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PATTERN AVOIDANCE IN SET PARTITIONS AND
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Adam M. Goyt

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PATTERN AVOIDANCE IN SET PARTITIONS AND COMPOSITIONS

By

Adam M. Goyt

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ABSTRACT

PATTERN AVOIDANCE IN SET PARTITIONS AND COMPOSITIONS

By

Adam M. Goyt

We consider pattern avoidance in set partitions and compositions as natural extensions of pattern avoidance in permutations. We enumerate pattern avoiding set-partitions, even and odd pattern avoiding set partitions, and generalized set partitions patterns. The generalized set partitions patterns are then used to encode set partition statistics. We continue our focus on set partitions statistics by considering distributions over pattern restricted sets. These distributions give nice q-analogues of the Fibonacci numbers, which are then shown to satisfy q-analogues of many Fibonacci identities. We finish by considering a poset of restricted compositions. We show that this poset is shellable and use shellability to determine its Möbius function. We then show that the Möbius function and the zeta function are both rational, and determine generating function in commuting variables according to the type of the composition.

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Introduction

The focus of this dissertation is pattern avoidance in set partitions and compositions. We wish to give some background on paterns in permutations, as this is the birthplace of our topic. To do this, we will introduce some notation and basic definitions, to be made more rigorous in the next chapter.

Let $[n] = \{1, 2, \ldots, n\}$ and let S_n be the symmetric group on $[n]$. We write our permutations in one line notation, so the permutation 132 represents the permutation $\sigma \in S_3$ with $\sigma(1) = 1$, $\sigma(2) = 3$, and $\sigma(3) = 2$. If $\pi \in S_k$ and $\sigma \in S_n$ then one says that a copy of π occurs in σ as a *pattern* if there is a subsequence σ' of σ such that replacing the smallest element of σ' by 1, the next smallest by 2, and so on, results in the permutation π. For example in the permutation 15243, 154 is a copy of the pattern 132. If there are no copies of a particular pattern in $\sigma \in S_n$ then we say σ *avoids* the pattern. For a set of permutations $R \subseteq S_k$, let $S_n(R)$ be the set of permutations in S_n, which avoid every permutation in R.

In 1979, Knuth published a proof that the cardinality $\#S_n(231) = C_n$, the n^{th} Catalan number. He was interested in the enumeration of this set because the permutations which avoid the pattern 231 are exactly the stack sortable sequences. The fact that such a lovely enumeration emerged from a natural computer science question sparked other people's interest. The next natural question was the enumeration of $S_n(\sigma)$ for any $\sigma \in S_3$. It was shown that $\#S_n(\sigma) = Cn$ for any $\sigma \in S_3$. In 1985, Simion and Schmidt published a paper, in which they enumerated $S_n(R)$ for any $R \subseteq S_3$ as well as the odd and even permutations avoiding patterns in R, and the number of involutions avoiding all patterns in R. The study of patterns in permutations exploded from this point, and is currently an area of very active research.

In 1996, Klazar defined a notion of pattern avoidance in set partitions involving restricted growth functions. More recently, Sagan gave a definition of pattern avoidance in set partitions, which uses a block form of partitions. Let Π_n be the set of partitions of the set $[n]$ and $R \subseteq \Pi_k$. Similar to the definition for permutations, let $\Pi_n(R)$ be the partitions of $[n]$, which avoid every partition in R. In a 2006 preprint, Sagan enumerates $\Pi_n(\pi)$ for any $\pi \in \Pi_3$. In the second chapter we enumerate $\Pi_n(R)$ for any $R \subseteq \Pi_3$, the even and odd partitions of $[n]$ avoiding R, and the partitions in Π_n avoiding generalized partition patterns. Generalized partition patterns are analogues of generalized permutation patterns introduced by Babson and Steingrímsson and are patterns in which certain blocks must be adjacent. Babson and Steingrímsson introduce the generalized patterns as a way to classify Mahonian statistics on permutations. See the next paragraph for a definition of statistic and Mahonian statistic. In most cases, generalizing a pattern does not change the enumeration of its avoidance. In two cases we were unable to obtain a closed form for the enumerations and hence used exponential generating functions to handle them. Finally, we show that set partition statistics can be written in terms of generalized partitions.

The second chapter deals with distributions of set partition statistics on pattern restricted sets of set partitions. A *statistic*, ρ, on a set S is a function $\rho : S \rightarrow \mathbb{P}$, where $\mathbb{P} = \{0, 1, 2, \ldots\}$. When studying statistics, we often consider the generating function

$$\sum_{\sigma \in S} q^{\rho(\sigma)}.$$

A *Mahonian statistic* τ has generating function

$$\sum_{\sigma \in S_n} q^{\tau(\sigma)} = [n]_q! := [n]_q[n-1]_q \cdots [1]_q,$$

where $[k]_q = 1 + q + q^2 + \ldots + q^{k-1}$. Notice that if we let $q = 1$ then $[k]_q = k$ and $[n]_q! = n!$. For this reason, we say that $[n]_q!$ is a *q-analogue* of $n!$. It is known that $\#\Pi_n(13/2) = 2^{n-1}$ and $\#\Pi_n(13/2, 123) = F_n$, the n^{th} Fibonacci number. In chapter three we consider the generating functions for distributions of statistics ls and rb introduced by Wachs and White and obtain nice q-analogues of 2^{n-1} and F_n.

It turns out that the q-analogue of 2^{n-1} is exactly the same as the generating function for the number of integer partitions with distinct parts of size at most $n-1$. We are able to prove this relationship by bijection. The q-analogues of F_n are closely related to q-analogues of F_n first discovered by Carlitz and Cigler. We also prove this relationship using a bijection. The exciting part is that we were able to take these q-Fibonacci numbers, $F_n(q)$, and prove many q-analogues of Fibonacci identities bijectively by adapting tiling scheme proofs of Benjamin-Quinn and Brigham et. al. We also use a method due to Linstrom and popularized by Gessel and Viennot to prove a q-analogue of the Euler-Cassini identity using lattice paths and determinants of the Toeplitz matrix for $F_n(q)$.

Recently, Stanley and Björner produced an analogue of Young's lattice for compositions, called C. In the last chapter, we consider a restricted version of this lattice. That is, we consider all compositions, which avoid the composition $d+1$. This simply restricts the part sizes of the compositions to at most n. We call this new lattice C_n. The lattice C_d has a nice Möbius function. We show that C_d is **CL**-shellable and show that $\mu(u, w) = (-1)^{|u|+|w|}\binom{w}{u}_{dn}$, where $|u|$ is the sum of the parts of u and $\binom{w}{u}_{dn}$ is the number of d-normal embeddings of u in w. We also provide a combinatorial proof of the same fact. We show that the series $Z_{\otimes} = \zeta(u, w)u \otimes w$, where ζ is the zeta function of C_d, and the series $M_{\otimes} = \mu(u, w)u \otimes w$, where μ is the Möbius function of C_d, are rational by showing that they are accepted by a finite state automota. Finally, we determine norm and length generation functions for ζ and μ.

Chapter 1
Avoidance of Partitions of a Three-element Set

1.1 Introduction

If $f : S \rightarrow T$ is a function from set S to set T, then f acts element-wise on objects constructed from S. For example, if $a_1a_2 \ldots a_n$ is a permutation of elements of S then $f(a_1a_2 \ldots a_n) = f(a_1)f(a_2) \ldots f(a_n)$. Also, define $[n]$ to be the set $\{1, 2, \ldots, n\}$ and $[k, n]$ to be the set $\{k, k+1, \ldots n\}$.

Suppose that $S \subseteq \mathbb{Z}$ is a set with $\#S = n$, then the *standardization* map corresponding to S is the unique order preserving bijection $St_S : S \rightarrow [n]$. For example if $S = \{2, 5, 7, 10\}$ then $St_S(2) = 1$, $St_S(5) = 2$, $St_S(7) = 3$, and $St_S(10) = 4$. When it is clear from context what set the standardization map is acting on, we will omit the subscript S.

Let $p = a_1a_2 \ldots a_k \in S_k$ be a given permutation, called the *pattern*, where S_k is the symmetric group on k letters. A permutation $q = b_1b_2 \ldots b_n \in S_n$ *contains* the pattern p if there is a subsequence $q' = b_{i_1}b_{i_2} \ldots b_{i_k}$ of q with $St(q') = p$. Otherwise q *avoids* p. For example the permutation $q = 32145$ contains 6 copies of the pattern 213, namely 324, 325, 314, 315, 214, and 215. On the other hand q avoids the pattern 132. For $R \subseteq S_k$, let

$$S_n(R) = \{q \in S_n : q \text{ avoids every pattern } p \in R\}.$$

The problem of enumerating $S_n(R)$ for $R \subseteq S_3$ was considered by Simion and Schmidt [37]. We will consider the analogous problem for patterns in partitions.

A *partition* π of set $S \subseteq \mathbb{Z}$, written $\pi \vdash S$, is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of S called *blocks* such that $\bigcup_{i=1}^{k} B_i = S$. We write $\pi = B_1/B_2/\ldots/B_k$ and define the *length* of π, written $\ell(\pi)$, to be the number of blocks. Since the order of the blocks does not matter, we will always write our partitions in the *canonical order* where

$$\min B_1 < \min B_2 < \ldots < \min B_k.$$

We will also always write the elements of each block in increasing order. For example, $137/26/45 \vdash [7]$ has length 3.

Let

$$\Pi_n = \{\pi \vdash [n]\}$$

be the set of all partitions of $[n]$. Suppose σ is a set partition of length m and π is a partition of length ℓ. Then σ *contains* π, written $\pi \subseteq \sigma$, if there are ℓ different blocks of σ each containing a block of π. For example $\sigma = 137/26/45$ contains $\pi = 2/37/5$ but does not contain $\pi' = 2/37/6$ because 2 and 6 are in the same block of σ.

Let $\pi \in \Pi_k$ be a given set partition called the *pattern*. A partition $\sigma \in \Pi_n$ *contains the pattern* π if there is some $\sigma' \subseteq \sigma$ with $St(\sigma') = \pi$. Otherwise π *avoids* σ. For example $\sigma = 137/26/45$ contains six copies of the pattern $\pi = 14/2/3$, namely 17/2/4, 17/2/5, 17/4/6, 17/5/6, 26/3/4, and 26/3/5. It is important to note here that when looking for a copy of π in σ, the order of the blocks does not matter. On the other hand consider the pattern $\pi' = 1/234$. To be contained in σ the copy of the block 234 of π' must be contained in a block of size three or larger. The only such block of σ is 137. It is impossible to find an element smaller than 1, so σ does not contain a copy of π'. For $R \subseteq \Pi_k$, let

$$\Pi_n(R) = \{\sigma \in \Pi_n : \sigma \text{ avoids every pattern } \pi \in R\}.$$

The extensively studied set of non-crossing partitions may be defined as the set $\Pi_n(13/24)$. It is known that $\#\Pi_n(13/24) = C_n$, where C_n is the n^{th} Catalan number [31], [39]. For a survey of results about non-crossing partitions see Simion's paper [36].

Sagan [34] has provided enumerative results for $\Pi_n(R)$ when $\#R = 1$. In the spirit of work done by Simion and Schmidt on permutation patterns [37], we will enumerate $\Pi_n(R)$ for $\#R \geq 2$. We then define the sign of a partition and enumerate the set of signed partitions of $[n]$ avoiding particular patterns. In section 5, we define generalized patterns analogous to the generalized permutation patterns of Babson and Steingrímsson [2], and provide enumerative results for those. Finally, we will show how these generalized partition patterns can be used to describe set partition statistics.

1.2 Double Restrictions

In this section we will consider the case of $\#\Pi_n(R)$ where $\#R = 2$. Given a set partition $\sigma = B_1/B_2/\ldots/B_k \vdash [n]$, let $\sigma^c = B_1^c/B_2^c/\ldots/B_k^c$ be the *complement* of σ where

$$B_i^c = \{n - a + 1 : a \in B_i\}.$$

For example if $\sigma = 126/3/45$ then $\sigma^c = 156/23/4$. The following result is obvious, so we omit the proof.

Proposition 1.2.1 (Sagan) *For $n \geq 1$,*

$$\begin{aligned} \Pi_n(\sigma^c) &= \{\pi^c : \pi \in \Pi_n(\sigma)\}, \\ \#\Pi_n(\sigma^c) &= \#\Pi_n(\sigma). \end{aligned}$$

The following Lemma is an immediate consequence of Proposition 1.2.1.

Lemma 1.2.2

$$\begin{aligned}
\#\Pi_n(12/3,123) &= \#\Pi_n(1/23,123)\\
\#\Pi_n(1/2/3,12/3) &= \#\Pi_n(1/2/3,1/23)\\
\#\Pi_n(12/3,13/2) &= \#\Pi_n(1/23,13/2).
\end{aligned}$$

There are 10 different sets R with elements from Π_3 and $\#R = 2$, so by Lemma 1.2.2 there are seven different cases to consider. Note that $\#\Pi_0 = 1$ by letting the empty set partition itself. Since any partition in Π_1 or Π_2 cannot possibly contain a partition of [3], we have $\#\Pi_0(R) = 1$, $\#\Pi_1(R) = 1$ and $\#\Pi_2(R) = 2$ for all $R \subseteq \Pi_3$. The fact that $\#\Pi_3 = 5$ implies that $\#\Pi_3(R) = 3$ for any $R \subset \Pi_3$, with $\#R = 2$. Hence, it suffices to consider $n \geq 4$ in the following results.

A partition $\sigma \vdash [n]$ is *layered* if σ is of the form $[1,i]/[i+1,j]/[j+1,k]/\ldots/[\ell+1,n]$. An example of a layered partition is $\sigma = 123/4/56/789$. A partition σ is a *matching* if $\#B \leq 2$ for every block B of σ.

We will use the following results of Sagan [34] repeatedly, so we state them now.

Proposition 1.2.3 (Sagan)

$$\Pi_n(1/2/3) = \{\sigma : l(\sigma) \leq 2\}, \tag{1.1}$$

$$\Pi_n(12/3) = \{\sigma = B_1/B_2/\ldots/B_k : \min B_i = i \textit{ for each } i, \textit{ and } [k+1,n] \subseteq B_i \textit{ for some } i\}, \tag{1.2}$$

$$\Pi_n(13/2) = \{\sigma : \sigma \textit{ is layered}\}, \tag{1.3}$$

$$\Pi_n(123) = \{\sigma : \sigma \textit{ is a matching}\}. \square \tag{1.4}$$

Proposition 1.2.4 *For all $n \geq 3$,*

$$\begin{aligned} \Pi_n(1/2/3, 12/3) &= \{12\ldots n,\ 1/23\ldots n,\ 13\ldots n/2\}, \\ \#\Pi_n(1/2/3, 12/3) &= 3. \end{aligned}$$

Proof: Let $\sigma \in \Pi_n(1/2/3, 12/3)$. By (1.1), σ may have at most two blocks. If $\ell(\sigma) = 1$ then $\sigma = 12\ldots n$. If $\ell(\sigma) = 2$ then by (1.2), we must have $[3, n] \subset B_i$ for $i = 1$ or 2. □

Proposition 1.2.5 *For all $n \geq 1$,*

$$\begin{aligned} \Pi_n(1/2/3, 13/2) &= \{\sigma : \sigma = 12\ldots k/(k+1)(k+2)\ldots n \\ &\qquad\qquad\qquad\qquad for\ some\ k \in [n]\}, \\ \#\Pi_n(1/2/3, 13/2) &= n. \end{aligned}$$

Proof: If $\sigma \in \Pi_n(1/2/3, 13/2)$ then σ is layered by (1.3), and $\ell(\sigma) \leq 2$ by (1.1). Hence σ is of the form described above. The enumeration follows immediately. □

Proposition 1.2.6

$$\begin{aligned} \Pi_n(1/2/3, 123) &= \begin{cases} \{12/34, 13/24, 14/23\} & n = 4, \\ \emptyset & n \geq 5. \end{cases} \\ \#\Pi_n(1/2/3, 123) &= \begin{cases} 3 & n = 4, \\ 0 & n \geq 5. \end{cases} \end{aligned}$$

Proof: If $n \geq 5$ and $\sigma \vdash [n]$, then $\ell(\sigma) \geq 3$ or σ has a block of size ≥ 3 by the Pigeonhole Principle. Thus by (1.1) and (1.4), $\Pi_n(1/2/3, 123) = \emptyset$ for $n \geq 5$. The case $n = 4$ is easy to check. □

Proposition 1.2.7 *For all $n \geq 3$,*

$$\begin{aligned} \Pi_n(1/23, 12/3) &= \{12\ldots n,\ 1/2/\ldots/n,\ 1n/2/3/\ldots/n-1\}, \\ \#\Pi_n(1/23, 12/3) &= 3. \end{aligned}$$

Proof: Let $\sigma = B_1/B_2/\ldots/B_k$ avoid 12/3. If $k = 1$ then $\sigma = 12\ldots n$, which avoids 1/23. Similarly, when $k = n$, we have $\sigma = 1/2/\ldots/n$, which avoids 1/23. If $k = n-1$ and $n \in B_i$ for $i \geq 2$ then B_1/B_i is a copy of 1/23. Thus $n \in B_1$ and $\sigma = 1n/2/3/\ldots/n-1$. If $1 < k < n-1$ then, by (1.2), we must have $\{n-1, n\} \subseteq B_i$ for some i, and there is at least one more block. Hence σ contains a copy of 1/23, and so this case can not occur. □

Proposition 1.2.8 *For all $n \geq 1$,*

$$\begin{aligned} \Pi_n(12/3, 13/2) &= \{\sigma = 1/2/\ldots/k-1/k(k+1)\ldots n, \\ &\qquad\qquad for\ some\ k \in [n]\}, \\ \#\Pi_n(12/3, 13/2) &= n. \end{aligned}$$

Proof: Suppose $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(12/3, 13/2)$. Then by (1.2) we have $i \in B_i$ for each i and exactly one of the B_i contains $[k+1, n]$. From (1.3) we have that σ must be layered. So $[k+1, n] \in B_k$, and $B_k = [k, n]$. Thus there is exactly one $\sigma \in \Pi_n(12/3, 13/2)$ of length k for each $k \in [n]$. □

Proposition 1.2.9 *For all $n \geq 1$,*

$$\begin{aligned}\Pi_n(12/3, 123) &= \{\sigma = B_1/B_2/\ldots/B_k : \min B_i = i, \\ &\qquad\qquad\qquad\qquad \textit{and } k = n-1 \textit{ or } n\}, \\ \#\Pi_n(12/3, 123) &= n.\end{aligned}$$

Proof: Assume $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(12/3, 123)$. Then by (1.2) and (1.4), $k = n-1$ or n. The result follows. □

Let F_n be the n^{th} Fibonacci number, initialized by $F_0 = 1$ and $F_1 = 1$. A *composition* of an integer n is an ordered collection of positive integers $n_1, n_2, \ldots, n_k$ such that $n = n_1 + n_2 + \ldots + n_k$. The n_i are called *parts*. It is easy to see that F_n counts the number of compositions of n with parts of size 1 or 2.

Proposition 1.2.10 *For all $n \geq 0$,*

$$\begin{aligned}\Pi_n(13/2, 123) &= \{\sigma : \sigma \text{ is a layered matching}\}, \\ \#\Pi_n(13/2, 123) &= F_n.\end{aligned}$$

Proof: Any $\sigma \in \Pi_n(13/2, 123)$ must be layered by (1.3) and a matching by (1.4).

There is a bijection between the compositions of n with parts of size 1 or 2 and the partitions of $[n]$ that are layered matchings. If $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(13/2, 123)$, then we map σ to the composition $n = n_1 + n_2 + \ldots + n_k$ with $n_i = \#B_i$. □

From the results above we know that

$$\#\Pi_n(1/2/3, 13/2) = \#\Pi_n(12/3, 13/2) = \#\Pi_n(12/3, 123) = n,$$

and we have a very nice description of the elements in each of these sets. It is interesting to note that one gets similar results when avoiding certain sets of permutations in S_3.

Proposition 1.2.11 (Simion, Schmidt) *For every $n \geq 1$,*

$$\begin{aligned} \#S_n(123,132,231) &= \#S_n(123,213,312) = n. \\ \#S_n(132,231,321) &= \#S_n(213,312,321) = n. \end{aligned}$$

And:

$$\begin{aligned} q \in S_n(123,132,231) &\iff q = (n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k), \\ q \in S_n(123,213,312) &\iff q = (n, n-1, \ldots, k+1, 1, 2, 3, \ldots, k), \\ q \in S_n(132,231,321) &\iff q = (n-1, n-2, \ldots k+1, n, k, k-1, \ldots, 2, 1), \\ q \in S_n(213,312,321) &\iff q = (k-1, \ldots, 3, 2, 1, n, n-1, \ldots, k). \square \end{aligned}$$

The Fibonacci numbers also occur when avoiding permutations.

