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ABSTRACT

Galois Structure of Modular Forms of Even Weight

By

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We calculate the equivariant Euler characteristics of an even power of the canonical sheaf on modular curves over $\mathbb{Z}$ with a tame action of a finite abelian group. As a consequence, we obtain information on the Galois module structure of "twisted" modular forms of even weight having Fourier coefficients in a ring of algebraic integers.

To my family.

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TABLE OF CONTENTS

Introduction

The normal basis theorem implies that if N/K is a finite Galois extension of number fields with Galois group G, then N is a free $K[G]$-module of rank one. In particular, N is a free $\mathbb{Q}[G]$-module. Let $\mathcal{O}_N$ and $\mathcal{O}_K$ be the ring of integers of N and K respectively. Then we can ask for the analogous statement, namely, "Is $\mathcal{O}_N$ a free module over the group ring $\mathbb{Z}[G]$?" The first observation regarding this question belongs to E. Noether.

Theorem 0.1 *(E. Noether) Let N/K be a finite Galois extension of number fields with Galois group G. Then the ring of integers, $\mathcal{O}_N$ is a projective $\mathbb{Z}[G]$-module if and only if N/K is at most tamely ramified.*

When N/K is tamely ramified, the obstruction to $\mathcal{O}_N$ to be a stably free $\mathbb{Z}[G]$-module is the class $(\mathcal{O}_N)$ in the class group $Cl(\mathbb{Z}[G])$. Fröhlich's conjecture, proved by M.Taylor in [T], gives an interesting description for this class:

Theorem 0.2 *(M. Taylor) We have the following equality,*

$$(\mathcal{O}_N) = W_{N/K} \tag{1}$$

in $Cl(\mathbb{Z}[G])$. Here $W_{N/K}$ is the "root number class"; the class $W_{N/K}$ has order two and is given by the signs of the ϵ-constants in the functional equation of the Artin L-functions of symplectic representations of G.

It was natural to try to extend Fröhlich's conjecture by relating the ϵ-constants with the Galois modules attached to a group action on an arithmetic scheme. However, the right formulation of the generalized conjecture was not clear until the work of Chinburg and of Chinburg, Erez, Pappas and Taylor ([CEPT], [CPT]) . Let $\pi : X \to Y = X/G$ be a geometric tame G-cover of projective flat regular schemes over $\mathbf{Z}$. In [CEPT], the authors define an equivariant deRham Euler characteristic class $\chi(X, G)$ in $Cl(\mathbb{Z}[G])$ using equivariant Euler characteristics of differential sheaves. When $X = \mathrm{Spec}(\mathcal{O}_N)$ this generalizes the class of the ring of integers $(\mathcal{O}_N)$. They also define a root number class $W_{X/Y}$ (similar to $W_{N/K}$) and introduced a ramification class $R_{X/Y}$ which depends on the ϵ-constants of the branch locus of the covering π. The definition of these classes was motivated by the prior work of Chinburg [C] who considered the same constructions for covers of varieties over a finite field. Under some additional technical assumptions on X and Y they show

Theorem 0.3 *([CPT]) We have*

$$\chi(X, G) = W_{X/Y} + R_{X/Y} \tag{2}$$

in $Cl(\mathbb{Z}[G])$.

This generalizes Fröhlich's conjecture to higher dimensional varieties over $\mathbb{Z}$.

It turns out that one can consider more general equivariant projective Euler characteristics: Suppose that X is a scheme projective and flat over $\mathbb{Z}$ which supports a tame action of the finite group G. For any coherent sheaf $\mathcal{F}$ on X which supports a G-action that is compatible with the action of G on X one can define following Chinburg [C] the equivariant projective Euler characteristics $\overline{\chi}^P(X, \mathcal{F}) \in Cl(\mathbb{Z}[G])$.

The calculation of these Euler characteristic often connects to other fundamental problems in Number Theory. A recent method, developed by Chinburg, Pappas and Taylor in [CPT1], shows how to calculate the Euler characteristic of coherent sheaves on projective flat schemes over $\mathbb{Z}$ on which a finite abelian group acts tamely. Unlike other techniques, this one does not neglect any torsion information if the base scheme has dimension less than 5. Roughly speaking, the idea in their paper was that Euler characteristic should differ by computable terms from classes in Grothendieck groups which have "n-cubic" structures. This idea was motivated by previous works of Pappas in [P] and [P1]. In this recent paper, a precise formula is given for the Euler characteristic. Furethermore, they determined the structure of the lattice of weight 2 cusp forms for $\Gamma_1(p)$ which have integral Fourier expansions as a module for the action of the finite group of diamond Hecke operators. This is done by calculating the equivariant Euler characteristic $\overline{\chi}^P(X, \mathcal{O}_X)$ where X is a certain integral model of the modular curve $X_1(p)$.

In this thesis, we calculate the equivariant Euler characteristic of k-th power of the "twisted" canonical sheaf over an integral module of the modular curve $X_1(p)$ (here some twists are allowed along a fibral divisor at p for some technical reasons). Moreover, we find a lower bound to the degree of the twist which guarantees that the first cohomology group vanishes. Consequently, the structure of the lattice of "twisted" cusp forms of weight $2k$ and Nebentypus character can be obtained as a module for the diamond Hecke operators. Here twist means that we allow the Fourier coefficients to have denominator a certain (bounded) power of the uniformizer over p

(see below).

More properly, let $p \equiv 1 \bmod 24$ be a prime and $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$. Suppose $\chi : \Gamma \longrightarrow \mu_r \subset \mathbb{Z}[\zeta_r]^*$ is a character of prime order $r|(p-1)$ with $r > 3$. Let $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi$ be the $\mathbb{Z}[\zeta_r]$-module of "twisted" cusp forms of weight $2k$, level p and of Nebentypus character χ (for some technical reasons some twists to the dualizing sheaf on the modular curve is allowed and this δ represents the order of the pole that we allow along the twist). In addition, we ask that the Fourier expansion of these modular forms is of the form $F(z) = \sum_{n\geq 1} \frac{a_n}{p^{\delta k}} e^{2\pi i n z}$ where the coefficients a_n belong to $\mathbb{Z}[\zeta_r]$. The locally free $\mathbb{Z}[\zeta_r]$-module $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi$ is of rank $n(\chi) = \frac{(2k-1)(p-25)}{12}$. For $a \in (\mathbb{Z}/r\mathbb{Z})^*$ let $\{a\}$ be the unique integer in the range $0 < \{a\} < r$ having residue class a, and let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ be the automorphism for which $\sigma_a(\zeta_r) = \zeta_r^{\{a\}}$. Define $\omega_r : (\mathbb{Z}/r\mathbb{Z})^* \longrightarrow \mathbb{Z}_r^*$ to be the Teichmuller character. The ring $\mathbb{Z}$ (resp. $\mathbb{Z}_r$) is embedded into the pro-finite completion $\hat{\mathbb{Z}} = \prod_{l \text{ prime}} \mathbb{Z}_l$ of $\mathbb{Z}$ diagonally (resp. via the factor $l = r$). Then a modified quadratic Stickelberger element of $\hat{\mathbb{Z}}[\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})]$ can be defined by

$$\theta_2 = \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^*} \frac{(p-1)}{24r^2} \left(\{a\}^2 - \omega_r(a)^2\right) \sigma_a^{-1}. \tag{3}$$

We also define truncated Stickelberger element $[\theta_1]$ by

$$[\theta_1] = \sum_{0<q\leq kr-1+\left[\frac{-2kr}{(p-1)}\right], (q,r)=1} m(k,r) q \sigma_q^{-1} \tag{4}$$

where $m(k,r)$ is an integer depending on k and r. The truncated sum-element $[\theta_0]$ can be also defined by

$$[\theta_0] = \sum_{0<q\leq kr-1+\left[\frac{-2kr}{(p-1)}\right],(q,r)=1} n(k,\delta,r)\sigma_q^{-1}. \tag{5}$$

where $n(k,\delta,r)$ is an integer depending on k,δ and r.

Since the ideal class group $Cl(\mathbb{Z}[\zeta_r])$ is finite, θ_2, $[\theta_1]$ and $[\theta_0]$ all act on this group. Let $\mathcal{P}_\chi$ be the prime ideal of $\mathbb{Z}[\zeta_r]$ over (p) with the property that the reduction of χ modulo $\mathcal{P}_\chi$ is the $\frac{p-1}{r}$ power of the identity character $(\mathbb{Z}/p\mathbb{Z})^* \to \mathbf{F}_p^*$.

Let X_1 be an integral model of the modular curve $X_1(p)$ and $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ be the group acting on X_1 faithfully. Let H be a subgroup of Γ of index r, we let X_H be the quotient X_1/H and let $\mu : X_1 \to X_H$ be the quotient map. The special fiber of X_1 over p has two irreducible components. The unramified component D_∞^1 is distinguished from the other component by the fact that D_∞^1 intersects the cuspidal section ∞.

Theorem 0.4 *Suppose $\mathfrak{A} \subset \mathbb{Z}[\zeta_r]$ is an ideal with ideal class $\theta_2 \cdot [\mathcal{P}_{\chi_0}] - [\theta_1] \cdot [\mathcal{P}_{\chi_0}] - [\theta_0] \cdot [\mathcal{P}_{\chi_0}]$. Then we have*

$$\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{A}$$

as $\mathbb{Z}[\zeta_r]$-modules.

Theorem 0.5 *Assuming the same terms as in the preceding theorem, for $\delta > 2 + r$,*

$$S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{A}$$

as $\mathbb{Z}[\zeta_r]$-modules.

This extends the corresponding theorem of [CPT1] to higher weight cusp forms.

CHAPTER 1

Definitions and Preliminaries

This chapter contains basic definitions and facts of tame covers of schemes, modular forms, Tate curves and moduli schemes of elliptic curves.

1.1 Tame covers of schemes

Let us recall the definition from [C].

