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AND THEIR GENERALIZATIONS
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Symplectic Structures, Lefschetz Fibrations and Their Generalizations on Smooth Four-manifolds

By

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ABSTRACT

Symplectic Structures, Lefschetz Fibrations and Their Generalizations on Smooth Four-manifolds

By

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In this thesis, we study symplectic structures, Lefschetz fibrations, and their various generalizations on smooth 4-manifolds along with the associated (smooth) invariants. Our results will be presented in separate chapters as follows:

In Chapter 3, we outline a general construction scheme to obtain minimal symplectic structures on simply-connected 4-manifolds with small Euler characteristics. Using this scheme, we illustrate how to obtain minimal symplectic 4-manifolds homeomorphic to $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$ for $k = 1, \ldots, 4$, or to $3\mathbb{CP}^2\#(2l+3)\overline{\mathbb{CP}}^2$, for $l = 2, \ldots, 6$. Secondly, for each of these homeomorphism types with $b^+ = 1$, we show how to produce an infinite family of pairwise nondiffeomorphic nonsymplectic 4-manifolds belonging to it. In particular, we prove that there are infinitely many irreducible nonsymplectic smooth structures on $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.

In Chapter 4, we study the 4-manifolds with nontrivial Seiberg-Witten invariants which are equipped with near-symplectic broken Lefschetz fibrations. We first study the topology of these fibrations and describe simple presentations of them. We then provide several examples using handlebody diagrams. We define a near-symplectic

operation that generalizes the symplectic fiber sum operation, together with its effect on the Seiberg-Witten invariants and Perutz's Lagrangian matching invariants. These techniques are then used to obtain several results on near-symplectic manifolds with non-trivial invariants.

In Chapter 5, we show that every closed oriented smooth 4-manifold can be decomposed into two codimension zero submanifolds (one with reversed orientation) so that both pieces are exact Kähler manifolds with strictly pseudoconvex boundaries and that induced contact structures on the common boundary are isotopic. Meanwhile, matching pairs of Lefschetz fibrations with bounded fibers are offered as the geometric counterpart of these structures. We also provide a simple topological proof of the existence of folded symplectic forms on 4-manifolds. Finally in the Addendum, we provide answers to two open questions stated by David Gay and Rob Kirby.

To my family who have become my friends and my friends who have become my family

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CHAPTER 1

Introduction

The world of smooth 4-manifolds has been explored using both analytical and topological tools. Several special structures which are the subject of complex, Riemannian, or symplectic geometries have been extensively used in this research, often in conjunction with the gauge theoretic smooth invariants of Donaldson to Seiberg-Witten. On the other hand, Kirby calculus and various surgery techniques have become classical tools to attack problems of 4-manifold topology. The last decade has witnessed a novel advance due to the intense collobaration of these two forces. During this period most attention has been given to symplectic 4-manifolds. This was mostly due to work of Taubes which related the Seiberg-Witten invariants (SW) to enumerative geometry [75], Donaldson's work which provided a description of symplectic 4-manifolds in terms of Lefschetz fibrations/pencils [17], and finally various surgeries introduced by Fintushel and Stern, Gompf, and several others (for example [38, 27, 28]).

An elegant bridge in this story was established by the work of Donaldson and Smith [19], who defined an invariant which, roughly speaking, associates a Gromov count to sections of fiberwise symmetric products corresponding to nicely embedded multisections of a given symplectic Lefschetz pencil. This invariant was shown to be equivalent to SW later by Usher. Remarkably, the most recent smooth 4-manifold invariant of all, Heegaard-Floer invariants of Ozsváth and Szabó were shown to compute nontrivially on symplectic 4-manifolds by using cobordisms that arise from underlying Lefschetz pencils/fibrations.

In a nutshell, this thesis work focuses on extending the territory of smooth 4-manifolds that similar techniques can be employed in the alliance of these forces, through *generalizations of symplectic structures and Lefschetz fibrations.* The major problem we have in mind is determining the number of smooth structures on a given 4-manifold.

In the past few years a lot of interest has been gathered around constructing smooth 4-manifolds which are homeomorphic but not diffeomorphic to the projective plane $\mathbb{CP}^2$ blown-up at n points ($n < 10$) as well as to the connected sum of three copies of $\mathbb{CP}^2$ blown-up at m points ($m < 20$). These manifolds are "small" in the sense that they have small Euler characteristics, whereas the construction of exotic smooth structures gets harder when the manifolds get smaller. The most recent history can be split into two periods. The first period was opened by J. Park's paper [60] which used the rational blowdown technique of Fintushel and Stern [27], and several constructions of small exotic manifolds relied on an artful use of rational blowdown techniques combined with improved knot surgery tricks [71, 30, 62]. More recently, Akhmedov's construction in [3] triggered the hope that using building blocks with nontrivial fundamental groups could succeed in obtaining exotic smooth structures on simply-connected 4-manifolds. These techniques were initially espoused by Fintushel and Stern in [29] and later discussed in [70] and in [25]. The common theme in the recent constructions ([3, 6, 10, 25]) is the manipulations to kill the fundamental group. These constitute the content of Chapter 3, and appeared in a joint article of the author with Akhmedov and Park [5].

Chapter 3 begins with an outline a general recipe to obtain small minimal symplectic 4-manifolds and to fit *all* the recent constructions in [3, 6, 10] in this construction scheme (Section 3.2). In particular, we aim to show that seemingly different examples are closely related through a sequence of Luttinger surgeries. The second goal is to calculate the basic classes and the Seiberg-Witten invariants of these small 4-manifolds. Using these calculations we show for instance how to obtain infinite families of pairwise nondiffeomorphic manifolds in the homeomorphism type of $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$, for $k = 1, \ldots, 4$ or or to $3\mathbb{CP}^2\#(2l+3)\overline{\mathbb{CP}}^2$, for $l = 2, \ldots, 6$. (Sections 3.5 and 3.4). We distinguish the diffeomorphism types of these 4-manifolds by comparing their SW invariants. Each of our families has exactly one symplectic member.

Recent research suggests the next target beyond the realm of symplectic topology to be the near-symplectic manifolds i.e. manifolds which admit a kind of singular symplectic form that is singular along an embedded 1-manifold. These are precisely the closed oriented smooth 4-manifolds with $b^+ > 0$. Taubes' program [77, 78] aims to obtain SW invariants as generalized Gromov invariants in this setting. It has motivated several parallel ideas. In [9], Auroux, Donaldson, and Katzarkov defined a generalization of Lefschetz fibrations, which we here call 'broken Lefschetz fibrations', and showed that they are to near-symplectic 4-manifolds what Lefschetz fibrations are to symplectic 4-manifolds. Perutz combined these approaches to define an invariant [64, P2], called *Lagrangian matching invariant* (LM). He conjectured that it is equivalent to SW. This invariant generalizes the Donaldson-Smith construction [19] to near-symplectic broken Lefschetz fibrations by considering pairs of sections over a splitting base that 'match' by satisfying certain Lagrangian boundary conditions which arise from the zero locus of the near-symplectic form. The very nature of LM invariants requires the study of broken Lefschetz fibrations. These topics constitute the content of Chapter 4.

The point of view we take is to consider 4-manifolds with nontrivial Seiberg-Witten invariants as an intermediate class that lies in between near-symplectic 4-manifolds and the symplectic ones. (When the manifolds in consideration have $b^+ = 1$, we always take the SW invariant computed in the Taubes' chamber of a symplectic or near-symplectic form.) Thus our work in Chapter 4 runs in two veins. We first study the topology of near-symplectic broken Lefschetz fibrations, describe simplified representations of them via Kirby diagrams and monodromies, and provide several examples (Section 4.2). Importantly, all possible round handle cobordisms that arise in this context are described in this section. Having the conjectural equivalence in mind, we define new operations on near-symplectic broken Lefschetz fibrations, and investigate their effect on both LM and SW invariants (Section 4.3). The *broken fiber sum* operation introduced in this section generalizes the symplectic fiber sum construction to the near-symplectic setting (Theorem 4.3.1).

We use these techniques to obtain various results regarding near-symplectic 4-manifolds with nontrivial Seiberg-Witten invariants (Section 4.4). Let (X, ω) be a near-symplectic 4-manifold with zero locus Z. Taubes has proved that if X has nonzero SW, then there is a J-holomorphic curve C in X with homological boundary Z, where J is an almost complex structure compatible with ω in the complement of Z [78]. In Theorem 4.4.1 we show that the converse of this statement cannot be true, and that an analogous result holds for the LM invariants. This is natural and expected, since it suggests that the moduli space that one would like to consider here can be nonempty while the count is zero. Another question we address is the behavior of near-symplectic 4-manifolds with nontrivial SW invariants under the symplectic fiber sum operation. Although the symplectic fiber sum operation preserves the class of symplectic 4-manifolds, we show that it does not preserve the class of SW nontrivial near-symplectic 4-manifolds (Theorem 4.4.2). In a comparison with symplectic Lefschetz fibrations , we determine the constraints on the self-intersection of sections of near-symplectic broken Lefschet fibrations on manifolds with nontrivial SW invariants (Theorem 4.4.4), and we describe the near-symplectic broken Lefschetz fibrations on knot surgered $E(n)$ (Proposition 4.4.5).

Further extension of these ideas takes us out of the usual range of SW invariants, and requires a new setting. (For instance to work with S^4 or $S^1 \times S^3$ which have $b^+ = 0$.) In Chapter 5 we search for 'nice' additional structures on general closed simply-connected oriented 4-manifolds. The results of this chapter, except for the Addendum, are contained in the article [11].

One possible strategy for understanding oriented smooth 4-manifolds is to break them up into more tractable classes of manifolds in a controlled manner. Situated in the intersection of complex, symplectic and Riemannian geometries, Kähler manifolds are the best known candidates to be pieces of such a decomposition. The main theorem of Chapter 5 (Theorem 5.4.2) shows that this can be achieved for any closed oriented smooth 4-manifold X. We decompose X into two exact Kähler manifolds with strictly pseudoconvex boundaries, up to orientation, such that contact structures on the common boundary induced by the maximal complex distributions are isotopic.

This decomposition gives rise to a globally defined 2-form on X, which we call a *(nicely) folded-Kähler structure*, and it belongs to a larger family of 2-forms: *folded-symplectic structures*. Cannas da Silva showed in [13] that any closed smooth oriented 4-manifold can be equipped with a folded-symplectic form, by using a version of the h-principle defined for folding maps by Eliashberg. In Section 5.2, we introduce a way to construct some simple examples of folded-symplectic 4-manifolds. Afterwards we reprove the existence fact by constructing a folded-symplectic form ω for a given handlebody decomposition of X, essentially by means of simple handle calculus and contact topology (Theorem 5.3.1). The main ingredient there is achiral Lefschetz fibrations, and recent work of Etnyre and Fuller [23] will play a key role in our construction.

Next, we switch gears, and using several results on compact Stein surfaces and Lefschetz fibrations with bounded fibers (mainly [44], [20], [39], [50], [2]), we prove the aforementioned decomposition theorem. In fact we obtain a stronger result, as the pieces of this decomposition are actually Stein manifolds with strictly pseudoconvex boundaries. It was first shown by Akbulut and Matveyev in [1] that any closed oriented smooth 4-manifold X can be decomposed into Stein pieces, but there was no particular information one could use to argue for matching the induced contact structures on the separating hypersurface. Our proof follows an alternative way via open book decompositions, and we conclude that the Stein structures can be chosen to agree on the common contact boundary.

In Section 5.5, we introduce folded-Kähler structures, and discuss some properties they enjoy, after showing that all closed oriented smooth 4-manifolds admit them (Theorem 5.5.2). This improves the folded-symplectic existence result, and indeed both structures we construct are shown to be equivalent on the symplectic level. The collection of these discussions yield us to define *folded Lefschetz fibrations* which are, roughly speaking, pairs of positive and negative Lefschetz fibrations over disks with bounded fibers which agree on the common boundary through induced open book decompositions. We prove that any nicely folded-Kähler 4-manifold, possibly after an orientation preserving diffeomorphism, admits compatible folded Lefschetz fibrations (Proposition 5.5.6).

In [34], David Gay and Rob Kirby proved that any closed smooth oriented 4-manifold can be equipped with a broken achiral Lefschetz fibration. In the Addendum (Section 5.6) we use our results in Chapters 4 and 5 to establish the corresponding symplectic generalization in this setting (Section 5.6.1), and show a way to avoid achirality (Section 5.6.2) in such a construction. These provide answers to two questions asked in [34].

CHAPTER 2

General background

In this preliminary chapter we review several definitions, notations and facts that are used in the later chapters but not contained in the background material given there. Thus this review is not intended to be complete. For the details or proofs of the quoted facts, the reader can turn to [40] and [53].

2.0.1 Topology of smooth 4-manifolds

Let X be a closed smooth oriented 4-manifold. We denote the same 4-manifold with the opposite orientation by $-X$, yet sometimes use the notation $\overline{X}$ for standard manifolds such as $\overline{\mathbb{CP}}^2$.

The *intersection form* on X is the symmetric bilinear form

$$Q_X : H_2(X;\mathbb{Z})/\mathrm{Tor} \times H_2(X;\mathbb{Z})/\mathrm{Tor} \longrightarrow \mathbb{Z}$$

defined by $(\alpha,\beta) \mapsto \alpha \cup \beta[X]$. It is unimodular on such X, and is diagonalizable over the rationals. The rank of the maximal positive eigenspace of Q_X is denoted by b^+ and that of the negative eigenspace by b^-. The *signature* of X is then understood to be the signature of this nondegenerate form, namely $\sigma(X) = b^+ - b^-$. Finally X is said to be of *even type* if every diagonalization of Q_X has even diagonal entries, and *odd type* otherwise. It turns out that these algebraic topological invariants completely classify the homeomorphism type of such 4-manifolds:

Theorem 2.0.1 (Freedman [31]; Donaldson [15]) *The homeomorphism type of a simply-connected closed smooth oriented 4-manifold X is captured by Q_X, which in turn is determined by the Euler characteristic $e(X)$, the signature $\sigma(X)$ and the type.*

On the other hand, there are infinitely many simply-connected 4-manifolds each of which admits infinitely many distinct smooth structures (see for example [28]). By contrast however, there are no *complete* invariants to classify the diffeomorphism types.

Herein the notation $mX_1\#nX_2$ is used to express the connected sum of m copies of X_1 and n copies of X_2. We say X is *reducible* if it can be written as a connected sum $X = X_1\#X_2$, where neither X_i is a homotopy 4-sphere. Otherwise, it is called *irreducible*. We view the *blow-up* of X at a point $x \in X$ as the result topological operation described by taking out a 4-ball around x and gluing in the complement of a regular neighborhood of the exceptional sphere in $\overline{\mathbb{CP}}^2$, so to obtain $X\#\overline{\mathbb{CP}}^2$. Conversely, if X contains an *exceptional sphere*, i.e. a smoothly embedded sphere S of self-intersection -1, then a tubular neighborhood of S can be replaced by a 4-ball to obtain a new closed smooth oriented 4-manifold Y with $X = Y\#\overline{\mathbb{CP}}^2$. The latter operation is called *blowing-down*. A 4-manifold X is called *minimal* if it does not contain any exceptional spheres. Irreducibility or minimality are not aspects of the underlying homeomorphism type of X, but of its smooth type.

2.0.2 Symplectic structures

A *symplectic structure* on a smooth $2n$-dimensional oriented manifold X is a closed 2-form ω such that $\omega^n > 0$. The pair (X, ω) is called a *symplectic manifold*. A diffeomorphism $\phi : X_1 \to X_2$ is called a *symplectomorphism* between (X_1, ω_1) and (X_2, ω_2) if $\phi^*(\omega_2) = \omega_1$. Two symplectic forms ω and ω' on a fixed manifold X are said to be *deformation equivalent* if there exists a family of symplectic forms ω_t, $t \in [0, 1]$, on X, with $\omega_0 = \omega$ and $\omega_1 = \omega'$.

The Euclidean space $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$ admits a *canonical symplectic form* $\omega_o = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$. Darboux's theorem states that every symplectic $2n$-manifold (X, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_o)$ [53]. It follows that symplectic manifolds do not have local invariants.

Symplectic structures are closely related to complex structures on the tangent bundle. An *almost complex structure* on X is a smooth, fiberwise linear map $J : TX \to TX$ covering id_X such that $J^2 = -id_{TX}$. The pair (X, J) is called an *almost complex manifold*. An almost complex structure J is said to be *compatible* with ω if $g(u, v) := \omega(u, Jv)$, $u, v \in TX$, is a Riemannian metric on X. For any Riemannian metric g on (X, ω) there exists a compatible almost complex structure given by $J(v) := \sqrt{2}|\omega|^{-1} g^{-1}\omega(v, \cdot)$ for all non-zero $v \in TX$. Moreover the space of compatible almost complex structures for a fixed ω is contractible [53]. Therefore, there is a unique class $c_1(X, \omega)$ associated to (X, ω) by taking the first chern class of *any* compatible almost complex structure on the bundle TX. This class' minus Poincaré dual $K_X = -PD(c_1(X, \omega))$ is called the *canonical class*.

A complex structure on X^{2n} induces an almost-complex structure given by $Jz = iz$. A symplectic manifold (X, ω) together with a Riemannian metric g and a compatible $J := \sqrt{2}|\omega|^{-1} g^{-1}\omega$ is called *Kähler* if J arises in this way from a complex structure on X. Such ω is called a *Kähler structure* and g a *Kähler metric*. Often times we simply say (X, ω) is a *Kähler manifold*; where it is implicitly assumed that there exist such compatible g and J on X.

Let us once again restrict our attention to closed smooth oriented 4-manifolds.

An orientable 4-manifold X admits an almost complext structures if and only if there is a characteristic class $h \in H^2(X)$ which satisfies $h^2 = 3\sigma(X) + 2e(X)$. More interestingly, if X is complex, then it is Kähler if and only if $b_1(X)$ is even. On the other hand the existence of complex and symplectic structures on 4-manifolds are much more involved and subtler problems to which algebraic topology can not fully answer. Once again, these are aspects of the smooth structure on X. Also note that no two of the families of closed 4-manifolds admitting complex, Kähler, symplectic or almost complex structures coincide (cf. [40]).

There are three types of surfaces of great importance in symplectic 4-manifolds. Let (X, ω) be a symplectic 4-manifold, and let Σ be an embedded surface in X. Then Σ is called a *symplectic surface* in X if $(\Sigma, \omega|_\Sigma)$ is a symplectic manifold. On the other extreme if $\omega|_\Sigma \equiv 0$, then Σ is called *Lagrangian*.

Similarly let (X, J) be an almost complex 4-manifold, and Σ be a possibly immersed surface in X. Then Σ is called a *J-holomorphic curve* or a *pseudoholomorphic curve* in (X, J) provided $(\Sigma, J|_\Sigma)$ is almost complex. i.e. when J is a bundle endomorphism on $T\Sigma \cup TX$.

If ω is a symplectic structure on X compatible with J, then every pseudoholomorphic curve is symplectic. Conversely, for every symplectic submanifold of (X, ω), one can choose a compatible almost complex structure that makes it pseudoholomorphic [53]. If Σ is an embedded surface in (X, ω, J), the self-intersection and genus of Σ are related through the formulae:

$$-\chi(\Sigma) = [\Sigma]^2 + K_X \cdot \Sigma (\textit{the adjunction equality}),$$

when Σ is symplectic or pseudoholomorphic; and

$$-\chi(\Sigma) = [\Sigma]^2,$$

when Σ is Lagrangian.

If X is the blow-up of a symplectic 4-manifold (Y, ω) at a point $y \in Y$, then it can be equipped with a symplectic form ω' which agrees with ω away from the exceptional sphere E in $X = Y \# \overline{\mathbb{CP}}^2$. Furthermore, the total transform of any symplectic surface containing y will also be symplectic in (X, ω'). Conversely, if S is an exceptional symplectic sphere in (X, ω'), then it can be *blown-down symplectically*, that is $Y = X \setminus N(S) \cup D^4 = Y$ admits a symplectic form ω which agrees with ω' on $X \setminus N(S)$, where $N(S)$ is a tubular neighborhood of S in X. Any symplectic surface that intersects S in X will also descend to a symplectic surface in Y. ([53])

We finish with a classical theorem of Thurston which not only provides us with a plethora of examples of closed symplectic 4-manifolds, but also motivates several other results proved and/or used in this work:

Theorem 2.0.2 (Thurston [80]) *Let $f : X \to B$ be an F-bundle where the fiber F is a closed Riemann surface and the base B is a compact Riemann surface. If F is nonzero in $H_2(X; \mathbb{R})$, then X can be equipped with a symplectic form ω such that all fibers are symplectic.*

2.0.3 Lefschetz fibrations

A smooth *Lefschetz fibration* on an oriented 4-manifold X, possibly with boundary, is a smooth map $f : X \to \Sigma$, where Σ is a compact oriented surface, such that f is a submersion everywhere but at finitely many points $C = \{p_1, \ldots, p_n\}$ contained in the interior of X, and conforming to local models: (i) $f(z_1, z_2) = z_1$ around each regular point, and (ii) $f(z_1, z_2) = z_1 z_2$ around each *Lefschetz critical point* $p_i \in C$; both given by orientation preserving charts on X and Σ. The preimage of a regular value is a Riemann surface F, called the *regular fiber*, whereas and the *singular fibers* containing the Lefschetz critical points locally have the model of a complex nodal singularity around those points. In a handlebody of X, these singularities are obtained by attaching 2-handles to regular fibers with framing -1 with respect to the framing induced by the fiber. The attaching circles of these 2-handles are called *vanishing cycles.*

A *Lefschetz pencil* is a map $f : X \setminus \{b_1, \dots, b_m,\} \to S^2$, such that around any *base point* b_i it has a local model $f(z_1, z_2) = z_1/z_2$, preserving the orientations, and that f is a Lefschetz fibration elsewhere. By convention, $B = \{b_1, \dots, b_m\}$ is always non-empty and called the base locus, and $C = \{p_1, \dots, p_n\}$ is called the critical locus. There is an obvious link between these two definitions. In a Lefschetz pencil, the closures of the fibers of the map f cut the 4-manifold X into a family of closed surfaces all passing through the b_i —locally like complex lines through a point in $\mathbb{C}^2$. Blowing up all the points in the bsae locus, the map f extends to the entire manifold and we obtain a Lefschetz fibration $\hat{f} : \hat{X} \to S^2$, with each exceptional sphere appearing as a section.

If F is a regular fiber of a Lefschetz fibration $f : X \to \Sigma$, then $F \hookrightarrow X \xrightarrow{f} \Sigma$ induce an exact sequence $\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma) \to \pi_0(F) \to 0$. It follows that if the base space is simply connected, then each fiber of f is connected and carries $\pi_1(X)$. If a fiber is not connected, then $\pi_1(X)$ maps to a finite-index subgroup of $\pi_1(\Sigma)$, and passing to the corresponding finite cover $\tilde{\Sigma}$ of Σ, we obtain a new Lefschetz fibration $\tilde{f} : X \to \tilde{\Sigma}$ with connected fibers. Thus without loss of generality we can assume that the fibers are connected, and in this case the *genus* of a generic fiber will be called the genus of the Lefschetz pencil or fibration.

Given a compact oriented genus g surface F with m boundary components and r marked points on it, the *mapping class group* of F is defined as the group of orientation preserving self-diffeomorphisms of F fixing marked points and ∂F pointwise, modulo isotopies of F fixing marked points and ∂F pointwise. It can be shown that this group is generated by positive (right handed) and negative (left handed) Dehn twists. Importantly, isotopy type of a surface bundle over S^1 with fiber closed oriented surface F is determined by the return map of a flow transverse to the fibers, which can be identified with an element of a mapping class group F, called the *monodromy* of this fibration.

Let $f : X \to D^2$ be a Lefschetz fibration, where the regular fiber F is an oriented genus g surface with m boundary components, and suppose all critical points of the fibration lie on various fibers. Select a regular value O in the interior of D^2, an identification of $f^{-1}(O) \cong F$, and a collection of arcs $a_1, \cdots, a_k$ in the interior of D^2 with each a_i connecting O to a distinct critical value, and all disjoint except at O. We index the critical values as well, so that each arc a_i is connected to a critical value y_i and that they appear in a counterclockwise order around the point O. Now if we take a regular neighborhood of each arc away from remaining critical points and consider the union of these, we obtain a disk V and an F-bundle over $\partial V = S^1$. The monodromy of this fibration is an element of the mapping class group of F, which is called the *global monodromy* of the Lefschetz fibration f. We call the ordered set of arcs $\{a_1, \cdots, a_k\}$ a *representation* of the Lefschetz fibration f. It is well-known that this data gives a handlebody description of X, and vice versa.

The next two theorems establish a beautiful connection between the main concepts of the last two sections:

Theorem 2.0.3 (Donaldson [17]) *If (X, ω) is a closed symplectic 4-manifold with ω integral, then it admits a Lefschetz pencil with symplectic regular fibers.*

Theorem 2.0.4 (Gompf; see [40]) *If $f : X \to \Sigma$ is a Lefschetz fibration such that the homology class of the regular fiber F is nonzero in $H_2(X; \mathbb{R})$, then X admits a deformation class of symplectic structures with respect to which the fibers are symplectic. Moreover, such a symplectic form can be chosen so that any prescribed finite set of sections are also symplectic.*

The proof of the latter theorem generalizes Thurston's construction to Lefchetz fibrations, using the local models around singular points (see Proposition 5.2.2 for complete details).

2.0.4 Seiberg-Witten invariants

We now review the basics of Seiberg-Witten invariant (cf. [85]). The *Seiberg-Witten invariant* of a smooth closed oriented 4-manifold X is an integer valued function which is defined on the set of Spin^c structures on X. If we assume that $H_1(X;\mathbb{Z})$ has no 2-torsion, then there is a one-to-one correspondence between the set of Spin^c structures on X and the set of characteristic elements of $H^2(X;\mathbb{Z})$ as follows: To each Spin^c structure $\mathfrak{s}$ on X corresponds a bundle of positive spinors $W_{\mathfrak{s}}^+$ over X. Let $c(\mathfrak{s}) = c_1(W_{\mathfrak{s}}^+) \in H^2(X;\mathbb{Z})$. Then each $c(\mathfrak{s})$ is a characteristic element of $H^2(X;\mathbb{Z})$; i.e. $c_1(W_{\mathfrak{s}}^+)$ reduces mod 2 to $w_2(X)$.

In this setup we can view the *Seiberg-Witten invariant* as an integer valued function

$$\mathrm{SW}_X : \{k \in H_2(X;\mathbb{Z}) \mid \mathrm{PD}(k) \equiv w_2(X) \pmod 2\} \longrightarrow \mathbb{Z},$$

where $\mathrm{PD}(k)$ denotes the Poincaré dual of k. The Seiberg-Witten invariant SW_X is a diffeomorphism invariant when $b_2^+(X) > 1$ or when $b_2^+(X) = 1$ and $b_2^-(X) \leq 9$ (see Remark 2.0.5 for the $b^+ = 1$ case). Its overall sign depends on our choice of an orientation of $H^0(X;\mathbb{R}) \times \det H_+^2(X;\mathbb{R}) \times \det H^1(X;\mathbb{R})$.

If $\mathrm{SW}_X(\beta) \neq 0$, then we call β (and its Poincaré dual $\mathrm{PD}(\beta) \in H^2(X;\mathbb{Z})$) a *basic class* of X. It was shown in [74] that the canonical class $K_X = -c_1(X,\omega)$ of a symplectic 4-manifold (X,ω) is a basic class when $b^+(X) > 1$ with $\mathrm{SW}_X(K_X) = 1$. It can be shown that, if β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(e(X)+\sigma(X))/4}\,\mathrm{SW}_X(\beta),$$

where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X. We say that

X is of *simple type* if every basic class β of X satisfies

$$\beta^2 = 2\mathrm{e}(X) + 3\sigma(X).$$

It was shown in [75] that symplectic 4-manifolds with $b_2^+ > 1$ are of simple type. Let $\Sigma \subset X$ be an embedded surface of genus $g(\Sigma) > 0$. If X is of simple type and β is a basic class of X, we have the following (generalized) *adjunction inequality* (cf. text [58]):

$$-\chi(\Sigma) = 2g(\Sigma) - 2 \geq [\Sigma]^2 + |\beta \cdot [\Sigma]|. \tag{2.1}$$

Remark 2.0.5 *When $b^+(X) = 1$, the (small-perturbation) Seiberg-Witten invariant $\mathrm{SW}_{X,H}(K) \in \mathbf{Z}$ is defined for every positively oriented element $H \in H^2_+(X;\mathbf{R})$ and every element $A \in \mathcal{C}(X)$ such that $A \cdot H \neq 0$. We say that H determines a* chamber. *It is known that if $\mathrm{SW}_{X,H}() \neq 0$ for some $H \in H^2_+(X;\mathbf{R})$, then $d(A) \geq 0$. The* wall-crossing formula *prescribes the dependence of $\mathrm{SW}_{X,H}(A)$ on the choice of the chamber (that of H): if $H, H' \in H^2_+(X;\mathbf{R})$ and $A \in \mathcal{C}(X)$ satisfy $H \cdot H' > 0$ and $d(A) \geq 0$, then*

$$\mathrm{SW}_{X,H'}(A) = \mathrm{SW}_{X,H}(A) + \begin{cases} 0 & \text{if } A \cdot H \text{ and } A \cdot H' \text{ have the same sign,} \\ (-1)^{\frac{1}{2}d(A)} & \text{if } A \cdot H > 0 \text{ and } A \cdot H' < 0, \\ (-1)^{1+\frac{1}{2}d(A)} & \text{if } A \cdot H < 0 \text{ and } A \cdot H' > 0. \end{cases}$$

These facts imply that $\mathrm{SW}_{X,H}(A)$ is independent of H in the case $b^-(X) \leq 9$ [73], so we simply talk about the Seiberg-Witten invariant of X in this case.

The Seiberg-Witten invariant of X with $b^+(X) \geq 1$ can be formulated as a map

$$\mathrm{SW}_X \colon \mathrm{Spin}^c(X) \to \mathbb{A}(X) = \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H^1(X;\mathbb{Z}).$$

where $\mathbb{A}(X)$ is the graded abelian group with $\deg(U) = 2$, and $\mathrm{SW}_X(\mathfrak{s})$ is homogeneous of degree $d(\mathfrak{s})$. In this formulation, $\mathrm{SW}_X(\mathfrak{s})$ is the fundamental homology class of the Seiberg-Witten moduli space in the ambient configuration space $\mathcal{B}^*_{X,\mathfrak{s}}$, under isomorphisms

$$H_*(\mathcal{B}^*_{X,\mathfrak{s}};\mathbb{Z}) = H_*(B\mathcal{G}_X;\mathbb{Z}) \cong \mathbb{A}(X).$$

where $\mathcal{G}_X = \mathrm{Map}(X, S^1)$ is the gauge group. Evaluating SW_X on monomials $U^a \otimes l_1 \wedge \cdots \wedge l_b$ of degree $d(\mathfrak{s})$, we obtain a map to $\mathbb{Z}$ as above.

We finish with some important results regarding the SW invariants. According to Taubes [75], the SW invariant of a symplectic 4-manifold (X, ω) can be computed as a Gromov invariant (Gr) enumerating embedded pseudo-holomorphic curves and their unramified coverings with respect to a generic J compatible with ω. When we have a symplectic broken Lefschetz pencil (X, f) of high enough degree, there is another invariant called the Donaldson-Smith invariant (DS) associated it, which counts nicely embedded pseudoholomoprhic mutlisections within a chosen homology class [19, 69]. In [82], Usher proves that DS and Gr counts agree when the degree of the pencil is high enough. Hence, under mild assumptions, Gr and DS invariants are seen to be independent of the symplectic structure or the Lefschetz pencil that are chosen, yielding equivalent smooth invariants.

Last two results to add are as follows:

If $X = X_1 \# X_2$ with $b^+(X_i) > 0$, $i = 1, 2$, then $\mathrm{SW}_X \equiv 0$. (*SW vanishing theorem for connected sums.*)

If $\tilde{X} = X \# \overline{\mathbb{CP}}^2$, then every basic class $\tilde{\beta}$ of $\tilde{X}$ is of the form $\tilde{\beta} = \beta \pm E$, where $H^2(\tilde{X};\mathbb{Z})$ is identified with $H^2(X;\mathbb{Z}) \oplus H^2(\overline{\mathbb{CP}}^2;\mathbb{Z})$, β is a basic class of X and E is the class of the exceptional sphere generating $H^2(\overline{\mathbb{CP}}^2;\mathbb{Z})$. (*The blow-up formula for SW invariants.*)

CHAPTER 3

New symplectic 4-manifolds

3.1 Background

3.1.1 Generalized fiber sum

Assume that two 4-manifolds X_1 and X_2 each contain a closed embedded genus g surface $F_i \subset X_i$ such that the normal bundles νF_i have opposite Euler numbers, i.e. $[F_1]^2 = -[F_2]^2$. Then there exists a fiber-orientation reversing isomorphism between the two normal bundles. If we canonically identify each νF_i with a tubular neighborhood N_i of F_i, then there exits an orientation reversing diffeomorphism $\varphi : N_1 \setminus F_1 \to N_2 \setminus F_2$ which turns each punctured normal disk inside out. Then we can define the *generalized fiber sum* of (X_1, F_1) and (X_2, F_2) as $X_1 \setminus N_1 \cup_\varphi X_2 \setminus N_2$ by identifying ∂N_i via φ. We denote this operation by $X = X_1 \#_\varphi X_2$.