Proposition 1.2.12 (Simion, Schmidt) *For every $n \geq 1$,*

$$\#S_n(123,132,213) = F_n. \square$$

There is a simple map $\Phi : \Pi_n \to S_n$, given by sending $\sigma = B_1/B_2/\ldots/B_k$ to $B_k B_{k-1} \cdots B_1$. For example, $\Phi(1/23/4/56) = 564231$.

Proposition 1.2.13 *The map Φ restricts to a bijection from the set $\Pi_n(13/2, 123)$ to the set $S_n(123, 132, 213)$.*

Proof: We may describe $q \in S_n(123, 132, 213)$ recursively. To avoid the patterns 123 and 213, we must have $q^{-1}(n) \leq 2$. If $q^{-1}(n) = 1$ then the remaining positions form a permutation in $S_{n-1}(123, 132, 213)$. If $q^{-1}(n) = 2$ then $q^{-1}(n-1) = 1$, otherwise there will be a copy of 132 in q. The remaining positions form a permutation in $S_{n-2}(123, 132, 213)$.

Suppose $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(13/2, 123)$, then $B_k = \{n\}$ or $\{n-1, n\}$. The permutation $\Phi(\sigma)$ thus begins with n or $n-1, n$. Inductively, one can see that this restriction of the map Φ is well defined.

To prove that the restricted Φ is a bijection we provide its inverse map. Let $q = q_1q_2\ldots q_n \in S_n(123, 132, 213)$ then we say that q_k is a *descent* if $q_k > q_{k+1}$. Let $D = \{q_{i_1}, q_{i_2}, \ldots, q_{i_\ell}\}$ be the set of descents of q, with $i_1 < i_2 < \ldots < i_\ell$. Then

$$\Phi^{-1}(q) = q_{i_\ell+1}q_{i_\ell+2}\cdots q_n/q_{i_{\ell-1}+1}\cdots q_{i_\ell}/\ldots/q_1\cdots q_{i_1}.$$

For example $\Phi^{-1}(564231) = 1/23/4/56$ because its descent set is $D = \{3, 4, 6\}$.

We now show that Φ^{-1} is well defined. Every $q \in S_n(123, 132, 213)$ must have a descent in at least one of its first two positions. After this initial descent there may be no more than one position between any two descents. Thus the blocks of $\Phi^{-1}(q)$ will have size at most 2, and from the description of the elements of $S_n(123, 132, 213)$ above $\Phi^{-1}(q)$ will be layered.

The fact that Φ and Φ^{-1} are inverses follows easily from the descriptions of the maps. □

1.3 Higher Order Restrictions

We begin, as with double restrictions, by reducing the number of cases. The following Lemma is a consequence of Proposition 1.2.1.

Lemma 1.3.1

$$\begin{aligned}\#\Pi_n(1/2/3, 12/3, 123) &= \#\Pi_n(1/2/3, 1/23, 123),\\ \#\Pi_n(1/2/3, 12/3, 13/2) &= \#\Pi_n(1/2/3, 1/23, 13/2),\\ \#\Pi_n(12/3, 13/2, 123) &= \#\Pi_n(1/23, 13/2, 123). \square\end{aligned}$$

The results for $\#\Pi_n(R)$ where $\#R = 3$ are easy to prove. Table 1.3.3 describes these sets and gives their enumeration for $n \geq 4$. The following proposition describes $\#\Pi_n(R)$ for $\#R \geq 4$. We omit the simple proof.

Proposition 1.3.2 *For $R \subseteq \Pi_3$ with $\#R \geq 4$ and $n \geq 4$,*

$$\#\Pi_n(R) = \begin{cases} 0 & \textit{if } \{1/2/3, 123\} \subseteq R, \\ 1 & \textit{else}. \square \end{cases}$$

Table 1: Enumeration of partitions restricted by 3 patterns

R	$\Pi_n(R)$	$\#\Pi_n(R)$
$\{1/2/3, 12/3, 13/2\}$	$\{12\ldots n, 1/23\ldots n\}$	2
$\{1/2/3, 12/3, 123\}$	$\emptyset$	0
$\{1/2/3, 13/2, 123\}$	$\{12/34\}$ $\emptyset$	1 if $n = 4$ 0 if $n \geq 5$
$\{1/2/3, 1/23, 12/3\}$	$\{12\ldots n\}$	1
$\{12/3, 13/2, 123\}$	$\{1/2/\ldots/n, 1/2/\ldots/n-2/(n-1)n\}$	2
$\{1/23, 12/3, 13/2\}$	$\{123\ldots n, 1/2/\ldots/n\}$	2
$\{1/23, 12/3, 123\}$	$\{1/2/\ldots/n, 1n/2/3/\ldots/n-1\}$	2

1.4 Even and Odd Set Partitions

In this section we will consider the number of even and odd partitions of the set $[n]$, which avoid a single pattern of length three. A partition $\sigma \vdash [n]$ with $l(\sigma) = k$ has *sign*,

$$\operatorname{sgn}(\sigma) = (-1)^{n-k}.$$

Even partitions σ satisfy $\text{sgn}(\sigma) = 1$, and *odd* partitions σ satisfy $\text{sgn}(\sigma) = -1$. We will use the following notation:

$$\begin{aligned} E\Pi_n(\pi) &= \{\sigma \vdash [n] : \text{sgn}(\sigma) = 1\}, \\ O\Pi_n(\pi) &= \{\sigma \vdash [n] : \text{sgn}(\sigma) = -1\}. \end{aligned}$$

The following follows directly from the definitions.

Lemma 1.4.1 *The sign of σ is the same as the sign of σ^c. Thus $\#E\Pi_n(12/3) = \#E\Pi_n(1/23)$ and*
$\#O\Pi_n(12/3) = \#O\Pi_n(1/23)$. □

We will use the following result of Sagan [34] repeatedly, so we state it now. Define the *double factorial* by

$$(2i)!! = 1 \cdot 3 \cdot 5 \cdots (2i-1).$$

Proposition 1.4.2 (Sagan)

$$\begin{aligned} \#\Pi_n(1/2/3) &= 2^{n-1}, & (1.5) \\ \#\Pi_n(12/3) &= \binom{n}{2} + 1, & (1.6) \\ \#\Pi_n(13/2) &= 2^{n-1}. & (1.7) \\ \#\Pi_n(123) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} (2i)!! \;\square & (1.8) \end{aligned}$$

We now consider single restrictions. By Lemma 1.4.1 there are only four cases.

Proposition 1.4.3 *For all odd $n \geq 1$,*

$$\begin{aligned} \#E\Pi_n(1/2/3) &= 1, \\ \#O\Pi_n(1/2/3) &= 2^{n-1} - 1. \end{aligned}$$

For all even $n \geq 2$,

$$\begin{aligned} \#E\Pi_n(1/2/3) &= 2^{n-1} - 1, \\ \#O\Pi_n(1/2/3) &= 1. \end{aligned}$$

Proof: By (1.1), any $\sigma \in \Pi_n(1/2/3)$ must have $\ell(\sigma) \leq 2$. If n is odd then a partition of length 1 will be even and a partition of length 2 will be odd. There is only one partition of length 1, and $\#O\Pi_n(\pi) + \#E\Pi_n(\pi) = \#\Pi_n(\pi)$ for any pattern π. Thus, the result holds for odd n by (1.5). The proof for even n is similar. □

Proposition 1.4.4 *For all odd* $n \geq 0$,

$$\begin{aligned} \#E\Pi_n(12/3) &= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1, \\ \#O\Pi_n(12/3) &= \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

Proof: By (1.2) we have, for n odd,

$$\#E\Pi_n(12/3) = 1 + \sum_{k=0}^{\frac{n-3}{2}} (2k+1) = 1 + \frac{(n-1)^2}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1,$$

and by (1.6)

$$\#O\Pi_n(12/3) = \binom{n}{2} + 1 - \frac{(n-1)^2}{4} - 1 = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

The proof for even n is similar. □

Proposition 1.4.5 *For all $n \geq 1$,*

$$\#O\Pi_n(13/2) = \#E\Pi_n(13/2) = 2^{n-2}.$$

Proof: By (1.7) it suffices to give a sign reversing involution $\psi : \Pi_n(13/2) \to \Pi_n(13/2)$. By (3), $\sigma \in \Pi_n(13/2)$ is layered, so it is of the form $\sigma = B_1/B_2/\ldots/B_k$, where either $B_k = \{n\}$ or $B_k \supset \{n\}$. Let

$$\psi(\sigma) = \begin{cases} B_1/B_2/\ldots/B_{k-1} \cup \{n\} & \text{if } B_k = \{n\}, \\ B_1/B_2/\ldots/B_k - \{n\}/n & \text{if } B_k \supset \{n\}. \end{cases}$$

Notice that $\psi(\sigma)$ is still layered for any $\sigma \in \Pi_n(13/2)$, so ψ is well defined. And, ψ is its own inverse because it either moves n into the block preceding it if $\{n\}$ is a block and into its own block otherwise. Also, ψ changes the sign of σ by either increasing or decreasing the length of σ by 1. □

Proposition 1.4.6 *For all $n \geq 1$,*

$$\#E\Pi_n(123) = \sum_{i=0}^{\left\lfloor \frac{n-2}{4} \right\rfloor} \binom{n}{4i+2} (4i+2)!!$$

$$\#O\Pi_n(123) = \sum_{i=0}^{\left\lfloor \frac{n}{4} \right\rfloor} \binom{n}{4i} (4i)!!$$

Proof: Any $\sigma \in \Pi_n(123)$ is a matching. If i blocks of σ have 2 elements each and the remaining blocks are singletons then σ has $i + (n - 2i) = n - i$ blocks. Thus $\text{sgn}(\sigma) = (-1)^{n - (n - i)} = (-1)^i$. So the even and odd counts are obtained by taking the appropriate terms from (1.8). □

Table 1.4.7 gives the results for $\#E\Pi_n(R)$ and $\#O\Pi_n(R)$ where $\#R \geq 2$ and $n \geq 4$. We prove the enumeration of $E\Pi_n(13/2, 123)$ and $O\Pi_n(13/2, 123)$ as an example and leave the rest to the reader.

Proposition 1.4.7

$$\#E\Pi_n(13/2, 123) = \begin{cases} \lceil F_n/2 \rceil & \text{for } n \equiv 0, 1 \pmod 6, \\ F_n/2 & \text{for } n \equiv 2, 5 \pmod 6, \\ \lfloor F_n/2 \rfloor & \text{for } n \equiv 3, 4 \pmod 6. \end{cases}$$

$$\#O\Pi_n(13/2, 123) = \begin{cases} \lfloor F_n/2 \rfloor & \text{for } n \equiv 0, 1 \pmod 6, \\ F_n/2 & \text{for } n \equiv 2, 5 \pmod 6, \\ \lceil F_n/2 \rceil & \text{for } n \equiv 3, 4 \pmod 6. \end{cases}$$

Proof: Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(13/2, 123)$. Then $B_k = \{n\}$ or $\{n - 1, n\}$. If $B_k = \{n\}$ then $B_1/B_2/\ldots/B_{k-1}$ is a layered matching of $[n - 1]$ and $\text{sgn}(B_1/B_2/\ldots/B_{k-1}) = \text{sgn}(\sigma)$. If $B_k = \{n - 1, n\}$ then $B_1/B_2/\ldots/B_{k-1}$ is a layered matching of $[n - 2]$ and $\text{sgn}(B_1/B_2/\ldots/B_{k-1}) = -\text{sgn}(\sigma)$. Thus we have that

$$\#E\Pi_n(13/2, 123) = \#E\Pi_{n-1}(13/2, 123) + \#O\Pi_{n-2}(13/2, 123).$$

Similarly,

$$\#O\Pi_n(13/2, 123) = \#O\Pi_{n-1}(13/2, 123) + \#E\Pi_{n-2}(13/2, 123).$$

Table 2: Enumeration of even and odd partitions restricted by at least 2 patterns.

R	$\#E\Pi_n(R)$	$\#O\Pi_n(R)$
$\{1/2/3, 12/3\}$	1 for n odd 2 for n even	2 for n odd 1 for n even
$\{1/2/3, 13/2\}$	1 for n odd $n-1$ for n even	$n-1$ for n odd 1 for n even
$\{1/2/3, 123\}$	3 for $n=4$ 0 for $n \geq 5$	0
$\{1/23, 12/3\}$	2 for n odd 1 for n even	1 for n odd 2 for n even
$\{12/3, 13/2\}$	$\lceil n/2 \rceil$	$\lfloor n/2 \rfloor$
$\{12/3, 123\}$	1	$n-1$
$\{13/2, 123\}$	$\lceil F_n/2 \rceil$ for $n \equiv 0, 1 \pmod 6$ $F_n/2$ for $n \equiv 2, 5 \pmod 6$ $\lfloor F_n/2 \rfloor$ for $n \equiv 3, 4 \pmod 6$	$\lfloor F_n/2 \rfloor$ for $n \equiv 0, 1 \pmod 6$ $F_n/2$ for $n \equiv 2, 5 \pmod 6$ $\lceil F_n/2 \rceil$ for $n \equiv 3, 4 \pmod 6$
$\{1/2/3, 1/23, 12/3\}$	1 for n odd 0 for n even	0 for n odd 1 for n even
$\{1/2/3, 12/3, 13/2\}$	1	1
$\{1/2/3, 12/3, 123\}$	0	0
$\{1/2/3, 13/2, 123\}$	1	0
$\{1/23, 12/3, 13/2\}$	2 for n odd 1 for n even	0 for n odd 1 for n even
$\{1/23, 12/3, 123\}$	1	1
$\{12/3, 13/2, 123\}$	1	1
$\{1/2/3, 1/23, 12/3, 13/2\}$	1 for n even 0 for n odd	0 for n odd 1 for n even
$\{1/2/3, 1/23, 12/3, 123\}$	0	0
$\{1/2/3, 12/3, 13/2, 123\}$	0	0
$\{1/23, 12/3, 13/2, 123\}$	1	0
$\{1/2/3, 1/23, 12/3,$ $13/2, 123\}$	0	0

Now induct on n. To show that the proposition is true when $0 \leq n \leq 5$ is easy.

This leaves us with twelve cases to check for the inductive step. We will show one of them. It is easy to see that F_n is odd unless $n \equiv 2, 5 \pmod 6$.

Suppose that $n \equiv 4 \pmod 6$. Then we have

$$\begin{aligned}
\#E\Pi_n(13/2, 123) &= \#E\Pi_{n-1}(13/2, 123) + \#O\Pi_{n-2}(13/2, 123) \\
&= \lfloor F_{n-1}/2 \rfloor + F_{n-2}/2 \\
&= \frac{F_{n-1} - 1 + F_{n-2}}{2} \\
&= \lfloor F_n/2 \rfloor. \square
\end{aligned}$$

1.5 Generalized Partition Patterns

Babson and Steingrímsson [2] defined generalized patterns for permutations. These were patterns in which certain elements were required to be consecutive. Generalized permutation patterns were used to describe permutation statistics and classify Mahonian statistics. In this section we will define a similar notion for set partition patterns and consider the avoidance case. In the next section we will show that generalized partition patterns can be used to describe set partition statistics.

Recall that if $\sigma = B_1/B_2/\ldots/B_k$ is a partition then the blocks are written in such a way that $\min B_1 < \min B_2 < \ldots < \min B_k$. This gives us a well defined notion of adjacency of blocks, where we consider B_i as being adjacent to both B_{i-1} and B_{i+1}. Consider the partition $\sigma = 147/25/36$ and the pattern $\pi = 13/2$. Suppose now that a copy of π must appear in adjacent blocks. Then $17/2$ is still a copy, but $17/3$ is not. We may also have the blocks in the restricted copy of $13/2$ in the opposite order making $25/4$ a copy of π in σ. We will denote π with the adjacency restriction by the *generalized pattern* $\rho = 13|2$. In general, we will denote block adjacency using a vertical bar.

Recall that the elements of a block are put in order by size, which gives us a way to consider adjacent elements. Now, suppose we want to find a copy of $13/2$

in $\sigma = 147/25/36$, but we require that the elements that represent 1 and 3 in this copy are adjacent. In this case 14/3 is a copy of 13/2, but 17/6 is not, since 1 and 7 are not adjacent in their block. We will denote this by the *generalized pattern* $\rho = \overset{\frown}{13}/2$. In general, we will denote element adjacency by placing an arc over the elements, which must be adjacent.

If ρ is a generalized pattern, then the notation $\Pi_n(\rho)$ denotes the set of partitions of $[n]$, which avoid ρ. Similarly, if R is any set of generalized patterns then $\Pi_n(R)$ is the set of partitions of $[n]$, which avoid all generalized patterns in R.

We are interested in enumerating the $\Pi_n(R)$ where R is a set of partitions of [3] at least one of which contains an adjacency restriction. It turns out that the adjacency restrictions do not actually restrict most of the original patterns. This is summed up in the next lemma.

Lemma 1.5.1 *The following are true for generalized patterns:*

$$\begin{array}{rcccccc}
\Pi_n(1/2/3) & = & \Pi_n(1|2/3) & = & \Pi_n(1/2|3) & = & \Pi_n(1|2|3), \\
\Pi_n(1/23) & = & \Pi_n(1|23) & = & \Pi_n(1/\overset{\frown}{23}) & = & \Pi_n(1|\overset{\frown}{23}), \\
\Pi_n(13/2) & = & \Pi_n(\overset{\frown}{13}/2) & = & \Pi_n(13|2) & = & \Pi_n(\overset{\frown}{13}|2), \\
\Pi_n(123) & = & \Pi_n(\overset{\frown}{12}3) & = & \Pi_n(1\overset{\frown}{23}) & = & \Pi_n(\overset{\frown}{1\,2}\!\overset{\frown}{\,3}), \\
\Pi_n(12/3) & = & \Pi_n(\overset{\frown}{12}/3). & & & & \\
\Pi_n(12|3) & = & \Pi_n(\overset{\frown}{12}|3). & & & &
\end{array}$$

Proof: We will only prove the second line as the others are very similar. First we show that $\Pi_n(1/23) = \Pi_n(1|23)$. It is obvious that if a partition $\sigma \vdash [n]$ contains a copy of $1|23$ then it contains a copy of $1/23$. So it will suffice to show the other containment holds. Let $\sigma = B_1/B_2/\ldots/B_k \vdash [n]$ contain a copy a/bc of 1/23. Suppose $a \in B_s$ and $b, c \in B_t$. If $s < t$ then the block B_{t-1} exists and $\min B_{t-1} < \min B_t \leq b < c$. Letting $d = \min B_{t-1}$ gives a copy d/bc of $1|23$ in σ. If $s > t$ then B_{t+1} exists and $\min B_{t+1} \leq a < b < c$. Letting $e = \min B_{t+1}$ gives a copy e/bc of $1|23$ in σ. We remind the reader that the adjacent blocks of the copy of $1|23$ may appear in either order in σ.

Now we will show that $\Pi_n(1/23) = \Pi_n(1/\overset{\frown}{23})$. Again, it suffices to show that if $\sigma \vdash [n]$ contains a copy of $1/23$ then it contains a copy of $1/\overset{\frown}{23}$. Given a copy a/bc of $1/23$ in σ, if b and c are not adjacent in their block B then let d be the minimum of all of the elements of B which are larger than b. Thus a/bd is a copy of $1/\overset{\frown}{23}$ in σ. These two observations can be used to prove the remaining equality. □

Let R be a set of generalized patterns, and let S be the same set with adjacency restrictions dropped. That is if, for example, $1|\overset{\frown}{23} \in R$ then $1/23 \in S$, and S only contains patterns without adjacency restrictions. Lemma 1.5.1 says that unless $12|3$ or $\overset{\frown}{12}|3 \in R$, we have that $\Pi_n(R) = \Pi_n(S)$. However, since we have $\Pi_n(12|3) = \Pi_n(\overset{\frown}{12}\,|3)$, we only need to consider cases when $12|3 \in R$. The sets $\Pi_n(S)$ were enumerated in sections 2 and 3, so we need only enumerate the sets $\Pi_n(S \cup \{12|3\})$ where $S \subseteq \Pi_3 - \{12/3\}$.

Proposition 1.5.2 *Let* $S \subseteq \Pi_3 - \{12/3\}$ *then* $\Pi_n(S \cup \{12|3\}) = \Pi_n(S \cup \{12/3\})$ *unless* $S = \emptyset$ *or* $\{123\}$.

