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Front Dynamics in Non-Smooth Ignition Systems in a Noisy Environment

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FRONT DYNAMICS IN NON-SMOOTH IGNITION SYSTEMS IN A NOISY ENVIRONMENT

By

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ABSTRACT

FRONT DYNAMICS IN NON-SMOOTH IGNITION SYSTEMS IN A NOISY ENVIRONMENT

By

Mohar Guha

Non-smooth excitable systems arise as models of combustion, nerve impulses, elastic displacements during phase transitions, and proton conduction in weakly hydrated polymer electrolyte membranes (PEMs). These systems often support front solutions which correspond to moving phase transitions. Typically the non-smoothness arises as an approximation to a system which is so sensitive to changes in state at critical values of internal variables, for example changes in temperature near the flash point of a mixture, that its response may be considered discontinuous. We consider a singularly perturbed, piece-wise smooth system with a discontinuity at an ignition threshold for which the noiseless system supports a unique, stable front solution. At spatial points where the front crosses criticality the system will be particularly sensitive to noise, leading the solution to lose monotonicity, and engendering multiple crossings of the ignition threshold. We identify regimes in which the front preserves its stability, showing that noise smears the front location and modifies the front propagation. In a neighborhood of the ignition threshold the system interacts strongly with noise, the front can loose monotonicity, resulting in multiple crossings of the

ignition threshold. We adapt the renormalization group methods developed for coherent structure interaction, a key step being to determine pairs of function spaces for which the the ignition function is Frechet differentiable, but for which the associated semi-group, $S(t)$, is integrable at $t = 0$. We parameterize a neighborhood of the front solution through a dynamic front position and a co-dimension one remainder. The front evolution and the asymptotic decay of the remainder are on the same time scale, the RG approach shows that the remainder becomes asymptotically small, in terms of the noise strength and regularity, and the front propagation is driven by a competition between the ignition process and the noise. The main result shows that the front retains its stability, but the ignition point smears into an "ignition set" over which the front can have multiple crossings of the ignition threshold. Moreover in the scalings for which we show the front retains its stability, the ignition set is thinner than the front and the impact of the noise on the front velocity is a correction to the noiseless front velocity.

To my family

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Last, but not least, I thank my family: my parents, for educating me with aspects from both arts and sciences, for unconditional support and encouragement to pursue my interests. My brother for sharing his experience of the dissertation writing endeavor with me, for listening to my complaints and frustrations, and for believing in me.

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CHAPTER 1

Introduction

Non-smooth excitable systems are important models of combustion, nerve impulses and elastic displacements during phase transitions. To this we add our model of proton conduction in hydrated polymer electrolyte membranes (PEMs). These systems support travelling wave solutions which correspond to moving phase transitions. When the underlying system properties depend sensitively on the system state then it may be appropriate to incorporate a non-smooth response to the system. This thesis studies the interaction of non-smooth systems with a noisy environment.

An example of a non-smooth model of nerve impulses is Henry McKean's caricature of the Fitzhugh Nagumo equations modelling nerve conduction by a tunnel diode network [42], [30] . If a nerve is stimulated below a threshold then the signal damps out and no information is transmitted, while if it is stimulated above a threshold then the signal undergoes a fast change into a train of pulses. Let $u(x,t)$ be the voltage difference actoss the membrane of the nerve, then u satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u(x,s) : 0 \leq s \leq t). \tag{1.0.1}$$

The nonlinearity f in the Fitzhugh Nagumo equations has the form

$$f(u) = u(1-u)(u-a) \quad 0 < a < 1, \tag{1.0.2}$$

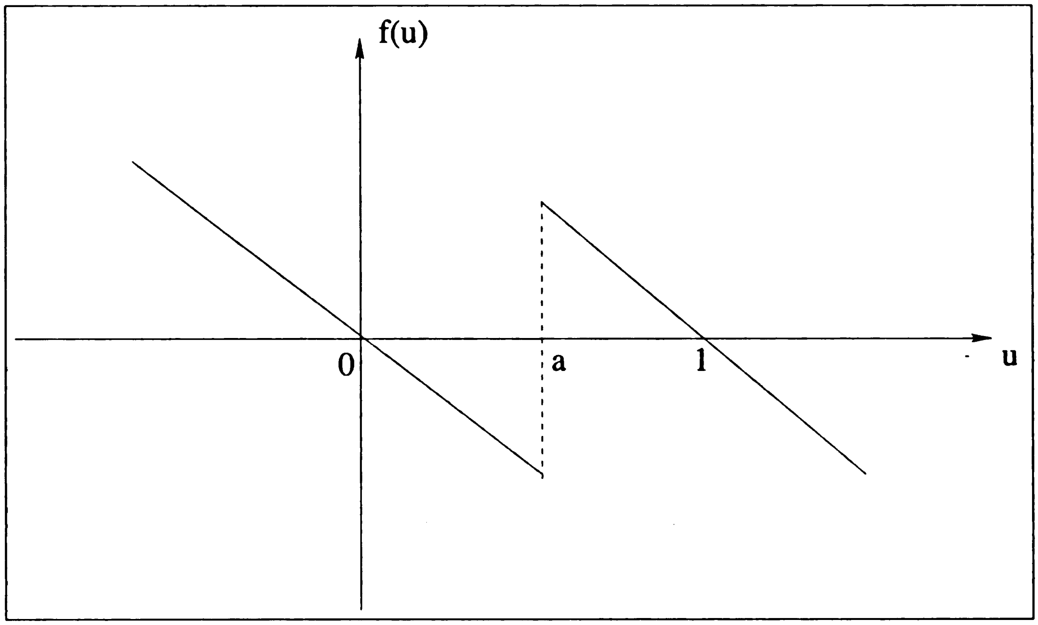

Figure 1.1. McKean's piecewise-linear caricature of Fitzhugh-Nagumo equations.

and McKean suggested a 'piecewise linear caricature' of the cubic polynomial [19]

$$f(u) = -u + \begin{cases} 0 & u < a \\ 1 & u > a. \end{cases} \tag{1.0.3}$$

A key issue in a non-smooth system of the form (1.0.3) arises from the structure of the wave. The model studied in this thesis has a similar structure, the nonlinearity is peice-wise linear but discontinuous with fixed jump. The problem of existence of travelling waves for the non-smooth nonlinearty (1.0.3) has been addressed by several authors [27],[26],[46], the final result being that model admits a diverse variety of waveforms depending on the number of crossings of the level a. The case of one-crossing was described by McKean [19] where he showed that, upto translation, (1.0.1), (1.0.3) has a single waveform which rises monotonically and moves at a definite speed. The other important problem is the existence of a threshold, that is, a level below which no information is transmitted. This was confirmed by McKean and Moll [20] in the next simplest case of two-crossings. In this case, there exists a standing wave solution of (1.0.1), ϕ_2, symmetrical about its peak at $x = 0$ and (1.0.1) preserves the shape of such data for $t > 0$. The separatrix surface, constructed in [49], divides the space of initial data X_2, into two regions: a region of *collapse* for which $u(x,t) \to 0$ and a region of *expansion* for which $u(x,t) \to 1$. By constructing the classical solution of (1.0.1) with nonlinearity (1.0.3), it was proved in [20] that any smooth, symmetric initial data which does not collapse to 0 nor expand to 1, stabilizes to the standing wave ϕ_2.

To model a bilinear thermoelastic material capable of undergoing solid-solid phase transition, Anna Vainchtein considered a non-monotone, up-down-up strain relation. In [3], a special thermoelastic material is studied which permits analytical calculation of travelling wave solutions for one-dimensional regularised problem. The model includes both thermal and heat dissipation. When the nondimensional parameter comparing the length scales associated to the viscosity and to heat conduction exceeds a threshold value, the kinetic relation becomes non-monotone. Vainchtein, considers an infinite homogeneous elastic bar, where $u(x,t)$ and $T(x,t)$ are the longitudinal deformation in the displacement and temperature fields, at a point x and time t. The local displacement of the bar is described by its displacement field and its spatial derivative, called the strain $w = u_x(x,t)$. The elastic property of the bar is determined by an elastic energy density, $\sigma(w,T)$, which is a function of strain, modelled as,

$$\sigma(w,T) = \begin{cases} \mu w & w < w_0(T) \\ \mu(w - e_T) & w > w_0(T), \end{cases} \tag{1.0.4}$$

where μ is the stress modulus. The two strain intervals where stress is linearly increasing represent two different material phases. In this model, the strain interval where the stress is compressed into a single point $w_0(T)$, is where the phase transition occurs. The distance between the linear branches is denoted by e_T. To investigate the kinetics of a phase boundary, the above model of σ enables analytic calculations. In this paper, the author derived an exact formula describing interface kinetics and its dependence on material parameters is studied.

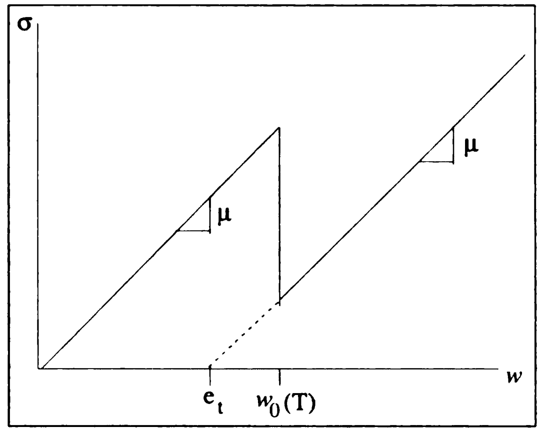

Figure 1.2. The nonlinearity $\sigma(w.T)$

This thesis considers a reduced model of slow transient behaviour and proton transport through Polymer-Electrolyte Membranes (PEM). The full model was introduced by Igor Nazarov and Keith Promislow in [21]. To motivate the model studied in my thesis, a brief discussion of PEM fuel cells is appropriate. PEM fuel cells use a polymer membrane as an electrolyte, porous carbon electrodes containing a platinum catalyst and need only hydrogen, oxygen from the air, and water to operate. PEM fuel cells generate electric potential by separating the oxidation of hydrogen into two catalysed steps performed on opposite sides of an electrolyte membrane. Hydrogen fuel is chanelled to the anode on one side while oxygen from air is chanelled to the cathode on the other side of the cell. At the anode the platinum catalyst causes the hydrogen to split into positively charged hydrogen ions and negatively charged electrons. The voltage gradient across the polymer electrolyte membrane drives the positively charged ions to pass through it to the cathode, while the negatively charged electrodes must pass through an external circuit thereby creating electric current. At the cathode the positively charged hydrogen ions combine with oxygen, with the end products of water, water vapor, and heat. The electrolyte membrane is a complex polymer comprised of Teflon spines. These are arranged in a nanoscale configura-

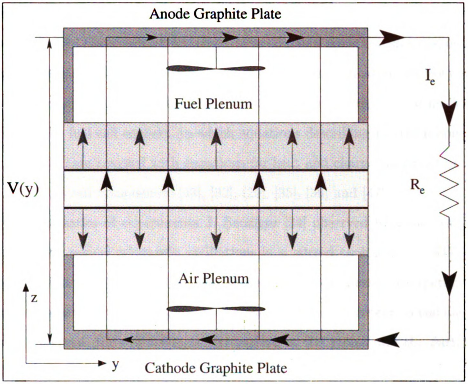

Figure 1.3. The anode and cathode bi-polar plates, gas plenums, membrane, and the gas diffusion layer. The Plenums are slowly replenished from an external feed. The black arrows indicate direction of electron flow and of proton counterflow.

tion which facilitates the selective diffusivity of the membrane, enabling the fuel cell to perform close to the thermodynamic limit for efficiency. The reactant gases are distributed to catalyst sites through a sheet of porous carbon fiber paper known as the gas diffusion layer, or GDL. Most efforts at modeling and simulating fuel cells in the literature to date have focused on the entire fuel cell, including charge, heat, and mass transport. Despite the complexity of these coupled models, extensive one and two-dimensional numerical simulations yield concentration profiles that vary linearly through the thickness of both the GDL and PEM [13], [48], [14], [32]. Models for convective-diffusive gas transport in porous media have been developed in [50], [40], [2], [11], [31], [28] and also for applications arising in a wide range of fields, including electrochemistry [15], flow in insulating materials [39] and groundwater transport [37],[5]. The standard approach is to couple a mass transport equation for the mixture (typically Darcy's law or some appropriate modification thereof) with an equation governing intercomponent diffusion within the mixture. Similar models have appeared in the fuel cell context, in which equations describing multicomponent gas flow in the GDL are coupled with equations for heat and charge transport occurring in the other fuel cell components [43], [32], [29], [35], [36] and [47].

In a seminal series of experiments J. Benziger [24] observed hystersis, slow transients, and long period relaxation oscillations in a stirred tank reactor (STR) PEM fuel cell feed from dry inlet gases. The key features of Beneziger's experiment are the external resistances at which the jump from high-water content to the low-water content is observed, the cell voltages and current at the jumps and the partitioning of product water into the cathode and anode plenums. Experimental work of Sone et al [44] shows that the membrane protonic conductivity drops by several orders of magnitude at a critical membrane hydration level. The membrane must be well hydrated to function, however overproduction of liquid water may saturate the surrounding porous electrodes and leads to oxygen transport limitations. The control of the motion and distribution of liquid water in both the nano-structure of the membrane and the surrounding fibrous electrodes is referred to as water management, and is critical to effective cell operation. I. Nazarov and K. Promislow, [21] proposed a model which coupled two degenerate parabolic PDEs for membrane water content to an elliptic equation for proton conduction. This model accounted for the impact of membrane water content on protonic conductivity and showed that the fuel cell system possessed two stable states, an ignited state in which the membrane has sufficient water to sustain the electrochemical reaction, and an extinguished state in which the high membrane resistence reduces the reaction rate to a point where the loss of water to the dry gas environment dehydrates the membrane and extinguishes the electro-

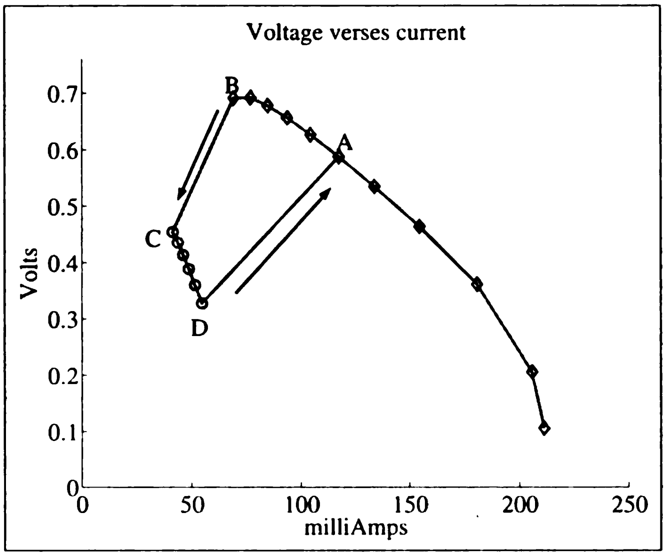

Figure 1.4. Numerical verification of experimental hysteresis from Benziger et al.

chemical reaction. Moreover they showed that the bifurcation from a uniformly hydrated/ignited state to a partially ignited/partially extinguished state could explain the observed experimental hysteresis, **Fig**(1,??) and the slow transients observed in Beneziger's experiments. In particular the transients corresponded to ignition waves slowly propagating by self-diffusion of water within the plane of the membrane. The ignition waves observed in the numerical work of Nazarov et al. are initiated when the water production within the fuel cell is so small that the approximately uniform membrane water content approaches the critical membrane hydration level for protonic conductivity. At this point small variations in membrane water content induced by fluctuations in the noisy fuel cell environment become nucleation points for the igniation wave. In this thesis, following the work of Sone et al, we model the water diffusivity as a function of the water content U, in the membrane

$$\sigma(U) = \begin{cases} 0 & U < \sigma_c \\ U & U > \sigma_c, \end{cases} \tag{1.0.5}$$

where σ_c is the conductivity threshold for the membrane.

A PEM fuel cell is a noisy environment. A key issue in this thesis is to study the behaviour of non-smooth excitable systems in noisy settings. We consider a thin membrane exposed on either side to air of prescribed humidity, which varies with position along the length of the membrane. The relative humidity of the air produces a local reference water content for the membrane, which is modified by the local production of water by the electrochemical reaction which occurs only in the regions of the membrane which are already sufficiently humidified. Averaging the water content through the plane of the membrane, we consider the diffusive transport of water content, $U(y,t)$ within the plane of the membrane.

$$U_t = \epsilon U_{yy} - (U - g(y)) + \mu\sigma(U) + \epsilon\eta(y,t), \tag{1.0.6}$$

where the ignition function σ is defined as in (1.0.5) and the noise $\eta \in L^{\infty}(R^+, H^{-\gamma_s})$. In a recent paper by Eric Vanden-Eijenden et. al. [41], analyses the effect of small amplitude noise on excitable systems. The authors consider a stochastically-forced reaction-diffusion equation in a medium such that the reaction terms sustain travelling pulses and the stochastic forcing is such that it acts at a neighbourhood of a particular point. The noise initiates pulses at this point at random times and the pulses then travel through the medium. The main result is, by appropriate scaling limits and properly matched timescales, stochastic forcing can generate spatially-periodic travelling wave trains. However, they avoid studying the impact of noise on the structure of the travellig wave by restricting the noise to a localized spatial domain.

Through detailed analysis of the shape of our wavefront in a neighbourhood of the ignition threshold σ_c, we quantify the impact of noise on the wave structure, specifically the possible number of crossings of the ignition threshold, the wave speed and the waves stability.

This thesis is broadly divided into four sections. In the first section we provide a description of the model equation (1.0.6), a singularly perturbed, non-smooth parabolic pde. A standard matched asymptotic expansion permits the construction of a $\mathcal{C}^1$

front, $\phi(y; y_0)$, parameterized by front position y_0 connecting ignited and extinguished states traveling with a position dependent velocity $v = v(y_0)$. The front is *not an exact traveling wave solution* of the underlying system, even in the absence of the noise term, the spatial inhomogeneity induced by the equilibrium water content $g(y)$ renders the construction of an asymptotically exact traveling wave solution nontrivial due to hysteretic effects. The front is monotonically increasing, but noise can break the monotonicity, and in a neighborhood of the ignition region $U(y,t) = \sigma_c$, this could possibly lead to ignition of new fronts and pulse splitting. The absence of noise for a front of constant shape, would allow us to apply a very developed body of invariant manifold theorems [6], [7], [45], including the blow-up, decay, meta-stable properties [4], [25], [9]. Our front shape evolves in shape as it travels and is subject to noise.

We use a renormalization technique to study the stability of the travelling front, breaking the evolution into a series of initial value problems, each with a frozen co-ordinate system including a fixed linearization corresponding to frozen front position $\bar{y_0}$, a frozen convective frame with velocity $\bar{v} = v(\bar{y_0})$ and fixed spectral dichotomy corresponding to the frozen linear operator. In the co-moving frame

$$z = \frac{y - \bar{y_0} - \sqrt{\epsilon} v(\bar{y_0})(t - t_0)}{\sqrt{\epsilon}}, \tag{1.0.7}$$

the evolution equation (2.0.1) becomes

$$U_\tau = F(U) = U_{zz} + v(z_0)U_z - (U - g_s(z)) + \mu\sigma(U) + \epsilon^{3/4-\gamma_s/2}\eta_s. \tag{1.0.8}$$

where the scaled noise satisfies $\|\eta_s\|_{H^{-\gamma_s}} = \mathcal{O}(1)$. The solution of the full equation (2.0.1) is decomposed as $U = \phi(z; z_0) + W(z,t)$, where the time dependent parameter $z_0(t)$ shadows the slow evolution of U along the family of quasi-steady front solutions, up to a small residual term W. The nonlinear evolution equation for the remainder W takes the form,

$$W_t + \frac{\partial \phi}{\partial z_0} z_0' = F(\phi) + L_{\bar{\phi}}W + (L_\phi - L_{\bar{\phi}})W + N(W), \tag{1.0.9}$$

where $L_{\bar{\phi}}$ is the linearized operator of F at the frozen point $\bar{\phi} = \phi(z; \bar{z}_0)$,

$$L_{\bar{\phi}} = \partial_z^2 + v(z_0)\partial_z - I + \mu\chi_{[\bar{z}_0,\infty]} + \mu\frac{\sigma_c}{\phi'(\bar{z}_0)}\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0}. \tag{1.0.10}$$

Here the rank-one tensor products of delta-functions, $\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0}$, acts on W by

$$(\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0})W \equiv \langle W, \delta_{\bar{z}_0}\rangle \delta_{\bar{z}_0} = W(\bar{z}_0)\delta_{\bar{z}_0}. \tag{1.0.11}$$

The price for freezing the linear operator in the evolution equation (3.2.3) is the introduction of the secular term, $(L_\phi - L_{\bar{\phi}})W$ which encodes the movement of the front location z_0 away from the frozen value $\bar{z}_0$. A central issue is the structure of the nonlinear operator N. The choice of the domain of the linearized operator $L_{\bar{\phi}}$, is the **"Goldilocks problem"**. The Sobolev space should not be too weak, so that the nonlinearity $\sigma : H^\gamma \to H^{-\beta}$ is Frechet differentiable at $\bar{\phi}$, but not too strong that the semigroup $S_{\bar{\phi}}(t) \equiv S(t)$ generated by $L_{\bar{\phi}}$, $S(t) : H^{-\beta} \to H^\gamma$, blows up too rapidly as $t \to 0^+$. The following theorem addresses the first side of the "Goldilocks Problem":

Theorem 1.0.1 *The Frechet derivative of F in H^γ for $\gamma > 1/2$, at the composite solution $\bar{\phi}$ is given by (3.2.15). Moreover for $W \in H^\gamma$, $\gamma > 1/2$, the nonlinearity $N(W) = F(\bar{\varphi} + W) - F(\bar{\phi}) - L_{\bar{\phi}}W$ satisfies:*

$$\|N(W)\|_{H^{-\beta}} \leq c\left(\|W\|_{H^\gamma}^{\beta+1/2} + \|W\|_{H^\gamma}^{\gamma+1/2}\right). \tag{1.0.12}$$

for any $1/2 < \beta < 1$.

The key step is to write the nonlinearity as in terms of the characteristic functions

$$\sigma(\phi + W) = \left(\chi_{E_+} + \chi_{[\bar{z}_0,\infty)} - \chi_{E_-}\right)(\phi + W), \tag{1.0.13}$$

where E_- denotes the set of "false negatives" for which $\bar{\phi}(z) > \sigma_c$ but $\bar{\phi}(z) + W < \sigma_c$, and similarly E_+ is the set of "false positives" for which $\bar{\phi}(z) < \sigma_c$ but $\bar{\phi}(x) + W > \sigma_c$. Using the Hölder continuity of W allows one to show that **false ignition sets are thin compared to their distance from the ignition point** $\bar{z}_0$. Although the perturbation W may nucleate many ignition regions, this is a small, nonlinear effect compared to its dominant, linear impact, which is to move the ignition point, as captured by the rank-one operator $\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0}$ in $L_{\bar{\phi}}$.

In the next step, we focus out attention on the spectral stability of the travelling front. The linearized operator $L_{\bar{\phi}}$ has a simple, small eigenvalue corresponding to the broken translation invariance of the travelling front. All the remaining spectrum is contained inside a closed angle lying in the left half of the complex plane.

The second side of the "Goldilocks problem" is to establish estimates on the semi-group $S(t)$ associated to the linear operator $L_{\bar{\phi}}$ of the form

$$\|S(t)F\|_{H^\gamma} \leq \frac{Ce^{-\nu t}}{t^{(\gamma+\beta)/2}} \|F\|_{H^{-\beta}}, \tag{1.0.14}$$

for $F \in X_{\bar{z}_0}$, where $X_{\bar{z}_0}$ is the eigenspace associated to the $\mathcal{O}(\sqrt{\epsilon})$ eigenvalue of $L_{\bar{\phi}}$. This requires estimates on the resolvent operator, $\left(L_{\bar{\phi}} - \lambda\right)^{-1}$ acting on $H^{-\beta}$ which is non-trivial since the rank-one operator is not defined on $H^{-\beta}$. We circumvent this by decomposing

$$L - \lambda = \mathcal{L} + \alpha\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0} + \mu\chi_{(\bar{z}_0,\infty)}, \tag{1.0.15}$$

where the linear operator $\mathcal{L}$ is given by,

$$\mathcal{L} = \partial_z^2 + \bar{v}\partial_z - I.$$

We observe that $\left(L_{\bar{\phi}} - \lambda\right)^{-1} F = w + \mathcal{L}^{-1}F$, where w solves

$$(L - \lambda)w = -\alpha v_s \left(\mathcal{L}^{-1}F\right)(\bar{z}_0)\delta_{\bar{z}_0} - \mu\left(\mathcal{L}^{-1}F\right)\chi_{(\bar{z}_0,\infty)},$$

This reduction establishes the invertibility of $L_{\bar{\phi}} - \lambda$ by applying the exponential dichotomy $\mathcal{L}$ as obtained by Dan Henry [12].

We adapt the renormalization group method developed in [10], [17], [18], [22], [23], [24], for the asymptotic stability of patterns. The architecture of the renormalization group lies in the construction of a family of decompositions $\{(\bar{z}_n, t_n)\}_{n=0}^{\infty}$, where $\bar{z}_n$ is the fixed coordintate system and t_n is an initial time. The decompositions permit the initial value problem satisfied by the remainder to be broken into a sequence of initial value problems over the time intervals $[t_n, t_{n+1}]$. The bounds on the semigroup enables us to develop estimates on the decay of W by writing the evolution equation for W in terms of the linearized operator $L_{\bar{\phi}}$ frozen at a point $\bar{z}_0$ in the convected frame. As the front location z_0 moves away from the front location $\bar{z}_0$ there is a natural secular growth in the estimates for W and after a finite time the control over the decay of W is lost. We remove this secular growth by renormalization of the evolution equations, updating the base point $\bar{z}_0$ through a nonlinear projection. Given the solution $U(\cdot, t_n)$ of (1.0.6) at any point, we choose $\bar{z}_n$ such that the remainder $W(\cdot, t_n) = U(\cdot, t_n) - \phi(\cdot; \bar{z}_n) \in X_{\bar{z}_n}$. This decomposition when applied to the abstract equation $U_t = F(U)$, yields an evolution equation on each time interval $[t_n, t_{n+1}]$,

$$
\begin{aligned}
W_t + \frac{\partial \phi}{\partial z_0} z_0' &= R + L_{\bar{z}_0} W + (L_{z_0} - L_{\bar{z}_0}) W + \mathcal{N}(W) + \epsilon^{3/4 - \gamma_s/2} \eta_s(\xi, t), && (1.0.16) \\
W(\xi, 0) &= W(t_n), && (1.0.17)
\end{aligned}
$$

and (5.2.2) admits a *mild solution* on the time interval (t_n, t_{n+1}). The long time evolution of the original PDE reduces to the study of the renormalization map $r_n : W(t_n) \longrightarrow W(t_{n+1})$. The series of initial value problems generated in this manner permit us to avoid the complication of studying the time-dependent linearized operator. The nonlinear stability of the system via RG methods exploits the fact that the evolution of the front location z_0 is on a slower time scale and the secularity $L_{zo} - L_{\bar{z}_0}$ is a lower order operator than either of L_{z_0} and $L_{\bar{z}_0}$.

Finally, for $\gamma \in \left(\frac{1}{2}, 1\right)$ and $\gamma_s > \frac{1}{2}$ satisfying $\gamma + \gamma_s < 2$, the system relaxes quickly in a small neighborhood of the manifold, and after that moves slowly along a thin tube about the manifold.

Theorem 1.0.2 *If the initial data U_0 can be written as $U(\xi, t) = \phi(y; y_*) + W_0(y, t)$, where $\|W_0\|_{H_y^\gamma}$ is sufficiently small, then the solution of the governing equation can be decomposed as, $U(y, t) = \phi(y; y_0(t)) + W(y, t)$, where*

$$
\|W\|_{H_y^\gamma} \leq M \left(e^{-\nu t} \|W_0\|_{H_y^\gamma} + \epsilon^{3/4 - \gamma_s/2} \right),
$$

and

$$
y_0'(t) = \sqrt{\epsilon} v(y_0) + \mathcal{O}(\epsilon^{5/4 - \gamma_s/2}).
$$

CHAPTER 2

Model Description and Construction of Travelling Wave Solution

The membrane water content is governed by the parabolic nonlinear partial differential equation, $U(y,t)$ with scaled time variable t and spatial variable y

$$U_t = F(U) = \epsilon U_{yy} - (U - g(y)) + \mu\sigma(U) + \epsilon\eta(y,t). \tag{2.0.1}$$

The nonlinear ignition functional σ, illustrated in **Fig(2)**, which models the conductivity of membrane as a function of membrane water content, is a discontinuous function with a fixed jump σ_c

$$\sigma(U) = \begin{cases} 0 & U < \sigma_c \\ U & U > \sigma_c. \end{cases} \tag{2.0.2}$$

The form of the ignition function highlights the phenomenon that the membrane is "ignited" if the water content crosses a specific threshold, while it is "extinguished" when water content is below the threshold. The equilibrium water content of the membrane, $g(y)$ satisfies the following conditions:

- $g \in \mathcal{C}^2(\mathbf{R})$,

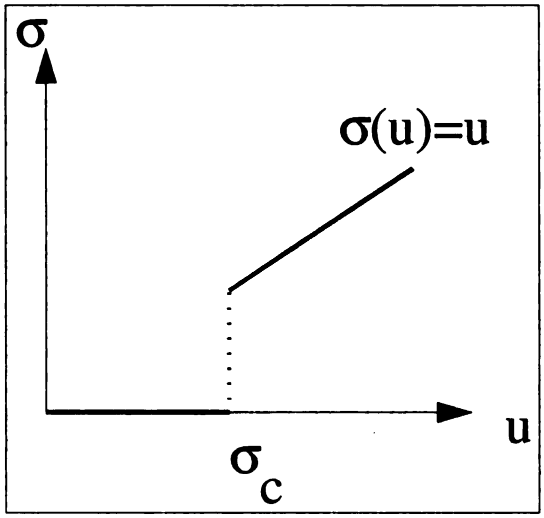

Figure 2.1. The nonlinear ignition functional σ

- g is monotone increasing,
- $g(y) \to g_{\pm}$ as $y \to \pm\infty$.

The noise function $\eta \in L^{\infty}(\mathbf{R}^{+}, H^{-\gamma_s})$, where $\gamma_s \in \left(\frac{1}{2}, 1\right)$. The diffusivity coefficient $\epsilon \ll 1$ is small and the parameter μ satisfies $0 < \mu < 1$. As a first step we construct travelling wave solutions of the autonomous version of the second order pde (2.0.1) via matched asymptotic expansion.

