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A DISCREPANCY PRINCIPLE FOR PARAMETER
SELECTION IN LOCAL REGULARIZATION OF LINEAR
VOLTERRA INVERSE PROBLEMS

presented by

Cara Dylyn Brooks

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Ph.D. degree in Mathematics

Patricia K Lamm

Major Professor's Signature

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A DISCREPANCY PRINCIPLE FOR PARAMETER SELECTION IN LOCAL REGULARIZATION OF LINEAR VOLTERRA INVERSE PROBLEMS

By

Cara Dylyn Brooks

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ABSTRACT

A DISCREPANCY PRINCIPLE FOR PARAMETER SELECTION IN LOCAL REGULARIZATION OF LINEAR VOLTERRA INVERSE PROBLEMS

By

Cara Dylyn Brooks

We consider the problem of solving of a linear, first kind Volterra convolution equation with finitely smoothing kernel. Deeming that the problem is ill-posed, we approximate its solution using local regularization. In [22], sufficient conditions were established for local regularization of the problem under which regularized approximations based on exact data in $C[0,T]$ were shown to converge uniformly to the problem's true solution. An *a priori* strategy was also provided for choosing the regularization parameter for which uniform convergence of approximations made with perturbed data in $C[0,T]$ was also guaranteed.

However, until now, no *a posteriori* regularization parameter selection criteria existed to be paired with local regularization and convergence of the resulting method proved. We supply this missing piece by defining a new discrepancy principle for selecting the regularization parameter constant-valued based on measured data and the known level of noise. We establish sufficient conditions for the local regularization scheme, based on those in [22], so that when paired with our discrepancy principle,

we are able to prove uniform convergence of approximations made with perturbed data in $C[0,1]$ to the true solution in $C[0,1]$ as the noise level shrinks to zero.

We also extend the theory of local regularization to address the case when the linear Volterra convolution operator with finitely smoothing kernel is defined on the space $L^p[0,1], 1 < p < \infty$. We amend our conditions slightly and prove them sufficient for L^p-convergence of approximations based on exact data in $L^p[0,1]$ and provide an *a priori* rule for selecting the regularization parameter given perturbed data in $L^p[0,1]$. We redefine our principle and again establish sufficient conditions on the local regularization scheme so that when paired with the principle, approximations based on perturbed data in $L^p[0,1]$ converge to the true solution in $L^p[0,1]$ as the noise level shrinks to zero. For both the $C[0,1]$ and $L^p[0,1]$ cases, we provide a rate of convergence. Numerical examples are provided to illustrate the method's effectiveness.

Our principle is found to be a natural complement to the existing theory in $C[0,1]$ as well as its extension to $L^p[0,1], 1 < p < \infty$. This is an initial, yet fundamental step in the development of *a posteriori* principles for use with local regularization in solving linear and eventually non-linear Volterra equations.

To G.M. and G.P.

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Introduction

Over the last few decades, a variety of applications emerged that require solving inverse problems and thus accurately approximating their solutions is of great interest to the scientific community. For example, the need to solve an inverse problem arises in areas such as tomography, image reconstruction, and remote sensing. The difficulty lies in that these problems are frequently ill-posed, either due to non-existence or non-uniqueness of a solution, or due to a lack of continuous dependence of a solution on data. Issues of existence and uniqueness can often be handled, though the problem's ill-posedness, due to the failure of solutions to depend continuously on data, is much more problematic. As is the case in practice, we are given measured data that always has an element of error associated with it, thought of as a slight perturbation of the true data or a version corrupted by noise. The problem's lack of stability means that a solution obtained using noisy data can have arbitrarily large error.

Let $A : X \to X$ be a compact linear operator from a Banach space X into itself. We may consider an inverse problem as the abstract operator equation

$$Au = f$$

which we would like to solve for $u \in dom(A)$, where $f \in X$ is the *data* and the operator A represents a known relationship between the variables derived from the physics of the problem.

