

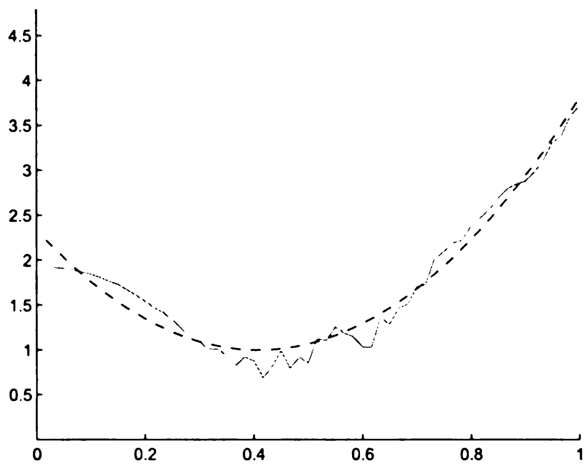

Figure 4.4. Example 2 (a 3-smoothing kernel): solution with regularization, $\delta = 5\%$, $N = 60$, $R = 20$.

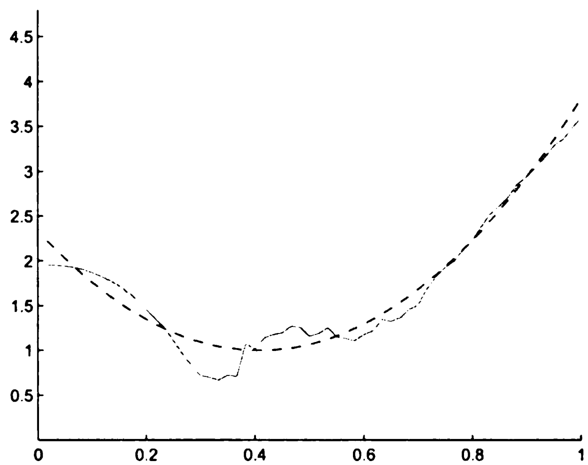

Figure 4.5. Example 2, continued, solution with regularization, $\delta = 1\%$, $N = 60$, $R = 11$.

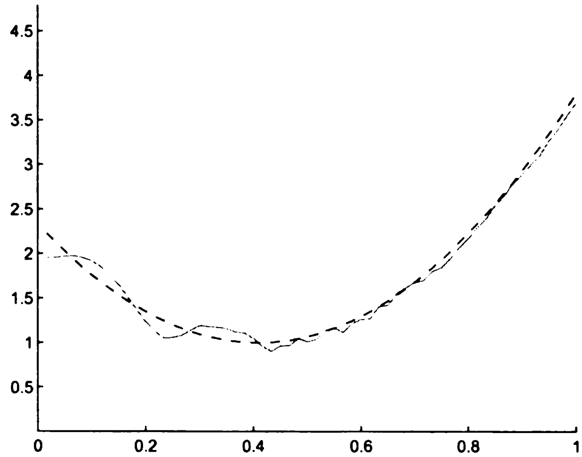

Figure 4.6. Example 2, continued, solution with regularization, $\delta = 0.1\%$, $N = 60$, $R = 7$.

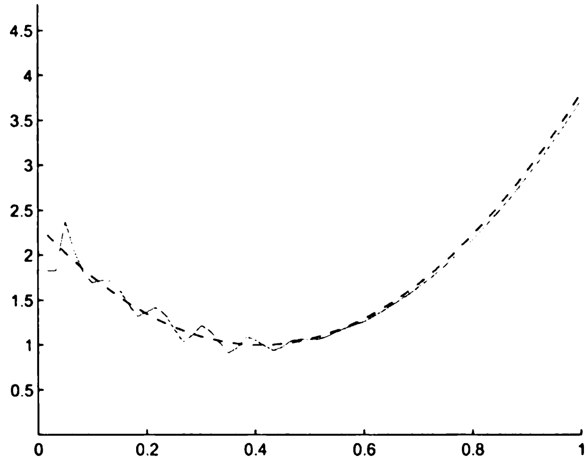

Figure 4.7. Example 2, continued, solution with regularization, $\delta = 0\%$, $N = 60$, $R = 3$.