2.1 One parameter family of outer solutions ϕ_0

At the leading order in ϵ, the outer expansion for the stationary solution of (2.0.1) takes the form,

$$(U - g(y)) - \mu\sigma(U) = 0, \tag{2.1.1}$$

Let $y_0 \in \mathbf{R}$ be the point where the solution of (2.1.1), denoted by ϕ_0 assumes the value σ_c. The functional σ acts on the outer solution ϕ_0 as

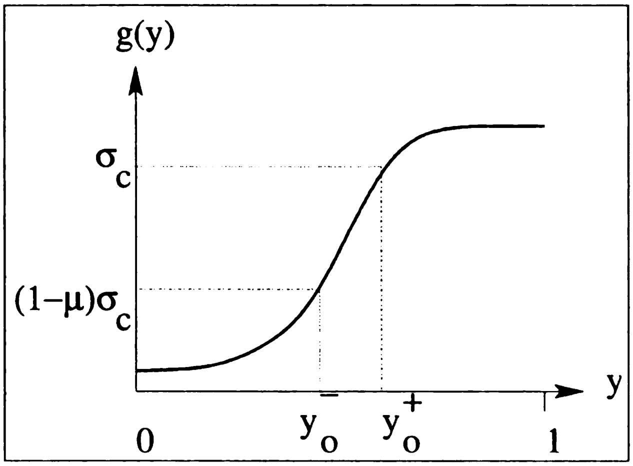

Figure 2.2. The function g

$$\sigma(\phi_0) = \begin{cases} 0 & y < y_0 \\ \phi_0 & y > y_0, \end{cases} \tag{2.1.2}$$

and the solution of (2.1.1) becomes

$$\phi_0 = \begin{cases} g(y) & y < y_0, \\ \frac{g(y)}{1-\mu} & y > y_0. \end{cases} \tag{2.1.3}$$

This is self consistent as long as

$$g(y_0) < \sigma_c < \frac{g(y_0)}{1-\mu}. \tag{2.1.4}$$

The self consistency condition determines an admissible range for y_0, $y_0 \in [y_0^-, y_0^+]$ shown in **Fig(2.2)** where

$$y_0^- = g^{-1}((1-\mu)\sigma_c) \text{ and } y_0^+ = g^{-1}(\sigma_c).$$

Fig(3) illustrates the outer solution ϕ_0. The jump in ϕ_0 is patched by an internal layer at $y = y_0$. The form of the stationary internal layer is determined by the existence of the heteroclinic solution of the inner equation.

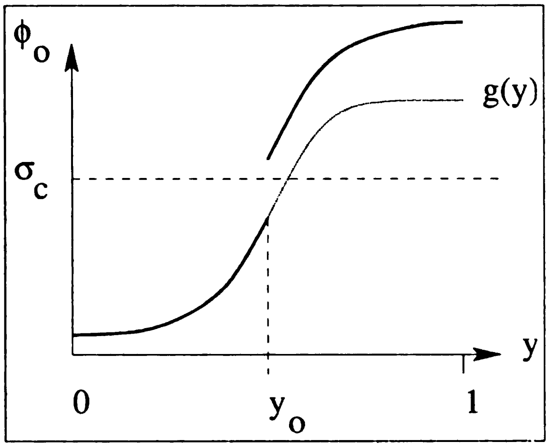

Figure 2.3. The outer function ϕ_0.

2.2 The inner solution

Using the transformation $\xi = \frac{y-y_0}{\sqrt{\epsilon}}$, we stretch the interval around y_0, and freeze the slowly varying background at y_0,

$$U_t = U_{\xi\xi} - (U - g(y_0)) + \mu\sigma(U). \tag{2.2.1}$$

We construct a travelling wave solution $U(\xi, t) = \phi_i(\xi - vt; y_0)$ of (2.2.1) which satisfies

$$\phi_i'' + v\phi' - (\phi_i - g(y_0)) + \mu\sigma(\phi_i) = 0. \tag{2.2.2}$$

The travelling wave ϕ_i approaches $g(y_0)$ as $\xi \to -\infty$ and $\frac{g(y_0)}{1-\mu}$ as $\xi \to \infty$. This condition is the matching condition between the outer and inner layers. The velocity v is to be determined. Here, y_0 serves as a parameter and the unknown wave velocity $v = v(y_0)$ depends on y_0. The wave speed v and the front location y_0 depend on the time t. We look for a heteroclinic solution of (2.2.2), recasting it as an autonomous first order system

$$\begin{pmatrix} \phi_i \\ \phi_i' \end{pmatrix}' = \begin{pmatrix} \phi_i' \\ (\phi_i - g(y_0)) - \mu\sigma(\phi_i) - v\phi_i' \end{pmatrix}, \tag{2.2.3}$$

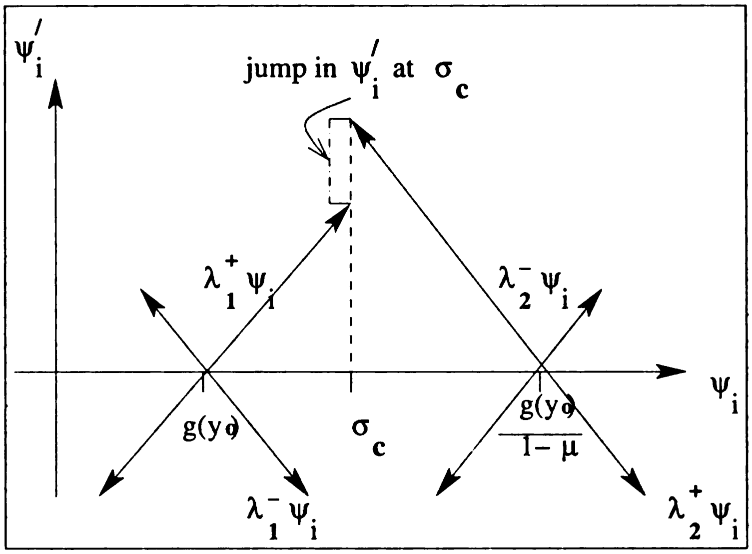

Figure 2.4. The phase diagram for ϕ_i

where $'$ denotes differentiation with respect to ξ. The fixed points of (2.2.3) are located at $Q^+ \equiv \begin{pmatrix} g(y_0) \\ 0 \end{pmatrix}$ and $Q^- \equiv \begin{pmatrix} \frac{g(y_0)}{1-\mu} \\ 0 \end{pmatrix}$, which correspond to the values of the outer solution on either side of the jump at y_0. For $v > 0$, the fixed points Q^+ and Q^- are both saddles with eigenvalues:

$$\lambda_1^{\pm} = \frac{-v \pm \sqrt{v^2+4}}{2}, \tag{2.2.4}$$

$$\lambda_2^{\pm} = \frac{-v \pm \sqrt{v^2+4(1-\mu)}}{2}. \tag{2.2.5}$$

We can observe that $\lambda_1^+ > 0$ and $\lambda_2^- < 0$ where λ_1^+ and λ_2^- are the eigenvalues corresponding to the stable manifold $\mathcal{W}^s(Q^+)$ of Q^+ and unstable manifold $\mathcal{W}^u(Q^-)$of Q^-. Since the governing equation is linear on each side of σ_c, **Fig(2.4)**, $\mathcal{W}^s(Q^+)$ and $\mathcal{W}^u(Q^-)$ are exactly the stable and unstable eigenspaces of (2.2.3). In the unscaled variables, y, the inner solution of (2.2.3) satisfying the boundary conditions at $\xi \to \pm\infty$ is given by

$$\phi_i = \begin{cases} g(y_0) + A_0 e^{\lambda_1^+ (y-y_0)/\sqrt{\epsilon}} & y < y_0 \\ \frac{g(y_0)}{1-\mu} + B_0 e^{\lambda_2^- (y-y_0)/\sqrt{\epsilon}} & y > y_0. \end{cases}$$

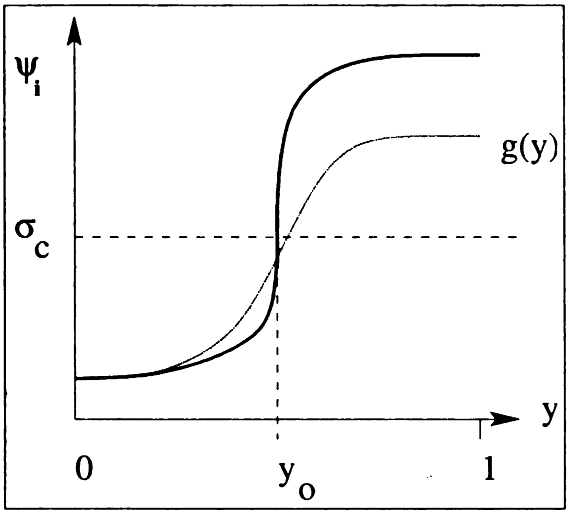

Figure 2.5. The inner solution ϕ_i

As in **Fig(2.5)**, imposing the continuity of ϕ_i, and that ϕ_i attains σ_c at y_0, we solve for A_0 and B_0 obtaining

$$\phi_i = \begin{cases} g(y_0) + (\sigma_c - g(y_0))e^{\lambda_1^+(y-y_0)/\sqrt{\epsilon}} & y < y_0 \\ \frac{g(y_0)}{1-\mu} + (\sigma_c - \frac{g(y_0)}{1-\mu})e^{\lambda_2^-(y-y_0)/\sqrt{\epsilon}} & y > y_0. \end{cases} \tag{2.2.6}$$

2.3 The composite solution

The general form of the composite solution ϕ, **Fig(2.6)**, is given by

$$\varphi = \begin{cases} g(y) + (\sigma_c - g(y_0))e^{\lambda_1^+(y-y_0)/\sqrt{\epsilon}} & y < y_0 \\ \frac{g(y)}{1-\mu} + (\sigma_c - \frac{g(y_0)}{1-\mu})e^{\lambda_2^-(y-y_0)/\sqrt{\epsilon}} & y > y_0. \end{cases} \tag{2.3.1}$$

where λ_1^+ and λ_2^- are functions of v as defined in equations (2.2.4) and (2.2.5). To reduce the size of the residual error $F(\phi_i)$, we tune the final free parameter, the velocity v, to render ϕ, a C^1 function. From (2.3.1) we observe that $\phi(y; y_0, v)$ does not have a continuous derivative as depicted in **Figure 2.4**. We define $E(v)$, which

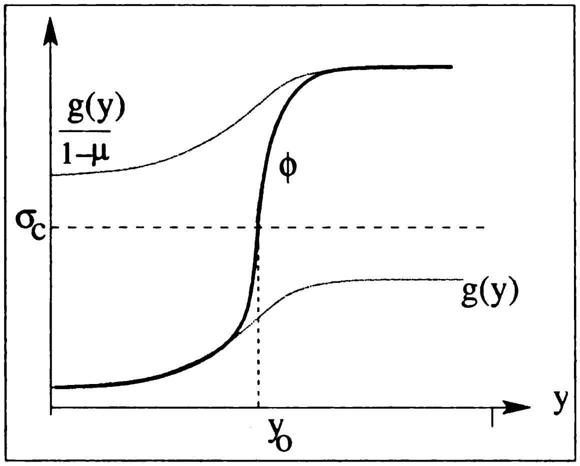

Figure 2.6. The composite solution ϕ

is proportional to the jump in derivative of ϕ at y_0

$$E(v) = \frac{2\sqrt{\epsilon}}{(\sigma_c - g(y_0))}[\![\phi']\!]_{y_0}. \tag{2.3.2}$$

Lemma 2.3.1 *There exists an unique $v = v(y_0)$ such that the composite solution $\phi(y; y_0, v)$ given by (2.3.1) is $C^1(\boldsymbol{R})$. Moreover,*

$$\phi \to \begin{cases} \frac{g_+}{1-\mu} & as \quad y \to \infty \\ g_- & as \quad y \to -\infty. \end{cases}$$

Proof: The jump in the derivative of ϕ at y_0 is given by

$$[\![\phi']\!]_{y_0} = \left(\left(\sigma_c - \frac{g(y_0)}{1-\mu}\right)\frac{\lambda_2^-(v)}{\sqrt{\epsilon}} + \frac{g'(y_0)}{1-\mu}\right) - \left((\sigma_c - g(y_0))\frac{\lambda_1^+(v)}{\sqrt{\epsilon}} + g'(y_0)\right). \tag{2.3.3}$$

An explicit formula for $E(v)$ is obtained by substituting for λ_1^+ and λ_2^- from (2.2.4) and (2.2.5) respectively,

$$E(v) = \frac{2\sqrt{\epsilon}\mu g'(y_0)}{(1-\mu)(\sigma_c - g(y_0))} + \frac{\frac{g(y_0)}{1-\mu} - \sigma_c}{\sigma_c - g(y_0)}\left[v + \sqrt{v^2 + 4(1-\mu)} - (-v + \sqrt{v^2+4})\right]. \tag{2.3.4}$$

Factoring $-(v + \sqrt{v^2 + 4(1-\mu)})$ from (2.3.4) we obtain the following form of $E(v)$

$$E(v) = -(v + \sqrt{v^2 + 4(1-\mu)})E_1(v). \tag{2.3.5}$$

where $E_1(v)$ has the form

$$E_1(v) = \frac{-v+\sqrt{v^2+4}}{v+\sqrt{v^2+4(1-\mu)}} - \frac{\frac{g(y_0)}{1-\mu}-\sigma_c}{\sigma_c - g(y_0)} - \frac{2\sqrt{c}g'(y_0)\mu}{(1-\mu)(v+\sqrt{v^2+4(1-\mu)})(\sigma_c-g(y_0))}. \tag{2.3.6}$$

Since $v+\sqrt{v^2+4(1-\mu)}$ has no real zeros for $v>0$, the zeros of $E(v)$ are given by those of $E_1(v)$. A simple calculation shows that $E_1'(v)<0$ and hence $E_1(v)$ is a strictly decreasing function of v. As $v\to\infty$ the first and third term on the right hand side of (2.3.6) tends to zero, thus $\lim_{v\to\infty} E_1(v) = -\frac{\frac{g(y_0)}{1-\mu}-\sigma_c}{\sigma_c-g(y_0)} < 0$. Similarly as $v\to-\infty$ the first term on the right hand side of (2.3.6) tends to ∞ and $\lim_{v\to-\infty} E_1(v)=\infty$. The intermediate value theorem and monotonicity of E_1 provide a unique solution to $E(v)=0$.

To derive the asymptotic states of ϕ we refer to its form (2.3.1). We take the limit $y\to-\infty$ of the left branch of ϕ. Since $\lambda_1^+>0$, the second term in the formula vanishes and we obtain

$$\lim_{y\to-\infty}\phi = \lim_{y\to-\infty} g(y) = g_-.$$

Simlarly, for the right branch

$$\lim_{y\to\infty}\phi = \lim_{y\to\infty}\frac{g(y)}{1-\mu} = \frac{g_+}{1-\mu},$$

since $\lambda_2^-<0$. ■

The composite solution ϕ does not generate an exact solution of (2.0.1). In the following lemma we compute the residual error, $F(\phi)$, of the composite solution $\phi(y;y_0)$ of (2.0.1), constructed in (2.3.1). The outer solution ϕ_0 is smooth except at y_0. We introduce $\left[\frac{\partial^2\phi_0}{\partial y^2}\right]$ to denote the second order piecewise derivative of ϕ_0.

$$\left[\frac{\partial^2\phi_0}{\partial y^2}\right] = \begin{cases} g_{yy} & y<y_0 \\ \frac{g_{yy}}{1-\mu} & y>y_0. \end{cases} \tag{2.3.7}$$

Lemma 2.3.2 *The residual $F(\phi)$ is given by*

$$F(\phi) = -v\sqrt{\epsilon}\frac{\partial\phi_i}{\partial y} + \epsilon\left[\frac{\partial^2\phi_0}{\partial y^2}\right], \tag{2.3.8}$$

*where v is given by **Lemma 1.1**.*

Proof: Substituting the composite solution ϕ from (2.3.1) into the expression of F from (2.0.1) we obtain

$$F(\phi) = \epsilon\phi_{yy} - (\phi - g(y)) + \mu\sigma(\phi).$$

Referring to the definition of σ from (2.0.2)

$$F(\phi) = \begin{cases} \epsilon\phi_{yy} - (\phi - g(y)) & y < y_0 \\ \epsilon\phi_{yy} - (\phi - g(y)) + \mu\phi & y > y_0. \end{cases} \tag{2.3.9}$$

From the formula for the outer solution ϕ_0 and the inner solution ϕ_i given by (2.1.3) and (2.2.6) ϕ can be expressed as

$$\phi = \phi_i - \phi_0 \tag{2.3.10}$$

We substitute this expression for ϕ into (2.3.9), yielding

$$F(\phi) = \epsilon\left[\frac{\partial^2\phi_0}{\partial y^2}\right] + \begin{cases} \epsilon\phi_i'' - (\phi_i - \phi_0 - g(y)) & y < y_0 \\ \epsilon\phi_i'' - (1-\mu)\phi_i - (1-\mu)\phi_0 - g(y) & y > y_0. \end{cases} \tag{2.3.11}$$

From the formula (2.1.3) for ϕ_0 and the (2.2.3) for ϕ_i we obtain

$$F(\phi) = -v\sqrt{\epsilon}\frac{\partial\phi_i}{\partial y} + \epsilon\frac{\partial^2\phi_0}{\partial y^2}.$$

■

CHAPTER 3

The Rescaled Problem and its Linearization

In this section we introduce the rescaled coordinates which form the backbone of the RG methodology developed of **Section 4**. We also address the Frechet differentiability of the nonlinearity at the composite solution and identify the point and essential spectrum of the linearized operator.

3.1 The Co-moving coordinate system

At a time t_0, we choose a fixed co-ordinate system based upon front position $\bar{y_0}$. We shift to a co-moving frame with fixed velocity $\sqrt{\epsilon}v(\bar{y_0})$, for which the composite solution ϕ is stationary at leading order. We scale the spatial variable, removing ϵ from the equation, hiding it in the velocity and the function g, which are now slowly varying. We transform the space coordinate to z where at time $t = t_0$, $y = \bar{y_0}$ is mapped to $z = \bar{z}_0 = 0$

$$z = \frac{y - \bar{y}_0 - \sqrt{\epsilon}v(\bar{y_0})(t - t_0)}{\sqrt{\epsilon}}, \tag{3.1.1}$$

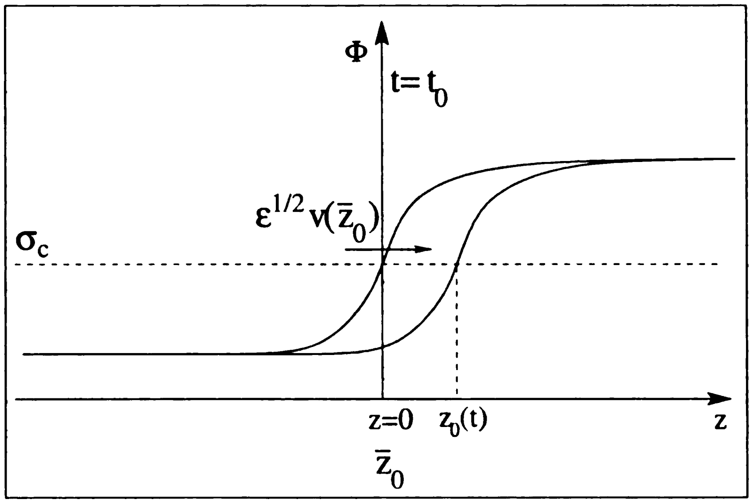

Figure 3.1. The coordinate system after each renormalization.

as illustrated in **Fig(3.1)**. After the transformation (3.1.1), the front location $z_0(t)$ at time t takes the form

$$z_0(t) = \frac{y_0(t) - \bar{y}_0 - \sqrt{\epsilon}v(\bar{y}_0)(t - t_0)}{\sqrt{\epsilon}}. \tag{3.1.2}$$

Throughout, y always refers to the spatial coordinate in the lab frame and z refers to a co-moving frame with a fixed speed $\sqrt{\epsilon}v(\bar{y}_0)$. We adapt the convention that an overbar denotes a quantity that is fixed in time. We denote $\bar{z}_0 \equiv 0$, $\bar{v} \equiv v(\bar{z}_0)$ and $\bar{\phi} \equiv \phi(z; \bar{z}_0)$ while $'$ denotes derivative w.r.t. z. Under the transformation (3.1.1) after rescaling, the pde (2.0.1) inherits a drift term and F takes the form

$$U_t = F(U) = U_{zz} + v_s U_z - (U - g_s(z)) + \mu\sigma(U) + \epsilon^{3/4-\gamma_s}\eta_s(z,t). \tag{3.1.3}$$

$v_s(z_0)$ and $g_s(z)$ denote scaled velocity v and background g

$$\begin{aligned} v_s(z_0) &= v(\sqrt{\epsilon}z_0 + \bar{y}_0 + v(\bar{y}_0)(t - t_0)) & (3.1.4) \\ g_s(z) &= g(\sqrt{\epsilon}z + \bar{y}_0 + v(\bar{y}_0)(t - t_0)). & (3.1.5) \end{aligned}$$

Indeed, both v_s and g_s are slowly varying functions of the front location z_0 and the spatial variable z respectively, i.e.

$$\begin{aligned}\frac{\partial v_s(z_0)}{z_0} &= \sqrt{\epsilon}\frac{\partial v(y)}{\partial z_0} = \mathcal{O}(\sqrt{\epsilon}), &(3.1.6)\\ g_s'(z) &= \sqrt{\epsilon}\frac{dg}{dy} = \mathcal{O}(\sqrt{\epsilon}). &(3.1.7)\end{aligned}$$

The noise term satisfies, $\|\eta_s(t)\|_{H^{\gamma n}} = 1$, with the corresponding reduction in its ϵ coefficient. The shift to the co-moving frame eliminates ϵ from the diffusive term. The inner solution, ϕ_i, (2.2.6) and the composite solution, ϕ, (2.3.1) are expressed in terms of the new variable z as

$$\phi_i(z) = \begin{cases} g_s(z_0) + (\sigma_c - g_s(z_0))e^{\lambda_1^+(z-z_0)} & z < z_0 \\ \frac{g_s(z_0)}{1-\mu} + (\sigma_c - \frac{g_s(z_0)}{1-\mu})e^{\lambda_2^-(z-z_0)} & z > z_0. \end{cases} \tag{3.1.8}$$

and

$$\phi(z; z_0) = \begin{cases} g_s(z) + (\sigma_c - g_s(z_0))e^{\lambda_1^+(z-z_0)} & z < z_0 \\ \frac{g_s(z)}{1-\mu} + (\sigma_c - \frac{g_s(z_0)}{1-\mu})e^{\lambda_2^-(z-z_0)} & z > z_0. \end{cases} \tag{3.1.9}$$

We show that $\frac{\partial \phi_i}{\partial z}$ and $\frac{\partial \phi}{\partial z_0}$ agree to leading order in the L^2 and L^∞ norms. From the explicit formula of the inner solution ϕ_i given by (3.1.8) and composite solution ϕ given by (3.1.9), the following relation holds

$$\frac{\partial \phi}{\partial z_0} = -\frac{\partial \phi_i}{\partial z} + r(z), \tag{3.1.10}$$

where the remainder r is given by

$$r(z) = \begin{cases} e^{\lambda_1^+(z-z_0)}\left(-\frac{\partial g_s(z;z_0)}{\partial z_0} + (\sigma_c - g_s(z_0))(z - z_0)\frac{\partial \lambda_1^+}{\partial z_0}\right) & z < z_0 \\ e^{\lambda_2^-(z-z_0)}\left(-\frac{1}{1-\mu}\frac{\partial g_s(z;z_0)}{\partial z_0} + (\sigma_c - \frac{g_s(z_0)}{1-\mu})(z - z_0)\frac{\partial \lambda_1^+}{\partial z_0}\right) & z > z_0. \end{cases} \tag{3.1.11}$$

Since the rescaled velocity $v_s(z_0)$ varies slowly with the front location z_0, see to (3.1.6), the eigenvalues vary slowly with z_0

$$\frac{\partial \lambda_1^+(v_s)}{\partial z_0} = \frac{\partial \lambda_1^+(v_s(z_0))}{\partial v_s}\frac{\partial v_s(z_0)}{\partial z_0} = O(\sqrt{\epsilon}). \tag{3.1.12}$$

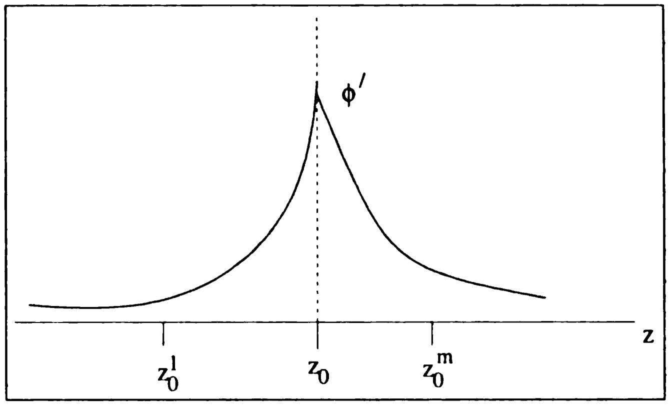

Figure 3.2. The function $\phi'(z; z_0)$.

and similarly

$$\frac{\partial \lambda_2^-(v_s(z_0))}{\partial z_0} = O(\sqrt{\epsilon}). \tag{3.1.13}$$

Combining (3.1.7), (3.1.12) and (3.1.13) we obtain the following estimate on r

$$\|r(z)\|_{L^2} + \|r(z)\|_{L^\infty} \leq c\sqrt{\epsilon}. \tag{3.1.14}$$

As proved in **Lemma 2.1**, the rescaled composite solution is monotone increasing in the front regime and C^1 as, $[\![\phi']\!]_{z_0}$ satisfies

$$[\![\phi']\!]_{z_0} = \sqrt{\epsilon}[\![\phi']\!]_{y_0} = 0, \tag{3.1.15}$$

and the relation (3.1.15) reduces to

$$\phi'(z_0; z_0) = \sqrt{\epsilon}\frac{dg(y_0)}{dy} + (\sigma_c - g_s(z_0))\lambda_1^+ = \frac{\sqrt{\epsilon}}{1-\mu}\frac{dg(y_0)}{dy} + (\sigma_c - \frac{g_s(z_0)}{1-\mu})\lambda_2^-. \tag{3.1.16}$$

In the following lemma we obtain a lower bound on the derivative ϕ', **Fig(3.1),** on a bounded interval containing the front location z_0. This property of ϕ' is used in proving future results.

Lemma 3.1.1 *For each bounded interval* $\left(z_0^l, z_0^r\right)$ *containing* z_0*, there exists* $\alpha_0 > 0$*, independent of* ϵ*, but depending on the length of the interval* $\Delta = \left(z_0^r - z_0^l\right)$ *such that* *the derivative of the composite solution $\phi'_z(z; z_0)$ satisfies,*

$$\phi'(z; z_0) > \alpha_0 > 0, \tag{3.1.17}$$

for all $z \in \left(z_0^l, z_0^r\right)$.

Proof: Differentiating ϕ given by (3.1.9) we obtain

$$\phi'(z; z_0) = \begin{cases} \sqrt{\epsilon} g'(z) + \lambda_1^+(\sigma_c - g_s(z_0))e^{\lambda_1^+(z-z_0)} & z < z_0, \\ \frac{\sqrt{\epsilon} g'(z)}{1-\mu} + \lambda_2^-(\sigma_c - \frac{g_s(z_0)}{1-\mu})e^{\lambda_2^-(z-z_0)} & z > z_0. \end{cases} \tag{3.1.18}$$

Since g is increasing we have

$$\phi'(z; z_0) \geq \begin{cases} \lambda_1^+(\sigma_c - g_s(z_0))e^{\lambda_1^+(z-z_0)} & z < z_0, \\ \lambda_2^-(\sigma_c - \frac{g_s(z_0)}{1-\mu})e^{\lambda_2^-(z-z_0)} & z > z_0, \end{cases} \tag{3.1.19}$$

Since $|z - z_0| \leq \Delta$, $\phi'(z)$ can be bounded below as

$$\phi'(z; z_0) \geq \begin{cases} \lambda_1^+(\sigma_c - g_s(z_0))e^{-\lambda_1^+\Delta} & z < z_0, \\ \lambda_2^-(\sigma_c - \frac{g_s(z_0)}{1-\mu})e^{\lambda_2^-\Delta} & z > z_0, \end{cases} \tag{3.1.20}$$

Choosing α_0 as

$$\alpha_0 = \min\left(\lambda_1^+(\sigma_c - g_s(z_0))e^{-\lambda_1^+\Delta}, \lambda_2^-(\sigma_c - \frac{g_s(z_0)}{1-\mu})e^{\lambda_2^-\Delta}\right).$$

we obtain (3.1.17), where $\alpha_0 > 0$ is independent of ϵ. ■

In the rescaled coordinates the piecewise defined second derivative of the outer solution ϕ_0 becomes

$$\left[\frac{\partial^2 \phi_0}{\partial z^2}\right] = \begin{cases} \partial_z^2 g_s & z < z_0 \\ \frac{\partial_z^2 g_s}{1-\mu} & z > z_0. \end{cases} \tag{3.1.21}$$

In the following lemma, we bound the residual in the new coordinate system.

Lemma 3.1.2 *The residual $R = F(\phi)$ in the convected frame is given by*

$$R = -(v_s(z_0) - v_s(\bar{z}_0))\frac{\partial \phi_i}{\partial z} + \epsilon\left[\frac{\partial^2 \phi_0}{\partial z^2}\right]. \tag{3.1.22}$$

Additionally we have the following bound

$$\|R\|_{L^2} + \|R\|_{L^\infty} \leq C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + \epsilon\right). \tag{3.1.23}$$

Proof: In the coordinate system given by (3.1.9), ϕ given by (2.3.10), when substituted in the revised form of F given by (3.1.3) yields

$$F(\phi) = v_s(\bar{z}_0)\phi_i' + \left[\frac{\partial^2 \phi_0}{\partial z^2}\right] + \begin{cases} \phi_i'' - (\phi_i - \phi_0 - g_s(y)) & z < z_0 \\ \phi_i'' - (1-\mu)\phi_i - (1-\mu)\phi_0 - g_s(y) & z > z_0. \end{cases} \tag{3.1.24}$$

Since ϕ_i satisfies (3.1.8), we obtain the following expression for the residual

$$R = -(v_s(z_0) - v_s(\bar{z}_0))\frac{\partial \phi_i}{\partial z} + \left[\frac{\partial^2 \phi_0}{\partial z^2}\right].$$

The rescaled velocity is a smooth function of the front location and changes at a $O(\sqrt{\epsilon})$ rate. The mean value theorem yields the estimate,

$$|v_s(z_0) - v_s(\bar{z}_0)| \leq C\sqrt{\epsilon}\,|z_0 - \bar{z}_0|. \tag{3.1.25}$$

From the definition of g_s in (3.1.5), we see, $\|g_s''\|_{L^\infty} = \mathcal{O}(\epsilon)$ which from (3.1.21) implies, $\left\|\left[\frac{\partial^2\phi_0}{\partial z^2}\right]\right\|_{L^\infty} = \mathcal{O}(\epsilon)$. Since $\left\|\frac{\partial\phi_i}{\partial z}\right\|_{L^\infty} = \mathcal{O}(1)$, substituting these estimates in (3.1.22) we acheive a bound for $\|R\|_{L^\infty}$

$$\|R\|_{L^\infty} \leq C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + \epsilon\right).$$

Since $\frac{\partial\phi_i}{\partial z}$ and $\left[\frac{\partial^2\phi_0}{\partial z^2}\right]$ decay at an $\mathcal{O}(1)$ rate at infinity, we obtain similar bounds for $\|R\|_{L^2}$. ∎

3.2 The decomposition of the solution U and the Frechet differentiability of F.

We decompose the solution $U(z,t)$ of our model equation (2.0.1) in a neighbourhood of the composite solution ϕ as

$$U(z,t) = \phi(z; z_0(t)) + W(z,t), \tag{3.2.1}$$

where $\|W\|_{H^\gamma} \ll 1$ and W lies in a tube of width ϖ. The width φ of the tube is prescribed by

$$\varpi = \frac{1}{2}\min\left(\sigma_c - g^-, \frac{g^+}{1-\mu} - \sigma_c\right), \tag{3.2.2}$$

where $\frac{g^+}{1-\mu}$ and g^- are the asymptotic states of $\bar{\phi}$. One of the fundamental results of this thesis is that the nonlinearity F is **Frechet differentiable** as a map from $H^\gamma(\mathbf{R}) \to H^{-\beta}(\mathbf{R})$, at the composite solution $\bar{\phi}$ and that the evolution for the solution U can be written in terms of the decomposed variables z_0 and W as,

$$W_t + \frac{\partial \phi}{\partial z_0} z_0' = F(\phi) + L_{\bar{\phi}} W + (L_\phi - L_{\bar{\phi}})W + \mathcal{N}(W). \tag{3.2.3}$$

Here $L_{\bar{\phi}}$ is the Frechet derivative of $F : H^\gamma \to H^{-\beta}$ at $\bar{\phi} = \phi(z; \bar{z}_0)$, and the nonlinear operator $\mathcal{N}$ is given by,

$$\mathcal{N} = F(\bar{\phi} + W) - F(\bar{\phi}) - L_{\bar{\phi}} W. \tag{3.2.4}$$

A first step in this direction is to show that functions in H^γ, for $\gamma \in \left(\frac{1}{2}, \frac{3}{2}\right]$, are infact Hölder continuous.

