IMAGES OF CERTAIN MANIFOLDS UNDER MAPPINGS OF DEGREE ONE

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ABSTRACT

IMAGES OF CERTAIN MANIFOLDS
UNDER MAPPINGS OF DEGREE ONE

By

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This thesis considers two distinct problems. In the second chapter a classification of the images of certain products of spheres under mappings of degree one is obtained. The principal results are the following theorems.

Theorem 2.1: Let $f: S^m \times S^{n-m} \to M$ be a mapping of degree one into a closed, connected, orientable n-manifold M. Then either M has the homotopy type of S^n, or f is a homotopy equivalence.

Theorem 2.3: Let M be a closed, connected, orientable 3-manifold and $f: S^1 \times S^1 \times S^1 \to M$ a mapping of degree one. Then either f is a homotopy equivalence, or M has the homotopy type of $S^1 \times S^2$ or S^3.

In the third chapter $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are characterized in the class $\mathfrak{M}$ of closed, connected 3-manifolds M having the property that each connected, finite-sheeted covering space over M is homeomorphic to M. Both $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are members of this class with non-zero fundamental groups; whether there are other such 3-manifolds remains unanswered. But $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ can be shown to be the only

members of $\mathfrak{M}$ satisfying certain additional conditions.

Theorem 3.1: Let M be a member of the class $\mathfrak{M}$ having a non-zero, nilpotent fundamental group. Then M is homeomorphic to $S^1 \times S^1 \times S^1$ or to $S^1 \times S^2$.

Theorem 3.6: Let M be a member of the class $\mathfrak{M}$ such that each double covering of M is proper. (A double covering $p: \tilde{M} \to M$ is said to be proper if the non-trivial covering transformation over p is homotopic to $1_{\tilde{M}}$.) If $H_1(M)$ is infinite, then M is homeomorphic to $S^1 \times S^1 \times S^1$ or to $S^1 \times S^2$.

IMAGES OF CERTAIN MANIFOLDS
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INTRODUCTION

This thesis considers two distinct problems. In chapter II a classification by homotopy type is obtained for those closed, connected, orientable n-manifolds M which admit a degree one mapping $S^m \times S^{n-m} \to M$. In the third chapter $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are characterized in the class $\mathfrak{M}$ of closed, connected 3-manifolds M having the property that each finite-sheeted covering space over M is homeomorphic to M.

There is a well-known result which states: For $n \geq 5$, any closed, orientable n-manifold M admitting a degree one map $S^n \to M$ is homeomorphic to S^n. The principal theorem of the second chapter generalizes this result to the case in which the domain consists of a product of two spheres; of necessity, the classification is by homotopy type rather than homeomorphism. Two additional results are obtained by taking as domain certain other products of spheres.

In [5] Kyung W. Kwun asks which closed, connected, orientable 3-mainfolds admit double coverings (or proper double coverings) of themselves. More recently, Jeffrey L. Tollefson has proved [12, Theorem 2] that a closed, connected, orientable 3-manifold properly covers itself k times for every prime k if and only if it is the product of a 2-manifold and S^1. The class $\mathfrak{M}$ described above consists of those closed, connected 3-manifolds which admit no finite-sheeted coverings other than

coverings of themselves. This class is examined in chapter III, and $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are shown to be the only manifolds in this class which satisfy certain additional conditions.

CHAPTER I

THE DEGREE OF A MAP

In this chapter two definitions of the degree of a proper mapping between n-manifolds will be given. These definitions will then be used to obtain two theorems (Theorem 1.1 and Theorem 1.4) which will be applied frequently in the next chapter.

The n^{th} sheaf cohomology group of X with compact supports will be denoted by $H_c^n(X; \mathcal{Z})$, where $\mathcal{Z}$ is the constant sheaf on X with stalk Z, the infinite cyclic group. For a connected, orientable n-dimensional manifold M, $H_c^n(M; \mathcal{Z}) = Z$. Each such manifold will be assumed to have a preferred free generator $\mu_M \in H_c^n(M; \mathcal{Z})$.

The (algebraic) degree of a mapping is defined for proper maps between connected, orientable n-manifolds. (A mapping is proper if the inverse image of each compact subset of the range is a compact subset of the domain.) If $f: (M, \partial M) \to (N, \partial N)$ is such a mapping, then the degree of f is the integer denoted by $\deg(f)$ which satisfies the equality $f^*(\mu_N) = \deg(f)\mu_M$.

This purely algebraic definition of degree has no geometric interpretation. In order to recognize the geometric significance of the degree of a map, it is necessary to introduce an alternate definition of degree. The geometric degree $G(f)$ of a proper mapping $f: (M, \partial M) \to (N, \partial N)$ between n-manifolds is defined to be infinite unless there exists an n-disk D in the interior of N such that $f^{-1}(D)$ is the union of a finite number of disjoint n-disks each mapped homeomorphically onto D under f. When such disks do exist, G(f) is defined to be the minimal number of components in the inverse image of each such disk.

The two definitions of degree are related by the inequality $|\deg(f)| \leq G(f)$, which is obvious if G(f) is infinite. If G(f) is a positive integer k, then there is an n-disk D in the interior of N such that $f^{-1}(D) = D_1 \cup D_2 \cup \cdots \cup D_k$ is the union of k disks each mapped homeomorphically onto D under f. Let ϵ_i equal 1 or -1 according to whether $f: D_i \to D$ is orientation preserving or orientation reversing. Then $\deg(f) = \Sigma_{i=1}^{k} \epsilon_i$ [3, Lemma 2.1b], so that $|\deg(f)| \leq G(f)$ in this case also.

The existence of a mapping $f: (M, \partial M) \to (N, \partial N)$ of degree one between two connected, orientable n-manifolds has important implications for the algebraic invariants of the manifolds, as the following fundamental result shows.