Example 3. In this example, we still consider the kernel $k(t) = 0.5t^2$, and function $g(u) = u^3$ with discontinuous true solution.

$$\bar{u}(t) = \begin{cases} \dfrac{0.9}{(0.3)^{1/2}} t^{\frac{1}{2}} & \text{if} \quad 0 \leq t < 0.3, \\ t + 1.2 & \text{if} \quad 0.3 \leq t < 0.6, \\ -\dfrac{15}{4} t + \dfrac{81}{20} & \text{if} \quad 0.6 \leq t \leq 1. \end{cases}$$

See Figures 4.8–4.10.

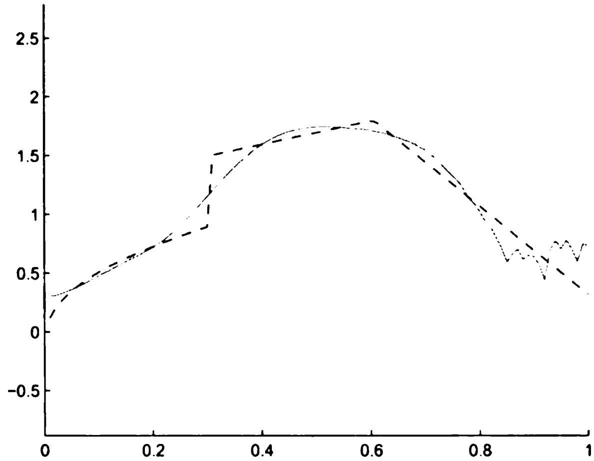

Figure 4.8. Example 3, (a 3-smoothing kernel): solution with regularization, $\delta = 1\%$, $N = 100$, $R = 16$.

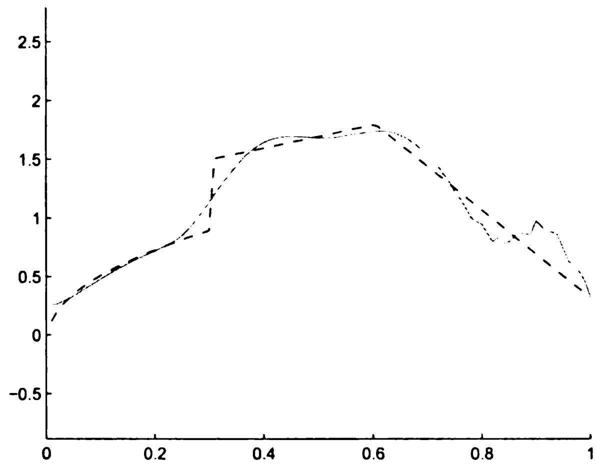

Figure 4.9. Example 3, continued, solution with regularization, $\delta = 0.3\%$, $N = 100$, $R = 12$.

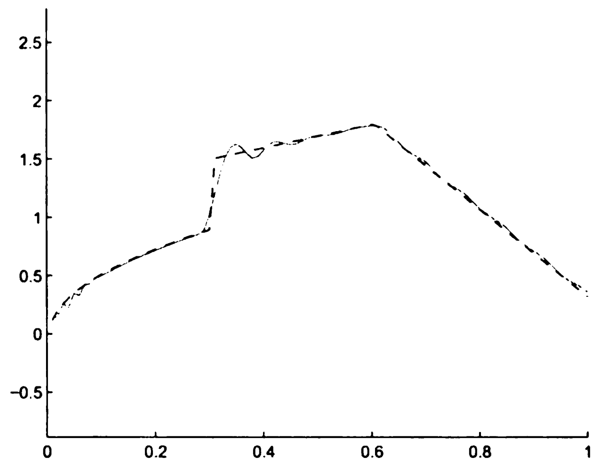

Figure 4.10. Example 3, continued, solution with regularization, $\delta = 0\%$, $N = 100$, $R = 4$.

Example 4. We consider a 1-smoothing kernel $k(t) = 1$ and the same nonlinear function $g(u) = u^3$ as in the above example. The true solution is a periodic function $\bar{u}(t) = sin(2t) + 2$, for $t \in [0, 10]$.