Note that the diffeomorphism type of X is determined by the embeddings of F_i together with the choice of φ up to 'fiber preserving isotopy' (of the corresponding fiber bundle isomorphisms between νF_i). When $[F_1]^2 = [F_2]^2 = 0$, the map ϕ can be taken as an orientation preserving self-diffeomorphism of F times a complex conjugation on the punctured unit disk $D^2 \setminus \{0\}$. In this simpler case, the operation is called *fiber sum*. Finally note that generalized fiber sum operation can be defined in higher dimensions as well (see for example [55] or [38]), but here we are solely interested in the 4-dimensional case.

The characteristic classes of X can easily be expressed in terms of those of X_i. We have $\mathrm{e}(X) = \mathrm{e}(X_1) + \mathrm{e}(X_2) + \mathrm{e}(\partial N_1) - \mathrm{e}(N_1) - \mathrm{e}(N_2)$, where $\partial N_1 = -\partial N_2$ is an oriented 3-manifold and each N_i deformation retracts to $F_i = \Sigma_g$. So $\mathrm{e}(\partial N_1) = 0$ and $\mathrm{e}(N_1) = \mathrm{e}(N_2) = \mathrm{e}(\Sigma_g)$. On the other hand, the signature of X can be computed by using Novikov's additivity. So we have:

$$\mathrm{e}(X) = \mathrm{e}(X_1) + \mathrm{e}(X_2) + 4g - 4\ , \qquad \sigma(X) = \sigma(X_1) + \sigma(X_2). \tag{3.1}$$

In addition the type is odd unless each F_i is characteristic in X_i, $i = 1, 2$.

Importantly, this operation can be performed symplectically in the following setting:

Theorem 1 (McCarthy and Wolfson [55], Gompf [38], also see Gromov [42]) *Let (X_i, ω_i) be symplectic 4-manifolds and $F_i \hookrightarrow X_i$ be symplectically embedded genus $g > 0$ surfaces, for $i = 1, 2$. If $[F_1]^2 = -[F_2]^2$, then $X = X_1 \#_\varphi X_2$ can be equipped with a symplectic form ω. Moreover, given arbitrarily small collar neighborhoods $\tilde{N}_i$ of $\partial(N_i)$ in X_i, we can choose ω so that $\omega|_{X_1 \setminus \tilde{N}_1} = \omega_1|_{X_1 \setminus \tilde{N}_1}$ and $\omega|_{X_2 \setminus \tilde{N}_2} = c\omega_2|_{X_2 \setminus \tilde{N}_2}$, where c is some constant.*

The last part of the theorem is immediate if we construct the symplectic fiber sum following Etnyre's symplectic cut-and-paste technique [24].

3.1.2 Minimality

Recall that a 4-manifold X is called *minimal* if it does not contain an embedded 2-sphere with self-intersection -1. Similarly a symplectic 4-manifold (X, ω) is called *symplectically minimal* if it does not contain such a *symplectic* sphere. In both smooth and symplectic categories we aim to construct minimal 4-manifolds.

A family of minimal 4-manifolds is the products of non-rational Riemann surfaces. Let Σ_g denote a closed Riemann surface of genus $g > 0$. Since the universal cover of Σ_g is contractible, Σ_g is acyclic. It follows from the long exact homotopy sequence of a fibration that any Σ_g bundle over Σ_h, with $g, h > 0$ is acyclic. In particular, $\pi_2(\Sigma_g \times \Sigma_h) = 0$ and hence $\Sigma_g \times \Sigma_h$ is minimal. So equipped with any symplectic form, $\Sigma_g \times \Sigma_h$ is symplectically minimal.

One new ingredient in our constructions that will follow is the following theorem of Michael Usher:

Theorem 2 (Usher [83]) *Let $X = Y\#_{\Sigma=\Sigma'}Y'$ be the symplectic sum, where the genus g of Σ and Σ' is strictly positive.*

(i) If either $Y \setminus \Sigma$ or $Y' \setminus \Sigma'$ contains an embedded symplectic sphere of square -1, then X is not minimal.

(ii) If one of the summands, say Y for definiteness, admits the structure of an S^2-bundle over a surface of genus g such that Σ is a section of this S^2-bundle, then X is minimal if and only if Y' is minimal.

(iii) In all other cases, X is minimal.

One final comment is on the close relationship between minimality and irreducibility when dealing with symplectic 4-manifolds:

Theorem 3 (Hamilton and Kotschick [43]) *Minimal symplectic 4-manifolds with residually finite fundamental groups are irreducible.*

Thus simply-connected minimal symplectic 4-manifolds are always irreducible: a fact that we will use repeatedly in this chapter.

3.1.3 Surgery along Lagrangian tori

Let Λ be a torus of self-intersetion zero inside a 4-manifold X. Choose a framing of the tubular neighborhood $\nu\Lambda$ of Λ in X, i.e. a diffeomorphism $\nu\Lambda \cong T^2 \times D^2$. Given a simple loop λ on Λ, let S^1_λ be a loop on the boundary $\partial(\nu\Lambda) \cong T^3$ that is parallel to λ under the chosen framing. Let μ_Λ denote a meridian circle to Λ in $\partial(\nu\Lambda)$. By the p/q surgery on Λ with respect to λ, or more simply by a $(\Lambda, \lambda, p/q)$ *surgery*, we mean the closed 4-manifold

$$X_{\Lambda,\lambda}(p/q) = (X \setminus \nu\Lambda) \cup_\varphi (T^2 \times D^2),$$

where the gluing diffeomorphism $\varphi : T^2 \times \partial D^2 \longrightarrow \partial(X \setminus \nu\Lambda)$ satisifies

$$\varphi_*([\partial D^2]) = p[\mu_\Lambda] + q[S^1_\lambda] \in H_1(\partial(X \setminus \nu\Lambda); \mathbb{Z}).$$

By Seifert-Van Kampen theorem, one easily concludes that

$$\pi_1(X_{\Lambda,\lambda}(p/q)) = \pi_1(X \setminus \nu\Lambda) \,/\langle\, [\mu_\Lambda]^p[S^1_\lambda]^q = 1\rangle.$$

In the symplectic case, we will be surgering Lagrangian tori. *Luttinger surgery* is a special case of p/q surgery on a self-intersection zero torus Λ described in the previous subsection. It was first studied in [51] and then in [8] in a more general setting. Assume that (X, ω) is a symplectic 4-manifold, and that the torus Λ is a Lagrangian submanifold of X. From the Weinstein tubular neighborhood theorem, there is a canonical framing of $\nu\Sigma \cong T^2 \times D^2$, called the *Lagrangian framing*, such that $T^2 \times \{x\}$ corresponds to a Lagrangian submanifold of X for every $x \in D^2$. Given a simple loop λ on Λ, let S^1_λ be a simple loop on $\partial(\nu\Lambda)$ that is parallel to λ under the Lagrangian framing. For any integer m, the $(\Lambda, \lambda, 1/m)$ *Luttinger surgery* on X will be $X_{\Lambda,\lambda}(1/m)$, the $1/m$ surgery on Λ with respect to λ and the Lagrangian framing. Note that our notation is different from the one in [8] wherein $X_{\Lambda,\lambda}(1/m)$ is denoted by $X(\Lambda, \lambda, m)$.

Theorem 4 (Auroux, Donaldson and Katzarkov [8]) *$X_{\Lambda,\lambda}(1/m)$ possesses a symplectic form that restricts to the original symplectic form ω on $X \setminus \nu\Lambda$.*

In this thesis, we will *only* deal with Luttinger surgeries where $m = \pm 1 = 1/m$, so there should be no confusion in notation.

Remark 3.1.1 *In Section 3.2.2 and Section 3.5, we will also be looking at* non-*Luttinger $(\Lambda, \lambda, -n)$ surgeries $X_{\Lambda,\lambda}(-n)$ for a Lagrangian torus Λ equipped with the Lagrangian framing and a positive integer $n \geq 2$.*

3.1.4 Surgeries and Seiberg-Witten invariants

In what follows, we will be frequently using the following theorem:

Theorem 5 (Fintushel, Park, Stern [25]) *Let X be a closed oriented smooth 4-manifold which contains a nullhomologous torus Λ with λ a simple loop on Λ such that S^1_λ is nullhomologous in $X \setminus \nu\Lambda$. If $X_{\Lambda,\lambda}(0)$ has nontrivial Seiberg-Witten invariant, then the set*

$$\{X_{\Lambda,\lambda}(1/n) \mid n = 1, 2, 3, \dots\} \tag{3.2}$$

contains infinitely many pairwise nondiffeomorphic 4-manifolds. Furthermore, if $X_{\Lambda,\lambda}(0)$ has just one Seiberg-Witten basic class up to sign, then every pair of 4-manifolds in (3.2) are nondiffeomorphic.

Here the Seiberg-Witten invariant is the small perturbation invariant whenever the 4-manifold has $b^+ = 1$.

Remark 3.1.2 *Note that $X = X_{\Lambda,\lambda}(1/0)$. Let T be the core torus of the 0 surgery $X_{\Lambda,\lambda}(0)$. If k_0 is a characteristic element of $H_2(X_{\Lambda,\lambda}(0); \mathbb{Z})$ satisfying $k_0 \cdot [T] =$*

0, then k_0 gives rise to unique characteristic elements $k \in H_2(X;\mathbb{Z})$ and $k_n \in H_2(X_{\Lambda,\lambda}(1/n);\mathbb{Z})$. The product formula in [52] then gives

$$\mathrm{SW}_{X_{\Lambda,\lambda}(1/n)}(k_n) = \mathrm{SW}_X(k) + n\sum_{i\in\mathbb{Z}} \mathrm{SW}_{X_{\Lambda,\lambda}(0)}(k_0 + 2i[T]). \tag{3.3}$$

Let us now assume that $X_{\Lambda,\lambda}(0)$ has only one basic class up to sign and this basic class is not a multiple of $[T]$. Under these assumptions, the infinite sum in (3.3) only contains at most one nonzero summand. If we further assume that X and $X_{\Lambda,\lambda}(0)$ are both symplectic, then the adjunction inequality implies that the only basic class of X and $X_{\Lambda,\lambda}(0)$ is the canonical class up to sign. Under all these assumptions, it follows that $X_{\Lambda,\lambda}(1/n)$ also has only one basic class up to sign for every $n \geq 1$.

3.2 Constructing small exotic symplectic 4-manifolds

3.2.1 The construction scheme for odd blow-ups of $\mathbb{CP}^2$ and $3\mathbb{CP}^2$

Here we outline a general construction scheme to construct simply-connected minimal symplectic 4-manifolds with small Euler characteristics. This is an incidence of the "reverse engineering" ([70, 25]) idea applied to certain symplectic manifolds. Any example using this scheme and homeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ (for $n > 0$) and $m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ (for $m > 0$) can be distinguished from the latter standard manifolds by comparing their symplectic structures or their Seiberg-Witten invariants, respectively. Recall that, $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ (for $n > 0$) are nonminimal, and $m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ (for $m > 0$) all have vanishing Seiberg-Witten invariants, unlike the minimal symplectic 4-manifolds that we produce. Our approach will allow us to argue easily how all

4-manifolds obtained earlier in [6, 10] arise from this construction scheme, and in particular we show how seemingly different examples rely on the very same idea.

The only building blocks we need are the products of two Riemann surfaces. In fact, it suffices to consider multiple copies of $S^2 \times T^2$ and $T^2 \times T^2$, since all the other product manifolds except for $S^2 \times S^2$ (which we will not use here) can be obtained by fiber summing copies of these manifolds appropriately. Note that any such manifold is a minimal symplectic manifold. Both $S^2 \times T^2$ and $T^4 = T^2 \times T^2$ can be equipped with product symplectic forms where each factor is a symplectic submanifold with self-intersection zero. Denote the standard generators of $\pi_1(T^4)$ by a, b, c and d, so that $H_2(T^4;\mathbb{Z}) \cong \mathbb{Z}^6$ is generated by the homology classes of two symplectic tori $a \times b$ and $c \times d$, and four Lagrangian tori $a \times c$, $a \times d$, $b \times c$ and $b \times d$ with respect to the product symplectic form on T^4 that we have chosen. The intersection form splits into three hyperbolic pairs: $a \times b$ and $c \times d$, $a \times c$ and $b \times d$, $a \times d$ and $b \times c$. Finally, note that all four Lagrangian tori can be pushed off to nearby Lagrangians in their Weinstein neighborhoods so that they lie in the complement of small tubular neighborhoods of the two chosen symplectic tori $T^2 \times \{pt\}$ and $\{pt\} \times T^2$. With a little abuse of notation (which will be remembered in our later calculations of fundamental groups), we will still denote these parallel Lagrangian tori with the same letters.

In order to produce an exotic copy of a target manifold Z, we first perform blow-ups and symplectic fiber sums to obtain an intermediate manifold X'. Whenever a piece is blown-up, we make sure to fiber sum that piece along a symplectic surface that intersects each exceptional sphere positively at one point. This allows us to employ Theorem 2 to conclude that X' is minimal. We want this intermediate manifold to satisfy the following two properties:

(I) X' should have the same signature and Euler characteristic as Z.

(II) If r is the rank of the maximal subspace of $H_2(X';\mathbb{Z})$ generated by homologically essential Lagrangian tori, then we should have $r \geq s = 2b_1(X') = b_2(X') - b_2(Z)$.

Moreover, we generally desire to have $\pi_1(X') = H_1(X';\mathbb{Z})$ for the reasons that will become apparent below. However, surprisingly one can also handle some examples where $\pi_1(X')$ is not abelian. (See for example the construction of a minimal symplectic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ in [6].)

Finally, we carefully perform $\frac{s}{2}$ Luttinger surgeries to kill $\pi_1(X')$ and obtain a simply-connected symplectic 4-manifold X. Since signatures of simply-connected spin 4-manifolds are always divisible by 16, all of our target manifolds among $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ (for $n > 0$) and $m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ (for $m > 0$) are of odd type. Observing that Luttinger surgeries do not change neither the Euler characteristics nor the signature, one concludes that X is homeomorphic to the target manifold.

Note that these surgeries can easily be chosen to obtain a manifold with $b_1 = 0$. However, determining the correct choice of Luttinger surgeries in this last step to kill the fundamental group completely is a much more subtle problem. This last part is certainly the hardest part of our approach, at least for the 'smaller' constructions. The reader might want to compare below the complexity of our fundamental group calculations for $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$, for $k = 1, \ldots, 4$ as k gets smaller.

In order to compute and effectively kill the fundamental group of the resulting manifold X, we will do the Luttinger surgeries in our building blocks as opposed to doing them in X'. This is doable, since the Lagrangian tori along which we perform Luttinger surgeries lie away from the symplectic surfaces that are used in any symplectic sum constructions, as well as the blow-up regions. In other words, one can change the order of these operations while paying extra attention to the π_1 identifications. Having the π_1 calculations of the pieces in hand, we can use Seifert-Van Kampen theorem repeatedly to calculate the fundamental group of our exotic candidate X.

Below, we will work out some concrete examples, where we construct minimal symplectic 4-manifolds homeomorphic to $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$, for $k = 1,\dots,4$, and $3\mathbb{CP}^2\#(2l+3)\overline{\mathbb{CP}}^2$, for $l = 2,\dots,6$ (See [5] for $l = 1$ case). We hope that the reader will have a better understanding of the recipe we have given here by looking at these examples. Another essential observation that is repeatedly used in our arguments below is the interpretation of some manifold pieces used in [3, 6] as coming from Luttinger surgeries on T^4, together with the description of their fundamental groups. This is proved in the Section 3.2.2. A concise history of earlier constructions will be given at the beginning of each subsection.

Remark 3.2.1 *The building blocks we used in the construction scheme described here do not suffice to get models for even number of blow-ups of $\mathbb{CP}^2$ or $3\mathbb{CP}^2$. By the time of writing, finding appropriate models for these manifolds has not been completely accomplished.*

3.2.2 Twist knots and Luttinger surgeries

Let $T^4 = a \times b \times c \times d \cong (c \times d) \times (a \times b)$, where we have switched the order of the symplectic T^2 components $a \times b$ and $c \times d$ just to have a comparable notation with earlier π_1 calculations (say in [6]). Let K_n be an n-twist knot (cf. Figure 3.3). Let M_{K_n} denote the result of performing 0 Dehn surgery on S^3 along K_n. Our goal here is to show that the 4-manifold $S^1 \times M_{K_n}$ is obtained from $T^4 = (c \times d) \times (a \times b) = c \times (d \times a \times b) = S^1 \times T^3$ by first performing a Luttinger surgery $(c \times \tilde{a}, \tilde{a}, -1)$ followed by a surgery $(c \times \tilde{b}, \tilde{b}, -n)$. Here, the tori $c \times \tilde{a}$ and $c \times \tilde{b}$ are Lagrangian and the second tilde circle factors in T^3 are as pictured in Figure 3.2. We use the Lagrangian framing to trivialize their tubular neighborhoods, so when $n = 1$ the second surgery is also a Luttinger surgery.

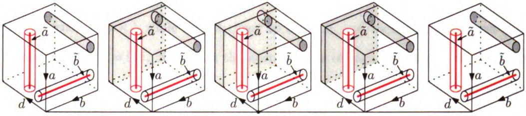

Figure 3.1: The 4-torus $c \times (d \times a \times b)$. The neighborhood of a fiber chosen in the 3-torus $d \times a \times b$ at $c = 0.5$ is given by fat slices parallel to $a \times b$ face, which get thinner while c gets closer to $0.5 \pm \epsilon$ and disappear when $0 < c < 0.5 - 2\epsilon$ or $0.5 + 2\epsilon < c < 1$. The neighborhood of the torus section is given by a cylindrical neighborhood in the direction of d lying in the 3-torus times c. Neighborhoods of the tori $c \times \tilde{a}$ and $c \times \tilde{b}$ are drawn similarly.

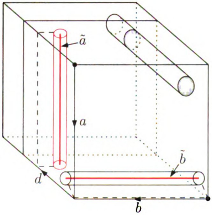

Figure 3.2: The 3-torus $d \times a \times b$ at $c = 0.5$

The figures should be self-explanatory. We view the 4-torus as the product $c\times(d\times a\times b)$, and excise the tubular neighborhoods of the tori $c\times\tilde{a}$, $c\times\tilde{b}$ and $c\times d$ as shown in the Figure 3.1. The tubular neighborhood of the torus $a\times b$ appears as a slice in the 3-torus $d\times a\times b$ while we get closer to $c=0.5$, and we have the thickest slice precisely when $c=0.5$. Note that the normal disks of each Lagrangian tori in their Weinstein neighborhoods lie completely in T^3 and are disjoint. Thus topologically, the result of these surgeries can be seen as the product of the first S^1 factor with the result of Dehn surgeries along $\tilde{a}$ and $\tilde{b}$ in T^3. Therefore we can restrict our attention to the effect of these Dehn surgeries in T^3 since the diffeomorphisms of the 3-manifolds induce diffeomorphisms between the product 4-manifolds.

The Kirby calculus diagrams in Figure 3.3 show that the result of these Dehn surgeries is the manifold M_{K_n}, where K_n is (the mirror of) the n-twist knot. In particular, note that for $n=1$ we get the trefoil knot K. Thus the effect of $(c\times\tilde{b}, \tilde{b}, -n)$ surgery with $n>1$ as opposed to the Luttinger surgery $(c\times\tilde{b}, \tilde{b}, -1)$ is equivalent to using the non-symplectic 4-manifold $S^1\times M_{K_n}$ instead of symplectic $S^1\times M_K$ in our symplectic sum constructions.

Next we describe the effect of these surgeries on π_1. First it is useful to view $T^3=d\times(a\times b)$ as a T^2 bundle over S^1 with fibers given by $\{\mathrm{pt}\}\times(a\times b)$ and sections given by $d\times\{\mathrm{pt}\}$. The complement of a fiber union a section in T^3 is the complement of 3-dimensional shaded regions in Figure 3.1.

It is not too hard to see that the Lagrangian framings give the following product decompositions of two boundary 3-tori (compare with [10, 25]):

$$\partial(\nu(c\times\tilde{a})) \cong c\times(dad^{-1})\times[d,b^{-1}], \tag{3.4}$$

$$\partial(\nu(c\times\tilde{b})) \cong c\times b\times[a^{-1},d]. \tag{3.5}$$

The Lagrangian pushoff of $\tilde{b}$ is represented by b, as a homotopy to b is given by the "diagonal" path (dotted lines emanating from the horizontal boundary cylinder

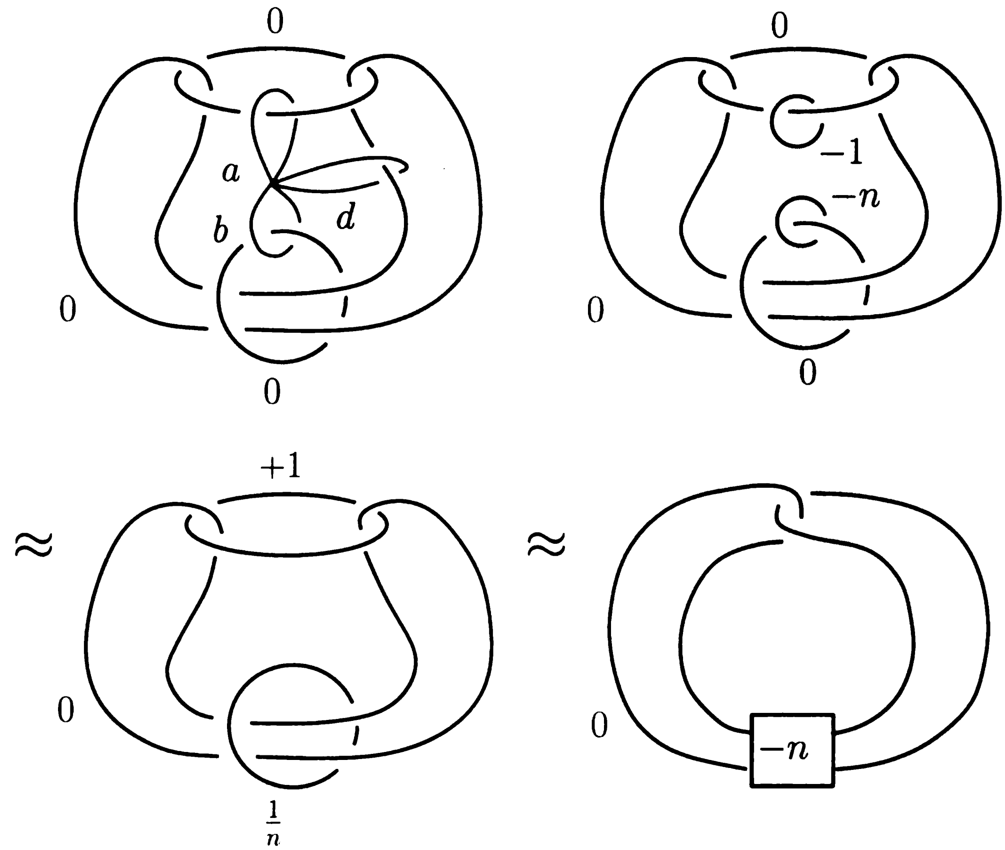

Figure 3.3: The first diagram depicts the three loops a, b, d that generate the $\pi_1(T^3)$. The curves $\tilde{a} = dad^{-1}$ and $\tilde{b}$ are freely homotopic to the two extra curves given in the second diagram. The third diagram is obtained from the second via two slam-dunk operations; wheras the last diagram is obtained after Rolfsen twists.

$\partial(\nu\tilde{b})$ in Figure 3.2). For decomposition (3.5), it is helpful to view the base point as the front lower right corner of the cube represented by a dot in Figure 3.2. It is comparatively more difficult to see that the Lagrangian pushoff of $\tilde{a}$ is represented by dad^{-1}. The Lagrangian pushoff of $\tilde{a}$ is represented by the dotted circle in Figure 3.4 and is seen to be homotopic to the composition $a[a^{-1}, d] = a(a^{-1}dad^{-1}) = dad^{-1}$. For decomposition (3.4), it is helpful to view the base point as the front upper left corner of the cube represented by a dot in Figure 3.2. The new relations in π_1 introduced

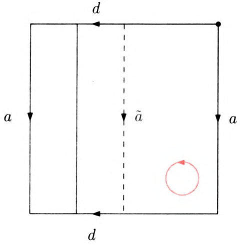

Figure 3.4: The face of cube where we can see the Lagrangian pushoff of $\tilde{a}$

by the two surgeries are

$$dad^{-1} = [d, b^{-1}] = db^{-1}d^{-1}b, \quad b = [a^{-1}, d]^n = (a^{-1}dad^{-1})^n. \tag{3.6}$$

From now on, let us assume that $n = 1$. Then the second relation in (3.6) gives

$$ab = dad^{-1}. \tag{3.7}$$

Combining (3.7) with the first relation in (3.6) gives $ab = dad^{-1} = db^{-1}d^{-1}b$, which can be simplified to $a = db^{-1}d^{-1}$. Thus we have

$$a^{-1} = dbd^{-1}. \tag{3.8}$$

Hence we see that (3.7) and (3.8) give the standard representation of the monodromy of the $T^2 = a \times b$ bundle over $S^1 = d$ that is the 0-surgery on S^3 along the trefoil $K = K_1$.

3.3 Minimal symplectic 4-manifolds with $b^+ = 1$

3.3.1 A new description of a minimal symplectic $E(1)$

The first example of an exotic smooth structure on the elliptic surface $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, and in fact the first exotic smooth structure on any closed topological 4-manifold, was constructed by Donaldson in [16]. Donaldson's example was the Dolgachev surface $E(1)_{2,3}$. Later on, Friedman showed that $\{E(1)_{p,q} \mid \gcd(p,q) = 1\}$ contains infinitely many nondiffeomorphic 4-manifolds (cf. [32]). In [28] Fintushel and Stern have shown that knot surgered manifolds $E(1)_K$ give infinitely many irreducible smooth structures on $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$.

Consider $S^2 \times T^2 = S^2 \times (S^1 \times S^1)$ equipped with its product symplectic form, and denote the last two circle factors by x and y. One can take the union of three symplectic surfaces $(\{s_1\} \times T^2) \cup (S^2 \times \{t\}) \cup (\{s_2\} \times T^2)$ in $S^2 \times T^2$, and resolve the two double points symplectically. This yields a genus two symplectic surface in $S^2 \times T^2$ with self-intersection four. Symplectically blowing up $S^2 \times T^2$ along these four intersection points and taking the proper transform, we obtain a symplectic genus two surface Σ in $Y = (S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$. Note that the inclusion induced homomorphism from $\pi_1(\Sigma) = \langle a, b, c, d \mid [a, b][c, d] = 1\rangle$ into $\pi_1(Y) = \langle x, y \mid [x, y] = 1\rangle$ maps the generators as follows:

$$a \mapsto x,\ b \mapsto y,\ c \mapsto x^{-1},\ d \mapsto y^{-1}.$$

Let us run the same steps in a second copy of $S^2 \times T^2$ and label every object with a prime symbol at the end. That is, $Y' = (S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$, Σ' is the same symplectic genus two surface described above with π_1 generators a', b', c', d', and finally let x', y' denote the generators of the $\pi_1(Y')$. Let X be the symplectic fiber sum of Y and Y' along Σ and Σ' via a diffeomorphism that extends the orientation-preserving diffeomorphism $\phi : \Sigma \to \Sigma'$, described by:

$$a \mapsto a'b', \ b \mapsto (a')^{-1}, \ c \mapsto c', \ d \mapsto d'.$$

The Euler characteristic of X can be computed as $e(X) = 4+4-2(2-2\cdot 2) = 12$, and the Novikov additivity gives the signature $\sigma(X) = -4 + (-4) = -8$, which are exactly the Euler characteristic and the signature of $Z = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. We claim that X is already simply-connected and thus no Luttinger surgery is needed. Note that $\pi_1(Y \setminus \nu\Sigma) = \pi_1(Y)$ since a meridian circle of Σ bounds a punctured exceptional sphere from one of the four blowups. Using Seifert-Van Kampen theorem, we see that

$$\begin{aligned}\pi_1(X) &= \langle x, y, x', y' \mid [x, y] = [x', y'] = 1, \\ &\qquad x = x'y', \ y = (x')^{-1}, \ x^{-1} = (x')^{-1}, \ y^{-1} = (y')^{-1}\rangle.\end{aligned}$$

We conclude that $x = x'$, $y = y'$, $y = x^{-1}$. Thus $x = x'y'$ implies $y = 1$, and in turn $x = 1$. So $\pi_1(X) = 1$. Hence by Theorem 2.0.1, X is homeomorphic to $E(1)$. However, X is irreducible by Theorem 3, and therefore X is not diffeomorphic to $E(1)$. The 4-manifold X we obtained here can be shown to be the knot surgered manifold $E(1)_K$, where the knot K is the trefoil (cf. [29]).

Alternatively we could construct the above manifold in the following way. First we symplectically sum two copies of $(S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$ along Σ and Σ' via a map that directly identifies the generators a, b, c, d with a', b', c', d' in that order. Call this symplectic 4-manifold X' and observe that while the characteristic numbers e and σ are the same as above, this manifold has $\pi_1(X') = H_1(X'; \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(X'; \mathbb{Z})$ has four additional classes that do not occur in X. These classes are as follows. Inside $((S^2 \times T^2)\#4\overline{\mathbb{CP}}^2) \setminus \nu\Sigma$, there are cylinders C_a and C_b with

$$\partial C_a = a \cup c, \quad \partial C_b = b \cup d.$$

Similiarly we obtain cylinders C'_a and C'_b in the second copy of $((S^2 \times T^2)\#4\overline{\mathbb{CP}}^2) \setminus \nu\Sigma'$. Thus we can form the following internal sums in X':

$$\Sigma_a = C_a \cup C'_a, \quad \Sigma_b = C_b \cup C'_b.$$

These are all tori of self-intersection zero. Let μ denote a meridian of Σ, and let $R_a = \tilde{a} \times \mu$, and $R_b = \tilde{b} \times \mu$ be the 'rim tori', where $\tilde{a}$ and $\tilde{b}$ are suitable parallel copies of the generators a and b. Note that $[R_a]^2 = [R_b]^2 = [\Sigma_a]^2 = [\Sigma_b]^2 = 0$, and $[R_a] \cdot [\Sigma_b] = 1 = [R_b] \cdot [\Sigma_a]$.

Observe that these rim tori are in fact Lagrangian. One can show that the effect of two Luttinger surgeries $(R_a, \tilde{a}, -1)$ and $(R_b, \tilde{b}, -1)$ is the same as changing the gluing map that we have used in the symplectic sum to the gluing map ϕ in the first construction. This second viewpoint is the one that will fit in with our construction of an infinite family of pairwise nondiffeomorphic smooth structures in Section 3.5.

3.3.2 A new construction of a minimal symplectic $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$

The first example of an exotic $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ was constructed by J. Park in [60] by using rational blowdown (cf. [27]), and the Seiberg-Witten invariant calculation in [59] shows that it is irreducible. Infinitely many exotic examples were later constructed by Fintushel and Stern in [30]. All of their constructions use the rational blowdown technique. Here we construct another irreducible symplectic 4-manifold homemorphic but not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ using our scheme, and thus without using any rational blowdown.

We equip $T^4 = T^2 \times T^2$ and $S^2 \times T^2$ with their product symplectic forms. The two orthogonal symplectic tori in T^4 can be used to obtain a symplectic surface of genus two with self-intersection two. Symplectically blowing-up at these self-intersection points we obtain a new symplectic surface Σ of genus two with trivial normal bundle in $Y = T^4\#2\overline{\mathbb{CP}}^2$. The generators of $\pi_1(T^4\#2\overline{\mathbb{CP}}^2)$ are the circles a, b, c, d, and the inclusion induced homomorphism from $\pi_1(\Sigma)$ to

$$\pi_1(Y) = \langle a, b, c, d \mid [a,b] = [a,c] = [a,d] = [b,c] = [b,d] = [c,d] = 1\rangle$$

is surjective. Indeed the four generators of $\pi_1(\Sigma)$ are mapped onto a, b, c, d in $\pi_1(Y)$, respectively.

On the other hand, as in Subsection 3.3.1, we can start with $S^2 \times T^2$ and get a symplectic genus two surface Σ' in $Y' = (S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$. Once again $\pi_1(Y') = \langle x, y \mid [x,y] = 1\rangle$ and the generators a', b', c', d' of $\pi_1(\Sigma')$ are identified with x, y, x^{-1}, y^{-1}, respectively.