<u>Theorem</u> 1.1: If $f: (M, \partial M) \to (N, \partial N)$ is a proper mapping of degree one between connected, orientable n-manifolds, then

(a.) the induced mapping of fundamental groups $f_{\#}: \pi_1(M) \to \pi_1(N)$ is epimorphic;

(b.) the induced mapping of homology groups $f_*: H_*(M, \partial M) \to H_*(N, \partial N)$ is a split epimorphism.

Proof of (a.): Let $p\colon \hat{N} \to N$ denote the covering space of N corresponding to the subgroup $f_{\#}(\pi_1(M))$ of $\pi_1(N)$. Then f can be lifted to a map $\hat{f}\colon M \to \hat{N}$ which is necessarily proper (because f is proper). Now

$$1 = \deg(f) = \deg(p\hat{f}) = \deg(p)\deg(\hat{f}),$$

and therefore $\deg(p) = \pm 1$. Thus $\hat{N} = N$ [3, §2], and hence $f_{\#}(\pi_1(M)) = \pi_1(N)$.

Proof of (b.): Since the Borel-Moore homology groups of n-manifolds coincide with the corresponding singular homology groups [1, V. 12.6], there is a commutative diagram

$$\begin{array}{ccc} H_c^q(M) & \xleftarrow{\quad f^* \quad} & H_c^q(N) \\ \downarrow & & \downarrow \\ H_{n-q}(M,\partial M) & \xrightarrow{\quad f_* \quad} & H_{n-q}(N,\partial N) \end{array}$$

in which the vertical maps are the Poincaré Duality isomorphisms induced by the cap product [1, V. 9.4 and V. 10.2]. That f_* is a split epimorphism follows from the cap product rule $f_*(\alpha \cap f^*(\beta)) = f_*(\alpha) \cap \beta$, which implies the commutativity of the diagram.

The homotopy results used in chapter II often require that the manifolds under consideration be simply connected. So when $f\colon (M, \partial M) \to (N, \partial N)$ is a mapping of degree one between manifolds which are not simply connected, it will be necessary to pass to the universal covering spaces of M and N. Thus it is important to know conditions under which f will induce a

mapping of degree one on the universal covering spaces of M and N. One such situation is described in Theorem 1.4, the proof of which requires two lemmas.

<u>Lemma</u> 1.2: Let $f: X \to Y$ be a continuous function which induces an epimorphism $f_{\#}: \pi_1(X) \to \pi_1(Y)$ of fundamental groups. If $p: \tilde{Y} \to Y$ is a fibration with unique path lifting such that $\tilde{Y}$ is path-connected, then P, the fibered product of f and p, is also path-connected.

Proof: In the diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

let $\tilde{f}$ and $\tilde{p}$ denote the maps induced by f and p, respectively. Then $\tilde{p}$ is a fibration with unique path lifting [10, 2.8.6].

In order to show that P is path-connected, it suffices to prove that there is a path between any two points of $\tilde{p}^{-1}(x)$ for an arbitrary $x \in X$; so let $(x,y_i) \in \tilde{p}^{-1}(x)$ $(i = 0,1)$. Since $\tilde{Y}$ is path-connected, there is a path $\omega: I \to \tilde{Y}$ from y_o to y_1. Now $p(y_i) = f(x)$ for $i = 0,1$, so that $[p\omega] \in \pi_1(Y,f(x))$. Because $f_{\#}$ is epimorphic, there is a loop $\lambda: I \to X$ based at x such that $f\lambda \simeq p\omega$ rel $\{0,1\}$. Let $\tilde{\lambda}: I \to P$ be a lifting of λ such that $\tilde{\lambda}(0) = (x,y_o)$. Now

$$p_{\#}[\tilde{f}\tilde{\lambda}] = [p\tilde{f}\tilde{\lambda}] = [f\tilde{p}\tilde{\lambda}] = [f\lambda] = [p\omega] = p_{\#}[\omega].$$

But since $p_{\#}\colon \pi_1(\tilde{Y}) \to \pi_1(Y)$ is a monomorphism [10, 2.3.4], $\tilde{f}\tilde{\lambda} \simeq \omega$ rel $\{0,1\}$. In particular, $\tilde{f}\tilde{\lambda}(1) = \omega(1) = y_1$. So $\tilde{\lambda}(1) = (x,y_1)$, and $\tilde{\lambda}$ is the required path.

<u>Lemma</u> 1.3: In the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & W \end{array}$$

let f and g be continuous maps of Hausdorff spaces, P be the fibered product of f and g, and $\tilde{f}$ and $\tilde{g}$ be the maps induced by f and g, respectively. If f is a proper map, then $\tilde{f}$ is also a proper map.

Proof: Recall that $P = \{(x,y) \in X \times Y\colon f(x) = g(y)\}$ and that $\tilde{f}$ and $\tilde{g}$ are defined by $\tilde{f}(x,y) = y$ and $\tilde{g}(x,y) = x$. If K is a compact subset of Y, then $\tilde{f}^{-1}(K)$ is a closed subset of $P \cap (f^{-1}(gK) \times K)$. Because W is a Hausdorff space, P is a closed subset of $X \times Y$ [2, V11.1.5]. Moreover, $f^{-1}(gK) \times K$ is compact since f is proper and g is continuous. Thus $\tilde{f}^{-1}(K)$ is a closed subset of a compact set in the Hausdorff space $X \times Y$ and hence is compact.

<u>Theorem</u> 1.4: Let M and N be compact, connected, orientable n-manifolds, and let $f\colon M \to N$ be a mapping of degree one which induces a monomorphism $f_{\#}\colon \pi_1(M) \to \pi_1(N)$ of fundamental groups. If $q\colon \tilde{N} \to N$ is the universal covering space of N and P is the fibered product of f and q, then:

(a.) The induced covering projection $p: P \to M$ is the universal covering space of M;

(b.) If $G(f) = 1$, then any map $\tilde{f}: P \to \tilde{N}$ induced by f has geometric degree one;

(c.) If $G(f) \neq 1$, there is a proper map $\tilde{f}: P \to \tilde{N}$ of degree one such that $q\tilde{f} = fp$.