Definition 1.1 Let Y be a normal scheme of finite type over R and let D be a closed subset of Y which is of codimension at least one. A morphism $f : X \to Y$ is tamely ramified covering of Y relative to D if the followings hold:

a. f is finite,

b. f is etale over $Y - D$,

c. Every irreducible component of X dominates an irreducible component of Y.

d. X is normal,

e. Let y on D have codimension one in Y and let x be a point of X over y, then $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is tamely ramified extension of DVR's.

Definition 1.2 Let $f : X \to Y$ and D be as in the previous definition and let G be a finite group. Then $f : X \to Y$ is tame G-cover relative to D if $X \times (Y - D) \mapsto Y - D$ is a G-torsor when G is regarded as a constant group scheme over $Y - D$.

Definition 1.3 The G-action on X is called tame if for every closed point $x \in X$, order of inertia subgroup $I_x \subset G$ is relatively prime to the charactetistic of the residue field $k(x)$.

Definition 1.4 Let $f : X \to Y$ be a tame G-cover. A quasi-coherent O_X-G-module F is quasi-coherent O_X-module having an action G which is compatible with the action of G on $\mathcal{O}_X$. i.e. suppose $x \in X$, $g \in G$ and let x^g be the image of x under g. The action of g on O_X and F gives homomorphism of stalks $O_{X,x^g} \mapsto O_{X,x}$ and $F_{x^g} \mapsto F_x$; both of these homomorphism is denoted by ϕ, and $\phi(am) = \phi(a)\phi(m)$ for all $a \in O_{X,x^g}$ and $m \in F_{x^g}$.

1.2 Modular Forms and Diamond Operators

Definition 1.5 Let k be an integer. We say a function f is *modular of weight* $2k$ if it is meromorphic on the upper half plane and ∞ also satisfying following condition

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z) \tag{1.1}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Definition 1.6 A modular function is called as *modular form* if it is holomorphic everywhere including ∞.

Definition 1.7 A modular form is called as *cusp form* if it is zero at ∞.

A modular form of weight 2k can also be written as a series,

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \tag{1.2}$$

where $q = e^{2\pi i z}$ and verifies the identity

$$f(1/z) = z^{2k} f(z) \tag{1.3}$$

So, $a_0 = 0$ when f is a cusp form.

Let $\mathfrak{H}$ denote upper half plane $\{z \in \mathbb{C} | \Im z > 0\}$ on which we have $SL_2(\mathbb{Z})$ action as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}. \tag{1.4}$$

When we extend the upper half plane by adding cusps $P^1(\mathbb{Q})(= \mathbb{Q} \bigcup \{\infty\})$ to $\mathfrak{H}^*$ we can extend this action on cusps using the same fractional transformation. We have two subgroups of $SL_2(\mathbb{Z})$, namely $\Gamma_0(N)$ and $\Gamma_1(N)$ defined as:

$$\Gamma_0(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \quad c \equiv 0 \mod N\} \tag{1.5}$$

$$\Gamma_1(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \quad c \equiv 0 \mod N, \quad a \equiv d \equiv 1 \mod N\} \tag{1.6}$$

One can easily see that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and when we take the quotient we get,

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^* \tag{1.7}$$

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mapsto \bar{d} \tag{1.8}$$

We can define space of modular forms (cusp forms) of weight $2k$ and level N by restricting $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\Gamma_1(N)$ and we can denote it by $M_{2k}(\Gamma_1(N))$ $(S_{2k}(\Gamma_1(N)))$. There is a $\Gamma_0(N)$ action on $S_{2k}(\Gamma_1(N))$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f := (cz+d)^{-2k} f(\frac{az+b}{cz+d}) \tag{1.9}$$

Since the action of $\Gamma_1(N)$ is trivial, we can define an action of $(\mathbb{Z}/N\mathbb{Z})^*$ using the isomorphism 1.7 as follows:

$$< d > f := \begin{pmatrix} a & b \\ c & d \end{pmatrix} f \tag{1.10}$$

This operator $< d >$ is called as *Diamond Operator.*

1.3 Tate Curves

Let R be a noetherian local ring which is complete with respect to ideal qR. The Tate curve E_q proper smooth scheme over $R[q^{-1}]$ defined (as in [CSS]) by following equation on an affine chart $z \neq 0$:

$$y^2 + xy = x^3 + a_4(q)x + a_6(q) \tag{1.11}$$

where

$$a_4(q) = -5\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad a_6(q) = -\sum_{n=1}^{\infty} \frac{(5n^3+7n^5)q^n}{12(1-q^n)} \tag{1.12}$$

The following series gives us parametrization of the curve.

$$x(u) = \sum_{n=1}^{\infty} \frac{uq^n}{(1-uq^n)^2} - 2\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \tag{1.13}$$

$$y(u) = \sum_{n=1}^{\infty} \frac{(uq^n)^2}{(1-uq^n)^3} + \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \tag{1.14}$$

They induce homomorphism

$$R[q^{-1}]^*/q^{\mathbf{Z}} \to E_q(R[q^{-1}]),\ u \mapsto \begin{cases} (x(u), y(u)) & \text{if } u \notin q^{\mathbf{Z}} \ ; \\ O, & \text{if } u \in q^{\mathbf{Z}} \ , \end{cases}$$

This map is bijective when R is a complete discrete valuation ring.

1.4 Coarse Moduli Scheme of Elliptic Curves

A moduli problem is roughly given by two ingredients. First, a class of objects together with a notion of being family of such objects over a scheme B. Second, an equivalence relation $\sim$ on the set of $S(B)$ of all such families over B. We can define a moduli functor F from the category of schemes to that of sets by $F(B) = S(B)/\sim$ for our moduli problem. The functor F is representable if there is a scheme $\mathfrak{M}$ and isomorphism ψ between F and the functor $Mor(\bullet, \mathfrak{M})$. If that is the case, then we say that $\mathfrak{M}$ is *fine moduli spcheme* for the moduli problem F. When a fine moduli exists, every family over B is the pullback of universal family $\mathfrak{C}$ via a unique map of B to $\mathfrak{M}$. This allows us to translate information about the geometry of families of our moduli problem to information about geometry of the moduli space $\mathfrak{M}$ itself. Unfortunately, most of the time it is not possible to have a fine moduli scheme.

Definition 1.8 A scheme $\mathfrak{M}$ and a natural transformation $\psi_{\mathfrak{M}}$ from the functor F to the functor of points $Mor(\bullet, \mathfrak{M})$ of $\mathfrak{M}$ is a *coarse moduli scheme* for the functor F if

i. The map $\psi_{Spec(\mathbb{F})} : F(Spec(\mathbb{F})) \to \mathfrak{M}(\mathbb{F})$ is a bijection for every algebraically closed field $\mathbb{F}$.

ii. Given another scheme $\mathfrak{M}'$ and a natural transformation $\psi_{\mathfrak{M}'}$ from F to $Mor(\bullet, \mathfrak{M}')$, there is a unique morphism $\phi : \mathfrak{M} \to \mathfrak{M}'$ such that the associated transformation $\Phi : Mor(\bullet, \mathfrak{M}) \to Mor(\bullet, \mathfrak{M}')$ satisfies $\psi_{\mathfrak{M}'} = \Phi \circ \psi_{\mathfrak{M}}$

Often the moduli functor is represented by a more general type of object, a *moduli stack*. In our case, we will use the moduli stack $\mathfrak{M}_{\Gamma_1(p)}$ that classifies triples (E, C, γ) where $E \to S$ is a generalized elliptic curve i.e. n-gons are allowed (see [DR], or [CES]), C a locally free rank p subgroup of the smooth locus E^{sm} and $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to C^D$ a "generator" (in the sense of [KM, Ch. 1]) of the Cartier dual of C; we require that C intersects every irreducible component of every geometric fiber of $E \to S$. (Notice that a group scheme embedding $\mu_p \hookrightarrow E^{\mathrm{sm}}$ gives data C and γ; in fact, if p is invertible on S, giving C together with γ as above exactly amounts to giving a group scheme embedding $\mu_p \hookrightarrow E^{\mathrm{sm}}$.) We denote by $X_1 = X_1(p)$ the corresponding coarse moduli scheme over $\mathrm{Spec}\,(\mathbb{Z})$. The group $(\mathbb{Z}/p\mathbb{Z})^*$ acts on $\mathfrak{M}_{\Gamma_1(p)}$ via

$$(a \bmod p) \cdot (E, C, \gamma) = (E, C, \gamma \circ a^{-1}). \tag{1.15}$$

(When p is invertible, this action sends the corresponding $j : \mu_p \hookrightarrow E^{\mathrm{sm}}$ to the composition $j \circ (z \mapsto z^a) : \mu_p \hookrightarrow E^{\mathrm{sm}}$.) This produces a faithful $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$-action on X_1. When H is a subgroup of Γ we let X_H be the quotient X_1/H, and set $X_0 = X_\Gamma$. The singularity structure of X_H is explicitly given in [CES] depending on the order of H. When they analyze non-regular points they use deformation theory. In our case we used it as follows: Let $\mathcal{R}_s$ be the formal deformation ring of the point $s = (E, \mathbb{Z}/p\mathbb{Z} \subset E, 0)$ in the moduli stack $\mathfrak{M}_{\Gamma_1(p)}$. Then $\mathcal{R}_s$ supports an action of $\mathrm{Aut}(E) \times (\mathbb{Z}/p\mathbb{Z})^*$. Let H' be the inverse image of H under the surjection

$(\mathbb{Z}/p\mathbb{Z})^* \to \Gamma$, and let s' be the image of s on X_H. The completion of the local ring is

$$\widehat{\mathcal{O}}_{X_H \otimes_{\mathbf{Z}} W, s'} \simeq (\mathcal{R}_s)^{\Delta \times H'} \tag{1.16}$$

as ring with $G = (\mathbb{Z}/p\mathbb{Z})^*/H'$-action. It tells us that we can get the deformation of coarse moduli scheme X_1/H from the deformation of the moduli stack $\mathfrak{M}_{\Gamma_1(p)}$. As it is mentioned in [CES], checking the regularity along the cusps can be done by using the Tate curve. We will have similar calculations using the Tate curve, therefore it is better to mention here what is the place of the Tate Curve in the Moduli scheme. The Tate curve $\bar{\mathcal{G}}_m/q^{\mathbf{Z}}$ over Spec $(\mathbb{Z}[[q]])$ together with the embedding $\mu_p \subset \mathcal{G}_m/q^{\mathbf{Z}}$ (see [DR, VII]) gives a morphism $\tau : \mathrm{Spec}\,(\mathbb{Z}[[q]]) \to X_H$. We call the support of the corresponding section Spec $(\mathbb{Z}) \to \mathrm{Spec}\,(\mathbb{Z}[[q]]) \to X_H$ the ∞ cusp. Over $\mathbb{C}$, provided we trivialize $\mu_p(\mathbb{C})$ via $\zeta_p = e^{2\pi i/p}$, this corresponds to the "usual" ∞ cusp and the parameter q to $e^{2\pi i z}$ with z in the upper half plane $\mathfrak{H}$. The morphism τ identifies Spec $(\mathbb{Z}[[q]])$ with the formal completion of X_H along ∞.