Proof: The cases where $\#S \geq 2$ follow automatically from those with $\#S = 1$ and Lemma 1.5.1. The three cases with $\#S = 1$ are very similar, so we will only prove the statement for $S = \{13/2\}$. Let $\sigma \in \Pi_n(13/2, 12|3)$, then σ must be layered. Thus any copy of $12/3$ in σ easily reduces to a copy of $12|3$ as in the proof of Lemma 1.5.1. □

The following lemma describes the elements of $\Pi_n(12|3)$.

Lemma 1.5.3 *We have* $\sigma \in \Pi_n(12|3)$ *if and only if whenever a block* B_t *of* σ *satisfies* $\#B_t \geq 2$*, then*

$$\#B_{t-1} = 1 \text{ and } \#B_{t+1} = 1.$$

Furthermore, if $B_{t+1} = \{a\}$ then $a < b$ for every $b \in B_t - \{\min B_t\}$.

Proof: First we show that $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(12|3)$ can be described as above. Let $\#B_t \geq 2$ and suppose that B_{t-1} contains at least 2 elements and let $a < b$ be the two smallest elements of B_{t-1}. Let $c < d$ be the two smallest elements of B_t. By the definition of canonical order, $a < c$. If $b < d$, then ab/d is a copy of 12|3. If $b > d$, then cd/b is a copy of 12|3 another contradiction. The proof that $\#B_{t+1} = 1$ is similar. The single element in B_{t+1} must be larger than c by definition. If it is larger than any other element of B_t we will again have an unwanted copy of 12|3.

Now, suppose that $\sigma \in \Pi_n$ has the structure described above. Then it is straight forward to show that σ cannot contain a copy of 12|3. □

First we will consider the case where $S = \emptyset$ in Proposition 1.5.2. Let $a_n = \#\Pi_n(12|3)$ and let

$$f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

be the corresponding exponential generating function.

Proposition 1.5.4 *For $n \geq 2$,*

$$a_n = a_{n-1} + 1 + \sum_{k=1}^{n-2} \binom{n-2}{k} a_{n-k-2}$$

with the initial conditions $a_0 = 1$ and $a_1 = 1$, and $f(x)$ satisfies the differential equation

$$y'' = y' + y(e^x - 1) + e^x.$$

Proof: That $\#\Pi_0(12|3) = \#\Pi_1(12|3) = 1$ is obvious. Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(12|3)$. Either $\#B_1 = 1$ or $\#B_1 \geq 2$. If $\#B_1 = 1$ then, by the definition of canonical order, $B_1 = \{1\}$. Clearly any 12|3 avoiding partition of the set $[2, n]$

will still avoid 12|3 if we prepend the block $\{1\}$. This gives the first term of the recursion.

Now suppose that $\#B_1 \geq 2$, then either $\sigma = 12\ldots n$ or not. The case where $\sigma = 12\ldots n$ is counted by the 1 in the recursion. If $\sigma \neq 12\ldots n$ then, by Lemma 1.5.3, we must have $B_2 = \{2\}$. If k of the elements from $[3, n]$ are in B_1, then the remaining $n-k-2$ elements must form a 12|3 avoiding partition. This establishes the recursion.

Using the recursion to produce the differential equation satisfied by $f(x)$ is routine and is left the reader. □

The substitution $y = ue^{x/2}$ simplifies the equation to

$$u'' = u(e^x - \frac{3}{4}) + e^{x/2}.$$

Using Maple, we obtain the solution

$$u = C_1 \cdot I_{\sqrt{-3}}(2e^{x/2}) + C_2 \cdot K_{\sqrt{-3}}(e^{x/2}) +$$

$$2I_{\sqrt{-3}}(2e^{x/2}) \int K_{\sqrt{-3}}(e^{x/2}e^{x/2})dx -$$

$$2K_{\sqrt{-3}}(e^{x/2}) \int I_{\sqrt{-3}}(2e^{x/2})e^{x/2}dx,$$

for certain constants C_1 and C_2, where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and second kinds respectively. There are known combinatorial interpretations for certain Bessel functions. See, for example, [3] and [26]. It is unlikely, however, that there is a combinatorial interpretation for the Bessel functions appearing in the exponential generating function $f(x) = ue^{-x/2}$, since $K_{\sqrt{-3}}(e^{x/2})$ is not well defined as a formal power series.

Now, we turn our focus to $\Pi_n(123, 12|3)$. Let $b_n = \#\Pi_n(123, 12|3)$ and

$$g(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$

be the corresponding exponential generating function.

The proof of the following proposition is very similar to the proof of Proposition 1.5.4 and is omitted.

Proposition 1.5.5 *For $n \geq 3$,*

$$b_n = b_{n-1} + (n-2)b_{n-3}$$

with the initial conditions $b_0 = 1$, $b_1 = 1$, and $b_2 = 2$. Also, $g(x)$ satisfies the differential equation

$$y''' = y'' + xy' + y. \square$$

Using Maple, we obtain the solution

$$y = D_1 e^{x/2} Ai(1/4 + x + D_2 e^{x/2} Bi(1/4 + x) +$$

$$D_3 e^{x/2} \left(Ai(1/4 + x) \int Bi(1/4 + x) e^{-x/2} dx - \right.$$

$$\left. \int Ai(1/4 + x) e^{-x/2} dx Bi(1/4 + x) \right),$$

for constants D_1, D_2, and D_3, where Ai and Bi are Airy functions.

It is not terribly surprising that Airy functions appear, since these functions are closely related to Bessel functions, and
$\Pi_n(123, 12|3)$ is a subset of the set $\Pi_n(12|3)$. There do not seem to be any existing combinatorial interpretations of Airy functions. There is also unlikely to be a combinatorial interpretation of this generating function due to the fact that $Ai(1/4 + x)$ is not well defined as a formal power series.

For completeness we will consider the cases where odd and even set partition avoid generalized set partitions. As before only the cases $O\Pi_n(R)$ and $E\Pi_n(R)$ where $R = \{12|3\}$ or $\{123, 12|3\}$ are new.

Let $oa_n = \#O\Pi_n(12|3)$ and $ea_n = \#E\Pi_n(12|3)$. Let $ob_n = \#O\Pi_n(123, 12|3)$ and $eb_n = \#E\Pi_n(123, 12|3)$. The following propositions easily follow from the recursions above. We let χ be the truth function, where χ of a statement is 1 if the statement is true and 0 if the statement is false.

Proposition 1.5.6 *For $n \geq 2$,*

$$oa_n = oa_{n-1} + \chi(n \text{ is even}) + \sum_{l=2,\ l\ even}^{n-2} \binom{n-2}{l} oa_{n-2-l} + \sum_{l=1,\ l\ odd}^{n-2} \binom{n-2}{l} ea_{n-2-l},$$

and

$$ea_n = ea_{n-1} + \chi(n \text{ is odd}) + \sum_{l=2,\ l\ even}^{n-2} \binom{n-2}{l} ea_{n-2-l} + \sum_{l=1,\ l\ odd}^{n-2} \binom{n-2}{l} oa_{n-2-l}. \square$$

Proposition 1.5.7 *For $n \geq 3$*

$$ob_n = ob_{n-1} + (n-2)eb_{n-3},$$

and

$$eb_n = eb_{n-1} + (n-2)ob_{n-3}. \square$$

1.6 Set Partition Statistics

Carlitz [16, 17] and Gould [27] were the first to give versions of the q-Stirling numbers of the second kind. In [33], Milne introduces an inversion and dual inversion statistic on set partitions, whose distributions over partitions of $[n]$ with k blocks produce these two q-Stirling numbers of the second kind. Later, Sagan [35] introduced the major index and dual major index of a set partition, whose distributions produced the same two q-Stirling numbers of the second kind. At around the same time, Wachs and White [40] investigated four natural statistics, which they called *lb*, *ls*, *rb*, and *rs*, again producing the same two q-Stirling numbers of the second kind. Other statistics of interest are the number of crossings, nestings and alignments of a partition, see for example [14], [20], or [29]. In this section we will show that all of these statistics can be described in the language of generalized partition patterns.

We will need some more notation. Consider the pattern $\pi = 1/23$. If we are looking for a copy of π in $\sigma = 137/26/45$, but we want the element representing 1 in the copy to be the minimum of its block then $1/45$ is a copy, but $3/45$ is not. We will represent this generalized pattern by $\overset{\frown}{1}/23$. And in general, we will denote such a generalized pattern by putting an arc over the first element of the block, in which we want the minimum to occur. In the same fashion, if we want the element representing 1 in a copy of $1/23$ to be the maximum in its block, then we denote the pattern by $1^{\frown}/23$. If we want the element representing 1 in a copy of $1/23$ to be both the minimum and the maximum of its block, then we denote the pattern by ${}^{\frown}1^{\frown}/23$.

In the sequel, if we say ρ is a pattern then ρ may or may not have adjacency restrictions. Let ρ be a pattern and $\sigma \in \Pi_n$. Then ρ will be treated as a function from Π_n to the nonnegative integers by letting $\rho(\sigma)$ be the number of copies of ρ in σ. If we have patterns $\rho_1, \rho_2, \ldots, \rho_\ell$ then

$$(\rho_1 + \rho_2 + \ldots + \rho_\ell)(\sigma) = \rho_1(\sigma) + \rho_2(\sigma) + \ldots + \rho_\ell(\sigma).$$

We begin with the inversion statistic. Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n$ and $b \in B_i$. We will say that (b, B_j) is an *inversion* if $b > \min B_j$ and $i < j$. Define the *inversion number* of σ, written $\mathrm{inv}(\sigma)$, to be the number of inversions in σ.

We may calculate $\text{inv}(\sigma)$ by summing, over all elements $b \in [n]$, the number of inversions of the form (b, B_j). This observation leads to the next Proposition.

Proposition 1.6.1 *For any* $\sigma \in \Pi_n$,

$$\text{inv}(\sigma) = (\overset{\frown}{1}3/\overset{\frown}{2})(\sigma).$$

Proof: We will show that there is a one-to-one correspondence between inversions and copies of $\overset{\frown}{1}3/\overset{\frown}{2}$. Let $\sigma = B_1/B_2/\ldots/B_k$. Let $b \in B_i$ and (b, B_j) be an inversion. If $a = \min B_i$ and $c = \min B_j$ then (b, B_j) corresponds to the copy ab/c of $\overset{\frown}{1}3/\overset{\frown}{2}$. Conversely, if ab/c is a copy of $\overset{\frown}{1}3/\overset{\frown}{2}$, then $a = \min B_i$ and $c = \min B_j$ where $i < j$ since $a < c$. Also, $b > c = \min B_j$. Thus, the copy ab/c yields the inversion (b, B_j). □

Let $\sigma = B_1/B_2/\ldots/B_k$ be a partition. We will say that (b, B_{i+1}) is a *descent* of σ if $b \in B_i$ and $b > \min B_{i+1}$. Let d_i be the number of descents of σ in block B_i. Then the major index of σ is

$$\text{maj}(\sigma) = \sum_{i=1}^{k-1} i d_i = d_1 + 2d_2 + \ldots + (k-1)d_{k-1}.$$

Notice that each descent (b, B_{i+1}) contributes i to the major index.

Proposition 1.6.2 *For any* $\sigma \in \Pi_n$,

$$\text{maj}(\sigma) = (\overset{\frown}{1}3|\overset{\frown}{2} + \overset{\frown}{1}/\overset{\frown}{2}4|\overset{\frown}{3})(\sigma).$$

Proof: Let $\sigma = B_1/B_2/\ldots/B_k$ and $b \in B_i$. Let $\rho_1 = \overset{\frown}{1}3|\overset{\frown}{2}$ and $\rho_2 = \overset{\frown}{1}/\overset{\frown}{2}4|\overset{\frown}{3}$. We will first show that (b, B_{i+1}) is a descent if and only if b represents the 3 in a copy of ρ_1, or, for $i \geq 2$, the 4 in a copy of ρ_2. Then we will show that each descent (b, B_{i+1}) contributes i to the right hand side.

Let (b, B_{i+1}) be a descent. If $a = \min B_i$ and $c = \min B_{i+1}$ then ab/c is a copy of ρ_1 where b represents the 3. If additionally $i \geq 2$ and we let $d = \min B_j$ where $j < i$ then $d/ab/c$ is a copy of ρ_2, in which b represents the 4. For the converse, let ab/c be a copy of ρ_1, then $c = \min B_{i+1}$ for some i, and (b, B_{i+1}) is a descent. Similarly, a copy $d/ab/c$ of ρ_2 with $c = \min B_{i+1}$ for some $i \geq 2$ produces the descent (b, B_{i+1}).

If (b, B_{i+1}) is a descent, then there is exactly one copy of ρ_2 with b representing 3, since the 1 in ρ_1 must be represented by $a = \min B_i$, and the 2 must be represented by $c = \min B_{i+1}$. Now, if b represents the 4 in a copy of ρ_2 then the 2 must be represented by $a = \min B_i$, and the 3 must be represented by $c = \min B_{i+1}$. But now the 1 may be represented by the minimum of any block appearing before B_i. So the total contribution of the two patterns is $1+(i-1) = i$. □

Let $\sigma = B_1/B_2/\ldots/B_k$ and $b \in B_i$. The dual of a descent is an *ascent*, which is a pair (b, B_{i-1}) with $b > \min B_{i-1}$. Note, it is true that each $b \in B_i$ forms an ascent because of the canonical ordering. So, we define the dual major index to be

$$\widehat{\mathrm{maj}}(\sigma) = \sum_{i=2}^{k} (i-1)(\#B_i).$$

The *dual inversion number* of σ, written $\widehat{\mathrm{inv}}(\sigma)$, is the number of pairs (b, B_j) such that $b \in B_i$, $b > \min B_j$, and $i > j$. We will call these pairs *dual inversions.* Clearly, $\widehat{\mathrm{inv}}(\sigma) = \widehat{\mathrm{maj}}(\sigma)$ for any $\sigma \in \Pi_n$, since every ascent causes $i-1$ dual inversions.

Proposition 1.6.3 *For any $\sigma \in \Pi_n$,*

$$\widehat{\mathrm{inv}}(\sigma) = \widehat{\mathrm{maj}}(\sigma) = (\overset{\frown}{1}/\overset{\frown}{2} + \overset{\frown}{1}/\overset{\frown}{2}3)(\sigma).$$

Proof: Let $\sigma = B_1/B_2/\ldots/B_k$. The proof that $\widehat{\mathrm{inv}}(\sigma) =$

$(\overset{\frown}{1}/\overset{\frown}{2}+\overset{\frown}{1}/\overset{\frown}{2}3)(\sigma)$ is similar to the proof of Proposition 1.6.1. The only difference here is that the minimum of a block can represent the b in a dual inversion (b, B_j). This is taken care of by the first pattern. □

Wachs and White [40] define four natural statistics on partitions by encoding the partitions as restricted growth functions. Their statistics are lb, ls, rb, and rs, which stand for left bigger, left smaller, right bigger and right smaller. For consistency, we will define these statistics without introducing restricted growth functions, and hence the names of the statistics may seem a little unusual.

Let $\sigma = B_1/B_2/\ldots/B_k$. If $b \in B_i$, then we will say that (b, B_j) is:

- a *left bigger pair* of σ if $i < j$, and $b > \min B_j$,

- a *left smaller pair* of σ if $i > j$ and $b > \min B_j$,

- a *right bigger pair* of σ if $i < j$ and $b < \max B_j$,

- a *right smaller pair* of σ if $i > j$ and $b < \max B_j$.

Let $lb(\sigma)$, $ls(\sigma)$, $rb(\sigma)$, and $rs(\sigma)$ be, respectively, the number of left bigger pairs, the number of left smaller pairs, the number of right bigger pairs, and the number of right smaller pairs in σ.

Notice that (b, B_j) is a left bigger pair if and only if it is an inversion of σ, and (b, B_j) is a left smaller pair if and only if (b, B_j) is a dual inversion of σ. Thus we have from Propositions 1.6.2 and 1.6.3 that

$$lb(\sigma) = (\overset{\frown}{1}3/\overset{\frown}{2})(\sigma),$$

$$ls(\sigma) = (\overset{\frown}{1}/\overset{\frown}{2}+\overset{\frown}{1}/\overset{\frown}{2}3)(\sigma).$$

We will now consider the other two statistics.

Proposition 1.6.4 *For any* $\sigma \in \Pi_n$,

$$rb(\sigma) = (\overset{\frown}{1}/\overset{\frown}{2}\overset{\frown}{3} + \overset{\frown}{1}3/\overset{\frown}{2}\overset{\frown}{4} + \overset{\frown}{1}/\overset{\frown\frown}{2} + \overset{\frown}{1}2/\overset{\frown\frown}{3})(\sigma).$$

Proof: Let $\sigma = B_1/B_2/\ldots/B_k$. The pattern $\overset{\frown}{1}/\overset{\frown}{2}\overset{\frown}{3}$ counts right bigger pairs (b, B_j) where $b = \min B_i$ and $\#B_j \geq 2$. The pattern $\overset{\frown}{1}3/\overset{\frown}{2}\overset{\frown}{4}$ counts those pairs where $b \neq \min B_i$ and $\#B_j \geq 2$. The other two patterns correspond to the same two cases when $\#B_j = 1$. □

The proof of the following proposition is similar to the proof of Proposition 1.6.4 and is omitted.

Proposition 1.6.5 *For any* $\sigma \in \Pi_n$,

$$rs(\sigma) = (\overset{\frown}{1}\overset{\frown}{3}/\overset{\frown}{2} + \overset{\frown}{1}\overset{\frown}{4}/\overset{\frown}{2}3)(\sigma). \square$$

There has long been interest in non-crossing partitions. Recall that the non-crossing partitions are those in the set $\Pi_n(13/24)$ for some n. Non-nesting partitions may be described as those in the set $\Pi_n(\overset{\frown}{14}/\overset{\frown}{23})$. Note that this definition of a non-nesting partition is not the only one. Klazar [30] defines non-nesting partitions as those in the set $\Pi_n(14/23)$.

Recently, however, there has been increasing interest in counting the number of crossings or nestings of a partition. In [20], Chen et al. show that the crossing number and nesting number are symmetrically distributed over Π_n by giving a bijection between partitions and vacillating tableaux. In [29], Kasraoui and Zeng give an involution of Π_n, which exchanges the crossing number and the nesting number while keeping another statistic, the number of alignments of two edges, fixed.

We will describe each of these statistics and show that they too may be translated into the language of patterns.

Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n$. We may rewrite σ as a set $P \subseteq [n] \times [n]$ in the following way. If $a, b \in B_i$ and there is no $c \in B_i$ such that $a < c < b$ then $(a, b) \in P$. If $B_i = \{d\}$ then $(d, d) \in P$. It's easy to see that P uniquely represents σ. We will call P the *standard representation* of σ.

Let A be a family $\{(i_1, j_1), (i_2, j_2)\} \subseteq P$. We will say that A is:

- a *crossing* if $i_1 < i_2 < j_1 < j_2$,

- a *nesting* if $i_1 < i_2 < j_2 < j_1$,

- an *alignment* if $i_1 < j_1 \leq i_2 < j_2$.

For example, the following diagram represents $\sigma = 137/26/45$, where an edge connects elements if they are adjacent in a block.

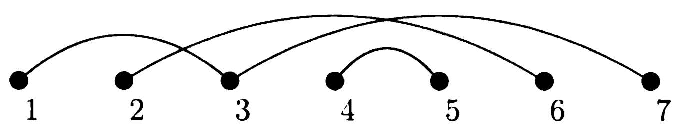


Figure 1: 137/26/45

Notice that the pair $\{(1,3),(2,6)\}$ forms a crossing, the pair $\{(2,6),(4,5)\}$ forms a nesting and the pairs $\{(1,3),(4,5)\}$ and $\{(1,3),(3,7)\}$ each form an alignment of two edges.

Let $\mathrm{cr}(\sigma)$ be the number of crossings in σ, $\mathrm{ne}(\sigma)$ the number of nestings, and $\mathrm{al}(\sigma)$ the number of alignments.

The following proposition is an easy consequence of the previous definitions.