Lemma 3.2.1 *For each $\gamma \in (1/2, 3/2]$, there exists $C > 0$ independent of x,y, for any $\varphi \in H^\gamma$*

$$|\varphi(y) - \varphi(x)| \leq C\,|y-x|^{\gamma - 1/2}\,\|\varphi\|_{H^\gamma}. \tag{3.2.5}$$

In particular, the L^∞ norm is controlled by the H^γ norm,

$$\|\varphi\|_{L^\infty} \leq C\,\|\varphi\|_{H^\gamma}. \tag{3.2.6}$$

Proof: The Fourier transform representation of φ

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ikx} \hat{\varphi}(k)\,dk, \tag{3.2.7}$$

implies that

$$
\begin{aligned}
|\varphi(y)-\varphi(x)| &\le \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \left|e^{iyk}-e^{ixk}\right| |\hat{\varphi}(k)|\, dk \\
&\le \frac{1}{\sqrt{2\pi}} \int_{k\le\frac{1}{2|y-x|}} \left|e^{iyk}-e^{ixk}\right| |\hat{\varphi}(k)|\, dk \\
&+ \frac{1}{\sqrt{2\pi}} \int_{k\ge\frac{1}{2|y-x|}} \left|e^{iyk}-e^{ixk}\right| |\hat{\varphi}(k)|\, dk \qquad (3.2.8)
\end{aligned}
$$

$$(3.2.9)$$

Applying the mean value theorem on e^{ikz} on the interval (x, y) we obtain the following bound

$$
\left|e^{iyk}-e^{ixk}\right| \le \begin{cases} |k|\,|y-x| & \text{for } |k|\,|y-x|<1 \\ 2 & \text{for } |k|\,|y-x|>1. \end{cases} \qquad (3.2.10)
$$

With the estimate in hand, the two integrals on the right side of (3.2.8) reduce to

$$
\begin{aligned}
|\varphi(y)-\varphi(x)| &\le \frac{1}{\sqrt{2\pi}} \int_{k\le\frac{1}{2|y-x|}} |k|\,|y-x|\,|\hat{\varphi}(k)|\, dk + \frac{1}{\sqrt{2\pi}} \int_{k\ge\frac{1}{2|y-x|}} |\hat{\varphi}(k)|\, dk, \\
&\le \frac{1}{\sqrt{2\pi}} |y-x| \int_{k\le\frac{1}{2|y-x|}} |k|\,(1+|k|^2)^{-\gamma/2} \left((1+|k|^2)^{\gamma/2}\,|\hat{\varphi}(k)|\right) dk \\
&+ \frac{1}{\sqrt{2\pi}} \int_{k\ge\frac{1}{2|y-x|}} (1+|k|^2)^{-\gamma/2} \left((1+|k|^2)^{\gamma/2}\,|\hat{\varphi}(k)|\right) dk. \qquad (3.2.11)
\end{aligned}
$$

Applying Hölders inequality to the two integrals in (3.2.11) we obtain

$$\begin{aligned}
|\varphi(y)-\varphi(x)| &\leq \frac{1}{\sqrt{2\pi}}|y-x|\,\|\varphi\|_\gamma \left(\int\limits_{k\leq \frac{1}{2|y-x|}} \frac{|k|^2}{(1+|k|^2)^\gamma} dk \right)^{1/2} \\
&+ \frac{1}{\sqrt{2\pi}}\|\varphi\|_\gamma \left(\int\limits_{k\geq \frac{1}{2|y-x|}} (1+|k|^2)^{-\gamma} dk \right)^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi}}|y-x|\,\|\varphi\|_\gamma \left(\int\limits_{k\leq \frac{1}{2|y-x|}} |k|^{2-2\gamma}\, dk \right)^{1/2} \\
&+ \frac{1}{\sqrt{2\pi}}\|\varphi\|_\gamma \left(\int\limits_{k\geq \frac{1}{2|y-x|}} |k|^{-2\gamma}\, dk \right)^{1/2} \qquad (3.2.12)
\end{aligned}$$

Performing the integrations in (3.2.12)

$$\begin{aligned}
|\varphi(y)-\varphi(x)| &\leq c\,\|\varphi\|_\gamma\,|y-x|\,\frac{1}{|y-x|^{(3-2\gamma)/2}} + c\,\|\varphi\|_\gamma\,|y-x|^{(2\gamma-1)/2}, \\
&\leq c\,|y-x|^{\gamma-1/2}\,\|\varphi\|_\gamma .
\end{aligned}$$

The L^∞ estimates follows from a similar estimate applied directly to (3.2.7). ■

With **Lemma 3.2.1** in hand, we show that $F: H^\gamma \to H^{-\beta}$ is Frechet differentiable for $\gamma \in (1/2, 3/2]$ and $\beta > 1/2$. For notational convenience, we introduce the tensor product

$$(f \otimes g)W = \langle W, g\rangle f, \qquad (3.2.13)$$

where $f \otimes g$ is a rank-one operator with range f. In particular, when f and g are delta functions centered at $\bar{z}_0$

$$(\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0})W \equiv \langle W, \delta_{\bar{z}_0}\rangle \delta_{\bar{z}_0} = W(\bar{z}_0)\delta_{\bar{z}_0}. \qquad (3.2.14)$$

A key step in proving the Frechet differentiability of the linearized oprator $L_{\bar{\phi}}$, is the choice of the Sobolev spaces H^{γ} and $H^{-\beta}$. The gap $\gamma+\beta$ must be large enough so that the non-smoothness in σ can be accomodated. However if $\gamma+\beta$ is too large the semigroup associated to the linear operator $L_{\bar{\phi}}$, as a map from $H^{-\beta} \to H^{\gamma}$, will not be integrable in t as $t \to 0^{+}$. Only the nonlinear ignition functional σ from F contributes to the nonlinearity. The primary challenge in proving the Frechet differentiability of F at $\bar{\phi}$ is to control the impact of the perturbation W on $\sigma(\bar{\phi})$, in particular the nucleation of new ignition sets on which $\bar{\phi}+\sigma > 0$ but $\bar{\phi} < \sigma_c$. We show that the primary impact of $W \in H^{\gamma}$ is to move the location of the ignition point, which is a linear effect defined by the rank-one operator. The spreading of the ignition point into an ignition set is a smaller, non-linear effect, which can be ignored at leading order.

Lemma 3.2.2 *Fix $\gamma \in \left(\frac{1}{2}, 1\right]$. The Frechet derivative of F, given by (3.1.3), as a map from H^{γ} to $H^{-\beta}$, at the composite solution $\bar{\phi}$, is given by,*

$$L_{\bar{\phi}} = \partial_z^2 + v_s\partial_z - I + \mu\chi_{[\bar{z}_0,\infty]} + \mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0;\bar{z}_0)}\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0}. \tag{3.2.15}$$

Moreover for $W \in H^{\gamma}$, the nonlinearity $\mathcal{N}(W)$ satisfies:

$$\|\mathcal{N}(W)\|_{H^{-\beta}} \le c\left(\|W\|_{H^{\gamma}}^{\beta+1/2} + \|W\|_{H^{\gamma}}^{\gamma+1/2}\right). \tag{3.2.16}$$

Proof : Writing $\mathcal{N}(W)$ explicitly using the form of F given by (3.1.3), and $L_{\bar{\phi}}$ as given by (3.2.15), we obtain the following expression

$$\begin{aligned} \mathcal{N}(W) &\equiv F(\bar{\phi}+W) - F(\bar{\phi}) - L_{\bar{\phi}}W \\ &= \mu\left(\sigma(\bar{\phi}+W) - \sigma(\bar{\phi})\right) - \mu\left(\chi_{[\bar{z}_0,\infty)}W + \frac{\sigma_c}{\bar{\phi}'(\bar{z}_0;\bar{z}_0)}W(\bar{z}_0)\delta_{\bar{z}_0}\right) \end{aligned} \tag{3.2.17}$$

To begin the study of the nonlinearity, we investigate the non-smooth term $\sigma(\bar{\phi}+W)$. We define, the set E of "ignition points" of the perturbation $\bar{\phi}+W$, to be the set of zeros of $\bar{\phi}+W-\sigma_c$,

$$E = \left\{z \middle| \bar{\phi}(z) + W(z) = \sigma_c\right\}. \tag{3.2.18}$$

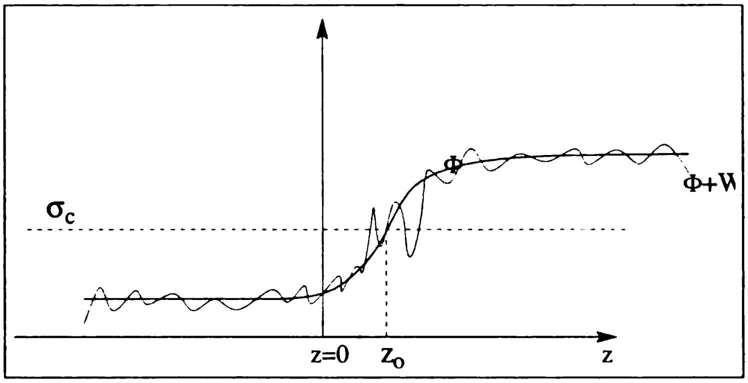

Figure 3.3. The graph of $\phi + W$.

We note that the set E is contained in the interval $(\bar{z}_0 - \varpi, \bar{z}_0 + \varpi)$,

$$|E| \leq 2\varpi. \tag{3.2.19}$$

Let $z \in E$ be an arbitrary ignition point. Since $\bar{\phi}$ attains σ_c at $\bar{z}_0$, the following relation holds

$$\bar{\phi}(z) + W(z) = \sigma_c = \bar{\phi}(\bar{z}_0). \tag{3.2.20}$$

Applying the Mean Value theorem to $\bar{\phi}$ on the interval (z, z_0) (w.l.o.g. $z < \bar{z}_0$), where $\bar{\phi}$ is $\mathcal{C}^1$ and monotone increasing, we obtain

$$W(z) = \bar{\phi}(\bar{z}_0) - \bar{\phi}(z) = (\bar{z}_0 - z)\,\bar{\phi}'(\xi_z), \tag{3.2.21}$$

where $\xi_z \in (z, \bar{z}_0)$. Solving the relation (3.2.21) for z we obtain the expression

$$z = \bar{z}_0 - \frac{W(z)}{\bar{\phi}'(\xi)}, \tag{3.2.22}$$

valid for each $z \in E$. We denote by $l(z)$ the distance of an arbitrary ignition point, $z \in E$ from the froxen front location $\bar{z}_0$, i.e.

$$l(z) \equiv |\bar{z}_0 - z| = \left|\frac{W(z)}{\bar{\phi}'(\xi_z)}\right|. \tag{3.2.23}$$

where $\xi_z \in (z, \bar{z}_0)$. From **Lemma 3.1.1** we know that

$$\min_{(z,\bar{z}_0)} \bar{\phi}'(z) > \min_{z\in E} \bar{\phi}'(z) \geq \alpha_0 > 0.$$

where α_0 depends on the length of set E and from (3.2.19), α_0 is independent of z. Thus, there exists $C > 0$ independent of z and ϵ such that

$$|l(z)| = |z - \bar{z}_0| \leq C \|W\|_{H^\gamma}. \tag{3.2.24}$$

The ignition points of the perturbation $\bar{\phi} + W$ are close to the ignition point $\bar{z}_0$ of $\bar{\phi}$ for $\|W\|_{H^\gamma} \ll 1$. The elements of E are localized as illustrated in **Figure3.3**. This implies the existence of a maximum and minimum element of E which we denote by z_l and z_r, i.e., $z_l < z < z_r$ for $z \in E$. Since E is closed, z_l and z_r are zeros of $\bar{\phi} + W - \sigma_c$,

$$\bar{\phi}(z_l) + W(z_l) = \sigma_c = \bar{\phi}(z_r) + W(z_r). \tag{3.2.25}$$

Applying the inequality (3.2.5) on Hölder continuity of $W \in H^\gamma$ from **Lemma 3.2.1**, with $x = z_l$ and $y = z_r$, we bound the difference in the values of $\bar{\phi}$ on (z_l, z_r)

$$\left|\bar{\phi}(z_l) - \bar{\phi}(z_r)\right| = |W(z_r) - W(z_l)| \leq C\, |z_l - z_r|^{\gamma - 1/2} \|W\|_{H^\gamma}. \tag{3.2.26}$$

Since $E \subseteq (\bar{z}_0 - \varpi, \bar{z}_0 + \varpi)$, we have $|z_l - z_r| \leq \varpi$. Then from **Lemma 3.1.1** there exists $\alpha_0 > 0$ independent of ϵ, z_l and z_r , such that $\phi'(z) > \alpha_0$ for $z \in (z_l, z_r)$. We apply the Mean Value theorem to $\bar{\phi}$ on $[z_l, z_r]$ to obtain

$$\left|\bar{\phi}(z_l) - \bar{\phi}(z_r)\right| = |z_l - z_r| \left|\bar{\phi}'(\xi)\right| \geq \alpha_0\, |z_l - z_r|. \tag{3.2.27}$$

The relations (3.2.26) and (3.2.27) together implies

$$\alpha_0\, |z_l - z_r| \leq |z_l - z_r|^{\gamma - 1/2} \|W\|_{H^\gamma}.$$

This gives us an estimate of the length of the interval containing E

$$|z_l - z_r| \leq C \|W\|_\gamma^{\frac{1}{3/2 - \gamma}}. \tag{3.2.28}$$

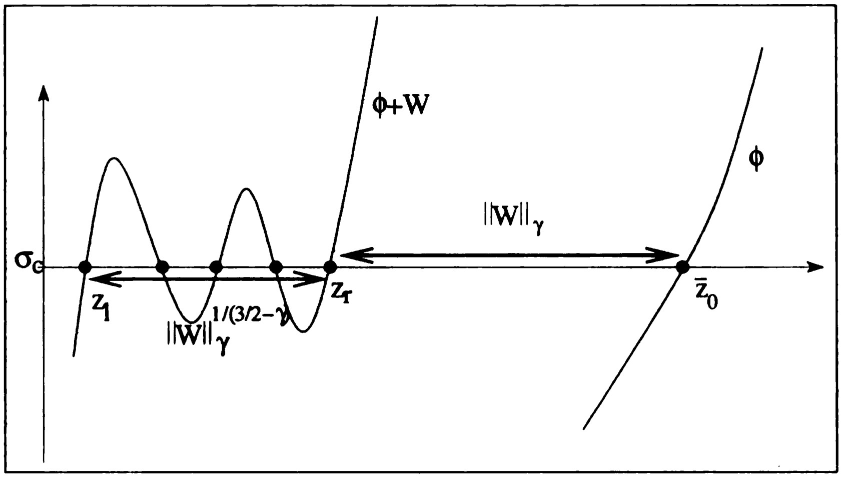

Figure 3.4. Measure of set E smaller that distance of E from $\bar{z}_0$.

The exponent on $\|W\|_{H^\gamma}$ varies from 1 at $\gamma = \frac{1}{2}$ to 2 at $\gamma = 1$, so that the set E shrinks super-linearly as $\|W\|_{H^\gamma} \to 0$. But for $\gamma \in (1/2, 1)$, $\frac{1}{3/2-\gamma} > \gamma + \frac{1}{2}$, and we simplify the inequaliy (3.2.28) to

$$|z_l - z_r| \leq C \, \|W\|_\gamma^{\gamma+1/2} . \tag{3.2.29}$$

However, since z_l and z_r belong to E, from relation (3.2.24), the distance of z_l, z_r from $\bar{z}_0$ is bounded above by $\|W\|_{H^\gamma}$

$$|\bar{z}_0 - z_l| \leq c \, \|W\|_{H^\gamma} , \tag{3.2.30}$$

$$|\bar{z}_0 - z_r| \leq c \, \|W\|_{H^\gamma} . \tag{3.2.31}$$

For $\|W\|_{H^\gamma}$ small, the length of the smallest interval containing E is smaller than the distance of E from the unperturbed front location $\bar{z}_0$, **Fig(3.2)**. **This** key observation, yields the form of the Frechet differential $F : H^\gamma \to H^{-\beta}$ at $\bar{\phi}$.

To study the ignition set E closely, we define the E_-, the set of "false negatives" and E_+, the set of "false positives"

$$E_- = \{z | \phi + W \leq \sigma_c\} \bigcap \{\bar{z}_0 \leq z \leq z_r\} , \tag{3.2.32}$$

$$E_+ = \{z | \phi + W \geq \sigma_c\} \bigcap \{z_l \leq z \leq \bar{z}_0\} . \tag{3.2.33}$$

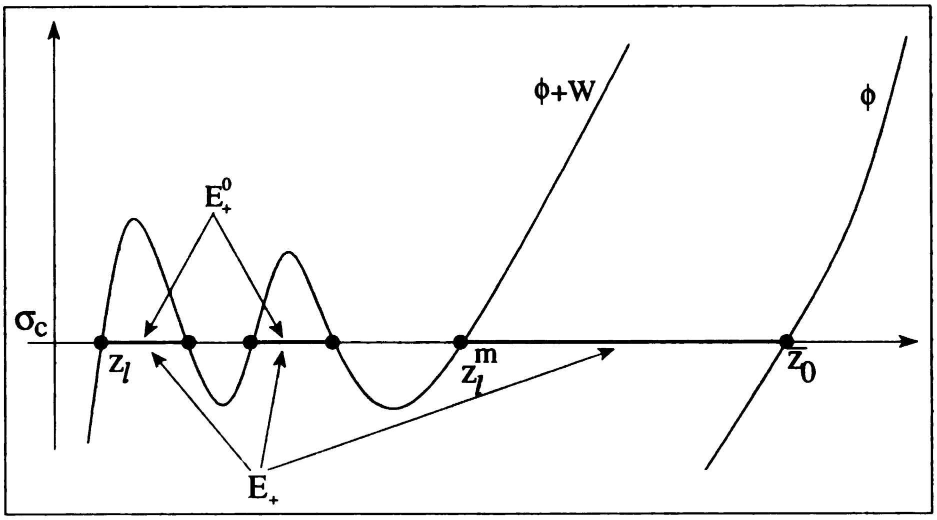

Figure 3.5. The graph of $\bar{\phi} + W$ in the neighbourhood of $\bar{z}_0$ where E_- is empty.

We remark that $z \in E_-$ implies $\phi(z) + W(z) \leq \sigma_c$ but $\phi(z) \geq \sigma_c$, while $z \in E_+$ implies $\phi(z) + W(z) \geq \sigma_c$ but $\phi(z) \leq \sigma_c$. In particular, we may decompose $\sigma(\bar{\phi} + W)$ in terms of E_+ and E_- as follows

$$\sigma(\bar{\phi} + W) = \left(\chi_{E_+} + \chi_{[\bar{z}_0,\infty)} - \chi_{E_-}\right)(\bar{\phi} + W). \tag{3.2.34}$$

The $H^{-\beta}$ norm of $\mathcal{N}(W)$ is formally defined as

$$\|\mathcal{N}(W)\|_{H^{-\beta}} = \sup_{v \in H^\beta} \frac{|\langle \mathcal{N}(W), v\rangle|}{\|v\|_{H^\beta}}. \tag{3.2.35}$$

We study three cases, E is to the left of $\bar{z}_0$, E to the right of $\bar{z}_0$, and E straddles $\bar{z}_0$.

Case-I: E_- **is empty**.

As depicted in **Fig(3.2)**, we denote by $z_l^m \equiv \max_{z \in E} z$, the maximum element of E, which lies to the left of $\bar{z}_0$. So, $\bar{\phi} + W - \sigma_c > 0$ on (z_l^m, ∞). We partition E_+ as

$$E_+ = E_+^0 \bigcup [z_l^m, \bar{z}_0], \tag{3.2.36}$$

where E_+^0 contains the intervals on the left of z_l^m, for which $\bar{\phi} + W - \sigma_c > 0$. Generically the interval $[z_l^m, z_0]$ is the largest share of E_+. In this case the decomposition

(3.2.34) of $\sigma(\bar{\phi}+W)$ reduces to

$$\sigma(\bar{\phi}+W) = (\bar{\phi}+W)(\chi_{E_+^0} + \chi_{[z_l^m,\infty)}). \tag{3.2.37}$$

Substituting the decomposition (3.2.37) into the equation (3.2.17), the nonlinearity takes the following form

$$\begin{aligned} \mathcal{N}(W) &= \mu\left(\left(\chi_{E_+^0} + \chi_{[z_l^m,\infty)}\right)(\bar{\phi}+W) - \bar{\phi}\chi_{[\bar{z}_0,\infty)}\right) \\ &- \mu\left(\chi_{[\bar{z}_0,\infty)}W + \frac{\bar{\phi}(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}W(\bar{z}_0)\delta_{\bar{z}_0}\right), \\ &= \mu\left(\chi_{E_+^0} + \chi_{[z_l^m,\bar{z}_0]}\right)(\bar{\phi}+W) - \mu\frac{\bar{\phi}(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}W(\bar{z}_0)\delta_{\bar{z}_0}. \end{aligned} \tag{3.2.38}$$

For notational convenience we introduce the length scale l_0

$$l_0 = \frac{W(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}, \tag{3.2.39}$$

which we will compare to $\left|z_l^m - \bar{z}_0\right|$, the dominant part of E_+. The nonlinearity $\mathcal{N}(W)$ from (3.2.38) can be rewritten in terms of l_0 as

$$\mathcal{N}(W) = \mu\left(\chi_{E_+^0} + \chi_{[z_l^m,\bar{z}_0]}\right)(\bar{\phi}+W) - \mu l_0\bar{\phi}(\bar{z}_0)\delta_{\bar{z}_0}. \tag{3.2.40}$$

Substituting the expression for $\mathcal{N}(W)$ from (3.2.40) into the dual pairing we find

$$\begin{aligned} |\langle\mathcal{N}(W),v\rangle| &= \mu\left|\left\langle(\bar{\phi}+W)\left(\chi_{E_+^0} + \chi_{[z_l^m,\bar{z}_0]}\right) - l_0\bar{\phi}(\bar{z}_0)\delta_{\bar{z}_0}, v\right\rangle\right| \qquad (3.2.41) \\ &= \mu\left|\int_{E_+^0}(\bar{\phi}+W)\,v dz + \int_{\bar{z}_0}^{z_l^m}\bar{\phi}v dz + \int_{\bar{z}_0}^{z_l^m}Wv dy - l_0\bar{\phi}(\bar{z}_0)v(\bar{z}_0)\right|. \end{aligned}$$

To isolate the dominant part of E_+, we add and subtract $l(z_l^m)\bar{\phi}(\bar{z}_0)v(\bar{z}_0)$ in (3.2.41), where $l(z_l^m)$ is defined in (3.2.23), to achieve

$$\begin{aligned} |\langle\mathcal{N}(W),v\rangle| &\le \mu\left|\int_{E_+^0}(\bar{\phi}+W)\,v dz\right| + \mu\left|\int_{\bar{z}_0}^{z_l^m}\bar{\phi}v - \bar{\phi}(\bar{z}_0)v(\bar{z}_0)dz\right| \\ &+ \mu\left|l(z_l^m) - l_0\right|\left|\bar{\phi}(\bar{z}_0)\right|\left|v(\bar{z}_0)\right| + \mu\left|\int_{\bar{z}_0}^{z_l^m}Wv dz\right|. \end{aligned} \tag{3.2.42}$$

We denote the four terms in the left hand side of the inequality in (3.2.42) by

$$\mathbf{A} = \mu \left| \int_{E_+^0} (\bar{\phi} + W)\, v dz \right|, \tag{3.2.43}$$

$$\mathbf{B} = \mu \left| \int_{\bar{z}_0}^{z_l^m} \bar{\phi} v - \bar{\phi}(\bar{z}_0) v(\bar{z}_0) dz \right|, \tag{3.2.44}$$

$$\mathbf{C} = \mu \left| l(z_l^m) - l_0 \right| \left| \bar{\phi}(\bar{z}_0) \right| \left| v(\bar{z}_0) \right|, \tag{3.2.45}$$

$$\mathbf{D} = \mu \left| \int_{\bar{z}_0}^{z_l^m} W v dz \right|. \tag{3.2.46}$$

To estimate $|\langle \mathcal{N}(W), v \rangle|$ we investigate each of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ in turn. We begin with $\mathbf{A}$, estimating $\bar{\phi} + W$ and v by their L^∞ norms

$$\mathbf{A} \leq C \left\| \bar{\phi} + W \right\|_{L^\infty} \|v\|_{L^\infty} m(E_+^0). \tag{3.2.47}$$

Since $E_+^0 \subseteq [z_l, z_l^m]$, using the estimate on the measure of E given by (3.2.29), we obtain,

$$m(E_+^0) \leq C \|W\|_{H^\gamma}^{\gamma+1/2}.$$

Substituting this estimate in (3.2.47) and controlling $\|v\|_{L^\infty}$ by H^β norm we obtain

$$\mathbf{A} \leq C \|W\|_{H^\gamma}^{\gamma+1/2} \|v\|_{H^\beta}. \tag{3.2.48}$$

To investigate $\mathbf{B}$, adding and subtracting $v(\bar{z}_0)\bar{\phi}$

$$\mathbf{B} \leq \mu \left| \int_{z_l^m}^{\bar{z}_0} \bar{\phi}\,(v - v(\bar{z}_0))\, dz \right| + \mu \left| \int_{z_l^m}^{\bar{z}_0} v(\bar{z}_0) \left(\bar{\phi} - \bar{\phi}(\bar{z}_0)\right) dz \right|.$$

Since $\bar{\phi}$ is uniformly bounded, $\bar{\phi} \in L^\infty$ and $v \in H^\beta$ is Hölder continuous for $1/2 < \beta < 1$, from **Lemma 3.2.1**, we have the upper bound

$$\mathbf{B} \leq C \int_{z_l^m}^{\bar{z}_0} |z - \bar{z}_0|^{\beta - 1/2} \|v\|_{H^\beta} dz + C \int_{z_l^m}^{\bar{z}_0} \|v\|_{H^\beta} |z - \bar{z}_0| \left\| \bar{\phi} \right\|_{H^1} dz. \tag{3.2.49}$$

Performing the integrations above, we arrive at the upper bound

$$\mathbf{B} \leq C \left|\bar{z}_0 - z_l^m\right|^{\beta+1/2} \|v\|_{H^\beta}. \tag{3.2.50}$$

Since $z_l^m \in E$ substituting the upper bound for $\left|\bar{z}_0 - z_l^m\right|$ from (3.2.24)

$$\mathbf{B} \leq C \|W\|_{H^\gamma}^{\beta+1/2} \|v\|_{H^\beta}. \tag{3.2.51}$$

The quantity $\mathbf{C}$, balances the dominate component of E_+ against the rank-one tensor product of the delta functions within the linear operator. We recall that $\bar{\phi}(\bar{z}_0) = \sigma_c$, hence

$$\begin{aligned} \mathbf{C} &= \mu \left|l(z_l^m) - l_0\right| \left|\bar{\phi}(\bar{z}_0)\right| \left|v(\bar{z}_0)\right| \\ &\leq \sigma_c \|v\|_{L^\infty} \left|l(z_l^m) - l_0\right| \leq \sigma_c \|v\|_{H^\beta} \left|l(z_l^m) - l_0\right| \end{aligned} \tag{3.2.52}$$

To derive a bound for $\left|l(z_l^m) - l_0\right|$ we substitute for $l(z_l^m)$ from (3.2.23) and for l_0 from (3.2.39) to achieve the equality

$$\left|l(z_l^m) - l_0\right| = \left|\frac{W(z_l^m)}{\bar{\phi}'(\xi)} - \frac{W(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}\right| = \frac{\left|W(z_l^m)\bar{\phi}'(\bar{z}_0) - W(\bar{z}_0)\bar{\phi}'(\xi)\right|}{\left|\bar{\phi}'(\xi)\bar{\phi}'(\bar{z}_0)\right|}. \tag{3.2.53}$$

where $\xi \in (z_l^m, \bar{z}_0)$. From **Lemma 3.1.1**, $\bar{\phi}'(\xi)$ and $\bar{\phi}'(\bar{z}_0)$ are uniformly bounded below. So, (3.2.53) can be rewritten as

$$\left|l(z_l^m) - l_0\right| \leq C \left|W(z_l^m)\bar{\phi}'(\bar{z}_0) - W(\bar{z}_0)\bar{\phi}'(\xi)\right|. \tag{3.2.54}$$

To control the right hand side of (3.2.54) by $\|W\|_{H^\gamma}$ we add and subtract $W(\bar{z}_0)\bar{\phi}'(\bar{z}_0)$,

$$\left|l(z_l^m) - l_0\right| \leq C \left|\left(W(z_l^m) - W(\bar{z}_0)\right)\bar{\phi}'(\bar{z}_0) + \left(\bar{\phi}'(\bar{z}_0) - \bar{\phi}'(\xi)\right) W(\bar{z}_0)\right|. \tag{3.2.55}$$

Applying the inequality (3.2.5) from **Lemma 3.2.1** to right hand side of (3.2.55), since $W, \bar{\phi}' \in H^\gamma(z_l^m, \bar{z}_0)$, we obtain

$$\left|l(z_l^m) - l_0\right| \leq C \left|z_l^m - \bar{z}_0\right|^{\gamma-1/2} \|W\|_{H^\gamma} + c \left|z_l^m - \bar{z}_0\right|^{\gamma-1/2} \|\bar{\phi}'\|_{H^\gamma} \|W\|_{H^\gamma}. \tag{3.2.56}$$

Bounding $\left|z_l^m - \bar{z}_0\right|$ from (3.2.24), which is valid since z_l^m is a zero of $\bar{\phi} + W - \sigma_c$, we obtain

$$\left|l(z_l^m) - l_0\right| \leq C \|W\|_{H^\gamma}^{\gamma+1/2}.$$

Finally, substituting for $\left|l(z_l^m) - l_0\right|$ to (3.2.52) yields

$$\mathbf{C} \leq C \|W\|_{H^\gamma}^{(\gamma+1/2)} \|v\|_{H^\beta}. \tag{3.2.57}$$

To derive an upper bound for $\mathbf{D}$, we estimate W and v by their L^∞ norms, which are controlled by their fractional Sobolev norms,

$$\mathbf{D} = \mu \left| \int_{z_l^m}^{\bar{z}_0} W v dz \right| \leq c \|v\|_{H^\beta} \|W\|_{H^\gamma} \int_{z_l^m}^{\bar{z}_0} 1 dz. \tag{3.2.58}$$

Evaluating the integral on the right-hand side of the inequality (3.2.58) and again using (3.2.24) we find

$$\mathbf{D} \leq C \|v\|_{H^\beta} \|W\|_{H^\gamma} \left|\bar{z}_0 - z_l^m\right| \leq c \|W\|_{H^\gamma}^2 \|v\|_{H^\beta}. \tag{3.2.59}$$

Combining the estimates for **A**, **B**, **C** and **D** from (3.2.49), (3.2.51), (3.2.57) and (3.2.59) respectively, the estimate on the pairing begun in (3.2.41) yields,

$$|\langle \mathcal{N}(W), v\rangle| \leq C \left(\|W\|_\gamma^{\beta+1/2} + \|W\|_{H^\gamma}^{\gamma+1/2} \right) \|v\|_{H^\beta}, \tag{3.2.60}$$

and returning to the definition of the $H^{-\beta}$ norm of $\mathcal{N}(W)$ from (3.2.35) we obtain the result (3.2.16) in this case.