In this thesis, we consider specifically the problem of solving a linear first-kind Volterra integral equation with finitely smoothing kernel. We take $A : X \to X$ to be a linear Volterra convolution operator from a Banach space X into itself defined as

$$Au(t) = \int_0^t k(t-s)u(s)ds, \qquad \text{a.e.} \qquad t \in [0,1], \tag{1}$$

where the kernel $k \in C^{\nu}[0,1]$ is ν-smoothing, $\nu \geq 1$, meaning

$$\frac{d^{\ell}k}{dt^{\ell}}(0) = 0, \quad \ell = 0, 1, ..., \nu - 2, \qquad \text{and} \qquad \frac{d^{(\nu-1)}k}{dt^{(\nu-1)}}(0) \neq 0.$$

We would like to solve

$$Au(t) := \int_0^t k(t-s)u(s)ds = f(t), \qquad \text{a.e.} \qquad t \in [0,1], \tag{2}$$

for $u \in dom(A)$ given $f \in X$.

The need to solve a linear first kind Volterra equation with finitely smoothing kernel arises in various applications. For example, determining the ν-th derivative of a given function f can be expressed as solving equation (2) with ν-smoothing kernel $k(t) = \dfrac{t^{\nu-1}}{(\nu-1)!}$. Determining the propagation rate for a simple population problem from measurements of total population can be viewed as a first kind Volterra problem with one-smoothing kernel, [44]. In [4], the author describes the problem of determining the density of a chain, given information about its motion as it slides down a known curved surface. This problem is modeled as a first kind Volterra equation with kernel determined by the shape of the surface, and in the case where the surface is a cycloid, the kernel is two-smoothing.

We will show that the ν-smoothing problem, a generalization of the derivative problem mentioned above, is ill-posed due to a lack of continuous dependence of solutions on data. It is worthwhile to note that the degree of ill-posedness of the problem depends on ν. Thus as ν increases, the ill-posedness of the problem becomes more like that of the very ill-posed inverse heat conduction problem, a first kind Volterra problem with infintely smoothing kernel.

In the 1960's, J.V. Beck developed a method for approximating the solution to the discretized inverse heat conduction problem that was used successfully in practice for years. In [23], Lamm established its convergence and stability, as well as generalizing the method to the continuous case. This work resulted in the development of a class of regularizations for solving (2) called *sequential local regularization* or just *local regularization* (motivation for the name to follow) for which Beck's method was a special case. We refer the reader to [23], [25], [22], [27], [28], and [26] for developments of the theory to date. With local regularization of (2). stable solutions of a particular parameter-dependent second kind Volterra equation are taken to be approximations to the problem's true solution.

Lamm extended the theory of local regularization further by providing sufficient conditions to prove convergence in $C[0,1]$ for the linear finitely smoothing problem in the noise-free case and provided an *a priori* rule for selecting the regularization parameter given noisy data for which convergence in $C[0,1]$ was guaranteed. A rate of convergence was also provided.

The theory for the linear finitely smoothing problem in $C[0,1]$ was however without an *a posteriori* parameter selection strategy. The drawback of *a priori* strategies is that they only provide an asymptotic result, specifying how to select the regularization parameter in the limiting case as the noise level goes to zero. In practice, one typically has a single value of δ, relatively small, and would like a rule to select a single value of the regularization parameter, something *a posteriori* criteria would provide. The work we present here supplies this missing piece. We also extend the results in [22] to include a convergence theory for the case when the underlying space is $L^p[0,1], 1 < p < \infty$.

We begin Chapter 1 with a brief introduction to the theory of regularization and a short survey of *a posteriori* parameter selection criteria that lead to convergent methods for solving (2) using the classical Tikhonov regularization and those used with the non-classical Lavrentiev regularization. In Chapter 2, we outline the existing theory of local regularization for the linear Volterra ν-smoothing problem when the underlying space is $C[0,1]$. We then define a discrepancy principle and prove uniform convergence in $C[0.1]$ for the regularization method. In Chapter 3, we extend the theory of local regularization to the linear Volterra ν-smoothing problem when the underlying space is $L^p[0,1]$. $1 < p < \infty$. We prove L^p-convergence of approximations in the noise-free case as well as redefine our discrepancy principle and prove L^p-convergence for the regularization method. In Chapter 4, numerical examples are presented to illustrate the success of the method in practice.