We take the symplectic sum of Y and Y' along Σ and Σ' given by a diffeomorphism that extends the identity map sending $a \mapsto a', b \mapsto b', c \mapsto c', d \mapsto d'$ to obtain an intermediate 4-manifold X'. The Euler characteristic can be computed as $e(X') = 2 + 4 + 4 = 10$, and the Novikov additivity gives $\sigma(X') = -2 + (-4) = -6$, which are the characteristic numbers of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Since exceptional spheres intersect Σ and Σ' transversally once, we have $\pi_1(Y \setminus \nu\Sigma) \cong \pi_1(Y)$ and $\pi_1(Y' \setminus \nu\Sigma') \cong \pi_1(Y')$. Using Seifert-Van Kampen theorem, we compute that

$$\begin{aligned}\pi_1(X') \;\; = \;\; & \langle a, b, c, d, x, y \mid [a,b] = [a,c] = [a,d] = [b,c] = [b,d] = [c,d] = 1, \\ & [x,y] = 1,\ a = x,\ b = y,\ c = x^{-1},\ d = y^{-1}\rangle.\end{aligned}$$

Thus $\pi_1(X') = \langle x, y \mid [x,y] = 1\rangle \cong \mathbb{Z}^2$, and it follows that $b_2(X') = 12$ from our Euler characteristic calculation above. The four homologically essential Lagrangian tori in T^4 are also contained in X', and thus one can see that condition (II) is satisfied.

The two Luttinger surgeries we choose are -1 surgery on $\tilde{a} \times c$ along $\tilde{a}$ and another -1 surgery on $\tilde{b} \times c$ along $\tilde{b}$. Here, $\tilde{a}$ and $\tilde{b}$ are suitable parallel copies of the generators a and b, respectively. We claim that the manifold X we obtain after these two Luttinger surgeries is simply-connected. To prove our claim, we observe that these two Luttinger surgeries could be first made in the T^4 piece that we had at the very beginning. This is because both Lagrangian tori $\tilde{a} \times c$ and $\tilde{b} \times c$ lie in the complement of Σ. By our observation in Section 3.2.2, the result of these two Luttinger surgeries in T^4 is diffeomorphic to $S^1 \times M_K$. Observe that $\pi_1((S^1 \times M_K)\#2\overline{\mathbb{CP}}^2 \setminus \nu\Sigma) \cong \pi_1(S^1 \times M_K)$, which is (cf. [6] and (3.6)–(3.8) in Section 3.2.2)

$$\langle a, b, c, d \mid [a, b] = [c, a] = [c, b] = [c, d] = 1,\ dad^{-1} = [d, b^{-1}],\ b = [a^{-1}, d] \rangle.$$

As before, $\pi_1(((S^2 \times T^2)\#4\overline{\mathbb{CP}}^2) \setminus \nu\Sigma') \cong \pi_1(S^2 \times T^2) = \langle x, y \mid [x, y] = 1 \rangle$. Therefore by Seifert-Van Kampen theorem,

$$\begin{aligned}\pi_1(X) \quad = \quad & \langle a, b, c, d, x, y \mid [a, b] = [c, a] = [c, b] = [c, d] = 1, \\ & dad^{-1} = [d, b^{-1}],\ b = [a^{-1}, d],\ [x, y] = 1, \\ & a = x,\ b = y,\ c = x^{-1},\ d = y^{-1} \rangle.\end{aligned}$$

Thus x and y generate the whole group, and by direct substitution we see that $y^{-1}xy = [y^{-1}, y^{-1}] = 1$ and $y = [x^{-1}, y^{-1}]$. The former gives $x = 1$, and the latter then yields $y = 1$. Hence $\pi_1(X) = 1$. Therefore by Theorem 2.0.1, X is homeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Since the latter is not irreducible, X is an exotic copy of it.

3.3.3 A new construction of a minimal symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$

The first example of an exotic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ was obtained by J. Park, Stipsicz and Szabó in [62], combining the double node neighborhood surgery technique discovered by Fintushel and Stern (cf. [30]) with rational blowdown. Fintushel and Stern also constructed similar examples using the same techniques in [30]. The first exotic symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ was constructed in [3]. Here, we present another construction with a much simpler π_1 calculation, using our construction scheme.

As in Subsection 3.3.2, we construct a symplectic surface Σ of genus two with trivial normal bundle in $Y = T^4\#2\overline{\mathbb{CP}}^2$. Let us use the same notation for the fundamental groups as above. Take another copy $Y' = T^4\#2\overline{\mathbb{CP}}^2$, and denote the same genus two surface by Σ', while using the prime notation for all corresponding fundamental group elements.

We obtain a new manifold X' by taking the symplectic sum of Y and Y' along Σ and Σ' determined by the map $\phi : \Sigma \to \Sigma'$ that satisfies:

$$a \mapsto c',\ b \mapsto d',\ c \mapsto a',\ d \mapsto b'. \tag{3.9}$$

By Seifert-Van Kampen theorem, one can easily verify that $\pi_1(X') \cong \mathbb{Z}^4$ generated by, say a, b, a', b'. The characteristic numbers we get are: $e(X') = 2 + 2 + 4 = 8$ and $\sigma(X') = -2 + (-2) = -4$, the characteristic numbers of $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. Finally the homologically essential Lagrangian tori in the initial T^4 copies can be seen to be contained in X' with the same properties. Thus $r \geq 8 = 2b_1(X') = b_2(X') - b_2(\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2)$, so our condition (II) is satisfied.

We perform the following four Luttinger surgeries on pairwise disjoint Lagrangian tori:

$$(\tilde{a} \times c, \tilde{a}, -1),\quad (\tilde{b} \times c, \tilde{b}, -1),\quad (\tilde{a}' \times c', \tilde{a}', -1),\quad (\tilde{b}' \times c', \tilde{b}', -1).$$

It is quite simple to see that the resulting symplectic 4-manifold X satisfies $H_1(X; \mathbb{Z}) = 0$. Using the observation in Section 3.2.2 again, after changing the order of operations and assuming that we have done the Luttinger surgeries at the very beginning, we can view X as the fiber sum of two copies of $(S^1 \times M_K)\#2\overline{\mathbb{CP}}^2$ along the identical genus two surface Σ where the gluing map switches the symplectic bases for Σ as in (3.9). Thus, using Seifert-Van Kampen's theorem as above, we can see that

$$\begin{aligned}\pi_1(X) &= \langle a, b, c, d, a', b', c', d' \mid [a, b] = [c, a] = [c, b] = [c, d] = 1, \\ &\quad dad^{-1} = [d, b^{-1}],\ b = [a^{-1}, d],\ [a', b'] = [c', a'] = [c', b'] = [c', d'] = 1, \\ &\quad d'a'(d')^{-1} = [d', (b')^{-1}],\ b' = [(a')^{-1}, d'], \\ &\quad a = c',\ b = d',\ c = a',\ d = b' \rangle.\end{aligned}$$

Now $b' = [(a')^{-1}, d']$ can be rewritten as $d = [c^{-1}, b]$. Since b and c commute, $d = 1$. The relations $dad^{-1} = [d, b^{-1}]$ and $b = [a^{-1}, d]$ then quickly implies that $a = 1$ and $b = 1$, respectively. Lastly, $d'a'(d')^{-1} = [d', (b')^{-1}]$ is $bcb^{-1} = [b, d^{-1}]$, so $c = 1$ as well. Since a, b, c, d generate $\pi_1(X)$, we see that X is simply-connected. By similar arguments as before, X is an irreducible symplectic 4-manifold that is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$.

3.3.4 A minimal symplectic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ in terms of Luttinger surgeries

The first irreducible symplectic smooth structures on $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ were constructed independently by Akhmedov and D. Park in [6] and by Baldridge and Kirk in [10]. Shortly after, a more elegant construction appeared in [25].

Let us demonstrate how the construction of an exotic symplectic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ in [6] fits into our recipe. We will use three copies of the 4-torus, T_1^4, T_2^4 and T_3^4. Symplectically fiber sum the first two along the 2-tori $a_1 \times b_1$ and $a_2 \times b_2$ of self-intersection zero, with a gluing map that identifies a_1 with a_2 and b_1 with b_2. Clearly we get $T^2 \times \Sigma_2$, where the symplectic genus 2 surface Σ_2 is obtained by gluing together the orthogonal punctured symplectic tori $(c_1 \times d_1) \setminus D^2$ in T_1^4 and $(c_2 \times d_2) \setminus D^2$ in T_2^4. Here, $\pi_1(T^2 \times \Sigma_2)$ has six generators $a_1 = a_2$, $b_1 = b_2$, c_1, c_2, d_1 and d_2 with relations $[a_1, b_1] = 1$, $[c_1, d_1][c_2, d_2] = 1$ and moreover a_1 and b_1 commute with all

c_i and d_i. The two symplectic tori $a_3 \times b_3$ and $c_3 \times d_3$ in T_3^4 intersect at one point, which can be smoothened to get a symplectic surface of genus two. Blowing up T_3^4 twice at the self-intersection points of this surface as before, we obtain a symplectic genus two surface Σ' of self-intersection zero.

Next we take the symplectic fiber sum of $Y = T^2 \times \Sigma_2$ and $Y' = T_3^4 \# 2\overline{\mathbb{CP}}^2$ along the surfaces Σ_2 and Σ', determined by a map that sends the circles c_1, d_1, c_2, d_2 to a_3, b_3, c_3, d_3 in the same order. By Seifert-Van Kampen theorem, the fundamental group of the resulting manifold X' can be seen to be generated by a_1, b_1, c_1, d_1, c_2 and d_2, which all commute with each other. Thus $\pi_1(X')$ is isomorphic to $\mathbb{Z}^6$. It is easy to check that $e(X') = 6$ and $\sigma(X') = -2$, which are also the characteristic numbers of $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Now we perform six Luttinger surgeries on pairwise disjoint Lagrangian tori:

$$(a_1 \times \tilde{c}_1, \tilde{c}_1, -1), \quad (a_1 \times \tilde{d}_1, \tilde{d}_1, -1), \quad (\tilde{a}_1 \times c_2, \tilde{a}_1, -1),$$
$$(\tilde{b}_1 \times c_2, \tilde{b}_1, -1), \quad (c_1 \times \tilde{c}_2, \tilde{c}_2, -1), \quad (c_1 \times \tilde{d}_2, \tilde{d}_2, -1).$$

Afterwards we obtain a symplectic 4-manifold X with $\pi_1(X)$ generated by a_1, b_1, c_1, d_1, c_2, d_2 with relations:

$$[b_1, d_1^{-1}] = b_1 c_1 b_1^{-1}, \quad [c_1^{-1}, b_1] = d_1, \quad [d_2, b_1^{-1}] = d_2 a_1 d_2^{-1},$$
$$[a_1^{-1}, d_2] = b_1, \quad [d_1, d_2^{-1}] = d_1 c_2 d_1^{-1}, \quad [c_2^{-1}, d_1] = d_2,$$

and all other commutators are equal to the identity. Since $[b_1, c_2] = [c_1, c_2] = 1$, $d_1 = [c_1^{-1}, b_1]$ also commutes with c_2. Thus $d_2 = 1$, implying $a_1 = b_1 = 1$. The last identity implies $c_1 = d_1 = 1$, which in turn implies $c_2 = 1$.

Hence X is simply-connected and since these surgeries do not change the characteristic numbers, we have it homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$. Since Y is minimal and the exceptional spheres in Y' intersect Σ', Theorem 2 guarantees that X' is minimal.

It follows from Theorem 3 that X is an irreducible symplectic 4-manifold which is not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.

3.4 Minimal symplectic 4-manifolds with $b^+ > 1$

Symplectic fiber sum operation can be effectively used to obtain several other new minimal symplectic 4-manifolds with bigger Euler characteristics from small minimal symplectic 4-manifolds. Here we will provide a sample result in this direction.

Theorem 6 *Let X be a simply-connected minimal symplectic 4-manifold which is not a sphere bundle over a Riemann surface and such that X contains a genus two symplectic surface of self-intersection zero. Then X can be used to construct simply-connected irreducible symplectic 4-manifolds Z' and Z'' satisfying:*

$$\begin{aligned}(b_2^+(Z'), b_2^-(Z')) &= (b_2^+(X)+2, b_2^-(X)+4),\\ (b_2^+(Z''), b_2^-(Z'')) &= (b_2^+(X)+2, b_2^-(X)+6).\end{aligned}$$

Proof: Let us denote the genus two symplectic surface of self-intersection 0 in X by Σ_2. By our assumptions, the complement $X\setminus\nu\Sigma_2$ does not contain any exceptional spheres. Take $T^4 = T^2\times T^2$ equipped with a product symplectic form, with the genus two symplectic surface that is obtained from the two orthogonal symplectic tori after resolving their singularities. After symplectically blowing up T^4 at two points on this surface, we get a symplectic genus two surface Σ_2' of self-intersection 0 in $T^4\#2\overline{\mathbb{CP}}^2$, and it is clear that $(T^4\#2\overline{\mathbb{CP}}^2)\setminus\nu\Sigma_2'$ does not contain any exceptional spheres either. Since we also assumed that X was not a sphere bundle over a Riemann surface, it follows from Theorems 2 and 3 that the 4-manifold Z' obtained as the symplectic sum of X with $T^4\#2\overline{\mathbb{CP}}^2$ along Σ_2 and Σ_2' is minimal and hence irreducible.

Next we take $S^2\times T^2$ with its product symplectic form, and as before consider the genus two symplectic surface obtained from two parallel copies of the symplectic torus component and a symplectic sphere component, after symplectically resolving their intersections. Symplectically blowing up $S^2 \times T^2$ on four points on this surface, we get a new symplectic genus 2 surface Σ_2'' with self-intersection 0 in $(S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$. Although this second piece $(S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$ is an S^2 bundle over a Riemann surface, the surface Σ_2'' cannot be a section of this bundle. Moreover, it is clear that $((S^2 \times T^2)\#4\overline{\mathbb{CP}}^2) \setminus \nu\Sigma_2''$ does not contain any exceptional spheres. Hence, applying Theorems 2 and 3 again, we see that the 4-manifold Z'' obtained as the symplectic sum of X with $(S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$ along Σ_2 and Σ_2'' is minimal and irreducible.

It is a straightforward calculation to see that $(e(Z'), \sigma(Z')) = (e(X)+6, \sigma(X)-2)$ and $(e(Z''), \sigma(Z'')) = (e(X)+8, \sigma(X)-4)$. Note that the new meridian in $X \setminus \nu\Sigma_2$ dies after the fiber sum since the meridian of Σ_2' in $T^4\#2\overline{\mathbb{CP}}^2$ can be killed along any one of the two exceptional spheres. The same argument works for the fiber sum with $(S^2 \times T^2)\#4\overline{\mathbb{CP}}^2$. Hence Seifert-Van Kampen's theorem implies that $\pi_1(Z') = \pi_1(Z'') = 1$. Our claims about b_2^+ and b_2^- follow immediately. □

Corollary 3.4.1 *There are exotic* $3\mathbb{CP}^2\#(2l+3)\overline{\mathbb{CP}}^2$, *for* $l = 2, \ldots, 6$, *which are all irreducible and symplectic.*

Proof: We observe that each one of the irreducible symplectic $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$ $(k = 1, \ldots, 4)$ we obtained above contains at least one symplectic genus two surface of self-intersection zero. (Also see Section 3.5 for more detailed description of these surfaces.) To be precise, let us consider the genus two surface Σ which is a parallel copy of the genus two surface used in the last symplectic sum in any one of our constructions. Since these exotic 4-manifolds are all minimal, they cannot be the total space of a sphere bundle over a Riemann surface with any blow-ups in the fibers. Also they cannot be homeomorphic to either $F \times S^2$ or $F \tilde{\times} S^2$ for some Riemann surface F, because of their intersection forms. Therefore we see that assumptions of

Theorem 6 hold. It quickly follows from Theorem 6 that we can obtain irreducible symplectic 4-manifolds homeomorphic to $3\mathbb{CP}^2\#(2l+3)\overline{\mathbb{CP}}^2$, for $l = 2, \dots, 6$. $\square$

Remark 3.4.2 *Using the generic torus fiber and a sphere section of self-intersection* -1 *in an elliptic fibration on* $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, *one can form a smooth symplectic torus* T_1 *of self-intersection* $+1$ *in* $E(1)$. *As each one of our exotic* $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$ *for* $k = 1, \dots, 4$ *contains at least one symplectic torus of self-intersection* -1 *(these tori are explicitly described in Section 3.5), we can symplectically fiber sum each exotic* $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$ *with* $E(1)$ *along a chosen torus of self-intersection* -1 *and* T_1 *to obtain irreducible symplectic 4-manifolds that are homeomorphic but not diffeomorphic to* $3\mathbb{CP}^2\#(2k+11)\overline{\mathbb{CP}}^2$ *for* $k = 1, \dots, 4$.

The crux of the above construction is that one can use simple minimal symplectic 4-manifold blocks, possibly with nontrivial fundamental groups, to produce new *simply-connected* minimal symplectic 4-manifolds. In a joint work with A. Akhmedov, S. Baldridge, B. D. Park and P. Kirk, we exploited this basic idea to fill in a large region of the geography plane [4]. Let us finish this section by quoting the main theorem from this work:

Theorem 3.4.3 (Akhmedov, Baldridge, Baykur, Kirk, Park [4]) *Let* σ *and* e *denote integers satisfying* $2e + 3\sigma \geq 0$, *and* $e + \sigma \equiv 0 \pmod 4$. *If, in addition,* $\sigma \leq -2,$, *then there exists a simply connected minimal symplectic 4-manifold with signature* σ *and Euler characteristic* e *and odd intersection form, except possibly for* (σ, e) *equal to* $(-3, 7)$, $(-3, 11)$, $(-5, 13)$, *or* $(-7, 15)$. *Moreover, for each integer* $k \geq 49$, *there exists a simply connected minimal symplectic 4-manifold* $X_{2k-1,2k}$ *with* $(e, \sigma) = (4k+1, -1)$, *and for each integer* $k \geq 45$, *there exists a simply connected minimal symplectic 4-manifold* $X_{2k+1,2k+1}$ *with* $(e, \sigma) = (4k+4, 0)$.

3.5 Infinite families of nonsymplectic irreducible smooth structures

In this section we will show how to construct an infinite family of pairwise nondiffeomorphic 4-manifolds that are homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. The very same idea will apply to the others, as we will discuss briefly. We begin by describing these families of 4-manifolds, showing that they all have the same homeomorphism type, and afterwards we will use the Seiberg-Witten invariants to distinguish their diffeomorphism types. The SW invariants will be distinguished via Theorem 5.

We first need to choose a null-homologous torus and peform $1/n$ surgery on it as in Subsection 3.1.4. We then prove that $\pi_1 = 1$ for the resulting infinite family of 4-manifolds. To apply Theorem 5 in its full strength, i.e. to obtain a family that consists of pairwise nondiffeomorphic 4-manifolds, we will show that we have exactly one basic class for $X_{\Lambda,\lambda}(0)$, up to sign, for each exotic X that we have constructed. We will do this check by straightforward calculations using adjunction inequalities.

In all the constructions in Section 3.3, we observe that there is a copy of $V = (S^1 \times M_K) \setminus (F \cup S)$ embedded in the exotic X we constructed, where F is the fiber and S is the section of $S^1 \times M_K$, viewed as a torus bundle over a torus. As shown in the Section 3.2.2, $S^1 \times M_K$ is obtained from T^4 after two Luttinger surgeries, which are performed in the complement of $F \cup S$. So we can think of $S^1 \times M_K$ as being obtained in two steps. Let V_0 be the complement of $F \cup S$ in the intermediate 4-manifold which is obtained from T^4 after the first Luttinger surgery. The next Luttinger surgery, say $(L, \gamma, -1)$, produces V from V_0. (In the Section 3.2.2, $L = c \times \tilde{b}$ and $\gamma = \tilde{b}$.) This second surgery on L in V_0 gives rise to a nullhomologous torus Λ in V. There is a loop λ on Λ so that the 0 surgery on Λ with respect to λ gives V_0 back. As the framing for this surgery must be the nullhomologous framing, we call it the '0-framing'. Note that performing a $1/n$ surgery on Λ with respect to λ and this 0-framing in V is the same as performing an $(L, \gamma, -(n+1))$ surgery in V_0 with respect to the Lagrangian framing. We denote the result of such a surgery by $V(n) = V_{\Lambda,\lambda}(1/n)$. In this notation, $V(\infty) = V_0$ and we see that $V(0) = V$.

We know that performing a $-n$ surgery on L with respect to γ and the Lagrangian framing, we obtain $V(n-1) = (S^1 \times M_{K_n}) \setminus (F \cup S)$, where K_n is the n-twist knot. It should now be clear that replacing a copy of V in X with $V(n-1) = (S^1 \times M_{K_n}) \setminus (F \cup S)$ (i.e. 'using the n-twist knot') has the same effect as performing a $1/(n-1)$ surgery in the 0-framing on Λ in $V \subset X$. We denote the result of such a surgery by $X_n = X_{\Lambda,\lambda}(1/(n-1))$. Clearly, $X_1 = X$. We claim that the family $\{X_n \mid n = 1, 2, 3, \ldots\}$ are all homeomorphic to X but have pairwise inequivalent Seiberg-Witten invariants. The first claim is proved in the following lemma.

Lemma 7 *Let X_n be the infinite family corresponding to a fixed exotic copy of $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ that we have constructed above. Then X_n are all homeomorphic to X.*

Proof: For a fixed exotic X, $\pi_1(X_n)$ only differs from $\pi_1(X)$ by replacing a single relation of the form $b = [a^{-1}, d]$ by $b = [a^{-1}, d]^n$ in the presentation of $\pi_1(X)$ we have used. Thus one only needs to check that raising the power of the commutator in one such relation does not effect our calculation of $\pi_1(X) = 1$. This is easily verified in all of our examples. Hence all the fundamental group calculations follow the same lines and result in the trivial group.

Since X_n's differ from X only by surgeries on a nullhomologous torus, the characteristic numbers remain the same. On the other hand, since none have new homology classes, the parity should be the same. By Theorem 2.0.1 again, they all should be homeomorphic to each other. $\square$

Below, let X be a 4-manifold obtained by fiber summing 4-manifolds Y and Y' along submanifolds $\Sigma \subset Y$ and $\Sigma' \subset Y'$. Let $A \subset Y$ and $B' \subset Y'$ be surfaces transversely intersecting Σ and Σ' positively at one point, respectively. Then we can form the internal connected sum $A\#B'$ inside the fiber sum X, which is the closed surface that is the union of punctured surfaces $(A \setminus (A \cap \nu\Sigma)) \subset (X \setminus \nu\Sigma)$ and $(B' \setminus (B' \cap \nu\Sigma')) \subset (Y' \setminus \nu\Sigma')$. It is not hard to see that the intersection number between $A\#B'$ and $\Sigma = \Sigma'$ in X is one, and thus they are both homologically essential. If all these manifolds and submanifolds are symplectic and the fiber sum is done symplectically, then $A\#B'$ can be made a symplectic submanifold of X as well. Also note that, if either A or B' has self-intersection zero, then their parallel copies in their tubular neighborhoods can also be used to produce such internal sums in X.

3.5.1 An infinite family of irreducible smooth structures on $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$

Let X be the exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ that we have described in Subsection 3.3.4. We begin by describing the surfaces that generate $H_2(X;\mathbb{Z})$. There is a symplectic torus $T = T^2 \times \{\mathrm{pt}\}$ of self-intersection zero in $Y = T^2 \times \Sigma_2$ intersecting $\Sigma = \{\mathrm{pt}\} \times \Sigma_2$ positively at one point. On the other side, in $Y' = T^4\#2\overline{\mathbb{CP}}^2$, there is a symplectic torus T_1' of self-intersection zero, and two exceptional spheres E_1' and E_2', each of which intersects Σ' positively at one point. (There is actually another symplectic torus T_2' in Y' satisfying $[\Sigma'] = [T_1'] + [T_2'] - [E_1'] - [E_2']$ in $H_2(Y';\mathbb{Z})$, but we will be able to express the homology class that T_2' induces in X in terms of the four homology classes below.)

Hence we have four homologically essential symplectic surfaces: two genus two surfaces $\Sigma = \Sigma'$, $G = T\#T_1'$, and two tori $R_i = T\#E_i'$, $i = 1, 2$. Clearly $[\Sigma]^2 = [G]^2 = 0$, and $[R_1]^2 = [R_2]^2 = -1$. It is a straightforward argument to see that these span $H_2(X;\mathbb{Z})$, and the corresponding intersection form is isomorphic to that of $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. (Note that $[T\#T_2'] = [\Sigma_2] - [G] + [R_1] + [R_2]$.)

The 0-surgery on Λ with respect to λ results in a 4-manifold $X_0 = X_{\Lambda,\lambda}(0)$ satisfying $H_1(X_0;\mathbb{Z}) \cong \mathbb{Z}$ and $H_2(X_0;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus \mathbb{Z}^2$, where the new 2-dimensional homology classes are represented by two Lagrangian tori L_1 and L_2. Both L_j have self-intersection zero. They intersect each other positively at one point, and they do not intersect with any other class. Thus the adjunction inequality forces this pair to not appear in any basic class of X_0. Denoting the homology classes in X_0 that come from X by the same symbols, let $\beta = a[\Sigma] + b[G] + \sum_i r_i[R_i]$ be a basic class of X_0. Since it is a characteristic element, a and b should be even, and r_1 and r_2 should be odd.

Since $b_2^+(X_0) > 1$, applying the (generalized) adjunction inequality for Seiberg-Witten basic classes (cf. [58]) to all these surfaces, we conclude the following.

(i) $2 \geq 0 + |\beta \cdot [G]|$, implying $2 \geq |a|$.

(ii) $2 \geq 0 + |\beta \cdot [\Sigma]|$, implying $2 \geq |b + \sum_i r_i|$.

(iii) $0 \geq -1 + |\beta \cdot [R_i]|$, implying $1 \geq |a - r_i|$ for $i = 1, 2$.

On the other hand, since X_0 is symplectic and $b_2^+(X_0) > 1$, X_0 is of simple type so we have $\beta^2 = 2e(X_0) + 3\sigma(X_0) = 2e(X) + 3\sigma(X) = 2 \cdot 6 + 3(-2) = 6$, implying:

(iv) $6 = 2a(b + \sum_i r_i) - \sum_i r_i^2$.

From (i), we see that a can only be 0 or ± 2. However, (iv) implies that $a \neq 0$. Let us take $a = 2$. Then by (iv) and (ii) we have

$$\left|6 + \sum_i r_i^2\right| = 4\left|b + \sum_i r_i\right| \leq 8,$$

which implies that $\sum_i r_i^2 \leq 2$. Therefore by (iii) we see that both r_i have to be 1. Finally by (iv) again, $b = 0$.

Similarly, if we take $a = -2$, we must have $r_1 = r_2 = -1$ and $b = 0$. Hence the only basic classes of X_0 are $\pm(2[\Sigma] + \sum_i [R_i]) = \pm K_{X_0}$, where K_{X_0} denotes the canonical class of X_0. By Theorem 5, all $X_n = X_{\Lambda,\lambda}(1/(n-1))$ are pairwise nondiffeomorphic.

Moreover, by Remark 3.1.2 we see that $X = X_1$ also has one basic class up to sign. It is easy to see that this is the canonical class $K_X = 2[\Sigma]+[R_1]+[R_2]$. Therefore the square of the difference of the two basic classes is $4K_X^2 = 24 \neq -4$, implying that X is irreducible (and hence minimal) by a direct application of Seiberg-Witten theory (cf. [26]). Furthermore, the basic class β_n of X_n corresponding to the canonical class K_{X_0} satisfies

$$\begin{aligned} \mathrm{SW}_{X_n}(\beta_n) &= \mathrm{SW}_X(K_X) + (n-1)\mathrm{SW}_{X_0}(K_{X_0}) \\ &= 1 + (n-1) = n. \end{aligned} \tag{3.10}$$

Thus every X_n with $n \geq 2$ is nonsymplectic. In conclusion, we have proved the following.

Theorem 8 *There is an infinite family of pairwise nondiffeomorphic 4-manifolds which are all homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. All of these manifolds are irreducible, and they possess exactly one basic class, up to sign. All except for one are nonsymplectic.*

3.5.2 Infinite families of irreducible $\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2$ for $k = 2, 3, 4$

For exotic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$'s, the second homology of X_0 will be generated by the following surfaces: two genus two surface of self-intersection zero, $\Sigma = \Sigma'$ and $G = T\#T'$, four tori of self-intersection -1, $R_i = E_i\#T'$ $(i = 1, 2)$ and $S_j = T\#E_j'$ $(j = 1, 2)$, and two Lagrangian tori L_1 and L_2 as before. A basic class of X_0 is of the form

$\beta = a[\Sigma] + b[G] + \sum_i r_i[R_i] + \sum_j s_j[S_j]$, where a and b are even and r_i and s_j are odd. The inequalities are:

(i) $2 \geq 0 + |\beta \cdot [G]|$, implying $2 \geq |a|$.

(ii) $2 \geq 0 + |\beta \cdot [\Sigma]|$, implying $2 \geq |b + \sum_i r_i + \sum_j s_j|$.

(iii) $0 \geq -1 + |\beta \cdot [R_i]|$, implying $1 \geq |a - r_i|$ for $i = 1, 2$.

(iv) $0 \geq -1 + |\beta \cdot [S_j]|$, implying $1 \geq |a - s_j|$ for $j = 1, 2$.

(v) $4 = 2a(b + \sum_i r_i + \sum_j s_j) - (\sum_i r_i^2 + \sum_j s_j^2)$.

By (i), a can only take the values $0, \pm 2$, where 0 is ruled out by looking at (v). If $a = 2$, then by (iii) and (iv) r_i and s_j are either 1 or 3. However, using (ii) and (v) as before, we see that none of these can be 3. It follows that $r_i = s_j = 1$ for all i and j, and $b = -2$. The case when $a = -2$ is similar, and we see that X_0 has only two basic classes $\pm(2[\Sigma] - 2[G] + \sum_i [R_i] + \sum_j [S_j])$.

For exotic $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$'s, the classes are similar except now we have four tori of the form S_j. For a basic class $\beta = a[\Sigma] + b[G] + \sum_{i=1}^2 r_i[R_i] + \sum_{j=1}^4 s_j[S_j]$ of X_0, we see that the coefficients have the same parity as above. The first four inequalities are the same (with (iv) holding for $j = 1, \dots, 4$), whereas the last equality (v) coming from X_0 being of simple type becomes:

(v′) $2 = 2a(b + \sum_{i=1}^2 r_i + \sum_{j=1}^4 s_j) - (\sum_{i=1}^2 r_i^2 + \sum_{j=1}^4 s_j^2)$.

Once again, (i) implies that a is 0 or ± 2, but by (v′) it cannot be 0. If $a = 2$, then by exactly the same argument as before we see that $r_i = s_j = 1$ for all i and j, and thus $b = -4$. The case $a = -2$ is similar. Therefore the only two basic classes of X_0 are $\pm(2[\Sigma] - 4[G] + \sum_{i=1}^2 [R_i] + \sum_{j=1}^4 [S_j])$.

For exotic $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$'s, the only difference is that the number of tori in total is eight. Let us denote two of the additional tori as R_3 and R_4 corresponding to say

R_a and Σ_a where the other two will be denoted by S_3 and S_4 corresponding to R_b and Σ_b as described in Subsection 3.3.1.

For a basic class $\beta = a[\Sigma] + b[G] + \sum_{i=1}^4 r_i[R_i] + \sum_{j=1}^4 s_j[S_j]$, once again a, b are even and r_i, s_j are odd. The inequalities (i)–(iv) remain the same. Finally (v′) becomes:

(v″) $0 = 2a(b + \sum_{i=1}^4 r_i + \sum_{j=1}^4 s_j) - (\sum_{i=1}^4 r_i^2 + \sum_{j=1}^4 s_j^2)$.

As before, a cannot be 0. If $a = 2$, then by the same argument we see that $r_i = s_j = 1$ for all i and j. Thus $b = -6$, and we get a basic class $\beta = 2[\Sigma] - 6[G] + \sum_{i=1}^4 [R_i] + \sum_{j=1}^4 [S_j]$. For $a = -2$ it is easy to check that we get the negative of this class.

Hence in all three examples X_0 has only two basic classes, and therefore by Theorem 5, all three families $\{X_n\}$ consist of pairwise nondiffeomorphic 4-manifolds homeomorphic to $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, or $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. Furthermore, as in the previous subsection, we see that each family $\{X_n\}$ consists of 4-manifolds with only one basic class, up to sign. In each family, all but one member are nonsymplectic as the only nonzero values of the Seiberg-Witten invariant of X_n are $\pm n$. Finally, each exotic X_n can be seen to be irreducible by a direct Seiberg-Witten argument as before (cf. [26]).

CHAPTER 4

Near-symplectic 4-manifolds

4.1 Background

4.1.1 Near-symplectic structures

Let ω be a closed 2-form on an oriented smooth 4-manifold X such that $\omega^2 \geq 0$, and Z_ω be the set of points where ω degenerates. Then ω is called a *near-symplectic structure* on X if it satisfies the following transversality condition at every point x in Z_ω: if we use local coordinates on a neighborhood U of x to identify the map $\omega : U \to \Lambda^2(T^*U)$ as a smooth map $\omega : \mathbb{R}^4 \to \mathbb{R}^6$, then the linearization $D\omega_x : \mathbb{R}^4 \to \mathbb{R}^6$ at x should have rank three —which is in fact independent of the chosen charts [9]. In particular, $Z = Z_\omega$ is a smoothly embedded 1-manifold in X, if not empty. We then call (X, ω) a near-symplectic 4-manifold, and Z the *zero locus of* ω.