Proposition 1.6.6 *For any $\sigma \in \Pi_n$,*

$$\begin{aligned}
\mathrm{cr}(\sigma) &= (\overset{\frown}{13}/\overset{\frown}{24})(\sigma),\\
\mathrm{ne}(\sigma) &= (\overset{\frown}{14}/\overset{\frown}{23})(\sigma),\\
\mathrm{al}(\sigma) &= (\overset{\frown}{12}/\overset{\frown}{34} + \overset{\frown}{12}\overset{\frown}{34} + \overset{\frown}{1\,2}\overset{\frown}{\,3})(\sigma). \square
\end{aligned}$$

Let $\sigma \in \Pi_n$ and P be the standard representation of σ. Consider the family $A = \{(i_1, j_1), (i_2, j_2), \ldots (i_k, j_k)\} \subseteq P$. Then A is a *k-crossing* if $i_1 < i_2 < \ldots < i_k < j_1 < j_2 < \ldots < j_k$. We say A is a *k-nesting* if $i_1 < i_2 < \ldots < i_k < j_k < j_{k-1} < \ldots < j_1$. Let $\mathrm{cr}_k(\sigma)$ be the number of k-crossings of σ and $\mathrm{ne}_k(\sigma)$ be the number of k-nestings of σ. Notice that $\mathrm{cr} = \mathrm{cr}_2$ and $\mathrm{ne} = \mathrm{ne}_2$. The following proposition describes these two statistics as patterns.

Proposition 1.6.7 *For any $\sigma \in \Pi_n$,*

$$\begin{aligned}
cr_k(\sigma) &= (\overset{\frown}{1(k+1)} \;/\; \overset{\frown}{2(k+2)} \;/ \ldots /\; \overset{\frown}{k(2k)})(\sigma),\\
ne_k(\sigma) &= (\overset{\frown}{1(2k)} \;/\; \overset{\frown}{2(2k-1)} \;/ \ldots /\; \overset{\frown}{k(k+1)})(\sigma). \square
\end{aligned}$$

Chapter 2
Set Partition Statistics and q-Fibonacci Numbers

2.1 Equidistribution of ls and rb

In this chapter, we are interested in the distributions of set partition statistics on $\Pi_n(13/2)$ and $\Pi_n(13/2, 123)$ and the resulting q-analogues of 2^n and F_n. Of the known set partition statistics, only the left smaller, ls, and right bigger, rb, statistics of Wachs and White [40] seem to give interesting distributions on these sets.

For a statistic ρ on a finite set S, the *distribution* of ρ over S is

$$\sum_{s \in S} q^{\rho(s)}.$$

Hence, the coefficient of q^k is the number of $s \in S$ with $\rho(s) = k$. The following theorem shows that the statistics rb and ls are equidistributed over the sets $\Pi_n(13/2)$ and $\Pi_n(13/2, 123)$.

Theorem 2.1.1 *For any n,*

$$\sum_{\pi \in \Pi_n(13/2)} q^{ls(\pi)} = \sum_{\pi \in \Pi_n(13/2)} q^{rb(\pi)},$$

and

$$\sum_{\pi \in \Pi_n(13/2,123)} q^{ls(\pi)} = \sum_{\pi \in \Pi_n(13/2,123)} q^{rb(\pi)}.$$

Proof: Given a set partition $\pi = B_1/B_2/\ldots/B_k \in \Pi_n(13/2)$, let the complement of π be the partition $\pi^c = B_k^c/\ldots/B_2^c/B_1^c$, where $B_i^c = \{n-b+1 : b \in B_i\}$.

Notice that taking the complement reverses the order of the blocks since π is layered. Clearly complementation is an involution, and so bijective. So it suffices to show that it exchanges ls and rb. This easily follows because the block order is reversed and minima are exchanged with maxima. Also, complementation does not alter the block sizes and so restricts to a map on $\Pi_n(13/2, 123)$. □

2.2 Distribution over $\Pi_n(13/2)$

Define

$$A_n(q) = \sum_{\pi \in \Pi_n(13/2)} q^{rb(\pi)}.$$

It will be useful to think of the rb statistic in the following way. Let $\pi = B_1/B_2/\ldots/B_k$ be a partition and $m_j = \max B_j$. For each element $b \in B_i$ with $i < j$ and $b < m_j$, we have that (b, B_j) is a right bigger pair. The number of right bigger pairs of the form (b, B_j) will be the *contribution* of B_j to rb. When restricted to layered partitions the contribution of B_j is

$$\sum_{i<j} \#B_i = \min B_j - 1.$$

The generating function $A_n(q)$ is closely related to integer partitions. An *integer partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of the integer d is a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{k} \lambda_i = d$; the λ_i are called *parts*. We let $|\lambda| = \sum_{i=1}^{k} \lambda_i$. Denote by D_{n-1} the set of integer partitions with distinct parts of size at most $n-1$. It is well known that

$$\sum_{\lambda \in D_{n-1}} q^{|\lambda|} = \prod_{i=1}^{n-1} (1 + q^i).$$

For the rest of this chapter we will refer to a set partition as just a partition and an integer partition by its full name.

For the following proof it will be more convenient for us to list the parts of an integer partition in weakly increasing order. Let $\phi : \Pi_n(13/2) \rightarrow D_{n-1}$ be the map defined by

$$\phi(B_1/B_2/\ldots/B_k) = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}),$$

where $\lambda_j = \sum_{i=1}^{j} \#B_i$.

Theorem 2.2.1 *The map ϕ is a bijection, and for $\pi \in \Pi_n(13/2)$, $rb(\pi) = |\phi(\pi)|$. Hence,*

$$A_n(q) = \prod_{i=1}^{n-1} (1 + q^i).$$

Proof: Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ consider, for $1 \leq j \leq k$, the differences $d_j = \lambda_j - \lambda_{j-1}$, where $\lambda_0 = 0$ and $\lambda_k = n$. If $\lambda \in D_{n-1}$ then we have $d_j > 0$ for all j. And if $\phi(\pi) = \lambda$ then the d_j give the block sizes of π. But since π is layered, it is uniquley determined by its block sizes. So ϕ is bijective. Since $\lambda_j = \sum_{i<j} \#B_i$ is the contribution of B_{j+1} to rb (and B_1 makes no contribution) we have $rb(\pi) = |\phi(\pi)|$ as desired. □

The proof of Theorem 2.2.1 will be useful for proving q-analogues of Fibonacci identities.

2.3 q-Fibonacci Numbers Past and Present

We turn our focus to the distribution of rb over $\Pi_n(13/2, 123)$. As remarked in the introduction

$$\#\Pi_n(13/2, 123) = F_n.$$

The distribution of rb over $\Pi_n(13/2, 123)$ gives a nice q-analogue of the Fibonacci numbers. Let

$$F_n(q) = \sum_{\pi \in \Pi_n(13/2,123)} q^{rb(\pi)}.$$

Proposition 2.3.1 *The generating function $F_n(q)$ satisfies the boundry condition $F_0(q) = 1$, $F_1(q) = 1$, and the recursion*

$$F_n(q) = q^{n-1} F_{n-1}(q) + q^{n-2} F_{n-2}(q).$$

Proof: Let $\pi \in \Pi_n(13/2, 123)$. Since π is a matching it must end in a block of size one or of size two. If π ends in a singleton then the singleton is $\{n\}$, which contributes $n-1$ to rb, and the remaining elements form a partition in $\Pi_{n-1}(13/2, 123)$. Similarly the doubleton case is counted by the second term of the recursion. □

We now introduce the q-Fibonacci numbers of Carlitz and Cigler and explore their relationship to the q-Fibonacci numbers as defined above. Let BS_n^1 be the set of binary sequences $\beta = b_1 \ldots b_n$ of length n without consecutive ones. It is well known that $\#BS_n^1 = F_{n+1}$. In [18, 19], Carlitz defined and studied a statistic on BS_n^1 as follows. Let $\rho : BS_n^1 \to \mathbb{N}$ be given by

$$\rho(\beta) = \rho(b_1 \ldots b_n) = b_1 + 2b_2 + \ldots + nb_n,$$

and define

$$F_n^K(q) = \sum_{\beta \in BS_{n-1}^1} q^{\rho(\beta)}.$$

Carlitz showed that $F_n^K(q)$ satisfies $F_0^K(q) = 1$, $F_1^K(q) = 1$, and

$$F_n^K(q) = F_{n-1}^K(q) + q^{n-1} F_{n-2}^K(q).$$

Cigler [22] defined his q-Fibonacci polynomials using Morse sequences. A *Morse sequence* of length n is a sequence of dots and dashes, where each dot has length 1 and each dash has length 2. For example, $\nu = \bullet\bullet - - \bullet -$ is a Morse sequence of length 9. Let MS_n be the set of Morse sequences of length n. Each Morse sequence corresponds to a layered partition where a dot is replaced by a singleton block and a dash by a doubleton. So, $\#MS_n = F_n$.

Define the weight of a dot to be x and the weight of a dash to be yq^{a+1} where a is the length of the portion of the sequence appearing before the dash. Also, define a weight $w : MS_n \to \mathbb{Z}[x, y, q]$ by letting $w(\nu)$ be the product of the weights of its dots and dashes. For example, the sequence above has weight $(x)(x)(yq^3)(yq^5)x(yq^8) = x^3y^3q^{16}$. Let

$$F_n^C(x, y, q) = \sum_{\nu \in MS_n} w(\nu).$$

Cigler shows that $F_n^C(x, y, q)$ satisfies $F_0^C(x, y, q) = 1$,
$F_1^C(x, y, q) = x$, and

$$F_n^C(x, y, q) = xF_{n-1}^C(x, y, q) + yq^{n-1}F_{n-2}(x, y, q).$$

Note that $F_n^C(1, 1, q) = F_n^K(q)$. In fact, Cigler [21, 22, 23, 24] studied more general q-Fibonacci numbers satisfying the above recursion with yq^{n-1} replaced by $t(yq^{n-1})$, where t is an arbitraty nonzero function. One could apply our method to such q-analogues, but we chose t to be the identity for simplicity.

Proposition 2.3.2 *For all $n \geq 0$,*

$$F_n(q) = q^{\binom{n}{2}} F_n^K(1/q).$$

Proof: It suffices to construct a bijection $\Pi_n(13/2, 123) \leftrightarrow BS_{n-1}$ such that if $\pi \leftrightarrow \beta$ then $rb(\pi) = \binom{n}{2} - \rho(\beta)$.

Let $\pi \in \Pi_n(13/2, 123)$ be mapped to the binary sequence $\beta = b_1 \ldots b_n$ where $b_i = 0$ if i and $i+1$ are in separate blocks and $b_i = 1$ otherwise. For example, $1/2/34/56 \leftrightarrow 00101$. We first show that this map is well defined. Suppose $\pi \mapsto b_1 \ldots b_n$, where $b_i = 1$ and $b_{i+1} = 1$ for some i, then i, $i+1$, and $i+2$ must be in a block together. This contradicts the fact that the blocks may only be of size at most 2. The fact that this map is a bijection is straightforward.

Now, suppose that $\pi \leftrightarrow \beta$ and $\beta = b_1 \ldots b_n$. If every element of π is a maximum of its block, which is the same as saying that every block is a singleton and that $b_i = 0$ for all i, then $rb(\pi) = \sum_{i=1}^{n-1} i = \binom{n}{2}$. If $b_i = 1$ for some i then i and $i+1$ are in the same block. Thus, i is no longer a block by itself and so no longer contributes rb. When i and $i+1$ were in different blocks, $\{i\}$ contributed $i-1$ to rb, so we have lost $i-1$. Also, since i and $i+1$ are now in a block together, the block containing $i+1$ cannot be put together with i to form a right bigger pair. This reduces the contribution of $\{i, i+1\}$ by 1. These are the only changes made when $b_i = 1$. Thus, each time $b_i = 1$ we reduce $rb(1/2/\ldots/n)$ by i and hence,

$$rb(\pi) = \binom{n}{2} - \sum_{i\,:\,b_i = 1} i = \binom{n}{2} - \rho(\beta). \square$$

In order to describe the relationship between $F_n(q)$ and $F_n^C(x, s, q)$ we will define a weight, ω on the partitions in $\Pi_n(13/2, 123)$. Let

$$\omega : \Pi_n(13/2, 123) \to \mathbb{Z}[x, s, q],$$

with $\omega(\pi) = \omega(B_1/B_2/\ldots/B_k) = \prod_{i=1}^{k} \omega(B_i)$, where

$$\omega(B_i) = \begin{cases} xq^{\min B_i - 1} & \text{if } \#B_i = 1, \\ yq^{\min B_i - 1} & \text{if } \#B_i = 2. \end{cases}$$

Now, let

$$F_n(x, y, q) = \sum_{\pi \in \Pi_n(13/2, 123)} \omega(\pi).$$

Let $s(\pi)$ be the number of singletons of π, $d(\pi)$ be the number of doubletons of π. It is easy to see directly from the definitions that

$$F_n(x, y, q) = \sum_{\pi \in \Pi_n(13/2, 123)} x^{s(\pi)} y^{d(\pi)} q^{rb(\pi)}.$$

The proof of the next proposition is omitted since it parallels that of Proposition 2.3.2 using the bijection $\Pi_n(13/2, 123) \leftrightarrow MS_n$ mentioned above.

Proposition 2.3.3 *We have that*

$$F_n(x, y, q) = q^{\binom{n}{2}} F_n^C(x, y, 1/q). \square$$

2.4 q-Fibonacci Identities

We now provide bijective proofs of q-analogues of Fibonacci identities. Many of the proofs in this paper are simply q-analogues of the tiling scheme proofs of Fibonacci identities given in [4, 15]. It is impresive that merely using the rb statistic on $\Pi_n(13/2, 123)$ gives so many identities with relatively little effort. We will state our identities for $F_n(x, y, q)$, but one can translate them in terms of $F_n^C(x, y, q)$ or $F_n^K(q)$ using Propositions 2.3.2 and 2.3.3.

Theorem 2.4.1 *For $n \geq 0$*

$$F_{n+2}(x,y,q) = x^{n+2}q^{\binom{n+2}{2}} + \sum_{j=0}^{n} x^j y q^{\binom{j+1}{2}} F_{n-j}(xq^{j+2}, yq^{j+2}, q).$$

Proof: There is exactly one partition in $\Pi_{n+2}(13/2, 123)$ with all singleton blocks and the weight of this partition is $x^{n+2}q^{\binom{n+2}{2}}$. The remaining partitions have at least one doubleton. Consider all partitions where the first doubleton is $\{j+1, j+2\}$. There are exactly j singletons preceding this doubleton contributing $x^j q^{\binom{j}{2}}$ to the weight of each partition. The doubleton contributes weight yq^j. The remaining blocks of these partitions form layered matchings of $[j+3, n+2]$. We may think of these as being layered matchings of $[n-j]$ where the contribution of each block to the rb statistic is increased by $j+2$. Hence, these contribute weight $F_{n-j}(xq^{j+2}x\, yq^{j+2}, q)$. Thus, the contributed weight of the partitions, whose first doubleton is $\{j+1, j+2\}$, is

$$x^j y q^{\binom{j+1}{2}} F_{n-j}(xq^{j+2}, yq^{j+2}, q).$$

Summing from $j = 0$ to n the weighted count of partitions whose first doubleton is $\{j+1, j+2\}$ and adding the weight of the partition with all singletons proves the identity.□

Theorem 2.4.2 *For $n \geq 0$,*

$$F_{2n+1}(x,y,q) = \sum_{j=0}^{n} xy^j q^{j(j+1)} F_{2n-2j}(xq^{2j+1}, yq^{2j+1}, q),$$

and

$$F_{2n} = \sum_{j=0}^{n-1} xy^j q^{j(j-1)} F_{2n-2j-1}(xq^{2j+1}, yq^{2j+1}, q).$$

Proof: If $\pi \in \Pi_{2n+1}(13/2, 123)$, then π must have at least one singleton. Consider all partitions with first singleton $\{2j+1\}$. This block must be preceded by j doubletons, which contribute $y^j q^{j(j-1)}$ to the weight. The singleton $\{2j+1\}$ contributes xq^{2j} to the weight. The remaining $2n-2j$ elements form a layered matching of $[2j+2, 2n+1]$. As in the previous proof, we may think of these as being elements of $\Pi_{2n-2j}(13/2, 123)$ where the contribution of each block to rb is increased by $2j+1$. This portion of our partition will contribute $F_{2n-2j}(xq^{2j+1}, yq^{2j+1}, q)$ to the weight.

This proves the first identity. The proof of the second identity is similar, and hence omitted. □

Theorem 2.4.3 *For all $n \geq 0$ and $m \geq 0$,*

$$\begin{aligned} F_{m+n}(x, y, q) &= F_m(x, y, q) F_n(xq^m, yq^m, q) \\ &+ yq^{m-1} F_{m-1}(x, y, q) F_{n-1}(xq^{m+1}, yq^{m+1}, q). \end{aligned}$$

Proof: Every $\pi \in \Pi_{m+n}(13/2, 123)$ has $\{m, m+1\}$ as a block or does not. If π has $\{m, m+1\}$ as a block then the blocks prior to this block form a partition in $\Pi_{m-1}(13/2, 123)$. This contributes $F_{m-1}(x, y, q)$ to the weight. The doubleton $\{m, m+1\}$ has weight yq^{m-1}. The remaining blocks form a partition in $\Pi_{[m+2, m+n]}(13/2, 123)$. The contribution of each block in this partition to the rb statistic is increased by $m+1$, so this portion of the partition contributes $F_{n-1}(xq^{m+1}, yq^{m+1}, q)$ to the weight. Thus, the sum of $\omega(\pi)$ over all π in $\Pi_{m+n}(13/2, 123)$ with doubleton $\{m, m+1\}$ is

$$yq^{m-1} F_{m-1}(x, y, q) F_{n-1}(xq^{m+1}, yq^{m+1}, q).$$

If $\pi \in \Pi_{m+n}(13/2, 123)$ does not have $\{m, m+1\}$ as a block, then we can split π into a partition of $[m]$ and a partition of $[m+1, m+n]$. By a similar argument, the sum of $\omega(\pi)$ over all π in $\Pi_{m+n}(13/2, 123)$ without the block $\{m, m+1\}$ is

$$F_m(x, y, q)F_n(xq^m, yq^m, q). \square$$

Theorem 2.4.4 *For $n \geq 1$,*

$$\begin{aligned} xq^n F_{2n+1}(x, y, q) &+ y^2 q^{2n-1} \\ &= F_{n+1}(x, y, q)F_{n+1}(xq^n, yq^n, q) \\ &- F_{n-1}(x, y, q)F_{n-1}(xq^{n+2}, yq^{n+2}, q). \end{aligned}$$

Proof: Notice that each partition $\pi \in \Pi_{2n+1}(13/2, 123)$ has one of three configurations: Either the element $n+1$ is in its own block, is in a block with n, or is in a block with $n+2$.

We put together the right hand side of the identity by considering the set $\Pi_{n+1}(13/2, 123) \times \Pi_{n+1}(13/2, 123)$. A weighted count of the elements (π_1, π_2) of this set by increasing by n the contribution to rb of each block in π_2 is $F_{n+1}(x, y, q)F_{n+1}(xq^n, yq^n, q)$.

Let $\Pi'_{n+1}(13/2, 123) \times \Pi'_{n+1}(13/2, 123)$ be the set of those ordered pairs (π_1, π_2) such that π_1 ends in a doubleton and π_2 begins with a doubleton. There is a bijection between $\Pi'_{n+1}(13/2, 123) \times \Pi'_{n+1}(13/2, 123)$ and $\Pi_{n-1}(13/2, 123) \times \Pi_{n-1}(13/2, 123)$ by removing the last block of π_1, removing the first block of π_2, and reducing each element of π_2 by 2. The weighted count of such pairs (π_1, π_2) is $y^2 q^{2n-1} F_{n-1}(x, y, q)F_{n-1}(xq^{n+2}, yq^{n+2}, q)$.

Thus,

$F_{n+1}(x,y,q)F_{n+1}(xq^n,yq^n,q)$ $-$
$y^2q^{2n-1}F_{n-1}(x,y,q)F_{n-1}(xq^{n+2},yq^{n+2},q)$
is the weighted count of $(\pi_1,\pi_2) \in \Pi_{n+1}(13/2,123) \times \Pi_{n+1}(13/2,123)$, where π_1 ends with a singleton or π_2 begins with a singleton or both. In any of these three cases we create a partition in $\Pi_{2n+1}(13/2,123)$ by removing the singleton at the end of π_1 or the singleton at the beginning of π_2, concatenating the partitions and increasing the elements of π_2 by n. The statement about the possible configurations of an element of $\Pi_{2n+1}(13/2,123)$ shows that this construction gives a bijection

$$\begin{aligned}\Pi_{2n+1}(13/2,123) &\leftrightarrow \Pi_{n+1}(13/2,123) \times \Pi_{n+1}(13/2,123)\\ &- \Pi'_{n+1}(13/2,123) \times \Pi'_{n+1}(13/2,123).\end{aligned}$$

In each case we lose a singleton block, whose weight is xq^n. Thus, we must multiply $F_{2n+1}(x,y,q)$ by xq^n to obtain an appropriate weighted count of $\Pi_{2n+1}(13/2,123)$.
□

Theorem 2.4.5 *For* $n \geq 0$, $F_n(x,y,q)F_{n+1}(x,y,q) =$
$\sum_{j=0}^{n} xy^jq^{\left\lfloor \frac{j^2}{2} \right\rfloor} F_{n-j}(xq^j,yq^j,q)F_{n-j}(xq^{j+1},yq^{j+1},q)$.