Case-II: E_+ **is empty.**

This analysis for this case is similar to Case-1, where E_- is empty. As depicted in **Fig(3.2)**, let we denote by z_r^m the $\min_{z \in E} z$, so $\bar{\phi} + W - \sigma_c < 0$ on $(-\infty, z_r^m]$. We partition E_- as

$$E_- = E_-^0 \bigcup [z_l^m, \bar{z}_0]. \tag{3.2.61}$$

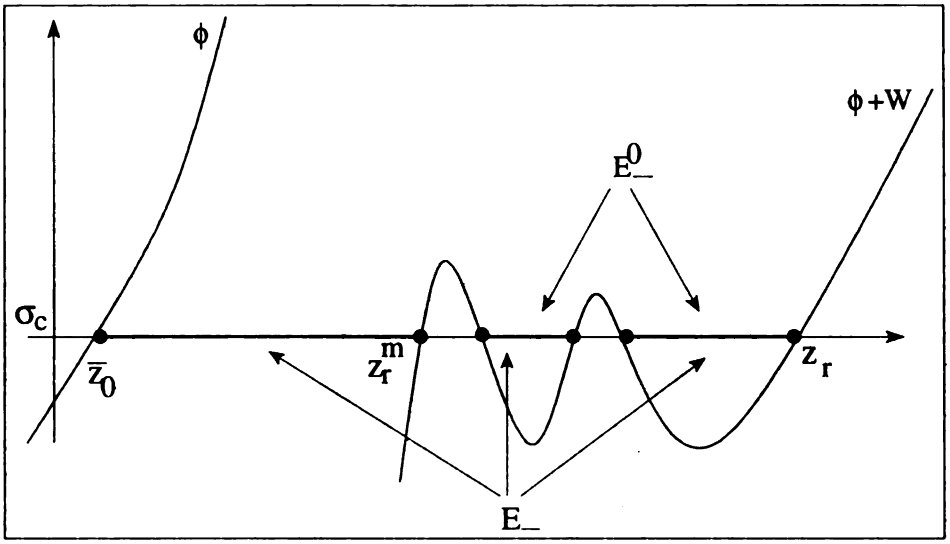

Figure 3.6. The graph of $\bar{\phi} + W$ in the neighbourhood of $\bar{z}_0$ where E_+ is empty.

Since, $\bar{\phi} + W < \sigma_c$ in E_-, $\sigma(\bar{\phi} + W)$ has the form

$$\sigma(\bar{\phi} + W) = \left(\chi_{[z_r^m,\infty)} - \chi_{E^0_-}\right)(\bar{\phi} + W). \tag{3.2.62}$$

The remainder of Case-II follows as in Case-I.

Case-III: Both E_+ and E_- are nonempty.

As in (3.2.34), we obtain the decomposition of $\sigma(\bar{\phi} + W)$

$$\sigma(\bar{\phi} + W) = \left(\chi_{E_+} + \chi_{[\bar{z}_0,\infty)} - \chi_{E_-}\right)(\bar{\phi} + W). \tag{3.2.63}$$

Substituting (3.2.63) in the equation (3.2.17), the nonlinearity takes the form

$$\begin{aligned} \mathcal{N}(W) &= \mu\left(\left(\chi_{E_+} - \chi_{E_-} + \chi_{[\bar{z}_0,\infty)}\right)(\bar{\phi} + W)\right. \\ &- \left.\bar{\phi}\chi_{[\bar{z}_0,\infty)}\right) - \mu\left(\chi_{[\bar{z}_0,\infty]}W + \frac{\bar{\phi}(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}W(\bar{z}_0)\delta_{\bar{z}_0}\right), \\ &= \left(\chi_{E_+} - \chi_{E_-}\right)(\bar{\phi} + W) - \mu\frac{\bar{\phi}(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}W(\bar{z}_0)\delta_{\bar{z}_0}. \end{aligned} \tag{3.2.64}$$

Substituting for $\mathcal{N}(W)$ from (3.2.72) into the dual pairing we find

$$\langle \mathcal{N}(W), v\rangle = \mu\int_{E_+}(\bar{\phi} + W)\,v dz - \mu\int_{E_-}(\bar{\phi} + W)\,v dz - \mu\frac{\bar{\phi}(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}W(\bar{z}_0)v(\bar{z}_0). \tag{3.2.65}$$

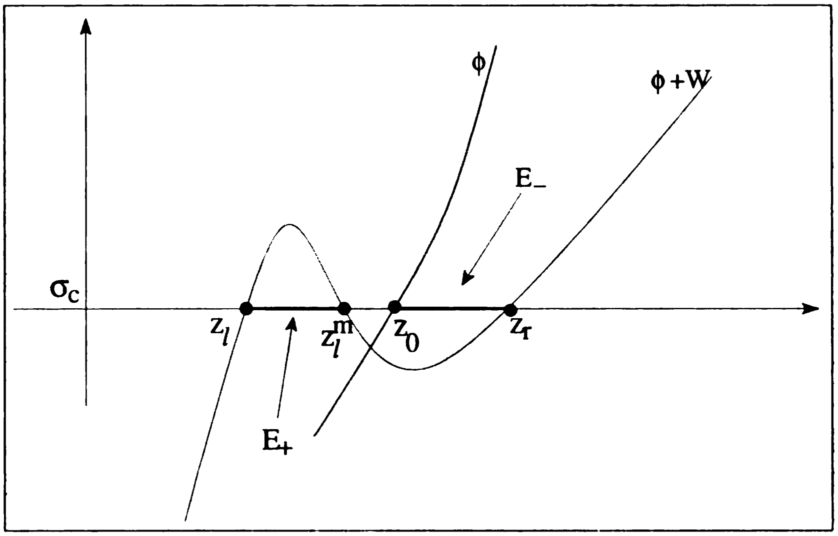

Figure 3.7. The graph of $\bar{\phi} + W$ in the neighbourhood of $\bar{z}_0$ where both E_+ and E_- are empty.

We bound $\bar{\phi} + W$ by its L^∞ norm and v by its H^β norm, inequality (3.2.72) reduces to

$$|\langle \mathcal{N}(W), v\rangle| \leq C \left\|\bar{\phi} + W\right\|_{L^\infty} \|v\|_{H^\beta} (m(E_+) + m(E_-)) + C\,|W(\bar{z}_0)|\,\|v\|_{H^\beta}\,. \tag{3.2.66}$$

From **Fig(3.2)**, we observe $E_+ \subseteq [z_l, z_r]$ and $E_- \subseteq [z_l, z_r]$, the estimate (3.2.29) affords the following bound

$$m(E_\pm) \leq |z_r - z_l| \leq C\,\|W\|_{H^\gamma}^{\gamma+1/2}\,. \tag{3.2.67}$$

For any $z_* \in E$, the following equality holds

$$\bar{\phi}(z_*) + W(z_*) = \sigma_c = \bar{\phi}(\bar{z}_0).$$

Applying the Mean Value theorem to $\bar{\phi}$ on $(z_*, \bar{z}_0) \subset (z_l, z_r)$, we obtain

$$|W(z_*)| = \left|\bar{\phi}(z_*) - \bar{\phi}(\bar{z}_0)\right| \leq C\,|z_* - \bar{z}_0| \leq C\,\|W\|_{H^\gamma}^{\gamma+1/2}\,, \tag{3.2.68}$$

where we have used the bound (3.2.29). Now, we bound $|W(\bar{z}_0)|$ as

$$|W(\bar{z}_0)| \leq |W(\bar{z}_0) - W(z_*)| + |W(z_*)|\,. \tag{3.2.69}$$

Applying the estimate (3.2.5) from **Lemma 3.2.1** to the first term on the right-hand side of (3.2.69) and the estimate (3.2.68) to the second term we obtain

$$|W(\bar{z}_0)| \leq C\left(|\bar{z}_0 - z_*|^{\gamma-1/2} \|W\|_{H^\gamma} + \|W\|_{H^\gamma}^{\gamma+1/2}\right). \tag{3.2.70}$$

Using (3.2.67) to bound $|\bar{z}_0 - z_*|$ we obtain

$$|W(\bar{z}_0)| \leq C\left(\|W\|_{H^\gamma}^{(\gamma-1/2)(\gamma+1/2)+1} + \|W\|_{H^\gamma}^{\gamma+1/2}\right) \leq c\,\|W\|_{H^\gamma}^{\gamma+1/2}. \tag{3.2.71}$$

since $\|W\|_{H^\gamma} < 1$. Returning to (3.2.66), we bound $m(E_\pm)$ and $|W(\bar{z}_0)|$ from (3.2.67) and (3.2.71) in (3.2.66) to obtain

$$|\langle \mathcal{N}(W), v\rangle| \leq C\,\|W\|_{H^\gamma}^{\gamma+1/2}\,\|v\|_{H^\beta}, \tag{3.2.72}$$

which leads to (3.2.16).

3.3 Bilinear Formulation of the Resolvent operator of $L_{\bar{\phi}}$

The bilinear form $B[\cdot\ ,\ \cdot] : H^1 \times H^1 \to R$, associated with the linearized operator $-L_{\bar{\phi}}$ from (3.2.15), is given by:

$$B[u,v] \equiv \left(-L_{\bar{\phi}}u, v\right) = \left(-u_{zz} - v_s(\bar{z}_0)u_z + u - \mu\chi_{[\bar{z}_0,\infty)}u + \mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}u(\bar{z}_0)\delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0}, v\right). \tag{3.3.1}$$

Integrating by parts on $u_{zz}v$ we obtain,

$$B[u,v] = \int_{-\infty}^{\infty} \left(u_z v_z - v_s u_z v + uv - \mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}u(\bar{z}_0)v(\bar{z}_0)\delta_{\bar{z}_0} - \mu u(z)v(z)\chi_{\bar{z}_0,\infty)}\right) dz. \tag{3.3.2}$$

The evaluation operator $\delta_{\bar{z}_0}$ is continuous on H^1 and we have the inequalities:

$$|u(\bar{z}_0)|^2 \leq \|u\|_\infty^2 \leq c\,\|u\|_{H^1}\,\|u\|_{L^2}. \tag{3.3.3}$$

We see that B is a bounded operator since,

$$\begin{aligned} |B(u,v)| &\leq \|u\|_{H^1}\|v\|_{H^1} + (1+\mu)\|u\|_{H^1}\|v\|_{H^1} + \frac{\mu\sigma_c}{\bar{\phi}'(\bar{z}_0)}|u(\bar{z}_0)v(\bar{z}_0)| \\ &\leq c\|u\|_{H^1}\|v\|_{H^1} \end{aligned}$$

for all $u, v \in H^1$. To see that B is coercive, we evaluate $B[u,u]$. The convective term $v_s u_z u$ vanishes upon integration by parts.

$$B[u,u] = \int_{-\infty}^{\infty} \left(u_z^2 + u^2 - \mu u(z)^2 \chi_{[\bar{z}_0,\infty)}\right) dz - \frac{\mu\sigma_c}{\bar{\phi}'(\bar{z}_0)} u(\bar{z}_0)^2. \tag{3.3.4}$$

We note that the coefficient $\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}$ of $u(\bar{z}_0)^2$ is positive. We obtain lower bound for $B[u,u]$ given by (3.3.4), by applying the inequality (3.3.3),

$$\begin{aligned} B[u,u] &= \int_{-\infty}^{\infty} u_z^2 dz + \int_{-\infty}^{-\infty} u^2 dz - \mu \int_{\bar{z}_0}^{\infty} u^2 dz - \frac{\mu\sigma}{\bar{\phi}'(\bar{z}_0)} u(\bar{z}_0)^2 \\ &\geq \|u\|_{H^1}^2 - \mu\|u\|_{L^2} - \alpha\|u\|_{H^1}\|u\|_{L^2}, \end{aligned}$$

where $\alpha = \frac{\mu\sigma}{\bar{\phi}'(\bar{z}_0)}$. Applying Young's inequality to the last term of (3.3.5), we obtain,

$$\begin{aligned} B(u,u) &\geq \|u\|_{H^1}^2 - \mu\|u\|_{L^2} - \frac{1}{2}\|u\|_{H^1}^2 - \frac{c^2}{2}\|u\|_{L^2}^2 \\ B(u,u) + \left(\frac{\alpha^2}{2} + \mu\right)\|u\|_{L^2}^2 &\geq \frac{1}{2}\|u\|_{H^1}^2. \end{aligned}$$

This inequality demonstrates the the coercivity of B, i.e. there exists $\gamma > 0$ such that

$$B(u,u) + \gamma\|u\|_{L^2}^2 \geq \frac{1}{2}\|u\|_{H^1}^2. \tag{3.3.5}$$

The Lax-Milgram theorem provides an upper bound for the Resolvent operator. Consider the following problem for $u \in H^1$,

$$L_{\bar{\phi}} u + \lambda u = f. \tag{3.3.6}$$

The bilinear form corresponding to (3.3.6) is

$$B_\lambda[u,v] = B[u,v] + \lambda(u,v) \tag{3.3.7}$$

Since B is coercive B_λ satisfies the Lax-Milgram conditions each $\lambda \geq \gamma$ and for each $f \in L^2$, which gives the existence and uniqueness of a weak solution in H^1. Hence, $(\lambda I - L_{\bar{\phi}})$ is one-one and onto for $\lambda \geq \gamma$. The resolvent operator $R_\lambda : L^2 \to L^2$ is defined by

$$R_\lambda f := (\lambda I - L_{\bar{\phi}})^{-1} f.$$

We have shown that, for each $f \in L^2$, there exists $u \in H^1$ such that $B_\lambda(u, v) = (f, v)$, for all $v \in H^1$. Defining $u = R_\lambda f$, then the coercivity of B_γ shows that,

$$\beta \|u\|_{H^1}^2 \leq B_\gamma(u, u) \leq B_\lambda(u, u) = (f, u) \leq \|f\|_{L^2} \|u\|_{H^1} .$$

We deduce that,

$$\|R_\lambda\|_{H^1} \leq c \|f\|_{L^2} .$$

■

3.4 Spectrum of the Linearized operator

The spectrum of the linearized operator $L_{\bar{\phi}}$, denoted by Σ_L, contains the singularity of the Resolvent. The spectrum of $L_{\bar{\phi}}$ is a disjoint union of the *essential spectrum* Σ_{ess} and *point spectrum* Σ_{pt} which consists of all isolated eigenvalues with finite multiplicities and hence is discrete. While determining the point spectrum requires a detailed knowledge of the full nonlinear front $\bar{\phi}$, the essential spectrum is entirely determined by the linearization of F about the asymptotic states of ϕ determined in **Lemma 2.3.1**. We formulate the eigenvalue problem for $\psi \in H^{\gamma}(\mathbf{R})$ as,

$$L_{\bar{z}_0}\psi = \lambda\psi. \tag{3.4.1}$$

Integrating the eigenvalue problem (3.4.1) about a small interval $[\bar{z}_0 - \delta, \bar{z}_0 + \delta]$, $L_{\bar{\phi}}$ given by (3.2.15), yields

$$\int_{\bar{z}_0-\delta}^{\bar{z}_0+\delta} \psi_{zz} + v_s\psi_z dz - (1+\lambda)\int_{\bar{z}_0-\delta}^{\bar{z}_0+\delta} \psi dz + \mu\int_{\bar{z}_0}^{\bar{z}_0+\delta} \psi dz + \mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}\psi(\bar{z}_0)\int_{\bar{z}_0-\delta}^{\bar{z}_0+\delta} \delta_{\bar{z}_0}(z)dz = 0.$$

Taking the limit $\delta \to 0$ we obtain

$$[\![\psi']\!]_{\bar{z}_0} - (1+\lambda)\,[\![\psi]\!]_{\bar{z}_0} = -\mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}\psi(\bar{z}_0). \tag{3.4.2}$$

Since $\psi \in H^{\gamma}$, it is continuous at $\bar{z}_0$, so $[\![\psi]\!]_{\bar{z}_0} = 0$, and the relation (3.4.2) reduces to a jump condition on ψ'.

$$[\![\psi']\!]_{\bar{z}_0} = -\mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}\psi(\bar{z}_0). \tag{3.4.3}$$

An eigenvector ψ solves (3.4.1) if and only if it also solves

$$\psi_{zz} + v_s\psi_z - (\lambda+1)\psi + \mu\psi\chi_{\bar{z}_0,\infty)} = 0, \tag{3.4.4}$$

$$[\![\psi']\!]_{\bar{z}_0} = -\mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}\psi(\bar{z}_0). \tag{3.4.5}$$

3.5 Essential Spectrum

Since the linearised operator is defined on $\mathbf{R}$, we do not expect the spectrum to consist of purely point spectrum. We define essential spectrum as the set of $\{\lambda \in \mathbf{C}\}$ for which $\lambda - L_{\bar{\phi}} : H^{\gamma} \to H^{-\beta}$ is not a Fredholm operator of index zero. We refer to the following theorem from Dan Henry [12], which is the most compact criterion in determining the essential spectrum.

Theorem A.2

Suppose $M(x)$, $\mathcal{N}(x)$ are bounded real matrix functions and $M(x), \mathcal{N}(x) \to M_{\pm}, n_{\pm}$ as $x \to \pm\infty$, and suppose D is constant symmetric and positive. Define,

$$Lu(x) = -Du_{xx} + M(x)u_x + \mathcal{N}(x)u, -\infty < x < \infty. \tag{3.5.1}$$

Let $S_{\pm} = \left\{\lambda \middle| \det(\tau^2 D + i\tau M_{\pm} + \mathcal{N}_{\pm} - \lambda I) = 0\right\}$ for some real τ, where $\infty < \tau < \infty$. Let P denote the union of regions inside or on the cureves S_{+}, S_{-}. Then the essential spectrum of L is contained in P, and in particular includes $S_{+} \bigcup S_{-}$.

Comparing $-L_{\bar{\phi}}$ given by (3.2.15) with the operator L (3.5.1) from Theorem A.2, we obtain, $D = 1$, $M = v_s$ and $N(x) = \left(I - \mu\chi_{[\bar{z}_0,\infty)}\right)(x)$. The asymptotic states of N are $N_{-} = I$ and $N_{+} = (1-\mu)I$. Hence the essential spectrum of $L_{\bar{\phi}}$ lies in the union of the essential spectra of the limiting operators

$$\begin{aligned} L_{-} &= -u_{zz} + v_s u_z + u, \\ L_{+} &= -u_{zz} + v_s u_z + (1-\mu)u, \end{aligned}$$

i.e., regions inside or on the curves $S_{\pm}$ where,

$$S_{-} = \left\{\lambda \middle| -k^2 - v_s ik - 1 = \lambda\right\}, \tag{3.5.2}$$

$$S_{+} = \left\{\lambda \middle| -k^2 - v_s ik - 1 + \mu = \lambda\right\}. \tag{3.5.3}$$

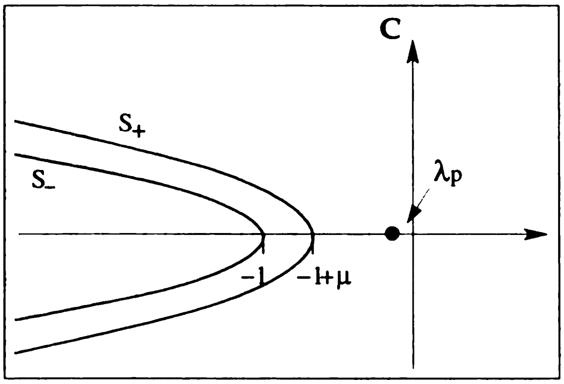

Figure 3.8. The spectrum of $L_{\bar{\phi}}$.

Furthermore, we observe that $|\arg\lambda| \to \pi$ as $|k| \to \infty$, in each connected component of the essential spectrum.

3.6 Point spectrum

Having determined the essential spectrum of $L_{\bar{\phi}}$ we now consider its point spectrum. We show that the with exception of a **single, simple** eigenvalue the entire spectrum lies an $\mathcal{O}(1)$ distance inside the left half complex plane, **Fig(3.8)**.

Lemma 3.6.1 *Let $L_{\bar{\phi}}$ be as given in (3.2.15).*

a) The point spectrum of $L_{\bar{\phi}}$ is ***real****.*

b) $\Sigma_{pt}(L_{\bar{\phi}}) \bigcap (1-\mu, \infty) = \{\lambda_p\}$ where $\lambda_p = -\sqrt{\epsilon}\lambda^1$, where $\lambda^1 > 0$ is given by (3.6.27). The principal eigenfunction ψ corresponding to λ_p, ***normalized*** *w.r.t. the weighted L^2_ν norm generated by the inner product defined in (3.6.13), is given by*

$$\psi(\xi) = \begin{cases} \frac{1}{\theta_1} e^{a(\xi-\bar{z}_0)} & \xi \leq \bar{z}_0, \\ \frac{1}{\theta_1} e^{b(\xi-\bar{z}_0)} & \xi \geq \bar{z}_0. \end{cases} \tag{3.6.1}$$

The coefficients a, b are given by (3.6.6), (3.6.7) and the constant θ_1 is defined as in (3.6.28).

Proof: We seek to determine whether there is any point spectrum satisfying, Reλ_p $\geq -1+\mu$. Suppose ψ is an eigenfunction with eigenvalue **Re**$\lambda_p \geq -1+\mu$. Thus, ψ satisfies

$$\psi'' + v_s\psi' - (\lambda_p + 1)\psi + \mu\chi_{[\bar{z}_0,\infty)}\psi = 0. \tag{3.6.2}$$

$$[[\psi']]_{\bar{z}_0} = -\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}\psi(\bar{z}_0). \tag{3.6.3}$$

The equation (3.6.2) has constant coefficients with associated characteristic equations:

$$r^2 + v_s r - (1+\lambda_p) \;=\; 0 \quad z > \bar{z}_0 \tag{3.6.4}$$

$$r^2 + v_s r - (1+\lambda_p) + \mu \;=\; 0 \quad z < \bar{z}_0 \tag{3.6.5}$$

Equation (3.6.4) has a positive root a, and (3.6.5) has a negative root b given by

$$a = \frac{-v_s + \sqrt{v_s^2 + 4(1+\lambda_p)}}{2}, \tag{3.6.6}$$

$$b = \frac{-v_s - \sqrt{v_s^2 + 4(1+\lambda_p - \mu)}}{2}. \tag{3.6.7}$$

The eigenfunction ψ takes the form

$$\psi(\xi) = \begin{cases} \psi(\bar{z}_0)e^{a(\xi-\bar{z}_0)} & \xi \leq \bar{z}_0, \\ \psi(\bar{z}_0)e^{b(\xi-\bar{z}_0)} & \xi \geq \bar{z}_0. \end{cases} \tag{3.6.8}$$

We turn to the proof of part(a), motivated by the example on page 130 Dan Henry [12], where he transforms the non-self-adjoint eigenvalue problem into a self-adjoint one. Examining the characteristic equation for the limits $\xi \to \pm\infty$, we will show that any eigenvector ψ must actually tend to zero at least as fast as $\mathcal{O}(e^{-v_s|\xi-\bar{z}_0|/2})$, when $\xi \to \pm\infty$. Indeed, we show that $\psi(\xi)e^{v_s(\xi-\bar{z}_0)/2}$ vanishes as $\xi \to \pm\infty$. From (3.6.8) we have

$$\psi(\xi)e^{v_s(\xi-\bar{z}_0)/2} = \begin{cases} \psi(\bar{z}_0)e^{a(\xi-\bar{z}_0)}e^{v_s(\xi-\bar{z}_0)/2} = \psi(\bar{z}_0)e^{(a+v_s/2)(\xi-\bar{z}_0)} & \xi \leq \bar{z}_0, \\ \psi(\bar{z}_0)e^{b(\xi-\bar{z}_0)}e^{v_s(\xi-\bar{z}_0)/2} = \psi(\bar{z}_0)e^{(b+v_s/2)(\xi-\bar{z}_0)} & \xi \geq \bar{z}_0, \end{cases} \tag{3.6.9}$$

while from (3.6.6) and (3.6.7) we see

$$\begin{aligned} a+\frac{v_s}{2} &= \frac{-v_s+\sqrt{v_s^2+4(1+\lambda_p)}}{2}+\frac{v_s}{2}=\frac{\sqrt{v_s^2+4(1+\lambda_p)}}{2}>0 \\ b-\frac{v_s}{2} &= \frac{-v_s-\sqrt{v_s^2+4(1+\lambda_p)}}{2}+\frac{v_s}{2}=-\frac{\sqrt{v_s^2+4(1+\lambda_p)}}{2}<0 \end{aligned}$$

Defining

$$w(\xi)=\psi(\xi)e^{v_s(\xi-\bar{z}_0)/2}, \tag{3.6.10}$$

we see that w decays at $\pm\infty$ and moreover, w satisfies

$$w''-\left(\frac{v_s^2}{4}+1+\lambda_p\right)w+\mu w\chi_{[\bar{z}_0,\infty)}=0, \tag{3.6.11}$$

$$[[w']]_{\bar{z}_0}=-\mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}w(\bar{z}_0). \tag{3.6.12}$$

This shows that the linearized operator is self-adjoint in the weighted space $L^2_\nu(\mathbf{R})$, with the exponential weight $\nu(z)=e^{v_s(z-\bar{z}_0)}$ and inner product defined as,

$$\langle u,v\rangle_\nu=\int_{\mathbf{R}} e^{v_s(z-\bar{z}_0)}u(z)v(z)dz, \tag{3.6.13}$$

for any $u,v\in H^\gamma$. Hence, all eigenvalues of $L_{\bar{\phi}}$ must be real and they form a countable sequence. The point spectrum is determined by a jump condition (3.6.3), which in light of (3.6.8) reduces to,

$$b-a=-\mu\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}. \tag{3.6.14}$$

Substituting for a and b from (3.6.6) and (3.6.7) in (3.6.17) any eigenvalue $\lambda_p\in[-1+\mu,\infty)$ satisfies,

$$\sqrt{v_s^2+4(1+\lambda_p-\mu)}+\sqrt{v^2+4(1+\lambda_p)}=\frac{2\mu\sigma_c}{\bar{\phi}'(\bar{z}_0)} \tag{3.6.15}$$

We substitute for $\phi'(\bar{z}_0)$ from (3.1.16) into the right-hand side of (3.6.15) to obtain the relations

$$\sqrt{v_s^2+4(1+\lambda_p-\mu)}+\sqrt{v_s^2+4(1+\lambda_p)}=\frac{2\mu\sigma_c}{(\sigma_c-g_s(\bar{z}_0))\lambda_1^+}[1-\sqrt{\epsilon}\frac{dg(\bar{y}_0)}{dy}+\mathcal{O}(\epsilon)] \tag{3.6.16}$$

By the continuity of ϕ' at $\bar{z}_0$, from (3.1.16) we also obtain the relation

$$\begin{aligned}\frac{2\mu\sigma_c}{(\sigma_c - g_s(\bar{z}_0))\lambda_1^+}(1 - \sqrt{\epsilon}\frac{dg(\bar{y}_0)}{dy} + \mathcal{O}(\epsilon)) &= \frac{2\mu\sigma_c}{(\sigma_c - \frac{g_s(\bar{z}_0)}{1-\mu})\lambda_2^-}[1 - \frac{\sqrt{\epsilon}}{1-\mu}\frac{dg(\bar{y}_0)}{dy} \\ &+ \mathcal{O}(\epsilon)] \end{aligned} \tag{3.6.17}$$

Substituting for λ_1^+ and λ_2^- from (2.2.4) and (2.2.5) the leading order terms in (3.6.17) satisfies

$$\frac{v_s + \sqrt{v_s^2 + 4}}{(\sigma_c - g_s(\bar{z}_0))} = \frac{v_s - \sqrt{v_s^2 + 4(1-\mu)}}{(\sigma_c(1-\mu) - g_s(\bar{z}_0))}. \tag{3.6.18}$$

The left-hand side of (3.6.17) is monotonically increasing for λ_p, the equation has atmost **one**, **simple** root. We consider an asymptotic expansion of λ_p

$$\lambda_p = \lambda^0 - \sqrt{\epsilon}\lambda^1 + \ldots \tag{3.6.19}$$

Then substituting for λ_p from (3.6.19), into the spectrum equation (3.6.17), we obtain the leading order equation for λ^0

$$\sqrt{v_s^2 + 4(1 + \lambda^0 - \mu)} + \sqrt{v_s^2 + 4(1 + \lambda^0)} = \frac{2\mu\sigma_c}{(\sigma_c - g_s(\bar{z}_0))\lambda_1^+}, \tag{3.6.20}$$

$$\frac{2\mu\sigma_c}{(\sigma_c - g_s(\bar{z}_0))\lambda_1^+} = \frac{2\mu\sigma_c}{(\sigma_c - \frac{g_s(\bar{z}_0)}{1-\mu})\lambda_2^-}. \tag{3.6.21}$$

Substituting for λ_1^+ and λ_2^- from (2.2.4) and (2.2.5) in the above, we obtain

$$\sqrt{v_s^2 + 4(1 + \lambda^0 - \mu)} + \sqrt{v_s^2 + 4(1 + \lambda^0)} = \frac{4\mu\sigma_c}{(\sigma_c - g_s(\bar{z}_0))(-v_s + \sqrt{v_s^2 + 4})} \tag{3.6.22}$$

Rationalizing the square root terms in the denominator of the right hand side of (3.6.22) yields

$$\sqrt{v_s^2 + 4(1 + \lambda^0 - \mu)} + \sqrt{v_s^2 + 4(1 + \lambda^0)} = \frac{\mu\sigma_c(v_s + \sqrt{v_s^2 + 4})}{(\sigma_c - g_s(\bar{z}_0))}. \tag{3.6.23}$$

Now, we show that $\lambda^0 = 0$ solves (3.6.23). Plugging $\lambda^0 = 0$ into the left-hand side of (3.6.23) we obtain

$$\begin{aligned}
\text{LHS} &= -v_s + \sqrt{v_s^2 + 4(1-\mu)} + v_s + \sqrt{v_s^2 + 4}, \\
&= -\frac{((1-\mu)\sigma_c - g(\bar{z}_0))}{(\sigma_c - g_s(\bar{z}_0))}\left(v_s + \sqrt{v_s^2+4}\right) + v_s + \sqrt{v_s^2+4} \quad \text{(from (3.6.18))}, \\
&= \frac{\mu\sigma_c(v_s + \sqrt{v_s^2+4})}{(\sigma_c - g_s(\bar{z}_0))}, \\
&= \text{RHS}. \qquad (3.6.24)
\end{aligned}$$

Hence the leading order term λ^0 in the asymptotic expansion (3.6.19) of λ_p is zero and we obtain,

$$\lambda_p = -\sqrt{\epsilon}\lambda^1 + \mathcal{O}(\epsilon). \qquad (3.6.25)$$