CHAPTER 1

BACKGROUND

1.1 The Linear ν-smoothing Volterra Problem

We now take $A : X \rightarrow X$ to be a linear Volterra convolution operator with ν-smoothing kernel $k \in C^{\nu}[0, 1]$, where X is a Banach space. We refer to solving

$$Au(t) := \int_0^t k(t-s)u(s)ds = f(t), \qquad \text{a.e.} \qquad t \in [0,1], \tag{1.1}$$

for $u \in dom(A)$ given $f \in X$ as the *ν-smoothing problem* or the *finitely smoothing problem.*

If $f \in R(A)$, then a solution to our problem exists. If $N(A) = \{0\}$, then A^{-1} exists and the solution is unique. However if the $\dim R(A) = \infty$, then $R(A)$ is not closed since A is a compact linear operator [15]. It follows from the closed graph theorem that A^{-1} fails to be continuous and so the problem is ill-posed due to the lack of continuous dependence of solutions on data. We will examine the nullspace and range space for A in the two cases of interest here, namely $X = C[0, 1]$ (Chapter 2) and $X = L^p[0, 1], 1 < p < \infty$ (Chapter 3).

Note that in the case when X is a Hilbert space, (for example $L^2[0,1]$), if the data $f \notin R(A)$ but instead contained in the dense subspace $R(A) + R(A)^{\perp} \subset X$, we are content to solve the problem in the *least squares sense*. More precisely, we solve $\min_u \|Au - f\|_X$ and let u^+ denote the (unique) solution of minimal norm (MNLS). We define $A^+ : R(A) + R(A)^{\perp} \to N(A)^{\perp}$ as the mapping of $f \in R(A) + R(A)^{\perp}$ to $u^+ \in N(A)^{\perp}$. Then $u^+ = A^+ f$, where the unbounded linear operator A^+ is called the generalized inverse of A.

In an effort to mimic the properties of Hilbert spaces, the concept of orthogonality and orthogonal subspaces can be defined for the Banach spaces $L^p[0,1], 1 < p < \infty, p \neq 2$ [11], or one may consider so called semi-inner product spaces [12]. However, exploring the possibility of extending the least squares solution concept to finding a best approximation to $u \in dom(A)$ for which u uniquely solves $\min_u \|Au - f\|_{L^p[0,1]}$ for $1 < p < \infty, p \neq 2$ is beyond the scope of our current study. It is worth noting that the extension of the theory of regularization of linear ill-posed problems from Hilbert spaces to Banach spaces is of relatively new interest to the inverse problems community and we refer the reader to [8], [35], and [41] for recent developments.

1.2 Regularization and *A Priori* Parameter Selection

In practice, one commonly does not have access to exact data, rather a measurement of the exact data which always has an element of error associated with it. The ill-posedness of the problem due to a lack of continuous dependence of solutions on data now becomes an issue. As previously mentioned, we think of the given measurement data, denoted f^δ, as being a slight perturbation of the "true" data f or a version of the true data corrupted by noise. And so for a given level of error δ, we assume that the given "noisy" data f^δ satisfies $\left\|f - f^\delta\right\|_X \leq \delta$.

We are then faced with approximating the "true" solution u of $Au = f$ given f^δ. Notice that even if we are given $f^\delta \in R(A)$, the ill-posedness of the problem (due to A^{-1} being unbounded) means that $A^{-1}f^\delta$ could be a very poor approximation of u even if the noise level is small. Moreover, there is no guarantee that solutions $A^{-1}f^\delta$ converge to $A^{-1}f$ in X as the noise level δ shrinks to zero. A similar argument can be made regarding the unbounded generalized inverse. It follows that $A^{+}f^\delta$ need not converge to u^{+} in X as the noise level shrinks to zero.

To handle this issue, we implement a *regularization method.* The idea is to construct parameter-dependent approximations that depend continuously on the noisy data f^δ in such a way that the approximations converge to the true solution as the noise level shrinks to zero, or for a given level of noise, so that the error made in approximating the solution is small.

Definition 1.1 *Let $\{T_\alpha\}_{\alpha > 0}$ be a family of continuous operators from X into itself such that for any $u \in X$,*

$$\lim_{\alpha \to 0} \|T_\alpha A u - u\|_{C[0,1]} = 0. \tag{1.2}$$

We say that $T_\alpha, \alpha > 0$ is a **regularization operator** *for A^{-1}. (We may define similarly a regularization operator for A^+.)*

A **regularization method** consists of a family of regularization operators $\{T_\alpha\}_{\alpha > 0}$ accompanied by criteria, which we denote as d_α, for choosing the **regularization parameter** α.