Proof: Consider an element $(\pi_1,\pi_2) \in \Pi_n(13/2,123) \times \Pi_{n+1}(13/2,123)$ with $\pi_1 = A_1/A_2/\ldots/A_\ell$, and $\pi_2 = B_1/B_2/\ldots/B_m$. Search through the blocks in the order $B_1, A_1, B_2, A_2, \ldots$ and find the first singleton block.

If the first singleton is some $A_i = \{j\}$ then $B_1,\ldots,B_i$ are all doubletons, and j is odd. There are $(j-1)/2$ doubletons at the beginning of π_1 and $(j+1)/2$ doubletons at the beginning of π_2 contributing $y^jq^{(j-1)^2/2}$ to the weight. The singleton block A_i has weight xq^{j-1}. The remaining $\ell - i$ blocks of π_1 form a layered matching of $[j+1,n]$, which has weighted count $F_{n-j}(xq^j,yq^j,q)$. The remaining $m - i$ blocks of π_2 are a layered matching of $[j+2, n+1]$, which has weighted count $F_{n-j}(xq^{j+1}, yq^{j+1}, q)$. So the weight contributed by all pairs (π_1, π_2) with $A_i = \{j\}$ as the first singleton is

$$q^{\left\lfloor \frac{j^2}{2} \right\rfloor} xy^j F_{n-j}(xq^j, yq^j, q) F_{n-j}(xq^{j+1}, yq^{j+1}, q).$$

If the first singleton is some $B_i = \{j+1\}$ then j is even, and by similar arguments, the weight contributed by all such pairs (π_1, π_2) with $B_i = \{j+1\}$ as the first singleton is

$$xy^j q^{\left\lfloor \frac{j^2}{2} \right\rfloor} F_{n-j}(xq^j, yq^j, q) F_{n-j}(xq^{j+1}, yq^{j+1}, q).$$

Summing over all possible cases gives the identity above. □

The identity $F_n = \sum_{k \geq 0} \binom{n-k}{k}$ relates the Fibonacci numbers F_n to the binomial coefficients $\binom{n}{k}$. For convenience we assume that $\binom{n}{k} = 0$ if $k > n$. To state a q-analogue of this identity, we need the q-binomial coefficients. The *q-binomail coefficients* are

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}.$$

Define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $k > n$ for convenience. Carlitz [18] derived the following identity using algebraic and operator methods. We will provide an alternate proof using the relationship between layered partitions and integer partitions.

Theorem 2.4.6 (Carlitz) *For* $n \geq 0$,

$$F_n(x, y, q) = \sum_{k \geq 0} x^{n-2k} y^k q^{\binom{n}{2} - k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Proof: Let $\Pi_n^k(13/2, 123)$ be the set of $\pi \in \Pi_n(13/2, 123)$ with exactly k doubletons. Thus we have

$$\Pi_n(13/2, 123) = \biguplus_{2k \leq n} \Pi_n^k(13/2, 123),$$

where $\biguplus$ is the disjoint union. This implies that

$$\sum_{\pi \in \Pi_n(13/2, 123)} \omega(\pi) = \sum_{k \geq 0} x^{n-2k} y^k \left(\sum_{\pi \in \Pi_n^k(13/2, 123)} q^{rb(\pi)} \right).$$

It suffices to show that

$$\sum_{\pi \in \Pi_n^k(13/2, 123)} q^{rb(\pi)} = q^{\binom{n}{2} - k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Let $\pi \in \Pi_n^k(13/2, 123)$ with doubleton blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$. As in the proof of Proposition 2.3.2 the maximum $rb(\pi)$ is $\binom{n}{2}$, and each doubleton B_{i_j} of π reduces rb by $\min B_{i_j}$. This gives us that

$$\sum_{\pi \in \Pi_n^k(13/2, 123)} q^{rb(\pi)} = \sum q^{\binom{n}{2} - \sum_{j=1}^{k} \min B_{i_j}},$$

where the first sum on the right hand side is over all possible choices of k doubletons $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$. The right hand side is equal to $q^{\binom{n}{2} - nk} \sum q^{\sum_{j=1}^{k} n - \min B_{i_j}}$.

Consider the map which sends π to the integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_j = n - \min B_{i_j}$. This is clearly a bijection $\Pi_n^k(13/2, 123) \to E_{n-1}^k$, where E_{n-1}^k is the set of integer partitions with exactly k parts of size $\leq n-1$ and consecutive part sizes differ by at least 2. It is well known that

$$\sum_{\lambda \in E_{n-1}^k} q^{|\lambda|} = q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

See Andrews' book [1] for a proof.

Thus, we have $\sum_{\pi \in \Pi_n^k(13/2,123)} q^{rb(\pi)} = q^{\binom{n}{2} - k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}$. □

Another identity involving binomial coefficients is $F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{n-k}$.

Theorem 2.4.7 *For $n \geq 0$,*

$$F_{2n}(x, y, q) = \sum_{k=0}^{n} q^{\binom{n+k}{2} - nk} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k F_{n-k}(q^{n+k}x, q^{n+k}y, q).$$

Proof: Let Δ_k be the set of partitions $\pi \in \Pi_{2n}(13/2, 123)$, which begin with a partition of $[n+k]$ having exactly k doubletons.

We claim that $\Pi_n(13/2, 123)$ is the disjoint union of the Δ_k. First, we show that each Δ_k is nonempty for $0 \leq k \leq n$. The proof is by induction on the number of doubletons in π. The case where π has 0 doubletons is obvious. If π has ℓ doubletons with the last doubleton $B = \{a, a+1\}$ then we split this doubleton forming a new partition π' with a and $a+1$ in singleton blocks. Now, by induction there must be some t such that π' has exactly t of its doubletons in the first $n+t$ elements. If $a > n+t$ then π also has exactly t doubletons in the first $n+t$ elements. If $a \leq t$ then π has exactly $t+1$ doubletons in the first $n+t+1$ elements.

It suffices to show that each $\pi \in \Pi_n(13/2, 123)$ is in exactly one Δ_k. Suppose that π has exactly j doubletons in the first $n+j$ elements. And suppose that for some $i > j$, π also has exactly i doubletons in the first $n + i$ elements. This means that there must be exactly $i - j$ doubletons consisting of elements from $[n+j+1, n+i]$. Since there are only $i-j$ elements in this set we have a contradiction. This also proves the case where $i < j$. Thus, each partition $\pi \in \Pi_{2n}(13/2, 123)$ is in exactly one Δ_k.

Using Theorem 2.4.6 it is easy to see that

$$\sum_{\pi \in \Delta_k} \omega(\pi) = q^{\binom{n+k}{2} - nk} x^{n-k} y^k \begin{bmatrix} n \\ k \end{bmatrix} F_{n-k}(xq^{n+k}, yq^{n+k}, q). \square$$

We conclude this section by finding a q-analogue of

$$F_n + F_{n-1} + \sum_{k=0}^{n-2} F_k 2^{n-2-k} = 2^n.$$

We provide a proof for a q-analogue involving $F_n^K(q)$ as this is the nicest proof.

Theorem 2.4.8 *For $n \geq 0$,*

$$F_n^K(q) + q^n F_{n-1}^K(q) + \sum_{k=0}^{n-2} F_k^K(q) q^{2k+3} \prod_{j=k+3}^{n} (1 + q^j) = \prod_{k=1}^{n} (1 + q^k).$$

Proof: Let $\beta = b_1 \dots b_n \in BS_n$, where BS_n is the set of binary sequences of length n. We consider the same statistic that Carlitz did. Namely, $\rho(\beta) = \sum_{k=1}^{n} kb_k$. Clearly,

$$\sum_{\beta \in BS_n} q^{\rho(\beta)} = \prod_{k=1}^{n} (1 + q^k).$$

Now, we count in a different way. Separate all binary sequences with no consecutive ones into those which end in a one and those which end in a zero. Those which end in a zero are counted by $F_n^K(q)$, and those which end in a one, must have a zero preceding the one and are counted by $q^n F^K_{n-1}(q)$.

Now, consider all sequences, in which the first time consecutive ones appear are in positions $k+1$ and $k+2$, for $0 \leq k \leq n-2$. The binary sequence preceding these ones is a binary sequence of length k, and it must end in a 0. Thus, the first part is counted by $F^K_k(q)$. The two ones contribute q^{2k+3} and the remaining $n-k+3$ positions form a binary sequence without restriction contributing $\prod_{j=k+3}^{n}(1+q^j)$. Putting these together and summing over k gives the identity. □

2.5 Determinant Identities

In [22], Cigler proves a q-analogue of the Euler-Cassini identity,

$$F_{n-1}F_{n+k} - F_n F_{n+k-1} = (-1)^n F_{k-1}.$$

We state his theorem without proof.

Theorem 2.5.1 (Cigler) *For $k \geq 1$, the q-Fibonacci polynomials $F^C_n(x,y,q)$ satisfy*

$$\begin{aligned} F^C_{n-1}(x,yq,q)F^C_{n+k}(x,y,q) &- F^C_n(x,y,q)F^C_{n+k-1}(x,yq,q) \\ &= (-1)^n q^{\binom{n}{2}} y^{n-1} F^C_{k-1}(x,yq^n,q). \square \end{aligned}$$

Cigler proves this identity twice, once by using determinants and once by adapting a bijective proof of Zeilberger and Werman [41]. We will prove a q-analogue of the Euler-Cassini identity for $F_n(q)$ by using weighted lattice paths and their relationship to minors of the Toeplitz matrix for the q-Fibonacci sequence. This is a well known method of Lindström [32], which was later popularized by Gessel and Viennot [25].

Consider the digraph $D = (V, A)$ where the verticies are labeled $0, 1, 2, \ldots$, and the only arcs are from vertex n to vertex $n+1$ and from vertex n to vertex $n+2$ for all nonnegative integers n. The portion of the digraph consisting of the vertitcies $0, 1, 2, \ldots, 6$ is pictured below. All arcs are directed to the right.

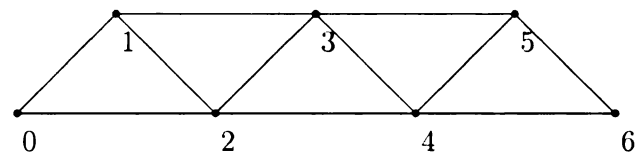


Figure 2: D

It is easy to see that the number of directed paths from a to b in D is F_{b-a}. Let the arc from n to $n+1$, written $\overrightarrow{n(n+1)}$, have wieght $\omega(\overrightarrow{n(n+1)}) = xq^n$. Let the arc from n to $n+2$ have weight $\omega(\overrightarrow{n(n+2)}) = yq^n$. Let p be a directed path from a to b, written $a \xrightarrow{p} b$. We define the weight of p, written $\omega(p)$, to be the product of the weights of its arcs. It is not hard to see that

$$\sum_p \omega(p) = F_{b-a}(xq^a, yq^a, q),$$

where the sum is over all paths p from a to b.

Suppose that $u_1 < u_2 < \ldots < u_k$ and $v_1 < v_2 < \ldots < v_k$ are verticies in D. We will think of the u_i as starting points of paths and the v_i as endpoints of paths. Let

$$P = \{u_1 \xrightarrow{p_1} v_{\alpha(1)}, u_2 \xrightarrow{p_2} v_{\alpha(2)}, \ldots, u_k \xrightarrow{p_k} v_{\alpha(k)}\}$$

be a k-tuple of paths from the u_i to the v_j, where $\alpha \in S_k$, the symmetric group on k elements. We will let the weight of such a k-tuple be $\omega(P) = \prod_{i=1}^{k} \omega(p_i)$. Let $\text{sgn}(P) = \text{sgn}(\alpha)$, where sgn denotes sign, and consider all such k-tuples of paths, P. We have that

$$\sum_P \text{sgn}(P)\omega(P)$$

is equal to the $k \times k$ minor formed from rows $u_1, u_2, \ldots, u_k$ and columns $v_1, v_2, \ldots, v_k$ of the Toeplitz matrix pictured below. We label the rows and columns starting with 0. For example, column 0 starts with $F_0(x, y, q)$ and the rest of the entries are 0.

$$F = \begin{bmatrix} F_0(x,y,q) & F_1(x,y,q) & F_2(x,y,q) & F_3(x,y,q) & \cdots \\ 0 & F_0(xq,yq,q) & F_1(xq,qy,q) & F_2(xq,yq,q) & \cdots \\ 0 & 0 & F_0(xq^2,yq^2,q) & F_1(xq^2,yq^2,q) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We will say that two paths are *noncrossing* if they do not share a vertex.

Theorem 2.5.3 *Let* $u_1 < u_2 < \ldots < u_k$ *and* $v_1 < v_2 < \ldots < v_k$ *be verticies in* D. *Let* m^+ *be the sum of the weights of the noncrossing* k*-tuples of paths,* P*, from* $u_1, u_2, \ldots, u_k$ *to* $v_1, v_2, \ldots, v_k$*, with* $\mathrm{sgn}(P) = 1$. *Let* m^- *be the sum of the weights of the noncrossing* k*-tuples of paths,* Q*, from* $u_1, u_2, \ldots, u_k$ *to* $v_1, v_2, \ldots, v_k$*, with* $\mathrm{sgn}(Q) = -1$. *The* $k \times k$ *minor given by rows* $u_1, u_2, \ldots, u_k$ *and columns* $v_1, v_2, \ldots, v_k$ *of the Toeplitz matrix is equal to* $m^+ - m^-$.

Proof: We prove this by giving a weight-preserving, sign-reversing involution on the k-tuples of paths where at least one pair of paths cross. Let k-tuple P have a crossing pair of paths. Let u_{i_1}, the starting point of path p_{i_1}, be the smallest starting point of any path which crosses another path. Let w be the first vertex shared by p_{i_1} and another path p_{i_j}. Let u_{i_2} be the starting point of path p_{i_j}. Exchange the portions of the p_{i_1} and p_{i_2} starting at w. This produces a new k-tuple of paths, Q, and is clearly an involution. Since the weight of the k-tuple of paths is just the product of the weights of all arcs appearing in the k-tuple, the weight is preserved. Finally, the permutation for P and Q differ by a transposition so $\mathrm{sgn}(P) = -\mathrm{sgn}(Q)$. □

The digraph D that we are working with is relatively restrictive as the next theorem shows.

Theorem 2.5.4 *Let m be the $k \times k$ minor of F consisting of rows $u_1 < u_2 < \cdots < u_k$ and columns $v_1 < v_2 < \cdots < v_k$. If $u_i > v_i$ for some i then $m = 0$. If $u_3 < v_1$ or if for some $i \geq 0$ and $j \geq 1$ we have $v_i < u_j$ and $u_{j+2} < v_{i+1}$ then $m = 0$.*

Proof: If $u_i > v_i$ for some i then there are no k-tuples of paths with startings points $u_1, u_2, \ldots, u_k$ and endpoints $v_1, v_2, \ldots, v_k$. Thus $m = 0$.

Suppose that for some i and j we have $v_i < u_j$ and $u_{j+2} < v_{i+1}$. Let p_j, p_{j+1}, and p_{j+2} be the paths starting at u_j, u_{j+1}, andu_{j+2} respectively. The endpoints of p_j, p_{j+1} and p_{j+2} must appear after u_{j+2}. We may assume that p_j ends before p_{j+1}. In order to avoid crossing the path p_{j+1}, the path p_j must contain the vertex u_{j+1}. Now, every vertex between u_{j+1} and the endpoint of p_j lies on either p_j or p_{j+1}. Since p_j ends at a vertex appearing after u_{j+2}, we must have that u_{j+2} appears on p_j or p_{j+1}. Hence, there is no noncrossing k-tuple of paths and $m = 0$. □

There are two more cases to consider. The case where we have a k-tuple of paths with starting points $u_1, u_2, \ldots, u_k$ and endpoints $v_1, v_2, \ldots, v_k$ with $u_1 < v_1 < u_2 < v_2 < \cdots < u_k < v_k$ is just the product

$$\prod_{i=1}^{k} F_{v_i - u_i}(q^{u_i}x, q^{u_i}y, q).$$

The last case to consider is the case where $u_1 < u_2 < v_1 < v_2 < \cdots < u_{k-1} < u_k < v_{k-1} < v_k$. This case produces a block matrix with 2×2 blocks along the diagonal, whose determinant is just the product of the determinants of the blocks. We may as well just consider the case of two paths with $u_1 < u_2 < v_1 < v_2$. We will prove the special case where $u_1 = 0$, $u_2 = 1$, $v_1 = n$, and $v_2 = n + k$, which will give us the q-analogue of the Euler-Cassini identity discussed above. The general case is proved similarly.

Theorem 2.5.5 *For $n \geq 0$ and $k \geq 1$,*

$$F_n(x,y,q)F_{n+k-1}(xq,yq,q) - F_{n-1}(xq,yq,q)F_{n+k}(x,y,q) = (-1)^n q^{\binom{n+1}{2}} y^n F_{k-1}(xq^{n+1}, yq^{n+1}, q).$$

Proof: Clearly the left hand side is the minor given by rows 0 and 1 and columns n and $n+k$. We will first show that the weighted count of the noncrossing pairs of paths is $q^{\binom{n+1}{2}} y^n F_{k-1}(xq^{n+1}, yq^{n+1}, q)$. Let p be the path starting at 0 and s be the path starting at 1. The portions of p and s between 0 and n are unique, since p must pass through all of the even verticites and s must pass through all of the odd verticies. Thus if n is even the path p must stop at n, and if n is odd then s must stop at n. Assume, without loss of generality, that p stops at n. Then s must contain the vertex $n+1$ and must continue to the vertex $n+k$. Since p stopped at n there is no restriction on the portion of s between $n+1$ and $n+k$. The paths between $n+1$ and $n+k$ contribute weight $F_{k-1}(xq^{n+1}, yq^{n+1}, q)$. The first portions of p and s contribute $q^{\binom{n+1}{2}} y^n$ to the weight. The product of these gives the weight of all possible noncrossing pairs.

Now, if n is even then p goes from u_1 to v_1 and s goes from u_2 to v_2, so the sign of all noncrossing pairs is 1. If n is odd then p goes from u_1 to v_2 and s goes from u_2 to v_1, so the sign of all noncrossing pairs is -1. □

2.6 Other Analogues

The q-Fibonacci numbers that are the focus of this paper come from two statistics, which are equidistributed over the set $\Pi_n(13/2, 123)$. The rb statistic proved most useful for proving identities. The next natural question is whether the bistatistic (ls, rb) also has nice propertites when considered on the set $\Pi_n(13/2, 123)$. The answer is yes. Because the proofs of the p, q-Fibonacci identies are very similar to the proofs above we will define the p, q-Fibonacci numbers and leave the proofs of the identites to the reader.

We define

$$F_n(x, y, p, q) = \sum_{\pi \in \Pi_n(13/2, 123)} x^{s(\pi)} y^{d(\pi)} p^{ls(\pi)} q^{rb(\pi)}.$$

We will also need the p, q-*binomial coefficient*,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i}.$$

One identity involving these p, q-Fibonacci numbers is

$$F_n(x, y, p, q) = \sum_{k \geq 0} x^{n-2k} y^k (pq)^{\binom{n}{2} - k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q}.$$

The proof is omitted as it is similar to that of Theorem 2.4.6

The following is a list of p, q-analogues of some well known Fibonacci identities. Letting $p = 1$ gives the identites involving $F_n(x, y, q)$. The proofs are very similar to the proofs given in this chapter. There are some cases where we may let $x = y = 1$ and obtain analogues involving just p and q. In the list we let $F_n(x, y) = F_n(x, y, p, q)$, and $F_n(p, q)$ be the case where $x = y = 1$.

There are other analogues of Fibonacci numbers which could be studied. For example, there is a restricted set of permutations which is counted by the Fibonacci numbers, see [37]. Given the plethora of permutation statistics, these are bound to be interesting analogues. One may also find another p, q-analogue of the Fibonacci numbers in the following way. It's not hard to see that the weight of a dash in a Morse sequence as defined by Cigler can be translated to give a statistic on layered matchings where the weight of a doubleton B is $q^{\min B} y$, and the weight of a singleton is just x. Call this statistic rc, for Cigler. In a similar way we can find a new statistic lc, which defines the relationship between the ls statistic and the q-analogues Carlitz and Cigler. The bistatistic (lc, rc) gives another p, q-analogue of the Fibonacci numbers.