To determine the sign of the coefficient λ^1, we equate the $\sqrt{\epsilon}$ terms on both sides in the spectrum equation (3.6.17) and solve for λ^1. Utilizing the expansion

$$\sqrt{y + \sqrt{\epsilon}x} = \sqrt{y} + \sqrt{\epsilon}\frac{x}{2\sqrt{y}} + \mathcal{O}(\epsilon),$$

in the left-hand side of (3.6.17), we obtain

$$-\lambda^1\left(\frac{1}{2\sqrt{v_s^2+4(1-\mu)}} + \frac{1}{2\sqrt{v_s^2+4}}\right) = -\frac{2\mu\sigma_c}{\sigma_c - g_s(\bar{z}_0)}\frac{dg(\bar{y}_0)}{dy}. \qquad (3.6.26)$$

Solving for λ^1

$$\lambda^1 = \frac{4\mu\sigma_c\sqrt{v_s^2+4(1-\mu)}\sqrt{v_s^2+4}}{(\sigma_c - g_s(\bar{z}_0))(\sqrt{v_s^2+4(1-\mu)} + \sqrt{v_s^2+4})}\frac{dg(\bar{y}_0)}{dy}. \qquad (3.6.27)$$

We observe that $\lambda^1 > 0$ since g is strictly increasing and $\sigma_c - g_s(\bar{z}_0) > 0$. We normalize the eigenfunction ψ from (3.6.8), w.r.t the inner product defined by (3.6.13), defining θ_1 as

$$\theta_1^2 \equiv \int_R e^{v_s(z-\bar{z}_0)}\psi^2(z)dz, \qquad (3.6.28)$$

and the corresponding eigenfunction satisfies,

$$\psi(\xi) = \begin{cases} \frac{1}{\theta_1}e^{a(\xi-\bar{z}_0)} & \xi \leq \bar{z}_0, \\ \frac{1}{\theta_1}e^{b(\xi-\bar{z}_0)} & \xi \geq \bar{z}_0. \end{cases} \qquad (3.6.29)$$

We expand the expressions of a and b given by (3.6.6) and (3.6.7) to express them in terms of the eigenvalues λ_1^+ and λ_2^- respectively. Substituting for λ_p from (3.6.25) into the expressions for a and b

$$\begin{aligned} a &= \frac{-v_s + \sqrt{v_s^2 + 4}}{2} - \frac{\lambda^1 \sqrt{\epsilon}}{2\sqrt{v_s^2 + 4}} + \mathcal{O}(\epsilon), \\ b &= \frac{-v_s - \sqrt{v_s^2 + 4(1-\mu)}}{2} + \frac{\lambda^1 \sqrt{\epsilon}}{2\sqrt{v_s^2 + 4(1-\mu)}} + \mathcal{O}(\epsilon). \end{aligned}$$

We recall the definition of λ_1^+ and λ_2^- from (2.2.4) and (2.2.5) to obtain,

$$a = \lambda_1^+ - \frac{\lambda^1 \sqrt{\epsilon}}{2\sqrt{v_s^2 + 4}} + \mathcal{O}(\epsilon), \tag{3.6.30}$$

$$b = \lambda_2^- - \frac{\lambda^1 \sqrt{\epsilon}}{2\sqrt{v_s^2 + 4(1-\mu)}} + \mathcal{O}(\epsilon). \tag{3.6.31}$$

We define the coefficients of $\sqrt{\epsilon}$ in a and b as l_1 and l_2,

$$l_1 = \frac{\lambda^1}{2\sqrt{v_s^2 + 4}}, \tag{3.6.32}$$

$$l_2 = \frac{\lambda^1}{2\sqrt{v_s^2 + 4(1-\mu)}}. \tag{3.6.33}$$

The principal eigenfunction corresponding to the eigenvalue $-\sqrt{\epsilon}\lambda^1 + \mathcal{O}(\epsilon)$, is given by,

$$\psi = \begin{cases} \frac{1}{\theta_1} e^{\left(\lambda_1^+ - \sqrt{\epsilon} l_1 + \mathcal{O}(\epsilon)\right)(z - \bar{z}_0)} & z \leq \bar{z}_0 \\ \frac{1}{\theta_1} e^{\left(\lambda_2^- - \sqrt{\epsilon} l_2 + \mathcal{O}(\epsilon)\right)(z - \bar{z}_0)} & z \geq \bar{z}_0 \end{cases} \tag{3.6.34}$$

■

The adjoint eigenvalue problem is given by,

$$\psi^\dagger_{zz} - v_s \psi^\dagger_z - (\lambda_p^\dagger + 1)\psi^\dagger + \mu\chi_{[\bar{z}_0, \infty)}\psi^\dagger = 0. \tag{3.6.35}$$

$$\left[\left[\psi_z^\dagger\right]\right]_{\bar{z}_0} = -\frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)}\psi^\dagger(\bar{z}_0). \tag{3.6.36}$$

where $\psi^\dagger$ is the eigenvector corresponding to $\lambda_p^\dagger$. The adjoint of $L_{\bar{\phi}}$ is denoted by $L_{\bar{\phi}}^\dagger$. We define the following inner product with the exponential weight $\nu^\dagger(z) = e^{-v_s(z-\bar{z}_0)}$

$$\langle u, v \rangle_{\nu^\dagger} = \int_{\mathbf{R}} e^{-v_s(z-\bar{z}_0)} u(z) v(z) dz. \tag{3.6.37}$$

Lemma 3.6.2 *a) The point spectrum of $L^{\dagger}_{\bar{\phi}}$ is contained in $(-\infty, -1+\mu]$ except for a **single, simple** and **real** eigenvalue λ_p which has the following expansion*

$$\lambda_p = -\lambda^1\sqrt{\epsilon} + \mathcal{O}(\epsilon),$$

where λ^1 is given by (3.6.27).

b) The principal eigenfunction of $L^{\dagger}_{\bar{\phi}}$ normalized w.r.t. the $L^2_{\nu^{\dagger}}$ norm, induced by the inner product (3.6.37) is given by,

$$\psi^{\dagger}(\xi) = \begin{cases} \frac{1}{\theta_1} e^{a^{\dagger}(\xi - \bar{z}_0)}, & \xi \leq \bar{z}_0 \\ \frac{1}{\theta_1} e^{b^{\dagger}(\xi - \bar{z}_0)}. & \xi \geq \bar{z}_0 \end{cases} \tag{3.6.38}$$

where $a^{\dagger}$ and $b^{\dagger}$ are defined according to (3.6.40) and (3.6.41), and satisfies

$$\int_{\boldsymbol{R}} \psi\psi^{\dagger} dz = 1. \tag{3.6.39}$$

Proof: The proof of part a) directly follows from **Lemma 3.6.1**. Since λ_p is real, the point spectrum $\lambda_p^{\dagger}$ of $L^{\dagger}_{\bar{\phi}}$ satisfies, $\lambda_p^{\dagger} = \lambda_p$. The characteristic equations associated to (3.6.35) has roots $a^{\dagger}$ and $b^{\dagger}$ given by.

$$a^{\dagger} = \frac{v_s + \sqrt{v_s^2 + 4(1+\lambda_p)}}{2}, \tag{3.6.40}$$

$$b^{\dagger} = \frac{v_s - \sqrt{v_s^2 + 4(1+\lambda_p - \mu)}}{2}, \tag{3.6.41}$$

and the corresponding family of eigenfunctions are,

$$\psi^{\dagger}(\xi) = \begin{cases} \frac{1}{\theta_1} e^{a^{\dagger}(\xi - \bar{z}_0)}, & \xi \leq \bar{z}_0 \\ \frac{1}{\theta_1} e^{b^{\dagger}(\xi - \bar{z}_0)}. & \xi \geq \bar{z}_0 \end{cases} \tag{3.6.42}$$

Comparing the forms of ψ and $\psi^{\dagger}$ given by (3.6.29) and (3.6.42), we find $e^{-v_s(z-\bar{z}_0)}\psi^{\dagger} = \psi$. This shows that $\psi^{\dagger}$ is normalized w.r.t $L^2_{\nu^{\dagger}}$ norm

$$\begin{aligned} \left\|\psi^{\dagger}\right\|_{L^2_{\nu^{\dagger}}} &= \int_{\mathbf{R}} e^{-v_s(z-\bar{z}_0)} \left(\psi^{\dagger}\right)^2 dz, \\ &= \int_{\mathbf{R}} e^{v_s(z-\bar{z}_0)} (\psi)^2 dz = 1. \end{aligned} \tag{3.6.43}$$

Now, we prove the orthonormality of ψ and $\psi^\dagger$, by studying their dual pairing

$$\begin{aligned}
\left\langle \psi, \psi^\dagger \right\rangle_{L^2} &= \int_{\mathbf{R}} \psi(z)\psi^\dagger(z)dz, \\
&= \int_{\mathbf{R}} e^{v_s(z-\bar{z}_0)}\psi(z)\left(e^{-v_s(z-\bar{z}_0)}\psi^\dagger(z)\right)dz, \\
&= \int_{\mathbf{R}} e^{v_s(z-\bar{z}_0)}\psi(z)\psi(z)dz, \\
&= 1
\end{aligned}$$

■

In the following lemma we prove that there is a nonzero angle between $\bar{\phi}_{\bar{z}_0}$ and the eigenvector $\psi(z;\bar{z}_0)$.

Lemma 3.6.3 *There exists $\theta_0 > 0$, independent of ϵ, such that,*

$$\theta(\bar{z}_0) \equiv \left\langle \frac{\partial\bar{\phi}(z)}{\partial\bar{z}_0}, \psi(z;\bar{z}_0) \right\rangle_{L^2} \geq \theta_0 \tag{3.6.44}$$

for any $\bar{z}_0 \in \boldsymbol{R}$.

Proof: Referring to the form of the composite solution from (3.1.9), the angle between $\frac{\partial\bar{\phi}(z)}{\partial\bar{z}_0}$ and the eigenfunction $\psi(z;\bar{z}_0)$ (using the form given by (3.6.34)), denoted by $\theta(\bar{z}_0)$ is given by,

$$\begin{aligned}
\theta(\bar{z}_0) &\equiv \left\langle \frac{\partial\bar{\phi}}{\bar{z}_0}, \psi \right\rangle_{L^2} \\
&= \frac{1}{\theta_1}\int_{-\infty}^{\bar{z}_0} e^{(2\lambda_1^+ - \sqrt{\epsilon}l_1)(z-\bar{z}_0)}\left((\sigma_c - g(\bar{z}_0))(-\lambda_1^+ + (z-\bar{z}_0)\frac{\partial\lambda_1^+(v_s)}{\partial\bar{z}_0}) - \sqrt{\epsilon}g'(\bar{z}_0)\right) \\
&+ \frac{1}{\theta_1}\int_{\bar{z}_0}^{\infty} e^{(2\lambda_2^- - \sqrt{\epsilon}l_2)(z-\bar{z}_0)}\left((\sigma_c - \frac{g(\bar{z}_0)}{1-\mu})(-\lambda_2^- + (z-\bar{z}_0)\frac{\partial\lambda_2^-(v_s)}{\partial\bar{z}_0}) - \frac{\sqrt{\epsilon}g'(\bar{z}_0)}{1-\mu}\right).
\end{aligned}$$

Performing integration and referring to relations (3.1.7), (3.1.12) and (3.1.13) we obtain the closed form of $\theta(\bar{z}_0)$ upto leading order,

$$\theta(\bar{z}_0) = -\lambda_1^+\frac{\sigma_c - g(\bar{z}_0)}{2\lambda_1^+ - \sqrt{\epsilon}l_1} + \lambda_2^-\frac{\sigma_c - \frac{g(\bar{z}_0)}{1-\mu}}{2\lambda_2^- - \sqrt{\epsilon}l_2} + \mathcal{O}(\sqrt{\epsilon}). \tag{3.6.45}$$

Simplifying the expression in the right-hand side,

$$\theta(\bar{z}_0) = \frac{\mu g(\bar{z}_0)}{2(1-\mu)} + \mathcal{O}(\sqrt{\epsilon}), \tag{3.6.46}$$

which is uniformly bounded away from zero for all $\bar{z}_0 \in \mathbf{R}$.

■

CHAPTER 4

Resolvent and Semigroup estimates

Having described the spectrum of the linearized operator, we study the resolvent operator, $(L_{\bar{\phi}} - \lambda)^{-1} : H^{-\beta} \to H^{\gamma}$ for $\gamma, \beta \in \left(\frac{1}{2}, 1\right]$. We prove that the linearized operator $L_{\bar{\phi}}$ is sectorial following the definition from Dan Henry [12]:

A linear operator L in a Banach space X is sectorial if it is a closed densly defined operator such that, for some $\theta \in (0, \pi/2)$ and some $M \geq 1$ and real a, the sector

$$S_{a,0} = \{\lambda | \theta \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

is in the resolvent set of L and

$$\left\| (\lambda - L)^{-1} \right\| \leq \frac{M}{|\lambda - a|} \quad \forall \lambda \in S_{a,\theta}. \tag{4.0.1}$$

We introduce the space, $X_{\bar{z}_0} = \left\{ U \middle| \|U\|_{H^\gamma} < \infty \text{ and } \pi_{\bar{z}_0} U = 0 \right\}$, where the spectral projection is defined as,

$$\pi_{\bar{z}_0} U = \frac{(U, \Psi^\dagger_{\bar{z}_0})}{(\Psi_{\bar{z}_0}, \Psi^\dagger_{\bar{z}_0})} \Psi^\dagger_{\bar{z}_0}, \tag{4.0.2}$$

which reduces to,

$$\pi_{\bar{z}_0} U = (U, \Psi^\dagger_{\bar{z}_0}) \Psi^\dagger_{\bar{z}_0}, \tag{4.0.3}$$

by the scaling of $\Psi_{\bar{z}_0}$ and $\Psi^\dagger_{\bar{z}_0}$. We denote the complimentary projection by

$$\tilde{\pi}_0 = I - \pi_0. \tag{4.0.4}$$

The space $\tilde{X}_{\bar{z}_0}$ corresponds to the $L_{\bar{z}_0}$ spectral set which is uniformly in the left-half complex plane, while $X_{\bar{z}_0} = \mathcal{R}(\tilde{X}_{\bar{z}_0})$ is the eigenspace associated to the $\mathcal{O}(\sqrt{\epsilon})$ eigenvalue arising from the broken translational invariance. We fix the contour $\mathcal{C}$ in $\rho(L_{\bar{\phi}})$ with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$, for some θ in $(\frac{\pi}{2}, \pi)$. The branches $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ of contour $\mathcal{C}$ are given by

$$\mathcal{C}_1 = \{\lambda | \lambda_r \le -l, \lambda_i = -m\lambda_r\}, \tag{4.0.5}$$

$$\mathcal{C}_2 = \{\lambda | \lambda_r = -l, -ml \le \lambda_i \le ml\}, \tag{4.0.6}$$

$$\mathcal{C}_3 = \{\lambda | \lambda_r \le -l, \lambda_i = m\lambda_r\}, \tag{4.0.7}$$

where λ_r and λ_i denoted the real and imaginary parts of $\lambda \in \mathcal{C}$. As illustrated in **Fig(4)**, $a - \mu > l > \frac{v_s^2}{4}$ determines the position of the contour and m determines the opening angle of the branches. We denote by $a \in \mathbf{R}$

$$a = 1 + \frac{v_s^2}{4}. \tag{4.0.8}$$

We employ a to obtain the decaying estimate (4.0.1) of the resolvent operator. For $\lambda \in \mathcal{C}$, $|\lambda + a|^2$ denoting the square of distance of $\mathcal{C}$ from $-a$ can be expressed as a map d

$$\begin{aligned} d: \mathbf{R}^+ &\to \mathbf{R}^+ \\ x &\mapsto \begin{cases} (a-l)^2 + x^2 & 0 \le x \le ml \\ (a - \frac{x}{m})^2 + x^2 & ml \le x < \infty \end{cases} \end{aligned} \tag{4.0.9}$$

We note that

$$d(x) \ge x^2 + (a-l)^2, \qquad x \in \mathbf{R}. \tag{4.0.10}$$

which provides a lower bound for $|\lambda + a|$, $\lambda \in \mathcal{C}$,

$$|\lambda + a| \ge a - l > 1. \tag{4.0.11}$$

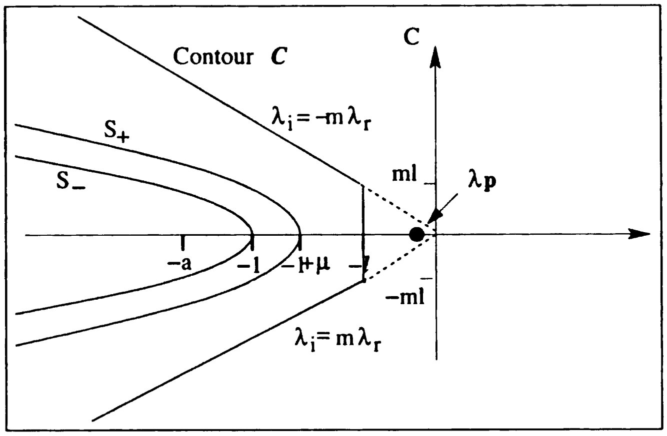

Figure 4.1. The contour $\mathcal{C}$.

4.1 Decomposition of the Linearized operator

For $F \in H^{-\beta}$, the resolvent $(L-\lambda)^{-1}$ finds the associated u solving

$$(L-\lambda)u = F, \tag{4.1.1}$$

where $L \equiv L_{\bar{\phi}}$. In this section we introduce the notation

$$\alpha = \mu \frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)} > 0, \tag{4.1.2}$$

and L from (3.2.15) can be written as

$$L = \partial_z^2 + v_s \partial_z - I + \alpha \delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0} + \mu \chi_{(\bar{z}_0, \infty)}. \tag{4.1.3}$$

We decompose $L - \lambda$ as

$$(L-\lambda) = \mathcal{L} + \alpha \delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0} + \mu \chi_{(\bar{z}_0, \infty)}, \tag{4.1.4}$$

where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L} = \partial_z^2 + v_s \partial_z - I. \tag{4.1.5}$$

In the following lemma we derive estimates on $\mathcal{L}^{-1}F$ for $F \in H^{-\beta}$.

Lemma 4.1.1 *Fix $\gamma + \beta < 2$. There exists $c > 0$ such that for all λ on the contour $\mathcal{C}$ and $F \in H^{-\beta}$ the following estimate holds*

$$\left\|\mathcal{L}^{-1}F\right\|_{H^\gamma} \leq c(|\lambda + a|)^{(\gamma+\beta-2)/2} \|F\|_{H^{-\beta}}. \tag{4.1.6}$$

Proof: Referring to the definition of $\mathcal{L}$ given by the equation (4.1.5), $\mathcal{L}u = F$ is equivalent to,

$$u_{zz} + v_s u_z - (1 + \lambda)u = F. \tag{4.1.7}$$

Taking the Fourier transform of (4.1.7) and solving for $\hat{u}(k) = \widehat{\mathcal{L}^{-1}F}(k)$, we obtain,

$$\hat{u}(k) = \frac{\hat{F}(k)}{-k^2 - ikv_s - (1 + \lambda)}. \tag{4.1.8}$$

To compute the H^γ norm of $\mathcal{L}^{-1}F$ we multiply both sides of equation (4.1.8) by $(1 + k^2)^{\gamma/2}$ and take the L^2 norm,

$$\begin{aligned} \left\|\mathcal{L}^{-1}F\right\|^2_{H^\gamma} &= \int_{\mathbf{R}} \frac{(1+k^2)^\gamma \left|\hat{F}(k)\right|^2}{\left|k^2 + ikv_s + 1 + \lambda\right|^2} dk, \\ &= \int_{\mathbf{R}} \frac{(1+k^2)^{\gamma+\beta}}{\left|k^2 + ikv_s + 1 + \lambda\right|^2} \left|(1+k^2)^{-\beta/2}\hat{F}(k)\right|^2 dk. \end{aligned} \tag{4.1.9}$$

Applying Holder's inequality to the integral on the left hand side of (4.1.9) with $p = \infty$ and $q = 1$, we obtain,

$$\left\|\mathcal{L}^{-1}F\right\|^2_{H^\gamma} \leq \left(\max_{k\in\mathbf{R}} g_k\right) \|F\|^2_{H^{-\beta}}, \tag{4.1.10}$$

where,

$$g_k = \frac{(1+k^2)^{\gamma+\beta}}{\left|k^2 + ikv_s + 1 + \lambda\right|^2}. \tag{4.1.11}$$

We refer to S_- given by (3.5.3), the leftmost branch of the essential spectrum of $L_{\bar{\phi}}$, and observe

$$\left|k^2 + ikv_s + 1 + \lambda\right| = \text{dist}(\lambda, -S_-).$$

From **Fig(4)** since S_- separates $\mathcal{C}$ from $-a$ we see that

$$\text{dist}(\lambda, -S_-) \geq \text{dist}(\lambda, -a), \tag{4.1.12}$$

and relation (4.0.10) yields

$$\left|k^2+1\right| \leq \left|k^2+(a-l)^2\right| \leq |\lambda+a|, \tag{4.1.13}$$

for $\lambda \in \mathcal{C}$ and $k \in \mathbf{R}$, since $a-l>1$. Thus, we obtain the following bound for $\max\limits_{k\in\mathbf{R}}(g_k)$,

$$\max_{k\in\mathbf{R}}(g_k) \leq \frac{C}{|\lambda+a|^{2-\gamma-\beta}}, \tag{4.1.14}$$

and substituting for $\max\limits_{k\in\mathbf{R}} g_k$ in inequality (4.1.10) we obtain,

$$\left\|\mathcal{L}^{-1}F\right\|_{H^\gamma} \leq C\left(|\lambda+a|\right)^{(\gamma+\beta-2)/2} \|F\|_{H^{-\beta}}.$$

■

4.2 Exponential Dichotomy

Referring to the formulation (4.1.4), we decompose u solving (4.1.1) as

$$u = \mathcal{L}^{-1}F + w \tag{4.2.1}$$

and refering to (4.1.4), w solves

$$(L-\lambda)w = F-(L-\lambda)\mathcal{L}^{-1}F = -\alpha\left(\mathcal{L}^{-1}F\right)(\bar{z}_0)\delta_{\bar{z}_0} - \mu\left(\mathcal{L}^{-1}F\right)\chi_{(\bar{z}_0,\infty)}. \tag{4.2.2}$$

For notational convenience we define

$$\kappa \equiv -\alpha\left(\mathcal{L}^{-1}F\right)(\bar{z}_0) \in \mathbf{R}, \tag{4.2.3}$$

and

$$G \equiv -\mu\left(\mathcal{L}^{-1}F\right)\chi_{(\bar{z}_0,\infty)} \in L^2(\mathbf{R}) \cap H^\gamma(\bar{z}_0,\infty), \tag{4.2.4}$$

where G has support in $(\bar{z}_0,\infty)$. The expression (4.2.2) can be rewritten in terms of κ and G as

$$(L-\lambda)w = \kappa\delta_{\bar{z}_0} + G. \tag{4.2.5}$$

In the following lemma we derive an upper bound for w in L^2 and L^∞ norms.

Lemma 4.2.1 *For $\gamma + \beta < 2$, $\exists C > 0$, independent of ϵ, such that for all λ on the contour $\mathcal{C}$ and $F \in H^{-\beta}$, w solving (4.2.5) satisfies*

$$\|w\|_{L^\infty} + \|w\|_{L^2} \leq C\,|\lambda + a|^{(\gamma+\beta-2)/2}\,\|F\|_{H^{-\beta}}. \tag{4.2.6}$$

Proof: We construct a continuous solution $w(z)$ of (4.2.5) which decays exponentially as $z \to \pm\infty$. We note that w' has a jump at $\bar{z}_0$ which can be prescribed by integrating both sides of (4.2.5) over a δ - neighbourhood about $\bar{z}_0$ and then taking limit $\delta \to 0$, which defines the jump as,

$$[\![w']\!]_{\bar{z}_0} = \kappa + \alpha w(\bar{z}_0), \tag{4.2.7}$$

We rewrite the ode (4.2.5), along with the continuity and jump in the derivative at $\bar{z}_0$ as,

$$\begin{aligned} w'' + v_s w' - (1+\lambda)w &= 0 \text{ for } z < \bar{z}_0, & (4.2.8)\\ w'' + v_s w' - (1+\lambda-\mu)w &= G \text{ for } z \geq \bar{z}_0, & (4.2.9)\\ [\![w]\!]_{z=\bar{z}_0} &= 0, & (4.2.10)\\ [\![w']\!]_{z=\bar{z}_0} &= \kappa + \alpha w(\bar{z}_0). & (4.2.11)\end{aligned}$$

For $z < \bar{z}_0$, the solution of (4.2.8) is given by

$$w(z) = A_1 e^{\nu_1^+(z-\bar{z}_0)} + B_1 e^{\nu_1^-(z-\bar{z}_0)}, \tag{4.2.12}$$

where

$$\nu_1^\pm = \frac{-v_s \pm \sqrt{v_s^2 + 4 + 4\lambda}}{2}. \tag{4.2.13}$$

for $\lambda \in \mathcal{C}$. To determine the subspaces E_s^+ and E_u^- we study the signs of the real parts of the eigenvalues $\nu_1^\pm$. Using the definition of a from (4.0.8),

$$\mathrm{Re}(\nu_1^\pm) = -\sqrt{a-1} \pm \sqrt{a}\,\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right).$$

We consider the principal branch-cut of the multi-valued complex function $\sqrt{1+\frac{\lambda}{a}}$, and its real part has the form

$$\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) = \frac{1}{\sqrt{2}}\left(\left|1+\frac{\lambda}{a}\right| + 1 + \frac{\lambda_r}{a}\right)^{1/2}. \tag{4.2.14}$$

From (4.0.6), $\lambda \in \mathcal{C}_2$ satisfies $\lambda_r = -l$ and $-ml \leq \lambda_i \leq ml$, and from (4.2.14) we obtain

$$\begin{aligned} \mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) &= \frac{1}{\sqrt{2}}\left[\sqrt{\left(1-\frac{l}{a}\right)^2+\left(\frac{\lambda_i}{a}\right)^2}+1-\frac{l}{a}\right]^{1/2} \\ &\geq \frac{1}{\sqrt{2}}\left[2\left(1-\frac{l}{a}\right)\right]^{1/2} \\ &= \sqrt{\frac{a-l}{a}}, \end{aligned}$$

where the lower bound is well-defined as $l < a$. For, $\lambda \in \mathcal{C}_2$, $l < 1$, implies

$$\mathrm{Re}(\nu_1^+) = -\sqrt{a-1}+\sqrt{a}\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) \geq -\sqrt{a-1}+\sqrt{a-l} > 0, \quad (4.2.15)$$

$$\mathrm{Re}(\nu_1^-) = -\sqrt{a-1}-\sqrt{a}\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) \leq -\sqrt{a-l}+\sqrt{a-1} < 0. \quad (4.2.16)$$

For $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_3$, substituting for $\lambda_i^2 = m^2\lambda_r^2$ from (4.0.5) and (4.0.7), in (4.2.14) we obtain,

$$\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) = \frac{1}{\sqrt{2}}\left[\left(\left(1+\frac{\lambda_r}{a}\right)^2+m^2\left(\frac{\lambda_r}{a}\right)^2\right)^{1/2}+1+\frac{\lambda_r}{a}\right]^{1/2}. \quad (4.2.17)$$

From **Fig(4.2)** we see that

$$\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) \geq 1 \geq \sqrt{\frac{a-1}{a}}, \quad (4.2.18)$$

for $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_3$ provided, $\lambda_r \leq \frac{-4a}{m^2}$. For $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_3$, λ_r is negative and bounded away from the origin by, $-\infty < \lambda_r < -l$. Hence, if we impose the condition

$$-a \leq -l \leq \frac{-4a}{m^2}, \quad (4.2.19)$$

then from (4.2.18)

$$\begin{aligned} \mathrm{Re}(\nu_1^+) &= -\sqrt{a-1}+\sqrt{a}\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) \geq \sqrt{a}-\sqrt{a-1} > 0, \quad (4.2.20) \\ \mathrm{Re}(\nu_1^-) &= -\sqrt{a-1}-\sqrt{a}\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) \leq -\sqrt{a}+\sqrt{a-1} < 0. \quad (4.2.21) \end{aligned}$$

We fix $\nu = \sqrt{a-l} - \sqrt{a-1} < \sqrt{a} - \sqrt{a-1}$, then

$$\begin{aligned} \mathrm{Re}(\nu_1^+) &> \nu, && (4.2.22) \\ \mathrm{Re}(\nu_1^-) &< -\nu. && (4.2.23) \end{aligned}$$

for $\lambda \in \mathcal{C}$ provided (4.2.19) holds. Now, refering back to the solution (4.2.12), we solve for the arbitrary constants A_1 and B_1 by expressing them in terms of the initial data $(w(\bar{z}_0), w'(\bar{z}_0^-))$,

$$\begin{aligned} w(\bar{z}_0) &= A_1 + B_1, && (4.2.24) \\ w'(\bar{z}_0^-) &= \nu_1^+ A_1 + \nu_1^- B_1. && (4.2.25) \end{aligned}$$

To construct a solution that decays as $z \longrightarrow -\infty$, we set the coefficient B_1 of the growing part of the solution (4.2.12) to zero, which reduces the solution (4.2.12) and (4.2.25) to

$$\begin{aligned} w(z) &= w(\bar{z}_0) e^{\nu_1^+ (z - \bar{z}_0)} \quad \text{for} \quad z < \bar{z}_0, && (4.2.26) \\ w'(\bar{z}_0^-) &= \nu_1^+ w(\bar{z}_0). && (4.2.27) \end{aligned}$$

For $z > \bar{z}_0$, we solve the inhomogeneous equation (4.2.9) to obtain

$$w(z) = A_2 e^{\nu_2^+ (z - \bar{z}_0)} + B_2 e^{\nu_2^- (z - \bar{z}_0)} + \int_{\bar{z}_0}^{z} R(z, \xi) G(\xi) d\xi, \qquad (4.2.28)$$

where $R(z, \xi)$ is defined as

$$R(z, \xi) = \frac{e^{\nu_2^- (z - \xi)} - e^{\nu_2^+ (z - \xi)}}{\nu_2^- - \nu_2^+}, \qquad (4.2.29)$$

and the eigenvalues $\nu_2^\pm$ are given by

$$\nu_2^\pm = \frac{-v_s \pm \sqrt{v_s^2 + 4(1 + \lambda - \mu)}}{2} = -\sqrt{a-1} \pm \sqrt{a-\mu} \sqrt{1 + \frac{\lambda}{a - \mu}}. \qquad (4.2.30)$$

We note that $\mu < 1 < a$. To examine the signs of the real part of $\nu_2^\pm$, we follow the similar steps as before. The real part of the principal square root of $1 + \frac{\lambda}{a - \mu}$ is given