CHAPTER 2

Coarse Moduli Schemes X_1,X_0 and X_H

We will assume that $p \equiv 1 \bmod 24$, and hence using the genus formula [pg.23, Sh],

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2} \tag{2.1}$$

where $\mu = p + 1$ and $\nu_2 = \nu_3 = \nu_\infty = 2$, then we have g_0 of $(X_0)_{\mathbf{C}}$ is $(p-13)/12$. The following theorem is deduced from the works of Deligne and Rapoport [DR], Katz and Mazur [KM] and Conrad, Edixhoven and Stein in [CES] by Chinburg, Pappas and Taylor and available in [CPT1]. We directly barrow from them.

Theorem 2.1

a. The scheme $X_H \to \operatorname{Spec}(\mathbb{Z})$ is a flat projective curve, X_H is normal Cohen-Macaulay and $X_H[1/p] \to \operatorname{Spec}(\mathbb{Z}[1/p])$ is smooth (where $X_H[1/p] = X_H \otimes_{\mathbf{Z}} \mathbb{Z}[1/p]$). The special fiber of X_H over p has two irreducible components D_∞^H and D_0^H distinguished by the fact that D_∞^H intersects the cuspidal section ∞; these have multiplicities 1 and $(p-1)/(2 \cdot \#H)$ respectively.

b. The scheme X_H has at most two non-regular points which are rational singularities and lie on $D_0^H - (D_0^H \cap D_\infty^H)$. Their exact number depends on $\#H$ mod 6: In particular, if 6 divides $\#H$ then there are no such points and X_H is regular. In particular, when $H = \{1\}$ there are two non-regular points on X_1. There is a morphism $b : X_1' \to X_1$ which is a rational resolution of those two singular points and a morphism $c : X_1' \to \mathcal{X}_1$ which is a sequence of blow-downs of exceptional curves such that $\mathcal{X}_1$ is regular and all the geometric fibers of $\mathcal{X}_1 \to \mathrm{Spec}(\mathbb{Z})$ are integral. Let $U = X_1 - D_0^{\{1\}} \subset X_1$. Then $U \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth, b and c are isomorphisms on $b^{-1}(U)$ and $\mathcal{X}_1 - c(b^{-1}(U))$ has dimension 0.

c. The special fiber of X_0 over p is reduced with simple normal crossings. Each of the two irreducible components $D_\infty = D_\infty^\Gamma$ and $D_0 = D_0^\Gamma$ are isomorphic to $\mathbf{P}^1_{\mathbf{F}_p}$ and $D_0 \cdot D_\infty = g_0 + 1 = (p-1)/12$.

d. Assume that 6 divides the order $\#H$.

i. The morphism $\pi_H : X_H \to X_0$ is a tame $G = \Gamma/H$ cover of regular projective curves and $\pi_H[1/p] : X_H[1/p] \to X_0[1/p]$ is a G-torsor.

ii. The morphism π_H is totally ramified over the generic point of D_0, and unramified over the generic point of D_∞. The irreducible components D_0^H and D_∞^H of $X_H \otimes_{\mathbb{Z}} \mathbf{F}_p$ are the (reduced) inverse images of D_0 and D_∞ under π_H The character $\chi_{D_0^H}$ giving the action of G on the cotangent space of the codimension 1 generic point of D_0^H equals $\omega^{-2\cdot\#H}$, where $\omega : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbf{F}_p^*$ is the Teichmuller (identity) character.

$\square$

Proof: Check Theorem 4.2 and Theorem 4.3 in [CPT1].

CHAPTER 3

Lattices of cusp forms

If R is a subring of $\mathbb{C}$ we will denote by $S_{2k}(\Gamma_1(p), R)$ the R-module of cusp forms $F(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$ of weight $2k$ for the congruence subgroup $\Gamma_1(p) \subset \mathrm{PSL}_2(\mathbb{Z})$ whose Fourier coefficients a_n belong to R. (These are the Fourier coefficients "at the cusp ∞"). For simplicity, we will also denote "twisted modular forms" by $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}R)$ when its Fourier coefficients are in this form $\frac{a_n}{p^{\delta k}}$ where a_n in R. Particularly, if $\delta = 0$, we assume $S_{2k,0}(\Gamma_1(p), R) = S_{2k}(\Gamma_1(p), R)$.

Proposition 3.1 There are Γ-equivariant isomorphisms

$$S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}) \simeq \mathrm{H}^0(X_1, \omega_{X_1/\mathbb{Z}}^{\otimes k}(k\delta D_\infty^1)) \tag{3.1}$$

where the Γ-action on $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z})$ is via the diamond operators and $\omega_{X_1/\mathbb{Z}}(\delta D_\infty^1)$ denotes the twisted canonical (dualizing) sheaf of $X_1 \to \mathrm{Spec}\,(\mathbb{Z})$ along the divisor δD_∞^1. $\square$

Proof. Let $G(q) = \sum_{n\geq 1} \frac{a_n}{p^{\delta k}} q^n \in S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z})$ with $q = e^{2\pi i z}$ and consider $G(q)(dq/q)^{\otimes k}$ as a regular differential over $\mathrm{Spec}\,(\mathbb{Z}[[q]])$. A standard argument using the Kodaira-Spencer map shows that $G(q)(dq/q)^{\otimes k}$ extends to a regular differential over $X_1[1/p]$ (cf. [Ma, II §4]). This extension must also be regular in an open neighborhood of the section at ∞. Hence there is an open subset U' of the set $U \subset X_1$

defined in Theorem 2.1 (b) such that $G(q)(dq/q)^{\otimes k}$ is regular on U' and $U - U'$ is a finite set of closed points. We obtain an injective Γ-equivariant homomorphism

$$\Phi : S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}) \to \mathrm{H}^0(U', \omega_{U'/\mathbb{Z}}^{\otimes k}(k\delta D_\infty^1)). \tag{3.2}$$

The equalities,

$$\mathrm{H}^0(U', \omega_{U'/\mathbb{Z}}^{\otimes k}(k\delta D_\infty^1)) = \mathrm{H}^0(\mathcal{X}_1, \omega_{\mathcal{X}_1/\mathbb{Z}}^{\otimes k}(k\delta D_\infty^1)) = \mathrm{H}^0(X_1, \omega_{X_1/\mathbb{Z}}^{\otimes k}(k\delta D_\infty^1)), \tag{3.3}$$

follow from the fact that $b : X_1' \to X_1$ and $c : X_1' \to \mathcal{X}_1$ are rational morphisms which are isomorphisms on $b^{-1}(U')$ and that $\mathcal{X}_1 - c(b^{-1}(U'))$ has codimension 2 in $\mathcal{X}_1$ and the following lemma, which proves that 0'th cohomology of the k'th power of the dualizing sheaf is preserved under blow-up of two singular points on X_1 and blow-down of a -1 curve. The surjectivity of Φ follows from pulling back elements of $\mathrm{H}^0(X_1, \omega_{X_1/\mathbb{Z}}^{\otimes k}(k\delta D_\infty^1))$ via $\tau : \mathrm{Spec}\,(\mathbb{Z}[[q]]) \to U$ and using the Kodaira-Spencer isomorphism.

We keep the same notation as in Theorem 2.1.

Lemma 3.1 *Let $\tilde{X}_1 \xrightarrow{f} X_1$ be the blow-up of the surface X_1 at some point* Q. *Let $\omega_{\tilde{X}_1}(\delta D_\infty^1)$ and $\omega_{X_1}(\delta D_\infty^1)$ be the twisted dualizing sheaves respectively. Then,*

$$\mathrm{H}^0(\tilde{X}_1, \omega_{\tilde{X}_1}^{\otimes k}(k\delta \tilde{D}_\infty^1)) = \mathrm{H}^0(X_1, \omega_{X_1}^{\otimes k}(k\delta D_\infty^1)) \tag{3.4}$$

PROOF. As it is explained in [pg.380, CES], X_1 has two singular points corresponding to $j = 0$ and $j = 1728$ which we call them $\mathrm{Q_E}$ and $\mathrm{Q_F}$ respectively. When we blow-up $\mathrm{Q_E}$ we get one exceptional curve E such that $E^2 = -2$. Similarly, when we blow-up $\mathrm{Q_F}$ we get another exceptional curve F such that $F^2 = -3$. Both of these curves are isomorphic to $\mathbb{P}^1$.

CASE 1. ($j = 0$) and ($j = 1728$)

Since we will follow the same routine, we consider both cases together. First, we will show that Q_E and Q_F are rational singularities in the sense of Artin [pg.268, Ar]: Recall that the point Q is a rational singularity if the stalk of $R^1 f_* \mathcal{O}_{\tilde{X}_1}$ at Q is zero for every desingularization $\tilde{X}_1 \xrightarrow{f} X_1$. We will show that every singular point Q with multiplicity α (i.e. when Q is blown-up we get an exceptional curve C, isomorphic to $\mathbb{P}^1$, whose self intersection is $-\alpha < -1$) is a rational singularity.