Table 3: List of p, q-Fibonacci Identites

$$F_{n+2}(x,y) = x^{n+2}(pq)^{\binom{n+2}{2}} + \sum_{j=0}^{n} x^j y (pq)^{\binom{j}{2}} p^{n-j-2} q^j F_{n-j}(xq^{j+2}, yq^{j+2})$$

$$F_{2n+1}(x,y) = \sum_{j=0}^{n} p^{2n(j+1)-j(j+2)} q^{j(j+1)} F_{2n-2j}(xq^{2j+1}, yq^{2j+1})$$

$$F_{2n}(x,y) = y^n (pq)^{n(n-1)} + \sum_{j=0}^{n-1} xy^j p^{2x(j+1)-j(j+3)-1} q^{j(j-1)} F_{2(n-j)-1}(xq^{2j+1}, yq^{2j+1})$$

$$F_{m+n}(x,y) = F_m(xp^n, yp^n) F_n(xq^m, yq^m) + yp^{n-1} q^m F_{m-1}(xp^{n+1}, yp^{n+1}) F_{n-1}(xq^{m+1}, yq^{m+1})$$

$$F_{2n+1}(x,y) = \frac{1}{x(pq)^n}\Big(F_{n+1}(xp^n, yp^n) F_{n+1}(xq^n, yq^n) - y^2 p^{2n-1} q^{2n+1} F_{n-1}(xp^n, yp^n) F_{n-1}(xq^{n+2}, yq^{n+2})\Big)$$

$$F_n(x,y) F_{n+1}(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} (xy^{2j} p^{j(2n-2j-1)} q^{\lfloor (2j)^2/2 \rfloor} \cdot F_{n-2j}(xq^{2j}, yq^{2j}) F_{n-2j}(xq^{2j+1}, yq^{2j+1})) + \sum_{j=0}^{\lfloor n/2 \rfloor} (xy^{2j} p^{j(2n-2j-2)} q^{\lfloor (2j+1)^2/2 \rfloor} \cdot F_{n-2j}(xq^{2j}, yq^{2j}) F_{n-2j}(xq^{2j+1}, yq^{2j+1}))$$

Table 3 (cont'd)

$$F_{2n-1}(x,y) = \sum_{i=1}^{n-1} \Big(xy^{2i} p^{2(n(2i+1) - i(i+3) - 1)} q^{2i^2} \cdot F_{2n-2i-2}(xq^{2i+2}, yq^{2i+2}) F_{2n-2i-1}(xq^{2i}, yq^{2i}) + xy^{2i+1} p^{(2i+2)(2n-4) - 2i^2 + 3} q^{2i(i+1)} \cdot F_{2n-2i-3}(xq^{2i+2}, yq^{2i+2}) F_{2n-2i-2}(xq^{2i+1}, yq^{2i+1}) \Big)$$

$$\begin{aligned} F_{3k-1}(x,y) = {} & F_k(xp^{2k-1}, yp^{2k-1}) \cdot F_k(xp^{k-1}q^k, yp^{k-1}q^k) F_{k-1}(xq^{2k}, yq^{2k}) \\ & + yp^{2k-2} q^{k-1} F_{k-1}(xp^{2k}, yp^{2k}) \cdot F_{k-1}(xp^{k-1}q^{k+1}, yp^{k-1}q^{k+1}) F_{k-1}(xq^{2k}, yq^{2k}) \\ & + yp^{k-2} q^{2k-1} F_k(xp^{2k-1}, yp^{2k-1}) \cdot F_{k-1}(x(pq)^k, y(pq)^k) F_{k-2}(xq^{2k+1}, yq^{2k+1}) \\ & + y^2 p^{3k+4} q^{3k-2} F_{k-1}(xp^{2k}, yp^{2k}) \cdot F_{k-2}(xp^k q^{k+1}, yp^k q^{k+1}) F_{k-2}(xq^{2k+1}, yq^{2k+1}) \end{aligned}$$

$$F_{2n+1}(x,y) = \sum_{i,j} \Big(x^{2n-2i-2j+1} y^{i+j} (pq)^{n(n-i-j-1)+i+j} \cdot (q^{n-j+i+1})^{n-i} (p^{n+j-i+1})^{n-j} \begin{bmatrix} n-j \\ i \end{bmatrix}_{p,q} \begin{bmatrix} n-i \\ j \end{bmatrix}_{p,q} \Big)$$

$$\prod_{k=1}^{n} (1 + p^{n-k} q^k) = q^n F_n(p,q) + pq^{n-1} F_{n-1}(p,q) + \sum_{k=0}^{n} F_{k-1}(p,q) p^{n-k+1} q^k \prod_{j=k+3}^{n} (1 + p^{n-j+1} q^j)$$

Chapter 3
The Möbius Function of a Compostion Poset

3.1 Introduction

We now turn our focus to compositions. A composition α is an element $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{P}^k$, where $\mathbb{P} = \{1, 2, 3, \ldots\}$ is the set of positive integers. The *length* of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is $\ell(\alpha) = k$, and the *norm* is n if $\sum_{i=1}^{k} \alpha_i = n$, also written $|\alpha| = n$. The α_i are called *parts*. For example, $\alpha = (1, 3, 2, 1, 4)$ is a composition of 11, and (1,3,1,2,4) is a different composition of 11. Define C_n to be the set of all compositions of n and let

$$C = \bigcup_n C_n.$$

We are interested in studying a particular partial ordering on a set of restricted compositions. Before we define this set, let's briefly remind the reader of some basic terminology about partially ordered sets (posets). A *poset* is a set P with relation $\leq$, which is reflexive, antisymmetric, and transitive. For example, if we consider the set [12] and we say that for $a, b \in [12]$, $a \leq b$ if and only if a divides b, then $[12]_{\leq}$ is a poset with this relation.

The poset $[12]_{\leq}$ is finite, but we could easily consider the set $\mathbb{P}$ under the same relation and have an infinite poset. For any $x, z \in P$ the set $[x, z]$ of elements $y \in P$ such that $x \leq y \leq z$ is called an *interval* of P. A poset P is *locally finite* if any interval is finite. A *subposet* of P is a set $Q \subseteq P$ under the same partial ordering as P. Finally, two posets P and Q are *isomorphic* if there is a bijection $\phi : P \to Q$ such that for $x, y \in P$, $x \leq y$ if and only if $\phi(x) \leq \phi(y)$.

The partial ordering on C that we are interested in was introduced by Björner and Stanley to produce a composition analogue of Young's Lattice. Unfortunately, their paper has since been withdrawn from the arXiv. For more information on Young's Lattice, we refer the reader to [38, 39]. To define the poset C we define its covering relation $\alpha \prec \beta$. In a poset an element y *covers* an element x, $y \succ x$, if $y > x$ and there is no z with $y > z > x$. In C, we say that $\beta \succ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ if β is of one of the two following forms:

$$\begin{aligned} \beta &= (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_k) \\ \beta &= (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i + 1 - h, h, \alpha_{i+1}, \ldots, \alpha_k). \end{aligned}$$

We will consider the poset C_d, which is the subposet of C consisting of compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_i \leq d$ for $1 \leq i \leq k$. C_d is the set of compositions, with part sizes at most d. Björner and Stanley use the fact that C is isomorphic to a poset of words to determine the Möbius function of C. We will follow in their footsteps and use the fact that the poset C_d is isomorphic to a restricted poset of words.

Let A^* be the free monoid under concatenation of $A = \{a, b\}$. We think of A^* as the set of all words that can be created from the *alphabet* A. The identity in A^* is the empty word ϵ. We say that the length of a word $u = u_1 u_2 \ldots u_k$ is $|u| = k$. Let $u = u_1 u_2 \ldots u_k$ and $w = w_1 w_2 \ldots w_\ell$ be words in A^*. We make A^* into a poset by letting $u \leq w$ if there exist $i_1 \leq i_2 \leq \ldots \leq i_k$ such that $u_j = w_{i_j}$ for $1 \leq j \leq k$. We call the set $\iota = \{i_1, i_2 \ldots, i_k\}$ an *embedding* of u in w and let $w_\iota = w_{i_1} w_{i_2} \ldots w_{i_k}$. If ι is an embedding of u in w, then we say that w_j is *supported* by u in ι if $j \in \iota$. For example, the word *abaab* is a subword of $w = aabbababb$, since $w_2 w_3 w_5 w_8 = abaab$, and w_3 is supported.

Use the notation $a^k = \underbrace{aa \ldots a}_{k}$. Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ let

$\phi(\alpha) = b^{\alpha_1 - 1}ab^{\alpha_2 - 1}a \ldots ab^{\alpha_k - 1}$. Clearly $\phi : C \to A^*$ is a bijection. Note that ϕ sends a part k of a composition to $a\underbrace{bb\ldots b}_{k-1}$ unless it is the first part of the composition, in which case the inital a is dropped.

Theorem 3.1.1 *The map ϕ is an isomorphism of C and A^* as partially ordered sets.*

Proof: Since we've already established ϕ as a bijection between A^* and **C**, it is enough to show that ϕ preserves the partial orderings.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \prec \beta = (\beta_1, \beta_2, \ldots, \beta_\ell) \in C$. If $\ell = k$ then, for some j, $\beta_j = \alpha_j + 1$ and $\beta_i = \alpha_i$ for $i \neq j$. It is not hard to see that in this case $\phi(\beta)$ is obtained from $\phi(\alpha)$ by inserting a b into $\phi(\alpha)$ anywhere between the j^{th} and $(j+1)^{st}$ occurrence of an a. Thus, $\phi(\alpha)$ is a subword of $\phi(\beta)$.

If $\ell = k + 1$ then $\beta_{j+1} = \alpha_j + 1 - h$ for some h, $\beta_j = h$ and $\beta_i = \alpha_i$ for $i \neq j$ or $j+1$. In this case $\phi(\beta)$ is obtained from $\phi(\alpha)$ by inserting an a between the j^{th} and $(j+1)^{st}$ occurrence of an a.

Thus $\beta \succ \alpha \Rightarrow \phi(\beta) \succ \phi(\alpha)$. The converse is proved similarly. $\Box$

Let A_d^* be the subposet of A^* consisting of all words that do not contain $d+1$ consecutive b's. Notice that ϕ restricts to an isomorphism between C_{d+1}, as defined above, and the subposet, A_d^*. Much is known about partially ordered sets on words and subword order, so it will be convenient to work with the poset A_d^* to understand the poset C_d.

3.2 The Möbius Function of C_d

To every locally finite poset, P, and field, K, we associate an incidence algebra, $I(P)$. The elements of $I(P)$ are functions, f, which assign a scalar, $f(a, b)$, to each interval $[a, b]$, and $f(a, b) = 0$ if $a \not\leq b$. For $f, g \in I(P)$ and $k \in K$ we have

$(kf)(a,b) = k(f(a,b))$ and $(f+g)(a,b) = f(a,b) + g(a,b)$. We define "multiplication" in this algebra as the convolution

$$(f * g)(a,b) = \sum_{a \leq x \leq b} f(a,x)g(x,b).$$

The multiplicative identity of $I(P)$ is

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in two special functions in $I(P)$. One of them is the zeta function,

$$\zeta(a,b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

That is to say $\zeta(a,b) = 1$ if the interval $[a,b]$ is not empty and $\zeta(a,b) = 0$ otherwise. It can be shown, [38], that ζ is invertible with respect to the multiplication defined above. That is, there is some μ such that $\zeta * \mu = \mu * \zeta = \delta$. The inverse of ζ is the Möbius function, μ. It is easy to see that, since μ is the inverse of ζ, μ is the unique function in $I(P)$ satisfying the following three properties.

- $\mu(x,x) = 1$
- $\sum_{x \leq z \leq y} \mu(z,y) = 0$
- $\mu(x,y) = 0$ for $x \not\leq y$

To describe the Möbius function of A_d^* we will need the concepts of d-normal embedding and right-most embedding. We will call a set $[r,s]$ a *run* if $w_r = w_{r+1} = \ldots = w_s$. Let the *repetition set* of w be $R(w) = \{j : w_j = w_{j-1}\}$. An embedding $\iota = \{i_1, i_2, \ldots, i_k\}$ of u in w is called *d-normal* if (a) $R(w) \subseteq \{i_1, i_2, \ldots, i_k\}$, and (b) if u has a run of d b's and the first b in this run corresponds to w_{i_j} in the embedding then $w_{i_j - 1} = a$ and $i_j - 1 \in \iota$. Let $\binom{w}{u}_{dn}$ be the number of d-normal embeddings of u in w. The *right-most embedding* of u in w is the unique embedding $\{j_1, j_2, \ldots, j_k\}$ such that $i_\ell \le j_\ell$, $1 \le \ell \le k$ for any other embedding $\{i_1, i_2, \ldots, i_k\}$ of u in w.

Theorem 3.2.1 *For any words $u, w \in A_d^*$*

$$\mu(u, w) = (-1)^{|w| + |u|} \binom{w}{u}_{dn}.$$

Proof: This proof is essentially the same as the proof of Theorem 3.1 from [7]. We will show that the function $f(u, w) = (-1)^{|w| + |u|} \binom{w}{u}_{dn}$ satisfies the three conditions above and hence, must be μ.

Clearly, $f(u, u) = 1$, and $f(u, w) = 0$ for $u \not\le w$. It remains to show that $\sum_{u \le v \le w} f(v, w) = 0$ for any $u \le w$.

Suppose that $|w| = n$ and $|u| = k$. Let N be the set of $\iota = \{i_1, i_2, \ldots, i_r\}$ such that ι is a d-normal embedding of w_ι in w. Let $N_e = \{\iota : \#\iota \text{ is even}\} \subseteq N$ and $N_o = \{\iota : \#\iota \text{ is odd}\} \subseteq N$. Then

$$\sum_{u \le v \le w} f(v, w) = (-1)^n (\#N_e - \#N_o).$$

A bijection between N_e and N_o will complete the proof.

For each embedding $\iota \in N$ let ι_f be the minimal number in $[n]$ such that ι_f is not in the right-most embedding of u in w_ι. Define $\psi : N \to N$ by

$$\psi(\iota) = \begin{cases} \iota \cup \{\iota_f\} & \text{if } \iota_f \notin \iota, \\ \iota - \{\iota_f\} & \text{if } \iota_f \in \iota. \end{cases}$$

We show that ψ is well-defined. If $\iota_f \notin \iota$ then, since $R(w) \subseteq \iota$, we must have $R(w) \subseteq \iota \cup \{\iota_f\}$. If $\iota_f \in \iota$ then we want to show that $\iota_f \notin R(w)$. If $\iota_f \in R(w)$

then $w_{i_f} = w_{i_f - 1}$ and $i_f - 1$ is in the right-most embedding of u in w_ι. This violates the definition of right-most embedding.

We must show that $w_{\psi(\iota)}$ is still an element of A_d^*. Supoose ψ removes an a creating a run of $d+1$ b's in $w_{\psi(\iota)}$. Since u can have a run of at most d b's, the first b in this newly formed run of b's cannot be in the right-most embedding. Thus, the a would never have been removed.

If ψ inserts a b creating a run of $d+1$ b's in $w_{\psi(\iota)}$ then w_ι has a run of d b's. Let the first b in this run be in position i_t. Since ι is a d-normal embedding of w_ι in w, $w_{i_t - 1} = a$, $i_t - 1 \in \iota$. Thus, ψ cannot insert a b at the beginning of the run. Inserting a b in the middle of the run violates the definition of right-most embedding.

Clearly ψ is its own inverse and changes the parity of $\#\iota$. □

3.3 Shellability and the Möbius Function

We now give an alternate proof that $\mu(u, w) = (-1)^{|w| + |u|}\binom{w}{u}_{dn}$ by showing that A_d^* is dual CL-shellable. Shellability of a poset is a condition on a simplicial complex related to the poset. We'll begin by giving the topological background necessary to develop the theory of shellability.

A *simplicial complex* Δ is a family of subsets σ of a vertex set V, called simplicies, such that if $\sigma \subseteq \tau \subseteq V$ and $\tau \in \Delta$ then $\sigma \in \Delta$. The *dimension* of a simplex σ is $\dim \sigma = |\sigma| - 1$. The *dimension* of a simplicial complex Δ is $\dim \Delta = n = \max\{\dim \sigma \in \Delta\}$. Each simplex in Δ is called a *face* and the simplices, which are maximal with respect to containment are called *facets.* We say that Δ is *pure* if all facets are of the same dimension.

We say that a simplicial complex Δ is *shellable* if the facets can be ordered, $F_1, F_2, \ldots F_\ell$, in such a way that for each $1 \leq j \leq \ell$, we have that $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ is a union of maximal subfaces of F_j.

Let $\tilde{H}_k(\Delta)$ be the k^{th} reduced homology group of Δ over $\mathbb{Z}$, and let $\dim \tilde{H}_k(\Delta)$ be its dimension. We refer the reader to [28] for background on homology and reduced homology of simplicial complexes.

Shellable complexes have very simple structure and homology as the next theorem [13] shows.

Theorem 3.3.1 *Let Δ be a pure, shellable simplicial complex of dimension k then Δ is homeomorphic to a wedge of k-spheres and hence* $\dim H_i(\Delta) = 0$ *for* $0 \le i < k$. □

Let P be a poset. A *chain* of *length* n in P is an ordered $(n+1)$-tuple $(x_0, x_1, \dots, x_n)$, with $x_i \le x_{i+1}$ for $0 \le i \le n-1$. A chain $c = (x_0, x_1, \dots, x_n)$ is said to be *saturated* if $x_i \prec x_{i+1}$ for $0 \le i \le n-1$. We will say that a chain $c = (x_0, x_1, \dots, x_n)$ in the interval $[x, y]$ is *maximal* if c is saturated, $x_0 = x$, and $x_n = y$. A locally finite poset P is *ranked* if every maximal chain in any interval has the same length.

There is a special simplicial complex associated to each poset P called the order complex. The *order complex* $\Delta(P)$ has the elements of P as its vertex set and all chains of P as its simplicies. Notice that each interval of a ranked poset gives rise to a pure order complex. A poset P is called *shellable* if its order complex $\Delta(P)$ is shellable.

Björner showed that the poset of words on any finite alphabet under subword order is dual CL-shellable [6]. We will show that the poset A_d^* admits a dual CL-shelling under the same labeling. First, let us define a chain lexicographic labeling of a poset as introduced by Björner and Wachs [11, 12].

Let P be a finite poset with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Let $E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the set of edges of the Hasse diagram of P, and $M(P)$ be the set of maximal chains in P. A *chain-edge labeling* of P is a map $\ell : \{(x, y, c) \in E(P) \times M(P) : x, y \in c\} \to \mathbb{Z}$ satisfying the following condition. If two maximal chains c_1 and c_2 share their first k edges then $\ell(x, y, c_1) = \ell(x, y, c_2)$ for the first k edges (x, y) of each chain.

The chain-edge labeling ℓ associates to each maximal chain, $c = (\hat{0} = x_0, x_1, \ldots, x_n = \hat{1})$ a unique n-tuple

$$\ell(c) = (\ell(x_0, x_1, c), \ell(x_1, x_2, c), \ldots, \ell(x_{n-1}, x_n, c)).$$

Notice that any saturated k-chain inherits a potentially different labeling for each maximal chain that includes it. To establish uniqueness, we use rooted intervals.

A *rooted interval* of P is an interval $[x, y]_r$, where r is a maximal chain in $[\hat{0}, x]$. If c is a maximal chain in $[x, y]_r$ then $r \cup c$ is a maximal chain in $[\hat{0}, y]$. Hence, the labels $\ell(m)$ and $\ell(m')$ of any two maximal chains m and m' of P containing $r \cup c$ must agree on the elements of $r \cup c$. Thus, for any maximal chain c in $[x, y]_r$, $\ell(c)$ is unique in $[x, y]_r$.

A maximal chain c in $[x, y]_r$ is called *ascending* (*descending*) if

$$\ell(x_0, x_1, c) < (>) \ell(x_1, x_2, c) < (>) \ldots < (>) \ell(x_{n-1}, x_n, c).$$

Given chains $c(x_0, x_1, \ldots, x_k)$ and $c' = (x'_0, x'_1, \ldots, x'_k)$ in $[x, y]_r$, we say that c is lexicographically smaller than c', $\ell(c) <_{lex} \ell(c')$, if $\ell(x_i, x_{i+1}, c) < \ell(x'_i, x'_{i+1}, c')$ in the first coordinate, where $\ell(c)$ and $\ell(c')$ differ.