Proof of (a.): There is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & \tilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N \end{array}$$

in which both vertical maps are covering projections [10, 2.8.6]. Since $f_{\#}p_{\#}: \pi_1(P) \to \pi_1(N)$ factors through $\pi_1(\tilde{N}) = 0$, $f_{\#}p_{\#}$ is the zero homomorphism. But both $p_{\#}$ and $f_{\#}$ are monomorphisms; so $\pi_1(P) = 0$. Because P is path-connected by Lemma 1.2, P is a simply connected covering space of M, and hence, the universal covering space of M [10, 2.5.7].

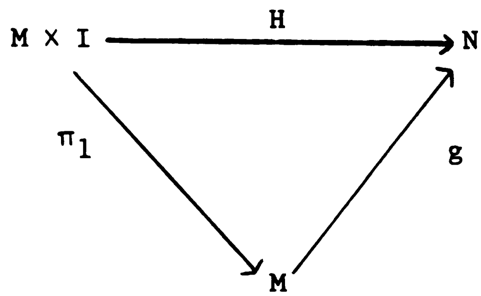
Proof of (b.): If $G(f) = 1$, there exists an n-disk D_1 in the interior of N such that $f^{-1}(D_1)$ is homeomorphic to D_1 under f. Choose an n-disk $D_2 \subset D_1$ so that D_2 lies in some open subset of N which is evenly covered by q and so that $f^{-1}(D_2)$ lies in some open subset of M which is evenly covered by p. Let D be any component of $q^{-1}(D_2)$; then q maps D homeomorphically onto D_2.

Since $\tilde{f}$ is a proper map (Lemma 1.3), $\tilde{f}^{-1}(D)$ is compact. Now $\tilde{f}^{-1}(D) \subset p^{-1}f^{-1}(D_2)$, and therefore $\tilde{f}^{-1}(D)$ is the union of a finite number of disjoint disks, each of which is mapped homeomorphically onto D by $\tilde{f}$. If $(x_i, y_i) \in \tilde{f}^{-1}(D)$ $(i = 1,2)$ lie in the same fiber, then $y_1 = \tilde{f}(x_1,y_1) = \tilde{f}(x_2,y_2) = y_2$ and hence $f(x_1) = p(y_1) = p(y_2) = f(x_2)$. But since $x_1, x_2 \in f^{-1}(D_2)$, it follows that $x_1 = x_2$. So each fiber of $\tilde{f}$ contains a single point, and therefore $G(\tilde{f}) = 1$.

Proof of (c.): There is a map $g: M \to N$ homotopic to f and having geometric degree one [3, Theorem 4.1]. If P_g denotes the fibered product of g and q, then the fibration $p': P_g \to M$ induced by q is fiber homotopy equivalent to $p: P \to M$ [10, 2.8.14], and the fiber homotopy equivalences between p and p' are easily seen to be homeomorphisms. Hence P and P_g may both be identified with the universal covering space $\tilde{M}$ of M by part (a.). Denote the covering projection $\tilde{M} \to M$ by π, and let $H: M \times I \to N$ be a homotopy from g to f. Let $\tilde{g}: \tilde{M} \to \tilde{N}$ denote any mapping induced by g; then $G(\tilde{g}) = 1$ by part (b.).

The homotopy lifting property guarantees the existence of a map $\tilde{H}: \tilde{M} \times I \to \tilde{N}$ such that $\tilde{H}(x,o) = \tilde{g}(x)$ and $q\tilde{H}(x,t) = H(\pi \times 1_I)(x,t)$ for all $x \in \tilde{M}$, $t \in I$. The desired map $\tilde{f}: \tilde{M} \to \tilde{N}$ is defined by $\tilde{f}(x) = \tilde{H}(x,1)$. In order to prove that $\tilde{f}$ is a proper map of degree one, it suffices to show that $\tilde{H}$ is a proper map (for the degrees of properly homotopic maps are equal). Since $\pi_1: M \times I \to M$, the projection onto the first factor, is a homotopy equivalence, the homotopy commutative

diagram



shows that H satisfies the hypotheses of part (a.). Thus $\tilde{M} \times I$ is the fibered product of H and q, and $\tilde{H}$ is proper by Lemma 1.3.

CHAPTER II

IMAGES OF CERTAIN MANIFOLDS UNDER MAPPINGS OF DEGREE ONE

In this chapter those manifolds which admit a mapping of degree one from certain products of spheres will be classified. The principal result (Theorem 2.1) gives a classification of such manifolds by homotopy type for the case in which the domain is a cartesian product of two spheres. This theorem generalizes the fact that for $n \geq 5$ the only n-manifold M which admits a mapping $S^n \to M$ of degree one is S^n itself. (This result follows immediately from Theorem 1.1 and the n-dimensional Poincaré Conjecture.) A similar result is obtained in Theorem 2.3 when the domain is $S^1 \times S^1 \times S^1$, and Theorem 2.2 proves that there are no mappings of degree one from the n-dimensional torus T^n (the product of n copies of S^1) into an n-manifold with fundamental group equal to $\bigoplus_{i=1}^{n-1} Z$.

<u>Theorem</u> 2.1: Let M be a closed, connected, orientable n-manifold and $f: S^m \times S^{n-m} \to M$ $(1 \leq m \leq n-m)$ a mapping of degree one. Then either M has the homotopy type of S^n, or f is a homotopy equivalence.