Remark 4.3.2. *Notice that this true solution $\bar{u}$ is similar to the true solution $\bar{u}$ as in Example 1 with the same kernel $k(t) = 1$ for $t \in [0, 10]$. However, this true solution is hard to recover and it is due to the fact that the nonlinear function g in Example 1 guarantees that $|g'(u(t))| \geq 1 > 0$ no matter what $u(t)$ is for $t \in [0, 10]$, while in Example 4 if u gets close to the t-axis due to measurement error, then g' gets close to 0. Therefore, in Example 4, really small errors are needed in order to keep g' bounded away from 0.*

See Figures 4.11–4.13.

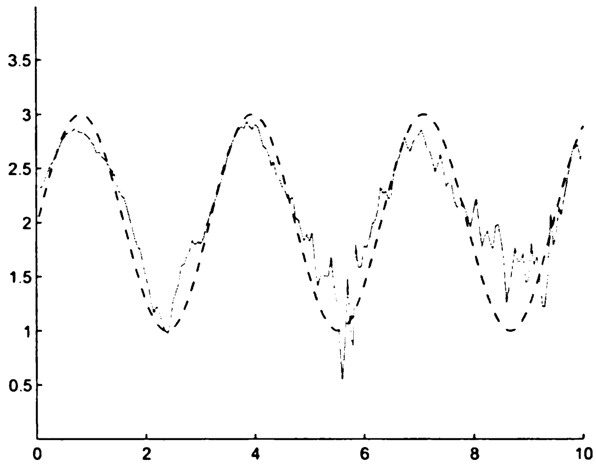

Figure 4.11. Example 4 (a 1-smoothing kernel): solution with regularization, δ = 0.05%, $N = 200$, $R = 11$.

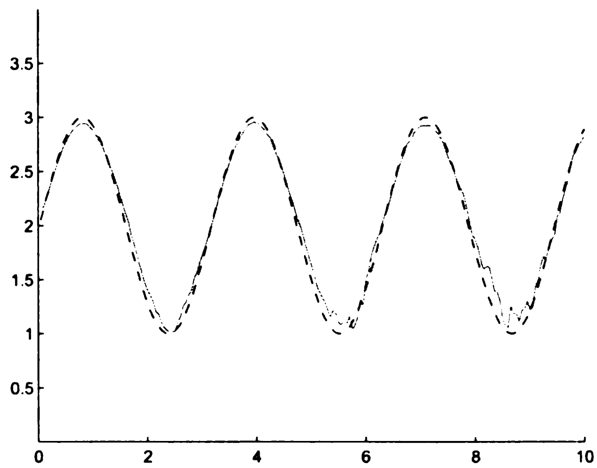

Figure 4.12. Example 4, continued, solution with regularization, δ = 0.005%, N = 200, $R = 7$.

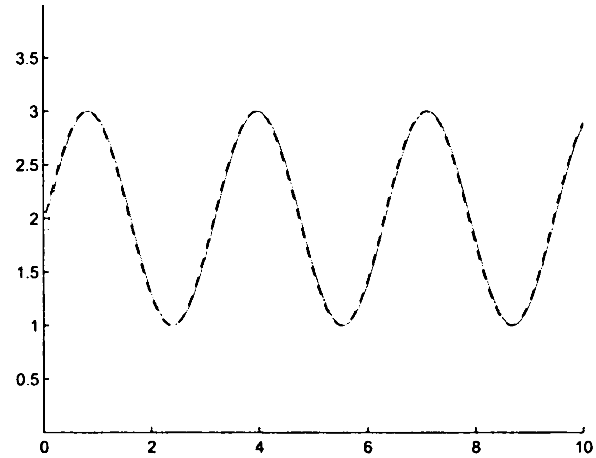

Figure 4.13. Example 4, continued, solution with regularization, $\delta = 0\%$, $N = 200$, $R = 2$.

Example 5. We consider a 2-smoothing kernel $k(t) = t$ and the same nonlinear function $g(u) = u^3$ as in the above example. The true solution is $\bar{u}(t) = sin(2t) + 2$, for $t \in [0, 10]$. See Figures 4.14–4.16.