If a given 4-manifold X admits a near-symplectic structure, then it is easy to see that $b^+(X) > 0$. One of the motivations for studying near-symplectic structures has been the converse observation. Namely, any closed smooth oriented 4-manifold X with $b^+(X) > 0$ can be equipped with a near-symplectic form, which was known to gauge theory afficianados since early 1980s and a written proof of which was first given by Honda through the analysis of self-dual harmonic 2-forms ([46], also see [9]). Thus the near-symplectic family is much broader than the symplectic family of 4-manifolds. For instance, connected sums of symplectic 4-manifolds can never be symplectic, due to the work of Taubes and the vanishing theorem for SW invariants. However these manifolds would still have $b^+ > 0$ and therefore are near-symplectic.

Example 4.1.1 Let M^3 be a closed 3-manifold and $f : M \to S^1$ be a circle valuded Morse function with only index 1 and 2 critical points. Then the 4-manifold $X = S^1 \times M$ can be equipped with a near-symplectic structure. To see this, first note that due to a theorem of Calabi there exists a metric g on M which makes df harmonic. Parametrize the first S^1 component by t, and consider the form $\omega = dt \wedge df + *(dt \wedge df)$, where the Hodge star operation is defined with respect to the product of the standard metric on S^1 and g on M. It is a straightforward check to verify that $\omega^2 \geq 0$ and that ω vanishes precisely on $Z = S^1 \times Crit(f)$. Finally using local charts one can see that ω vanishes transversally at every point on Z (also see the next subsection).

The attentive reader will realize that some of the symplectic building blocks extensively used in the previous chapter are in fact specific examples of this type. For consider a 3-manifold M_K obtained from S^3 after a 0-surgery on an arbitrary knot K, which comes with a circle valued Morse function as before. (Note that $\mathbb{Z} \cong H^1(M_K;\mathbb{Z}) \cong [M_K, S^1]$.) Then ω defined as above yields a *symplectic* form on $X = S^1 \times M_K$ if and only if K is fibered so that f can be assumed to have no critical points: i.e when $Z = \emptyset$.

4.1.2 Local models

Using a generalized Moser type of argument for harmonic self-dual 2-forms, Honda showed in [47] that there are exactly two local models around each connected component of Z_ω. To make this statement precise, let us consider the following local model: Take $\mathbb{R}^4$ with coordinates (t, x_1, x_2, x_3) and consider the 2-form $\Omega = dt \wedge dQ + *(dt \wedge dQ)$, where $Q(x_1, x_2, x_3) = x_1^2 - \frac{1}{2}(x_2^2 + x_3^2)$ and $*$ is the standard Hodge star operator on $\Lambda^2\mathbb{R}^4$. Restrict Ω to $\mathbb{R}$ times the unit 3-ball. Define two orientation preserving affine automorphisms of $\mathbb{R}^4$ by $\sigma_+(t, x_1, x_2, x_3) = (t + 2\pi, x_1, x_2, x_3)$ and $\sigma_-(t, x_1, x_2, x_3) = (t + 2\pi, -x_1, x_2, -x_3)$. Since both maps preserve Ω, they induce near-symplectic forms $\omega_\pm$ on the quotient spaces $N_\pm = \mathbb{R} \times D^3/\sigma_\pm$. Honda shows that given any near-symplectic (X, ω) with zero locus Z_ω, there is a Lipschitz self-homeomorphism φ on X which is identity on Z_ω, smooth outside of Z_ω and supported in an arbitrarily small neighborhood of Z_ω, such that around each circle in Z_ω, the form $\varphi^*(\omega)$ agrees with one of the two local near-symplectic models $(N_\pm, \omega_\pm)$. For our purposes, we can always replace the near-symplectic form ω with such a form $\varphi^*(\omega)$. Herein the zero circles which admit neighborhoods (N_+, ω_+) are called of *even* type, and the others of *odd* type.

On each local model $N_\pm \cong S^1 \times D^3$, one can consider fibration-like maps: $F_\pm \colon N_\pm \to S^1 \times I$ defined by

$$F_\pm(t, x_1, x_2, x_3) = (t, Q(x_1, x_2, x_3)) = (t, x_1^2 - \frac{1}{2}(x_2^2 + x_3^2)). \tag{4.1}$$

In either case, for a fixed t, we observe that on the complement of $S^1 \times 0$ we have fibrations with fibers composed of two disjoint disks in D^3 for the preimages of points with $Q > 0$, whereas the fibers are annuli for $Q < 0$. In the preimages of $(t, 0)$ we have conical singularities which amounts to attaching a 1-handle with feet on the two separate disks so to obtain the annulus on the other side. Dually it is a 2-handle attachment in the opposite direction which separates the annulus into two disks. Now if we let $t \in S^1$ vary, this amounts to doing this handle attachment fiberwise as we pass the middle circle $S^1 \times 0$ in $S^1 \times I$. The difference between the two local models N_+ and N_- manifests itself here. The model is even if and only if for a fixed $Q > 0$

the two disks are switched after one travel in the t direction, and odd otherwise.

Example 4.1.2 Once again let $X = S^1 \times M_K$ for nonfibered K. For simplicity, assume that $f : M_K \to S^1$ in Example 4.1.1 is injective on its critical points. Then the preimage of any regular value of f is a Seifert surface of K capped off with a disk, i.e a closed orientable surface. While passing an index k critical point ($k = 1, 2$), a k-handle is attached to get one Seifert surface from another. It follows that $F : M_K \to S^1$ is a fibration-like map, where the genera of fibers are increased or decreased by one at every critical point, depending on $k = 1$ or $k = 2$, respectively. When crossed with S^1, this yields a fibration-like map $F : id \times f : X \to T^2$. The base torus $T^2 = S^1 \times S^1$ can be parametrized by (t, s) where t traces the outer circle factor and s traces the base circle of f. Thus the monodromy of this fibration is trivial in the t direction and is prescribed by the knot monodromy in the s direction. Choosing local charts on a tubular neighborhood $S^1 \times D^3$ of each component of $S^1 \times Crit(f)$ and on the image $S^1 \times I$, one can see that F is locally the same as F_+ map we have defined above. This implies that all circles of Z_ω, where ω is the near-symplectic structure on X described in Example 4.1.2, are even.

4.1.3 Broken Lefschetz fibrations

In [9], Auorux, Donaldson, and Katzarkov defined a generalization of Lefschetz fibrations called "singular Lefschetz fibrations", where they allowed the maps to have singularities along embedded circles ("indefinite quadratic singularities" [9]) which are subject to the two local models described in the previous subsection (Equation 4.1) in addition to the usual nodal singularities *on the complement* of these. Here we refer to these fibrations as *broken Lefschetz fibrations* as in [64, 65], and the new type of singularities as odd or even *round singularities* depending on whether the local model around the singular circle is odd or even. A *broken Lefschetz pencil* is defined similarly, where we allow round singularities in the complement of the Lefschetz critical points *and* the base locus.

Tha main theorem of [9] states that broken Lefschetz fibrations are to near-syplectic 4-manifolds what Lefschetz fibrations are to symplectic 4-manifolds:

Theorem 4.1.3 (Auroux, Donaldson, Katzarkov [9]) *Suppose Γ is a smooth 1-dimensional submanifold of a compact oriented 4-manifold X. Then the following two conditions are equivalent:*

- *There is a near-symplectic form ω on X, with $Z_\omega = \Gamma$,*
- *There is a broken Lefschetz pencil f on X which has round singularities along Γ, with the property that there is a class $h \in H^2(X)$ such that $h(\Sigma) > 0$ for every fiber component Σ of f.*

Moreover, the implications in each direction can be obtained in a compatible way. That is, given a near-symplectic form ω, a corresponding broken Lefschetz pencil (BLP) can be obtained so that all the fibers are symplectic on the complement of the singular locus. Conversely, from a broken Lefschetz pencil (BLF) satisfying the indicated cohomological condition, one constructs a uniqe deformation class of ω which is symplectic on the fibers, away from the singularities.

As in the Lefschetz fibration case, blowing-up the base locus of a broken Lefschetz pencil results in a Lefschetz fibration. When the BLP supports a near-symplectic structure, these blow-ups/downs are understood to be made symplectically. If we have in hand a broken Lefschetz fibration over a Riemann surface B (which we will mostly take as $B = S^2$) that satisfies the same cohomological condition in the statement of the theorem, then we can construct compatible near-symplectic forms with respect to which a chosen set of sections are symplectic [9]. From now on we will refer to such a fibration f on X as *a near-symplectic broken Lefschetz fibration*, and say that the pair (X, f) is *near-symplectic.* Implicit in this notation is that the near-symplectic form on X is chosen from the unique deformation class of near-symplectic forms compatible with f obtained via Theorem 4.1.3.

Clearly one can define broken Lefschetz fibrations over any Riemann surface. The Example 4.1.2 gives such an example of a broken fibration (with no Lefschetz singularities) over T^2. In general, a broken Lefschetz fibrations over a Riemann surface can be split into Lefschetz fibrations over surfaces with boundaries, and fibered cobordisms between them relating the surface fibrations over the boundary circles. Round singularities of a broken Lefschetz fibration are contained in these cobordisms. We study these cobordisms more rigorously in the next subsection, but a brief discussion of their use beforehand might be helpful. For now, the reader is invited to convince himself/herself that our discussion of the local models around each round singular circle in the previous subsection implies that these cobordisms are given by fiberwise handle attachments, all with the same index (either 1 or 2).

If we fiberwise attach 1-handles to a fibered 3-manifold Y_0 to obtain a new fibered 3-manifold Y_1, the attaching region is necessarilly a bisection so that the handle attachment is compatible with the monodromy of the fibration on Y_0. That is, such a cobordism W is given by a fiberwise 1-handle attachment at the two intersection points of this bisection with the fibers of Y_0. The fibrations on the two ends of W uniquely extend to a broken fibration over $S^1 \times I$, with only one round singularity given by the centers of the cores of 1-handles attached to the fibers of Y_0. Similarly, we can fiberwise attach 2-handles to a fibered Y_0' to describe a cobordism to a new surface fibration Y_1' over a circle. This is obtained by a fiberwise 2-handle attachment along a curve γ_s on each fiber F_s, where s parametrizes the base $I/\{0 \sim 1\} = S^1$ of the fibration on Y_0'. Once again we obtain a broken fibration from this cobordism W' to $S^1 \times I$, with a single round singularity corresponding to the centers of the cores of the 2-handles attached fiberwise.

The diffeomorphism types of the fiber components of the fibrations on the two ends of such cobordisms can easily be deduced from each other by looking at whether the bisection intersects with only one fiber component or two, or in the other direction to whether γ_s is a separating curve or not (for one s and all). Relating the monodromies of these fibrations is a more elusive issue which we will address later in Subsections 4.2.1 and 4.2.2. However, the types of singular circles that arise from these round handle attachements is determined easily. In such a cobordism with 1-handles, the type is even if the attachings trace out an oriented link with two components and odd if they trace out an oriented knot. On the other hand, a cobordism with 2-handles gives rise to an even circle if the monodromy of the fibration on Y_0' maps γ_1 to γ_0 with the same orientation, and is odd otherwise.

Remark 4.1.4 *Roughly speaking, such cobordisms with 1-handle attachments increase the genus of a fiber component, or connect two different fiber components, whereas cobordisms with 2-handle attachments either decrease the genus or disconnect a fiber component. In [9] it was shown that for any given near-symplectic form ω on X, a compatible broken Lefschetz fibration $f : X\#b\overline{\mathbb{CP}}^2 \to S^2$, where b is the number of base points, can in fact be arranged in the following way: The base S^2 breaks into three pieces $D_l \cup A \cup D_h$, where A is an annular neighborhood of the equator of the base S^2 which does not contain the image of any Lefschetz critical point, D_l and D_h are disks, so that (i) On $X_l = f^{-1}(D_l)$ and $X_h = f^{-1}(D_h)$ we have genuine Lefschetz fibrations; and (ii) The cobordism $W = f^{-1}(A)$ is given by only fiberwise 1-handle attachments if one travels from the X_l side to X_h side. We call these kind of broken Lefschetz fibrations/pencils* directed, *X_l the* lower side *and X_h the* higher side.

4.1.4 Lagrangian matching invariants

We will now discuss an invariant due to Perutz [63, 64, 65] associated to any given (X, f) where X is a closed smooth oriented 4-manifold and $f : X \to B$ is an injective near-symplectic broken Lefschetz fibration. The injectivity condition is needed to guarantee that f maps components of the round singular locus to disjoint circles on B, which can be achieved by perturbing any given broken Lefschetz fibration. Perutz's work generalizes the Donaldson-Smith construction [19] to near-symplectic broken Lefschetz fibrations. It relies on a count of pseudo-holomorphic sections of the associated families of symmetric products over a splitting base that 'match' by satisfying certain 'Lagrangian boundary conditions' [64, 65]. This aspect of the construction suggests the name *Lagrangian matching invariants* for these invariants. The construction of Lagrangian matching invariants (LM) are quite tedious, and the reader is asked to turn to [64, 65] for the details which will be ignored below.

LM invariants are designed to be comparable to SW invariants of the underlying 4-manifold, and were conjectured by Perutz [64] to be equal to SW. When the round locus is empty, the equality of the Donaldson-Smith and Seiberg-Witten invariants for *symplectic Lefschetz fibrations* of high degree was proved by Usher [82] through Taubes' work on the correspondence between Gr invariants and SW invariants on symplectic 4-manifolds. More evidence in this direction were gathered in [65], including the equality of LM and SW invariants on the near-symplectic family of 4-manifolds $S^1 \times M_K$, for any knot K, described in Example 4.1.2. The conjecture in particular proposes the LM invariants to be independent of the choice of fibrations (possibly after imposing some constraints), even though the calculations make use of the fibration structure. Hence the nature of the invariant requires the study of near-symplectic broken Lefschetz fibrations, which will be the main theme of the next section, whereas the aforementioned conjecture motivates us to look at SW and

LM invariants simulatenously in the rest of this chapter.

Let $f : X \to S^2$ be a near-symplectic broken Lefschetz fibration. Let $\mathrm{Spin}^c(X)$ denote the $H^2(X;\mathbb{Z})$-torsor of isomorphism classes of Spin^c-structures on X, and $F \in H_2(X;\mathbb{Z})$ be the class of a regular fiber of f. Then

$$\mathrm{Spin}^c(X)_k = \{\mathfrak{s} \in \mathrm{Spin}^c(X) \mid \langle c_1(\mathfrak{s}), F\rangle = 2k, (*)\}.$$

where $(*)$ is the condition that for any connected component Σ of a regular fiber, one has $\langle c_1(\mathfrak{s}), [\Sigma]\rangle \geq \chi(\Sigma)$.

Definition 4.1.5 *$k \in \mathbb{Z}$ is* admissible *for (X, f) if either (i) the fibers are all connected and $k > 0$, or (ii) $\chi(X_s)/2 < k < -\chi(X_s)/2$ for all regular values s. A* Spin^c*-structure $\mathfrak{s}$ is admissible if $\mathfrak{s} \in \mathrm{Spin}^c(X)_k$ with k admissible.*

Then the *Lagrangian matching invariant* is a map

$$\bigcup_{k \text{ admissible}} \mathrm{Spin}^c(X)_k \to \mathbb{A}(X), \quad \mathfrak{s} \mapsto LM_{(X,f)}(\mathfrak{s}).$$

where $\mathbb{A}(X)$ is the graded abelian group $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H^1(X;\mathbb{Z})$, $\deg(U) = 2$. The element $LM_{(X,\pi)}(\mathfrak{s})$ is homogeneous of degree

$$d(LM_{(X,f)}(\mathfrak{s})) = \frac{1}{4}(c_1(\mathfrak{s})^2 - 3\sigma(X) - 2e(X)) \tag{4.2}$$

and derived from a moduli space [64, 65], whose construction in turn uses the broken Lefschetz fibration f on X as well as several auxiliary choices. It is invariant under isotopies of f through fibrations of the same type, and equivariant under automorphisms of (X, f).

Remark 4.1.6 *The above definition can be generalized to any base surface B, after replacing $\mathbb{A}(X)$ by $\mathbb{A}(X, f) = \mathbb{Z}[U] \times_{\mathbb{Z}} \Lambda^*\mathrm{Hom}(K_\pi, \mathbb{Z})$, where*

$$K_\pi = \ker(\pi_* : H_1(X;\mathbb{Z}) \to H_1(B;\mathbb{Z})) \subset H_1(X).$$

In the level of homology, the Spin^c-structures that the Lagrangian matching invariants are parametrized over correspond to multisections of a near-symplectic broken Lefchetz fibration (X, f) which have (homological) boundaries equal to the round locus. (This is *analogous* to the tautological correspondence between the multisections (called 'standard surfaces') of a symplectic Lefschetz fibration and the sections of the Hilbert schemes in the construction of Donaldson-Smith invariants [69].) What follows is a brief review of this:

The 'Taubes map' τ_X is a bijection (as proved in [77])

$$\mathrm{Spin}^c(X) \to \delta^{-1}([Z]) \subset H_2(X, Z; \mathbb{Z}),$$

where $\delta \colon H_2(X, Z; \mathbb{Z}) \to H_1(Z; \mathbb{Z})$ is the boundary homomorphism, and Z is oriented by a vector field v such that $i f_*(v)$ points into the higher side of $f(Z) \subset B$. The map τ_X arises from the canonical Spin^c-structure $\mathfrak{s}_{\mathrm{can}}$ on the almost complex manifold $X \setminus Z$. It is characterized by

$$\tau_X(\mathfrak{s}) = \beta \quad \text{if} \quad \mathfrak{s}|_(X \setminus Z) = \mathrm{PD}(\beta) \cdot \mathfrak{s}_{\mathrm{can}}. \tag{4.3}$$

That is $c_1(\mathfrak{s}) = c_1(\mathfrak{s}_{\mathrm{can}}) + 2PD(\beta)$. Thus a multisection in question is obtained by $\beta = \tau_X^{-1}(\mathfrak{s}) \in H_2(X, Z; \mathbb{Z})$.

To finish with we would like to note another aspect of LM invariants established by Perutz: that they fit in a 'fibered field theory'. This is achieved by assigning symplectic Floer homology groups to 3-manifolds fibered over circles, and relative invariants assigned to 4-manifolds fibered over Riemann surfaces with boundaries. A chosen multisection of a near-symplectic broken Lefschetz fibration of (X, f) restricts to multisections of these fibrations, which in turn is used to compute these Floer homologies. To simplify our discussion here, assume that the base $B = S^2$ splits as $B = B_1 \cup \ldots \cup B_n$, where B_1 and B_n are disks and the rest are annuli, with each one containing the image of the singular locus of f in their interiors only. Let

$X_i = f^{-1}(B_i)$ so that $X = X_1 \cup_\partial \cdots \cup_\partial X_n$. Then we get a map

$$LM_{X,f} = LM^\vee_{X_1, f|_{X_1}} \circ LM_{X_2, f|_{X_2}} \circ \cdots \circ LM_{X_n, f|_{X_n}} : \mathrm{Spin}^c(X)_{admissible} \longrightarrow \mathbb{A}(X)$$

where $\mathrm{Spin}^c(X)_{admissible} = \bigcup_{k \text{ admissible}} \mathrm{Spin}^c(X)_k$ and $LM^\vee$ is a dual map. (See [64, 65].) Then this map is evaluated on monomials $U^a \otimes l_1 \wedge \cdots \wedge l_b$ of degree $d(\mathfrak{s})$ to obtain a map into $\mathbb{Z}$ as in the SW setting.

4.2 Topology of broken Lefschetz fibrations

Handlebody diagrams of Lefschetz fibrations over S^2 are well-understood and proved to be useful in the study of topology of smooth 4-manifolds. The reader is advised to turn to [40] for the details of this *by now* classical theory and its several applications. In this section, we would like to extend these techniques to the study of broken Lefschetz fibrations. For this purpose, we will describe and study *round handles* that arise naturally in the context of 4-dimensional broken Lefschetz fibrations thoroughly.

An n-dimensional round k-handle is topologically $S^1 \times D^k \times D^{n-1-k}$. The first comprehensive study of round handles is due to Asimov [7], and more on 4-dimensional round 1-handles can be found in [34]. However, both articles assume a restriction on the way these handles are attached. Namely, these round handles $S^1 \times D^k \times D^{n-1-k}$ are attached along $S^1 \times S^{k-1} \times D^{n-1-k}$. As the work of [9] implicitly suggests, we shall also consider a different type of attachment. To keep the following discussion simple, let us define this other way of gluing in the case of 4-dimensional round $1-$ and $2-$ handles only, the ones which interest us in this work.

Take a 3-disk bundle over S^1 with total space $S^1 \times D^3$, and look at the splitting of this bundle into two subbundles of rank 1 and 2. These splittings are classified by homotopy classes of mappings from S^1 into $\mathbb{RP}^2$. Since $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ there are two possible splittings up to isotopy. These two splittings can be realized by the ones given in Subsection 4.1.2. Namely, these are determined by the two orientation preserving self-diffeomorphisms of $\mathbb{R}^3$, where one is the identity map, and the other one is given by $(x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3)$. Our second type of round handle attachment arises from the latter model. To distinguish the two type of round handles, let us denote this new one by $S^1 \tilde{\times} D^3$ just to emphasize the splitting we consider. Clearly, $S^1 \tilde{\times} D^3$ is diffeomorphic to $S^1 \times D^3$. We call the round handles attached in the usual way (as in [7]) *even round handles*, whereas the others are called *odd round handles* —apparently corresponding to the even and odd local models in Subsection 4.1.2.

Let us describe the attachments in the odd case more explicitly. The attachment of an odd round 1-handle $S^1 \tilde{\times} (D^1 \times D^2)$ is made along $S^1 \tilde{\times} (S^0 \times D^2)$, which is topologically the D^2 neighborhood of a circle $(= S^1 \tilde{\times} (S^0 \times 0))$. If we restrict our attention to the rank 1-bundle (parametrized by x_1) over S^1, both even and odd round 1-handles can be seen to have attaching regions given by the restriction of this bundle to its boundary (which gives a bisection of the rank 1-bundle) times the rank 2 bundle. Then the odd and even cases correspond to this bisection having one or two components, respectively. Similarly, an odd round 2-handle $S^1 \tilde{\times} (D^2 \times D^1)$ is attached along $S^1 \tilde{\times} (S^1 \times D^1)$. This is topologically a collar neighborhood of a Klein Bottle, whereas in the even case we would be gluing along a collar neighborhood of a torus.

4.2.1 Round 1-handles

Expressing the circle factor of an even round 1-handle $S^1 \times D^1 \times D^2$ as the union of a 0-handle $I_0 = D^0 \times D^1$ and a 1-handle $I_1 = D^1 \times D^0$, we can express an even round 1-handle as the union $(I_0 \cup I_1) \times D^1 \times D^2 = (D^0 \times D^1 \cup D^1 \times D^0) \times D^1 \times D^2 = (D^0 \times D^1) \times (D^1 \times D^2) \cup (D^1 \times D^0) \times (D^1 \times D^2) \cong (D^0 \times D^1) \times (D^1 \times D^2) \cup (D^1 \times$

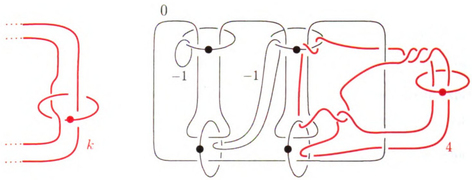

Figure 4.1: A general odd round 1-handle (left), and an even round 2-handle attachment to a genus two Lefschetz fibration over a disk (right). Red handles make up the round 1-handle.

$D^1) \times (D^0 \times D^2) \cong D^1 \times D^3 \cup D^2 \times D^2$, a 4-dimensional 1-handle H_1 and a 2-handle H_2. Note that we exchange and rewrite the factors simultaneously. It is not too hard to see that H_2 goes over H_1 geometrically twice but algebraically zero times.

In the same way, we can realize an odd round 1-handle as the union of a 1-handle H_1 and a 2-handle H_2. However this time the underlying splitting implies that H_2 goes over H_1 both geometrically and algebraically twice.

We are ready to discuss the corresponding Kirby diagrams. Recall that our aim is to study the round handle attachments to Lefschetz fibrations. Let F denote the 2-handle corresponding to the regular fiber. Both in even and odd cases, the 2-handle H_2 of the round 1-handle links F *geometrically and algebraically* twice and can attain any framing k. Both 'ends' of the H_2 are allowed to go through any one of the 1-handles of the fiber before completely wrapping once around F. In addition, these two ends might twist around each other as in Figure 4.2.1. (Caution! The "twisting" discussed in [9] is not this one; what corresponds to it is the framing k.) The difference between even and odd cases only show-up in the way H_2 goes through H_1. In Figure 4.2.1 we depict both types of handle attachments.

4.2.2 Round 2-handles

The handle decomposition of round 2-handles is analogous to that of round 1-handles. Expressing the circle factor of an even round 2-handle $S^1 \times D^2 \times D^1$ as the union of a 0-handle $I_0 = D^0 \times D^1$ and a 1-handle $I_1 = D^1 \times D^0$, this time we can express an even round 2-handle as the union $(I_0 \cup I_1) \times D^2 \times D^1 \cong D^2 \times D^2 \cup D^3 \times D^1$, a 4-dimensional 2-handle H_2' and a 3-handle H_3' through a similar rewrite as before. For an odd round 1-handle we get a similar decomposition. However the splittings once again imply the difference: the 3-handle goes over the 2-handle geometrically twice and algebraically zero times in the even case, and both geometrically and algebraically twice in the odd case. One can also conclude this from the previous subsection since a round 2-handle is dual to a round 1-handle.

We are now ready to discuss the corresponding Kirby diagrams for attaching round 2-handles to Lefschetz fibered 4-manifolds with boundary. Recall that the round 2-handle attachment to a surface fibration Y_0' over a circle that bounds a Lefschetz fibration is realized as a fiberwise 2-handle attachment. The attaching circle of the 2-handle H_2' of a round 2-handle is then a simple closed curve γ on a regular fiber, which is preserved under the monodromy of this fibration up to isotopy. Since this attachment comes from a fiberwise handle attachment, H_2' should have framing zero with respect to the fiber. As usual, we do not draw the 3-handle H_3' of the round 2-handle, which is forced to be attached in a way that it completes the fiberwise 2-handle attachment. The difference between the even and odd cases is then somewhat implicit; it is distinguished by the two possible ways that the curve γ might be mapped onto itself by a self-diffeomorphism of the fiber determined by the monodromy. If γ is mapped onto itself with the same orientation, we have an even round 2-handle, and an odd round 2-handle if the orientation of γ is reversed. The reader can also refer to the relevant monodromy discussion after the proof of Theorem 4.2.3.

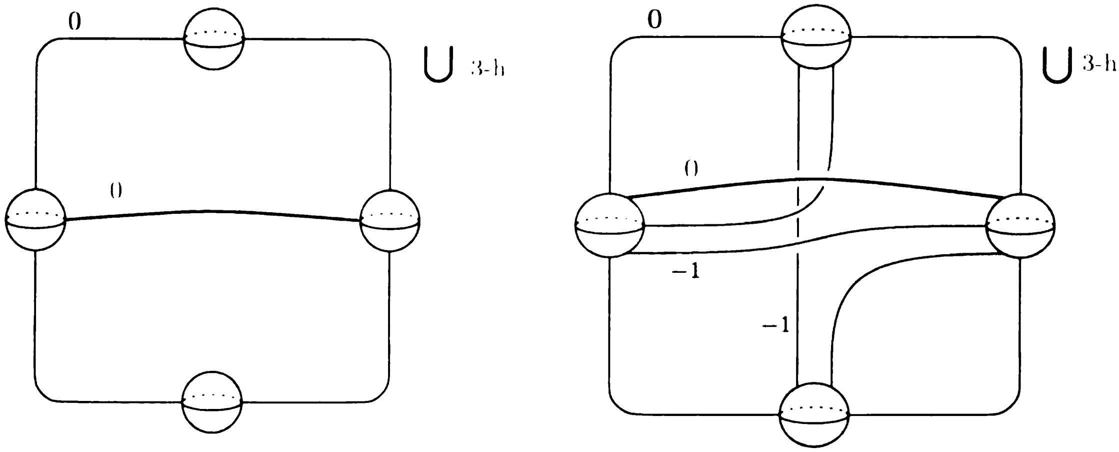

Figure 4.2: Left: an even round 2-handle attachment to $D^2 \times T^2$. Right: an odd round 2-handle attachment to an elliptic Lefschetz fibration over a disk with two Lefschetz singularities. Red handles make up the round 2-handle.

The upshot of using round 2-handles is that one can depict any Lefschetz fibration over a disk together with a round 2-handle attachment via Kirby diagrams explicitly as in the Lefschetz case [40]. One first draws the Lefschetz 2-handles following the monodromy data on a regular diagram of $D^2 \times \Sigma_g$ (where g is the genus of the fibration) with fiber framings -1, then attaches H'_2 with fiber framing 0 and includes an extra 3-handle. We draw the Kirby diagram with standard 1-handles so to match the fiber framings with the blackboard framings, which can then carefully be changed to the dotted notation if needed. Importantly, it suffices to study only these type of diagrams when dealing with broken Lefschetz fibrations on near-symplectic 4-manifolds, as we will prove in the next section.

To illustrate what we have stated above, let us look at the following two simple examples in Figure 4.2. Since the first round 2-handle is attached to a trivial fibration, γ is certainly mapped onto itself with the same orientation, and therefore it is an even round 2-handle. For the second one, we express the self-diffeomorphism of the 2-torus fiber induced by the monodromy μ by the matrix:

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

and the curve γ by the matrix $[1\ 0]^T$. Thus μ maps γ to $-\gamma$, and this yields an odd round 2-handle attachment. Both of these examples will be revisited later.

4.2.3 Simplified broken Lefschetz fibrations

The complexity of the topology of broken Lefschetz fibrations lies in round cobordisms. Our goal is to establish an existence result of much simpler broken fibrations, which can be associated to any near-symplectic 4-manifold.

Definition 4.2.1 *A* simplified broken Lefschetz fibration *on a closed 4-manifold X is a broken Lefschetz fibration over S^2 with only one round singularity and with all critical points on the higher side.*

Since the total space of the fibration is connected, the "higher side" always consists of connected fibers. The fibers on the higher side have higher genus whenever all the fibers are connected, while in general the term refers to the direction of the fibration. We shall need the following lemma:

Lemma 4.2.2 *Let X admit a directed broken Lefschetz fibration $f : X \to S^2$, then there exists a new broken Lefschetz fibration on $f' : X \to S^2$, where all the Lefschetz singularities are contained in the higher side.*

Proof: To begin with, we can perturb the directed fibration so to guarantee that it is injective on the circles of the round locus. Thus the fibration can be split into a Lefschetz fibration over a disk (the lower side), to which we consecutively attach round 1-handles, and then we close the fibration by another Lefschetz fibration over a disk (the higher side).

To simplify our discussion, for the time being assume that the fibers are all connected, so there is the lower genus side X_l with regular fiber F_l, the round handle cobordism W, and the higher genus side X_h with regular fiber F_h. Let the genus of the regular fibers in the lower side be g. The standard handlebody decomposition of X_l consists of a 0-handle, $2g$ 1-handles and some 2-handles one of which corresponds to the fiber, and the rest to the Lefschetz handles in X_l [40]. By our assumption, W is composed of ordered round 1-handle cobordisms $W_1 \cup W_2 \cup \cdots \cup W_k$, where k is the number of circle components in the round locus. Let us denote the lower side boundary of W_i by $\partial_- W_i$ and the higher side by $\partial_+ W_i$.

Consider $X_l \cup W_1$, which is obtained by adding a round 1-handle R_1 composed of a 1-handle H_1 and a 2-handle H_2. The $\partial(X_l \cup W_1) = \partial_+ W_1 = \partial_- W_2$ is the total space of a genus $g+1$ surface bundle over a circle. We can make sure that the vanishing cycles of the Lefschetz 2-handles in X_l sit on the fibers of the genus g fibration on ∂X_l. Moreover, we can assume that the bisection which is the attaching region of R_1 misses these vanishing cycles. This means that H_1 and H_2 do not link with any one of the Lefschetz 2-handles in X_l but only with the 2-handle corresponding to the fiber and possibly with some of the 1-handles corresponding to the genera of the fiber. We can rearrange the handlebody prescribed by the broken Lefschetz fibration on $X_l \cup W_1$ by another one where first H_1 and H_2 are attached to the standard diagram of $D^2 \times F_l$, and the Lefschetz 2-handles are attached afterwards. Having modified the diagram this way, now we can assume that the Lefschetz 2-handles are attached to $\partial(X_l \cup W_1)$, which can be pulled to $\partial_- W_2$ via the fiber preserving diffeomorphism between $\partial_+ W_1$ and $\partial_- W_2$. The fiber framings of these 2-handles remain the same, and therefore they are still Lefschetz.