Let $\hat{F^i} = \varprojlim \mathrm{H}^i(C_n, \mathcal{O}_{C_n})$ where C_n is closed subscheme of $\tilde{X}_1$ defined by $\mathcal{I}^n$ and $\mathcal{I}$ is the ideal sheaf of the exceptional curve C.

We have,

$$0 \rightarrow \mathcal{I}^n/\mathcal{I}^{n+1} \rightarrow \mathcal{O}_{C_{n+1}} \rightarrow \mathcal{O}_{C_n} \rightarrow 0 \tag{3.5}$$

We also know that $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_C(\alpha)$ then $\mathcal{I}^n/\mathcal{I}^{n+1} \simeq \mathrm{S}^n(\mathcal{I}/\mathcal{I}^2) \simeq \mathcal{O}_C(\alpha n)$.

Since C is just $\mathbb{P}^1$, $\mathrm{H}^i(C, \mathcal{O}_C(\alpha n)) = 0$ for $i > 0$ and $n > 0$. By writing the long exact sequence, we will get the following for $i > 0$

$$\mathrm{H}^i(C, \mathcal{O}_{C_{n+1}}) = \mathrm{H}^i(C, \mathcal{O}_{C_n}). \tag{3.6}$$

When $n = 1$, $\mathcal{O}_{C_1} = \mathcal{O}_C$, therefore $\mathrm{H}^i(C, \mathcal{O}_{C_n}) = 0$ for $i > 0$. So, $\hat{F^i} = 0$ for $i > 0$. Since it is supported just at Q, $F^i = 0$ for $i > 0$. So, when we take $\alpha = 2$ and $\alpha = 3$ we prove that Q_E and Q_F are rational singularities respectively. If the point Q with multiplicity $\alpha > 1$ is a rational singularity, then $\omega_{\tilde{X}_1} = f^* \omega_{X_1}$ by proposition 5.1 in [Ar].

Remark 3.2 Since Q_E is a double point singularity, the canonical sheaf of the X_1 is locally around Q_E a line bundle and we could get the equality $\omega_{\tilde{X}_1} = f^* \omega_{X_1}$ by a calculation which uses the adjunction formula. However, it is not true for the point Q_F; the canonical sheaf of X_1 is no longer a line bundle in a neighborhood of Q_F.

We need to show that

$$\mathrm{H}^0(\tilde{X}_1, f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))) \simeq \mathrm{H}^0(X_1, \omega_{X_1}^{\otimes k}(k\delta D_\infty^1)). \tag{3.7}$$

Using the projection formula, we get

$$R^i f_*(f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))) \simeq \omega_{X_1}^{\otimes k}(k\delta D_\infty^1) \otimes R^i f_* \mathcal{O}_{\tilde{X}}. \tag{3.8}$$

By taking $i = 0$, we get

$$R^0 f_*(f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))) \simeq \omega_{X_1}^{\otimes k}(k\delta D_\infty^1) \tag{3.9}$$

because, $\mathcal{O}_{X_1} = f_*\mathcal{O}_{\tilde{X}_1}$ which simply follows from the fact that X_1 is normal and f is birational. Now, using the Leray spectral sequence,

$$\mathrm{H}^i(X_1, R^j f_*(f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))) \Rightarrow \mathrm{H}^{i+j}(\tilde{X}_1, f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))) \tag{3.10}$$

and by choosing $i = j = 0$ we get the desired result.

CASE 2. ($j \neq 0, 1728$)

Let $\tilde{X}_1 \xrightarrow{f} X_1$ be the blow-up of the surface X_1 at a regular point Q. We have an exact sequence

$$0 \to \mathcal{O}_{\tilde{X}_1} \to \mathcal{O}_{\tilde{X}_1}(kE) \to \mathcal{O}_{kE}(kE) \to 0\ . \tag{3.11}$$

By tensoring the sequence by $f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))$, we obtain an exact sequence

$$\begin{aligned} 0 \to f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)) \otimes \mathcal{O}_{\tilde{X}_1} \to f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)) \otimes \mathcal{O}_{\tilde{X}_1}(kE) \to \\ \to f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)) \otimes \mathcal{O}_{kE}(kE) \to 0 \end{aligned} \tag{3.12}$$

Since

$$\omega_{\tilde{X}_1}(\delta D^1_\infty) \simeq f^*(\omega_{X_1}(\delta D^1_\infty))(E) \tag{3.13}$$

we have

$$\omega^{\otimes k}_{\tilde{X}_1}(k\delta D^1_\infty) \simeq f^*(\omega^{\otimes k}_{X_1}(k\delta D^1_\infty))(kE)\ . \tag{3.14}$$

By restricting to kE we get

$$\omega^{\otimes k}_{\tilde{X}_1}(k\delta D^1_\infty)|_{kE} \simeq f^*(\omega^{\otimes k}_{X_1}(k\delta D^1_\infty))(kE)|_{kE} \simeq \mathcal{O}_{kE}(kE) \tag{3.15}$$

Let's try to show that $\mathrm{H}^0(E, \mathcal{O}_{kE}(kE))$ is trivial. We have

$$0 \to \mathcal{I}^{k-1}/\mathcal{I}^k \to \mathcal{O}_{\tilde{X}_1}/\mathcal{I}^k \to \mathcal{O}_{\tilde{X}}/\mathcal{I}^{k-1} \to 0 \tag{3.16}$$

where $\mathcal{I}$ is the ideal sheaf of E. After twisting the sequence by the divisor (kE) and since E is just $\mathbb{P}^1$, $\mathcal{I}^{l-1}/\mathcal{I}^l \approx \mathcal{O}_E(l)$ therefore degree in the each summand becomes negative. By induction on k we obtain, $\mathrm{H}^0(\tilde{X}_1, \mathcal{O}_{kE}(kE))$ is trivial.

Using the above calculation on the following cohomology long exact sequence,

$$0 \to \mathrm{H}^0(\tilde{X}_1, f^*(\omega^{\otimes k}_{X_1}(k\delta D^1_\infty))) \to \mathrm{H}^0(\tilde{X}_1, \omega^{\otimes k}_{\tilde{X}_1}(k\delta D^1_\infty)) \to \mathrm{H}^0(E, \mathcal{O}_{kE}(kE)) \to \cdots \tag{3.17}$$

We conclude, $\mathrm{H}^0(\tilde{X}_1, f^*(\omega^{\otimes k}_{\tilde{X}_1}(k\delta D^1_\infty))) \simeq \mathrm{H}^0(\tilde{X}_1, \omega^{\otimes k}_{\tilde{X}_1}(k\delta D^1_\infty))$. Now, the only thing that we need to show is that, $\mathrm{H}^0(\tilde{X}_1, f^*(\omega^{\otimes k}_{X_1}(k\delta D^1_\infty))) \simeq \mathrm{H}^0(X_1, \omega^{\otimes k}_{X_1}(k\delta D^1_\infty))$.

Again, using the Projection Formula,

$$R^i f_*(f^*(\omega^{\otimes k}_{X_1}(k\delta D^1_\infty))) \simeq \omega^{\otimes k}_{X_1}(k\delta D^1_\infty) \otimes R^i f_* \mathcal{O}_{\tilde{X}_1}\ . \tag{3.18}$$

The standard argument from Hartshorne (pg. 387, Prop.3.4) gives $R^i f_* \mathcal{O}_{\tilde{X}_1} = 0$

if $i > 0$, and $R^0 f_* \mathcal{O}_{\tilde{X}_1} = \mathcal{O}_{X_1}$ for $i = 0$. Now, using the Leray spectral sequence,

$$\mathrm{H}^i(X_1, R^j f_*(f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))) \Rightarrow \mathrm{H}^{i+j}(\tilde{X}_1, f^*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))) \tag{3.19}$$

and by choosing $i = j = 0$ we get the desired result.

Proposition 3.1 gives us explicit relation between R-module of the "twisted" cusp forms of weight $2k$ and the global sections of the k'th power of the dualizing sheaf. If we follow the same argument as in the proof of the proposition, we can get the similar relation for cusp forms. This is given in the following:

Corollary 3.3 There are Γ-equivariant isomorphisms

$$S_{2k}(\Gamma_1(p), \mathbb{Z}) \simeq \mathrm{H}^0(X_1, \omega_{X_1/\mathbf{Z}}^{\otimes k}). \tag{3.20}$$

$\square$

CHAPTER 4

Galois structure of modular forms

Let's start this section by defining the module of the "twisted" modular forms of weight $2k$ on X_H which will be called as $S_{2k,\delta}(\Gamma_H(p), p^{-\delta k}\mathbb{Z})$. We can define it as follows:

$$S_{2k,\delta}(\Gamma_H(p), p^{-\delta k}\mathbb{Z}) := S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z})^H$$

We will try to calculate it here. Let $\mu : X_1 \to X_H$ be the quotient morphism. Since μ is a finite morphism, $R^j\mu_* = 0$ for $j > 0$. Now, using the Leray spectral sequence,

$$\mathrm{H}^i(X_H, R^j\mu_*((\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))) \Rightarrow \mathrm{H}^{i+j}(X_1, \omega_{X_1}^{\otimes k}(k\delta D_\infty^1)) \tag{4.1}$$

for $i = 0, j = 0$ we get:

$$\mathrm{H}^0(X_H, \mu_*\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)) \simeq \mathrm{H}^0(X_1, \omega_{X_1}^{\otimes k}(k\delta D_\infty^1)). \tag{4.2}$$