Definition 1.2 *A regularization method $\left(\{T_\alpha\}_{\alpha > 0}, d_\alpha\right)$ is said to be* **convergent** *if for any $u \in X$, and given data $f^\delta \in X$ such that $\left\|f - f^\delta\right\|_X \leq \delta$, with $f = Au$, the regularization parameter $\alpha = \alpha(\delta)$, selected via the criteria d_α, satisfies*

$$\lim_{\delta \to 0} \alpha(\delta) = 0 \qquad \textit{and} \qquad \lim_{\delta \to 0} \left\|T_{\alpha(\delta)} f^\delta - u\right\|_X = 0. \tag{1.3}$$

(We note that generalizations of the above are used in the case of least squares problems when X is Hilbert.)

For $\alpha > 0$, let T_α be a regularization operator and consider the bound on the error made in approximating the solution,

$$\begin{aligned}
\left\|T\alpha f^{\delta} - u\right\|_X &\leq \left\|T\alpha f^{\delta} - T\alpha f\right\|_X + \|T\alpha f - u\|_X \\
&\leq \|T\alpha\| \left\|f^{\delta} - f\right\|_X + \|T\alpha f - u\|_X \\
&\leq \|T\alpha\| \, \delta + \|T\alpha f - u\|_X \, . \qquad (1.4)
\end{aligned}$$

The second term on the right-hand side of (1.4) represents the error due to regularization and tends to zero as $\alpha \to 0$. The first term on the right-hand side of (1.4) represents the error due to regularization accompanied by noise in the data and as $\alpha \to 0$, $\|T\alpha\|$ tends to infinity in the case of A^{-1} unbounded. Thus **regularization parameter selection strategies** are devised in an effort to minimize this upper bound or to guarantee that this upper bound shrinks to zero and the noise level goes to zero. Note that the value of α dictates the amount of regularization; in classical methods, a choice of α too small tends to lead to highly oscillatory approximations whereas if α is chosen too large, approximations tend to be overly flat.

Tikhonov Regularization

Classical regularization methods, such as Tikhonov regularization, have been fully developed in the context of underlying Hilbert spaces. Since our purpose is only to briefly introduce regularization methods commonly used, we will assume for the moment that X is a Hilbert space. Then classical regularization methods involve regularization operators of the form

$$T_\alpha = g_\alpha(A^*A)A^*, \qquad \alpha > 0,$$

where A^* denotes the Hilbert adjoint of A and for each $\alpha > 0$, $g_\alpha : [0, \|A\|^2] \to \mathbb{R}$ is a continuous function defined on the spectrum of the self-adjoint compact operator A^*A. If g_α has the properties

i) $g_\alpha(t) \to \dfrac{1}{t}$ as $\alpha \to 0$ for each $t > 0$,

ii) $|tg_\alpha(t)|$ is uniformly bounded for each $\alpha > 0$ and $t > 0$,

then for each $\alpha > 0$, the approximation

$$T_\alpha f = g_\alpha(A^*A)A^*f \tag{1.5}$$

depends continuously on data and $\|T_\alpha f - u\|_X \to 0$ as $\alpha \to 0$.

The function $g_\alpha(t) = \dfrac{1}{\alpha + t}$ corresponds to Tikhonov regularization. With this choice of g_α, for each $\alpha > 0$, the approximation $T_\alpha f = g_\alpha(A^*A)A^*f$ is the unique minimizer of

$$J_\alpha(u) := \|Au - f\|^2 + \alpha \|u\|^2.$$

Equivalently, considering the Volterra operator defined in (1.1),

$$T_\alpha f = g_\alpha(A^*A)A^*f = (\alpha I + A^*A)^{-1}A^*f,$$

which solves the *second-kind Fredholm* integral equation

$$\alpha u(t) + \int_t^1 k(\tau - t) \int_0^\tau k(\tau - s) u(s) ds d\tau = \int_t^1 k(s - t) f(s) ds, \qquad \text{a.e.} \qquad t \in [0, 1]. \tag{1.6}$$