A chain edge labeling ℓ is called a *CL-labeling* (chain-lexicographic labeling) if for each rooted interval $[x, y]_r$ there exists a unique ascending maximal chain c, and $c <_{lex} c'$ for any other maximal chain $c' \in [x, y]_r$. If P has a CL-labeling then we say that P is *chain-lexicoraphically shellable.*

Theorem 3.3.2 (Björner,Wachs) *If a poset P is chain-lexicographically shellable then its order complex $\Delta(P)$ is shellable.* □

The shelling of the complex $\Delta(P)$ is given by putting a linear ordering $<_\Lambda$

on the set of maximal chains $c_1, c_2, \ldots, c_n$ such that $\ell(c_i) < \ell(c_j)$ implies that $c_i <_\Lambda c_j$. This odering on the maximal faces of $\Delta(P)$ is a shelling.

The *dual*, P^*, of a poset P is a poset on the same elements as P, in which $x \leq y$ in P^* if $y \leq x$ in P. A poset is called dual CL-shellable if its dual is CL-shellable. Without considering the dual of the entire poset, we simply reverse the labelings on our chains. That is to say, for $c = (x_0, x_1, \ldots, x_k)$ in $[x, y]_r$, we consider the labeling $\ell^*(c) = (\ell(x_{k-1}, x_k, c), \ell(x_{k-2}, x_{k-1}, c), \ldots, \ell(x_0, x_1, c))$. Now, ℓ^* is a dual CL-shelling of P if ℓ^* satisfies the same two conditions above. Let $\bar{P}$ be the poset $P - \{\hat{0}, \hat{1}\}$. Clearly, P is shellable if and only if $\bar{P}$ is.

Shellability allows us to find the Möbius function of a poset using the topology of its order complex. Let κ_i be the number of chains of length i starting at $\hat{0}$ and ending at $\hat{1}$. Thus, $\kappa_0 = 0$ and $\kappa_1 = 1$. A proof of the following theorem can be found in [38].

Theorem 3.3.3 (Hall) *For any finite poset P with $\hat{0}$ and $\hat{1}$,*

$$\mu_P(\hat{0}, \hat{1}) = \sum_{i \geq 0} (-1)^i \kappa_i. \square$$

It is easy to see that κ_i is the number of chains of length $i - 2$ in the poset $\bar{P}$. This essentially says that $\kappa_i = c_{i-2}$ where c_{i-2} is the number of $i - 2$-simplicies in $\Delta(\bar{P})$. Define the *reduced Euler Characteristic* of a finite simiplicial complex Δ to be

$$\tilde{\chi}(\Delta) = -1 + \sum_{j \geq 0} (-1)^j c_j.$$

Thus, we have that

$$\mu_P(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(\bar{P})).$$

A proof of the following theorem can be found in [28].

Theorem 3.3.4 *For any simplicial complex* Δ,

$$\tilde{\chi}(\Delta) = \sum_{k \geq -1} (-1)^k \dim H_k(\Delta). \square$$

This gives us that $\mu_P(\hat{0}, \hat{1}) = \sum_{k \geq -1} (-1)^k \dim \tilde{H}_k(\Delta(\bar{P}))$. Thus, by our remarks above, if P is shellable then $\Delta(\bar{P})$ is shellable and hence $\mu_P(\hat{0}, \hat{1}) = (-1)^k \dim \tilde{H}_k(\Delta(\bar{P}))$. This simplifies the computation of the Möbius function of a shellable poset.

The question now is how to determine $\dim \tilde{H}_k(\Delta(\bar{P}))$ from the dual CL-shelling. Let $D(x, y)$ be the number of descending maximal chains in the interval $[x, y]_r$. In their paper [8] Björner, Garsia, and Stanley give the following corollary, stated here as a Theorem.

Theorem 3.3.5 (A. Björner, A. Garsia, R. Stanley) *Let P be a ranked poset, and $[x, y]$ be an interval of P. If $\ell : P \to \mathbb{Z}$ is a CL-labeling of P then*

$$\mu(x, y) = (-1)^k D(x, y). \square$$

We apply this information to the poset A^*_d. We will use the same labeling that Björner used in [6] to determine the Möbius function of A^*.

Let $[u, w]_r$ be a rooted interval of A^*, and let each maximal chain $c = (u = x_1, x_2, \ldots, x_k = w)$ in $[u, w]$ be assigned a label $\ell(m) = (\ell_1(m), \ell_2(m), \ldots, \ell_k(m))$ as follows. Label the edge (x_{k-1}, x_k, c) by $\ell(x_{k-1}, x_k, c) = i_1$, where i_1 is the least element of $[k]$ such that $[k] - \{i_1\}$ is an embedding of x_{k-1} in x_k. Label the edge (x_{k-2}, x_{k-1}, c) by $\ell(x_{k-2}, x_{k-1}, c) = i_2$, where i_2 is the least element of $[k] - \{i_1\}$ such that $[k] - \{i_1, i_2\}$ is an embedding of x_{k-2} in x_k. Repeat this process for the remaining edges. It is clear that if two maximal chains have the same first s edges then the labels on these edges will be the same. Figure 3 shows this chain-edge labeling for $[abb, aabbab]$.

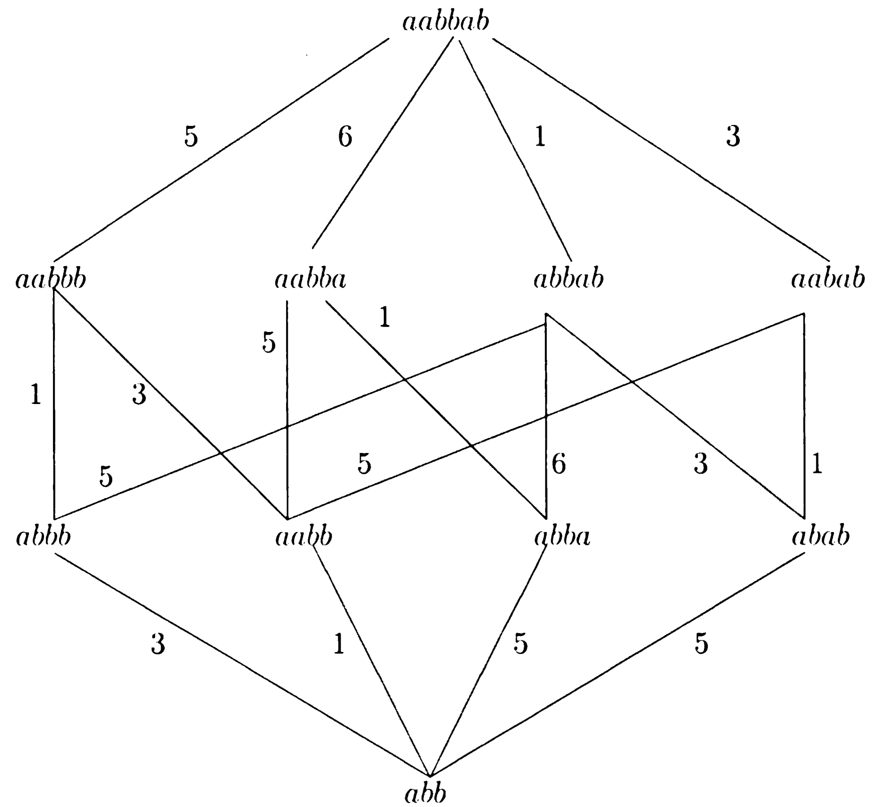


Figure 3: CL-labeling of [abb,aabbab]

Theorem 3.3.6 (Björner) *The map $\ell : A^* \to \mathbb{Z}$ described above is a dual CL-labeling for each interval of A^*.* □

In Björners proof [6] he shows that the unique ascending maximal chain is one corresponding to the right-most embedding of u in w. Let $\{i_1, i_2, \ldots, i_k\}$ be the right-most embedding of u in w, then the ascending maximal chain is the maximal chain labeled by the elements of $[n] - \{i_1, i_2, \ldots, i_k\}$ from smallest to largest. The unique ascending chain in Figure 3 is

$$abbaba \overset{1}{-} abbab \overset{3}{-} abab \overset{5}{-} abb.$$

It is also clear that since we are working with the right-most embedding, this chain must have the lexicographically smallest possible entries.

Each maximal chain of an interval $[u, w]$ in A_d^* is also a maximal chain of the same interval in A^*. We will denote by $[u,w]_d$ the interval $[u,w]$ in A^* restricted to A_d^*. We will label the edges of A_d^* by $\ell_d : A_d^* \to \mathbb{Z}$ in exactly the same way as we did for A^*.

Theorem 3.3.7 *The map $\ell_d : A_d^* \to \mathbb{Z}$ is a dual CL-labeling for each interval of A_d^*, and hence A_d^* is CL-shellable.*

Proof: We already know that there is exactly one ascending maximal chain, m_0, in the labeling of an interval $[u,w]$ in A^*, and that m_0 corresponds to the right-most embedding of u in w. This means that there is at most one ascending maximal chain among those in the interval $[u,w]_d$. We must show that this chain is still in $[u,w]$. Unless a chain passes through an element with a run of b's of length at least $d+1$, that chain is in $[u,w]_d$.

Note that u and w both do not contain such a run. The only way to pass through an element with such a run is by removing an a that was originally separating $d+1$ or more b's. Recall that at each step the left-most element is removed, maintaining an embedding. Since u does not contain a run of $d+1$ or more b's and we are considering the right-most embedding of u, such an a cannot be the left-most removable element. □

Corollary 3.3.8 *For words u and w in A_d^* we have $\mu(u,w) = (-1)^{|w|+|u|}\binom{w}{u}_{dn}$.*

Proof: By Theorem 3.5 it suffices to show that the number of descending chains in $[u,w]$ is the same as the number of d-normal embeddings of u in w.

Suppose $\iota = \{i_1, i_2 \ldots, i_k\}$ is an embedding of u in w such that there is a maximal chain m with $[n] - \{\ell_1(m), \ell_2(m), \ldots, \ell_{n-k}(m)\} = \iota$. Then m is descending if m is obtained by deleting the entries w_j, with $j \in [n] - \iota$, from right to left. This is possible only if for each $j_1 < j_2 < j_3$ with $j_1, j_3 \in [n] - \iota$ and $j_2 \in \iota$, we have that $w_{j_1} \neq w_{j_2}$. And if u contains a run of d b's and the first b in the run corresponds to w_{j_t} then $w_{j_t - 1} = a$ and $j_t - 1 \in \iota$, otherwise this maximal chain will pass through an element not in A_d^*. These two conditions show that $\{i_1, i_2, \ldots, i_k\}$ is a d-normal embedding of u in w.

Similarly, if $\iota = \{i_1, i_2, \ldots, i_k\}$ is a d-normal embedding of u in w then there is a descending maximal chain corresponding to ι given by deleting the elements w_j, with $j \in [n] - \iota$, from right to left. □

3.4 Rationality

Let S a finite set. The formal power series algebra in the noncommuting variables of S over the field $\mathbb{Z}$, denoted $\mathbb{Z}\langle\langle S\rangle\rangle$, is the set of all infinite series $f = \sum_i c_i x_i$, where $c_i \in \mathbb{Z}$ and x_i is some word in S^*. We define the $*$ operation on $\mathbb{Z}\langle\langle S\rangle\rangle$ to be $f^* = \epsilon + f + f^2 + \ldots$ where ϵ is the the *empty word* in S^*. A series $f \in \mathbb{Z}\langle\langle S\rangle\rangle$ is called *rational* if it can be constructed from a finite number of monomials under a finite number of the usual algebraic operations in $\mathbb{Z}\langle\langle S\rangle\rangle$ and the $*$ operation. Let $f^+ = f^* - \epsilon$. Clearly, f^+ is rational if f is.

The ring $\mathbb{Z}\langle\langle A\rangle\rangle$, where $A = \{a, b\}$ as before, is simply the set of series of the form $f = \sum_{w \in A^*} c_w w$, with $c_w \in \mathbb{Z}$. We may also consider series of the form $f = \sum_{u \leq w} c_{(u,w)} u \otimes w$ where $u \otimes w$ just represents the ordered pair $(u, w) \in A^* \times A^*$. Rationality of such a series is defined similarly.

In [9] Björner and Reutenauer showed that the following four series are rational:

$$
\begin{aligned}
Z(u) &= \sum_{w \in A^*} \zeta(u, w) w, \\
M(u) &= \sum_{w \in A^*} \mu(u, w) w, \\
Z_\otimes &= \sum_{w \in A^*} \zeta(u, w) u \otimes w, \\
M_\otimes &= \sum_{w \in A^*} \mu(u, w) u \otimes w.
\end{aligned}
$$

In this section we will use methods similar to those used by Björner and Sagan [10] to do the same for the series $Z_d(u)$, $M_d(u)$, $Z^d_{\otimes}$, and $M^d_{\otimes}$, where these series are defined exactly the same way as those above replacing A^* by A^*_d in each. For the remainder of this section we will assume that $d = 3$ to avoid cumbersome formulas and definitions. Everything that is used to prove these facts for $d = 3$ can be generalized to any d.

We begin with $Z_3(u)$ and $M_3(u)$. We remind the reader that the bijection $\phi : C \to A^*$ was given by $\phi(k) = a\underbrace{bb\ldots b}_{k-1}$. A function $f : S^* \to T^*$ where S and T are finite sets is *multiplicative* if for $u = u_1 u_2 \ldots u_m \in S^*$, we have that $f(u) = f(u_1)f(u_2)\cdots f(u_m)$. Let $A_3 = \{a, ab, abb, abbb\}$. Let $B = \epsilon + b + bb + bbb$, and notice that $(\epsilon + b + bb + bbb)(A_3)^* = A_3^*$. Notice that each word $u \in A_3^*$ can be broken uniquely into its maximal runs of a's and b's. Define the multiplicative function $z : A_3^* \to \mathbb{Z}\langle\langle A_3 \rangle\rangle$, where z acts on maximal runs, by

$$\begin{aligned}
z(a^k) &= (A_3)^{k-1}a, \\
z(b) &= (B - \epsilon)a^*, \\
z(bb) &= (B - \epsilon)a^+ba^* + (B - b - \epsilon)a^*, \\
z(bbb) &= (B - \epsilon)a^+ba^*ba^* + (B - b - \epsilon)a^+ba^* + bbba^*.
\end{aligned}$$

If a run of a's is at the end of the word then let $z(a^k) = z(a^k)B$ for this run of a's. Let $p_z(u)$ be the *prefix* of $Z(u)$ where,

$$p_z(u) = \begin{cases} B(A_3)^* & u \text{ begins with } a, \\ (B(A_3)^*a + \epsilon) & u \text{ begins with } b. \end{cases}$$

Define the multiplicative function $m : A_3^* \to \mathbb{Z}\langle\langle A_3 \rangle\rangle$, where m acts on maximal runs, by

$$
\begin{aligned}
m(a^k) &= ((ab)^* - a)(a + (ab)^+(\epsilon - a)a)^{k-1}(a + (ab)^+ a) \\
m(b) &= b((ab)^* - a - b), \\
m(bb) &= (b(ab)^*)^2(\epsilon - b), \\
m(bbb) &= (b(ab)^*)^3.
\end{aligned}
$$

Let $p_m(u)$ be the *prefix* of $M(u)$ where,

$$
p_m(u) = \begin{cases} (ab)^* & u \text{ begins with } a, \\ (ab)^* a & u \text{ begins with } b. \end{cases}
$$

Lemma 3.4.1 *For any $u \in A_3^*$ we have that*

$$Z_3(u) = p_z(u)z(u),$$

and

$$M_3(u) = p_m(u)m(u).$$

Proof: We begin by proving the statement for $Z_3(u)$. We must show that the function z will produce each word that contains u exactly once. This is done by showing that for $w \geq u$, $z(u)$ produces the right most embedding of u in w. We will explain how this works for $z(a^k)$, how these go together with $z(b^\ell)$ and how the prefixes work. The remaining cases are similar.

The last a in the run a^k is given by the a at the end of $z(a^k)$. This is clearly under the right-most possible a if we focus on $z(a^k)$. The preceding $k-1$ a's are each followed by $B = \epsilon + b + bb + bbb$. If there were an a between any of the k

a's, then the preceding a's could be shifted to the right, and hence the embedding would not be the right-most one.

The only place that any a's could be placed is at the beginning of the run. The reader will notice that this is not encorporated in $z(a^k)$. This issue is resolved in the following two ways. If this a^k is preced by a run of b's, then the a^* at the end of each of $z(b)$, $z(bb)$, and $z(bbb)$ places as many a's as one would like before the first a of a^k. If this a^k is at the beginning of the word the the prefix $p_Z(u)$ takes care of the problem.

The same arguments as above, minding the fact that there can be no more than three b's in a run explain $z(b)$, $z(bb)$, and $z(bbb)$. The prefix $p_Z(u)$, when u begins with a, is clear. Let's briefly discuss the prefix, when u begins with b. If we are considering the right most embedding of u in w and we are trying to generate any w, then each w can begin with any word from A_3^*. Hence, $p_Z(u)$ must begin with $B(A_3)^*$ regardless of the first letter of u. If u begins with b, we must be careful not to create a run of more than three b's. To avoid this, we put a at the end of $B(A_3)^*$ and add ϵ to take into consideration the fact that the first letter of u could be the first letter of w.

After a close look at $z(b^\ell)$ for $1 \leq \ell \leq 3$, the reader will see that $z(u)$ produces w only as a right-most embedding, and hence each $w \geq u$ is generated once.

Now, we turn our attention to $m(u)$. Again, we will explain $m(a^k)$, how it works together with $m(b^\ell)$ and how the prefixes work. We show that each $w \in A_3^*$ appears $(-1)^{|w|+|u|}\binom{w}{u}_{3n}$ times in $m(u)$.

We begin with an explanation of $m(a^k)$. At the beginning is $((ab)^* - a)$. Notice that an a (respectively b) does not have to be part of a 3-normal embedding if it is immediately preceded by a b (respectively a). This means that we can begin with a run of alternating a's and b's. Since we are appending two letters at a time, the sign of our word isn't changing. The $-a$ in this first part describes the option of placing an unsupported a at the beginning of a run of a's.

Each of the first $k - 1$ a's in this run is taken care of by $(a + ab^{+}(\epsilon - a)a)$. The first a covers the option of just having an a. The next portion places at least one copy of ab followed by nothing or an extra a. This takes care of the fact that we can place a run of alternating a's and b's before each a in our run. As long as there is at least one copy of ab before the a then we could place another $-a$ before the next a. The final portion forces $m(a^k)$ to end in a so as not to create a run of more than three b's.

This description shows that each word w produced by $m(u)$ is produced with respect to a unique 3-normal embedding of u in w. Also, any possible 3-normal embedding of u in a word w is produced. The prefix takes care of the beginning of a word in which u is 3-normally embedded. □

The fact that $z(u)$, $m(u)$, $p_z(u)$ and $p_m(u)$ are rational for each $u \in A_3^*$ proves the following theorem.

Theorem 3.4.2 *The series $Z_3(u)$ and $M_3(u)$ are rational.* □

The techniques used above to prove the rationality of $Z_3(u)$ and $M_3(u)$ are a bit cumbersome. We will use finite state automata to prove that $Z_{\otimes}^3$ and $M_{\otimes}^3$ are rational.

Let S be an alphabet. A finite state automaton is a digraph D, with vertex set V and arc set E, allowing loops and multi-arcs. There are unique verticies α and ω, where α is the inital vertex and ω is the final vertex. Each arc $e \in E$ is labeled by a monomial $f(e) \in \mathbb{Z}\langle\langle S \rangle\rangle$. A finite walk W with arcs $e_1, e_2, \ldots, e_k$ is given the monomial label

$$f(W) = \prod_{i=1}^{k} f(e_i).$$

The series *accepted* by D is

$$f(D) = \sum_{W} f(W),$$

where the sum us over all walks in D from α to ω.

If $e_1, \ldots e_j$ are all arcs from one vertex to another, replacing them by a single arc e and labeling this arc

$$\sum_{i=1}^{j} f(e_i)$$

does not change the series accepted by D. For simplification we will use this procedure.

It is a well-known fact that a series is rational if and only if it is accepted by a finite state automaton [5]. We will construct finite state automata accepting $Z_{\otimes}^{3}$ and $M_{\otimes}^{3}$ to prove the following theorem for $d = 3$. The pattern in the automata will be obvious and generalizable.