Proof: Since $\pi_1(M)$ is abelian (Theorem 1.1 (a.)), $\pi_1(M) = H_1(M)$ is a direct summand of $H_1(S^m \times S^{n-m}) = \pi_1(S^m \times S^{n-m})$ by Theorem 1.1 (b.). Thus, because the infinite cyclic group is indecomposable, $\pi_1(M)$ is a free abelian group.

If $n = 2$, so that $m = n-1 = 1$, the conclusion follows easily from the classification theorem for closed, connected 2-manifolds [6, 1.5.1]. In fact, M must be homeomorphic to either S^2 or $S^1 \times S^1$. Therefore it will be assumed that $n \geq 3$.

Suppose first that $m = 1$. Since $\pi_1(S^1 \times S^{n-1}) = Z$, either $\pi_1(M) = 0$ or $\pi_1(M) = Z$.

Consider first the case that $\pi_1(M) = 0$. In this case $H_1(M) = 0$, and $H_{n-1}(M) = H^1(M) = 0$ by Poincaré Duality and the universal-coefficient theorem for cohomology [10, 5.5.3]. Thus, by Theorem 1.1 (b.), $H_k(M)$ is trivial except for $k = 0$ or $k = n$, in which case it is infinite cyclic. The absolute Hurewicz isomorphism theorem [10, 7.5.5] then implies that the Hurewicz homomorphism $\varphi: \pi_n(M) \to H_n(M)$ is an isomorphism. Let $\mu_M \in H_n(M)$ and $\nu_n \in H_n(S^n)$ be the preferred generators, and select a map $g: S^n \to M$ representing the class $\varphi^{-1}(\mu_M)$. The definition of φ shows that $\mu_M = \varphi[g] = g_*(\nu_n)$; hence g is a mapping of degree one. So $g_*: H_*(S^n) \to H_*(M)$ is epimorphic by Theorem 1.1. Since every epimorphic endomorphism of the infinite cyclic group is an isomorphism, g_* is actually an isomorphism. It follows that g is a weak homotopy equivalence [10, 7.6.25] and hence, a homotopy equivalence [10, 7.6.24].

Now assume that $\pi_1(M) = Z$. As above, Theorem 1.1 implies that $f_\#: \pi_1(S^1 \times S^{n-1}) \to \pi_1(M)$ is an isomorphism. Thus, in order to prove that f is a homotopy equivalence, it suffices to show that $f_\#: \pi_k(S^1 \times S^{n-1}) \to \pi_k(M)$ is an isomorphism for

$k \geq 2$. It follows from Theorem 1.4 that there is a commutative diagram

$$\begin{array}{ccc} R \times S^{n-1} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S^1 \times S^{n-1} & \xrightarrow{f} & M \end{array}$$

in which $q: \tilde{M} \to M$ is the universal covering space of M and $\tilde{f}$ is a proper mapping of degree one. If $H_{n-1}(\tilde{M}) \neq Z$, then $H_{n-1}(\tilde{M}) = 0$ (Theorem 1.1). So the absolute Hurewicz isomorphism theorem implies that $\tilde{M}$ is contractible and thus implies that M is a space of type $(Z,1)$ [10, 7.2.11]. But then M is homotopically equivalent to S^1 [15, 2.10.4], contradicting that $H_n(M) = Z$. Therefore $H_{n-1}(\tilde{M}) = Z$, and $\tilde{f}_*: H_*(R \times S^{n-1}) \to H_*(\tilde{M})$ is an isomorphism. As before, it follows that $\tilde{f}_\#: \pi_k(R \times S^{n-1}) \to \pi_k(\tilde{M})$ is an isomorphism for $k \geq 2$, and so $f_\#: \pi_k(S^1 \times S^{n-1}) \to \pi_k(M)$ is an isomorphism for $k \geq 2$ [10, 7.2.11]. This completes the proof of the case that $m = 1$.

For $m \geq 2$ $S^m \times S^{n-m}$ is simply connected, and therefore M is simply connected. Since the only non-trivial homology groups of $S^m \times S^{n-m}$ occur in dimensions 0, m, $n-m$, and n, $H_k(M) = 0$ except possibly for $k = 0$, $k = m$, $k = n-m$, and $k = n$. Moreover, Poincaré Duality implies that $H_o(M) = H_n(M) = Z$ and that $H_m(M) = H_{n-m}(M)$. Because $H_m(S^m \times S^{n-m}) = Z$ if $m \neq n-m$, either $H_m(M) = 0$ or $H_m(M) = Z$ if $m \neq n-m$.

If $H_m(M) = 0$, then the Hurewicz homomorphism $\varphi: \pi_n(M) \to H_n(M)$ is again an isomorphism. As before, any representative of the

class $\varphi^{-1}(\mu_M)$ is a mapping of degree one from S^n to M, and such a map is necessarily a homotopy equivalence.

When $H_m(M) = Z$, $f_*: H_*(S^m \times S^{n-m}) \to H_*(M)$ is an isomorphism. Hence $f: S^m \times S^{n-m} \to M$ is a homotopy equivalence.