It can be shown that for the Tikhonov regularization operator, $\|T_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$, [15]. In the case of noisy data f^δ, referring back to (1.4) leads to the error bound

$$\left\| T_\alpha f^\delta - u \right\| \leq \frac{\delta}{\sqrt{\alpha}} + \|T_\alpha f - u\|.$$

If we use the *a priori* parameter selection criteria

(P_1) Choose $\alpha = \alpha(\delta)$ such that $\frac{\delta^2}{\alpha} \to 0$ as $\delta \to 0$,

then $\left(T_\alpha = g_\alpha(A^*A)A^*, P_1\right)$ is a convergent regularization method. Further, if the true solution satisfies an additional smoothness assumption, we can obtain a rate of convergence. For instance, a common **source condition** is to assume $u \in R((A^*A)^\mu)$, for $0 < \mu \leq 1$ and so $u = (A^*A)^\mu v$ for some $v \in X$. Then choosing $\alpha = K\delta^{2/(2\mu + 1)}$ (for example as in [15]), it follows that

$$\left\| T_\alpha f^\delta - u \right\| = O\left(\delta^{(2\mu)/(2\mu + 1)}\right).$$

In fact, with Tikhonov regularization the fastest rate of convergence that can be obtained for non-degenerate kernels in (1.1) is $O(\delta^{2/3})$. For detailed proofs, see [7], [15], or [21].

Lavrentiev Regularization

Lavrentiev (or simplified) regularization involves regularization operators of the form

$$T_\alpha = (\alpha I + A)^{-1}, \quad \alpha > 0.$$

Then for the Volterra operator defined in (1.1), we see that $T_\alpha f = (\alpha I + A)^{-1} f$ solves the *second-kind Volterra* integral equation

$$\alpha u(t) + \int_0^t k(t-s)u(s)ds = f(t), \quad \text{a.e.} \quad t \in [0,1]. \tag{1.7}$$

When X is Hilbert and A is monotone, it was shown that for the Lavrentiev regularization operator, $\|T_\alpha\| \le \dfrac{1}{\alpha}$, [42]. With noisy data f^δ, referring back to (1.4) leads to the error bound

$$\left\|T_\alpha f^\delta - u\right\| \le \frac{\delta}{\alpha} + \|T_\alpha f - u\|.$$

If we use the *a priori* parameter selection criteria

(P_2) Choose $\alpha = \alpha(\delta)$ such that $\dfrac{\delta}{\alpha} \to 0$ as $\delta \to 0$,

then if $\bar{u}(0) = 0$, (and possibly higher order derivatives of $\bar{u}$ are zero at $t = 0$ depending on ν) it can be shown for the one-smoothing problem that $\left(T_\alpha = (\alpha I + A)^{-1}, P_2\right)$ is a convergent regularization method [24]. If the true solution satisfies the additional smoothness assumption, $u \in R(A^\mu)$ for $0 < \mu \le 1$, and so $u = A^\mu v$ for some $v \in X$, we can obtain a rate of convergence. Choosing $\alpha = K\delta^{1/(\mu+1)}$, as shown in [43], it follows that

$$\left\|T_\alpha f^\delta - u\right\| = O\left(\delta^{\mu/(\mu+1)}\right).$$

1.3 *A Posteriori* Parameter Selection

Recall that a regularization method consists of a means of constructing parameter dependent stable approximations of u accompanied by a rule for selecting the regularization parameter. And so to say that the method is *convergent*, the rule for choosing α must satisfy

i) $\alpha = \alpha(\delta) \to 0 \quad \text{as} \quad \delta \to 0.$

ii) $\left\|T_\alpha f^\delta - u\right\|_X \to 0 \quad \text{as} \quad \delta \to 0,$

where $T_\alpha, \alpha > 0$ is any regularization operator.

There are many drawbacks of *a priori* strategies. First of all, to make the most proper choice of the regularization parameter requires knowledge of the smoothness parameter μ and $\|v\|$ appearing in each of the rate estimates in the previous section [43]. They also only provide an asymptotic result, specifying how to select the regularization parameter given a sequence of δ's approaching zero. However one usually does not know the the level of smoothness μ of the unknown true solution or $\|v\|$. Furthermore, one typically has a single value of δ and would like a rule to select a single value of the regularization parameter. As we will find, *a posteriori* parameter selection criteria are much more useful in practice since they give direction on how to select α for a given value of δ and do not rely heavily on knowledge μ or $\|v\|$.