Therefore

$$S_{2k,\delta}(\Gamma_H(p), p^{-\delta k}\mathbb{Z}) = \mathrm{H}^0(X_1, \omega_{X_1}^{\otimes k}(k\delta D_\infty^1))^H \tag{4.3}$$

and

$$\mathrm{H}^0(X_1, \omega_{X_1}^{\otimes k}(k\delta D_\infty^1))^H \simeq \mathrm{H}^0(X_H, \mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq$$

$$\simeq \mathrm{H}^0(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H). \quad (4.4)$$

Since $\mu^*(\omega_{X_H}(\delta D_\infty^H))$ and $\omega_{X_1}(\delta D_\infty^1)$ are line bundles, one of them can be written as a twist of the other. This twist is supported along the ramification locus since the line bundles are isomorphic on the complement of the ramification locus. Now we can write

$$\omega_{X_1}(\delta D_\infty^1) \simeq \mu^*(\omega_{X_H}(\delta D_\infty^H))(R^1) \quad (4.5)$$

where R^1 is supported on the ramification locus. Also, when $p \equiv 1 \bmod 24$ then the ramification locus of the map $\pi : X_1 \to X_0$ is D_0^1, $j = 0$ and $j = 1728$. Their ramification degrees are $\frac{p-1}{2r}$, 2 and 3 respectively. A local calculation like in [pg.74 Ma], shows that $R^1 = (\frac{p-1}{2r})D_0^1 + \overline{\{j=0\}}^1 + 2\overline{\{j=1728\}}^1$ where $\overline{\{j=0\}}^1$ and $\overline{\{j=1728\}}^1$ are closure of the generic point of each lines $j = 0$ and $j = 1728$ on X_1. When we take the k-th power of the sheaves we get

$$\omega_{X_1}^{\otimes k}(k\delta D_\infty^1) \simeq \mu^*(\omega_{X_H}^{\otimes k}(k\delta D_\infty^H))(kR^1)\ . \quad (4.6)$$

After taking the H-invariants, we obtain

$$(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))^H \simeq (\mu^*(\omega_{X_H}^{\otimes k}(k\delta D_\infty^H)(kR^1))^H\ . \quad (4.7)$$

We know that

$$(\mu^*\omega_{X_H}^{\otimes k}(k\delta D_\infty^H))^H := (\omega_{X_H}^{\otimes k}(k\delta D_\infty^H) \otimes_{O_{X_H}} \mathcal{O}_{X_1})^H \quad (4.8)$$

$$(\omega_{X_H}^{\otimes k}(k\delta D_\infty^H) \otimes_{\mathcal{O}_{X_H}} \mathcal{O}_{X_1})^H = \omega_{X_H}^{\otimes k}(k\delta D_\infty^H) \otimes_{\mathcal{O}_{X_H}} \mathcal{O}_{X_1}^H = \omega_{X_H}^{\otimes k}(k\delta D_\infty^H)\ . \tag{4.9}$$

Therefore,

$$(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq \omega_{X_H}^{\otimes k}(k\delta D_\infty^H) \otimes_{\mathcal{O}_{X_H}} (\mu_*\mathcal{O}_{X_1}(kR^1))^H \tag{4.10}$$

Notice that $\mathcal{O}_{X_1}(kR^1)$ is allowing poles of at most order k times along $\overline{\{j=0\}}^1$, at most $2k$ along $\overline{\{j=1728\}}^1$ and $k(\frac{p-1-2r}{2r})$ along D_0^1. The group H acts on the sheaf. By writing the sheaf as a direct sum of the eigenspaces with respect to the different characters of H we can see that it is a direct sum of an invertible sheaf and its powers. When we take H-invariants, only the powers that correspond to multiples of ramification degree remain. This happens when $2, 3$, or $\frac{p-1}{2r}$ divides k. Therefore, we can write explicitly,

$$\begin{aligned}&(\mu_*\mathcal{O}_{X_1}(kR^1))^H = \\ &\quad = \mathcal{O}_{X_H}\left(\left[\frac{k(p-1-2r)}{(p-1)}\right] D_0^H + \left[\frac{k}{2}\right]\overline{\{j=0\}}^H + \left[\frac{2k}{3}\right]\overline{\{j=1728\}}^H\right)\end{aligned} \tag{4.11}$$

where $[t]$ means the rational number t is rounded to the next smaller integer number. Thus,

$$\begin{aligned}&(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq \\ &\quad \simeq \omega_{X_H}^{\otimes k}\left(k\delta D_\infty^H + \left[\frac{k(p-1-2r)}{(p-1)}\right] D_0^H + \left[\frac{k}{2}\right]\overline{\{j=0\}}^H + \left[\frac{2k}{3}\right]\overline{\{j=1728\}}^H\right).\end{aligned} \tag{4.12}$$

Now, we will state a key proposition that allows us to understand how we can relate the lattice of modular forms and $\mathrm{Cl}(\mathbb{Z}[\zeta_r])$. This is based on results of [Ri] (see [CPT1]).

Proposition 4.1 Let $\chi : \Gamma \to \mathbb{Z}[\zeta_r]^*$ be a 1-dimensional character of prime order $r \geq 5$ with kernel H. Let $G = \Gamma/H$ and suppose M is a finitely generated torsion-free $\mathbb{Z}[G]$-module. Define M^χ to be the $\mathbb{Z}[\zeta_r]$-module $(M \otimes \mathbb{Z}[\zeta_r]\chi^{-1})^G$.

a. There is a unique homomorphism $e'_\chi : \mathrm{G}_0(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[\zeta_r])$ such that for all M as above, either $M^\chi = \{0\}$ and $e'_\chi([M]) = 0$ or M^χ is isomorphic to $\mathbb{Z}[\zeta_r]^s \bigoplus \mathfrak{U}$ for some integer $s \geq 0$ and a $\mathbb{Z}[\zeta_r]$-ideal $\mathfrak{U}$ in the ideal class $e'_\chi([M])$.

b. There is a unique isomorphism $t_\chi : \mathrm{K}_0(\mathbb{Z}[G]) \to \mathbb{Z} \bigoplus \mathrm{Cl}(\mathbb{Z}[\zeta_r])$ such that $t_\chi([P]) = (\mathrm{rank}_{\mathbb{Z}[G]}(P), e_\chi(\overline{[P]}))$ if P is a projective $\mathbb{Z}[G]$-module, where $\overline{[P]}$ is the image of P in $\mathrm{Cl}(\mathbb{Z}[G])$ and $e_\chi : \mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[\zeta_r])$ is the unique homomorphism such that $e_\chi(\overline{[P]}) = e'_\chi(f([P]))$ for all projective P, where $f : \mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{G}_0(\mathbb{Z}[G])$ is the forgetful homomorphism.

$\square$

CHAPTER 5

Proof of Main Theorem

We now compute the image of $\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ under the injective homomorphism $\Theta = \Theta_3 : \mathrm{Cl}(\mathbb{Z}[G]) \to C_{\mathbb{Z}}(G, 3)$, which is defined in [CPT1], by applying their main result and using the isomorphism (4.11). This result allows us to calculate the equivariant Euler characteristic of a sheaf if there is a tame cover and if the sheaf can be written as a pullback from the quotient. In our case, let $\pi_H : X_H \to X_0$ be our cover. Since the index of H in Γ is the prime $r \geq 5$, the order of H is divisible by 6. By Theorem (2.1) π_H is ramified only at the fiber over p. By (4.11) we already get the sheaf $(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H$ in terms of $\omega_{X_H}^{\otimes k}(k\delta D_\infty^H)$ twisted by a certain divisor. Also, $\omega_{X_H}^{\otimes k}(k\delta D_\infty^H)$ can be written as pull back of the sheaf $\omega_{X_0}^{\otimes k}(k\delta D_\infty)$ with some twist. Therefore, $(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H$ can be written as pull back of the sheaf $\omega_{X_0}^{\otimes k}(k\delta D_\infty)$ with some twist. Let's try to see the relation between them.

Making the similar local calculations in [pg.74 Ma], we can say

$$\omega_{X_H}^{\otimes k}(k\delta D_\infty^H) = \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + k(r-1)D_0^H).$$

And hence,

$$(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H). \tag{5.1}$$

Here, we denote by R^H the divisor $\left[\frac{k}{2}\right] \overline{\{j=0\}}^H + \left[\frac{2k}{3}\right] \overline{\{j=1728\}}^H$ on X_H , which can be identified as pull back of the divisor $R^0 = \left[\frac{k}{2}\right] \overline{\{j=0\}}^0 + \left[\frac{2k}{3}\right] \overline{\{j=1728\}}^0$ on X_0. We also note $\eta = k(r-1) + \left[\frac{k(p-1-2r)}{(p-1)}\right]$.

We have this fundamental sequence ;

$$0 \to \mathcal{O}_{X_H}(-\eta D_0^H) \to \mathcal{O}_{X_H} \to \mathcal{O}_{\eta D_0^H} \to 0 \tag{5.2}$$

and tensoring by $\pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H)$ we get,

$$0 \to \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H) \to \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H) \to \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H)|_{\eta D_0^H} \to 0. \tag{5.3}$$

Since $(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H)$, then our sequence becomes,

$$0 \to \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^1 + R^H) \to (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \to \mathcal{C} \to 0 \tag{5.4}$$

where $\mathcal{C} = \pi_H^*(\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H))|_{\eta D_0^H}$ is also supported on D_0^H.