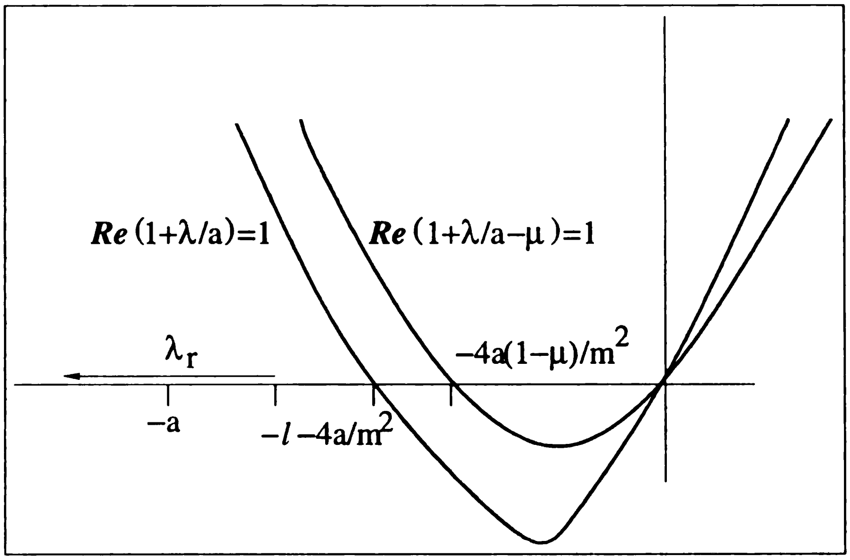

Figure 4.2. The graphs of $\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a}}\right) = 1$ and $\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a-\mu}}\right) = 1$ for $\lambda \in \mathcal{C}$.

by

$$\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a-\mu}}\right) = \frac{1}{\sqrt{2}}\left(\left|1+\frac{\lambda}{a-\mu}\right| + 1 + \frac{\lambda_r}{a-\mu}\right)^{1/2}. \tag{4.2.31}$$

From (4.0.6), $\lambda_r = -l > -(a-\mu)$ and $-ml \le \lambda_i \le ml$, for $\lambda \in \mathcal{C}_2$ yields

$$\begin{aligned} \mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a-\mu}}\right) &= \frac{1}{\sqrt{2}}\left[\sqrt{\left(1-\frac{l}{a-\mu}\right)^2 + \left(\frac{\lambda_i}{a-\mu}\right)^2} + 1 - \frac{l}{a-\mu}\right]^{1/2}, \\ &\ge \sqrt{1-\frac{l}{a-\mu}}. \end{aligned}$$

Hence $\lambda \in \mathcal{C}_2$ satisfies

$$\mathrm{Re}(\nu_2^+) \ge -\sqrt{a-1} + \sqrt{a-\mu}\sqrt{1-\frac{l}{a-\mu}} > 0, \tag{4.2.32}$$

$$\mathrm{Re}(\nu_2^-) \le \sqrt{a-1} - \sqrt{a-\mu}\sqrt{1-\frac{l}{a-\mu}} < 0. \tag{4.2.33}$$

For $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_3$, substituting for $\lambda_1^2 = m^2\lambda_r^2$ from the definition of $\mathcal{C}$ given by (4.0.5) and (4.0.7), into (4.2.31) we obtain

$$\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a-\mu}}\right) = \frac{1}{\sqrt{2}}\left[\left(\left(1+\frac{\lambda_r}{a-\mu}\right)^2 + m^2\left(\frac{\lambda_r}{a-\mu}\right)^2\right)^{1/2} + 1 + \frac{\lambda_r}{a-\mu}\right]^{1/2}. \tag{4.2.34}$$

From **Fig(4.2)**, for $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_3$

$$\mathrm{Re}\left(\sqrt{1+\frac{\lambda}{a-\mu}}\right) \geq 1,$$

provided $\lambda_r \leq \frac{-4a(1-\mu)}{m^2}$. For $\lambda \in \mathcal{C}$, the real part of λ satisfies, $-\infty < \lambda_r < -l$. Hence from (4.2.34) we can achieve

$$\mathrm{Re}(\nu_2^+) \geq -\sqrt{a-1} + \sqrt{a-\mu}, \tag{4.2.35}$$

$$\mathrm{Re}(\nu_2^-) \leq \sqrt{a-1} - \sqrt{a-\mu}, \tag{4.2.36}$$

if we impose

$$-a+\mu \leq -l \leq \frac{-4a(1-\mu)}{m^2}. \tag{4.2.37}$$

which is less stringent than the condition (4.2.19) imposed on l. Also note that for (4.2.37) to be consistent we require $-a+\mu < \frac{-4a(1-\mu)}{m^2}$, which gives a lower bound on the slope m of the branches

$$m^2 \geq \frac{4a(1-\mu)}{a-\mu}. \tag{4.2.38}$$

We fix $\nu' = -\sqrt{a-1} + \sqrt{a-\mu-l} > -\sqrt{a-1} + \sqrt{a-\mu} > 0$, then

$$\mathrm{Re}(\nu_2^+) \leq \nu', \tag{4.2.39}$$

$$\mathrm{Re}(\nu_2^-) \geq -\nu'. \tag{4.2.40}$$

Thus, for any $\lambda \in \mathcal{C}$, satisfying the condition (4.2.37) and (4.2.38) we obtain the relations (4.2.22), (4.2.23) and (4.2.39), (4.2.40) on $\mathbf{Re}(\nu_1^\pm)$ and $\mathbf{Re}(\nu_2^\pm)$. Refering back to the solution (4.2.28) for $z > \bar{z}_0$, substituting the expression for the kernel $R(z,\xi)$ into (4.2.28) and combining the decaying and growing parts, we obtain

$$\begin{aligned} w(z) &= \left[B_2 e^{\nu_2^-(z-\bar{z}_0)} + \int_{\bar{z}_0}^{z} e^{\nu_2^-(z-\xi)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi \right] \\ &+ \left[A_2 e^{\nu_2^+(z-\bar{z}_0)} - \int_{\bar{z}_0}^{z} e^{\nu_2^+(z-\xi)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi \right]. \end{aligned} \tag{4.2.41}$$

Since w is continuous at $\bar{z}_0$, we see that,

$$w(\bar{z}_0) = A_2 + B_2. \tag{4.2.42}$$

Evaluating the derivative of $w(z)$ given by (4.2.41) at $\bar{z}_0$,

$$w'(\bar{z}_0^+) = A_2\nu_2^+ + B_2\nu_2^-. \tag{4.2.43}$$

Substituting for $w'(\bar{z}_0^-)$ and $w'(\bar{z}_0^+)$ from (4.2.27) and (4.2.43) into the jump condition (4.2.11) we obtain,

$$A_2\nu_2^+ + B_2\nu_2^- = \kappa + (\nu_1^+ + \alpha)w(\bar{z}_0). \tag{4.2.44}$$

Solving for the arbitrary constants A_2 and B_2 from (4.2.42) and (4.2.44) we obtain,

$$A_2(w(\bar{z}_0)) = \frac{(\nu_1^+ - \nu_2^- + \alpha)w(\bar{z}_0) + \kappa}{\nu_2^+ - \nu_2^-}, \tag{4.2.45}$$

$$B_2(w(\bar{z}_0)) = \frac{w(\bar{z}_0)(\nu_1^+ - \nu_2^+ + \alpha) + \kappa}{\nu_2^+ - \nu_2^-}. \tag{4.2.46}$$

It remains to tune $w(\bar{z}_0)$ so that $w(z) \to 0$ as $z \to \infty$. This is acheived by balancing the coefficient A_2 against the inhomogeneous term in (4.2.41)

$$A_2(w(\bar{z}_0)) = \frac{1}{\nu_2^- - \nu_2^+} \int_{\bar{z}_0}^{\infty} e^{-\nu_2^+(\xi-\bar{z}_0)} G(\xi) d\xi. \tag{4.2.47}$$

Since $\mathrm{Re}(\nu_2^+ > \nu)$, the integral in the right hand side of (4.2.47) is convergent and we denote its value by M

$$M = \int_{\bar{z}_0}^{\infty} e^{-\nu_2^+(\xi-\bar{z}_0)} G(\xi) d\xi, \tag{4.2.48}$$

which modifies (4.2.47) to

$$A_2(w(\bar{z}_0)) = \frac{M}{\nu_2^- - \nu_2^+}. \tag{4.2.49}$$

Using (4.2.39), M is bounded above by the following

$$|M| \leq \frac{\|G\|_{L^2}}{\sqrt{\mathrm{Re}(\nu_2^+)}} \leq \frac{\|G\|_{L^2}}{\nu'}. \tag{4.2.50}$$

Substituting (4.2.45) into (4.2.47), the solution $w^*(\bar{z}_0)$ of (4.2.47) is given by

$$w^*(\bar{z}_0) = \frac{M + \kappa}{\nu_2^- - \nu_1^+ - \alpha}. \tag{4.2.51}$$

From (4.2.22), (4.2.23), (4.2.39) and (4.2.40) we obtain

$$\left|\nu_2^- - \nu_1^+ - \alpha\right| \geq \left|\mathrm{Re}(\nu_2^-) - \mathrm{Re}(\nu_1^+) - \alpha\right| \geq \nu' + \nu + \alpha, \tag{4.2.52}$$

$$\left|\nu_2^+ - \nu_2^-\right| \geq \left|\mathrm{Re}(\nu_2^+) - \mathrm{Re}(\nu_2^-)\right| \geq 2\nu'. \tag{4.2.53}$$

Using and (4.2.50) and (4.2.52), $|w^*(\bar{z}_0)|$ given by (4.2.51) satisfies the bound

$$|w^*(\bar{z}_0)| \leq C\left(\|G\|_{L^2} + |\kappa|\right). \tag{4.2.54}$$

The inequalities (4.2.50) and (4.2.53) provides a bound for A_2 given by (4.2.49)

$$|A_2| \leq C\,\|G\|_{L^2}. \tag{4.2.55}$$

Denoting $B^* = B_2(w^*(\bar{z}_0)) = w^*(\bar{z}_0) - A_2$, and using (4.2.55) and (4.2.54), we have the following bound

$$|B^*| \leq C\left(\|G\|_{L^2} + |\kappa|\right). \tag{4.2.56}$$

Substituting for A_2 from (4.2.47) into (4.2.41), the solution of (4.2.9) for $z > \bar{z}_0$, is

$$\begin{aligned} w(z) &= \left[B^* e^{\nu_2^-(z-\bar{z}_0)} + \int_{\bar{z}_0}^{z} e^{\nu_2^-(z-\xi)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi\right] \\ &+ \left[\int_{\bar{z}_0}^{\infty} e^{-\nu_2^+(\xi-z)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi - \int_{\bar{z}_0}^{z} e^{-\nu_2^+(\xi-z)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi\right], \\ &= \left[B^* e^{\nu_2^-(z-\bar{z}_0)} + \int_{\bar{z}_0}^{z} e^{\nu_2^-(z-\xi)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi\right] + \left[\int_{z}^{\infty} e^{-\nu_2^+(\xi-z)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi\right]. \end{aligned} \tag{4.2.57}$$

Hence the solution of (4.2.5) is given by

$$w(z) = \begin{cases} w^*(\bar{z}_0)e^{\nu_1^+(z-\bar{z}_0)} & z \leq \bar{z}_0 \\ B^* e^{\nu_2^-(z-\bar{z}_0)} + \int\limits_{\bar{z}_0}^{z} e^{\nu_2^-(z-\xi)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi + \int\limits_{z}^{\infty} e^{-\nu_2^+(\xi - z)} \frac{G(\xi)}{\nu_2^- - \nu_2^+} d\xi & z \geq \bar{z}_0 \end{cases} \tag{4.2.58}$$

Computing the L^2 norm of w, we find

$$\|w\|_{L^2} \leq \frac{|w^*(\bar{z}_0)|}{\sqrt{\mathrm{Re}(\nu_1^+)}} + \frac{|B^*|}{\sqrt{\mathrm{Re}(-\nu_2^-)}} + \frac{\|G\|_{L^2}}{\sqrt{\mathrm{Re}(-\nu_2^-)}\,|\nu_2^- - \nu_2^+|} + \frac{\|G\|_{L^2}}{\sqrt{\mathrm{Re}(\nu_1^+)}\,|\nu_2^- - \nu_2^+|}. \tag{4.2.59}$$

Using relations (4.2.22), (4.2.23), (4.2.39), (4.2.40) and (4.2.53) determining the lower bound of the real parts of $\nu_1^\pm$ and $\nu_2^\pm$, and substituting for $|w^*(\bar{z}_0)|$ and $|B^*|$ from (4.2.54) and (4.2.56) into (4.2.59) we obtain

$$\|w\|_{L^2} \;\leq\; C\left(\|G\|_{L^2} + |\kappa|\right). \tag{4.2.60}$$

From Lemma 4.1.1 we obtain the following estimates for $\|G\|_{L^2}$ and $|\kappa|$,

$$\begin{aligned} \|G\|_{L^2} &= \left\|\left(\mathcal{L}^{-1}F\right)\chi_{(\bar{z}_0,\infty)}\right\|_{L^2} \leq C\frac{\|F\|_{H^{-\beta}}}{|\lambda + a|^{(2-\gamma-\beta)/2}}, \\ |\kappa| &= \left|\alpha v_s \left(\mathcal{L}^{-1}F\right)(\bar{z}_0)\right| \leq C\frac{\|F\|_{H^{-\beta}}}{|\lambda + a|^{(2-\gamma-\beta)/2}}, \end{aligned}$$

which yields an upper bound for $\|w\|_{L^2}$

$$\|w\|_{L^2} \leq C\frac{\|F\|_{H^{-\beta}}}{|\lambda + a|^{(2-\gamma-\beta)/2}}. \tag{4.2.61}$$

From (4.2.58), we compute $\|w\|_{L^\infty}$. Using the lower bounds on the real parts of $\nu_1^\pm$ and $\nu_2^\pm$ we obtain

$$\|w\|_{L^\infty} \;\leq\; C\left(|w^*| + |B^*| + \|G\|_{L^2}\right). \tag{4.2.62}$$

Substituting for $|w^*(\bar{z}_0)|$ and $|B^*|$ from (4.2.54) and (4.2.56) into (4.2.62) we find

$$\|w\|_{L^\infty} \;\leq\; C\left(|\kappa| + \|G\|_{L^2}\right) \leq c\frac{\|F\|_{H^{-\beta}}}{|\lambda + a|^{(2-\gamma-\beta)/2}}. \tag{4.2.63}$$

■

We define the semigroup $S_{\bar{\phi}}(t) \equiv S(t)$ generated by $L_{\bar{\phi}}$, as an operator $S(t) : H^{-\beta} \to H^{\gamma}$ which maps $F \in H^{-\beta}$ to the solution at time t of the initial-value problem

$$W_t = L_{\bar{\phi}} W, \tag{4.2.64}$$

$$W(0) = F. \tag{4.2.65}$$

If $L_{\bar{\phi}}$ is sectorial, then for $F \in H^{-\beta}$, $S(t)$ has the following representation

$$S(t)F = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\lambda t} (L - \lambda)^{-1} F d\lambda. \tag{4.2.66}$$

By the aid of the previous lemma, we obtain the bound on the H^{γ} norm of the solution u of the resolvent problem (4.1.1).

Lemma 4.2.2 *For all $\gamma + \beta < 2$ $\exists$ $C > 0$ such that for all λ on the contour $\mathcal{C}$ and $F \in H^{-\beta}$*

$$\|u\|_{H^{\gamma}} \leq \frac{C \|F\|_{H^{-\beta}}}{(|\lambda + a|)^{(2-\gamma-\beta)/2}}. \tag{4.2.67}$$

where $u = \mathcal{L}^{-1} F$. Refering to $\tilde{\pi}_0$ defined in (4.0.4), the semigroup action $\|S(t)\tilde{\pi}_0 F\|_{H^{\gamma}}$ satisfies

$$\|S(t)\tilde{\pi}_0 F\|_{H^{\gamma}} \leq \frac{Ce^{-\nu t}}{t^{(\gamma+\beta)/2}} \|F\|_{H^{-\beta}}, \tag{4.2.68}$$

for all $F \in H^{-\beta}$.

Proof: Refering to the decomposition (4.2.1) of the solution u of (4.1.1)

$$u = \mathcal{L}^{-1} F + w,$$

we obtain an upper bound $\|u\|_{L^2}$ using (4.1.6) and (4.2.6),

$$\|u\|_{L^2} \leq \left\|\mathcal{L}^{-1} F\right\|_{L^2} + \|w\|_{L^2} \leq \frac{c \|F\|_{H^{-\beta}}}{(|\lambda + a|)^{(2-\gamma-\beta)/2}}. \tag{4.2.69}$$

Note that $\left\|\mathcal{L}^{-1}F\right\|_{L^\infty} \leq \left\|\mathcal{L}^{-1}F\right\|_{H^\gamma}$. Using (4.2.63) and (4.1.6), $\|u\|_{L^\infty}$ can be bounded as

$$\|u\|_{L^\infty} \leq \left\|\mathcal{L}^{-1}F\right\|_{H^\gamma} + \|w\|_{L^\infty} \leq \frac{C\,\|F\|_{H^{-\beta}}}{(|\lambda+a|)^{(2-\gamma-\beta)/2}}. \tag{4.2.70}$$

From formulation (4.1.4), $(L-\lambda)u = F$ can be written as,

$$\mathcal{L}u = F - \alpha u(\bar{z}_0)\delta_{\bar{z}_0} - \mu u\chi_{(\bar{z}_0,\infty)}. \tag{4.2.71}$$

Inverting $\mathcal{L}$ yields

$$u = \mathcal{L}^{-1}F - \alpha u(\bar{z}_0)\mathcal{L}^{-1}\delta_{\bar{z}_0} - \mu\mathcal{L}^{-1}u\chi_{(\bar{z}_0,\infty)}. \tag{4.2.72}$$

Now, taking the H^γ norm of u given as in (4.2.72) and using the estimate (4.1.6) from Lemma 4.1.1,

$$\begin{aligned}
\|u\|_{H^\gamma} &\leq \left\|\mathcal{L}^{-1}F\right\|_{H^\gamma} + \alpha\,|u(\bar{z}_0)|\left\|\mathcal{L}^{-1}\delta_{\bar{z}_0}\right\|_{H^\gamma} + \mu\left\|\mathcal{L}^{-1}u\chi_{[\bar{z}_0,\infty)}\right\|_{H^\gamma} \\
&\leq \frac{c\,\|F\|_{H^{-\beta}}}{|\lambda+a|)^{(2-\gamma-\beta)/2}} + \frac{c\alpha\,\|u\|_{L^\infty}\left\|\delta_{\bar{z}_0}\right\|_{H^{-\beta}}}{|\lambda+a|^{(2-\gamma-\beta)/2}} + \frac{c\,\|u\|_{L^2}}{|\lambda+a|^{(2-\gamma-\beta)/2}}
\end{aligned}$$

Using the bounds from (4.2.69) and (4.2.70) we obtain

$$\|u\|_{H^\gamma} \leq \frac{c\,\|F\|_{H^{-\beta}}}{|\lambda+a|^{(2-\gamma-\beta)/2}}.$$

for all $\lambda \in \mathcal{C}$. As per definition (4.0.1) introduced in the begining of this Section, $L_{\bar{\phi}}$ is a sectorial operator and the semigroup $S(t)$ generated by $L_{\bar{\phi}}$ can be espressed as (4.2.66). Taking the H^γ norm on both sides of (4.2.66) and using the bound for $\|u\|_{H^\gamma}$ from (4.2.67) we obtain,

$$\begin{aligned}
\|S(t)\tilde{\pi}_0 F\|_{H^\gamma} &\leq c\int_{\mathcal{C}} \left|e^{\lambda t}\right| \left\|(L-\lambda)^{-1}\tilde{\pi}_0 F\right\|_{H^\gamma} d\lambda, \\
&\leq c\int_{\mathcal{C}} \left|e^{\lambda t}\right| \frac{\|\tilde{\pi}_0 F\|_{H^{-\beta}}}{|\lambda+a|^{(2-\gamma-\beta)/2}} d\lambda.
\end{aligned} \tag{4.2.73}$$

We prove that there exists constants $\nu > 0$ and $\omega > 0$ such that,

$$\lambda_r \leq -(\nu + \omega\,|\lambda|), \tag{4.2.74}$$

for $\lambda \in \mathcal{C}$, where $\lambda_r = Re(\lambda)$. We define $\nu = \frac{l}{2}$ and $\omega = \frac{1}{2\sqrt{1+m^2}}$. For $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_3$, where $\mathcal{C}_1$ and $\mathcal{C}_3$ are given by (4.0.5) and (4.0.7),

$$\begin{aligned} \lambda_r + \omega\,|\lambda| &= \lambda_r + \frac{1}{2\sqrt{1+m^2}}\sqrt{\lambda_r^2 + m^2\lambda_r^2} \\ &= \frac{\lambda_r}{2} \leq \frac{-l}{2} = -\nu. \end{aligned}$$

For $\lambda \in \mathcal{C}_2$, from (4.0.6) we obtain

$$\begin{aligned} \lambda_r + \omega\,|\lambda| &= -l + \frac{1}{2\sqrt{1+m^2}}\sqrt{l^2 + \lambda_i^2} \\ &\leq -l + \frac{1}{2\sqrt{1+m^2}}\sqrt{l^2 + m^2 l^2} = \frac{-l}{2} = -\nu. \end{aligned} \tag{4.2.75}$$

Using (4.2.74) to bound λ in the integral (4.2.73), we obtain

$$\|S(t)\tilde{\pi}_0 F\|_{H^\gamma} \leq Ce^{-\nu t}\left(\int_{\mathcal{C}} \frac{e^{-\omega|\lambda|t}}{|\lambda + a|^{(2-\gamma-\beta)/2}} d\lambda\right)\|\tilde{\pi}_0 F\|_{H^{-\beta}}.$$

For large time the integral is uniformly bounded. For $t \ll 1$ we set $\tilde{\lambda} = |\lambda|\,t$ and the integral transforms to,

$$\begin{aligned} \left\|S(t)\pi_{\bar{z}_0}F\right\|_{H^s} &\leq Ce^{-\nu t}\left(\int_{\mathcal{C}} \frac{e^{-\omega\tilde{\lambda}}}{\left(a + \left(\frac{\tilde{\lambda}}{t}\right)^{(2-\gamma-\beta)/2}\right)} t^{-1} d\tilde{\lambda}\right)\|\tilde{\pi}_0 F\|_{H^{-\beta}}, \\ &\leq \frac{Ce^{-\nu t}}{t^{(\gamma+\beta)/2}}\left(\int_{\mathcal{C}} \frac{e^{-a\tilde{\lambda}}}{\left(at^{(2-\gamma-\beta)/2} + \tilde{\lambda}^{(2-\gamma-\beta)/2}\right)} d\tilde{\lambda}\right)\|\tilde{\pi}_0 F\|_{H^{-\beta}} \end{aligned} \tag{4.2.76}$$

For $\gamma + \beta < 2$, the integral is uniformly bounded since the singularity at $t = 0$ is integrable. We also have

$$\begin{aligned} \|\tilde{\pi}_0 F\|_{H^{-\beta}} &\leq \|F\|_{H^{-\beta}} + \left|\left(F, \Psi_{\bar{z}_0}^{\dagger}\right)\right| \left\|\Psi_{\bar{z}_0}^{\dagger}\right\|_{H^{-\beta}}, \\ &\leq C\,\|F\|_{H^{-\beta}}, \end{aligned}$$

Hence, we obtain the semigroup estimate (4.2.68) for $F \in H^{-\beta}$ and $\beta > 1/2$. ■

CHAPTER 5

The RG methodology

We employ the modification of RG method developed in [1, 33, 34, 38] for asymptotic decay of dissipative pde.. In this section we employ the smoothing bounds on the semigroup $S(t)$ from **Section 4** to obtain decay estimates on the remainder W given by (3.2.1).

5.1 Overview

We decompose the solution of the initial value problem (IVP)

$$U_t = F(U), \tag{5.1.1}$$

$$U(0) = U_0, \tag{5.1.2}$$

where F is given by (3.1.3), as

$$U(z,t) = \phi(z; z_0) + W(z,t), \tag{5.1.3}$$

where the front position $z_0 = z_0(t)$ is time dependent. The IVP (5.1.1) becomes

$$W_t + \frac{\partial \phi}{\partial z_0} z_0' = F(\varphi) + L_{z_0} W + \mathcal{N}(W), \tag{5.1.4}$$

$$W(z,0) = U_0 - \phi(z; z_0(0)). \tag{5.1.5}$$

The operator L_{z_0} is time-dependent. To eliminate this obstacle, we recast the IVP (5.1.4) as a series of IVP's on $[t_n, t_{n+1}]$ each with a fixed coordinate system determined by $\bar{z}_n$. Given the solution U at time t_n, we first determine the frozen front location $\bar{z}_n$ so that

$$W_n(0) \equiv U(\cdot, t_n) - \phi(z; \bar{z}_n) \in \tilde{X}_{\bar{z}_n}. \tag{5.1.6}$$

W_n now evolves according to

$$\begin{aligned} W_t + \frac{\partial \phi}{\partial z_0} z_0' &= F(\phi) + L_{\bar{z}_n} W + (L_{z_0} - L_{\bar{z}_n}) W + \mathcal{N}(W) && (5.1.7) \\ W(z, t_n) &= W_n && (5.1.8) \end{aligned}$$

with z_0' chosen to enforce $W(t) \in \mathcal{R}(\pi_{\bar{z}_n})$ for $t \in (t_n, t_{n+1})$. Note that the front location z_0 is discontinuous at the renormalization times. The diagram **Fig(5.1)** shows that $z_0(t)$ jumps from $z_0(t_n^-)$ to $\bar{z}_n$ at $t = t_n$.The RG equation is then the map from $W_n \to W_{n+1}$. The nonlinear stability of the system via RG methods exploits the fact that, the evolution of the front location z_0 is on a slower time scale and the secularity $L_{zo} - L_{\bar{z}_0}$ is a lower order operator than either of L_{z_0} or $L_{\bar{z}_0}$. In this manner we attain uniform decay estimates without uniform smoothing bounds on the semigroup $S(t)$ for $t > t_0$.

Throughout this section a subscript $\bar{z}_n$ indicates a quantity associated to the coordinate system centered about the base point $z = \bar{z}_n$. For example, $\Psi_{\bar{z}_n} = \Psi(z; \bar{z}_n)$ denotes the principle eigenfunction of $L_{\bar{z}_n}$ where $L_{\bar{z}_n}$ denotes the linearization of F at $\phi(z; \bar{z}_n)$. We assume at time $t = t_0$, the initial data U_0 satisfies,

$$\left\| \phi_{z_0^*} - U_0 \right\|_{H^\gamma} \le \delta, \tag{5.1.9}$$

for some $z_0^* \in \mathbf{R}$, where $\delta > 0$ will be specified later. The following proposition allows us to choose our base point $\bar{z}_0$ about which we develop the local coordinate system for the RG method.

Proposition 5.1.1 *Fix $\delta \ll 1$. Assume U_0 and $z_0^* \in \boldsymbol{R}$ satisfy $\|W_*\|_{H^\gamma} \le \delta$, where the remainder $W_* \equiv U_0 - \phi_{z_0^*}$. There exists $M > 0$, independent of U_0 and z_0^*, and*

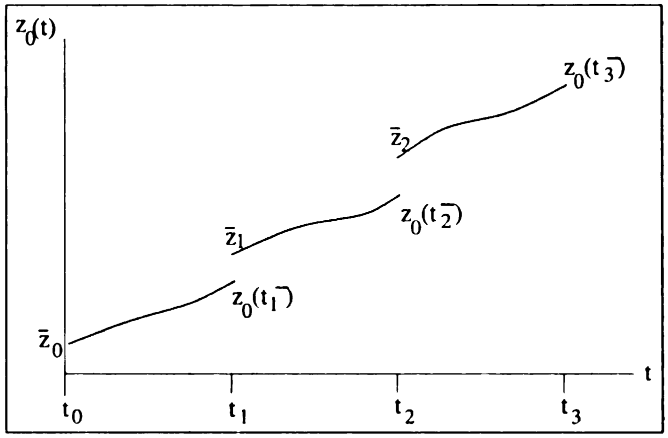

Figure 5.1. Evolution of front location due to the iterative scheme.

a smooth function $\mathcal{H} : X_{\bar{z}_0} \to \boldsymbol{R}$ such that $\bar{z}_0 = z_0^* + \mathcal{H}(W_*)$ satisfies,

$$W_0 \equiv U_0 - \phi_{\bar{z}_0} \in X_{\bar{z}_0}. \tag{5.1.10}$$

Moreover, if $W_* \in X_{\tilde{z}_0}$ for some $\tilde{z}_0 \in \boldsymbol{R}$ then

$$|\bar{z}_0 - z_0^*| \leq M_0 \, |\bar{z}_0 - \tilde{z}_0| \, \|W_*\|_{H^\gamma} . \tag{5.1.11}$$

Proof: Substituting for $U_0 = \phi_{z_0^*} + W_*$ in the definition of W_0 given by (5.1.10) we obtain

$$W_0 = W_* + \phi_{z_0^*} - \phi_{\bar{z}_0}. \tag{5.1.12}$$

The condition (5.1.10) is equivalent to the orthogonality of W_0 to $\Psi^\dagger_{\bar{z}_0}$

$$\pi_{\bar{z}_0} W_0 = \pi_{\bar{z}_0} \left(W_* + \phi_{z_0^*} - \phi_{\bar{z}_0} \right) = 0. \tag{5.1.13}$$

We prove by the Implicit function theorem that, for given W_* there exists $\bar{z}_0$ satisfying the equation (5.1.13). Defining $\Lambda(\bar{z}_0, W_*)$ as

$$\Lambda(\bar{z}_0, W_*) = \left(W_* + \phi_{z_0^*} - \phi_{\bar{z}_0}, \Psi^\dagger_{\bar{z}_0} \right) . \tag{5.1.14}$$

we observe that $\Lambda(\bar{z}_0 = z_0^*, W_* = 0) = 0$. The partial derivative of Λ w.r.t $\bar{z}_0$ is given by,

$$\frac{\partial \Lambda}{\partial \bar{z}_0} = \left(\frac{\partial \phi_{\bar{z}_0}}{\partial \bar{z}_0}, \Psi^{\dagger}_{\bar{z}_0}\right) + \left(W_* + \phi_{z_0^*} - \phi_{\bar{z}_0}, \frac{\partial \Psi^{\dagger}_{\bar{z}_0}}{\partial \bar{z}_0}\right),$$

and refering to **Lemma 3.6.3**, the inner product of $\frac{\partial \phi_{\bar{z}_0}}{\partial \bar{z}_0}$ and $\Psi^{\dagger}_{\bar{z}_0}$ is uniformly bounded away from zero for all $\bar{z}_0 \in \mathbf{R}$. Thus

$$\left.\frac{\partial \Lambda}{\partial \bar{z}_0}\right|_{\bar{z}_0 = z_0^*, W_* = 0} = \left.\left(\frac{\partial \phi_{z_0}}{\partial z_0}, \Psi^{\dagger}_{z_0}\right)\right|_{z_0 = z_0^*, W_* = 0} \geq \theta_0 > 0. \tag{5.1.15}$$

The implicit function theorem guarantees the existence of a smooth function $\mathcal{H}$ which provides a solution of (5.1.13) for U_0 in a neighbourhood of the manifold,

$$\mathcal{M} = \left\{\phi_{z_0} | z_0 \in \mathbf{R}\right\}. \tag{5.1.16}$$

In addition if $W_* \in X_{\bar{z}_0}$ then $\left(W_*, \Psi^{\dagger}_{\bar{z}_0}\right) = 0$. By the result above, there exists $\bar{z}_0 \in \mathbf{R}$ such that

$$\left(W_* + \phi_{z_0^*} - \phi_{\bar{z}_0}, \Psi^{\dagger}_{\bar{z}_0}\right) = 0.$$