Suppose now that $m = n-m$. Since $H_m(S^m \times S^m) = Z \oplus Z$, it follows that $H_m(M) = 0$, $H_m(M) = Z$, or $H_m(M) = Z \oplus Z$. When $H_m(M) = 0$ or $H_m(M) = Z \oplus Z$, the preceding arguments prove that M has the homotopy type of S^n or that f is a homotopy equivalence, respectively. So it remains to show only that $H_m(M) = Z$ is impossible. Assume that $H_m(M) = Z$, and choose a generator $\alpha \in H^m(M) = H_m(M)$. Poincaré Duality gives $\langle\alpha, \alpha \cap \mu_M\rangle = \pm 1$, where μ_M is the preferred generator of $H_n(M)$. If $\epsilon: H_o(M) \to Z$ is the augmentation, then $\langle\alpha \cup \alpha, \mu_M\rangle = \epsilon((\alpha \cup \alpha) \cap \mu_M) = \epsilon(\alpha \cap (\alpha \cap \mu_M)) = \langle\alpha, \alpha \cap \mu_M\rangle = \pm 1$. Hence $\alpha \cup \alpha$ generates $H^{2m}(M)$, and therefore $f^*(\alpha \cup \alpha)$ generates $H^{2m}(S^m \times S^m)$. If β denotes a generator of $H^o(S^m)$ and γ denotes a generator of $H^m(S^m)$, then $u_1 = \beta \times \gamma$ and $u_2 = \gamma \times \beta$ generate $H^m(S^m \times S^m)$. So there are integers a and b with $f^*(\alpha) = au_1 + bu_2$. But then $f^*(\alpha \cup \alpha) = 0$ if m is odd [4, 24.8] and $f^*(\alpha \cup \alpha) = (f^*\alpha) \cup (f^*\alpha) = 2ab(u_1 \cup u_2)$ if m is even, contradicting that $f^*(\alpha \cup \alpha)$ generates $H^{2m}(S^m \times S^m)$. This argument proves that $H_m(M) \neq Z$ and completes the proof of the theorem.

Both of the possibilities mentioned in the conclusion of Theorem 2.1 can actually occur. The map which collapses the equators of S^m and S^{n-m} to a point is a mapping

$\varkappa\colon S^m \times S^{n-m} \to S^n$ of geometric degree one, for any disk D in $S^m \times S^{n-m}$ which is disjoint from both equators is the complete inverse image of $\varkappa(D)$. And clearly the identity map of $S^m \times S^{n-m}$ is also a mapping with geometric degree one.

The proof of Theorem 2.1 shows that when M has the homotopy type of S^n, then M admits a mapping $S^n \to M$ of degree one. Thus for $n \geq 5$ the theorem actually proves either that M is homeomorphic to S^n or that f is a homotopy equivalence.

Theorem 2.3 is the analogue of the preceding result when the domain of f is $S^1 \times S^1 \times S^1$. The proof will depend upon the fact that if $f\colon S^1 \times S^1 \times S^1 \to M$ is a mapping of degree one, then $\pi_1(M) \neq Z \oplus Z$. A similar result is true for n-manifolds, as the following theorem shows.

<u>Theorem</u> 2.2: Let M be a closed, connected, orientable n-manifold with $\pi_1(M) = \overset{n-1}{\underset{i=1}{\oplus}} Z$. Then there exists no mapping $f\colon T^n \to M$ of degree one.

Proof: Assume that $f\colon T^n \to M$ is a mapping of degree one. Since the Kernel of $f_{\#}\colon \pi_1(T^n) \to \pi_1(M)$ is Z, the covering space of T^n corresponding to the Kernel of $f_{\#}$ is homeomorphic to $R^{n-1} \times S^1$. Let $p\colon R^{n-1} \times S^1 \to T^n$ denote this covering space, and let $q\colon \tilde{M} \to M$ be the universal covering space of M. An argument similar to that used in the proof of Theorem 1.4 (c.) gives a commutative diagram

$$\begin{array}{ccc} R^{n-1} \times S^1 & \xrightarrow{\tilde{f}} & \tilde{M} \\ \Big\downarrow p & & \Big\downarrow q \\ T^n & \xrightarrow{f} & M \end{array}$$

in which $\tilde{f}$ is a mapping of degree one. Thus Theorem 1.1 implies that $\tilde{f}_*: H_*(R^{n-1} \times S^1) \to H_*(\tilde{M})$ is an epimorphism. So $\tilde{M}$ is homologically trivial and hence contractible. But then [10, 7.2.11] implies that M is a space of type $(\bigoplus_{i=1}^{n-1} Z, 1)$ - an impossibility.

This chapter concludes with the previously mentioned analogue of Theorem 2.1.

<u>Theorem</u> 2.3: Let M be a closed, connected, orientable 3-manifold and $f: S^1 \times S^1 \times S^1 \to M$ a mapping of degree one. Then one of the following is true:

(a.) M has the homotopy type of S^3;

(b.) M has the homotopy type of $S^1 \times S^2$;

(c.) f is a homotopy equivalence.

Proof: It follows from Theorem 1.1 that $H_1(M) = 0$, $H_1(M) = Z$, $H_1(M) = Z \oplus Z$, or $H_1(M) = Z \oplus Z \oplus Z$; thus, in view of Theorem 2.2, either $\pi_1(M) = 0$, $\pi_1(M) = Z$, or $\pi_1(M) = Z \oplus Z \oplus Z$. Moreover, $H_1(M) = H_2(M)$ by Poincaré Duality and the universal coefficient theorem for cohomology.

If $\pi_1(M) = 0$, then the Hurewicz homomorphism $\varphi: \pi_3(M) \to H_3(M)$ is an isomorphism. As before, any representative of the class $\varphi^{-1}(\mu_M)$, where μ_M is the preferred generator of

$H_3(M)$, is a homotopy equivalence between M and S^3.

When $\pi_1(M) = Z$, then M is homotopically equivalent to a prime manifold, for M has a decomposition as a connected sum $P_1 \# P_2 \# \cdots \# P_k$ of prime manifolds [7, Theorem 1], and $\pi_1(P_1 \# P_2 \# \cdots \# P_k) = \pi_1(P_1) * \pi_1(P_2) * \cdots * \pi_1(P_k)$. Hence M is homotopically equivalent to $S^1 \times S^2$ or to an irreducible 3-manifold [7, Lemma 1]. If the latter were true, then the universal covering space of M would be contractible. But then M would be a space of type $(Z,1)$ and would be homotopically equivalent to S^1. This contradiction shows that M is homotopically equivalent to $S^1 \times S^2$.