Inductively, one slides the Lefschetz 2-handles so to have them attached to $\partial(X_l \cup W_1 \cup W_2 \cup \cdots \cup W_k) = \partial(X_l \cup W) = -\partial X_h$. Higher side X_h together with these 2-handles is equipped with a new Lefschetz fibration of genus $g+k$ (which is the same as the genus of F_h) over a disk. Hence we obtain a new handlebody decomposition which describes a new broken Lefschetz fibration on X, with all the Lefschetz singularities contained in the new higher side. It is left to the reader as an excercise to verify that a similar line of arguments work when X_l has disconnected fibers. $\square$

Given a near-symplectic form on a closed 4-manifold X, Perutz [66] and Taubes [79] independently showed that one can obtain a new near-symplectic form on X in the same cohomology class but with connected round locus. The meat of the next theorem is this observation and the Theorem 4.1.3.

Theorem 4.2.3 *On any closed near-symplectic 4-manifold (X, ω), possibly after replacing ω with a near-symplectic form ω' within the same cohomology class, one can find a near-symplectic broken Lefschetz pencil, which yields a simplified near-symplectic broken Lefschetz fibration on a blow-up $(\hat{X}, \hat{\omega}')$ of (X, ω').*

Proof: Replace ω with a near-symplectic ω' with connected $Z_{\omega'}$. Theorem 4.1.3 shows that there is a broken Lefschetz pencil compatible with this near-symplectic form, so it should have only one round handle singularity. Symplectically blow-up the base points to obtain a near-symplectic broken Lefschetz fibration f on the blow-up $\hat{X}$ of X. Apply the above lemma to get a simplified Lefschetz fibration on $\hat{X}$, which also supports the near-symplectic structure since the fibers are unchanged and still symplectic under the modification described in the proof of Lemma 4.2.2.

The exceptional spheres appear as 2-handles linked to the higher genus fiber component, all with framing -1, and not linking to each other or to any other handle. The modification in Lemma 4.2.2 is performed without involving these handles, so their linkings and framings remain the same. Since these represent the exceptional spheres, we can symplectically blow them down to obtain a new Lefschetz pencil on X, with the desired properties. $\square$

The simplified broken Lefschetz fibrations now can be represented by using the handlebody diagrams described in Subsection 4.2.2. Examples are given in the next subsection.

It is no surprise that the monodromy representations of these fibrations are also simpler than usual. Here we include a brief digression on this topic: Let $Map_\gamma(F_g)$ be the subgroup of $Map(F_g)$ that consists of elements that fix an embedded curve γ, up to isotopy. Then there is a natural homomorphism: ϕ_γ from $Map_\gamma(F_g)$ to $Map(F_{g-1})$ or to $Map(F_{g_1}) \times Map(F_{g_2})$ depending on whether γ is nonseperating or separating F_g into two closed oriented surfaces of genera g_1 and g_2. Define S_g to be the set of pairs (μ, γ) such that $\mu \in Map_\gamma(F_g)$ and $\mu \in Ker\,(\varphi_\gamma)$. Recall that when the fiber genus is at least two, the gluing map that preserves the fibers is determined uniquely upto isotopy. Hence, given any tuple $(\mu, \gamma) \in S = \bigcup_{g \geq 3} S_g$, we can construct a unique simplified broken Lefschetz fibration unless γ is separating and there is a $g_i \leq 1$. Otherwise, one needs to include the data regarding the gluing of the low genus pieces carrying genus 0 or genus 1 fibrations.

If the fibers are connected, the map $\phi_\gamma : Map(F_g) \rightarrow Map(F_{g-1})$ above factors as $\psi_\gamma : Map(F_g) \rightarrow Map(F_g \setminus N)$ and $\varphi_\gamma : Map(F_g \setminus N) \rightarrow Map(F_{g-1})$, where N is an open tubular neighborhood of γ away from the other vanishing cycles. (The middle group does not need to fix the boundaries.) The map ψ has kernel isomorphic to $\mathbb{Z}$ —the framing of the 2-handle of a round 1-handle. When we have a simplified BLF, the kernel of φ is isomorphic to the braid group on F_{g-1} with 2-strands, by definition. This gives an idea about the cardinality of S, and in turn about the cardinality of the family of broken Lefschetz fibrations on smooth 4-manifolds.

Remark 4.2.4 *If one has more than one round 2-handle involved in a broken Lef-*

schetz fibration, we may or may not be able to draw the Kirby diagrams as above. This is due to the fact that after each round cobordism, we obtain a new fiber, which does not need to simply 'sit on the blackboard'. If one draws the diagram from the lower side; the 2-handle of a round 1-handle might link with the 1− and 2− handles of other round 1-handles. To have a complete diagram, one would also need to pull the Lefschetz handles from the higher side to this diagram; but framings of both 2-handles of round 2-handles and those of the Lefschetz handles coming from the higher side all together are harder to determine.

4.2.4 Examples

In this subsection we provide examples of simplified broken Lefschetz fibrations. The examples are chosen to span various types of fibrations: with even round locus, odd round locus, connected fibers, disconnected fibers (on the lower side), and finally those which do not support any near-symplectic structure. The near-symplectic examples we present here are used in later sections.

Example 4.2.5 The Figure 4.3 describes a near-symplectic broken Lefschetz fibration on $S^2 \times \Sigma_g \# S^1 \times S^3$, with lower side genus equal to g and higher side genus increased by one via an even round 1-handle cobordism. We call this fibration the *step fibration* for genus g. To identify the total space, first use the 0-framed 2-handle of the round 2-handle to separate the 2-handle corresponding to the fiber. Then eliminate the obvious canceling pair, and note that the remaining 1-handle together with the 3-handle of the round 2-handle describes an $S^1 \times S^3$ summand. As the rest of the diagram gives $S^2 \times \Sigma_g$, we see that the total space is as claimed.

In several aspects, the round handle cobordism W in the step fibration is the simplest possible cobordism. Here not only $\partial_\pm W$ are products of Riemann surfaces Σ_g and Σ_{g+1} with S^1, but also W itself is the product of S^1 with a 3-dimensional

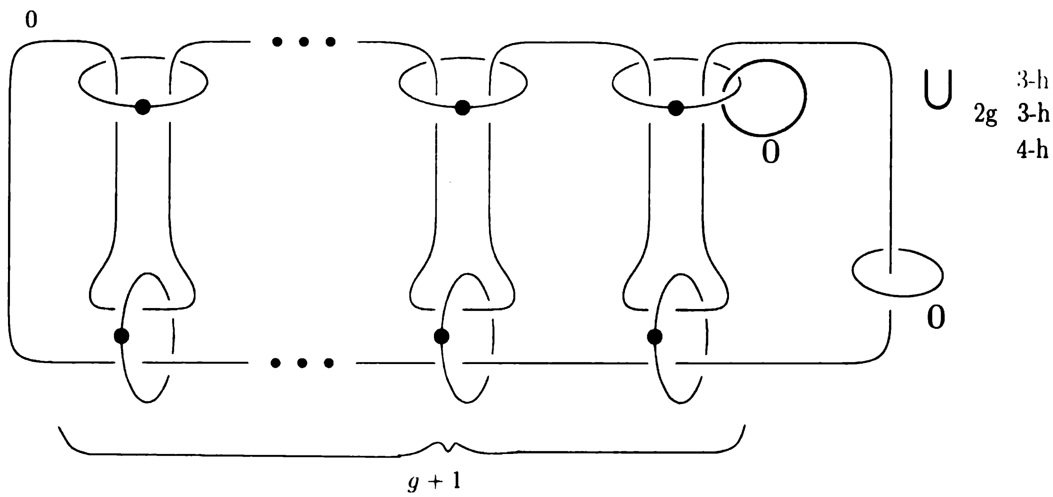

Figure 4.3: The step fibration on $S^2 \times \Sigma_g \# S^1 \times S^3$.

cobordism from Σ_g to Σ_{g+1} given by only one handle attachment. We refer to these type of cobordisms as *elementary cobordisms.* The round handle cobordisms in Example 4.1.2 are *all* elementary.

When $g = 0$ we can obtain a more general family as in Figure 4.4. These describe broken Lefschetz fibrations obtained from a trivial torus fibration and a trivial sphere fibration over disks and an elementary round handle cobordism between them. The fibrations we get are precisely the near-symplectic examples of [9], and historically the first examples of near-symplectic broken Lefschetz fibrations over S^2. After simple handle slides and cancellations, one ends up getting a diagram of the connected sum of an S^2 bundle over S^2 with Euler class k and an $S^1 \times S^3$. Thus for even k we get $S^2 \times S^2 \# S^1 \times S^3$ and $S^2 \tilde{\times} S^2 \# S^1 \times S^3$ for odd k.

Example 4.2.6 In Figure 4.5 we describe a family of simplified broken Lefschetz fibrations with odd round singularity. We claim that for even k the total space is $S^2 \times S^2$ and for odd k it is $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. In order to verify this we prefer to use the diagram with dotted notation on the right of the Figure 4.5. Let H_2 be the

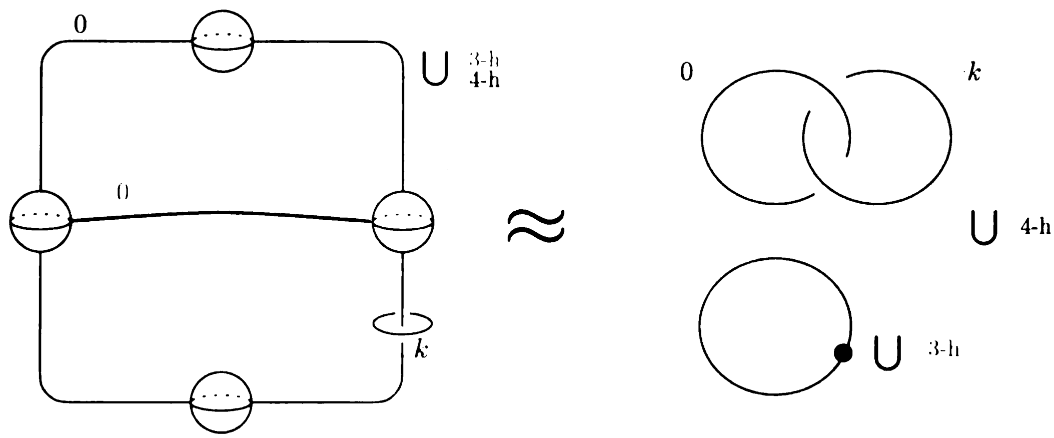

Figure 4.4: A family of near-symlectic BLFs over S^2 (left), and the diagram after the handle slides and cancelations (right).

2-handle of the round 2-handle, given in red and with fiber framing 0. Using H_2, first unlink all the 2-handles from the top 1-handle, and cancel this 1-handle against H_2. Then slide the $+1$-framed 2-handle over the -1-framed 2-handle to obtain the third diagram in the Figure 4.6, and cancel the surviving 1-handle against the (-1)-framed 2-handle. Finally cancel the remaining unlinked 0-framed 2-handle against the 3-handle. The result follows.

For $k = 0$ this is Perutz's *button* example in [64]. Moreover, when $k = -1$ the blow-down of this exceptional sphere yields a near-symplectic broken Lefschetz pencil on $\mathbb{CP}^2$.

All the examples we discussed so far had nonseparating round 2-handles; in other words, in all examples all the fibers were connected. However separating round 2-handles arise quite naturally when studying broken fibrations on connected sums of near-symplectic 4-manifolds, as illustrated in the next example.

Example 4.2.7 Since $b^+(2\mathbb{CP}^2) = 2$, there exists a near-symplectic form on this

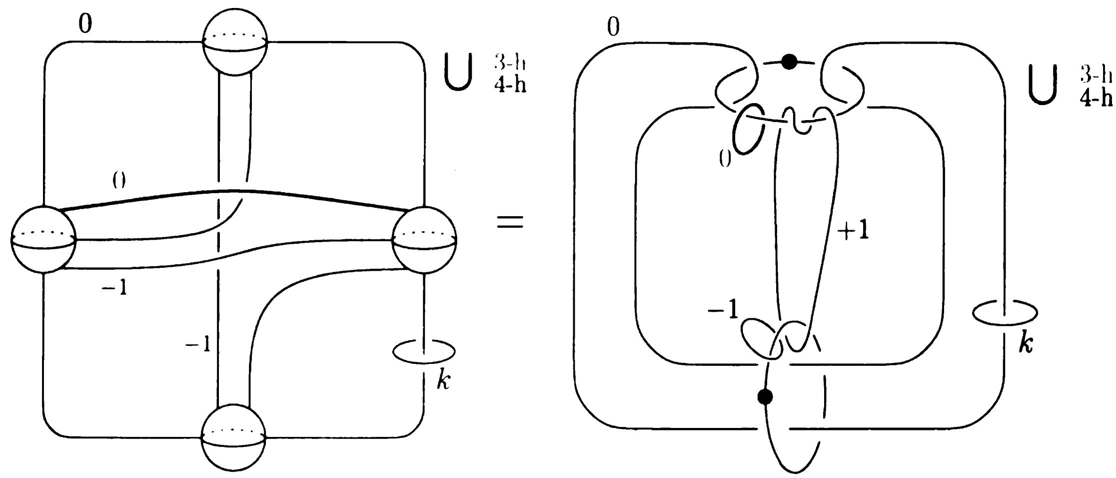

Figure 4.5: A near-symplectic BLF for an S^2 bundle over S^2 with Euler class k. On the right: 1-handles are replaced by dotted circles.

non-symplectic 4-manifold. We will construct a near-symplectic structure which restricts to a symplectic structure on each $\mathbb{CP}^2$ summand away from the connected sum region, through broken Lefschetz fibrations. Take the rational fibrations f_i, $i = 1, 2$ on two copies of $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, with -1 sections. Consider a fibration $f = f_1 \cup f_2$ on the disjoint union of these two, by simply imagining them 'on top of each other'. Now in a regular neigborhood of a fiber of f, introduce a round 1-handle so to connect the disjoint sphere fibers. The result is a broken Lefschetz fibration $f' : 2\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2 \to S^2$ with two exceptional spheres. Let h be the Poincaré dual of the sum of -1 sections. Then h evaluates positively on each fiber component of this fibration, so there exists a near-symplectic structure compatible with f' with respect to which the two -1 sections are symplectic. Blowing-down these two sections we obtain a near-symplectic broken Lefschetz fibration on $2\mathbb{CP}^2$ with the proposed properties. A diagram of this fibration is given in Figure 4.7.

Remark 4.2.8 *The very same idea can be applied to connected sums of any two near-*

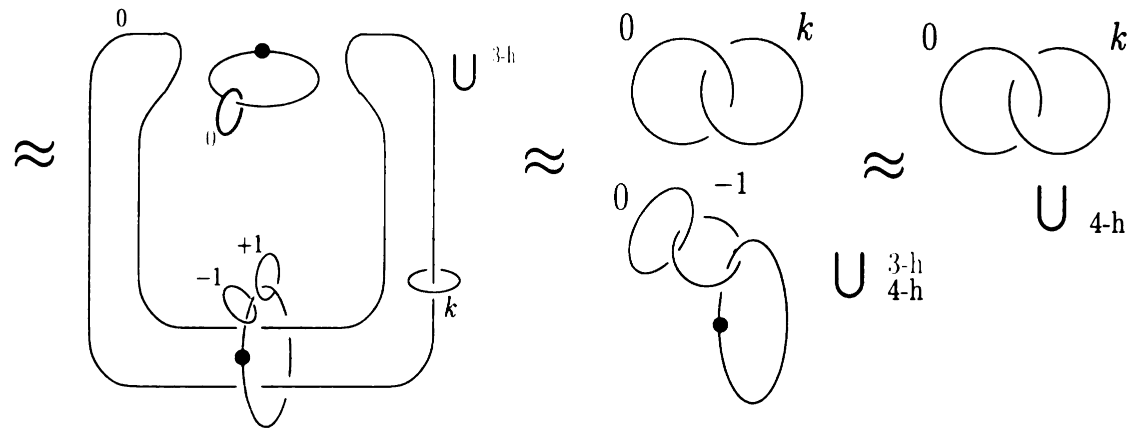

Figure 4.6: Identifying the total space of the BLF in Figure 4.5.

symplectic broken Lefschetz fibrations over the same base, say by connect summing in the higher genus sides (also see [65]). For the diagrams of such fibrations over S^2, abstractly, first slide a 2-handle F_1 corresponding to a fiber component over the 2-handle F_2 corresponding to the other fiber component. Then regard F_2 as the 2-handle of a round 2-handle, and add an extra 3-handle to the union of two fibration diagrams. This way we obtain a connected sum model for our (broken) Lefschetz fibration diagrams.

Using similar techniques, we can also depict diagrams of broken Lefschetz fibrations which do not necessarilly support near-symplectic structures. We finish with a few examples of this sort:

Example 4.2.9 As discussed in [9], a modification of $g = 0$ case in Example 4.2.5, yields a broken Lefschetz fibration on S^4. This can be realized by gluing the round cobordism W to the higher side fibration over D^2 by twisting the fibration on $\partial_+ W = T^3$ by a loop of diffeomorphisms of the T^2 fiber corresponding to a unit translation in the direction transverse to the vanishing cycle γ of the round 2-handle [9]. As a

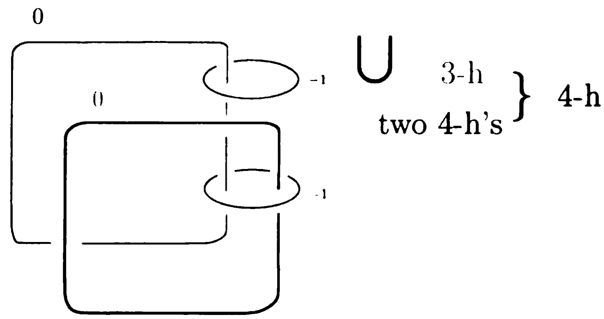

Figure 4.7: A near-symplectic BLF on $2\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. The round 2-handle separates the sphere fiber on the higher side into two spheres on the lower side.

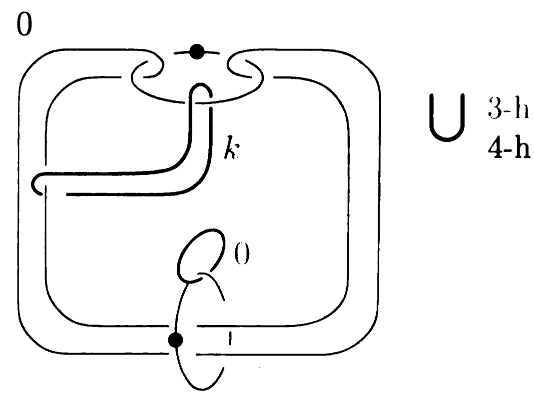

Figure 4.8: A broken Lefschetz fibration on S^4.

result of this, the 2-handle corresponding to the S^2 fiber of the lower side is pulled to the blue curve in Figure 4.8. The diagram then can be simplified as before: Use the 2-handle of the round 2-handle to separate the 2-handle corresponding to the fiber, and then proceed with the obvious handle cancelations.

It would also be interesting to note the existence of a broken Lefschetz fibration on $\#_n S^1 \times S^3$, for any $n \geq 1$, which do not admit achiral Lefschetz fibrations for $n \geq 2$ [40]. Taking the product of the Hopf fibration $S^3 \to S^2$ with S^1, we get a

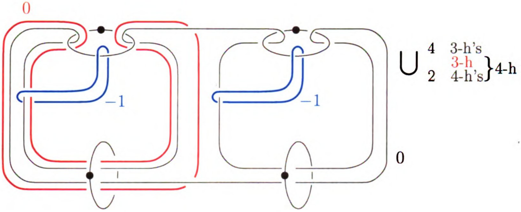

Figure 4.9: A broken Lefschetz fibration on $S^1 \times S^3 \# S^1 \times S^3$.

fibration $S^1 \times S^3 \to S^2$ with inessential torus fibers. Then the connected sum model discussed in the previous example allows us to construct a fibration on any number of connected sums of $S^1 \times S^3$s. In Figure 4.9 we give a diagram for the $n = 2$ case.

4.3 Some near-symplectic operations

We move on to presenting some surgical operations that give new near-symplectic broken Lefschetz fibrations from old. The first one generalizes the symplectic fiber sum operation Theorem 1 to the near-symplectic case, which can be set as a fibered operation. The second operation relies on an idea of Perutz [64], who modifies the near-symplectic broken Lefschetz fibration on the same 4-manifold. Both can be performed in general as near-symplectic operations, without any mention of broken fibrations.

4.3.1 Broken fiber sum

Let (X_i, f_i) be broken Lefschetz fibrations, and F_i be chosen regular fibers of genus $g_i > 0$, $i = 1, 2$. Choose regular neighborhoods $N_i = f_i^{-1}(D_i)$ of F_i, and without loss of generality, assume $g_1 - g_2 = k$ is a non-negative integer. Then we can obtain a new 4-manifold $X = X_1 \setminus N_1 \cup W \cup X_2 \setminus N_2$, where W is a composition of k elementary round 2-handle cobordisms. These cobordisms being elementary implies that the 2-handles of the round 2-handles can all be pushed onto a regular fiber F_1. The resulting manifold is uniquely determined by an unordered tuple of attaching circles $(\gamma_1, \cdots, \gamma_k)$ of the round 2-handles involved in W, together with the gluing maps $\phi_1 : \partial X_1 \to \partial_+ W$ and $\phi_2 : \partial X_2 \to \partial_- W$ preserving the fibrations. (Recall that these gluings are unique up to isotopy when the fiber genus is at least two.) Hence we obtain a new broken Lefschetz fibration (X, f) that extends the fibrations $(X_i \setminus N_i, f_i|_{X_i \setminus N_i})$ by standard broken fibrations over the elementary cobordisms. We say (X, f) is the *broken fiber sum* of (X_1, f_1) and (X_2, f_2) along F_1 and F_2, determined by $\gamma_1, \cdots, \gamma_k$ and ϕ_1, ϕ_2.

Theorem 4.3.1 *If (X_i, f_i) are near-symplectic broken Lefschetz fibrations, then (X, f) is a near-symplectic broken Lefschetz fibration. Moreover, given arbitrarily small collar neighborhoods $\tilde{N}_i$ of $\partial(N_i)$ in X_i, we can choose ω so that $\omega|_{X_1 \setminus \tilde{N}_1} = \omega_1|_{X_1 \setminus \tilde{N}_1}$ and $\omega|_{X_2 \setminus \tilde{N}_2} = c\,\omega_2|_{X_2 \setminus \tilde{N}_2}$, where c is some constant.*

Proof: Let k be as above. Take step fibrations on $S^2 \times \Sigma_g \# S^1 \times S^3$ described in Example 4.2.6 with $g = g_2, g_2 + 1, \ldots, g_2 + k = g_1$. Take the fiber sum $S^2 \times \Sigma_{g_2} \# S^1 \times S^3$ along a high genus fiber with $S^2 \times \Sigma_{g_2+1} \# S^1 \times S^3$ along a low genus fiber. Then take the fiber sum of this new broken fibration along a high genus fiber with $S^2 \times \Sigma_{g_2+2} \# S^1 \times S^3$ along a low genus fiber, and so on, until $g = g_2 + k$. Denote this manifold by $\tilde{W}$. Since the broken Lefschetz fibration on $\tilde{W}$ admits a section, it can be equipped with a near-symplectic structure. Hence the broken fiber sum of (X_1, f_1) and (X_2, f_2) along F_1 and F_2 is obtained by fiber summing the former along F_1 with $\tilde{W}$ along a lower side fiber, and the latter along F_2 with $\tilde{W}$ along a higher side fiber. Using Theorem 1, we can make these fiber sums symplectically, after possibly rescaling one of the near-symplectic forms ω_i, $i = 1, 2$. It is clear that when $k = 0$ this is the usual symplectic fiber sum. □

Remark 4.3.2 *If (X_i, f_i) for $i = 1, 2$ are Lefschetz fibrations over S^2, then one can depict the Kirby diagram of the broken fiber sum (X, f) in terms of these two by using Lemma 4.2.2. Since the round cobordism in the broken fiber sum consists of elemenatry cobordisms, all the 2-handles of the round 2-handles and the Lefschetz handles of the lower genus fibration can be drawn on the higher genus fiber directly.*

Remark 4.3.3 *Forgetting the fibrations, we can describe the above construction for any near-symplectic (X_i, ω_i) containing symplectically embedded surfaces F_i with $F_1^2 = F_2^2 = 0$. Moreover it is possible to form a cobordism similar to W in general when $F_1^2 = -F_2^2 \neq 0$ to handle the most general situation.*

Topological invariants of X are easily determined. For example if X_i are simply-connected and at least one of them admits a section, then using Seifert-Van Kampen theorem we conclude that X is also simply-connected. The Euler characteristic and signature of X can be expressed in terms of those of X_1 and X_2 as:

$$e(X) = e(X_1) + e(X_2) + 2(g_1 + g_2) - 4\ , \quad \sigma(X) = \sigma(X_1) + \sigma(X_2). \quad (4.4)$$

where g_i is the genus of F_i, for $i = 1, 2$. Therefore the holomorphic Euler characteristic $\chi_h(X) = \chi_h(X_1) + \chi_h(X_2) - 1 - (g_1 + g_2)/2$. It follows that if X_1 and X_2 are almost complex manifolds, then X obtained as their broken fiber sum along F_1 and F_2 is almost complex if and only if $k \equiv g_1 + g_2 \equiv 0 \pmod 2$. Lastly note that the broken fiber sum operation might introduce second homology classes in X that do not come

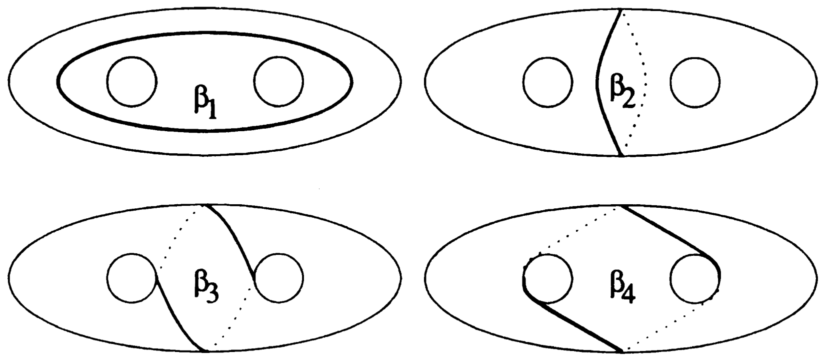

Figure 4.10: Vanishing cycles in the Matsumoto fibration.

from X_i in addition to the usual Rim tori. This phenomenon occurs for instance when some γ_i match with relative disks in $X_2 \setminus N_2$ to form an immersed sphere S_i. Then the torus T_i, which corresponds to a submanifold $\alpha_i \times S^1 \subset \partial(X_2 \setminus N_2) \cong F_2 \times S^1$, where α_i is the dual circle to γ_i on F_2, intersects with S_i at one point.

Example 4.3.4 Take $X_1 = S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2$ with the Matsumoto fibration $f_1 : X_2 \to S^2$, and $X_2 = S^2 \times S^2$ with the trivial rational fibration $f_2 : X_2 \to S^2$. The former is a genus two fibration and has the global monodromy: $(\beta_1\beta_2\beta_3\beta_4)^2 = 1$, where the curves β_1, β_2, β_3 and β_4 are as shown in Figure 4.10.

If we denote the standard generators of the fundamental group of the regular fiber Σ_2 as a_1, b_1, a_2, b_2, then the curves β_i are base point homotopic to: $\beta_1 = b_1 b_2$, $\beta_2 = a_1 b_1 a_1{}^{-1} b_1{}^{-1} = a_2 b_2 a_2{}^{-1} b_2{}^{-1}$, $\beta_3 = b_2 a_2 b_2{}^{-1} a_1$, $\beta_4 = b_2 a_2 a_1 b_1$.
Hence $\pi_1(X_1) = \pi_1(\Sigma_2) \,/\, \langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle$ is isomorphic to

$$\pi_1(X_1) \;=\; \langle a_1, b_1, a_2, b_2 \mid b_1 b_2 = [a_1, b_1] = [a_2, b_2] = b_2 a_2 b_2^{-1} a_1 = 1 \rangle.$$

Now take the broken fiber sum of (X_1, f_1) and (X_2, f_2) along regular fibers F_1 and F_2, where $\gamma_1 = a_1$, $\gamma_2 = b_2$. The gluing map φ_1 is unique, and we take φ_2 as the identity. Thus we get a new 4-manifold X and a near-symplectic broken

Lefschetz fibration $f : X \to S^2$ with two even round singular circles. Note that $\pi_1(X_1 \setminus N(F_1)) \cong \pi_1(X_1)$, and $\pi_1(X_2 \setminus N(F_2)) = 1$, since there are spheres orthogonal to each fiber F_i in X_i. From Seifert-Van Kampen's theorem and from the choice of γ_i in the broken sum, we see that

$$\pi_1(X) \quad = \quad \langle a_1, b_1, a_2, b_2 \mid b_1 b_2 = [a_1, b_1] = [a_2, b_2] = b_2 a_2 b_2^{-1} a_1 = a_1 = b_2 = 1 \rangle.$$

Thus $\pi_1(X) = 1$. On the other hand, $e(X) = e(X_1) + e(X_2) + 2(g_1 + g_2) - 4 = 8$, and $\sigma(X) = \sigma(X_1) + \sigma(X_2) = -4$. Hence, X is homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$ by Theorem 2.0.1. Moreover we obtain four distinct symplectic sections of self-intersection -1 in (X, f) which arise from the internal connected sum of four parallel copies of the self-intersection zero section of $S^2 \times S^2 \cup W$ and the four -1-sections in the Matsumoto fibration in the broken fiber sum. Symplectically blowing-down these sections, we get a near-symplectic structure with two even round circles on a *homotopy* $S^2 \times S^2$, together with a broken Lefschetz pencil supporting it.

Different choices of ϕ_2 would simply change the self-intersection of these sections in X, but the homeomorphism type of X would not change. (Alternatively we could take (X_2, f_2) as a rational fibration on a Hirzebruch surface with section of self-intersection k and fix the gluing ϕ_2 as the identity.)

What makes the broken fiber sum operation interesting is that, apriori, gluing formulae can be given for the invariants.

Proposition 4.3.5 *Let (X, f) be the broken fiber sum of (X_1, f_1) and (X_2, f_2) along F_1 and F_2 with $g_1 - g_2 = k \geq 0$, determined by the tuple $(\gamma_1, \cdots, \gamma_k)$ of circles on F_1, for $j = 1, \ldots, k$. Denote the j-th elementary cobordism corresponding to γ_j by W_j, and Poincaré-Lefschetz duals of γ_j on F_1 by c_j. Then we have*

$$LM_{X,f} = LM^{\vee}_{X_1 \setminus N_1, f_1|_{X_1 \setminus N_1}} \circ L_1 \circ \cdots \circ L_k \circ LM_{X_2 \setminus N_2, f_2|_{X_2 \setminus N_2}}$$

where L_j corresponds to wedging with c_j under the Piunikhin-Salamon-Schwarz isomorphism (defined for a given admissible Spin^c*) between Floer homologies and singular homology.*

Proof: The broken Lefschetz fibration (X, f) can be decomposed as

$$(X, f) = (X_1 \setminus N_1,\ f_1|_{X_1\setminus N_1}) \cup (W_1, p_1) \cup \cdots \cup (W_k, p_k) \cup (X_2 \setminus N_2,\ f_2|_{X_2\setminus N_2}).$$

where each W_i is equipped with the elementary broken fibration p_i. In [65] Perutz shows that on each (W_i, f_i), the LM invariant acts as described in the statement of the proposition. Thus the above formula follows from the fact that LM invariants fit in a fibered field theory. □

It should be possible to formulate a similar statement for the Seiberg-Witten invariants of X, using Seiberg-Witten monopole Floer homology [49].

For what follows we will be interested in a particular case where the result of a broken fiber sum (X, f) (resp. X) has trivial LM invariants (resp. SW invariants):

Proposition 4.3.6 *Let (X, f) be the result of a broken fiber sum of (X_1, f_1) and (X_2, f_2) along F_1 and F_2 with genera $g_1 > g_2$. If any round 2-handle introduced in the broken sum is attached to a nonseparating curve on F_1 which is also a vanishing cycle for a Lefschetz handle in f_1, then LM invariants of (X, f) are all zero. If $b^+(X) > 1$, then the SW invariants of X are also trivial.*

Proof: The fibration f_1 is isotopic to identity on $X_1 \setminus N_1$. The assumption, provides an essential sphere S obtained from the 2-handle of the round 2-handle and the Lefschetz handle mentioned in the statement. The 'equator' γ is an essential curve on F_1, so there is a dual circle α that intersects it positively at one point. Since the monodromy is trivial, this α sweeps out a torus in $F_1 \times S^1 = \partial(X_1 \setminus N_1)$, which has self-intersection zero. If we blow-down S, downstairs we get an embedded torus

with self-intersection $+1$, violating the adjunction inequality for Seiberg-Witten. It follows from the blow-up formula that $\mathrm{SW}_X \equiv 0$. For the LM invariants of (X, f), obseve that the map on the elementary cobordism is equivalent to contracting along γ, but the continuity (from γ to the nodal point) argument for quantum cap product [63] shows that this map is trivial —as the sections of the Hilbert scheme miss the nodal points. $\square$

However, there are examples when the result of a broken fiber sum has nontrivial LM and SW invariants:

Example 4.3.7 Let $X_1 = S^2 \times \Sigma_{g+1}$ and $X_2 = S^2 \times \Sigma_g$ with projections f_i on the S^2 components. The broken fiber sum (X, f) of (X_1, f_1) and (X_2, f_2) along the fibers Σ_{g+1} and Σ_g is the same as $S^2 \times \Sigma_g \# S^1 \times S^3$ equipped with the step fibration. Adapting the Example 5.1.3 from [65], we see that (X, f) has nontrivial LM invariants. It also has nontrivial SW invariants (cf. [58]), calculated in the Taubes chamber of a compatible near-symplectic form. (Since both $S^2 \times pt$ and $pt \times \Sigma_2$ are symplectic with respect to these near-symplectic structures, the near-symplectic forms can be chosen so that they are homologous to the product symplectic form. Therefore SW invariants are computed nontrivially in the *same* chamber.)