Applying the mean value theoem to $\phi_{\bar{z}_0}$ on the interval $(z_0^*, \bar{z}_0)$ and observing that $\left\|\frac{\partial \phi_{\bar{z}_0}}{\partial \bar{z}_0}\right\|_{L^2} = \mathcal{O}(1)$ and $\left\|\Psi^{\dagger}_{z_0}\right\|_{L^2} = \mathcal{O}(1)$ we find,

$$\begin{aligned}
\left|\left(W_*, \Psi^{\dagger}_{z_0}\right)\right| &= \left|\left(\phi_{z_0^*} - \phi_{z_0}, \Psi^{\dagger}_{z_0}\right)\right|, \\
&= |z_0^* - \bar{z}_0| \left|\left(\frac{\partial \phi(z; z_0^{\xi})}{\partial z_0^{\xi}}, \Psi^{\dagger}_{z_0}\right)\right|, \\
&= \theta(z_0^{\xi}) |z_0^* - \bar{z}_0| + \mathcal{O} |z_0^* - \bar{z}_0|^2.
\end{aligned} \tag{5.1.17}$$

where $z_0^{\xi} \in (\bar{z}_0, z_0^*)$ and from Lemma 3.6.3, $\theta(z_0^{\xi}) \neq 0$. Alternatively we estimate $\left|\left(W_*, \Psi^{\dagger}_{\bar{z}_0}\right)\right|$ as,

$$\left|\left(W_*, \Psi^{\dagger}_{\bar{z}_0}\right)\right| = \left|\left(W_*, \Psi^{\dagger}_{\bar{z}_0} - \Psi^{\dagger}_{\bar{z}_0}\right)\right| \leq C |\bar{z}_0 - \bar{z}_0| \, \|W_*\|_{H^{\gamma}}. \tag{5.1.18}$$

This gives the desired bound,

$$|z_0^* - \bar{z}_0| \le C\,|\bar{z}_0 - \tilde{z}_0|\,\|W_*\|_{H^\gamma}\,.$$

■

In application of Proposition 4.1 at time $t = t_n$ we take

$$\begin{aligned} \tilde{z}_0 &= \bar{z}_{n-1}, & (5.1.19)\\ z_* &= z_0(t_{n-1}), & (5.1.20)\\ \text{and construct } \bar{z}_0 &= \bar{z}_n. & (5.1.21)\end{aligned}$$

5.2 Control of Residual

We now estimate the decay in the remainder over the interval $[t_{n-1}, t_n]$. Without loss of generality we consider $n = 1$. For the underdetermined decomposition,

$$U(z,t) = \phi(z; z_0) + W(z,t), \tag{5.2.1}$$

the evolution for the remainder W takes the form,

$$\begin{aligned} W_t + \frac{\partial \phi}{\partial z_0} z_0' &= R + L_{\bar{z}_0} W + \Delta L\, W + \mathcal{N}(W) + \epsilon^{3/4-\gamma_s/2}\eta_s(\xi,t), & (5.2.2)\\ W(\xi,0) &= W_0. & (5.2.3)\end{aligned}$$

where $\bar{z}_0$ and hence W_0 are provided by **Proposition 4.1**. The term $\Delta L = L_{z_0} - L_{\bar{z}_0}$ describes the secular growth implicit in z_0 sliding away from the frozen front location $\bar{z}_0$ and takes the form,

$$\Delta L\, W = (v_s(z_0) - v_s(\bar{z}_0))\partial_z W + \sigma_c\Big(\frac{W(z_0)}{\phi'(z_0; z_0)}\delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0; \bar{z}_0)}\delta_{\bar{z}_0}\Big) + \mu\chi_{(\bar{z}_0, z_0)} W. \tag{5.2.4}$$

The residual R is given by (3.1.22). The decomposition (5.2.1) is made determinate by requiring $W \in X_{\bar{z}_0}$ which is achieved by imposing the non-degeneracy condition

$\pi_{\bar{z}_0} W_t = 0$, which is equivalent to the following equation,

$$\left(\frac{\partial \phi_{z_0}}{\partial z_0}, \Psi^\dagger_{\bar{z}_0}\right)_{L^2} z_0' = \left(R + L_{\bar{z}_0} W + \Delta L\, W + \mathcal{N}(W) + \epsilon^{3/4-\gamma_s/2}\eta_s(\cdot,t), \Psi^\dagger_{\bar{z}_0}\right)_{L^2}. \tag{5.2.5}$$

Solving for z_0' we obtain the front evolution equation,

$$z_0' = \frac{\left(R + L_{\bar{z}_0} + \Delta L\, W + \mathcal{N}(W) + \epsilon^{3/4-\gamma_s/2}\eta_s(\cdot,t), \Psi^\dagger_{\bar{z}_0}\right)_{L^2}}{\left(\frac{\partial \phi}{\partial z_0}, \Psi^\dagger_{\bar{z}_0}\right)_{L^2}}, \tag{5.2.6}$$

which is well-defined since $\theta = \left(\frac{\partial \phi}{\partial z_0}, \Psi^\dagger_{\bar{z}_0}\right)_{L^2}$ is uniformly bounded away from zero by **Lemma 3.6.3**. We observe that,

$$\left(L_{\bar{z}_0} W, \Psi^\dagger_{\bar{z}_0}\right) = \left(W, L^\dagger_{\bar{z}_0} \Psi^\dagger_{\bar{z}_0}\right) = \left(W, \lambda_0^\dagger \Psi^\dagger_{\bar{z}_0}\right) = 0,$$

since $W \in X_{\bar{z}_0}$. This reduces equation (5.2.6) for front velocity to,

$$z_0' = \frac{\left(R + \Delta L\, W + \mathcal{N}(W) + \epsilon^{3/4-\gamma_s/2}\eta_s(\cdot,t), \Psi^\dagger_{\bar{z}_0}\right)_{L^2}}{\left(\frac{\partial \phi}{\partial \bar{z}_0}, \Psi^\dagger_{\bar{z}_0}\right)_{L^2}}. \tag{5.2.7}$$

To estimate the terms in the inner product we examine the residual R. From the bound on the residual given by (3.1.23) and the observation that $\left\|\Psi^\dagger_{\bar{z}_0}\right\|_{L^1} = \mathcal{O}(1)$, we find,

$$\begin{aligned} \left|\left(R, \Psi^\dagger_{\bar{z}_0}\right)\right| &\leq \|R\|_{L^\infty} \left\|\Psi^\dagger_{\bar{z}_0}\right\|_{L^1}, \\ &\leq C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + \epsilon\right). \end{aligned} \tag{5.2.8}$$

From (5.2.4) we see that of the projection of secularity $\Delta L\, W$ onto $\Psi^\dagger_{\bar{z}_0}$ takes the form,

$$\begin{aligned} \left|\left(\Delta L\, W, \Psi^\dagger_{\bar{z}_0}\right)\right| &\leq |v_s(z_0) - v_s(\bar{z}_0)| \left|\left(\partial_z W, \Psi^\dagger_{\bar{z}_0}\right)\right| \\ &+ \sigma_c \left|\frac{W(z_0)}{\phi'(z_0)} \Psi^\dagger_{\bar{z}_0}(z_0) - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)} \Psi^\dagger_{\bar{z}_0}(\bar{z}_0)\right| \\ &+ \mu \left|\left(\chi_{(\bar{z}_0, z_0)} W, \Psi^\dagger_{\bar{z}_0}\right)\right|. \end{aligned} \tag{5.2.9}$$

To estmate the first term in the left hand side of (5.2.9) we integrate $\left(\partial_z W, \Psi^\dagger\right)$ by parts and estimate the velocity terms from (3.1.25) and rewrite this term as,

$$|v_s(z_0) - v_s(\bar{z}_0)| \left|\left(\partial_z W, \Psi^\dagger\right)\right| \le C\sqrt{\epsilon}\,|z - z_0| \left|\left(W, \partial_z \Psi^\dagger_{\bar{z}_0}\right)\right|.$$

From the form of $\Psi^\dagger_{\bar{z}_0}$ given by (3.6.38) we observe that $\left\|\partial_z \Psi^\dagger_{\bar{z}_0}\right\|_{L^1} = \mathcal{O}(1)$, and we achieve the following estimate,

$$|v_s(z_0) - v_s(\bar{z}_0)| \left|\left(\partial_z W, \Psi^\dagger\right)\right| \le C\sqrt{\epsilon}\,|z_0 - \bar{z}_0|\, \|W\|_{H^\gamma}. \tag{5.2.10}$$

The second term on the right of (5.2.9) involving the point evaluations can be expressed as,

$$\begin{aligned} \left| W(z_0)\frac{\Psi^\dagger_{\bar{z}_0}(z_0)}{\phi'(z_0; z_0)} - W(\bar{z}_0)\frac{\Psi^\dagger_{\bar{z}_0}(\bar{z}_0)}{\phi'(\bar{z}_0; \bar{z}_0)} \right| \;\le\;& |W(z_0) - W(\bar{z}_0)| \frac{\left|\Psi^\dagger_{\bar{z}_0}(z_0)\right|}{\phi'(z_0; z_0)} \\ &+ \frac{|W(\bar{z}_0)|}{\phi'(\bar{z}_0; \bar{z}_0)} \left|\Psi^\dagger_{\bar{z}_0}(z_0) - \Psi^\dagger_{\bar{z}_0}(\bar{z}_0)\right| \\ &+ \left|W(\bar{z}_0)\Psi^\dagger_{\bar{z}_0}(\bar{z}_0)\right| \left|\frac{\phi'(z_0) - \phi'(\bar{z}_0)}{\phi'(z_0; z_0)\phi'(\bar{z}_0; \bar{z}_0)}\right|. \end{aligned} \tag{5.2.11}$$

We apply the inequality (3.2.5) to the three terms on the right hand side of (5.2.11), since ϕ', $\Psi^\dagger_{\bar{z}_0}$ and $W \in H^\gamma$. Also from **Lemma 3.1.1**, ϕ' is bounded away from zero in any bounded interval containing z_0, reduces (5.2.11) to

$$\left| W(z_0)\frac{\Psi^\dagger_{\bar{z}_0}(z_0)}{\phi'(z_0; z_0)} - W(\bar{z}_0)\frac{\Psi^\dagger_{\bar{z}_0}(\bar{z}_0)}{\phi'(\bar{z}_0; \bar{z}_0)} \right| \;\le\; C|z_0 - \bar{z}_0|^{\gamma - 1/2}\, \|W\|_{H^\gamma}. \tag{5.2.12}$$

The last term on the right hand side of (5.2.9) has the following bound

$$\left|\left(\chi_{(\bar{z}_0, z_0)} W, \Psi^\dagger_{\bar{z}_0}\right)\right| \le \int_{\bar{z}_0}^{z_0} |W| \left|\Psi^\dagger_{\bar{z}_0}\right| dz \le |z_0 - \bar{z}_0|\, \|W\|_{H^\gamma} \left\|\Psi^\dagger_{\bar{z}_0}\right\|_{L^\infty} \le C\,|z_0 - \bar{z}_0|\, \|W\|_{H^\gamma}. \tag{5.2.13}$$

Finally substituting the estimates developed in (5.2.10), (5.2.12), (5.2.13) into (5.2.9) we obtain

$$\begin{aligned} \left|\left(\Delta L\, W, \Psi^\dagger_{\bar{z}_0}\right)\right| \;&\le\; C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + |z_0 - \bar{z}_0|^{\gamma - 1/2} + |z_0 - \bar{z}_0|\right) \|W\|_{H^\gamma}, \\ &\le\; C\left(|z_0 - \bar{z}_0|^{\gamma - 1/2}\right) \|W\|_{H^\gamma}, \end{aligned} \tag{5.2.14}$$

where we have neglected higher powers of $|z_0 - \bar{z}_0|$. Addressing the nonlinearity $\mathcal{N}(W)$, from (3.2.16) with $\beta = \gamma$, we obtain the following estimate

$$\begin{aligned}\left|\left(\mathcal{N}(W), \Psi^{\dagger}_{\bar{z}_0}\right)\right| &\leq \|N(W)\|_{H^{-\gamma}} \left\|\Psi^{\dagger}_{\bar{z}_0}\right\|_{H^{\gamma}}, \\ &\leq C\|W\|_{H^{\gamma}}^{\gamma+1/2}. \end{aligned} \tag{5.2.15}$$

Finally, for the noise term

$$\begin{aligned}\epsilon^{3/4-\gamma s/2}\left|\left(\eta_s(\cdot,t), \Psi^{\dagger}_{\bar{z}_0}\right)\right| &\leq \epsilon^{3/4-\gamma s/2}\|\eta_s(t)\|_{H^{-\gamma s}} \left\|\Psi^{\dagger}_{\bar{z}_0}\right\|_{H^{\gamma}_s}, \\ &\leq C\epsilon^{3/4-\gamma s/2}\|\eta_s(t)\|_{H^{-\gamma s}}. \end{aligned} \tag{5.2.16}$$

Turning to (5.2.7) we see that the denominator is uniformly bounded away from zero by **Lemma 3.6.3**. Employing the estimates developed in (5.2.8), (5.2.14), (5.2.15) and (5.2.16), we obtain an upper bound on the front velocity,

$$|z_0'| \leq C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + |z_0 - \bar{z}_0|^{\gamma-1/2}\|W\|_{H^{\gamma}} + \|W\|_{H^{\gamma}}^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\|\eta_s\|_{H^{-\gamma s}} + \epsilon\right). \tag{5.2.17}$$

To control the front dynamics we introduce the quantities,

$$T_0(\tau) = \sup_{t_0<s<t_0+\tau} e^{\nu(s-t_0)}\|W(s)\|_{\gamma}, \tag{5.2.18}$$

$$T_1(\tau) = \sup_{t_0<s<t_0+\tau} |z_0(s) - \bar{z}_0|. \tag{5.2.19}$$

where the $\mathcal{O}(1)$ exponential decay rate ν is motivated by the semigroup estimates (4.2.68). Using the definition of T_0 and T_1, for $t_0 < s < t_0 + \tau$, the expression for front velocity z_0' given by (5.2.17) reduces to

$$\begin{aligned}|z_0'(s)| \leq\ & C\ \Big(\sqrt{\epsilon}T_1(\tau) + T_1(\tau)^{\gamma-1/2}T_0(\tau)e^{-\nu(s-t_0)} \\ &+\ T_0(\tau)^{\gamma+1/2}e^{-\nu(\gamma+1/2)(s-t_0)} + \epsilon^{3/4-\gamma s/2}\|\eta_s(s)\|_{H^{-\gamma s}} + \epsilon\Big) \end{aligned} \tag{5.2.20}$$

Throughout this section we follow the convention that $T_0 \equiv T_0(\tau)$ and $T_1 \equiv T_1(\tau)$. In the following lemma we develop a bound for the change in position of the front location, T_1, in terms of T_0 and the length of time, τ, that the coordinate system is frozen by.

Lemma 5.2.1 *Fix $\gamma \in \left(\frac{1}{2}, 1\right)$. For ϵ and T_0 sufficiently small there exists $C > 0$, independent of τ and ϵ satisfying*

$$\sqrt{\epsilon}\tau \ll 1, \tag{5.2.21}$$

such that

$$T_1 \leq C\left(T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau\right), \tag{5.2.22}$$

is true.

Proof: Integrating z_0' over the interval $(t_0, t_0+\tau)$, and using definition (5.2.19) for T_1, and the estimate (5.2.20) we find

$$\begin{aligned} T_1(\tau) &\leq \int_{t_0}^{t_0+\tau} \left|z_0'(s)\right| ds = \int_0^{\tau} \left|z_0'(t_0+s)\right| ds \\ &\leq C\int_0^{\tau} \Big(\sqrt{\epsilon}T_1 + T_1^{\gamma-1/2}T_0 e^{-\nu s} + T_0^{\gamma+1/2}e^{-\nu(\gamma+1/2)s} \\ &\quad + \epsilon^{3/4-\gamma s/2}\left\|\eta_s(s)\right\|_{H^{-\gamma s}} + \epsilon\Big)\, ds. \end{aligned}$$

We evaluate the integral on the right hand side of inequality (5.2.23), obtaining

$$T_1 \leq C\left(\sqrt{\epsilon}\tau T_1 + T_1^{\gamma-1/2}T_0 + T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\left\|\eta_s\right\|_{L^\infty}\tau + \epsilon\tau\right) \tag{5.2.23}$$

Here, for notational convenience, $\|\eta_s\|_{L^\infty}$ denotes $\|\eta_s\|_{L^\infty(\mathbf{R}^+, H^{-\gamma s}(\mathbf{R}))}$. Applying Young's inequality to $CT_1^{\gamma-1/2}T_0$ with $p = \frac{1}{\gamma-1/2}$ and $q = \frac{1}{3/2-\gamma}$ we obtain,

$$CT_1^{\gamma-1/2}T_0 \leq \frac{1}{2}T_1 + (CT_0)^{\frac{1}{3/2-\gamma}}, \tag{5.2.24}$$

for $\gamma \in (1/2, 1)$. Substituting (5.2.24) in the estimate for T_1 in (5.2.23) yields

$$T_1 \leq C\sqrt{\epsilon}\tau T_1 + \frac{1}{2}T_1 + (CT_0)^{\frac{1}{3/2-\gamma}} + CT_0^{\gamma+1/2} + C\left(\epsilon^{3/4-\gamma s/2}\left\|\eta_s\right\|_{L^\infty} + \epsilon\right)\tau. \tag{5.2.25}$$

Since we have rescaled the noise term so that $\|\eta_s\|_{L^\infty}$ is $\mathcal{O}(1)$, we combine the two terms in the coefficient of τ in the right hand side of the above inequality (5.2.25). We neglect higher order powers of T_0 and solve for T_1

$$T_1 \leq C\frac{T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau}{\frac{1}{2} - C\sqrt{\epsilon\tau}}. \tag{5.2.26}$$

Since the constant C depends only on γ we have

$$T_1 \leq C\left(T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau\right).$$

so long as the following conditions hold

$$\sqrt{\epsilon\tau} \ll 1. \quad \blacksquare$$

To estimate the decay of the remainder W, we return to the evolution equation (5.2.2) and apply $\tilde{\pi}_0$ to both sides. Since $\pi_0 W_t = \pi_0 L_{\bar{z}_0} W = 0$, (5.2.2) reduces to,

$$\begin{aligned} W_t &= \tilde{R} + L_{\bar{z}_0} W + \tilde{\pi}_0\left(\Delta L\, W + \mathcal{N}(W) + \epsilon^{3/4-\gamma s/2}\eta_s(\xi, t)\right) \\ W(\xi, t_0) &= W_0 \end{aligned} \tag{5.2.27}$$

where we have introduced the reduced residual $\tilde{R}$ defined as,

$$\tilde{R} = \tilde{\pi}_0\left(R - z_0'\frac{\partial\phi}{\partial z_0}\right). \tag{5.2.28}$$

Using the frozen coefficient semigroup $S = S(\tau; \bar{z}_0)$ associated to $L_{\bar{z}_0}$, introduced in **Section 4**, the variation of constants formula applied to (5.2.27) yields the solution

$$\begin{aligned} W(\xi, t_0 + \tau) &= S(\tau)W_0 \\ &+ \int_{t_0}^{t_0+\tau} S(t_0 + \tau - s)\left(\tilde{R} + \tilde{\pi}_0\left(\Delta L\, W + \mathcal{N}(W) + \epsilon^{3/4-\gamma s/2}\eta_s(\xi, s)\right)\right) ds. \end{aligned} \tag{5.2.29}$$

Changing the limits of integration and taking the H^γ norm on both sides of equation (5.2.29) we obtain,

$$\|W(t_0+\tau)\|_{H^\gamma} \leq \|S(\tau)W_0\|_{H^\gamma} + \int_0^\tau \left\| S(\tau-s)\left(\tilde{R}+\tilde{\pi}_0(\Delta L\, W+\mathcal{N}(W)+\epsilon^{3/4-\gamma_s/2}\eta_s(\xi,s))\right)\right\|_{H^\gamma} \tag{5.2.30}$$

In section 4 we derived estimates S acting on $F \in H^{-\beta}$. In the following lemma we develop semigroup estimates on the terms in the right hand side of the inequality (5.2.30).

Lemma 5.2.2 : *For $\sqrt{\epsilon\tau} \ll 1$ we have the following estimates on each of the terms on the right-hand side of (5.2.30).*

*(**a**) Semigroup estimate on the reduced residual $\tilde{R}$ is:*

$$\left\|S(\tau-s)\tilde{R}\right\|_{H^\gamma} \leq Me^{-\nu(\tau-s)}\left(\sqrt{\epsilon}T_1+\epsilon^{3/4-\gamma_s/2}+\|W\|_{H^\gamma}\left(T_1^{\gamma-1/2}+\|W\|_{H^\gamma}^{\gamma-1/2}\right)\right). \tag{5.2.31}$$

*(**b**) Semigroup estimate on the secularity $\Delta L(W)$ is:*

$$\|S(\tau-s)\tilde{\pi}_0\Delta L\, W\|_{H^\gamma} \leq Me^{-\nu(\tau-s)}\|W\|_{H^\gamma}\left(\frac{\sqrt{\epsilon}T_1}{\sqrt{\tau-s}}+\frac{T_1^{\gamma-1/2}}{(\tau-s)^\gamma}\right). \tag{5.2.32}$$

*(**c**) Semigroup estimate on the nonlinearity $\mathcal{N}(W)$ is:*

$$\|S(\tau-s)\tilde{\pi}_0\mathcal{N}(W)\|_{H^\gamma} \leq M\frac{e^{-\nu(\tau-s)}}{(\tau-s)^\gamma}\|W\|_{H^\gamma}^{\gamma+1/2}. \tag{5.2.33}$$

*(**d**) Semigroup estimate on the noise term η_s is:*

$$\|S(\tau-s)\tilde{\pi}_0\eta_s\|_{H^\gamma} \leq M\frac{e^{-\nu(\tau-s)}}{(\tau-s)^{\gamma_s}}. \tag{5.2.34}$$

(e) Semigroup estimate on the initial condition $W_0 \in H^\gamma$ is:

$$\|S(\tau)W_0\|_{H^\gamma} \le Me^{-\nu\tau} \|W_0\|_\gamma . \tag{5.2.35}$$

Proof (a): We estimate the action of semigroup on the reduced residual $\tilde{R}$. Since $\tilde{R} \in H^\gamma$, substituting $\beta = -\gamma$ in (4.2.68) we obtain,

$$\left\|S(\tau - s)\tilde{R}\right\|_{H^\gamma} \le Ce^{-\nu(\tau-s)} \left\|\tilde{R}\right\|_{H^\gamma} . \tag{5.2.36}$$

From the form of $\tilde{R}$ we obtain,

$$\left\|\tilde{R}\right\|_{H^\gamma} \le C \|R\|_{H^\gamma} + C\left|z_0'\right| \left\|\frac{\partial\phi}{\partial z_0}\right\|_{H^\gamma} .$$

We estimate for $\|R\|_{H^\gamma}$ from (3.1.22) and $\left|z_0'\right|$ from (5.2.20) to obtain,

$$\begin{aligned} \left\|\tilde{R}\right\|_{H^\gamma} &\le C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + \epsilon^{3/4-\gamma s/2}\right) \\ &+ C\left(|z_0 - \bar{z}_0| + |z_0 - \bar{z}_0|^{\gamma-1/2} + \|W\|_{H^\gamma}^{\gamma-1/2}\right) \|W\|_{H^\gamma} . \end{aligned} \tag{5.2.37}$$

Using the definition of T_1, and neglecting higher powers of T_1, in (5.2.37) we obtain the following estimate,

$$\left\|S(\tau - s)\tilde{R}\right\|_\gamma \le Me^{-\nu(\tau-s)} \left(\sqrt{\epsilon}T_1 + \epsilon^{3/4-\gamma s/2} + \|W\|_{H^\gamma} \left(T_1^{\gamma-1/2} + \|W\|_{H^\gamma}^{\gamma-1/2}\right)\right).$$

(b) To prove part (b), using (??)

$$\|S(\tau - s)\tilde{\pi}_0 \Delta L\, W\|_{H^\gamma} \le \|S(\tau - s)\Delta L\, W\|_{H^\gamma} . \tag{5.2.38}$$

refering to the secular term ΔL from (5.2.4) we obtain,

$$\begin{aligned} \|S(\tau - s)\Delta L\, W\|_{H^\gamma} &\le |v_s(z_0) - v_s(\bar{z}_0)|\, \|S(\tau - s)\tilde{\pi}_0 \partial_z W\|_{H^\gamma} \\ &+ \sigma_c \left\|S(\tau - s)\tilde{\pi}_0 \left(\frac{W(z_0)}{\phi'(z_0)}\delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)}\delta_{\bar{z}_0}\right)\right\|_{H^\gamma} \\ &+ \mu \left\|S(\tau - s)\tilde{\pi}_0 \chi_{(\bar{z}_0, z_0)} W\right\|_{H^\gamma} \end{aligned} \tag{5.2.39}$$

Since $W \in H^\gamma$, $\partial_z W \in H^{\gamma-1}$ substituting for $\beta = 1-\gamma$ in (4.2.68) we have the following bound,

$$\|S(\tau-s)\partial_z W\|_{H^\gamma} \leq M\frac{e^{-\nu(\tau-s)}}{\sqrt{\tau-s}}\,\|\partial_z W\|_{H^{\gamma-1}} \tag{5.2.40}$$

The formal pulse speed v_s is continuous and varies slowly in space satisfying the bound (3.1.25). The derivative operator D is continuous as a map from H^γ to $H^{\gamma-1}$, and we have $\|W_z\|_{\gamma-1} \leq \|W\|_\gamma$. Substituting these estimates and using the definition of T_1, in the first term of the right hand side of (5.2.39) we obtain

$$|v_s(z_0) - v_s(\bar{z}_0)|\,\|S(\tau-s)\partial_z W\|_\gamma \leq M\frac{e^{-\nu(\tau-s)}}{\sqrt{\tau-s}}\sqrt{\epsilon}T_1\,\|W\|_{H^\gamma} \tag{5.2.41}$$

To obtain the semigroup estimate for the second term in (5.2.39) involving the difference of delta functions, we compute its $H^{-\gamma}$ norm. By definition,

$$\begin{aligned}\left\|\frac{W(z_0)}{\phi'(z_0)}\delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)}\delta_{\bar{z}_0}\right\|_{H^{-\gamma}} &= \sup_{\substack{\varphi\in H^\gamma\\ \varphi\neq 0}} \frac{\left|\left(\frac{W(z_0)}{\phi'(z_0)}\delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)}\delta_{\bar{z}_0},\varphi\right)_{L^2}\right|}{\|\varphi\|_{H^\gamma}}\\ &= \sup_{\substack{\varphi\in H^\gamma\\ \varphi\neq 0}} \frac{\left|\frac{W(z_0)}{\phi'(z_0)}\varphi(z_0) - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)}\varphi(\bar{z}_0)\right|}{\|\varphi\|_{H^\gamma}}\end{aligned} \tag{5.2.42}$$

Since $\phi'(z)$ is uniformly bounded away from zero on $(z_0, \bar{z}_0)$ by **Lemma 3.1.1**), the numerator of (5.2.42) can be bounded by

$$\begin{aligned}&\frac{|W(z_0)-W(\bar{z}_0)|\,\phi'(\bar{z}_0)\varphi(z_0)}{|\phi'(z_0)\phi'(\bar{z}_0)|} + \frac{|\phi'(z_0)-\phi'(\bar{z}_0)|\,W(\bar{z}_0)\varphi(z_0)}{|\phi'(z_0)\phi'(\bar{z}_0)|}\\ +\; &\frac{|\varphi(z_0)-\varphi(\bar{z}_0)|\,W(\bar{z}_0)\phi'(z_0)}{|\phi'(z_0)\phi'(\bar{z}_0)|} \leq c\,|z_0-\bar{z}_0|^{\gamma-1/2}\,\|W\|_{H^\gamma}\,\|\varphi\|_{H^\gamma}\,.\end{aligned}$$

where we have used the Hölder continuity of W and $\varphi \in H^\gamma$. Recalling the definition of T_1 from (5.2.19), we arrive at the estimate

$$\left\|\frac{W(z_0)}{\phi'(z_0)}\delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)}\delta_{\bar{z}_0}\right\|_{H^{-\gamma}} \leq T_1^{\gamma-1/2}\,\|W\|_{H^\gamma}\,. \tag{5.2.43}$$

Substituting $\beta = \gamma$ in the semigroup estimate given by (4.2.68) we obtain,

$$\left\| S(\tau - s) \left(\frac{W(z_0)}{\phi'(z_0)} \delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)} \delta_{\bar{z}_0} \right) \right\|_{H^\gamma} \leq M \frac{e^{-\nu(\tau - s)}}{\tau - s)^\gamma} T_1^{\gamma - 1/2} \|W\|_{H^\gamma} . \quad (5.2.44)$$

Turning to the third term on the right-hand side of (5.2.39), the characteristic function $\chi_{(\bar{z}_0, z_0)} W \in H^{-\gamma}$ plugging $\beta = \gamma$ in semigroup estimate (4.2.68) gives,

$$\left\| S(\tau - s) \chi_{(\bar{z}_0, z_0)} W \right\|_\gamma \leq M \frac{e^{-\nu(\tau - s)}}{(\tau - s)^\gamma} \left\| \chi_{(\bar{z}_0, z_0)} W \right\|_{H^{-\gamma}} . \quad (5.2.45)$$

We obtain the following bound for $\left\| \chi_{(\bar{z}_0, z_0)} W \right\|_{H^{-\gamma}}$,

$$\begin{aligned} \left\| \chi_{(\bar{z}_0, z_0)} W \right\|_{H^{-\gamma}} &= \sup_{\substack{\varphi \in H^\gamma \\ \varphi \neq 0}} \frac{\left| \left(\chi_{(\bar{z}_0, z_0)} W, \varphi \right)_{L^2} \right|}{\|\varphi\|_{H^\gamma}} \\ &\leq \sup_{\substack{\varphi \in H^\gamma \\ \varphi \neq 0}} \frac{\int_{\bar{z}_0}^{z_0} |W| \, |\varphi| \, dz}{\|\varphi\|_{H^\gamma}} \\ &\leq |z_0 - \bar{z}_0| \, \|W\|_{H^\gamma} \leq T_1 \, \|W\|_{H^\gamma} . \end{aligned} \quad (5.2.46)$$

Substituting this estimate into (5.2.45) we achieve,

$$\left\| S(\tau - s) \chi_{(\bar{z}_0, z_0)} W \right\|_\gamma \leq M \frac{e^{-\nu(\tau - s)}}{(\tau - s)^\gamma} T_1 \, \|W\|_\gamma . \quad (5.2.47)$$

Combining the estimates (5.2.41), (5.2.44) and (5.2.47) into (5.2.39), we the secularity as,

$$\|S(\tau - s) \Delta L \, W\|_{H^\gamma} \leq M e^{-\nu(\tau - s)} \, \|W\|_{H^\gamma} \left(\frac{\sqrt{\epsilon} T_1}{\sqrt{\tau - s}} + \frac{T_1^{\gamma - 1/2}}{(\tau - s)^\gamma} + \frac{T_1}{(\tau - s)^\gamma} \right) . \quad (5.2.48)$$

Neglecting the higher order terms we obtain

$$\|S(\tau - s) \Delta L \, W\|_{H^\gamma} \leq M e^{-\nu(\tau - s)} \, \|W\|_{H^\gamma} \left(\frac{\sqrt{\epsilon} T_1}{\sqrt{\tau - s}} + \frac{T_1^{\gamma - 1/2}}{(\tau - s)^\gamma} \right) .$$

(c) To derive semigroup estimate on the nonlinearity we use the estimate (3.2.16) and substitute $\beta = \gamma$ in (4.2.68) to obtain,

$$\|S(\tau - s)\mathcal{N}(W)\|_{H^\gamma} \leq M \frac{e^{-\nu(\tau - s)}}{(\tau - s)^\gamma} \|W\|_{H^\gamma}^{\gamma+1/2}.$$