Finally, if $\pi_1(M) = Z \oplus Z \oplus Z$, then f induces a mapping of degree one $\tilde{f}: R^3 \to \tilde{M}$ on the universal covering spaces of $S^1 \times S^1 \times S^1$ and M. Thus $\tilde{M}$ is contractible, and M is a space of type $(Z \oplus Z \oplus Z, 1)$. Hence there are isomorphisms $f_{\#}: \pi_k(S^1 \times S^1 \times S^1) \to \pi_k(M)$ for all k, and therefore f is a homotopy equivalence.

Here, as in Theorem 2.1, the three possibilities in the conclusion of the theorem actually occur, for two copies of S^1 can be collapsed to give S^2, and the resulting $S^1 \times S^2$ can be collapsed as before to give S^3.

CHAPTER III

A CHARACTERIZATION OF $S^1 \times S^1 \times S^1$ AND $S^1 \times S^2$

Kyung W. Kwun [5] and Jeffrey L. Tollefson [12, 13] have investigated conditions under which a 3-manifold admits a finite-sheeted covering of itself. In this chapter a study will be made of those closed, connected 3-manifolds M with the property that every connected finite-sheeted covering space over M is homeomorphic to M. The class of such 3-manifolds will be denoted by $\mathfrak{M}$.

Notice first that any closed, simply connected 3-manifold trivially belongs to $\mathfrak{M}$, for a simply connected manifold has no non-trivial, connected, finite-sheeted coverings. In fact, the fundamental group of a manifold in $\mathfrak{M}$ must be either zero or infinite, else the universal covering space is homeomorphic to the manifold. Moreover, since every manifold has an orientable double covering, each manifold in $\mathfrak{M}$ is orientable.

The only 3-manifold which is not prime and admits a k-fold covering $(k \geq 2)$ of itself is $P^3 \# P^3$, the connected sum of two copies of projective 3-space [13, Theorem 1]. Hence any manifold in $\mathfrak{M}$ with non-zero fundamental group is a prime manifold (since $S^1 \times S^2$ double covers $P^3 \# P^3$) and thus is homeomorphic to $S^1 \times S^2$ or is irreducible. Furthermore, if $M \in \mathfrak{M}$ is irreducible and $H_1(M)$ is infinite, then [5, Theorem 2] shows that M fibers over S^1 since the hypothesis that

$H_1(M)$ contain no element of order 2 is required only for the proof of [5, Proposition 5.1], where the assumption that $H_1(M)$ is infinite is clearly sufficient.

Both $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are members of $\mathfrak{M}$ having non-zero fundamental groups. Whether other such 3-manifolds exist remains unanswered. In this chapter two theorems will be obtained which characterize $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ in the class $\mathfrak{M}$. The first of these appears below.

<u>Theorem</u> 3.1: Let M be a manifold in the class $\mathfrak{M}$ with non-zero, nilpotent fundamental group. Then M is homeomorphic either to $S^1 \times S^1 \times S^1$ or to $S^1 \times S^2$.

Proof: If M is not homeomorphic to $S^1 \times S^2$, then M is irreducible. Since $\pi_1(M)$ must be infinite, the universal covering space of M is non-compact and hence, contractible. Thus $\pi_k(M) = 0$ for $k \geq 2$, and $\pi_1(M)$ has no elements of finite order. It follows from [11, Theorem N] that $\pi_1(M) = Z$, $\pi_1(M) = Z \oplus Z \oplus Z$, or $\pi_1(M)$ is a split extension of $Z \oplus Z$ by Z in which the action of Z is defined by the matrix

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

where m is a non-zero integer. Now $\pi_1(M) \neq Z$, since otherwise M would be a space of type $(Z,1)$, and $\pi_1(M) = Z \oplus Z \oplus Z$ implies that M is homeomorphic to $S^1 \times S^1 \times S^1$ [8, Theorem 1].

So it suffices to show that no 3-manifold in $\mathfrak{M}$ has as its fundamental group a split extension of the type described above. In view of [8, Theorem 1] there is a unique irreducible 3-manifold N with fundamental group isomorphic to the split extension

$$1 \to Z \oplus Z \to \pi_1(N) \to Z \to 1$$

in which the action of Z is defined by the matrix

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

where m is a non-zero integer. From [11] it can be seen that this manifold can be obtained from $S^1 \times S^1 \times I$ by identifying $(e^{2\pi i\alpha}, e^{2\pi i\beta}, 0)$ with $(e^{2\pi i\alpha}, e^{2\pi i(\beta+m\alpha)}, 1)$. Let $\hat{N}$ be the 3-manifold obtained from $S^1 \times S^1 \times I$ by identifying $(e^{2\pi i\alpha}, e^{2\pi i\beta}, 0)$ with $(e^{2\pi i\alpha}, e^{2\pi i(\beta+2m\alpha)}, 1)$. A double covering $p: \hat{N} \to N$ is defined by $p(e^{2\pi i\alpha}, e^{2\pi i\beta}, t) = (e^{4\pi i\alpha}, e^{2\pi i\beta}, t)$. (That p is well-defined follows from the equality

$$p(e^{2\pi i\alpha}, e^{2\pi i(\beta+2m\alpha)}, 1) = (e^{4\pi i\alpha}, e^{2\pi i(\beta+2m\alpha)}, 1) =$$

$$(e^{2\pi i(2\alpha)}, e^{2\pi i(\beta+m(2\alpha))}, 1) \sim (e^{2\pi i(2\alpha)}, e^{2\pi i\beta}, 0) =$$

$$p(e^{2\pi i\alpha}, e^{2\pi i\beta}, 0).)$$

Since $H_1(N) = Z \oplus Z \oplus Z_m$ and $H_1(\hat{N}) = Z \oplus Z \oplus Z_{2m}$, N and $\hat{N}$ are not homeomorphic. Therefore N does not belong to the class $\mathfrak{M}$, and the proof is complete.