Remark 4.3.8 *A similar argument can be used to calculate SW nontrivially, in general for the broken fiber sum of any symplectic Lefschetz fibration (Y, f) of genus g and $b^+(Y) > 1$ with the trivial fibration on $S^2 \times \Sigma_{g+1}$. The same type of handle calculus shows that the resulting manifold is $Y \# S^1 \times S^3$. Since Y has nontrivial SW, so does $Y \# S^1 \times S^3$ [58]. Moreover in [58], the authors shows that the dimension of the moduli space for such a nontrivial solution increases to one, thus $Y \# S^1 \times S^3$ is not of simple type. Hence, it is an intriguing question to determine whether the broken fiber sum of two simply-connected 4-manifolds can result in a 4-manifold with nontrivial*

Seiberg-Witten invariants, which is likely to be of non-simple type. We currently do not have such an example or a proof that shows this can not happen.

4.3.2 Button addition

In [64] Perutz discusses a local modification, called *button addition*, around a regular fiber of a broken fibration which locally increases the genus by one, while introducing two new Lefschetz singularities and an odd round handle singularity, and resulting in a homology equivalent 4-manifold with the same fundamental group. We will first show that this modification can be made indeed without changing the underlying smooth 4-manifold X.

The construction makes use of the fibration described in terms of Kirby diagrams in Example 4.2.6 with $k = 0$. Taking out a regular neighborhood of a sphere fiber from the lower side, we are left with a broken Lefschetz fibration over D^2, which precisely has the diagram given in Figure 4.2 on the right. Let us denote this piece by $\tilde{B}$, and call it *the button*. Now given any broken Lefschetz fibration f on X, take a regular neighborhood N of any regular fiber F, fibered trivially over D^2. Locally there exist self-intersection zero disk sections both in N and in B. We simply take the section sum of these two fibrations so to obtain the obvious broken Lefschetz fibration $N \amalg \tilde{B} \to D^2$, which can be glued back in $X \setminus N$ to obtain a new broken Lefschetz fibration f' over S^2. Furthermore, if X has a section of self-intersection s, we can choose the local section in N as the restriction of this one so f' also admits a section with self-intersection s. Using our handlebody diagrams and analyzing this operation a bit carefully, for a general X we see that:

Theorem 4.3.9 *Let $f : X \to S^2$ be a broken Lefschetz fibration compatible with a near-symplectic structure ω, and F be a chosen fiber around which we attach a button. The button addition does not change the diffeomorphism type of X, and the* *resulting fibration $f' : X \longrightarrow S^2$ supports a near-symplectic form ω' which restricts to the original near-symplectic structure ω away from F. Conversely, if there is a button in a simplified near-symplectic broken Lefschetz fibration, one can recover a genuine symplectic structure on X.*

Proof: In Example 4.2.6 we have shown that the total space of the button fibration is $S^2 \times S^2$, where $k = 0$. When we take out a regular neighborhood of an S^2 fiber and a regular neighborhood of the section, the remaining piece B can easily be seen to be D^4. The button addition amounts to taking out the local disk section and gluing in B. Trivializing N as $D^2 \times \Sigma_g$, where g is the genus of F, we express the gluing region as the union $\partial D^2 \times D^2 \cup D^2 \times \partial D^2 = S^3$. The horizontal gluing along $D^2 \times \partial D^2$ is determined uniquely by the self-intersection of the section, whereas the vertical gluing is determined by the fact that the monodromies of both fibrations are isotopic to the identity on $\partial D^2 \times \Sigma_g$. These certainly agree on the corners, so the operation boils down to taking out a D^4 in the original manifold X, and putting it back in by a diffeomorphism of $\partial D^4 = S^3$ which we have argued to be isotopic to the identity. This extends over the D^4 to give back X.

Alternatively, take $S^2 \times \Sigma_g$ with the projection map onto the first component. We can then take the section sum of this fibration with the button fibration $S^2 \times S^2 \longrightarrow S^2$ along self-intersection zero sections. The handlebody diagram of the resulting 4-manifold and the broken Lefschetz fibration on it is similar to the one given in Figure 4.5 before, except that the higher side fiber now has genus $g+1$. The same calculus as in Example 4.2.6 verifies that the total space is diffeomorphic to $S^2 \times \Sigma_g$. Since there is a section, this fibration admits a compatible near-symplectic structure. The button addition is equivalent to fiber summing this broken Lefschets fibration along a regular fiber in the lower side (which has genus g) with the broken Lefschetz fibration f on X along F. Since the fibers are symplectic, we can alter the near-symplectic structure on $S^2 \times \Sigma_g$ so that the fiber sum can be made symplectically (see Theorem 1). Hence we obtain a new near-symplectic form ω' supporting the new broken Lefschetz fibration $f' : X \to S^2$, and restricting to ω on the complement of a chosen neighborhood of F.

The last assertion follows from the definition of a simplified near-symplectic broken Lefschetz fibration. □

Using consecutive button additions one can locally increase the genus of any fiber of a given broken Lefschetz fibration without changing the ambient 4-manifold. This allows us to define another interesting way to generalize the symplectic fiber sum operation as follows: Let $f_i : X_i \to \Sigma_i$, F_i and k be as in the previous subsection. We repeatedly introduce k buttons in a regular neighborhood N_2 of F_2, such that the images of round handle singularities are arranged as a nest of ovals. Take a regular fiber F_2' of genus $g_2 + k$ with a small enough regular neighborhood N_2' contained in the very center of these ovals, and take the symplectic fiber sum of X_1 and X_2 along F_1 and F_2' to form $X = X_1 \setminus N_1 \cup X_2 \setminus N_2' = X_1 \setminus N_1 \cup W \cup X_2 \setminus N_2$. Then we obtain a broken fibration $f : X \to \Sigma_1 \# \Sigma_2$ which restricts to the fibrations $f_i : X_i \setminus N_i \to \Sigma_i \setminus D_i$, but now on W it is the trivial fibration on $F_1 \times D_1$ extended by 'button fibrations' – introducing k new round handle singularities and $2k$ new Lefschetz singularities. Call (X, f) the *buttoned fiber sum* of (X_1, f_1) and (X_2, f_2) along F_1 and F_2, which is uniquely determined by the choice of gluings $\phi_1 : \partial(X_1 \setminus N_1) \to \partial_+ W$ and $\phi_2 : \partial(X_2 \setminus N_2) \to \partial_- W$ preserving the fibrations as in the broken fiber sum.

Thus if (X_i, f_i) are symplectic Lefschetz fibrations with regular fiber genus $g_1 \neq g_2$, then buttoned fiber sum allows us to still take the fiber sum, after replacing one of the symplectic forms by a near-symplectic form. Similar vanishing results as in Proposition 4.3.6 works in this case as well, but we do not know if the resulting 4-manifold would always have trivial LM or SW invariants.

4.4 Applications to near-symplectic 4-manifolds with non-trivial invariants

We now turn our attention to near-symplectic 4-manifolds with nontrivial SW invariants (resp. LM invariants whenever fibrations are present). Let us refer to these as *nontrivial near-symplectic 4-manifolds* for a shorthand, even though we do not claim that the SW calculation makes use of the near-symplectic forms. However when $b^+ = 1$ we *always* consider the SW invariant computed in the chamber of the near-symplectic form.

Let (X, ω) be a near-symplectic 4-manifold with zero locus Z. One of the key observations that Taubes made in his programme is that if SW of X is nontrivial, then there is a finite energy J-holomorphic curve C in $X \setminus Z$ which homologically bounds Z (more precisely, C has the intersection number one with every linking 2-sphere of Z), where J is an almost complex structure compatible with ω in the complement of Z [78]. We call this *Taubes' curve*. Below we show that the converse to this theorem is not true, together with an analogous result for LM invariants:

Theorem 4.4.1 *There are infinitely many pairwise nonhomeomorphic closed oriented near-symplectic* 4*-manifolds* (X_m, ω_m), $m > 0$, *equipped with broken Lefschetz fibrations* $f_m : X \to S^2$ *that induce* ω_m, *such that:*

(i) Each (X_m, ω_m) *admits a Taubes' curve, but* $\mathrm{SW}_{X_m} \equiv 0$.

(ii) For each (X_m, f_m) *there is an admissible* Spin^c *structure* $\mathfrak{s}$ *such that the associated moduli space of Lagrangian matching invariants has non-empty moduli with non-negative dimension, but* $LM_{(X_m, f_m)} \equiv 0$.

Proof: Take $S^2 \# \Sigma_g$, $g \geq 1$ with the step fibration. Then use the connected sum model in Remark 4.2.8 to equip $X_g = \#2(S^2 \# \Sigma_g)$ with a near-symplectic broken

Lefschetz fibration $f_g : X_g \to S^2$. Let $F_h \cong \Sigma_{2g}$ be the higher genus side of this fibration. Take a regular neighborhood $D^2 \times F_h$ of F_h, where the fibration restrics as projection $pr_1 : D^2 \times F_h \to D^2$ on the first component. Let γ_s, for $s \in \partial D^2 = S^1$ be the attaching circles of the fiberwise attached 2-handles, and Z be the corresponding round singularity. One can find a parallel disk section D of $(D^2 \times F_h, pr_1)$, so that ∂D intersects γ_s at one point for all s. One can extend each D to a disk section $\tilde{D}$ into the round cobordism from the higher side, so that $\partial\tilde{D} = Z$. If necessary, we can perturb the near-symplectic form on X to make $\tilde{D}$ symplectic on $X \setminus Z$, and therefore it is J-holomorphic with respect to a compatible almost complex structure in $X \setminus Z$. Clearly $\tilde{D}$ is a finite energy curve, and the way we constructed it implies that each C intersects with every linking sphere of Z at one point. Setting $C = \tilde{D}$, we obtain the desired curve. However for any $g \geq 1$, by the connected sum theorem for SW invariants, $\mathrm{SW}_{X_g} \equiv 0$.

To show the second part, let us label the fiber components of the lower side regular fiber F_l and the two distinct self-intersection zero sections of f_g on $X_g = \#2(S^2\#\Sigma_g)$ by F_j and S_j $(j = 1, 2)$, respectively. Then a straightforward calculation shows that the canonical Spin^c structure associated to the fibration on $X \setminus Z$ has

$$c_1(X \setminus Z) = 2F_1 + 2F_2 - (2g-2)S_1 - (2g-2)S_2 - 2\,D\,.$$

Then the Spin^c structure associated to the class $\beta = -F_1 - F_2 + (g-1)S_1 + (g-1)S_2 + D$ has $c_1(\mathfrak{s}) = c_1(X \setminus Z) + 2PD(\beta) = 0$. So for every fiber component Σ (i.e. F_1, F_2 or F_h), we have $\langle c_1(\mathfrak{s}), \Sigma\rangle \geq \chi(\Sigma)$. Moreover $\frac{\chi(\Sigma)}{2} < \langle \beta, \Sigma\rangle < -\frac{\chi(\Sigma)}{2}$ is satisfied when $g > 1$. Therefore $\mathfrak{s}$ is an admissible Spin^c structure. However $LM_{X_g, f_g} \equiv 0$ as shown in [65]. Lastly, $d(LM_{(X_g, f_g)}(\mathfrak{s}_\mathfrak{g})) = \frac{1}{4}(c_1(\mathfrak{s})^2 - 3\sigma(X) - 2e(X)) = \frac{1}{4}(0 - 0 - 2(6-8g)) = 4g - 3 \geq 0$.

Setting $m = g + 1$, we get the infinite families by varying $g > 0$. $\square$

In general any Gromov type of invariant might vanish even if the associated moduli space is nonempty. Thus the above result should be regarded as an explicit demonstration of this phenomenon.

One might wonder if the class of nontrivial near-smplectic 4-manifolds is closed under the symplectic fiber sum operation, as it is the case for both near-symplectic and symplectic classes. We show that this is too much to hope:

Theorem 4.4.2 *There are infinitely many topologically distinct pairs of closed near-symplectic 4-manifolds with nontrivial SW invariants whose symplectic fiber sum results in trivial near-symplectic 4-manifolds. The same holds for LM invariants.*

Proof: As discussed in Example 4.3.7 and the succeeding paragraph, if Y has nontrivial SW, then so does $Y \# S^1 \times S^3$. Take $E(n)$ (say with $n > 1$) with an elliptic fibration, and equip it with a symplectic form making the regular torus fiber T symplectic. Also take $S^2 \times \Sigma_2$ with the product symplectic form. Look at the broken fiber sum of $E(n)$ with $n \geq 2$ along a regular torus fiber T with $S^2 \times \Sigma_2$ along a genus two surface $\{pt\} \times \Sigma_2$, where boundary gluings φ_1 and ϕ_2 are chosen to be identity, and γ is chosen to be some fixed standard generator of Σ_2. The result is the nontrivial near-symplectic 4-manifold $X_n \cong E(n) \# S^1 \times S^3$.

We can then take the symplectic fiber sum of such X_n and X_m along the higher side genus two fibers to get $X_{n,m}$. There are families of disks with their boundaries on $\partial(X_n \setminus N(\Sigma_2))$ and $\partial(X_m \setminus N(\Sigma_2))$, coming from the broken fiber sum construction in each piece. Matching pairs of these disks give spheres S_s with zero self-intersection, where s is parametrizeed by the base S^1 in the gluing region $S^1 \times \Sigma_2$ of the fiber sum. Denote the equator of S_s sitting on the fiber sum region by γ_s, and consider a dual circle α_s on the same fiber. Varying s along S^1 we obtain a Lagrangian torus T, which intersects each S_s at one point. Thus S_0 is an essential sphere in $X_{n,m}$. Since $b^+(X_{n,m}) > 1$, the existence of such a sphere implies that $\mathrm{SW}_{X_{n,m}} \equiv 0$. Infinite families are obtained by varying $n, m > 1$. For the second part of the statement, let us use $Y_g = S^2 \times \Sigma_g \# S^1 \times S^3$ equipped with the step fibration f_g over S^2. Then (Y_g, f_g) has nontrivial LM invarints. Taking the fiber sum of two copies of (Y_g, f_g) along higher genus regular fibers, we obtain a near-symplectic broken Lefschetz fibration (X'_g, f'_g). Observe that

$$LM_{X'_g, f'_g} = LM^{\vee}_{S^2\times\Sigma_g\setminus N(\Sigma_g), f_g|} \circ LM_{W, pr_1} \circ LM_{S^2\times\Sigma_g\setminus N(\Sigma_g), f_g|}$$

where W is a cobordism that consists of an elementary round 1-handle cobordism W_1 followed by an elementary round 2-handle cobordism W_2. So $LM_{W,pr_1} = LM_{W_2,pr_1} \circ LM_{W_1,pr_1}$. However, under Piunikhin-Salamon-Schwarz isomorphism, LM_{W_2,pr_2} corresponds to wedging with the Poincaré-Lefschetz dual of γ, the attaching circle of the 2-handle of the round 2-handle. Since the round 1-handle cobordism W_1 is constructed in the same way, this γ can be contracted along W_1, and therefore LM_{W_2,pr_2} is trivial. It follows that $LM_{X'_g,f'_g} \equiv 0$. Taking $g = 0, 1, 2, \ldots$ we obtain the desired infinite family. $\square$

Remark 4.4.3 *For the same examples in the proof of Theorem 4.4.2 if one indeed takes the fiber sum along lower genus fibers, the result is* $E(n+m)\# 2\, S^1 \times S^3$*, which again has nontrivial* SW*. Thus the choice of the fibers in a near-symplectic fiber sum affects the outcome drastically. A natural question that follows is:*

Question: *If* X_i *are nontrivial near-symplectic 4-manifolds and* F_i *are symplectically embedded surfaces in* F_i *with minimal genus, is the (symplectic) fiber sum* X *of* X_1 *and* X_2 *along* F_1 *and* F_2 *nontrivial?*

It is known that Lefschetz fibrations over S^2 do not admit sections of nonnegative self-intersections, and the self-intersection can be zero only when the fibration is trivial. In general near-symplectic broken Lefschetz fibrations are not subject to this constraint. Even when we restrict our attention to near-symplectic broken Lefschetz fibrations on nontrivial 4-manifolds, there appears a difference:

Theorem 4.4.4 *There are closed simply-connected* 4*-manifolds which admit near-symplectic broken Lefschetz fibrations over* S^2 *with sections of any self-intersection. More precisely, for any integer* k *and positive integer* n*, there is a near-symplectic* $(X_{n,k}, f_{n,k})$ *fibered over* S^2*, with a section of self-intersection* k *and with* $b^+(X_{n,k}) = n$*. If* $f : X \to S^2$ *is a nontrivial broken Lefschetz fibration over a nontrivial near-symplectic* 4*-manifold* X *with* $b^+(X) > 1$*, then any section* S *of* f *has negative self-intersection. There are simply-connected examples with sections of any self-intersection when* $b^+ = 1$*.*

Proof: In Example 4.2.6 we have constructed near-symplectic broken Lefschetz fibrations over S^2 which admit sections of any self-intersection k. As the total space of these fibrations are either $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, the SW invariants are nontrivial. (Since the near-symplectic forms can be chosen so that they determine the same chamber with the usual symplectic structures, and therefore SW invariants are computed nontrivially in the chamber of the near-symplectic forms.) These provide examples for the very last part of the theorem. As described in the Example 4.2.7, we can obtain a near-symplectic broken Lefschetz fibration on connected sums of these fibrations. Using n such copies, we obtain a 4-manifold with $b^+ = n$, which proves the first statement. For the remaining assertion, we simply employ the SW adjunction inequality as in the Lefschetz fibration case (see for instance [72]). □

There are various examples of nonsymplectic 4-manifolds which have nontrivial SW invariants. All these examples have $b^+ > 0$, which means that they admit near-symplectic broken Lefschetz pencils but not symplectic Lefschetz fibrations or pencils. This can be made explicit in Fintushel-Stern's knot surgered $E(n)$ examples [28, 29]. The below result gives near-symplectic broken Lefschetz fibrations on an infinite family of pairwise nondiffeomorphic closed simply-connected smooth 4-manifolds which can not be equipped with Lefschetz fibrations or pencils.

Proposition 4.4.5 *For any knot K, $E(n)_K$ admits a near-symplectic broken Lefschetz fibration over S^2.*

Proof: Think of $E(n)$ as the branched double cover of $S^2 \times S^2$ with branch set composed of four disjoint parallel copies of $S^2 \times \{pt\}$ and $2n$ disjoint parallel copies of $\{pt\} \times S^2$, equipped with the locally holomorphic 'horizontal fibration' [29]. The regular torus fiber F of the usual vertical fibration is a bisection with respect to this fibration. We have exactly four singular fibers each with multiplicity two. On the other hand, if M_K is obtained by a 0-surgery on a nonfibered knot K in S^3, then there is a broken fibration (no Lefschetz singularities) from $S^1 \times M_K$ to T^2 as discussed in Example 4.1.2. One can compose this map with a degree two branched covering map from the base T^2 to S^2, such that the branching points are not on the images of the round handle singularities. What we get is a broken fibration with four multiple fibers of multiplicity two, which are obtained from collapsing two components from all directions. An original torus section T of $S^1 \times M_K \to T^2$ is now a bisection of this fibration, intersecting each fiber *component* at one point. Both F and T have self-intersection zero, and thus we can take the symplectic fiber sum of $E(n)$ and $S^1 \times M_K$ along them to get $E(n)_K$. The multiplicity two singular fibers can be matched so to have a locally holomorphic broken fibration with four singular fibers of multiplicity two. This fibration can be perturbed to be Lefschetz as argued in [29]. When K is fibered, we obtain genuine Lefschetz fibrations. $\square$

CHAPTER 5

Folded-symplectic 4-manifolds

5.1 Background

5.1.1 Achiral Lefschetz fibrations and PALFs

An *achiral Lefschetz fibration* is defined in the same way a Lefschetz fibration is defined, except that the given charts around critical points are allowed to reverse orientation. In other words, the 2-handles can be glued with framing $+1$ with respect to fiber framing, too. Also recall that a Lefschetz pencil is a map $f\colon X\setminus\{b_1,\ldots,b_k\}\to S^2$, such that around any *base point* b_i it has a local model $f(z_1, z_2) = z_1/z_2$, preserving the orientations, and that f is a Lefschetz fibration elsewhere. An *achiral Lefschetz pencil* is then defined by allowing orientation reversing charts around the base points as well. Critical points or base points with orientation reversing charts are called *negative critical points* or *negative base points*, whereas the other critical points or base points are *positive*. For a detailed treatment of this topic and proofs of some facts quoted below, the reader is advised to turn to [40].

A Lefschetz fibration is said to be *allowable* if all its vanishing cycles are homologically nontrivial in the fiber. Particularly, we will be interested in allowable Lefschetz fibrations over D^2 with bounded fibers. In the literature, this type of Lefschetz fibration having only positive critical points is called a *PALF*. Similarly, when the critical points are instead all negative, we will call the fibration a *NALF*. Lastly note that the monodromy representation for an achiral Lefschetz fibration can be described in the exact same way as in Section 2.0.3; so we can talk about the global monodromy and representations of any given achiral Lefschetz fibration $f : X \to D^2$.

Next is a standard fact which was first observed by Harer:

Theorem 5.1.1 (Harer [44]) *Let X be a 4-manifold with boundary. Then X admits an achiral Lefschetz fibration over D^2 with bounded fibers if and only if it admits a handlebody decomposition with no handle of index greater than two.*

5.1.2 Open book decompositions

An *open book decomposition* of a 3-manifold M is a pair (B, f) where B is an oriented link in M, called the *binding*, and $f: M \setminus B \to S^1$ is a fibration such that $f^{-1}(t)$ is the interior of a compact oriented surface $F_t \subset M$ and $\partial F_t = B$ for all $t \in S^1$. The surface $F = F_t$, for any t, is called the *page* of the open book. The *monodromy* of an open book is given by the return map of a flow transverse to the pages and meridional near the binding, which is an element $\mu \in \Gamma_{g,m}$, where g is the genus of the page F, and m is the number of components of $B = \partial F$.

Suppose we have an achiral Lefschetz fibration $f: X \to D^2$ with bounded regular fiber F, and let p be a regular value in the interior of the base D^2. Composing f with the radial projection $D^2 \setminus \{p\} \to \partial D^2$ we obtain an open book decomposition on ∂X with binding $\partial f^{-1}(p)$. Identifying $f^{-1}(p) \cong F$, we can write $\partial X = (\partial F \times D^2) \cup f^{-1}(\partial D^2)$. Thus we view $\partial F \times D^2$ as the tubular neighborhood of the binding $B = \partial f^{-1}(p)$, and the fibers over ∂D^2 as its *truncated pages*. The monodromy of this open book is prescribed by that of the achiral fibration [44]. In this case, we say the open book $(B, f|_{\partial X \setminus B})$ *bounds* or *is induced by* the achiral Lefschetz fibration

$f\colon X \to D^2$. Recalling that any closed oriented 3-manifold can be bounded by a 4-manifold with only 0-, 1- and 2- handles, it is fairly easy to see that any open book decomposition bounds such an achiral Lefschetz fibration over a disk.

We would like to describe an elementary modification of these structures: Let $f\colon X \to D^2$ be an achiral Lefschetz fibration with bounded regular fiber F. Attach a 1-handle to ∂F to obtain F', and then attach a positive (resp. negative) Lefschetz 2-handle along an embedded loop in F' that goes over the new 1-handle exactly once. This is called a *positive stabilization* (resp. *negative stabilization*) of f. A positive (resp. negative) Lefschetz handle is attached with framing -1 (resp. $+1$) with respect to the fiber, and thus it introduces a positive (resp. negative) Dehn twist on F'. If the focus is on the 3-manifold, one can totally forget the bounding 4-manifold and view all the handle attachments in the 3-manifold. Either way, stabilizations correspond to adding canceling handle pairs, so diffeomorphism types of the underlying manifolds do not change, whereas the achiral Lefschetz fibration and the open book decomposition change in the obvious way. It turns out that stabilizations preserve more than the underlying topology, as we will discuss shortly.

5.1.3 Contact structures and compatibility

A 1-form $\alpha \in \Omega^1(M)$ on a $(2n-1)$-dimensional oriented manifold M is called a *contact form* if it satisfies $\alpha \wedge (d\alpha)^{n-1} \neq 0$. An *oriented contact structure* on M is then a hyperplane field ξ which can be globally written as kernel of a contact 1-form α. In dimension three, this is equivalent to asking that $d\alpha$ be nondegenerate on the plane field ξ.

A contact structure ξ on a 3-manifold M is said to be *supported by an open book* (B, f) if ξ is isotopic to a contact structure given by a 1-form α satisfying $\alpha > 0$ on positively oriented tangents to B and $d\alpha$ is a positive volume form on every page.

When this holds, we say that the open book (B, f) is *compatible with the contact structure* ξ on M.

Improving results of Thurston and Winkelnkemper [81], Giroux proved the following groundbreaking theorem regarding compatibilty of open books and contact structures:

Theorem 5.1.2 (Giroux [36]) *Let M be a closed oriented 3-manifold. Then there is a one-to-one correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilizations and isotopy.*

Considering contact 3-manifolds as boundaries of certain 4-manifolds together with some compatibility conditions is a current focus of research in low dimensional topology. From the contact topology point of view, it is the study of different types of *fillings* of a fixed contact manifold. In dimension four, there are essentially two considerations, yet we formulate them for all dimensions: Let (X^{2n}, ω) be a symplectic manifold with cooriented nonempty boundary $M = \partial X$. If there exists a *Liouville vector field* (aka *symplectic dilation*) ν defined on a neighborhood of ∂X pointing out along ∂X, then we obtain a positive contact structure ξ on ∂X, which can be written as the kernel of contact 1-form $\alpha = \iota_\nu \omega|_{\partial X}$. When this holds, we say (M, ξ) is the *ω-convex boundary* or *strongly convex boundary* of (X, ω). For the sake of entirety, note when ν points inside, we obtain a negative contact structure instead, and in this case we say (M, ξ) is the *ω-concave boundary of* (X, ω).

Now if (X^{2n}, J) is almost-complex, then the complex tangencies on $M = \partial X$ give a uniquely defined oriented hyperplane field. It follows that there is a 1-form α on M such that $\xi = Ker\alpha$. We define the *Levi form* on M as $d\alpha|_\xi(\cdot, J\cdot)$. If this form is positive definite then (M, ξ) is said to be *strictly J-convex boundary of* (X, J), and if it is J-convex for an unspecified J (for instance when J is tamed by a given symplectic form), we say (M, ξ) is *strictly pseudoconvex boundary*. If (X, ω, J)

is an almost-Kähler manifold, i.e. a manifold equipped with a symplectic form ω and a compatible almost-complex structure J, then it can be shown that strict pseudoconvexity of the boundary is equivalent to the condition that $\omega|_{\xi} > 0$ in dimenson $2n = 4$. We would like to remark that all these definitions can be formulated in more generality for hypersurfaces in X^{2n}, not necessarily for ∂X only.

For detailed and comparative discussions of these concepts, as well as proofs of some facts mentioned in the next subsection, the reader can turn to [22] and [24]. Also for further basic notions from contact topology of 3-manifolds such as Legendrian knots, Thurston-Bennequin framing, or convex surfaces, which we will occasionally use in this paper, see for example [56].

5.1.4 Stein manifolds

A smooth function $\psi\colon X \to \mathbb{R}$ on a complex manifold X of real dimension $2n$ is called *strictly plurisubharmonic* if ψ is strictly subharmonic on every holomorphic curve in X. We call a complex manifold X *Stein* if it admits a proper strictly plurisubharmonic function $\psi\colon X \to [0,\infty)$ (after Grauert [41]). Thus a compact manifold X with boundary which is equipped with a complex structure in its interior is called *compact Stein* if it admits a proper strictly plurisubharmonic function which is constant on the boundary.

Given a function $\psi\colon X \to \mathbb{R}$ on a Stein manifold, we can define a 2-form $\omega_\psi = -dJ^*d\psi$. It turns out that ψ is a strictly plurisubharmonic function if and only if the symmetric form $g_\psi(\cdot,\cdot) = \omega_\psi(\cdot, J\cdot)$ is positive definite. So every Stein manifold X admits a Kähler structure ω_ψ, for any strictly plurisubharmonic function $\psi\colon X \to [0,\infty)$. It is easy to see that the restriction of ω_ψ to each level set $\psi^{-1}(t)$ gives a Levi form on $\psi^{-1}(t)$, implying that all nonsingular level sets of ψ are strictly pseudoconvex hypersurfaces. Thus in this article, we equivalently call a Stein manifold a *strictly* *pseudoconvex manifold.* Moreover, it was observed in [22] that the gradient vector field of ψ defines a (global) Liouville vector field $\nu = \nabla_\psi$, making all nonsingular level sets ω_ψ-convex. Hence, Stein manifolds exhibit strongest filling properties for a contact manifold which can be realized as their boundary.

In this article, we are mainly interested in compact Stein surfaces. Another characterization of these manifolds, which might be called "the topologist's fundamental theorem of compact Stein surfaces", is due to Eliashberg, and was made explicit by Gompf in dimension four:

Theorem 5.1.3 (Eliashberg [20]; Gompf [39]) *A smooth oriented compact 4-manifold with boundary is a Stein surface, up to orientation preserving diffeomorphisms, if and only if it has a handle decomposition $X_0 \cup h_1 \cup \ldots \cup h_m$, where X_0 consists of 0- and 1-handles and each h_i, $1 \leq i \leq m$, is a 2-handle attached to $X_i = X_0 \cup h_1 \cup \ldots \cup h_i$ along a Legendrian circle L_i with framing $tb(L_i) - 1$.*

All structures we have introduced so far meet in the following theorem:

Theorem 5.1.4 (Loi and Piergallini [50], also see [2]) *An oriented compact 4-manifold with boundary is a Stein surface, up to orientation preserving diffeomorphisms, if and only if it admits a PALF.*

Throughout the article, we give ourselves the freedom of using the prefix 'anti' as a shorthand, whenever an oriented manifold X admits a structure when the orientation on X is reversed; like anti-symplectic, anti-Kähler, or anti-Stein. For Lefschetz fibrations and open books though, we use 'positive' and 'negative' adjectives to distinguish two possible cases.

5.2 Simple folded-symplectic structures

The definition of symplectic (or anti-symplectic) structures can be enlarged as follows in order to cover a larger family of manifolds, which was shown in [13] to contain entire family of closed oriented smooth 4-manifolds:

Definition 5.2.1 *A* folded-symplectic form *on a smooth $2n$-dimensional manifold X is a closed 2-form ω such that ω^n is transverse to the 0-section of $\Lambda^{2n}T^*X$, and whenever this intersection is nonempty, ω^{n-1} does not vanish on the hypersurface $H = (\omega^n)^{-1}(0)$, called the* fold.

For an oriented X, the kernel of ω on H integrates to a foliation called *null-foliation*. Martinet's singular form $x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + ... + dx_n \wedge dy_n$ on $\mathbb{R}^{2n}$ defines the standard folded-symplectic structure, as every folded-symplectic form can be expressed in this way in an appropriate Darboux coordinate system around any point on the fold. There is also a simple folded-structure that every even dimensional sphere carries: We think of S^{2n} sitting in $\mathbb{R}^{2n+1}$, then pull back the standard symplectic form $dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ on the unit disk bounded by the equator in $\mathbb{R}^{2n}$ to S^{2n} by the projection maps along the last coordinate, and finally glue them along the fold S^{2n-1} to obtain ω_0. This is equivalent to doubling the unit disk equipped with its standard symplectic form (by reversing the orientation on one of the disks). We call this form the *standard folded-symplectic form on S^{2n}*.

For more on folded-symplectic structures, the reader is referred to [14], [13]. Here we only consider these forms on Riemann surfaces and compact 4-manifolds, possibly with boundaries. For the former class, folded-symplectic forms form an open and dense set in the space of 2-forms, whereas in dimension four openness remains but the nonvanishing condition implies that they are nongeneric. We say an embedded surface $\Sigma \subset X^4$ is a *folded-symplectic submanifold* of (X, ω) if $\omega|_\Sigma$ is a folded-symplectic form on Σ. Observe that S^2 equipped with the standard form obtained by pulling back $dx_1 \wedge dy_1$ embeds as a folded-symplectic submanifold of S^4 with the standard folded-symplectic form defined as the pullback of $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ as above.