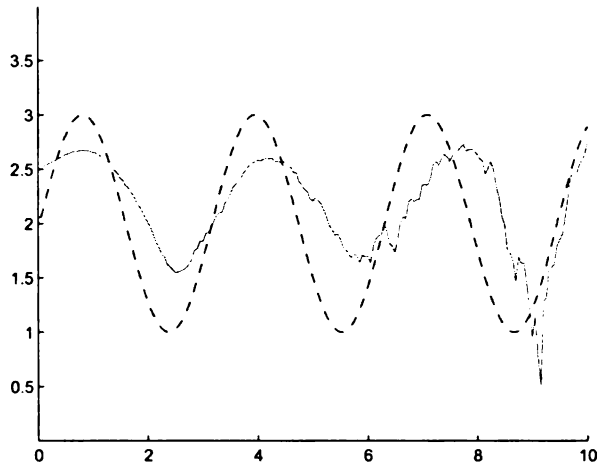

Figure 4.14. Example 5 (2-smoothing kernel): solution with regularization, $\delta = 0.005\%$, $N = 200$, $R = 27$.

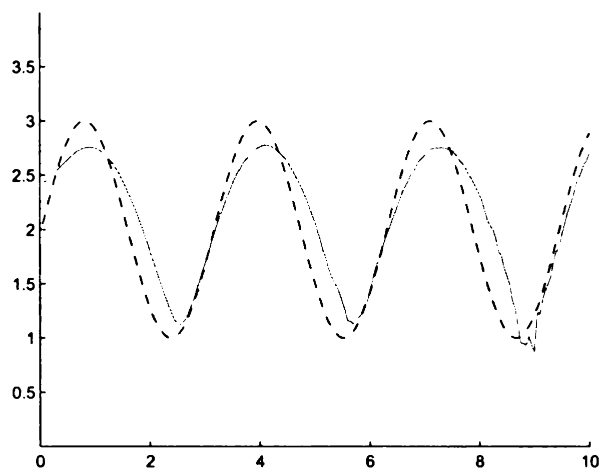

Figure 4.15. Example 5, continued: solution with regularization, $\delta = 0.0005\%$, $N = 200$, $R = 21$.

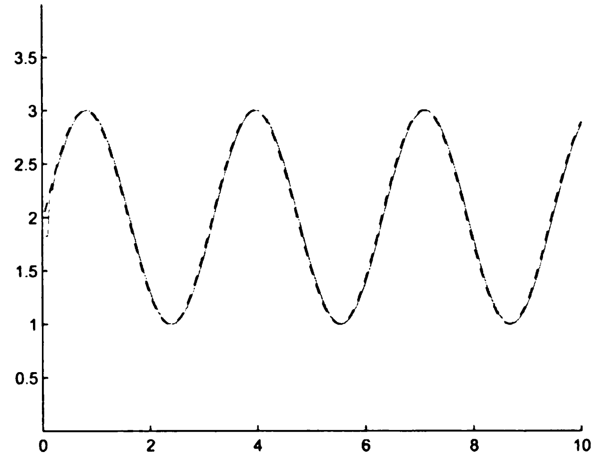

Figure 4.16. Example 5, continued, solution with regularization, $\delta = 0\%$, $N = 200$, $R = 3$.

Example 6. We consider a 3-smoothing kernel $k(t) = 0.5t^2$ and the same nonlinear function $g(u) = u^3$ as in the above example. The true solution is $\bar{u}(t) = sin(2t) + 2$, for $t \in [0, 10]$. See Figures 4.17–4.19.

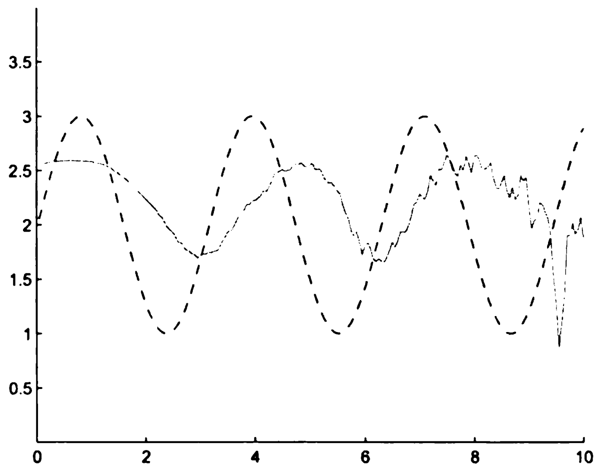

Figure 4.17. Example 6 (3-smoothing kernel): solution with regularization, $\delta = 0.001\%$, $N = 200$, $R = 42$.