It must be noted first that convergence cannot be guaranteed for any strategy with which the regularization parameter is chosen independent of the noise level (for a proof see [1]). Therefore we will focus on theoretically sound parameter selection strategies that depend on the noise level δ and in some cases the noisy data f^{δ}. *Heuristic* parameter choices rules, those not relying on the noise level, are still used in practice with much success. L-curve is an example of one such rule and a counterexample to its convergence can be found in [19]. More examples of heuristic parameter selection strategies may be found in [7] and [20].

A number of *a posteriori* parameter selection strategies, a few of which we list below, were developed originally for use with classical regularizations of linear problems. Paired with a suitable regularization, each leads to a convergent method when A is linear, see [7], [15], [16], and [34]. When A is non-linear, slight modifications of these strategies were made in [9] and [43] to obtain convergent methods.

Most *a posteriori* parameter selection strategies require the selection of the regularization parameter "online," i.e. the user computes an approximate solution $T_{\alpha}f^{\delta}$ at various values of α and selects the value of α for which $T_{\alpha}f^{\delta}$ satisfies a set of criteria. Morozov's Discrepancy Principle is the most popular with which one chooses the regularization parameter so that the norm of the discrepancy, $\left\|A(T_{\alpha}f^{\delta}) - f^{\delta}\right\|$, is of the order of the noise in the data. The heuristic motivation of the principle is that the method should not produce results more accurate than the level of error in the given data [7].

With this idea in mind, we present the following strategies that are based upon defining a functional d to represent the "discrepancy" resulting from the approximation and then choosing the regularization parameter $\alpha > 0$ so that d is of the same order as the noise in the data on which the approximation was based. Assume that for $\alpha > 0$, T_α is a regularization operator and let $u_\alpha^\delta := T_\alpha f^\delta$. Let $\tau \in (1,2)$ be fixed and let $d : (0,\infty) \to (0,\infty)$ be defined according to one of:

(d_1) Morozov's Discrepancy Principle - $d(\alpha) = \left\| Au_\alpha^\delta - f^\delta \right\|$

(d_2) Archangeli's Method - $d(\alpha) = \sqrt{\alpha} \left\| Au_\alpha^\delta - f^\delta \right\|$

(d_3) Rule of Raus - $d(\alpha) = \left\| [I - AT_\alpha]^{1/(2p_0)} [Au_\alpha^\delta - f^\delta] \right\|, \quad p_0 > 0$

(d_4) Modified discrepancy principle - $d(\alpha) = \alpha^q \left\| Au_\alpha^\delta - f^\delta \right\|, \quad q > 0.$

The rule for each is to choose the regularization parameter $\alpha > 0$ so that

$$d(\alpha) = \tau \delta^s,$$

where $s = 1$ in d_1, d_2, d_3 and $s > 0$ in d_4. For a detailed development of each rule see [9], [16], [30], [31], and [34].

Although Morozov's principle is by far the most well-known, it is not best suited for use with all regularization operators. For instance, with Tikhonov regularization, the rate of convergence obtained with this principle under standard source conditions is at best $O(\delta^{1/2})$ [7]. Bakushinski also proved that Lavrentiev regularization with Morozov's discrepancy principle is not a convergent method [1]. An example of this failure of approximations to converge can be found in [16].

Convergent methods can be obtained however using the other principles listed. Tautenhahn proved convergence of Lavrentiev regularization using the Rule of Raus in [43]. The class of modified discrepancy principles, a generalization of Arcangeli's method ($s = 1$ and $q = \frac{1}{2}$) considered originally by Engl, Groetsch, and Schock in [5], [14], [39], and [40] was used to improve the rate of convergence with regularization operators that already lead to convergent methods when paired with Morozov's discrepancy principle. For example, an optimal rate was given for Tikhonov regularization for specified choices of s and q. Convergence of a modified discrepancy principle with Lavrentiev regularization was shown in [16] and [31], as well as values of s and q determined that lead to optimal rates under various source conditions in [9] and [32].