The following proposition provides several examples of folded-symplectic 4-manifolds:

Proposition 5.2.2 *Let X be a closed oriented smooth 4-manifold and Σ be a closed oriented surface. If $f\colon X \to \Sigma^2$ is an achiral Lefschetz fibration such that the regular fiber is a closed oriented surface F which is nonzero in $H_2(X;\mathbb{R})$, then X admits a folded-symplectic structure ω such that fibers are symplectic and the fold H is an F-bundle over S^1. The fold H splits X into pieces X_+ and X_-, and f induces symplectic Lefschetz fibrations on $(X_+, \omega|_{X_+})$ and on $(-X_-, \omega|_{X_-})$, respectively. Furthermore, any finite set of sections can be made folded-symplectic for an appropriate choice of ω. This form is canonical up to deformation equivalence of folded-symplectic forms.*

We will call this type of folded-symplectic structures *simple* (after Thurston [80]). Base spaces of the fibrations defined on X_+ and $-X_-$ are determined by an arbitrary splitting $\Sigma = \Sigma_+ \cup \Sigma_-$. Here we take $\Sigma_- = D^2$ for simplicity. Observe that the fibration induces an exact sequence

$$\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma) \to \pi_0(F) \to 0$$

It follows that fibers are connected if the base is simply-connected. Otherwise we can define a new achiral Lefschetz fibration from X to the finite cover of Σ corresponding to the finite-index subgroup $f_\#(\pi_1(X))$ in $\pi_1(\Sigma)$, which has connected fibers. Finally, one can perturb f to get a fibration which has at most one critical point on each fiber. Hence, without loss of generality, we will assume that the fibers of f are connected and critical values are distinct.

Proof: [Proof of Proposition 5.2.2] Start by connecting all negative critical points in the base by an embedded arc in the complement of positive critical points, and cover it by the images of orientation reversing charts so that we get a closed neighborhood $\Sigma_- \cong D^2$ of this arc away from the positive critical points. This can be done because around the regular points we have freedom to take charts of either orientation. After we reverse the orientation on $f^{-1}(\Sigma_-)$, the map $f\colon\ f^{-1}(\Sigma_-) \to \Sigma_-$ defines a negative Lefschetz fibration. Set $\Sigma_+ = \Sigma\backslash\Sigma_-$, $C = \Sigma_+\cap\Sigma_-$, $X_+ = f^{-1}(\Sigma_+)$, $X_- = f^{-1}(\Sigma_-)$, and $H = f^{-1}(C)$. If there are no negative critical points, we can choose Σ_- as a small disk around a regular value which does not contain any critical values. Now let β be a folded-symplectic form on Σ which folds over C, such that it is a positive area form on Σ_+ and a negative area form on Σ_-. These forms always exist: For example take S^2 with its standard folded form ω_0, and suppose $\Sigma_\pm$ has genus $g_\pm$. Symplectic connect sum the upper-hemisphere of S^2 with a closed genus g_+ surface equipped with a positive symplectic form, and the lower-hemisphere with a closed genus g_- surface equipped with a negative symplectic form. This yields a folded-symplectic form on Σ, folded along C.

We will construct a folded-symplectic form on X by mimicking Gompf's proof which generalizes Thurston's result for symplectic fibrations to symplectic Lefschetz fibrations ([80] , [40]). Let ζ be a closed 2-form on X which evaluates positively on any closed surface contained in a fiber with the induced orientation. (We have not made any assumptions on the type of vanishing cycles, so one might have more than one closed surface on a fiber if there are separating vanishing cycles.) First we wish to define a closed 2-form η on all over X which is symplectic on each $F_y = f^{-1}(y)$, for all $y \in \Sigma$.

Let A be a tubular neighborhood of C in Σ which does not contain any critical values. Choose disjoint open balls $U_{+,k}$ around each positive and $V_{-,l}$ around each negative critical point so that these sets do not intersect $f^{-1}(A)$ in X and that in appropriate charts the fibration map can be written as $f(z_1, z_2) = z_1 z_2$ and $f(z_1, z_2) = \overline{z_1} z_2$, respectively. Take the standard forms

$$\omega_{+,k} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = -\frac{i}{2} dz_1 \wedge d\overline{z}_1 - \frac{i}{2} dz_2 \wedge d\overline{z}_2$$

on $U_{+,k}$ and

$$\omega_{-,l} = -dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = \frac{i}{2} dz_1 \wedge d\overline{z}_1 - \frac{i}{2} dz_2 \wedge d\overline{z}_2$$

on $V_{-,l}$ for all k, l. For any $y \in f(U_{+,k})$, $F_y \cap U_{+,k}$ is a $J_{+,k}$-holomorphic curve, where $J_{+,k}$ is an almost-complex structure compatible with $\omega_{+,k}$. Similarly for any $y \in f(V_{-,l})$, $F_y \cap V_{-,l}$ is $J_{-,l}$-holomorphic curve, where $J_{-,l}$ is an almost-complex structure compatible with $\omega_{-,l}$. Having expressed $\omega_{+,k}$ and $\omega_{-,l}$ in terms of Kähler forms, we can take these almost-complex structures as (i, i) and $(-i, i)$, respectively. It follows that $\omega_{+,k}|_{F_y \cap U_{+,k}}$ is symplectic, so we can extend it to a symplectic form ω_y on the entire fiber and get ω_y defined for all points in each $f(U_{+,k})$ this way. Do the same for all points in $f(V_{-,l})$, for every j. Finally, for all remaining $y \in \Sigma$ take any symplectic form ω_y on the fiber, and rescale every ω_y we have defined away from all $U_{+,k}$ and $V_{-,l}$ so that they are in the same cohomology class as the restriction of ζ to each F_y. Next, cover Σ with finitely many balls B_s containing at most one critical value, and whenever they do contain a critical value, assume they are centered at that point. Reindex $U_{+,k}$ and $V_{-,l}$, and shrink them if necessary to make sure they lie in $f^{-1}(B_s)$ for some s. Define η_s on each $f^{-1}(B_s)$ as the pullback of $\omega_{+,S}$, $\omega_{-,S}$, or ω_y by r_s, where r_s is the retraction of $f^{-1}(B_s)$ to the fiber F_y over the center of B_s, or the union of F_y either with closure of $U_{+,S}$ or with closure of $V_{-,S}$, whenever B_s contains a positive or negative critical value, respectively. Now we can glue these forms to construct the 2-form η we wanted, by using a partition of unity and that each η_s is cohomologous to $\zeta|_{f^{-1}(B_s)}$ as in [40].

We claim that $\omega_\kappa = \kappa\eta + f^*(\beta)$ is a folded-symplectic form on X, where κ is a small enough positive real number. ω_κ is clearly closed and symplectic in the fiber direction. It follows that for any noncritical point $x \in F_y$, $T_xX = T_xF_y \oplus (T_xF_y)^{\perp\eta}$. Here $f^*(\beta)$ is nondegenerate over $(T_xF_y)^{\perp\eta}$ for all $x \notin H$, implying that for sufficiently small κ, ω_κ is nondegenerate on $X \setminus (H \bigcup_s (U_{+,S} \cup V_{-,S}))$. On the other hand $\omega_\kappa|_{U_{+,S}} = \kappa\omega_{+,S} + f^*(\beta)$ and $\omega_\kappa|_{V_{-,S}} = \kappa\omega_{-,S} + f^*(\beta)$. Therefore for any nonzero $v \in TU_{+,S}$, we have

$$\omega(v, J_{+,S}\, v) = \kappa g(v,v)^2_{+,S} + \beta(f_*(v), if_*(v)) > 0\,,$$

where $g(-,-)_{+,S}$ is the metric induced from $\omega_{+,S}$ and $J_{+,S}$. Likewise, for any nonzero $v \in TV_{-,S}$, we will have

$$\omega(v, J_{-,S}\, v) = \kappa g(v,v)^2_{-,S} + \beta(f_*(v), -if_*(v)) > 0,$$

$g(-,-)_{-,S}$ being the metric induced from $\omega_{-,S}$ and $J_{-,S}$. (Recall that β is negative on Σ_-.) Hence ω_κ is symplectic everywhere on X except H, where it vanishes transversely. Moreover, $f^*(\beta)$ is a folded-symplectic form on any section, so taking κ even smaller, we can as well assume that any finite collection of sections of f are folded-symplectic. It is easy to check that the folded-symplectic form we get satisfies all the other declared properties. (Also see Remark 5.2.3). □

The homological assumption in the theorem is a very mild one. If S is the set of critical points of the achiral fibration $f\colon X \to \Sigma$, then the tangencies of the fibers define a complex line bundle $L = Ker(df)$ on $X \setminus S$, which extends uniquely over X. It follows that unless we have a torus fibration, the regular fiber F is essential, since $< c_1(L), F >= \chi(F)$. Also if the fibration is obtained from a pencil by blowing up the base points, the exceptional spheres will become sections of the fibration, guaranteeing that the fibers are essential in the homology.

Remark 5.2.3 *Alternatively, the folded-symplectic form in Proposition 5.2.2 can be constructed by using the* folding *operation described in [14]. Restrictions of* β *on* Σ_+ *and on* Σ_- *give well-defined area forms* β_+ *and* β_-, *respectively. Gompf's method can be used to define a symplectic form* $\kappa_+\eta + f^*(\beta_+)$ *on* X_+, *where* η *is a 2-form on* X *that restricts to the fibers as a (positive) symplectic form and* κ_+ *is a small enough positive real number. The orientation on the base together with the orientation on the regular fiber determines the orientation of the total space, and thus by taking the opposite orientation on* Σ_- *but keeping the orientation on* F, *one orients* $-X_-$. *Let* $\overline{f}\colon -X_- \longrightarrow \Sigma_-$ *be the fibration defined by taking orientation-preserving charts for* $f\colon X_- \longrightarrow \Sigma_-$, *then we can define a symplectic form* $\kappa_-\eta + \overline{f}^*(-\beta_-)$ *on* $-X_-$ *(as* $-\beta_-$ *is the area form on* Σ_-*) by following the same construction method. Observe that* $\overline{f}^*(-\beta_-) = f^*(\beta_-)$. *Hence, setting* $\kappa = min\{\kappa_+, \kappa_-\}$, *we obtain two symplectic manifolds* (X_+, ω_+) *and* $(-X_-, \omega_-)$, *where* $\omega_\pm = \kappa\eta + f^*(\beta_\pm)$. *Let* $\iota_\pm$ *be the inclusions of boundaries into* $\pm X_\pm$, *then* $\iota_+^*(\omega_+) = \kappa\eta = \iota_-^*(\omega_-)$ *and the orientations of both null-foliations agree. Thus we can glue these pieces to obtain a folded-symplectic structure on* $X_+ \cup X_- = X$, *which agrees with* ω_+ *and* ω_- *in the complement of a tubular neighborhood of the fold* $\partial X_+ = H = -\partial X_-$. *(See [14] for details.) This form is deformation equivalent to the form* $\kappa\eta + f^*(\beta)$ *in Proposition 5.2.2.*

5.3 Existence of folded-symplectic structures on closed oriented 4-manifolds

Here we show that any closed oriented smooth 4-manifold X can be equipped with a folded-symplectic form. For the sake of completeness, we start by outlining Etnyre and Fuller's proof that every 4-manifold admits an achiral Lefschetz fibration after a surgery along a framed circle [23] : Take a handlebody decomposition of X with one $0-$ and one 4-handle, let X_1 denote the union of the 0-handle, 1-handles and 2-handles, and X_2 denote the union of the 3-handles and the 4-handle. By Theorem 5.1.1 there exist achiral Lefschetz fibrations $f_i\colon X_i \to D^2$, which necessarily have bounded fibers, and stabilizing both fibrations we may as well assume the fibers have connected boundaries. After a possible slight modification of the handlebody decomposition, Etnyre and Fuller manipulate the contact structures on the boundaries so that they are both overtwisted and homotopic as plane fields. Then it follows from results of Eliashberg and Giroux that we have isotopic contact structures, and thus the induced open books are the same, possibly after some stabilizations and isotopies. Denoting the final manifolds and fibrations with X_i and f_i again, we may therefore assume that the open book decompositions induced by these fibrations on the common boundary $H = \partial X_1 = -\partial X_2$ are the same, so we can glue both pieces of X back along the truncated pages, and obtain an achiral Lefschetz fibration

$$f_1 \cup f_2\colon W = X_1 \bigcup_{f_1^{-1}(\partial D^2)=f_2^{-1}(\partial D^2)} X_2 \longrightarrow S^2.$$

To recover X we need to glue $S^1 \times D_1^2$ to $S^1 \times D_2^2$, where

$$S^1 \times D_i^2 = \partial X_i \setminus f_i^{-1}(\partial D^2).$$

Filling the boundary of W with an $S^1 \times D^3$ gives the same result, so we can view W as $X \setminus N$ where N is a neighborhood of an embedded curve $\gamma \subset X$. Now, if we instead add on a $D^2 \times S^2$ so that each $\partial D^2 \times \{pt\}$ is identified with $S^1 \times \{pt\}$, we can extend the fibration on W' by the projection on the S^2 component of $D^2 \times S^2$. Hence, we obtain an achiral Lefschetz fibration over S^2 on the resulting manifold Y, where the section S of this fibration discussed in [23] can be taken as $0 \times S^2$ coming from the glued in $D^2 \times S^2$, implying S has trivial normal bundle in Y.

We will refer the following as the *standard model*: Consider S^4 with the standard folded-symplectic structure ω_0 described before, and take $S^4 \cap \{x_4 = 0\}$ vertical to the fold $H_0 = S^4 \cap \{x_5 = 0\}$. Take $S_0 = S^4 \cap \{x_4 = 0 = x_3\} \cong S^2$ which intersects the fold along the circle $C_0 = \{x_1^2 + x_2^2 = 1 \,|\, x_3 = x_4 = x_5 = 0\}$. It is easy to see that ω_0 restricts to this S_0 as the standard folded-symplectic form on S^2, folded along C_0, and symplectic on the normal disks to S_0. Fix a disk neighborhood M_0 of S_0 so that ω_0 evaluates as 1 on each normal disk. That is, each normal disk projects onto unit disk $\{x_3^2 + x_4^2 \leq 1 \,|\, x_1 = x_2 = x_5 = 0\}$ symplectomorphically. By restricting ω_0, we get two folded-symplectic manifolds $M_0 \cong S^2 \times D^2$ and $N_0 = S^4 \setminus M_0 \cong D^3 \times S^1$, with folds $S^1 \times D^2$ and $D^2 \times S^1$, respectively.

The existence of the section $s\colon S^2 \longrightarrow S \subset X$ guarantees that the fiber of the achiral Lefschetz fibration $f\colon Y \longrightarrow S^2$ is homologically essential and therefore there exists a folded-symplectic form ω as described in Proposition 5.2.2. This restricts to $Y \setminus M$, where $M \cong S^2 \times D^2$ is a neighborhood of S. We may assume ω is constructed such that M is identified with M_0 in the standard model above as follows: Let $\phi\colon M \longrightarrow M_0$ be an orientation preserving diffeomorphism such that ϕ is orientation preserving on the spheres (and on the normal directions as well), and that it maps the upper-hemisphere of S_0 (where ω_0 is positive) to the positive part of S. Then one can start the construction in the proof of Proposition 5.2.2 with the folded-symplectic form $s^*\phi^*(\omega_0)$ on the base sphere, which naturally restricts to an area form on each hemisphere. We can also modify the symplectic form $\kappa\eta$ on the fibers so that it is symplectomorphic to $\phi^*(\omega_0)$ on the normal disks to S, each of which lies on a fiber.

Hence we obtain a folded-symplectic form ω on X such that $(M, \omega|_M)$ is folded symplectomorphic to $(M_0, \omega_0|_{M_0})$. This allows us to trade M for $N \cong S^1 \times D^3$ and extend the folded-symplectic structure to $(Y \setminus M) \cup N \cong X$. The effect of this surgery on the fold of Y is to turn the surface fibration over S^1 into an open book decomposition on the resulting fold. The core curve of N sits in the 3-manifold as the binding of this open book and therefore it carries a canonical framing. We have proved:

Theorem 5.3.1 *Every closed oriented smooth 4-manifold X admits a folded-symplectic structure. Furthermore, there exist folded-symplectic forms on X with connected folds, such that a surgery along a framed curve which lies in the fold results in a simple folded-symplectic manifold.*

Remark 5.3.2 *Away from the framed curve γ in X, the folded-symplectic model we have constructed is the restriction of the simple model discussed in the previous section, and as we will see shortly, the pieces are Stein and anti-Stein. So for any sort of pseudo-holomorphic curve counting with respect to this folded-symplectic structure, the focus would be understanding the limit behaviors around γ of the curves in the moduli space, where we do have a standard model, namely $(N_0, \omega_0|_{N_0})$ above. (For a digression on this topic, see [84].) We would like to point out that both the knot type of γ in the fold and its framing depend on the achiral Lefschetz fibrations we use in the construction, so does the simple model we get. The following example illustrates this phenomenon.*

Example 5.3.3 If we construct S^4 following the recipe given in the proof of Theorem 5.3.1, we get $W = D^2 \times D^2 \bigcup_{C \times D^2} D^2 \times D^2 = S^2 \times D^2$, which can be identified with M_0, and the simple folded-symplectic form on $Y = S^2 \times S^2 = M_0 \bigcup_{S^2 \times \partial D^2} M_0$

can be constructed so that its restriction to each copy of M_0 is indeed the standard form ω_0. Note that here both open books already agree, so we do not need to alter the contact structures and change the initial fibrations. Now if we undo the surgery, that is if we trade $N = N_0$ and M in the proof, what we get is the standard folded-symplectic form ω_0 on S^4.

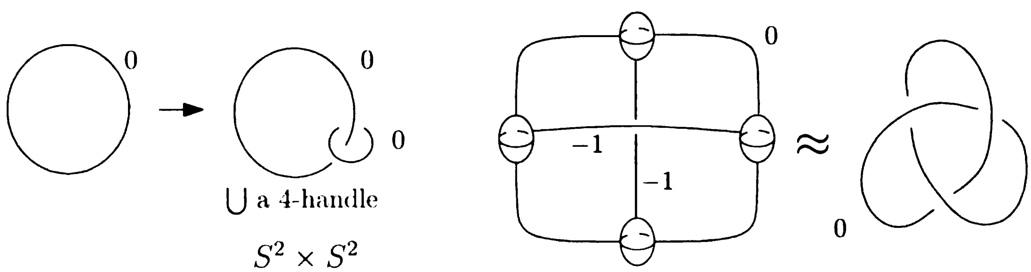

Figure 5.1: On the left: 0-surgery along the binding yields a trivial S^2 fibration over D^2 on each piece, which make up $S^2 \times S^2$. On the right: 0-surgery along the *new* binding yields a cusp neighborhood on both sides.

It is a standard fact that surgery along a framed curve in a simply-connected 4-manifold will result in connect summing with either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, depending on the framing, which can be thought as an element of $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. In [23] (also see [44]) it is described how one can homotope the framed knot in the 3-manifold to another framed knot, which is isotopic in the ambient 4-manifold to the original one, so that their framings differ by one and that surgering the new curve yields an achiral Lefschetz fibration on the resulting manifold as well. Applying this trick to our example, we can instead pass to an achiral Lefschetz fibration on $S^2 \tilde{\times} S^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, which is a torus fibration with two cusp fibers of opposite signs (Figures 5.1 and 5.2). The monodromy of this achiral fibration is $t_a\, t_b\, t_b^{-1}\, t_a^{-1}$, and the corresponding Kirby diagram is depicted in Figure 5.2 (see [37]). To verify that this manifold is $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$,

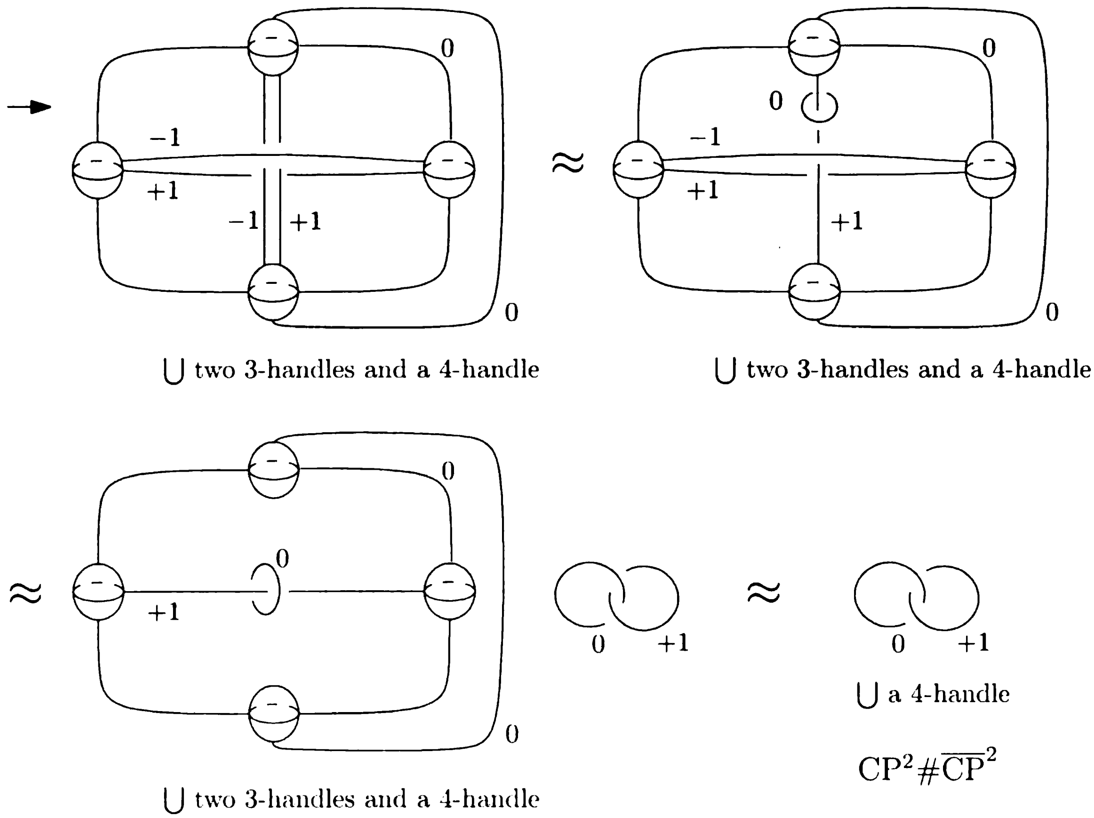

Figure 5.2: The achiral Lefschetz fibration on the second associated model. The total space is shown to be $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.

we first slide one of the vertical 2-handles over the other one, and then separate this pair from the rest of the diagram by sliding over the 0 framed 2-handle. Now the rest of the diagram can be shown to be S^4 after obvious handle cancelations. It is possible to see that our construction method will give a folded-symplectic structure on S^4 equivalent to the standard one again. Note that the first simple model above is obtained by surgering the unknot, whereas the second comes from surgering the right trefoil in S^3. Surgery framings on them differ by one in S^4.

5.4 Kähler decompostion theorem

While we shift our attention to Stein structures, we would like to have only non-separating vanishing cycles in our constructions, as it is suggested by the correspondence between PALFs and compact Stein surfaces established in [50] and [2]. We start with the following lemma:

Lemma 5.4.1 *Let X be a closed oriented 4-manifold. Then it can be decomposed into two handlebodies, each of which admits an* allowable *achiral Lefschetz fibration over D^2, such that the fibers have connected boundaries and that the induced open books are the same.*

Proof: We follow the construction of Etnyre and Fuller with more care given to having fibrations allowable. Take a handlebody decomposition of X with one 0– and one 4-handle, let X_1 be the union of the 0-handle, 1-handles and 2-handles, and X_2 be the union of 3-handles and the 4-handle. As it was implicitly present in [44], and was also observed in [2], one can always build an achiral Lefschetz fibration on a given 2-handlebody so that all vanishing cycles are non-separating. Therefore, we can start with allowable fibrations and then proceed with stabilizations as described in [23] to match the induced open books. A stabilization is given by gluing a positive or a negative Lefschetz 2-handle along a new 1-handle added to a regular fiber, and in order to keep the binding connected, we always introduce another adjacent stabilization. Therefore, all vanishing cycles introduced during stabilizations are also nonseparating. Induction concludes the proof. $\square$

Theorem 5.4.2 *Let X be a closed oriented smooth 4-manifold. Then X admits a decomposition into two codimension zero submanifolds X_+ and X_-, such that X_+ and $-X_-$ are both compact Stein manifolds with strictly pseudoconvex boundaries. These Stein structures can be chosen so that the induced contact structures ξ_+ on*

∂X_+ and ξ_- on $-\partial X_-$ are isotopic. Furthermore, there are PALFs on each piece such that the open book decompositions they induce on ∂X_+ and $-\partial X_-$ are compatible with ξ_+ and ξ_-, respectively, and they coincide. In short, all data match on the hypersurface $H = \partial X_+ = -\partial X_-$.

Proof: The lemma above gives us a decomposition of X into two pieces X_1 and X_2 with allowable achiral Lefschetz fibrations f_1 and f_2 on them, such that induced open books on the boundaries match. As in the proof of Theorem 5.3.1, we glue these two pieces along the truncated pages to get:

$$W = X_1 \bigcup_{f_1^{-1}(\partial D^2)=f_2^{-1}(\partial D^2)} X_2 .$$

Next we glue in $S^2 \times D^2$ to cap off the fibers and establish an achiral Lefschetz fibration $\hat{f}\colon Y \to S^2$ with closed fibers.

We wish to split the base of this fibration into two disks D_+ and D_- so that all the positive critical values lie in the interior of D_+ and all the negative ones lie in the interior of D_-. As discussed earlier, restrictions of f give a positive Lefschetz fibration on $X_+ = f^{-1}(D_+)$ and a negative Lefschetz fibration on $X_- = f^{-1}(D_-)$, respectively. It also prescribes a surface bundle over $S^1 = \partial D_+ = -\partial D_-$ on the hypersurface separating Y_+ and Y_-. This time we would like to describe the splitting more carefully by taking into consideration how the global monodromies of the new fibrations are related to the original ones.

Let μ_1 be the monodromy of the achiral Lefschetz fibration f_1 on X_1 and μ_2 be the monodromy of the fibration f_2 on X_2. Fix a representation of μ_1 by using arcs $s^1_1, \cdots, s^1_{k_1}$ and a representation of μ_2 by using arcs $s^2_1, \cdots, s^2_{k_2}$. Corresponding critical values are denoted by y^1_i and y^2_j. Monodromies of the open book decompositions bounding each fibration are given by μ_1 and μ_2 as well, and they coincide under an orientation reversing diffeomorphism, so $\mu_1 = (\mu_2)^{-1}$. Let V be a small neighborhood of a regular value in the base S^2. We obtain an achiral Lefschetz fibration

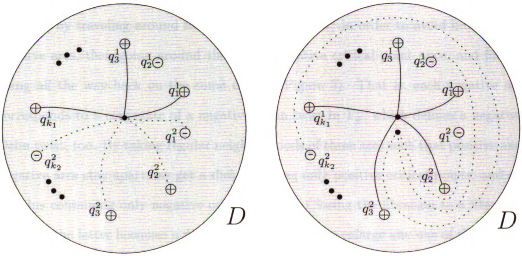

Figure 5.3: New monodromies from old ones. On the left: μ_1 is given by solid arcs, and μ_2 by dotted ones. On the right: Solid arcs are the *positive arcs* representing μ_+, whereas conjugated dotted arcs are *negative*, providing a representation of μ_- after closing the base back to S^2.

$f\colon W \setminus f^{-1}(V) \longrightarrow D = S^2 \setminus V$, which closes up to a fibration over S^2. If g is the genus of the page, then this fibration is determined by the relation $\mu_1\mu_2 = 1$ in $\Gamma_{g,1}$ and is mapped under the maps $\Gamma_{g,1} \to \Gamma_g^1 \to \Gamma_g$ to the relation that describes the achiral Lefschetz fibration $\hat{f}\colon Y \setminus \hat{f}^{-1}(V) \to D$. Since this map factors through Γ_g^1, the achiral Lefschetz fibration $\hat{f}$ comes naturally with a section S of self-intersection zero. We denote images of the elements in $\Gamma_{g,1}$ under this map with the same elements, so $\mu_1\mu_2 = 1$ is the global monodromy of the achiral Lefschetz fibration on $Y \setminus \hat{f}^{-1}(V)$. Note that we can use the same arcs s_i^1 and s_j^2 to represent the global monodromy of this fibration. Now, if we move counterclockwise and choose only the arcs that run through positive Dehn twists, we establish a monodromy μ_+. Call these arcs *positive*. Next, we choose a nearby base point, and move counterclockwise by running through the negative Dehn twists only, while avoiding intersecting any positive arc. This way, we obtain a monodromy μ_-. The new set of arcs involved in this monodromy will be referred as *negative* arcs. Observe that each negative arc is obtained by traveling around some old arcs s_i^1 and s_j^2 in order to avoid intersecting positive arcs, then going around the aimed negative critical point once, and finally going all the way back on the same detour (Figure 3). That is, each negative arc corresponds to a conjugate of a negative Dehn twist in Γ_g, which defines a negative Dehn twist, too. By taking regular neighborhoods of these arcs such that positive and negative arcs stay apart, we get a disk enclosing only positive critical points, and an annulus containing only negative critical points. Closing the fibration to a fibration over S^2, the latter becomes a disk as well. Now we can enlarge any one of these disks so both disks share the same boundary, and call the one containing positive values D_+, and the other one D_-. So we have a new factorization of the global monodromy of $\hat{f}$, given by the relation $\mu_+\mu_- = 1$. The section S prescribes how to lift the new elements $\mu_\pm$ of Γ_g to Γ_g^1 uniquely.

We proceed with taking out the tubular neighborhood $S^2 \times D^2$ of the section from $Y = Y_+ \cup Y_-$, and we get an inherited splitting $X_+ \cup X_-$. The discussion above shows that μ_+ defines a positive Lefschetz fibration on X_+ and μ_- defines a negative fibration on X_-. To recover the original 4-manifold X we need to put back in $S^1 \times D^3$, which has the same effect as gluing each other the tubular neighborhoods $S^1 \times D_+^2$ and $S^1 \times D_-^2$ of the bindings of open books on ∂X_+ and ∂X_-, respectively. Therefore we can think of X as decomposing into X_+ and X_-. We claim that this decomposition possesses the desired properties.

When we take out a tubular neighborhood of S from Y, we turn the positive and negative Lefschetz fibrations on Y_+ and Y_- into a PALF on X_+ and a NALF on X_-, respectively. In the meantime the surgery converts the surface fibration that separates Y_+ and Y_- to an open book decomposition on the common boundary $H = \partial X_+ = -\partial X_-$. The binding of this open book is the identified bindings of ∂X_+

and $-\partial X_-$, the page F is the bounded closed surface obtained by cutting off a disk from the regular fiber of $\hat{f}$, and the monodromy is induced from the fibration on either side. Noting that the NALF on X_- becomes a PALF on $-X_-$, we see that both PALFs induce the same open book decomposition on their boundaries.