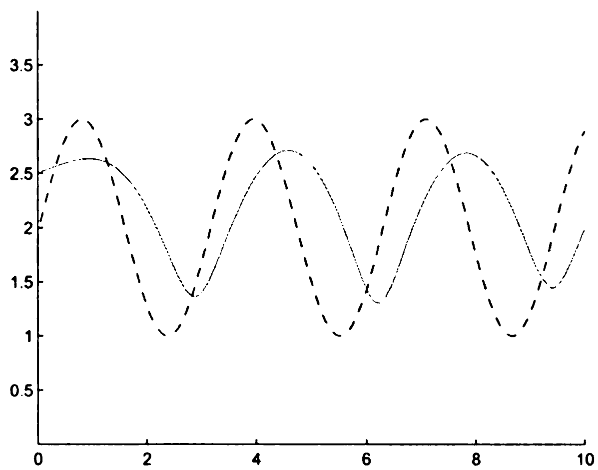

Figure 4.18. Example 6 (3-smoothing kernel): solution with regularization, $\delta = 0.0001\%$, $N = 200$, $R = 35$.

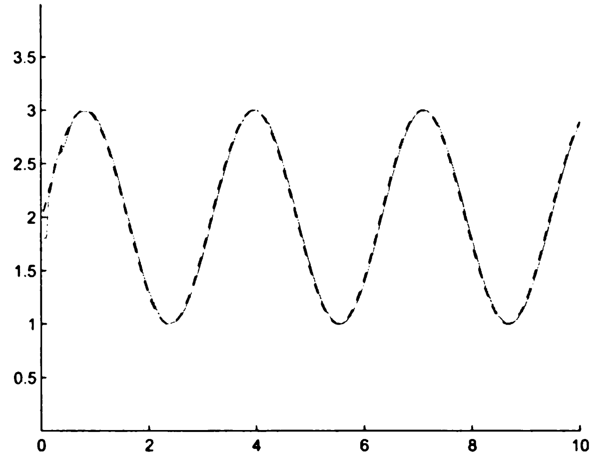

Figure 4.19. Example 6 (3-smoothing kernel): solution with regularization, $\delta = 0\%$, $N = 200,\ R = 4$.

Example 7. We consider the 1-smoothing *nonconvolution* kernel $k(t,s) = ts + 1$, and $g(u) = u^3$ with true solution

$$\bar{u}(t) = \begin{cases} 1/0.15t, & 0 \leq t < 0.15, \\ -10/3t + 1.5, & 0.15 \leq t < 0.3, \\ 0.5, & 0.3 \leq t < 0.5, \\ 7.5t - 3.25, & 0.5 \leq t < 0.7, \\ -20(t - 0.8), & 0.7 \leq t \leq 1. \end{cases}$$

See Figures 4.20–4.22.

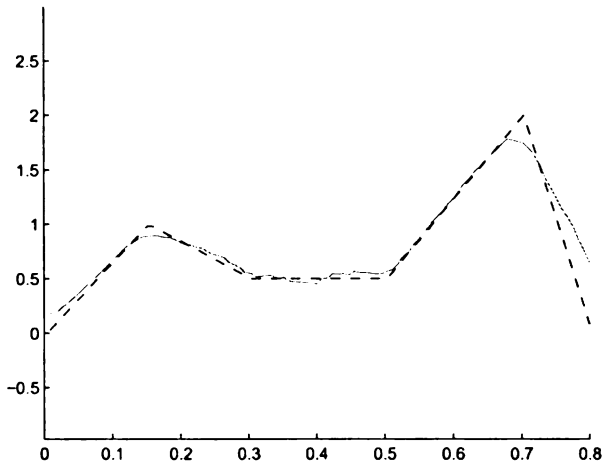

Figure 4.20. Example 7 (1-smoothing nonconvolution kernel): solution with regularization, $\delta = 5\%$, $N = 100$, $R = 7$.