A double covering $p: M \to M$ is said to be proper if the non-trivial covering transformation over p is homotopic to the identity map on M. Since $\pi_1(S^1 \times S^2)$ has only one subgroup of index 2, it follows that all double coverings of $S^1 \times S^2$ are equivalent to the map $(e^{2\pi i\alpha}, x) \to (e^{4\pi i\alpha}, x)$. For this map the non-trivial covering transformation is $a \times 1$, where $a: S^1 \to S^1$ is the antipodal map and $1: S^2 \to S^2$ is the identity map, and hence this double covering of $S^1 \times S^2$ is proper. The next three results lead to the proof that every double covering of $S^1 \times S^1 \times S^1$ is also proper.

Proposition 3.2: There are exactly seven subgroups of $Z \oplus Z \oplus Z$ with index 2, and each is completely determined by those elements of the basis

$$\{(1,0,0), (0,1,0), (0,0,1)\}$$

which are contained in the subgroup.

Proof: Let $H_1 = \{(a,b,c) \in Z \oplus Z \oplus Z: a \text{ is even}\}$, $H_2 = \{(a,b,c) \in Z \oplus Z \oplus Z: b \text{ is even}\}$, $H_3 = \{(a,b,c) \in Z \oplus Z \oplus Z: c \text{ is even}\}$, $H_4 = \{(a,b,c) \in Z \oplus Z \oplus Z: a + b \text{ is even}\}$, $H_5 = \{(a,b,c) \in Z \oplus Z \oplus Z: b + c \text{ is even}\}$, $H_6 = \{(a,b,c) \in Z \oplus Z \oplus Z: a + c \text{ is even}\}$, and $H_7 = \{(a,b,c) \in Z \oplus Z \oplus Z: a + b + c \text{ is even}\}$. Each of the sets above is clearly a subgroup of $Z \oplus Z \oplus Z$ having index 2. Let K be any subgroup of $Z \oplus Z \oplus Z$ having index 2. Since $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $Z \oplus Z \oplus Z$, at least one of the basis elements does not belong to K.

Suppose first that (0,1,0) and (0,0,1) lie in K but (1,0,0) does not. Since $K \cup [(1,0,0) + K] = Z \oplus Z \oplus Z$, either $(x,y,z) \in K$ or $(x-1,y,z) \in K$ for each $(x,y,z) \in Z \oplus Z \oplus Z$. Now $(-1,0,0) \notin K$, and therefore $(2,0,0) \in K$. So K contains

(2,0,0), (0,1,0), and (0,0,1) and hence contains H_1. Because both K and H_1 have index 2, it follows that $K = H_1$.

Similar arguments prove that when K contains (1,0,0) and (0,0,1), but not (0,1,0), or when K contains (1,0,0) and (0,1,0), but not (0,0,1), then $K = H_2$ or $K = H_3$, respectively.

Let K contain (0,0,1), but not (1,0,0) or (0,1,0). Since $K \cup [(0,1,0) + K] = Z \oplus Z \oplus Z$, either $(x,y,z) \in K$ or $(x,y-1,z) \in K$ for each element (x,y,z) in $Z \oplus Z \oplus Z$. Because $(1,0,0) \notin K$ and $(0,1,0) \notin K$, it follows that $(1,-1,0) \in K$ and $(0,2,0) \in K$. Thus K contains a generating set for H_4, so that $K = H_4$.

If K contains (1,0,0) but not (0,1,0) and (0,0,1), or if K contains (0,1,0) but not (1,0,0) and (0,0,1), then similar arguments show that $K = H_5$, or that $K = H_6$, respectively.

Finally, suppose that none of the basis elements lie in K; then (-1,0,0) and (0,-1,0) are not elements of K. As before, either $(x,y,z) \in K$, or $(x+1,y,z) \in K$ and $(x,y+1,z) \in K$, for each $(x,y,z) \in Z \oplus Z \oplus Z$. So K contains the elements (1,1,0), (1,0,1), and (0,1,1). But the set

$$\{(1,1,0),\ (1,0,1),\ (0,1,1)\}$$

generates H_7; for if $(a,b,c) \in H_7$, then $m = a+b-c$ is an even integer, and $\frac{m}{2}(1,1,0) + (a - \frac{m}{2})(1,0,1) + (b - \frac{m}{2})(0,1,1) = (a,b,c)$. Therefore $K = H_7$.

<u>Proposition</u> 3.3: If H and K are subgroups of $Z \oplus Z \oplus Z$ having index 2, then there exists an automorphism α of $Z \oplus Z \oplus Z$

with $\alpha(H) = K$.

Proof: It suffices to exhibit automorphisms of $Z \oplus Z \oplus Z$ which carry H_1 onto H_i $(i = 1,2,\ldots,7)$, where H_i is the subgroup defined in the proof of Proposition 3.2. Let α_i be the endomorphism of $Z \oplus Z \oplus Z$ defined for each $(a,b,c) \in Z \oplus Z \oplus Z$ by

$$\alpha_1(a,b,c) = (a,b,c)$$
$$\alpha_2(a,b,c) = (b,a,c)$$
$$\alpha_3(a,b,c) = (c,b,a)$$
$$\alpha_4(a,b,c) = (a+b,b,c)$$
$$\alpha_5(a,b,c) = (b,a+c,c)$$
$$\alpha_6(a,b,c) = (a+c,b,c)$$
$$\alpha_7(a,b,c) = (a+b+c,b,c).$$

Since each of these endomorphisms has an obvious inverse, each α_i is an automorphism of $Z \oplus Z \oplus Z$. Moreover, $\alpha_i(H_1) = H_i$; so $\alpha_1,\alpha_2,\ldots,\alpha_7$ are the required automorphisms.