Theorem 3.4.3 *For any d, $Z_{\otimes}^{d}$ and $M_{\otimes}^{d}$ are rational.*

Proof: The automata in Figures 4 and 5 are for $Z_{\otimes}^{3}$ and $M_{\otimes}^{3}$ respectively. We will use them to explain why these automata accept $Z_{\otimes}^{3}$ and $M_{\otimes}^{3}$ respectively and how they are generalizable to any d. For clarity, we left some arcs off of the diagrams. In Figure 4, there is an arc labelled $\epsilon \otimes \epsilon$ from each node to ω. In Figure 5, there is an arc labelled $\epsilon \otimes \epsilon$ from each of α_2, α_3, β_2, β_3, β_4, γ_5, γ_6, γ_7, δ_8, δ_9, and δ_{10} to ω. There is also an arc labelled $a \otimes a$ from each of β_2, β_3, β_4, γ_5, γ_6, γ_7, δ_8, δ_9, and δ_{10} to each of α_2, β_1, γ_1, and δ_1. Also in Figure 5, we separated the node α from the rest of the digraph.

We claim that for any $u = u_1 u_2 \ldots u_k \leq w = w_1 w_2 \ldots w_n$ there is a unique walk from α to ω in the automaton for $Z_{\otimes}^{3}$ labelled $u \otimes w$. First, we show that there is a walk for any $u \otimes w$. Consider the right-most embedding $\iota = \{i_1, i_2, \ldots, i_k\}$ of u in w. We'll build $u \otimes w$ from this embedding. The first $i_1 - 1$ letters in w are not supported, so this portion of w is constructed using the nodes α, α_1, α_2, and α_3. Notice that any word in A_3^* can be built uniquely by using the nodes α, α_1, α_2, and α_3. Now, if $u_1 = a$ and w_{i_1} is preceded by an a then we must follow the arc from α to α_4. If w_{i_1} is preceded by a run of b's then we must follow the arc from α to δ_2, construct this run of b's and then go to α_4. If $u_1 = b$ and w_{i_1} is preceded by an a then at this point the walk follows the arc from α to β. If $u_1 = b$ and w_{i_1} is preceded by a run of b's then the walk must follow the arc from α to δ_2, build up the preceding run of unsupported b's and then go on to either γ or δ from δ_2 or δ_3 respectively.

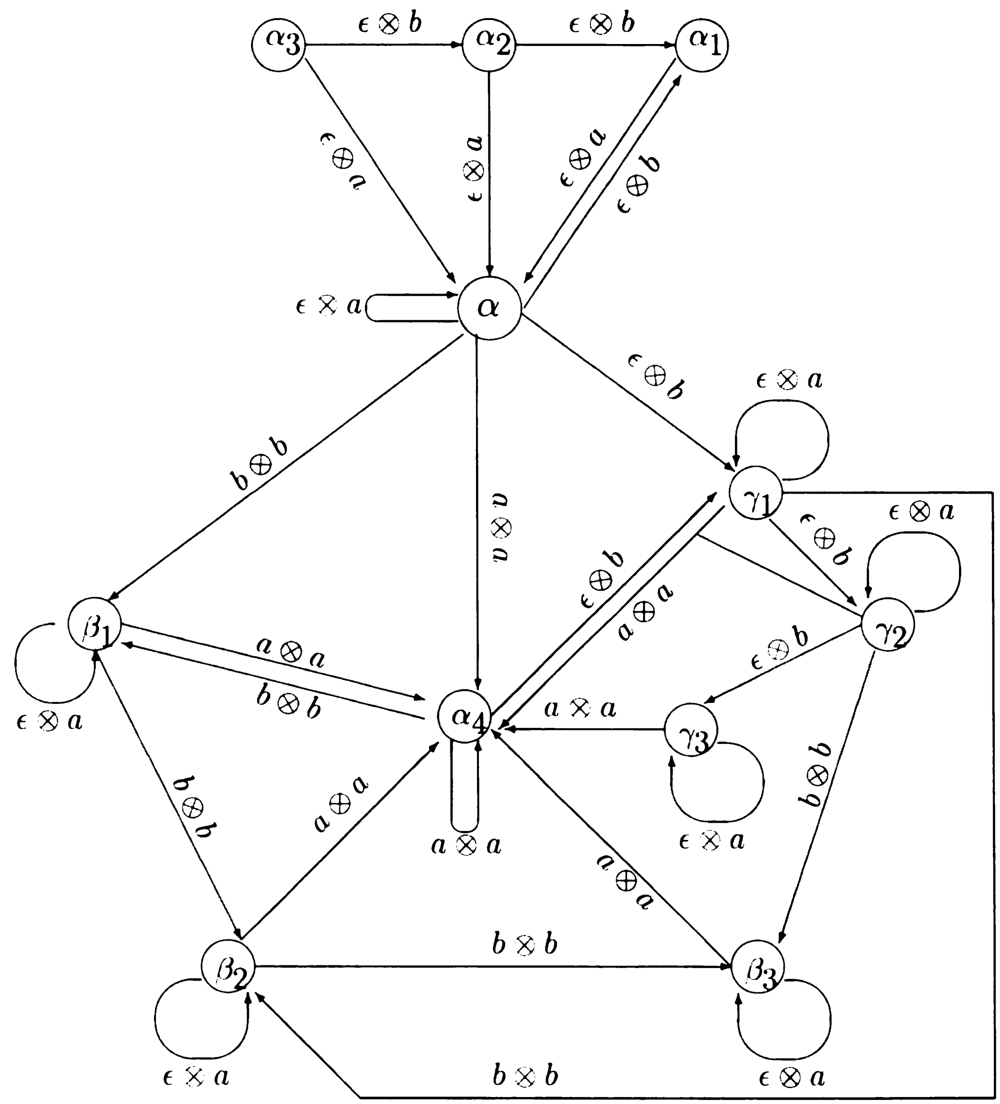


Figure 4: $Z^3_{\otimes}$

Now, there is no way to get back to the α, α_1, α_2, and α_3 part of the digraph.

Notice that if $u_1 = a$ then the unsupported part of w between w_{i_1} and w_{i_2} can only be a run of b's. If $u_1 = b$ then the unsupported part of w between w_{i_1} and w_{i_2} can only be a run of a's. This is forcing the embedding ι to be the right-most embedding. The construction continues as above for the remainder of $u \otimes w$.

It is important to note here that the complication of the digraph develops from the fact that we must be careful to avoid producing a run of more than three b's in u or w. The portion of the digraph involving the β's controls the supported b's in w, and the portion involving the γ's controls the unsupported b's.

Our comments above about the automaton forcing ι to be the right-most embedding proves that each $u \otimes w$ is produced by a unique walk, and hence the automaton accepts $Z^3_{\otimes}$. To generalize this to any $Z^d_{\otimes}$, we would merely extend the α, β and γ portions of the digraph appropriately.

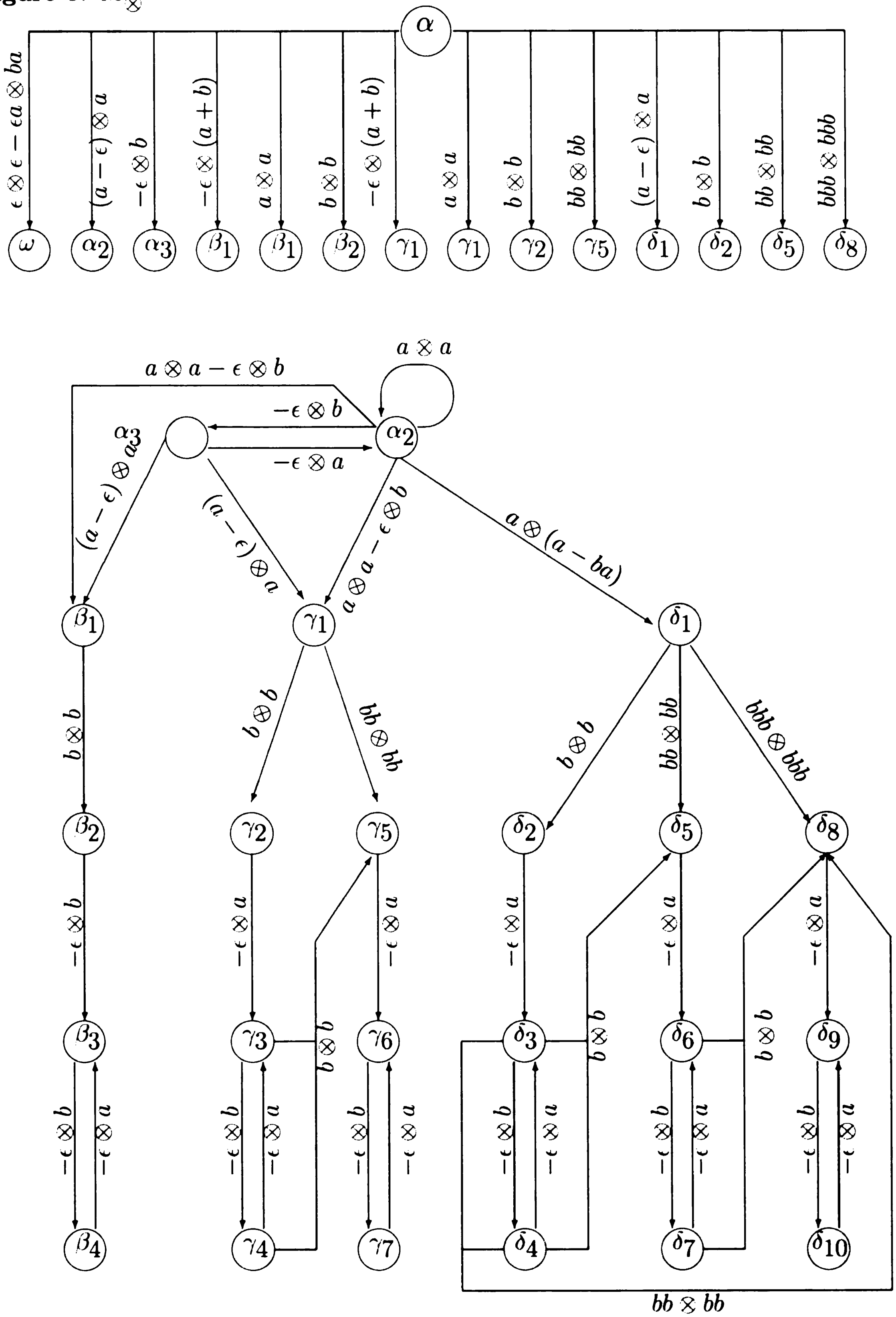
We turn our focus to the automaton for $M^3_{\otimes}$. We claim that for any walk from α to ω the monomial $u \otimes w$ associated to this walk is a 3-normal embedding. The walk must begin at α. Depending on the beginning of the embedding we select our first destination from α. The reader will see that each possibility is represented. Without explaining every possibility, we will begin by looking at what happens at α_2. The node α_2 is building runs of a's. Between any two a's in a run can be a run of alternating unsupported a's and b's. This is controled by the exchange between α_2 and α_3. Notice that the only way to create a run of two or more a's in w is to make at least one loop around α_2. The automaton is assuring that an a preceded by an a in w is supported. The automaton is forcing the embedding that a walk generates to be 3-normal.

Now, the three legs of the diagram labeled with β's, γ's and δ's respectively, are controlling the three possibilites for a run of b's in u. If u has a run of three b's, the walk producing $u \otimes w$ must pass through the δ part of the diagram. Each of the three different legs of the δ portion of the diagram represents a supported b. So once the walk is on any of the nodes δ_8, δ_9, or δ_{10} all three b's in this run have been produced. Notice that in this section between any two supported b's there can be a run of alternating a's and b's, given by the exchange between the bottom two nodes of each leg. Also, all but one arc going into δ_1 is labeled with a supported a immediately preceding the portion of the word to be produced. The only arc that does not is the arc from α to δ_1. This again is assuring that the embedding of any walk is 3-normal.

The portion marked with β's controls situations where u has a run of just one b and the portion marked with γ's takes care of two b's. This shows that every walk from α to ω in this automaton produces $u \otimes w$ according to a 3-normal embedding. It's not hard to see from the above explanation that any normal embedding ι of u in w is given by a unique walk from α to ω.

Finally, notice that anytime a letter in a label on an arc is unsupported the sign of the monomial label changes. This takes care of the sign of $\mu(u, w)$. □

Figure 5: $M^3_{\otimes}$



3.5 Generating Functions in Commuting Variables

The generating functions above are quite beautiful and are rational, but they don't emphasize the connection between words in A_d^* and the compositions in C_{d+1}. To see how this all ties together we will use what we know from the previous section to produce generating functions in commuting variables for ζ and μ. This will allow us to consider the generating functions in terms of the length, $\ell(\alpha)$, of a compostion α or the norm of α, $|\alpha|$. If n_k is the number of k's in α then we may redefine $\ell(\alpha) = \sum_{k \geq 1} nk$ and $|\alpha| = \sum_{k \geq 1} n_k k$. Let $\alpha \leftrightarrow u$, then the *type* of α is $\tau(\alpha) = (n_1, n_2, \ldots, n_n, r)$, where r is the number of runs of a's in u.

Each time k appears in α replace k by x^k, where x is a commuting variable. Then we obtain the norm generating function

$$Z_d(\alpha; x) = \sum_{\alpha \leq \beta} x^{|\beta|}.$$

If we replace each element in α by the variable t then we obtain the length generating function

$$Z_d(\alpha; t) = \sum_{\alpha \leq \beta} t^{\ell(\beta)}.$$

To avoid cumbersome formulas and because these generalize simply to any d we will focus on the case when $d = 3$. If $\alpha \in C_4$ corresponds to $u \in A_3^*$ then any a that is not immediately followed by a b represents a 1 from α and b, bb, and bbb represent 2,3, and 4 respectively. Let $[k]_x$ be the polynomial $1 + x + \ldots + x^{k-1}$.

Let $z(u; x)$ and $p_z(u; x)$ be the formal power series in $\mathbb{Z}[[x]]$ obtained from $z(u)$ and $p_z(u)$ respectively by replacing each letter in $z(u)$ by x. Then it's easy to see that $Z_3(u; x) = p_z(u; x) z(u; x)$. Defining $m(u; x)$ and $p_m(u; x)$ in a similar way gives us that $M_3(u; x) = p_m(u; x) m(u : x)$

Suppose u has type $\tau(u) = (n_1, n_2, n_3, n_4, r)$ then Lemma 3.4.1 gives us that $Z_3(u;x)$ is one of the following rational functions. The first two correspond to u beginning with a and the last two correspond to u beginning with b. The first and third correspond to u ending with b and the second and fourth correspond to u ending with a.

- $$\frac{1}{1-x[3]_x}\left(x^{n_1}([4]_x)^{n_1-r+1}\right)\left(\frac{[3]_x}{1-x}\right)^{n_2}\left(\frac{x^2[3]_x}{(1-x)^2}+\frac{x^2[1]_x}{1-x}\right)^{n_3} \cdot\left(\frac{x^3[3]_x}{(1-x)^3}+\frac{x^4[2]_x}{(1-x)^2}+\frac{x^3}{1-x}\right)^{n_4}$$
- $$\frac{1}{1-x[3]_x}\left(x^{n_1}([4]_x)^{n_1-r}\right)\left(\frac{[3]_x}{1-x}\right)^{n_2}\left(\frac{x^2[3]_x}{(1-x)^2}+\frac{x^2[1]_x}{1-x}\right)^{n_3} \cdot\left(\frac{x^3[3]_x}{(1-x)^3}+\frac{x^4[2]_x}{(1-x)^2}+\frac{x^3}{1-x}\right)^{n_4}$$
- $$\frac{1}{1-x[4]_x}\left(x^{n_1}([4]_x)^{n_1-r+1}\right)\left(\frac{[3]_x}{1-x}\right)^{n_2}\left(\frac{x^2[3]_x}{(1-x)^2}+\frac{x^2[1]_x}{1-x}\right)^{n_3} \cdot\left(\frac{x^3[3]_x}{(1-x)^3}+\frac{x^4[2]_x}{(1-x)^2}+\frac{x^3}{1-x}\right)^{n_4}$$
- $$\frac{1}{1-x[4]_x}\left(x^{n_1}([4]_x)^{n_1-r}\right)\left(\frac{[3]_x}{1-x}\right)^{n_2}\left(\frac{x^2[3]_x}{(1-x)^2}+\frac{x^2[1]_x}{1-x}\right)^{n_3} \cdot\left(\frac{x^3[3]_x}{(1-x)^3}+\frac{x^4[2]_x}{(1-x)^2}+\frac{x^3}{1-x}\right)^{n_4}$$

Again, if u has type $\tau(u) = (n_1, n_2, n_3, n_4, r)$ then $M_3(u;x)$ is one of the following. The first corresponds to u beginning with a, b, or bb. The last corresponds to those u beginning with bbb.

- $$\frac{1}{1-x^2}\left(\frac{x^3}{(1-x^2)^2}\right)^{n_1}(1-x+x^2)^r(1-x+x^3)^{n_1-r}\left(\frac{x(1-x+x^3)}{1+x}\right)^{n_2} \cdot\left(\frac{x^2}{1+x-x^2-x^3}\right)^{n_3}\left(\frac{x}{1-x^2}\right)^{3n_4}$$

- $\frac{x}{1-x^2}\left(\frac{x^3}{(1-x^2)^2}\right)^{n_1}(1-x+x^2)^r(1-x+x^3)^{n_1-r}\left(\frac{x(1-x+x^3)}{1+x}\right)^{n_2}$

$$\cdot\left(\frac{x^2}{1+x-x^2-x^3}\right)^{n_3}\left(\frac{x}{1-x^2}\right)^{3n_4}$$

The following theorem is now an immediate consequence of Lemma 4.1.

Theorem 3.5.1 *The norm generating fuctions $Z_3(u;x)$ and $M_3(u;x)$ for u with type $\tau(u) = (n_1, n_2, n_3, n_4, r)$ are as stated above.* □

Now, we want to consider the length generating functions for ζ and μ. Let $z(u;t)$ and $p_z(u;t)$ be the formal power series in $\mathbb{Z}[[t]]$ obtained from $z(u)$ and $p_z(u)$ respectively in the following way. Replace each a that is not immediately followed by a b by t and each maximal run of b's by t. Then it's easy to see that $Z_3(u;t) = p_z(u;t)z(u;t)$. A similar definition gives that $M_3(u;t) = p_m(u;t)m(u;t)$.

Suppose u has type $\tau(u) = (n_1, n_2, n_3, n_4, r)$ then Lemma 3.4.1 gives us that $Z_3(u;t)$ is one of the following with the same correspondance as in the list for $Z_3(u;x)$.

- $\frac{3t+1}{1-4t}(4t)^{n_1-r}\left(\frac{3t}{1-t}\right)^{n_2}\left(\frac{2t+t^2}{(1-t)^2}\right)^{n_3}\left(\frac{t^2(4t+1)(2-t)}{(1-t)^3}\right)^{n_4}$
- $\frac{1}{1-4t}(4t)^{n_1-r}\left(\frac{3t}{1-t}\right)^{n_2}\left(\frac{2t+t^2}{(1-t)^2}\right)^{n_3}\left(\frac{t^2(4t+1)(2-t)}{(1-t)^3}\right)^{n_4}$
- $\frac{(3t+1)(2-t)}{1-4t}(4t)^{n_1-r}\left(\frac{3t}{1-t}\right)^{n_2}\left(\frac{2t+t^2}{(1-t)^2}\right)^{n_3}\left(\frac{t^2(4t+1)(2-t)}{(1-t)^3}\right)^{n_4}$
- $\frac{2-t}{1-4t}(4t)^{n_1-r}\left(\frac{3t}{1-t}\right)^{n_2}\left(\frac{2t+t^2}{(1-t)^2}\right)^{n_3}\left(\frac{t^2(4t+1)(2-t)}{(1-t)^3}\right)^{n_4}$

Again, if u has type $\tau(u) = (n_1, n_2, n_3, n_4, r)$ then

$$M_3(u;t) = \left(\frac{1}{1-t}\right)t^{n_1}\left(\frac{2-3t+3t^2-t^3}{(1-t)^2}\right)^r\left(\frac{2-3t+2t^2}{(1-t)^2}\right)^{n_1-r}\left(\frac{t-t^2+t^3}{1-t}\right)^{n_2}$$

$$\cdot\left(\frac{t-t^2+t^3}{1-t}\right)^{n_3}\left(\frac{t}{1-t}\right)^{2n_3}\left(\frac{t}{1-t}\right)^{3n_4}.$$

The following theorem is now an immediate consequence of Lemma 3.4.1.

Theorem 3.5.2 *The norm generating fuctions $Z_3(u; t)$ and $M_3(u; t)$ for u with type $\tau(u) = (n_1, n_2, n_3, n_4, r)$ are as stated above.* □

We would like to extend our gratitude to Björner and Stanley for their work on the composition poset C, as this chapter would not have been possible without it.

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