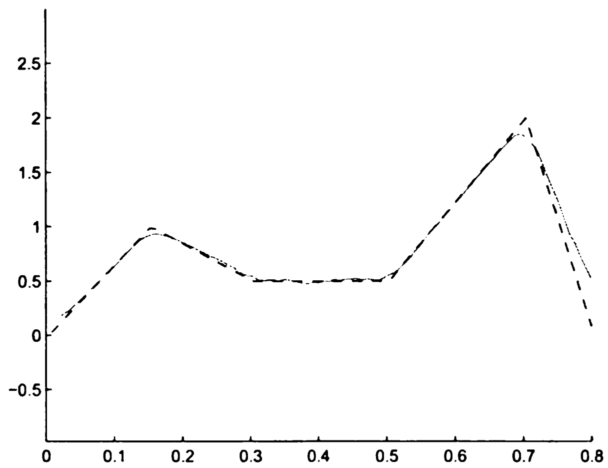

Figure 4.21. Example 7, continued: solution with regularization, $\delta = 1\%$, $N = 100$, $R = 5$.

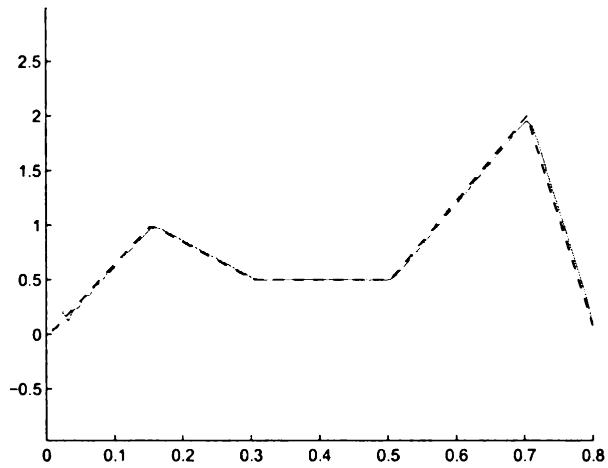

Figure 4.22. Example 7, continued, solution with regularization, $\delta = 0\%$, $N = 100$, $R = 2$.

Example 8. We consider the same 1-smoothing nonconvolution kernel $k(t,s) = ts + 1$, as in the above example with continuous true solution $\bar{u}(t) = -3t + 5$, for $t \in [0,1]$, and $g(u) = e^u$. See Figures 4.23–4.26.

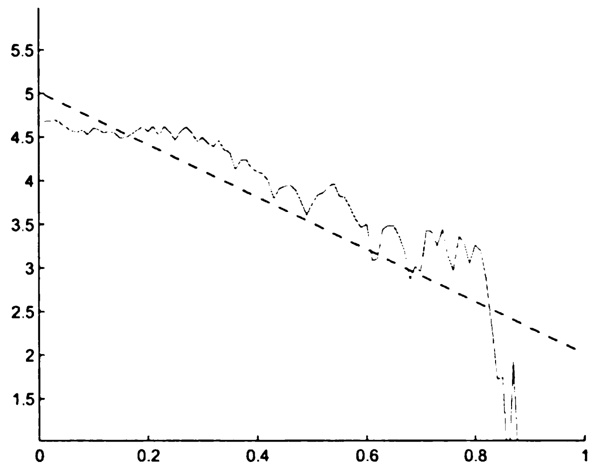

Figure 4.23. Example 8 (1-smoothing nonconvolution kernel): solution with regularization, $\delta = 5\%$, $N = 100$, $R = 65$.

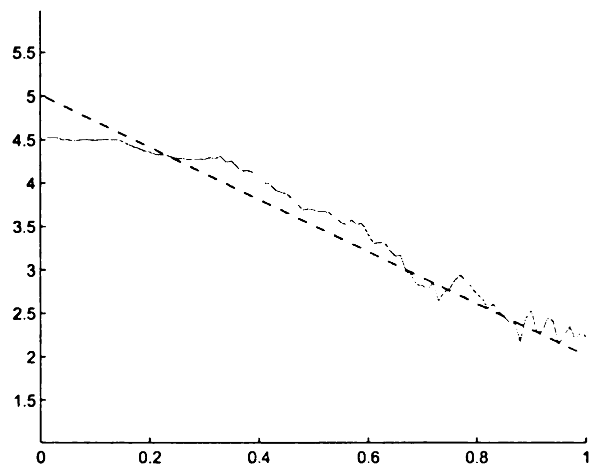

Figure 4.24. Example 8, continued, solution with regularization, $\delta = 1\%$, $N = 100$, $R = 53$.