Lemma 3.4: There is a proper double covering $p'\colon S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$.

Proof: Consider S^1 as the set of complex numbers having modulus 1, and define p' by $p'(x,y,z) = (x^2,y,z)$. The non-trivial covering transformation $h'\colon S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ of p' is $a \times 1 \times 1$, where $a\colon S^1 \to S^1$ is the antipodal map and $1\colon S^1 \to S^1$ is the identity map. The required homotopy $F\colon S^1 \times S^1 \times S^1 \times I \to S^1 \times S^1 \times S^1$ from the identity map to h'

is given by $F(x,y,z,t) = (e^{\pi it}x,y,z)$.

Theorem 3.5: Every double covering of $S^1 \times S^1 \times S^1$ is proper.

Proof: Let $p\colon S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ be a double covering and h the corresponding non-trivial covering transformation, and let p' and h' be defined as in the preceding lemma. There exists an automorphism α on $Z \oplus Z \oplus Z$ mapping $p'_{\#}\pi_1(S^1 \times S^1 \times S^1)$ onto $p_{\#}\pi_1(S^1 \times S^1 \times S^1)$ (Proposition 3.3); thus there is a homeomorphism $h''\colon S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ such that $h''_{\#} = \alpha$ [14, Corollary 6.5]. Lift $h''p'$ over p to a map $\tilde{h}\colon S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$. Since $h''p'$ and p are both double coverings of $S^1 \times S^1 \times S^1$, $\tilde{h}$ is a covering projection [10, 2.5.1] each fiber of which consists of a single point. Therefore $\tilde{h}$ is a homeomorphism. Because $\tilde{h}$ maps the fibers of p' onto the fibers of p, and because h' and h are both non-trivial homeomorphisms which preserve the fibers of p' and p, respectively, it follows that $h\tilde{h} = \tilde{h}h'$, and thus that $h = \tilde{h}h'\tilde{h}^{-1}$. Since h' is homotopic to the identity map on $S^1 \times S^1 \times S^1$, this equality implies that h is also homotopic to the identity map. So p is proper.

Both $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are 3-manifolds which satisfy all of the following properties:

(a.) The fundamental group of the manifold is nilpotent.

(b.) The first homology group of the manifold is infinite.

(c.) All double coverings of the manifold are proper.

Theorem 3.1 characterizes $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ among the manifolds in $\mathfrak{M}$ satisfying condition (a.). The final result shows that $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ are also the only manifolds in $\mathfrak{M}$ which satisfy both (b.) and (c.).

<u>Theorem</u> 3.6: Let M be a manifold in the class $\mathfrak{M}$ such that all double coverings of M are proper. If $H_1(M)$ is infinite, then M is homeomorphic to $S^1 \times S^1 \times S^1$ or to $S^1 \times S^2$.

Proof: Previous comments show that either M is homeomorphic to $S^1 \times S^2$ or that the conclusion of [5, Theorem 2] holds. Thus M can be obtained from $N \times R$, where N is a closed, connected, orientable 2-manifold, by identifying (x,t) with $(h(x), t+1)$ for all $x \in N$ and $t \in R$, where h is a homeomorphism from N onto itself with h^k isotopic to 1_N for some odd integer k. It follows from [9] that h may be assumed to satisfy $h^k = 1_N$. Denote by $[x,t]$ the equivalence class of (x,t) in M under the identifications described above.

Let $N \times S^1$ be considered as $N \times R$ with (x,t) identified with $(x,t+1)$ for each $x \in N$ and $t \in R$, and define $p: N \times S^1 \to M$ by $p(x,t) = [x, -kt]$. That p is well-defined follows from the equality

$$p(x,t+1) = [x, -kt - k] = [h^k(x), -kt] = [x, -kt] = p(x,t),$$

and p is easily seen to be a k-fold covering of M. (The fiber over $[x,t]$ is $\{(h^{-m}(x), \frac{m-t}{k}): m = 1,2,\ldots,k\}$.)

Therefore, if M belongs to the class $\mathfrak{M}$, then M is homeomorphic to $N \times S^1$. But if N is neither $S^1 \times S^1$ nor S^2, then finite-sheeted coverings of N can be constructed in which the total space is not homeomorphic to N. So $N \times S^1 \notin \mathfrak{M}$ unless N is homeomorphic to either $S^1 \times S^1$ or S^2.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] G.E. Bredon, Sheaf Theory, McGraw-Hill, New York, 1967.

[2] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[3] D.B.A. Epstein, The degree of a map, Proc. London Math. Soc. (3) 16 (1966), 369-383.

[4] M. Greenberg, Lectures on Algebraic Topology, W.A. Benjamin, New York, 1967.

[5] K.W. Kwun, 3-manifolds which double-cover themselves, Amer. J. Math. 91 (1969), 441-452.

[6] W.S. Massey, Algebraic Topology: An Introduction, Harcourt, Brace and World, New York, 1967.

[7] J. Milnor, A unique decomposition theorem for 3-manifolds, Amer. J. Math. 84 (1962), 1-7.

[8] L. Neuwirth, A topological classification of certain 3-manifolds, Bull. Amer. Math. Soc. 69 (1963), 372-375.

[9] J. Nielsen, Abbildungsklassen enlicher ordnung, Acta Math. 75 (1943), 23-115.

[10] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

[11] C. Thomas, Nilpotent groups and compact 3-manifolds, Proc. Cambridge Philos. Soc. 64 (1968), 303-306.

[12] J.L. Tollefson, A characterization of 3-manifolds that are products, to appear in Amer. J. Math.

[13] J.L. Tollefson, On 3-manifolds that cover themselves, Michigan Math. J. 16 (1969), 103-109.

[14] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. 87 (1968), 56-88.

[15] G.W. Whitehead, Homotopy Theory, The M.I.T. Press, Cambridge, 1966.