The sequence implies the following relation between equivariant Euler characteristics,

$$\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) = \overline{\chi}^P(X_H, \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H)) + \overline{\chi}^P(X_H, \mathcal{C}) \tag{5.5}$$

On the right hand side, the first equivariant Euler characteristic will be calculated easily using the main theorem of [CPT1], and the second one will be calculated using the adjunction formula as follows:

We have the following exact sequence ;

$$0 \to I^{n-1}/I^n \to \mathcal{O}_{X_H}/I^n \to \mathcal{O}_{X_H}/I^{n-1} \to 0 \tag{5.6}$$

where I is the ideal sheaf of D_0^H. By induction on η and tensoring each sequence by

$\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H)$ we get the following sum for the equivariant Euler characteristic of $\mathcal{C}$

$$\overline{\chi}^P(X_H, \mathcal{C}) = \sum_{q=0}^{\eta-1} \overline{\chi}^P(X_H, [\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H) \otimes (I^q/I^{q+1})]|_{D_0^H}). \tag{5.7}$$

Let us denote by $N_{D_0^H}^\vee$, the conormal bundle of D_0^H. Then (5.7) gives

$$\overline{\chi}^P(X_H, \mathcal{C}) = \sum_{q=0}^{\eta-1} \overline{\chi}^P(X_H, [\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H) \otimes ({N_{D_0^H}^\vee}^{\otimes q})]|_{D_0^H}). \tag{5.8}$$

Let us revisit the calculation of the Euler characteristic $\overline{\chi}^P(X_H, \mathcal{C})$. Since the group G acts trivially on D_0^H, the Euler characteristic $\overline{\chi}^P(X_H, \mathcal{C})$ is just the numerical Euler characteristic times the class of the character of $[\chi_0]$ of the group G. Here the class of a character $[\chi_0]$ is defined as follows: As we know $[\chi_0]$ is a homomorphism from G to field $\mathbb{Q}$. We can extend this homomorphism to $\mathbb{Z}[G]$ the kernel of this homomorphism, which is an ideal in $\mathbb{Z}[G]$ gives a class in $Cl(\mathbb{Z}[G])$. This class is referred as a class of a character. The numerical Euler characteristic of the sheaf $\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H) \otimes ({N_{D_0^H}^\vee}^{\otimes q})]|_{D_0^H}$ on D_0^H for any q can be calculated by using Riemann-Roch. We obtain

$$\begin{aligned}\chi^P(X_H, [\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^1 + R^H) \otimes ({N_{D_0^H}^\vee}^{\otimes q})]|_{D_0^H}) = \\ = deg(\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H) \otimes ({N_{D_0^H}^\vee}^{\otimes q})|_{D_0^H}) + 1 - g(X_H)\end{aligned} \tag{5.9}$$

where degree of the sheaf in the formula above can be calculated by adjunction formula: we get,

$$
\begin{aligned}
deg(\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H) \otimes ({N_{D_0^H}^\vee}^{\otimes q})|_{D_0^H}) = \\
= -\frac{kr(p-1)}{12} + 2k - \frac{k(r-1)(p-1)}{12r} + \\
+ \frac{kr\delta(p-1)}{12} + (\left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right])r - q\,\eta\frac{(p-1)}{12}.
\end{aligned}
\tag{5.10}
$$

So, we can write the Euler characteristic as,

$$
\overline{\chi}^P(X_H, \mathcal{C}) = \sum_{q=0}^{\eta-1}(m(k,r)q + n(k,\delta,r))[\chi_0^q] \tag{5.11}
$$

for integers

$$
m(k,r) = -\eta\frac{(p-1)}{12} \tag{5.12}
$$

and

$$
n(k,\delta,r) = -\frac{kr(p-1)}{12} + 2k - \frac{k(r-1)(p-1)}{12r} + \frac{kr\delta(p-1)}{12} + (\left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right])r \tag{5.13}
$$

depending on k,δ and r.

Now let's turn to the calculation of $\overline{\chi}^P(X_H, \pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H))$. Since we will use the main result of [CPT1] for this calculation, it is better to recall the theorem. For the convenience of the reader we will report the main theorem and some of the arguments for our calculation from [CPT1].

5.0.1 Main Theorem of [CPT1]

Let R be the ring of integers of a number field K. Let $\pi : X \to Y$ be a G-cover, which is *tame*, i.e. for every closed point $x \in X$, order of inertia subgroup $I_x \subset G$ is relatively prime to the charactetistic of the residue field $k(x)$, and also *domestic*, i.e the residue field characteristic of each point of Y which ramifies in $\pi : X \to Y$ is relatively prime to the order of the group G. Denote by S the finite set of rational primes such that the cover $\pi : X \to Y$ is only ramified at points above S. By our assumption assumption, $p \in S$ implies $p \not| \#G$. Denote by S_K the set of places of K that lie above S. Suppose $\mathcal{G}$ is a locally free coherent $\mathcal{O}_Y$-sheaf on Y and consider the G-sheaf $\mathcal{F} = \pi^*\mathcal{G}$ on X. Consider the injective homomorphism

$$\Theta = \Theta_{d+2} : \mathrm{Cl}(R[G]) = \mathrm{Pic}(R[G]) \to C_R(G; d+2)$$

where $C_R(G; d+2)$ is the isomophism classes of the objects which has cubic structure. For a finite place v of K, we denote by ϖ_v a uniformizer of the completion R_v and fix an algebraic closure $\bar{K}_v$ of its fraction field K_v. Also any finite idele $(a_v)_v \in \mathbf{A}^*_{f,K[G^{d+2}]}$ gives the element $(\cap_v(R_v[G^{d+2}]a_v \cap K[G^{d+2}]), 1)$ of $C_R(G; d+2)$. Now let $v \in S_K$ (then $(v, \#G) = 1$) and denote by R'_v the complete discrete valuation ring $R'_v \subset \bar{K}_v$ obtained by adjoining to R_v a primitive root of unity of order equal to $\#G$. Then ϖ_v is also a uniformizer for R'_v.Let us consider the cover $\pi'_v : X \otimes_R R'_v \to Y \otimes_R R'_v$ obtained from π by base change. Since R'_v has residue field characteristic prime to $\#G$ and contains a primitive $\#G$-th root of unity. There is an isomorphism

$$K_v[G^{d+2}]^* \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\bar{K}_v/K_v)}(\mathrm{Ch}(G^{d+2})_v, \bar{K}_v^*) \tag{5.14}$$

given by evaluating characters of G^{d+2}. Also if $\phi = \phi_1 \otimes \cdots \otimes \phi_{d+2}$ is a character of G^{d+2} given by a $d+2$-tuple $(\phi_i)_i$ of 1-dimensional $\bar{K}_v$-valued characters of G, we have $\Theta^D(\phi) = (\phi_1 - 1) \cdots (\phi_{d+2} - 1) \in \mathrm{Ch}(G^{d+2})_v$. The function $T_{v,\mathcal{G}}$ defined on R-valued characters is defined in [CPT1] by an explicit formula and its definition implies that, for all $\sigma \in \mathrm{Gal}(\bar{K}_v/K_v)$ and $\psi \in \mathrm{Ch}(G)_v$, we have

$$T_{v,\mathcal{G}}(\psi) = T_{v,\mathcal{G}}(\psi^\sigma).$$

Hence, the map $\phi \mapsto \varpi_v^{-T_{v,\mathcal{G}}(\Theta(\phi))}$ gives a function in $\mathrm{Hom}_{\mathrm{Gal}(\bar{K}_v/K_v)}(\mathrm{Ch}(G^{d+2})_v, \bar{K}_v^*)$.

Theorem 5.1 With the above assumptions and notations,

$$\Theta(2 \cdot \bar{\chi}^P(X, \mathcal{F})) = (\cap_v (R_v[G^{d+2}]\lambda_v \cap K[G^{d+2}]), 1),$$

where $(\lambda_v)_v \in \mathbf{A}^*_{f,K[G^{d+2}]}$ is the (unique) finite idele which is such that

$$\phi(\lambda_v) = \begin{cases} 1, & \text{if } v \notin S_K \ ; \\ \varpi_v^{-2 \cdot T_{v,\mathcal{G}}(\Theta^D(\phi))}, & \text{if } v \in S_K \ , \end{cases} \tag{5.15}$$

for all $\bar{K}_v$-valued characters ϕ of G^{d+2}.

If the "usual" Euler characteristic $\chi(Y, \mathcal{G}) = \sum_i (-1)^i \mathrm{rank}_R[\mathrm{H}^i(Y, \mathcal{G})]$ is *even*, then we can eliminate both occurrences of the factor 2 from the statement: $\Theta(\bar{\chi}^P(X, \mathcal{F}))$ is then given by the idele $(\lambda'_v)_v$ with $\phi(\lambda'_v) = 1$ if $v \notin S_K$, $\phi(\lambda'_v) = \varpi_v^{-T_{v,\mathcal{G}}(\Theta^D(\phi))}$ if $v \in S_K$. □

The field $\mathbb{Q}_p$ already contains a primitive $p-1$-st root of unity. Hence, we may take $R'_{(p)} = \mathbb{Z}_p$. We find that $\Theta(\overline{\chi}^P(X_H, \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H))) \in C_{\mathbf{Z}}(G; 3)$ is given by the idele $(b_v)_v \in \mathbf{A}^*_{f,\mathbb{Q}[G^3]}$ which is 1 at all places $v \neq (p)$ and is such that

$$(\chi \otimes \phi \otimes \psi)(b_{(p)}) = p^{-T((\chi-1)(\phi-1)(\psi-1))} \tag{5.16}$$

with $T : \mathrm{Ch}(G)_p \to \mathbb{Q}$ the function associated to the cover $X_H \otimes_{\mathbf{Z}} \mathbb{Z}_p \to X_0 \otimes_{\mathbf{Z}} \mathbb{Z}_p$ in the main theorem . For $a \in \mathbb{Z}/r\mathbb{Z}$ let $\{a\}$ be the unique integer in the range $0 \leq \{a\} < r$ having residue class a. Using the eqn (3.15) in [CPT1], T becomes

$$T(\psi) = \frac{p-1}{12} \cdot \left(-\frac{g(\psi, D_0)^2}{2} + (1 - 2k(1+\delta)) \frac{g(\psi, D_0)}{2} \right) + (1 - 2k(1+\delta)) g(\psi, D_0) \tag{5.17}$$

$$= \frac{p-1}{12}\cdot\left(-\frac{\{a\}_r^2}{2r^2} - (1-2k(1+\delta))\frac{\{a\}_r}{2r}\right) + (1-2k(1+\delta))\frac{\{a\}_r}{r}. \tag{5.18}$$

where $\psi = \chi_0^{-a}$, $\chi_0 = \omega^{\frac{(p-1)}{r}}$ and $\omega : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}_p^*$ is the Teichmuller character.