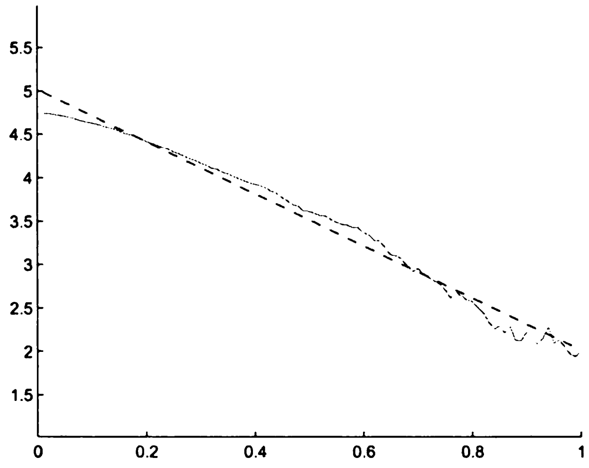

Figure 4.25. Example 8 continued: solution with regularization, $\delta = 0.1\%$, $N = 100$, $R = 25$.

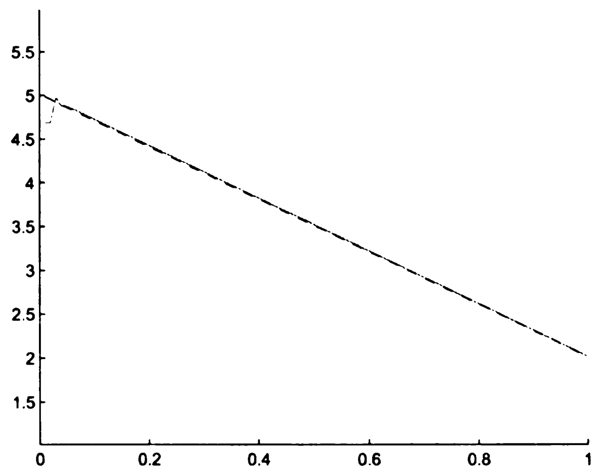

Figure 4.26. Example 8, continued, solution with regularization, $\delta = 0\%$, $N = 100$, $R = 2$.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] J.V. Beck, B. Blackwell and C.R. St. Clair, *Inverse Heat Conduction*, Wiley-Interscience, New York, 1985.

[2] R. Miller, Nonlinear Volterra integral equations, W. A. Benjamin, Menlo Park, CA, 1971.

[3] W. Ring and J. Prix, Sequential predictor-corrector regularization methods and their limitations, *Inverse Problems* **16** (2000) 619-634.

[4] P. K. Lamm, Approximation of ill-posed Volterra problems via predictor-corrector regularization methods, *SIAM J. Appl. Math.*, **195** (1995) 469-494.

[5] P. K. Lamm, Future-sequential regularization methods for ill-posed Volterra equations: applications to the inverse heat conduction problem, *J. Math. Anal. Appl.*, **56** (1996) 524-541.

[6] P. K. Lamm, Regularized inversion of finitely smoothing Volterra operators: predictor-corrector regularization methods, *Inverse Problems* **13** (1997) 375-402.

[7] P. K. Lamm and T. L. Scofield, Sequential predictor-coreector methods for the variable regularization of Volterra problems, *Inverse Problems* **16** (2000) 373-399.

[8] P. K. Lamm, A survey of regularization methods for first-kind Volterra equations, Surveys on Solution Methods for Inverse Problems, ed. by D. Colton, et al, (2000) 53-82.

[9] P. K. Lamm and T. L. Scofield, Local regularization methods for the stabilization of linear ill-posed equations of Volterra type, *Numerical Functional Analysis and Optimization* **22** (2001) 913-940.

[10] P. K. Lamm, Full convergence of sequential local regularization methods for Volterra inverse problems, *Inverse Problems* **21** (2005) 785-803.