(d) For the noise $\eta_s(\cdot, t) \in H^{-\gamma_s}$, substituting $\beta = \gamma_s$ in (4.2.68) yields,

$$\|S(\tau - s)\eta_s\|_{H^\gamma} \leq M \frac{e^{-\nu(\tau - s)}}{(\tau - s)^{\gamma_s}} \|\eta_s\|_{H^{-\gamma_s}}.$$

Since $\|\eta_s\|_{H^{-\gamma_s}} = \mathcal{O}(1)$ we obtain,

$$\|S(\tau - s)\eta_s\|_{H_s^\gamma} \leq M \frac{e^{-\nu(\tau - s)}}{(\tau - s)^{\gamma_s}}.$$

(e) For $W_0 \in H^\gamma$ from (4.2.68) we obtain,

$$\|S(\tau - s)W_0\|_\gamma \leq M e^{-\nu(\tau - s)} \|W_0\|_\gamma. \tag{5.2.49}$$

■

Combining the estimates developed in (5.2.36)-(5.2.49), the solution W given by (5.2.30) satisfies

$$\begin{aligned}
\|W(t_0 + \tau)\|_{H^\gamma} \;\leq\;& M e^{-\nu\tau} \|W_0\|_{H^\gamma} + M \int_0^\tau e^{-\nu(\tau - s)} \Big[\sqrt{\epsilon} T_1 + \epsilon^{3/4 - \gamma_s/2} \\
+\;& \|W\|_{H^\gamma}^{\gamma+1/2} + \|W\|_{H^\gamma} T_1^{\gamma - 1/2} \\
+\;& \|W\|_{H^\gamma} \left(\frac{\sqrt{\epsilon} T_1}{\sqrt{\tau - s}} + \frac{T_1^{\gamma-1/2} + \|W\|_{H^\gamma}^{\gamma-1/2}}{(\tau - s)^\gamma} \right) \\
+\;& e^{-\nu(\tau - s)} \frac{\epsilon^{3/4 - \gamma_s/2}}{(\tau - s)^{\gamma_s}} \Big] ds.
\end{aligned} \tag{5.2.50}$$

To estimate the decay of $\|W\|_{H^\gamma}$ we multiply the above equation (5.2.50) by $e^{\nu\tau}$ and use the definition of T_0 from (5.2.18)

$$\begin{aligned}
e^{\nu\tau} \;& \|W(t_0 + \tau)\|_{H^\gamma} \leq M \|W_0\|_{H^\gamma} + M \int_0^\tau e^{\nu s} \Big[\left(\sqrt{\epsilon} T_1 + \epsilon^{3/4 - \gamma_s/2} \right) \\
+\;& \|W\|_{H^\gamma} \left(\|W\|_{H^\gamma}^{\gamma - 1/2} + T_1^{\gamma - 1/2} \right) \\
+\;& \|W\|_{H^\gamma} \left(\frac{\sqrt{\epsilon} T_1}{\sqrt{\tau - s}} + \frac{T_1^{\gamma - 1/2} + \|W\|_{H^\gamma}^{\gamma - 1/2}}{(\tau - s)^\gamma} \right) + e^{\nu s} \frac{\epsilon^{3/4 - \gamma_s/2}}{(\tau - s)^{\gamma_s}} \Big] ds.
\end{aligned} \tag{5.2.51}$$

We take the supremum of both sides of (5.2.52), over $\tau \in (0, \tau)$ to obtain,

$$\begin{aligned} T_0(\tau) &\leq MT_0(t_0) + M\int_0^\tau \Big[e^{\nu s}\left(\sqrt{\epsilon}T_1 + \epsilon^{3/4-\gamma s/2}\right) \\ &+ T_0\left(e^{-(\gamma-1/2)s}T_0^{\gamma-1/2} + T_1^{\gamma-1/2}\right) \\ &+ T_0\left(\frac{\sqrt{\epsilon}T_1}{\sqrt{\tau - s}} + \frac{T_1^{\gamma-1/2} + e^{-(\gamma-1/2)s}T_0^{\gamma-1/2}}{(\tau-s)^\gamma}\right) + e^{\nu s}\frac{\epsilon^{3/4-\gamma s/2}}{(\tau-s)^{\gamma s}}\Big]ds. \end{aligned} \tag{5.2.52}$$

Performing integration on the right hand side of inequality (5.2.52), we obtain

$$\begin{aligned} T_0(\tau) &\leq M\big[T_0(t_0) + e^{\nu\tau}\left(\sqrt{\epsilon}T_1 + \epsilon^{3/4-\gamma s/2}\left(1 + \tau^{1-\gamma s}\right)\right) \\ &+ \left(T_0^{\gamma+1/2} + T_0T_1^{\gamma-1/2}\right)(\tau + \tau^{1-\gamma}) + \sqrt{\epsilon\tau}T_0T_1\big]. \end{aligned} \tag{5.2.53}$$

Since $\tau > 1$, τ satisfies $\tau > \tau^{1-\gamma s}$, $\tau > \tau^{1-\gamma}$ and the constant term 1 is dominated by τ, which reduces the right hand side of (5.2.53) to

$$T_0(\tau) \leq M\left[T_0(t_0) + e^{\nu\tau}\left(\sqrt{\epsilon}T_1 + \epsilon^{3/4-\gamma s/2}\tau\right) + \left(T_0^{\gamma+1/2} + T_0T_1^{\gamma-1/2}\right)\tau + \sqrt{\epsilon\tau}T_0T_1\right]. \tag{5.2.54}$$

From the inequality (5.2.24), the term $T_0T_1^{\gamma-1/2}$ can be bounded above by,

$$T_0T_1^{\gamma-1/2} \leq C(T_1 + T_0^{1/(3/2-\gamma)}).$$

Now substituting for T_1 from (5.2.22) we obtain,

$$\begin{aligned} T_0T_1^{\gamma-1/2} &\leq C(T_0^{\gamma+1/2} + T_0^{1/(3/2-\gamma)} + \epsilon^{3/4-\gamma s/2}\tau), \\ &\leq C(T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau), \end{aligned} \tag{5.2.55}$$

for $\gamma \in (1/2, 1)$. Bounding T_1 from (5.2.22) and $T_0T_1^{\gamma-1/2}$ from (5.2.55) in the right hand side of (5.2.54) we obtain,

$$\begin{aligned} T_0(\tau) &\leq M\Big[T_0(t_0) + e^{\nu\tau}\left(\sqrt{\epsilon}(T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau) + \epsilon^{3/4-\gamma s/2}\tau\right) \\ &+ \left(T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau\right)\tau + \sqrt{\epsilon\tau}T_0\left(T_0^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2}\tau\right)\Big] \end{aligned} \tag{5.2.56}$$

We combine the terms in (5.2.56) to obtain

$$\begin{aligned} T_0(\tau) &\leq M\Big[T_0(t_0) + \sqrt{\epsilon}e^{\nu\tau}T_0^{\gamma+1/2} + e^{\nu\tau}\epsilon^{3/4-\gamma s/2}\tau\left(\sqrt{\epsilon}+1+e^{-\nu\tau}\tau+e^{-\nu\tau}\sqrt{\epsilon\tau}T_0\right) \\ &+ T_0^{\gamma+1/2}(\tau+\sqrt{\epsilon\tau}T_0)\Big]. \end{aligned} \tag{5.2.57}$$

The term $\tau e^{-\nu\tau}$ is bounded above for $\tau > 0$. From the condition (5.2.21) on τ, $\sqrt{\epsilon}\tau < 1$ and, neglecting higher orders terms in T_0, we simplify (5.2.57) obtaining,

$$T_0(\tau) \leq M\left[T_0(t_0) + \epsilon^{3/4-\gamma s/2}e^{\nu\tau}\tau + \sqrt{\epsilon}e^{\nu\tau}T_0^{\gamma+1/2} + T_0^{\gamma+1/2}\tau\right]. \tag{5.2.58}$$

We impose a second constraint on τ, requiring

$$\tau T_0^{\gamma-1/2} \ll 1, \tag{5.2.59}$$

we can make $M\tau T_0^{\gamma+1/2} < \frac{T_0}{2}$. The estimate (5.2.58) for T_0 reduces to,

$$T_0 \leq M\left(T_0(t_0) + \epsilon^{3/4-\gamma s/2}e^{\nu\tau}\tau\right) + \sqrt{\epsilon}e^{\nu\tau}T_0^{\gamma+1/2}. \tag{5.2.60}$$

To simplify 5.2.60) further, we note that $\tau \leq e^{\nu\tau}$, provided $\nu \geq \frac{\ln\tau}{\tau}$, which reduces (5.2.60) to

$$T_0 \leq M\left(T_0(t_0) + \epsilon^{3/4-\gamma s/2}e^{\nu\tau}\right) + \sqrt{\epsilon}e^{\nu\tau}T_0^{\gamma+1/2}. \tag{5.2.61}$$

We solve the above inequality (5.2.61) for T_0 with the aid of the following lemma. The motivation of the lemma being, if the increasing function $f(x) = a + bx^{\alpha}, 1 < \alpha < 2$, is small initially, $a \ll 1$, and grows slowly, i.e. $f'(a) \ll 1$, then it intersects $y = x$ exactly twice with one fixed point lying in a neighbourhood of $x = a$.

Lemma 5.2.3 *Fix $\gamma \in (1/2, 1)$. Let $T_0 : (0,\tau) \to \boldsymbol{R}^+$ be continuous. Then if $T_0(t_0)$ is sufficiently small and T_0 satisfies (5.2.61) then*

$$T_0 \leq M\left(T_0(t_0) + \epsilon^{3/4-\gamma s/2}e^{\nu\tau}\right). \tag{5.2.62}$$

for all τ satisfying

$$\tau \leq \beta\,(3/4-\gamma_s/2)\,\frac{|\ln\epsilon|}{\nu}, \tag{5.2.63}$$

for any $\beta < 1$

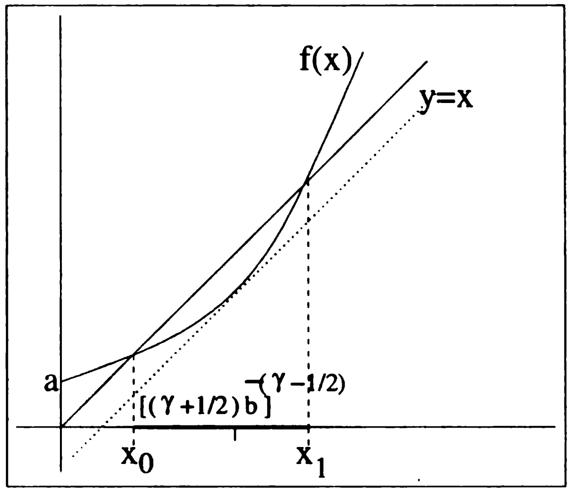

Figure 5.2. The graphical illustration of solution of the inequality for T_0.

Proof: To simplify the computation we define the constant term and the coefficient of $T_0^{\gamma+1/2}$ in the right hand side of (5.2.61) as,

$$a = M\left(T_0(t_0) + \epsilon^{3/4-\gamma s/2} e^{\nu\tau} \tau^{1-\gamma}\right), \tag{5.2.64}$$

$$b = \sqrt{\epsilon} e^{\nu\tau}, \tag{5.2.65}$$

which reduces the inequality (5.2.61) to,

$$T_0 \leq a + bT_0^{\gamma+1/2}. \tag{5.2.66}$$

We fix τ and investigate the fixed points of the function

$$f(x) = a + bx^{\gamma+1/2} \tag{5.2.67}$$

We note that

$$f'(x) = b(\gamma + 1/2)x^{\gamma-1/2} > 1, \quad \text{if} \quad x > \left(\frac{1}{b(\gamma + 1/2)}\right)^{\frac{1}{\gamma-1/2}}, \tag{5.2.68}$$

as illustrated in **Fig(5.2)**. The function $f(x)$ has a fixed point at

$$x_0(\tau) = a + ba^{\gamma+1/2} + \mathcal{O}(b^2 a^{2\gamma+1}), \tag{5.2.69}$$

and another at

$$x_1(\tau) > \left(\frac{1}{b(\gamma + 1/2)}\right)^{\frac{1}{\gamma - 1/2}}. \tag{5.2.70}$$

Thus T_0 satisfying (5.2.61) also satisfies

$$T_0 < x_0(\tau) \quad or \quad T_0 > x_1(\tau). \tag{5.2.71}$$

Thus if $a \ll 1$ and $b \ll 1$ there is an **excluded region**, either $0 < T_0 < x_0$ or $x_1 < T_0 < \infty$. Since $x_0(\tau$ and T_0 are continuous and $T_0(t_0) < x_0(0)$, then $T_0 < x_0(\tau)$ so long as $x_0(\tau) < x_1(\tau)$ which is true if $a \ll 1$ and $b \ll 1$. This holds only if τ satisfies (5.2.63). This condition on τ prevents the secularity from dominating in the linear operator, in particular it is a stronger condition on τ than imposed after (5.2.21) so long as, (5.2.59) holds true, i.e.

$$T_0 \ll 1/\left|\ln \epsilon\right|. \tag{5.2.72}$$

■

5.3 The RG equation

The definition of $T_0(t)$ in (5.2.18) we substitute $\|W(t)\|_{H^\gamma} \leq e^{-\nu(t-t_0)}T_0(t)$, so that (5.2.62) reads

$$\|W(t)\|_{H^\gamma} \leq M\left(\|W_0\|_{H^\gamma} e^{-\nu(t-t_0)} + \epsilon^{3/4-\gamma_s/2}\right). \tag{5.3.1}$$

for $t \in \left(t_0, t_0 + \beta\,(3/4 - \gamma_s/2)\,\frac{|\ln \epsilon|}{\nu}\right)$. Defining t_1 as

$$t_1 = t_0 + \beta\,(3/4 - \gamma_s/2)\,\frac{|\ln \epsilon|}{\nu}, \tag{5.3.2}$$

we find

$$\begin{aligned} \|W(t_1)\|_{H^\gamma} &\leq M\left(\epsilon^{\beta(3/4-\gamma_s/2)}\,\|W_0\|_{H^\gamma} + \epsilon^{3/4-\gamma_s/2}.\right) \\ &\leq M\epsilon^{\beta(3/4-\gamma_s/2)}\left(\|W_0\|_{H^\gamma} + \epsilon^{(1-\beta)(3/4-\gamma_s/2)}\right). \end{aligned} \tag{5.3.3}$$

To complete the RG procedure, we fix $\beta < 1$ and $\tau = \beta(3/4 - \gamma/2)\frac{|\ln\epsilon|}{\nu}$. The renormalization times are then defined sequentially by,

$$t_n = t_{n-1} + \beta(3/4 - \gamma_s/2)\frac{|\ln\epsilon|}{\nu}. \tag{5.3.4}$$

We break $\mathbf{R}^+$ into equal disjoint intervals $I_n = [t_n, t_{n+1}]$ as shown in **Fig 5.1**. On each interval I_n, we solve the initial value problem (5.2.27) with the initial data $W_n \equiv W(t_n) \in X_{z_n}$, with the quantities $T_{0,n}$ and $T_{1,n}$ defined in (5.2.18) and (5.2.19) over I_n. The renormalization map $\mathcal{G}$ takes the initial data W_{n-1} for the initial value problem on interval I_{n-1} and returns the initial data W_n for the initial value problem on the interval I_n

$$\mathcal{G}W_{n-1} = W_n \tag{5.3.5}$$

Argueing inductively, the initial data and the new base point $\bar{z}_n$ are obtained from $W(t_n^-)$, the end-value of the evolution of W over I_{n-1}, by applying Proposition 4.1. We see that $W(t_n^-) \in X_{z_{n-1}}$ and so from (5.1.11) we have

$$\left|\bar{z}_n - z(t_n^-)\right| \le M_0 \left\|W(t_n^-)\right\|_{H^\gamma} \left|z(t_n^-) - \bar{z}_{n-1}\right| \le M_0 \left\|W(t_n^-)\right\|_{H^\gamma} T_{1,n-1}(\tau). \tag{5.3.6}$$

Substituting the estimate of $T_{1,n-1}$ from (5.2.22), we bound the z jump, see **Fig 5.1**, by

$$\begin{aligned} \left|z_n - z(t_n^-)\right| &\le M_0 \left\|W(t_n^-)\right\|_{H^\gamma} \left(T_{0,n-1}^{\gamma+1/2} + \epsilon^{3/4-\gamma_s/2}\tau\right) \\ &\le M_0 \left\|W(t_n^-)\right\|_{H^\gamma} \left(T_{0,n-1}^{\gamma+1/2} + \epsilon^{3/4-\gamma_s/2}\left|\ln(\epsilon)\right|\right) \end{aligned} \tag{5.3.7}$$

The solution at time $t = t_n$ is independent of the decomposition

$$U(t_n) = \phi_{z(t_n^-)} + W(t_n^-) = \phi_{\bar{z}_n} + W_n, \tag{5.3.8}$$

so we can bound the jump in W at each renormalization by controlling the jump of ϕ,

$$\begin{aligned} \left\|W(t_n^-) - W_n\right\|_{H^\gamma} &= \left\|\phi_{z(t_n^-)} - \phi_{\bar{z}_n}\right\|_{H^\gamma} \le c\left|\bar{z}_n - z(t_n^-)\right| \\ &\le M_0 \left\|W(t_n^-)\right\|_{H^\gamma} \left(T_{0,n-1}^{\gamma+1/2} + \epsilon^{3/4-\gamma_s/2}\left|\ln(\epsilon)\right|\right) \end{aligned} \tag{5.3.9}$$

Observe that the jump in the front location and remainder W are controlled by the same estimate. From the definition of T_0 using the equality $T_{0,n-1}(t_{n-1}) = \|W_{n-1}\|_{H^\gamma}$, and the inequality (5.2.62) we have the estimate,

$$T_{0,n-1}(\tau) \leq M_1 \left(\|W_{n-1}\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} e^{\nu(\tau - t_{n-1})} \right) \tag{5.3.10}$$

for $\tau \in [t_{n-1}, t_n]$. Since the length of each renormalization step satisfies (5.2.63), which yields $e^{\nu(\tau-t_{n-1})} \leq \epsilon^{-\beta(3/4-\gamma s/2)}$, and (5.3.10) satisfies

$$T_{0,n-1}(\tau) \leq M_1 \left(\|W_{n-1}\|_{H^\gamma} + \epsilon^{(1-\beta)(3/4-\gamma s/2)} \right) \tag{5.3.11}$$

To derive an upper bound for $\|W(t_n^-)\|_{H^\gamma}$, we refer to (5.3.1) and (5.3.4) to obtain

$$\begin{aligned} \|W(t_n^-)\|_{H^\gamma} &\leq M \left(e^{-\nu(t_n - t_{n-1})} \|W_{n-1}\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} \right) \\ &\leq M \left(\epsilon^{\beta(3/4-\gamma s/2)} \|W_{n-1}\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} \right). \end{aligned} \tag{5.3.12}$$

To obtain a bound for $\mathcal{G}W_{n-1} = W_n$, we apply triangle inequality and (5.3.9),

$$\begin{aligned} \|\mathcal{G}W_{n-1}\|_{H^\gamma} &\leq \|W(t_n^-) - W_n\|_{H^\gamma} + \|W(t_n^-)\|_{H^\gamma}, \\ &\leq M_0 \|W(t_n^-)\|_{H^\gamma} \left(T_{0,n-1}^{\gamma+1/2} + \epsilon^{3/4-\gamma s/2} |\ln(\epsilon)| \right) + \|W(t_n^-)\|_{H^\gamma}, \\ &\leq \|W(t_n^-)\|_{H^\gamma} \left(1 + M_1 T_{0,n-1}^{\gamma+1/2} \right),. \end{aligned} \tag{5.3.13}$$

using $\epsilon^{3/4-\gamma s/2} |\ln(\epsilon)| < 1$. Substituting the estimates for $T_{0,n-1}$ from (5.3.11) and $\|W(t_n^-)\|_{H^\gamma}$ from (5.3.12) into (5.3.13) we achieve an estimate for $\|\mathcal{G}W_{n-1}\|_{H^\gamma}$,

$$\begin{aligned} \|\mathcal{G}W_{n-1}\|_{H^\gamma} \leq\ & M \left(\epsilon^{\beta(3/4-\gamma s/2)} \|W_{n-1}\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} \right) \\ & \left(1 + M_1^{\gamma+1/2} \left[\left(\|W_{n-1}\|_{H^\gamma} + \epsilon^{(1-\beta)(3/4-\gamma s/2)} \right) \right]^{\gamma+1/2} \right) \end{aligned} \tag{5.3.14}$$

Neglecting higher powers of $\|W_{n-1}\|_{H^\gamma}$ and positive powers of ϵ within the second parentheses on the left-hand side of (5.3.14), yields

$$\|\mathcal{G}W_{n-1}\|_{H^\gamma} \leq M \left(\epsilon^{\beta(3/4-\gamma s/2)} \|W_{n-1}\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} \right). \tag{5.3.15}$$

We define the renormalization series r_n which bounds $\|W(t_n)\|_{H^\gamma}$ as,

$$r_n = M\left(\epsilon^{(3/4-\gamma s/2)} + \epsilon^{\beta(3/4-\gamma s/2)} r_{n-1}\right). \tag{5.3.16}$$

We solve the recursive equation (5.3.16) for r_n

$$r_n = M\epsilon^{3/4-\gamma s/2}\left[\frac{1-\left(M\epsilon^{\beta(3/4-\gamma s/2)}\right)^n}{1-M\epsilon^{\beta(3/4-\gamma s/2)}} + \left(M\epsilon^{\beta(3/4-\gamma s/2)}\right)^n r_0\right], \tag{5.3.17}$$

where $r_0 = \|W_0\|_{H^\gamma}$. For ϵ small enough so that $M\epsilon^{\beta(3/4-\gamma s/2)} < 1$ then $\|W_n\|_{H^\gamma} \leq r_n$, implies the estimate

$$\|W_n\|_{H^\gamma} \leq M\epsilon^{3/4-\gamma s/2}\left(\left(M\epsilon^{\beta(3/4-\gamma s/2)}\right)^n \|W_0\|_{H^\gamma} + 1\right). \tag{5.3.18}$$

The term M^n may be interpreted as a logarithmic correction to the exponential decay rate ν. This correction arises from the time dependent modulation of the linearized operator induced by the slow fold on the manifold. Defining ν_1 as

$$\nu_1 = \frac{\nu \ln M}{\beta\left(3/4-\gamma s/2\right)|\ln\epsilon|}, \tag{5.3.19}$$

for $t > t_n$, we can bound M^n by

$$M^n \leq e^{\nu_1(t-t_0)}. \tag{5.3.20}$$

From (5.3.4) $t_n - t_0 = \frac{n\beta(3/4-\gamma s/2)|\ln\epsilon|}{\nu}$, which along with (5.3.20) modifies (5.3.18) to

$$\|W_n\|_{H^\gamma} \leq M\epsilon^{3/4-\gamma s/2}\left(e^{-\nu(t_n-t_0)}e^{\nu_1(t-t_0)}\|W_0\|_{H^\gamma} + 1\right), \tag{5.3.21}$$

for any time $t > t_n$. From (5.3.1) for $t \in I_n = [t_n, t_{n+1}]$, $\|W(t)\|_{H^\gamma}$ can be bounded by,

$$\|W(t)\|_{H^\gamma} \leq M\left(e^{-\nu(t-t_n)}\|W_n\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)}\right). \tag{5.3.22}$$

Substituting $\|W_n\|_{H^\gamma}$ from (5.3.21) into (5.3.22) we obtain,

$$\begin{aligned}\|W(t)\|_{H^\gamma} &\leq M\left(e^{-\nu(t-t_n)}e^{-\nu(t_n-t_0)}e^{\nu_1(t-t_0)}\|W_0\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)}\right),\\ &\leq M\left(e^{-(\nu-\nu_1)(t-t_0)}\|W_0\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)}\right).\end{aligned} \tag{5.3.23}$$

Defining the new rate ν' as,

$$\nu' = \nu - \nu_1 = \nu \left(1 - \frac{\ln M}{\beta \left(3/4 - \gamma_s/2\right) |\ln \epsilon|} \right) > 0, \tag{5.3.24}$$

we obtain an uniform bound for $\|W(t)\|_{H^\gamma}$ for $t > t_0$,

$$\|W(t)\|_{H^\gamma} \leq M \left(e^{-\nu'(t-t_0)} \|W_0\|_{H^\gamma} + \epsilon^{(3/4-\gamma_s/2)} \right). \tag{5.3.25}$$

To recover the asymptotic pulse motion, we consider the situation where t satisfying (5.2.63), is sufficiently large that $\|W\|_{H^\gamma} < M\epsilon^{(3/4-\gamma_s/2)}$. In this regime we see from (5.2.23), $T_1 \leq \epsilon^{3/4-\gamma_s/2} |\ln \epsilon|$. From (5.2.14), we obtain the bound

$$\left| \left(\Delta L\, W, \Psi^\dagger_{\bar{z}_0} \right) \right|_{L^2} \leq C T_1 \|W\|_{H^\gamma} \leq \epsilon^{(3/2-\gamma_s)} |\ln \epsilon|. \tag{5.3.26}$$

The inequality (5.2.15) yields,

$$\left| \left(\mathcal{N}(W), \Psi^\dagger_{\bar{z}_0} \right) \right|_{L^2} \leq c\epsilon^{(\gamma+1/2)(3/4-\gamma/2)}. \tag{5.3.27}$$

Substituting these estimates in the equation (5.2.7) for z_0' we get,

$$z_0' = \frac{\left(R, \Psi^\dagger_{\bar{z}_0} \right)_{L^2}}{\left(\frac{\partial \phi}{\partial z_0}, \Psi^\dagger_{\bar{z}_0} \right)_{L^2}} + \mathcal{O}(\epsilon^{(5/4-\gamma_s/2)}). \tag{5.3.28}$$

From the form of residual R in (3.1.22) and the relation between the derivatives of ϕ_i and ϕ given by (3.1.10), we obtain

$$\left(R, \Psi^\dagger_{\bar{z}_0} \right)_{L^2} = \left(v_s(z_0) - v_s(\bar{z}_0) \right) \left(\frac{\partial \phi_{z_0}}{\partial z_0} + r(z), \Psi^\dagger_{\bar{z}_0} \right) + \mathcal{O}(\epsilon),$$

and (5.3.31) reduces to

$$z_0' = \left(v_s(z_0) - v_s(\bar{z}_0) \right) - \left(v_s(z_0) - v_s(\bar{z}_0) \right) \frac{\left(r(z), \Psi^\dagger_{\bar{z}_0} \right)_{L^2}}{\left(\frac{\partial \phi}{\partial z_0}, \Psi^\dagger_{\bar{z}_0} \right)_{L^2}} + \mathcal{O}(\epsilon^{(5/4-\gamma_s/2)}). \tag{5.3.29}$$

Using (3.1.25) and (3.1.14), we bound the second term in the left-hand side of (5.3.31) as

$$\begin{aligned} |v_s(z_0(t)) - v_s(\bar{z}_0)| \left| \left(r(z), \Psi^\dagger_{\bar{z}_0} \right)_{L^2} \right| &\leq C\sqrt{\epsilon} |z_0(t) - \bar{z}_0| \|r\|_{L^2} \left\| \Psi^\dagger_{\bar{z}_0} \right\|_{L^2}, \\ &\leq C\epsilon T_1(t) \leq \epsilon^{7/4-\gamma_s/2} |\ln \epsilon|. \end{aligned} \tag{5.3.30}$$

which modifies the expression (5.3.31) for z_0' to

$$z_0' = (v_s(z_0) - v_s(\bar{z}_0)) + \mathcal{O}(\epsilon^{(5/4-\gamma s/2)}). \tag{5.3.31}$$

To complete the proof of our main result note that the curve $z_0(t)$ given by (5.2.6) is smooth except for the jumps from $z_i(t_i^-)$ to $\bar{z}_i$. We may replace $z_0(t)$ with a smooth enough curve $\tilde{z}_0(t)$, which is close enough to $z_0(t)$ such that the estimates on W are not effected. For $M > 0$ and $\tilde{z}_0(t)$ which satisfy

$$|z_0(t) - \tilde{z}_0(t)| \leq M \left(e^{-\nu'(t-t_0)} \|W_0\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} \right), \tag{5.3.32}$$

we define $\tilde{W}(z,t) = U(z,t) - \phi(z; \tilde{z}_0(t))$. The following holds true

$$\left\| W(\cdot,t) - \tilde{W}(\cdot,t) \right\|_{H^\gamma} \leq C \left| z_0(t) - \tilde{z}_0(t) \right|, \tag{5.3.33}$$

for $C > 0$, which leads to the bound

$$\left\| \tilde{W} \right\|_{H^\gamma} \leq \left\| W(\cdot,t) - \tilde{W}(\cdot,t) \right\|_{H^\gamma} + \|W\|_{H^\gamma} \leq \tilde{M} \left(e^{-\nu'(t-t_0)} \|W_0\|_{H^\gamma} + \epsilon^{(3/4-\gamma s/2)} \right) \tag{5.3.34}$$

for some $\tilde{M} > 0$. In order to be able to choose such a curve $\tilde{z}_0$ to be smooth we need to verify that the jumps suffered by $z_0(t)$ at $t = t_n$, lie below the error bound (5.3.32). This holds true since the jump in the front location and remainder W are controlled by the same estimate and in the asymptotic regime, the jumps in z_0 at $t = t_n$ are $\mathcal{O}(\epsilon^{(5/4-\gamma s/2)})$ with an $\mathcal{O}(|\ln(\epsilon)|)$ time interval between the jumps. Thus we may choose the smoothed curve $\tilde{z}_0$ in such a manner such that $|z_0 - \tilde{z}_0| = \mathcal{O}(\epsilon^{(5/4-\gamma s/2)})$, for $t \neq t_i$, $i = 1, 2, \ldots$. Dropping the tilde notation we may then write the evolution for the smoothed curve z_0 as in (5.3.31).

To revert back to the original coordinate system we introduce the rescaled H_y^γ norm which uniformly controls the L^∞ norm,

$$\|W\|_{H_y^\gamma} = \left(\int_{\mathbf{R}} \epsilon^{-1/2} |W|^2 + \epsilon^{\gamma-1/2} |D^\gamma W|^2 \, dy \right)^{1/2}. \tag{5.3.35}$$

Hence we have proved the following theorem,

Theorem 5.3.1 *Given $\gamma, \gamma_s \in \left[\frac{1}{2}, 1\right)$, then for ϵ sufficiently small there exists positive constants M and t_0 such that for all initial data of the form*

$$U_0(y,t) = \phi(y; y_*) + W_0(y,t),$$

where $\|W_0\|_{H_y^\gamma} \leq \frac{1}{|\ln \epsilon|}$, the solution of the governing equation (2.0.1) can be decomposed as

$$U(y,t) = \phi(y; y_0(t)) + W(y,t) \qquad \textit{for} \quad t > t_0,$$

where the residual W satisfies

$$\|W\|_{H_y^\gamma} \leq M\left(e^{-\nu(t-t_0)}\|W_0\|_{H_y^\gamma} + \epsilon^{3/4-\gamma_s/2}\right).$$

In particular, after the perturbation W has decayed to $\mathcal{O}(\epsilon^{3/4-\gamma_s/2})$, the pulse evolution is given by the reduced equation

$$y_0'(t) = \sqrt{\epsilon}v(y_0) + \mathcal{O}(\epsilon^{5/4-\gamma_s/2}).$$

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