For $a \in \mathbb{Z}/r\mathbb{Z}$ define $\omega_r(a) = 0$ if $a = 0$, and otherwise let $\omega_r(a) \in \mathbb{Z}_r \subset \hat{\mathbb{Z}}$ be the Teichmuller character associated to r. Define

$$T_1(\psi) = -\frac{p-1}{12}\left(\frac{\omega_r(a)^2}{2r^2}\right) \quad \text{and} \quad T_2(\psi) = (1-2k(1+\delta))\frac{\{a\}_r}{r} \tag{5.19}$$

where $\psi = \chi_0^{-a}$ as above. We extend $\psi \to T_i(\psi)$ to a function on the character ring $\mathrm{Ch}(G)_p$ by additivity. Since $p \equiv 1 \bmod 24$ and $r|\frac{1-p}{24}$, we can define $\beta = (\beta_v)_v$ with $\beta_v \in \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_v[G]^*$ by

$$\psi(\beta_v) = \begin{cases} 1, & \text{if } v \neq (p); \\ p^{-T(\psi)+T_1(\psi)+T_2(\psi)}, & \text{if } v = (p). \end{cases} \tag{5.20}$$

Since $\mathrm{Cl}(\mathbb{Z}[G])$ is a torsion group, β defines a unique class $[\beta]$ in $\mathrm{Cl}(\mathbb{Z}[G])$.

We now show

$$\overline{\chi}^P(X_H, \pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H)) = [\beta]. \tag{5.21}$$

Define $D = [\beta] - \overline{\chi}^P(X_H, \pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H))$, and let $R = \hat{\mathbb{Z}}$ if $i = 1$ and $R = \mathbb{Z}$ if $i = 2$. From (5.19) one has $rT_i(\psi) \in R$ and $rT_i(\psi) \equiv a \bmod rR$. It follows that for all triples (χ, ϕ, ψ) elements of $Ch(G)_p$, $T_i(\chi\phi\psi - \chi\phi - \phi\psi - \chi\psi + \chi + \phi + \psi - 1)$ lies in R. Hence there are elements $a_i = (a_{i,v})_v \in \prod_{v \text{ finite}}(R \otimes_{\mathbb{Z}} \mathbb{Q}_v[G^3]^*)$ for which

$$(\chi \otimes \phi \otimes \psi)(a_{i,v}) = \begin{cases} 1, & \text{if } v \neq (p); \\ p^{T_i(\chi\phi\psi - \chi\phi - \phi\psi - \chi\psi + \chi + \phi + \psi - 1)}, & \text{if } v = (p). \end{cases} \tag{5.22}$$

We now conclude from (5.16), (5.17) and (5.20) that

$$1 \otimes \Theta(D) = c_1 + c_2 \tag{5.23}$$

in $\hat{\mathbb{Z}} \otimes_{\mathbf{Z}} C_{\mathbf{Z}}(G;3)$, where c_i is the class associated to the element a_i.

Let us first show

$$c_2 = 0. \tag{5.24}$$

For this it will suffice to show that there is a cubic element $\lambda \in \mathbb{Q}[G^3]^*$ such that λa_2 is a unit idele of $\mathbb{Q}[G^3]$. Fix a primitive p-th root of unity $\zeta_p \in \overline{\mathbb{Q}}^*$, and let

$$\tau(\psi) = \sum_{j \in (\mathbb{Z}/p)^*} \psi(j)\zeta_p^j$$

be the usual Gauss sum associated to ψ. Let τ be the unique extension of the map $\psi \to \tau(\psi)$ to a homomorphism from R_G to $\overline{\mathbb{Q}}^*$. We let $\tau^{(3)}$ be the element of $\mathrm{Hom}(R_{G^3}, \overline{\mathbb{Q}}^*)$ which sends (χ, ϕ, ψ) to

$$\tau(\chi\phi\psi - \chi\phi - \phi\psi - \chi\psi + \chi + \phi + \psi - 1)$$

From the behavior of Gauss sums under automorphisms of $\overline{\mathbb{Q}}$, and the factorization of the ideals they generate (c.f. [La, §IV.3]), it follows that $\tau^{(3)}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, and corresponds to an element $\lambda \in \mathbb{Q}[G^3]^*$ of the required kind. This shows (5.24).

Turning now to c_1, let $\sigma(s)$ be the automorphism of G which sends $g \in G$ to g^s for $s \in (\mathbb{Z}/r\mathbb{Z})^*$. By (CPT1) the action of $\mathrm{Aut}(G)$ on $\mathbb{Z}_p[G]^*$ corresponds to the action of $\mathrm{Aut}(G)$ on $f \in \mathrm{Hom}(\mathrm{Ch}(G)_p, \mathbb{Q}_p^*)$ defined by $(\sigma(s)(f))(\chi) = f(\sigma(s)^{-1}(\chi)) = f(\chi^s)$ for $\chi \in \mathrm{Ch}(G)_p$. From the definition of the T_1 in (5.19) and the multiplicativity of the Teichmuller character we have $(\sigma(s)T_1)(\psi) = \omega_r(s)^2 T_1(\psi)$. It follows that the element $\alpha(s) = \sigma(s) - \omega_r(s)^2$ sends a_1 to the identity function, so

$$\alpha(s)c_1 = 0 \tag{5.25}$$

Because Θ is $\mathrm{Aut}(G)$-equivariant, we can now conclude from (5.23), (5.24) and (5.25) that

$$1 \otimes (\Theta(\alpha(s) \cdot D)) = 0 \quad \text{in} \quad \hat{\mathbb{Z}} \otimes_{\mathbf{Z}} C_{\mathbf{Z}}(G; 3). \tag{5.26}$$

If $A \to B$ is an injection of abelian groups, and A is finite, then $A = \hat{\mathbb{Z}} \otimes_{\mathbf{Z}} A \to \hat{\mathbb{Z}} \otimes_{\mathbf{Z}} B$ is injective, as one sees by reducing to the case in which $A \to B$ is the inclusion $n^{-1}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ for some $n \geq 1$. So (5.26) and the injectivity of Θ implies

$$\alpha(s) \cdot D = 0 \quad \text{in} \quad \mathrm{Cl}(\mathbb{Z}[G]). \tag{5.27}$$

Similarly, since $r^2 T_1(\psi)$ is in $\mathbb{Z}_r \subset \hat{\mathbb{Z}}$ and $\mathrm{Cl}(\mathbb{Z}[G])$ is a torsion group, we see from the injectivity of Θ that D is in the r-Sylow subgroup of $\mathrm{Cl}(\mathbb{Z}[G])$.

We now use the fact that $\mathrm{Cl}(\mathbb{Z}[G])$ is isomorphic to $\mathrm{Cl}(\mathbb{Z}[\zeta_r])$. Define C_j to be the group of classes c in the r-Sylow subgroup of $\mathrm{Cl}(\mathbb{Q}(\zeta_r))$ for which $(\sigma(s) - \omega_r(s)^j)(c) = 0$ for all $s \in (\mathbb{Z}/r)^*$. We have shown a_1 corresponds to a class in C_2. By the Spiegelungsatz (c.f. [Wa, Theorem 10.9]), $C_2 = 0$ if $C_{-1} = 0$. Herbrand's Theorem ([Wa, Thm. 6.17]) shows that if $C_{-1} \neq 0$, then the Bernoulli number $B_{r-(r-2)} = B_2$ is congruent to 0 mod $r\mathbf{Z}_r$. This is impossible since $B_2 = \frac{1}{6}$ and $r \geq 5$, so we have shown 5.21.

To complete the proof of our theorem, we will find $-e_\chi(\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H))$ after calculating $-e_\chi(\overline{\chi}^P(X_H, \mathcal{C}))$ and adding with our result 5.21. By choosing a suitable element of $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ to apply, we can reduce to the case in which

$$\chi = \chi_0 = \omega^{\frac{(p-1)}{r}}. \tag{5.28}$$

With these conventions from the definition of e_χ in Proposition 4.1 we have $-e_\chi([\chi_0]) = [\mathcal{P}_\chi]$ and hence

$$-e_\chi([\chi_0^a]) = [\sigma_a^{-1}(\mathcal{P}_\chi)], \text{ if } (a,r) = 1 \ , \ e_\chi([\chi_0^a]) = 0 \text{ otherwise.} \tag{5.29}$$

So, when we apply $-e_\chi$ to $\overline{\chi}^P(X_H, \mathcal{C})$ we will get;

$$\begin{aligned} -e_\chi(\overline{\chi}^P(X_H,\mathcal{C})) &= \sum_{q=0}^{\eta-1}(m(k,r)q + n(k,\delta,r))\sigma_q^{-1}\cdot[\mathcal{P}_{\chi_0}] = \\ &= \sum_{q=0}^{\eta-1} m(k,r)q\,\sigma_q^{-1}\cdot[\mathcal{P}_{\chi_0}] + \sum_{q=0}^{\eta-1} n(k,\delta,r)\sigma_q^{-1}\cdot[\mathcal{P}_{\chi_0}] \end{aligned} \tag{5.30}$$

From the definition of all terms, we get the following equality,

$$-e_\chi(\overline{\chi}^P(X_H,(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)) = \theta_2\cdot[\mathcal{P}_{\chi_0}]\frac{(1-p)(2k-1)}{24r} - \theta_1\cdot[\mathcal{P}_{\chi_0}] - [\theta_1]\cdot[\mathcal{P}_{\chi_0}] - [\theta_0]\cdot[\mathcal{P}_{\chi_0}]$$

where

$$\theta_1 = \sum_{a\in(\mathbb{Z}/r)^*}\{a\}\sigma_a^{-1} \in \mathbb{Z}[\Delta], \tag{5.31}$$

$$[\theta_1] = \sum_{0<q\le kr-1+\left[\frac{-2kr}{(p-1)}\right],(q,r)=1} m(k,r)q\sigma_q^{-1} \tag{5.32}$$

and

$$[\theta_0] = \sum_{0<q\leq kr-1+\left[\frac{-2kr}{(p-1)}\right],(q,r)=1} n(k,\delta,r)\sigma_q^{-1}. \qquad (5.33)$$

Again, by Stickelberger's Theorem, θ_1 annihilates $\mathrm{Cl}(\mathbb{Z}[\zeta_r])$, so the proof is complete.