[11] P. K. Lamm, Z. Dai, On local regularization methods for linear Volterra equations and nonlinear equations of Hammerstein type, *Inverse Problems* **21** (2005) 1773-1790.

[12] Z. Dai and P. K. Lamm , Local regularization for the nonlinear inverse autoconvolution problem, *SIAM J. Num. Analy.*, Submitted 2006.

[13] Gripenberg G, London S O and Staffens O 1990, *Volterra Integral and Functional Equations* (Cambridge: Cambridge University Press)

[14] H. ZEGEYE, Iterative solution of nonlinear equations of Hammerstein type, *Journal of Inequalities in Pure and Applied Mathematics* **4** Issue 5, Article 92, (2003).

[15] J. M. Bownds, On numerically solving nonlinear Volterra integral equations with fewer computations, *SIAM J. Numer. Anal.* **13** No.5 (1976)

[16] K. Deimling, Nonlinear Volterra integral equations of the first kind, *Nonlinear Analysis: Theory, Methods, and Applications* **25** (1995) 951-957.

[17] H. W. Engl, M. Hanke and A. Neubauer, Regularization of Inverse Problems, *Kluwer Academic Publishers, Dordrecht, The Netherlands* (1996)

[18] F. Perri and l. Pandolfi, Input identification to a class of nonlinear input-output causal systems, *Comput. Math. Appl.* **51** (2006) 1773-1788.

[19] C. W. Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind, Pitman, Boston (1984).

[20] C. Corduneanu, Integral equations and applications, *Cambridge University Press*, Cambridge, 1991.

[21] H. W. Engl and P. Kügler, Nonlinear inverse problems: theoretical aspects and some industrial applications, Multidisciplinary Methods for Analysis, Optimization and Control of Complex Systems, V. Capasso, J. Periaux, eds., Springer-Velag Berlin Heidelberg, New York, Series Mathematics in Industry, (2005), 3-48

[22] M. M. Lavrentiev, V. G. Romanov and S. P. Shishatiskii, Ill-posed problems of mathematical physics and analysis, *Translations of Mathematical Monographs* **64**, Amercan Mathematical Society, Providence, RI, (1986).

[23] G. Gripenberg, Periodic solutions of an epidemic model, *Journal of Mathematical Biology*, **10** No. 3, (1980).

[24] F. Brauer and C. Castillo-Chavez, Mathematical Models in Population Biology and Epidemiology, *Springer.*

[25] J. A. Dixon, A nonlinear weakly singular Volterra integral equation arising from a reaction-diffusion study in a small cell *J. Comput. Appl. Math.*, **18** (1987) 289-305.

[26] M. Hanke, A. Neubauer and O. Scherzer, A convergence analysis of the Landweber iteration for nonliknear ill-posed problems, *Numer. Math.*, **72** (1995) 21-37

[27] M. Hanke, Regularizing properties of a truncated Newton-CG algorithm for nonlinear inverse problems, *Numer. Funct. Anal. Optim.*, **18** No. 9-10, (1997) 971-993.

[28] M. Hanke, A regularizing Levenberg-Marquardt scheme, with applications to inverse groundwater filtration problems, *Inverse Problems*, **13** No. 1, (1997) 79-95.

[29] R. Ramlau, A modified Landweber-method for inverse problems *Numer. Funct. Anal. Optim.*, **20** No. 1-2, (1999) 79-98

[30] B. Kaltenbacher, Some Newton-type methods for the regularization of nonlinear ill-posed problems, *Inverse Problems*, **13** No. 3, (1997) 729-753.

[31] Bakushinskii, The problem of the convergence of the iteratively regularized Gauss-Newton method, *Comput. Math. Math. Phys.*, **32** (1992) 1353-1359.

[32] Bakushinskii, On convergence rates for the Iteratively regularized Gauss-Newton method ,*IMA J. Numer. Anal.*, **17** No. 3 (1997) 421-436.

[33] F. Liu and M. Z. Nashed, Convergence of regularized solutions of nonlinear ill-posed problems with monotone operators, Partial Differential Equations and Applications, ed. by P. Marcellini et al, Dekker, NY, (1996) 353-361.