By Theorem 5.1.4, both X_+ and $-X_-$ admit Stein structures. We will construct these Stein structures using Eliashberg's characterization so that they match on the common boundary. The technique we are going to use is the same as the one which was presented in [2]: The PALF on X_+ is obtained by attaching positive Lefschetz handles $h_1 \cdots h_m$ to $X_0 = F \times D^2$, which has the obvious PALF defined by projection onto D^2 component. The same is true for the PALF on $-X_-$. $F \times D^2$ has a natural Stein structure by Theorem 5.1.3. We can assume all vanishing cycles (coming from either side) sit in various pages of the open book on H. Read backwards, we can think of the fibrations as being constructed by attaching positive and negative Lefschetz handles to H on either side in a sequence following the monodromy of the open book. Thus we can induct on the number of handles. Assume that the PALF on $X_{i-1} = X_0 \cup h_1 \cup \ldots \cup h_{i-1}$ $(i \leq m)$ induces an open book decomposition on its boundary, and it carries a Stein structure such that the contact structure induced on the boundary is compatible with this open book. Let C be the vanishing cycle of the positive Lefschetz handle h_i lying on a page F of ∂X_{i-1}. We open up the open book decomposition and choose a page against F, and glue them together along the binding B to form a smooth closed convex surface Σ in the 3-manifold ∂X_{i-1}. As C is non-separating, $\Sigma \setminus C$ is connected and it contains the dividing set, namely B. So we can use the *Legendrian realization principle* ([35], [45]) to isotope Σ through convex surfaces to make C Legendrian. Note that this is done by a small C^∞ isotopy of the contact structure supported in a neighborhood of Σ, which fixes the binding pointwise. Hence the framing of C relative to the fiber F is the same as its contact framing, implying that the Lefschetz handle h_i is attached along a Legendrian curve with framing $tb - 1$. By Theorem 5.1.3 the Stein structure extends over this handle, and as shown by Gay in [33], the new open book on ∂X_i will be compatible with the new induced contact structure on ∂X_i. This completes the induction. Repeating the same argument dually for $-X_-$, we see that the compatible open books on ∂X_+ and $\partial(-X_-)$ are isotopic, and therefore the induced contact structures ξ_+ on ∂X_+ and ξ_- on $\partial(-X_-) = -\partial X_-$ are isotopic as well. So we fulfill all the matching conditions listed in the statement of the theorem. □

Remark 5.4.3 *In [1] it was asked if one could decompose a given closed oriented smooth 4-manifold into Stein pieces so that the induced contact structures on the separating 3-manifold coincide. Our theorem gives an affirmative answer to this question. In the same article authors remark that it is possible to alter their Stein decomposition to make the induced contact plane distributions homotopic, but the tightness of the contact structure precludes the use of Eliashberg's celebrated theorem on overtwisted contact structures to conclude more. Considering the underlying PALFs and isotopies of open books gives a way around this difficulty, thanks to Giroux's Theorem.*

Remark 5.4.4 *In [68], Quinn studied so-called* dual decompositions *of 4-manifolds: descriptions of 4-manifolds as a union of two 2-handlebodies. The author formulates the same question as in [1] in terms of necessary sequence of Kirby moves to relate a possibly nonmatching Stein decomposition. Theorem 5.4.2 provides an implicit answer to this question, and we would like to take this as an opportunity to summarize the handle calculus behind our construction: An arbitrary Stein decomposition $X = X_1 \cup X_2$ comes with some PALF pair. Using the stabilization moves of Etnyre and Fuller, we first change*

this PALF pair with a matching pair. This corresponds to adding canceling $1-$ *and* $2-$ *handles to each* X_i, *or in other words, we add canceling handle pairs of index* $1-$, $2-$ *and* $3-$ *in the original handlebody decomposition of* X. *In the next step, we pass to a cobordant* 4*-manifold* Y *so that we can split the positive and negative Lefschetz handles. Then we 'undo' the surgery and get the decomposition* $X = X_+ \cup X_-$ *with Stein structures on each piece that coincide on the common boundary. Having the simply-connected case in mind, this intermediate step can be seen as a stabilization. Let* $W \cong X \setminus S^1 \times D^3$ *be the complement of a regular neighborhood of the framed knot* γ *in* X, *then the first surgery defines a cobordism*

$$[0,2] \times W \bigcup_{[0,2]\times S^1\times S^2} ([0,1]\times S^1 \times D^3) \cup_{1\times S^1\times S^2} ([1,2]\times D^2\times S^2)),$$

which is identity on the first component. We trade 2*-handles of* X_1 *and* X_2 *in* Y *by making use of the two extra handles of index* 2. *Finally, the composition of two cobordisms that gives back* X *can be seen as the double of the cobordism above, and thus it deformation retracts to*

$$W \bigcup_{S^1\times S^2} (S^4 \cup_{D^2\times S^2} S^4).$$

This cobordism is built by attaching cells to $\partial W = S^1 \times S^2$, *where* $D^2\times S^2$ *is attached uniquely and the framing of* γ *indicates in which one of the two ways we shall glue the other two* $S^1 \times D^3$ *pieces. Although here we started with a (nonmatching) Stein decomposition, it is clear that the same discussion can be carried out in our main construction as well. Therefore we have a well-defined process, during which we first inflate the number of handles in a given decomposition of* X, *and then trade some of the* 2*-handles through a cobordism to achieve the desired decomposition at the end.*

5.5 Folded-Kähler structures and folded Lefschetz fibrations

Unlike symplectic structures, random folded-symplectic structures do not need to bear any information about the geometry or topology of the manifold they are defined on. In order to specify more meaningful members of this family, one first of all needs to impose some boundary conditions on the folding hypersurface. We would like to acknowledge a result of Kronheimer and Mrowka: In [48], the authors prove that a compact symplectic 4-manifold (Y, ω) with strictly pseudoconvex boundary has $\mathrm{SW}_Y(K) = 1$, where K is the canonical class of ω. This motivates us to see such manifolds as building blocks of 4-manifolds, and yields a good boundary constraint for folded-symplectic structures, at least in this dimension. Henceforth, we assume that the fold $H = (\omega^n)^{-1}(0)$ of a given folded-symplectic manifold (X^{2n}, ω) is always connected and nonempty. We will generalize the notion of a Kähler structure on a smooth $2n$-manifold by considering a distinguished subset of the family of folded-symplectic structures, and we then present some properties of these structures:

Definition 5.5.1 *A folded-symplectic form ω on an oriented $2n$-dimensional manifold X is called a* folded-Kähler structure, *if there is a tubular neighborhood N of H such that:*

1. *The closure of each component of $X \setminus N$ is a compact Kähler manifold $(\pm X_{\pm}, \omega|_{\pm X_{\pm}})$ with strictly pseudoconvex boundary.*
2. *$(N, \omega|_N)$ is folded symplectomorphic to $([-1,1] \times H, d((t^2+1)\,\pi^*(\alpha)))$, where α is a contact 1-form on the fold H, and π is the projection $\pi\colon [-1,1] \times H \to H$.*

In addition, if each $(\pm X_{\pm}, \omega|_{\pm X_{\pm}})$ is strictly pseudoconvex, we say ω is a nicely folded-Kähler structure *on X.*

In the definition above, *nice folding* can be reformulated as folding Stein manifolds along matching strictly pseudoconvex boundaries. Recall that if ψ is a proper strictly plurisubharmonic function on a complex manifold S, then the associated 2-form $\omega_\psi = -dJ^* d\psi$ is Kähler, and importantly, the symplectic class of (X, ω_ψ) is independent of the choice of ψ [22]. Therefore, to complete our alternative formulation, we ask that each piece $\pm X_\pm$ should admit some proper strictly plurisubharmonic function $\psi_\pm$, so that $\omega|_{\pm X_\pm} = \omega_{\psi_\pm}$. In short, it is built in the definition that a nicely folded-Kähler manifold is folded-Kähler. Finally note that, due to a theorem of Bogomolov [12], any compact folded-Kähler manifold X can be made nicely folded after deforming the complex structure and blowing down any exceptional curves. Even though these definitions narrow the family of folded-symplectic structures quite a lot, it is important to note, at least in dimension four, that we still have an adequately large family in the light of the following result:

Theorem 5.5.2 *Any closed oriented 4-manifold X admits a nicely folded-Kähler structure.*

Proof: By Theorem 5.4.2, X can be decomposed into two compact Stein manifolds X_+ and $-X_-$ with strictly pseudoconvex boundaries such that both induce the same contact structure on the common boundary $H = \partial X_+ = -\partial X_-$. We begin with adding collars $\pm U_\pm$ to $(\pm X_\pm, \omega_\pm)$, and extending the symplectic structures to $\omega'_\pm$ on $\pm X'_\pm = \pm(X_\pm \cup U_\pm)$ so that new boundaries $\partial(\pm X'_\pm)$ are still convex and contactomorphic. Let $\xi_\pm$ be the induced contact structures on $\partial(\pm X'_\pm)$ and ψ be a contactomorphism between them. Using the symplectic cut-and-paste argument of Etnyre [24], we can add a symplectic collar to $(\partial X'_+, \omega'_+)$ so that the new boundary is not only contactomorphic to $-\partial X'_-$ but also the induced *contact forms* agree up to a multiple $k \in \mathbb{R}^+$. For the sake of brevity, let us assume that U_+ above contains this collar part as well. So after rescaling ω'_- (and ω_-) by k if necessary, we see that restrictions of symplectic forms $\omega'_+|_{\partial X'_+}$ and $k\omega'_-|_{-\partial X'_-}$ agree via ψ, and orientations of null-foliations (which correspond to Reeb directions) are the same. Therefore, once again we can apply the folding technique of [14] to obtain a folded-symplectic structure ω on $X'_+ \cup X'_-$ such that ω agrees with $\omega'_\pm$ on the complement of a small tubular neighborhood of the fold H. We enlarge this neighborhood to include U_+ and U_- and call it N. It follows that $X = X_+ \cup N \cup X_- \cong X_+ \cup X_-$, and $\omega|_{X_+} = \omega_+$, whereas $\omega|_{X_-} = k\omega_-$. Also note that, the folding operation provides us with the desired local model on N, that is, $(N, \omega|_N)$ is folded symplectomorphic to $([-1,+1] \times H,\, d((t^2+1)\,\pi^*(\alpha)))$ by construction [14].

Lastly, suppose $\varphi_\pm\colon \pm X_\pm \to [0,\infty)$ are proper strictly plurisbuharmonic functions such that $\pm\partial X_\pm$ correspond to the maximum points of $\varphi_\pm$, and $\omega_\pm = -dJ^*d\varphi_\pm$, respectively. If $k \neq 1$, we can replace φ_- with $k\varphi_-$ and obtain $k\omega_-$ above as a Kähler form of a strictly pseudoconvex manifold. Equipped with these properties, ω is a nicely folded-Kähler form on X. $\square$

Remark 5.5.3 *It is clear that Theorem 5.5.2 is a refinement of Theorem 5.3.1. Since the folded forms we have constructed in both proofs are obtained through similiar steps, one expects that these structures are actually equivalent. Next, we verify this fact, and this way we get an insight of how folded-Kähler forms are 'supported' (precise definition is given below) by Lefschetz fibrations as was illustrated in Proposition 5.2.2:*

Take the PALF on X_+ in the proof of Theorem 5.3.1, and attach a symplectic 2-handle along the binding of the induced open book on ∂X_+ as described by Eliashberg in [21]. This yields a symplectic Lefschetz fibration over D^2_+. Dually the same argument for the NALF on X_- gives an anti-symplectic Lefschetz fibration over D^2_-, and these handle attachments can be done so that two fibrations agree on the common boundary. Moreover, we can assume that these fibrations have genus at least two, so the fibrations can be matched as symplectic surface fibrations over a circle, *as it was pointed out in [21]. At the end we get a simple folded-symplectic manifold Y obtained from X after a surgery along a framed curve γ. However, any two simple folded-symplectic forms compatible with a fixed achiral Lefschetz fibration are deformation equivalent by Proposition 5.2.2. Moreover, we can normalize both forms on the disks which are parallel copies of cocores of new 2-handles that were used to cap off the fibers. Hence, these two folded forms are deformation equivalent on $Y \setminus S^2 \times D^2$. As the folded-symplectic structure on $D^3 \times S^1$ which is glued back in to recover X is standard, the folded-symplectic form constructed in the proof of Theorem 5.3.1 and the folded-Kähler form obtained in Theorem 5.5.2 are indeed* equivalent as folded-symplectic structures. $\square$

Motivated by symplectic and near-symplectic cases ([17], [9]), we can conclude our discussion above by defining the Lefschetz fibration analogue for our structures:

Definition 5.5.4 *Let X be a closed oriented 4-manifold, and decompose S^2 as the union of the upper-hemisphere D_+ and the lower-hemisphere D_- which are glued along the equator $C = \partial D_+ = -\partial D_-$. Then a smooth map $f\colon X \to S^2$ is said to be a* folded Lefschetz fibration *on X, if it restricts to a PALF over D_+, to a NALF over D_-, and to an open book over C bounding both fibrations.*

Definition 5.5.5 *Let X be a closed oriented 4-manifold equipped with a nicely folded-Kähler form ω. Then a folded Lefschetz fibration $f\colon X \to S^2$ is said to be* compatible with ω *if each Stein piece $X_\pm$ corresponds to $f^{-1}(D_\pm)$, and if the contact structure they induce on $H = f^{-1}(C)$ is compatible with the open book decomposition coming from the fibration f. In this case, we also say that nicely folded-Kähler manifold (X, ω) is* supported by the folded Lefschetz fibration f.

The compatibility in the above definition is completely *on the symplectic level.* This becomes more visible if once again we recall that surgering the binding γ of the open book $f|_{H\setminus\gamma}\colon H\setminus\gamma\to S^1$, we pass to a *simple model* where the folded Lefschetz fibration can be extended to a folded *symplectic* achiral Lefschetz fibration $\hat{f}$ with closed fibers. Also note that, since Stein manifolds harbor less topological obstructions in complex dimensions >2, it is very likely that they admit higher dimensional analogues of PALFs with similar topological correspondences. If that is established, last two definitions, as well as several results in this paper, can be generalized to all $2n$-dimensions.

The complete statement of Theorem 5.4.2 combined with Theorem 5.5.2 shows that, given a closed oriented 4-manifold X, one can always find a nicely folded-Kähler structure ω on X together with a compatible folded Lefschetz fibration. Next, we prove that this property in fact holds for any nicely folded-Kähler structure:

Proposition 5.5.6 *Any nicely folded-Kähler structure ω on X, up to orientation preserving diffeomorphism, admits a compatible folded Lefschetz fibration.*

Proof: Each Stein piece X_+ and $-X_-$ admits a PALF by Theorem 5.1.4. If we construct these fibrations following the algorithm of [2] and keep track of the associated open books, the work of Plamenevskaya [67] shows that we can establish PALFs $f_\pm\colon \pm X_\pm \to D^2$ with the property that the open book decomposition on the boundaries are compatible with the contact structures induced from the Stein structures on $\pm X_\pm$, respectively. As the contact structures are assumed to be the same, Theorem 5.1.2 tells us that we can match these open books after positive stabilizations. Consequently, we get a compatible folded Lefschetz fibration. □

Remark 5.5.7 *A folded Lefschetz fibration that supports a given folded-Kähler structure fails to be unique. In fact, one can find infinitely many pairwise inequivalent such fibrations. This can be seen for example from the construction of [2], by using different (p,q) torus knots in their algorithm which we adopt for building our achiral Lefschetz fibrations.*

Example 5.5.8 The easiest examples are doubles. If Y^4 is a compact Kähler manifold with strictly pseudoconvex boundary, then $X = Y \cup -Y$ is equipped with a folded-Kähler structure. When Y is indeed Stein, we get a nicely folded structure. The first folded structure constructed in Example 5.3.3 is the double of standard $D^4 \subset \mathbb{C}^2$, whereas the latter is a 'monodromy double' of a cusp neighborhood minus a section. Here by 'monodromy double' we mean that the pieces are first glued along the pages of the open books, and if the monodromy of the folded Lefschetz fibration on one piece is μ, then the monodromy on the other one is μ^{-1}.

Example 5.5.9 There is a construction which also allows us to see the nicely folded-Kähler structure together with a compatible folded Lefschetz fibration. Take a contact 3-manifold (H, ξ), and fix a positive open book decomposition (B, f) compatible with ξ. Different PALFs bounding this open book describe (possibly) different Stein fillings of (H, ξ). Indeed there are examples of infinitely many pairwise non-diffeomorphic contact 3-manifolds each of which admit infinitely many pairwise non-diffeomorphic Stein fillings constructed this way [57]. Thus for every pair of PALFs X_1 and X_2 that fill the same open book, we can construct a nicely folded-Kähler form on $X = X_1 \cup -X_2$, as designated in the proof of Theorem 5.5.2.

Example 5.5.10 The main steps of our construction are depicted in the following simple, albeit instructive example: Start with classical handlebody decomposition of $X = \#_8 S^2 \times S^2$ with one 0-handle, sixteen 2-handles, and a 4-handle. Let X_1 be the union of 0-, 2- handles, and X_2 be the 4-handle. Each piece admits a D^2 fibration over D^2. However we wish to construct allowable fibrations, so we introduce two 1− and 2− canceling handle pairs and two 2- and 3- canceling handle pairs in the original handlebody decomposition of X. We start building the fibrations from the scratch: Add the 1-handles to the 0-handle and 3-handles to the 4-handle. Attach the two canceling

2-handles with framing -1 to the union of the 0- handle and 1- handles. Attach the other two the same way to the handlebody X_2, which is the union of 3- handles and the 4- handle. To simplify our description, we will label the 1-handles of the first handlebody as a and b, which generate π_1 of the torus fiber with one boundary component, and we do the same for the 1-handles of X_2 under the obvious identification. So we obtain two achiral Lefschetz fibrations over disks with bounded torus fibers; one with monodromy $t_a^{-1} t_b^{-1}$, and one with $t_b t_a$. One can verify by Kirby calculus that each time we insert a pair of Lefschetz handles prescribed by $t_a t_a^{-1}$ or $t_b t_b^{-1}$, we introduce an $S^2 \times S^2$. (See Figure 5.2, and observe that here we slide off the 2-handle pair over a $+1$ framed 2-handle instead.) Doing this consecutively, we attach all the remaining 2-handles to the first handlebody, and obtain an achiral Lefschetz fibration on X_1 with monodromy

$$\mu_1 = t_a^{-1} t_b^{-1} t_b t_a t_b t_a t_b t_a t_b t_a t_a^{-1} t_b^{-1} t_a^{-1} t_b^{-1} t_a^{-1} t_b^{-1} t_a^{-1} t_b^{-1}.$$

whereas X_2 still has the monodromy

$$\mu_2 = t_b t_a = (t_a^{-1} t_b^{-1})^{-1}.$$

Both open books that bound these fibrations contain negative Dehn twists (recall that on $-\partial X_2$, the monodromy is μ_2^{-1}), and therefore the contact structures they support are overtwisted. As we have already manipulated the monodromy that way, contact structures and open books are isotopic, so we can glue X_1 and X_2 along the truncated pages. Putting in $S^2 \times D^2$ we pass to a torus fibration $\hat{f}\colon Y \to S^2$ with global monodromy $\mu_1 \cdot \mu_2$. (Applying the handle slides given in Example 5.3.3 repeatedly, and proceeding with the same handle cancelations, one can indeed check that $Y = \#_8 S^2 \times S^2 \# \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.) Now the monodromy splits easily as explained in the proof of Theorem 5.4.2, and we get $\mu_+ = (t_b t_a)^5$ and $\mu_- = (t_a^{-1} t_b^{-1})^5$. It is not hard to see that when we take out the section now, we get pieces X_+ and X_-, which are diffeomorphic to $-E_8$ and E_8, respectively. So X decomposes into a Stein piece $-E_8$ and an anti-Stein piece E_8. This defines a nicely folded-Kähler structure ω on X, folded along the Poincaré homology sphere $\Sigma(2,3,5)$, and it is supported by a folded Lefschetz fibration which is the monodromy double of a torus fibration over D^2 minus a section.

5.6 Addendum: Interactions between the two generalizations

From the very definitions of the two different symplectic generalizations we know that folded-symplectic structures and near-symplectic structures on smooth 4-manifolds can *not* be generalized from each other. It is also apparent that achiral Lefschetz fibrations and broken Lefschetz fibrations are not generalizations of each other either. However, one can consider a simultaneous generalization of symplectic structures so to deal with both of them as follows:

Definition 5.6.1 *Let ω be a closed 2-form on a smooth 4-manifold X such that there exists a smooth embedded 1-manifold Z in X, and such that ω^2 intersects the 0-section of $\Lambda^4 T^* X$ transversally at every point on $X \setminus Z$ and $\omega = 0$ at every point on Z. We then call ω a* general symplectic structure *on X provided that at each point $x \in Z$, if we use local coordinates on a neighborhood U of x to identify the map $\omega : U \longrightarrow \Lambda^2(T^* U)$ as a smooth map $\omega : \mathbb{R}^1 \longrightarrow \mathbb{R}^6$, then its linearization at x, $D\omega_x : \mathbb{R}^1 \longrightarrow \mathbb{R}^6$, has rank 3. We call $H = \omega^2(0) \setminus Z$ the* fold singularity *and Z the* round singular *loci. When $H = \emptyset$, (X, ω) is a near-symplectic manifold if $\omega^2 \geq 0$ and an* anti-near-symplectic *manifold if $\omega^2 \leq 0$. If $Z = \emptyset$ and $H \neq \emptyset$, then we obtain a folded-symplectic manifold. Lastly, when $H \cup Z = \emptyset$ we have a symplectic 4-manifold if $\omega^2 > 0$ and an anti-symplectic 4-manifold if $\omega^2 < 0$.*

Note that around every component of Z, we either have $\omega^2 \geq 0$ or $\omega^2 \leq 0$, so for an appropriate choice of metric g, the map $\omega : U \to \Lambda^2(T^*U)$ above is either onto the subspace of self-dual 2-forms, or onto the subspace of anti-self-dual 2-forms with respect to g —both of which have rank 3. (Compare [9])

In a similar manner, we define a *general Lefschetz fibration* to be a fibration where we allow both achiral Lefschetz singularities and round singularities. In fact, what motivates us to introduce the above notion of general symplectic structures is the study of these fibrations by David Gay and Rob Kirby in [34] (under the name "broken achiral Lefschetz fibration"), where the authors proved:

Theorem 5.6.2 (Gay-Kirby [34]) *Let X be an arbitrary closed oriented 4-manifold and let F be a closed surface in X with $F \cdot F = 0$. Then there exists a general Lefschetz fibration from X to S^2 with F as a fiber.*

This theorem suggests an alternative way to study general 4-manifolds *through generalizations of Lefschetz fibrations.*

5.6.1 General symplectic structures on broken achiral Lefchetz fibrations

An open question stated in [34] was to give a meaningful formulation of a cohomological condition that would allow one to obtain a 2-form ω on a given general Lefschetz fibration, as in Theorem 2.0.4 and Theorem 4.1.3. The fibrations constructed in [34] can always be arranged so that the round handle singularities all project to the tropics of Cancer and Capricorn, with their high genus sides towards the equator and with all Lefschetz and negative Lefschetz singularities over the equator. For what follows, it would be convenient to introduce yet another name to refer to general fibrations arranged in this peculiar way. Let us call these "simplified general Lefschetz fibrations" for this purpose.

Proposition 5.6.3 *Let X be a closed oriented smooth 4-manifold and $f\colon X \to S^2$ be a simplified general Lefschetz fibration such that the regular fiber is a closed oriented surface F which is nonzero in $H_2(X;\mathbb{R})$, then X admits a general symplectic structure ω such that fibers are symplectic away from the critical points, the fold H is an F-bundle over S^1, and the round singular locus Z_ω coincides with the round singular locus of f. The fold H splits X into pieces X_+ and X_-, and f induces near-symplectic Lefschetz fibrations on $(X_+, \omega|_{X_+})$ and on $(-X_-, \omega|_{X_-})$, respectively. Furthermore, any finite set of sections can be made folded-symplectic for an appropriate choice of ω. This form is canonical up to deformation equivalence of general symplectic forms.*

Proof: We can perturb the equator circle of the base S^2 to an arc which passes from the south of the image of each Lefschetz singularity, and from the north of each negative Lefschetz singularity. Let this arc C be the new equator of the base S^2. It is easy to see that C splits off a disk D_+ from S^2 that contains only the images of positive Lefschetz singularities, the complement of which in S^2 is another disk D_- that contains only the images of negative Lefschetz singularities. Both might contain images of round handle singularities.

The rest of the proof is very much the same as the proof of Proposition 5.2.2 except on the pieces $X_+ = f^{-1}(D_+)$ and $X_- = f^{-1}(D_-)$ one now also needs to deal with the round singularities. However this can be done as in the proof of Theorem 4.1.3, where the authors generalize Gompf's construction to the case of broken Lefschetz fibrations [9]. □

It is easy to generalize the above proposition to any (not necessarilly *simplified*) general Lefschetz fibration $f\colon X \to \Sigma$. For that though, one will need to prefer one splitting of the base over another, since in general there is no canonical way of separating the images of negative and positive Lefschetz singularities in Σ.

5.6.2 From achiral to broken Lefschetz fibrations

In [34], another question that the authors ask is whether one can avoid achirality in their construction. In this subsection we will show that this can be achieved after blowing-up the ambient manifold sufficiently many times. Note that blow-ups do not kill standard smooth 4-manifold invariants and in fact their effect on the invariants is well-understood. Therefore, as far as the smooth invariants are concerned, this modification is very welcome.

To get rid of the negative Lefschetz singularities, we will consider a local modification around the image of an isolated negative singularity to obtain a new general Lefschetz fibration where this negative Lefschetz singularity is traded with an additional round singularity. We then show that this amounts to blowing-up the 4-manifold at that critical point. Our modification can be seen to be equivalent to performing the local operation described in the third example of [9], page 113, but *in orientation reversing charts*. For the convenience of the reader, we briefly describe this modification below.

Consider an isolated negative Lefschetz critical point of a general fibration f on X, with vanishing cycle a loop γ in the nearby generic fiber. We remove a neighbourhood of this singular fiber and insert in its place a configuration where γ is now taken as the vanishing cycle of a round 2-handle cobordism. The critical values form a simple closed loop δ. The inner most part of this round handle cobordism is a trivial fibration with a fiber of one less genus, or otherwise with two fiber components, depending on whether γ is nonseparating or separating. This way we add a new component to the round locus. See the Figure 5.6.2 which is taken from [9] after a slight modification for our case.

This modification yields a new general fibration f'. Let δ be the new round singular circle. The fibers outside δ are obtained from those inside by attaching a

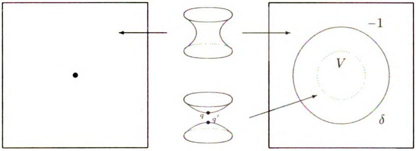

Figure 5.4: Replacing a negative Lefschetz singularity by a round singularity: f (left) and f' (right)

handle joining two points q, q' as shown in the Figure 5.6.2. Along δ the points q, q' describe a trivial braid, but the relative framing differs from the trivial one by -1, so that on the outer side the monodromy around δ consists of a single negative Dehn twist along γ, which compensates for the loss of the isolated singular fiber.

Next we would like to understand the total space X' that f' is defined on. The local model for f is simply a 4–ball. On the other hand, the total space of the new local model for f' contains a smoothly embedded sphere S, obtained by considering the two points q and q' in each of the fibers inside δ together with the equator δ. Since the monodromy around δ is a negative Dehn twist along γ, we deduce that S has self-intersection -1. Also observe that the preimage of the interior region V is the disjoint union of two $D^2 \times D^2$'s, giving a disk bundle over $S \cap f'^{-1}(V)$. The preimage of the outer region is again a disk bundle over a neighbourhood of the equator in S, and it is diffeomorphic to $S^1 \times D^3$. Hence, the total space of f' is a disk bundle over the sphere S with self-intersection -1. Therefore our operation locally (and thus globally) amounts to blowing-up X. That is, $X' = X\#\overline{\mathbb{CP}}^2$.

We can also depict this operation in terms of handle diagrams. For simplicity, assume that γ is a nonseparating curve. Clearly the vanishing cycle γ can be very complicated in general. However, there exists a self-diffeomorphism of the fiber which takes γ to any nonseparating curve. This self-diffeomorphism can be extended to an orientation preserving self-diffeomorphism ϕ of the piece $\partial f^{-1}(V)$. So it suffices to study our modification in the local model in Figure 5.5, and glue the new piece back via the same diffeomorphism ϕ on the boundary, which matches the boundary monodromies as indicated by the negative Dehn twist along the original γ.

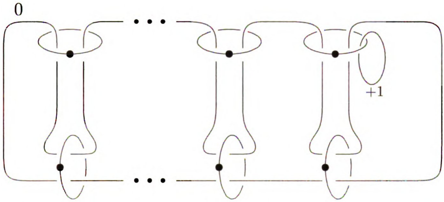

Figure 5.5: Neighborhood of a negative nodal fiber which has a simple nonseparating vanishing cycle.

After blowing-up in this piece, one can obtain a new diagram with no Lefschetz singularity but with a new round handle as shown in Figures 5.6 and 5.7. In Figure 5.6, we first slide the $+1$-framed 2-handle over the -1-framed 2-handle so that its framing becomes 0. Then the two strands of the 0-framed 2-handle can be slid off the 1-handle using the new 0-framed 2-handle, and now they go through the -1 framed 2-handle as shown in the third diagram. The new 0-framed 2-handle and the 1-handle becomes a canceling pair, which we remove from the diagram to get to Figure 5.7. The last step is just an isotopy which puts the diagram in the standard form of a trivial fibration with a fiber of one less genus, and a round 1-handle attached to it. Observe that the framing of the 2-handle of the round 1-handle is -1, compensating

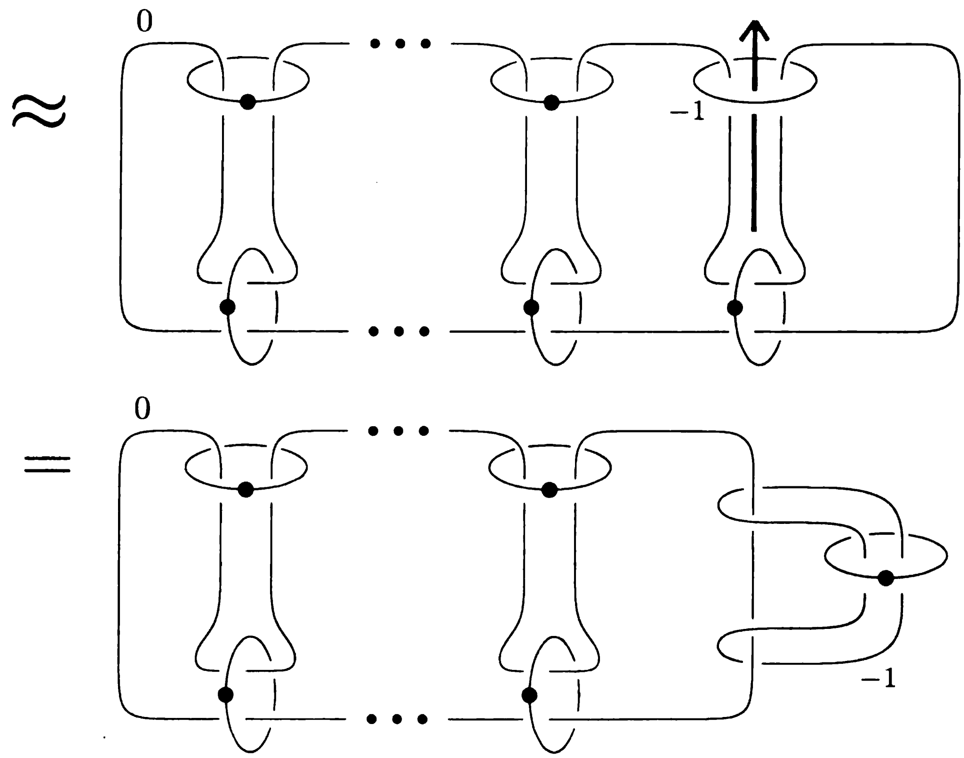

Figure 5.7: After an isotopy, we obtain a Kirby diagram of a round 1-handle attachment to a product neighborhood of a fiber with one less genus.

for the loss of the singular fiber on the boundary monodromy.

Since the modification is made locally around a critical point, it works for any general fibration containing negative Lefschetz singularities. In particular our folded Lefschetz fibrations in Section 5.5 can be replaced with folded fibrations with only broken and positive Lefschetz singularites on blow-ups of the given manifold. The same argument applies to any achiral Lefschetz fibration as well; for instance to those that Etnyre and Fuller obtained in [23].

5.6.3 Comments on describing invariants on general 4-manifolds

A question that remains unanswered is if one can define smooth invariants in the most general setups discussed in this chapter. We finish with a few rather speculative comments regarding this issue.

One might hope a Kähler decomposition for a given 4-manifold to be what a Heegaard decomposition is for a given 3-manifold. For this to work, one needs to relate any given two Kähler decompositions by a finite set of 'moves', i.e. some relative calculus which would take us from one decomposition to another. To reveal the difficulty in this task we shall point out that our 'construction' of a Kähler decomposition on a given 4-manifold is far from being explicit. This is due to the two results we have utilized: Eliashberg's theorem on the existence of *some* isotopy between homotopic overtwisted contact structures, as well as Giroux's theorem on the existence of common stabilizations of two open books supporting isotopic contact structures. Neither one of these theorems provide us with explicit algorithms.

On the other hand, it is a curious question to determine whether one can define an invariant for a *fixed* Kähler decomposition. The difficulty lies in the fact that gauge theoretic invariants are very sensitive to the orientation change. Even though the invariants (Seiberg-Witten or Heegaard-Floer) of compact Stein manifolds are well-known, it is unclear how one can make use of this information on the piece with the reversed orientation. A possible approach of course is to go beyond the gauge theory setting, as one can not avoid for example while dealing with S^4. The work of Jens von Bergmann in [84] runs in this vein, but various technical details prevent us from adapting the same arguments for our case in any straightforward way. Nevertheless, if this task together with the previos one can be accomplished in a compatible way, then one can derive a (hopefully nontrivial) invariant of general 4-manifolds.

In a different direction, and motivated by the Theorem 5.6.2 of Gay and Kirby, we might start with a generalized fibration compatible with a generalized symplectic structure on a given closed smooth oriented 4-manifold. By homological reasons, these exist precisely on 4-manifolds with nontrivial second homology. One can then try to generalize Perutz's invariant to this setting. The crucial step is to describe a meaningful Lagrangian matching condition along the fold. Since the fold is away from the round singularities, it would suffice to define a Lagrangian matching condition for achiral fibrations, which then can be used to define an invariant of general symplectic fibrations through Perutz's work.

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