5.0.2 Lower bound for δ

In this part we try to find a lower bound on δ that allows us to calculate the Galois structure of lattice of twisted cusp forms explicitly. In our main calculation, we calculated the equivariant Euler characteristic for any value of δ, which is namely,

$$\overline{\chi}^P(X_H,(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) = [H^0(X_H,(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1))))] - [H^1(X_H,\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))] \qquad (5.34)$$

in $\mathrm{Cl}[\mathbb{Z}(G)]$. If we arrange so that the first cohomology group vanishes then, we obtain a precise formula for the twisted cusp forms. Details are given in the following corollary.

Corollary 5.1 *There is δ_0 such that for every $\delta > \delta_0$, we have the following. Suppose $\mathfrak{A} \subset \mathbb{Z}[\zeta_r]$ is an ideal with ideal class $\theta_2 \cdot [\mathcal{P}_{\chi_0}] - [\theta_1] \cdot [\mathcal{P}_{\chi_0}] - [\theta_0] \cdot [\mathcal{P}_{\chi_0}]$. Then,*

$$S_{2k,\delta}(\Gamma_1(p),\mathbb{Z}[\zeta_r])_\chi \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{A} \qquad (5.35)$$

as $\mathbb{Z}[\zeta_r]$-modules.

Proof. With the notations of the theorem, recall that $S_{2k,\delta}(\Gamma_1(p),p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi$ is the $\mathbb{Z}[\zeta_r]$-submodule of $S_{2k,\delta}(\Gamma_1(p),p^{-\delta k}\mathbb{Z}[\zeta_r])$ consisting of twisted cusp forms of weight $2k$ and of Nebentypus character χ whose n'th Fourier coefficients at ∞ are in the form of $\frac{r}{p^{\delta k}}$ where r in $\mathbb{Z}[\zeta_r]$. Proposition 3.1 and its proof together with the fact that formation of the canonical sheaf commutes with the base change $\mathbb{Z} \to \mathbb{Z}[\zeta_r]$ implies that $S_{2k,\delta}(\Gamma_1(p), \mathbb{Z}[\zeta_r]) \simeq S_{2k,\delta}(\Gamma_1(p), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_r]$. Propositions 3.1 and ?? now give an isomorphism of (torsion free) $\mathbb{Z}[\zeta_r]$-modules

$$S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi \simeq \mathrm{H}^0(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\chi.$$

The projective class $\chi^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) \in \mathrm{K}_0(\mathbb{Z}[G])$ has the property that

$$\begin{aligned} f(\chi^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)) &= \\ &= [\mathrm{H}^0(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)] - [\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)] \quad (5.36) \end{aligned}$$

where $f : \mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{G}_0(\mathbb{Z}[G])$ is the forgetful homomorphism. If P is a projective $\mathbb{Z}[G]$-module, then $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is a free $\mathbb{Q}[G]$-module, so $\mathrm{rank}_{\mathbb{Z}[G]}(P) = \mathrm{rank}_{\mathbb{Z}}(P)/r$. Therefore, using Riemann-Roch we get

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}[\zeta_r]}\mathrm{H}^0(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) - \mathrm{rank}_{\mathbb{Z}[\zeta_r]}\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) &= \\ = \frac{(2k-1)(g(X_H)-1)}{r} + \left[\frac{k(p-1-2r)}{(p-1)}\right] + \left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right] \quad (5.37) \end{aligned}$$

where $g(X_H)$ is the genus of X_H.

Because the generic fiber of $X_H \to X_0$ is étale of degree r, by the Hurwitz Theorem, we can say $(g(X_H)-1)/r = g(X_0) - 1$ and we know that $g(X_0) = \frac{(p-13)}{12}$ hence,

$$n(\chi) = \frac{(2k-1)(p-25)}{12} + \left[\frac{k(p-1-2r)}{(p-1)}\right] + \left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right]. \quad (5.38)$$

Let $\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ be the image of $\chi^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ in $\mathrm{Cl}(\mathbb{Z}[G])$. If we prove that $\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ vanishes when $\delta > \delta_0$ for some δ_0, we can easily conclude from (5.36) and 4.1 that there is an isomorphism of $\mathbb{Z}[\zeta_r]$-modules

$$S_{2k,\delta}(\Gamma_1(p), \mathbb{Z}[\zeta_r])_\chi \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \bigoplus \mathfrak{U} \tag{5.39}$$

where $\mathfrak{U}$ is a $\mathbb{Z}[\zeta_r]$-ideal having ideal class $-e_\chi(\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H))$.

The only thing left is to prove first cohomology group is trivial when $\delta > \delta_0$ for some δ_0. This is done in following lemma.

Lemma 5.1 $\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ *is trivial when* $\delta > \delta_0$ *for some* δ_0.

PROOF. If we can show that $\mathrm{H}^1(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee)$ is torsion free (which is necessary condition for duality), then the result will follow by duality as follows:

$$\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}^0(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee), \mathbb{Z}) \tag{5.40}$$

Here $\mathrm{H}^0(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee)$ is trivial because of the degree of the sheaf is negative.

To show that $\mathrm{H}^1(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee)$ is torsion free it is enough to check that $\mathrm{H}^0((X_H)_\beta, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)_\beta^\vee)$ it is trivial on each fiber β. If the fiber $\beta \neq p$, then it is just $\mathbb{P}^1$ and degree of the sheaf is negative implies result. Otherwise ($\beta = p$), we have two component namely, D_0^H and D_∞^H one of them is totally ramified and the other one is unramified. Let s be a global section of our sheaf, then its restriction to D_∞^H is zero since D_∞^H is reduced. Let's call W for the non-reduced component. So, $W^{red} = D_0^H$ and

$$O_W = O_{D_0^H} \oplus N \oplus N^{\otimes 2} \oplus \cdots \oplus N^{\otimes r-1} \tag{5.41}$$

where $N = O_{X_H}(-D_0^H)|_{D_0^H}$.

Recall the following equation,

$$(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq \pi_H^* \omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H) \tag{5.42}$$

.

Therefore, s is given by r-tuple of sections s_i of the sheaf

$$N^{\otimes i}((1-k)K_H \cdot D_0^H - k\delta D_\infty^H \cdot D_0^H - \eta D_0^H \cdot D_0^H - R^H \cdot D_0^H) \tag{5.43}$$

We know that

$$N^{\otimes r} = O_{X_H}(-rD_0^H)|_{D_0^H} = O_{X_H}(D_\infty^H)|_{D_0^H} \tag{5.44}$$

then

$$deg(N^{\otimes r}) = D_0^H \cdot D_\infty^H = \frac{(p-1)}{12} \tag{5.45}$$

which implies

$$deg(N) = \frac{p-1}{12r} \tag{5.46}$$

We also know that

$$K_H \cdot (D_\infty^H + rD_0^H) = 0 \Rightarrow K_H \cdot D_\infty^H = -rK_H \cdot D_0^H \tag{5.47}$$

Adjunction formula gives,

$$(K_H + D_\infty^H) \cdot D_\infty^H = 2g_{D_\infty^H} - 2 \tag{5.48}$$

and Hurwitz formula gives,

$$2g_{D_\infty^H} - 2 = r(2g_{D_\infty} - 2) + \frac{(r-1)(p-1)}{12} \tag{5.49}$$

both together imply that

$$K_H \cdot D_\infty^H = -D_\infty^H \cdot D_\infty^H - 2r + \frac{(r-1)(p-1)}{12} \tag{5.50}$$

Also,

$$D^H_\infty \cdot (D^H_\infty + rD^H_0) = 0 \Rightarrow D^H_\infty \cdot D^H_\infty = -rD^H_0 \cdot D^H_\infty = \frac{-r(p-1)}{12} \tag{5.51}$$

So,

$$K_H \cdot D^H_\infty = \frac{r(p-1)}{12} + \frac{(r-1)(p-1)}{12} - 2r \tag{5.52}$$

and

$$K_H \cdot D^H_0 = -\frac{(p-1)}{12} - \frac{(r-1)(p-1)}{12r} + 2 \tag{5.53}$$

If we plug all these into the degree calculation of our sheaf we get;

$$\begin{aligned}\frac{i(p-1)}{12r} - \frac{k\delta(p-1)}{12} + (1-k)(-\frac{(p-1)}{12} - \frac{(r-1)(p-1)}{12r} + 2) + \\ + \left\lceil \frac{k(p-1-2r)}{(p-1)} \right\rceil (\frac{p-1}{12r}) - \left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{2k}{3} \right\rceil \\ \leq \frac{r(p-1)}{12} - \frac{k\delta(p-1)}{12} + (2k-1)\frac{(p-1)}{12} + \frac{(p-1)}{12r} + 2 - \frac{10k}{3}\end{aligned} \tag{5.54}$$

We want to find a lower bound to δ_0 which guarantees that this term is negative, we say

$$\frac{r(p-1)}{12} - \frac{k\delta(p-1)}{12} + (2k-1)\frac{(p-1)}{12} + \frac{(p-1)}{12r} + 2 - \frac{10k}{3} \leq 0 \tag{5.55}$$

$$\delta \geq 2 + \frac{1}{k}(r - 1 + \frac{1}{r} + \frac{24}{(p-1)}) - \frac{40}{(p-1)} \tag{5.56}$$

and remember that $r > 3$ and $24r$ divides $p-1$, therefore $r+2$ is going to be enough for δ_0.

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