Regularization Methods for Volterra Problems

In general, classical regularization methods are deemed unsuitable for Volterra problems [22], [23], [29] since as was previously discussed in the Hilbert space setting, they require use of the adjoint A^* of the operator A in (1.1). In making use of the adjoint, the regularized equation one solves is no longer Volterra [29] thus one loses the causal nature of the original Volterra problem. This was seen when Tikhonov regularization was applied to the ν-smoothing problem leading to the Fredholm equation in (1.6). This loss is also evident when the regularized equation is discretized, for instance by collocation with piecewise constants. The numerical discretization leads to solving a system $Dx = b$ where D is a full matrix which leads to numerical methods requiring $O\left(N^3\right)$ floating point operations where D is $N \times N$. Whereas with non-classical methods, such as Lavrentiev regularization and *local regularization* (to be defined), the regularized equation is Volterra, (second-kind in the examples mentioned), and thus the causal nature of the problem remains intact. The same discretization leads again to solving a system $Dx = b$ instead with a lower triangular matrix D (in the convolution problem, D is Toeplitz). These methods are faster and more efficient, of order $O\left(N^2\right)$ floating point operations, since the resulting system can be solved sequentially.

CHAPTER 2

THE THEORY OF LOCAL REGULARIZATION IN $C[0,1]$

In this Chapter, we begin our study focusing on the case when $A : C[0,1] \to C[0,1]$ is a linear Volterra convolution operator of the form

$$Au(t) = \int_0^t k(t-s)u(s)ds, \qquad \text{for all} \qquad t \in [0,1], \tag{2.1}$$

and the kernel $k \in C^{\nu}[0,1]$ is ν-smoothing, $\nu \geq 1$. Recall

$$\frac{d^{\ell}k}{dt^{\ell}}(0) = 0, \quad \ell = 0, 1, ..., \nu - 2 \qquad \text{and} \qquad \frac{d^{(\nu-1)}k}{dt^{(\nu-1)}}(0) \neq 0,$$

and without loss of generality, assume that $k^{(\nu-1)}(0) = 1$.

Our goal is to solve the ν-smoothing problem

$$Au(t) := \int_0^t k(t-s)u(s)ds = f(t), \qquad \text{for all} \qquad t \in [0,1], \tag{2.2}$$

for $u \in C[0,1]$ given $f \in C[0,1]$. Note that the range of A is given by

$$R(A) = G_{\nu,0} := \left\{ g \in C^{\nu}[0,1] | g(0) = g'(0) = \cdots = g^{(\nu-1)}(0) = 0 \right\},$$

which we verify as follows.

For any $u \in C[0,1]$, we may write $Au(t) = k * u(t)$ for all $t \in [0,1]$. Since $k \in C[0,1]$, then

$$k * u \in C^1[0,1] \qquad \text{and} \qquad k * u(0) = 0.$$

(see Theorem 3.5 of [13].) We may repeatedly differentiate with respect to t to obtain for all $t \in [0,1]$,

$$\frac{d^{\ell}}{dt^{\ell}}[Au(t)] = k^{(\ell)} * u(t) \in C^1[0,1] \qquad \text{and} \qquad k^{(\ell)} * u(0) = 0,$$

using that $k^{(\ell)} \in C[0,1]$ and $k^{(\ell)}(0) = 0$, for $\ell = 1, \cdots, \nu - 1$. Differentiating the ν-th time, we obtain

$$u(t) + k^{(\nu)} * u(t) \in C[0,1],$$

for all $t \in [0,1]$. Thus

$$Au \in C^{\nu}[0,1] \quad \text{and} \quad Au(0) = \left.\frac{d}{dt}[Au(t)]\right|_{t=0} = \cdots = \left.\frac{d^{(\nu-1)}}{dt^{(\nu-1)}}[Au(t)]\right|_{t=0} = 0.$$

This proves that $R(A) \subseteq G_{\nu,0}$.

On the other hand, if $f \in G_{\nu,0}$, then consider the second kind equation

$$u(t) + k^{(\nu)} * u(t) = f^{(\nu)}(t),$$

for all $t \in [0, 1]$. There exists a solution $\hat{u} \in C[0, 1]$ to this equation since $f^{(\nu)} \in C[0, 1]$ and $k \in C^{\nu}[0, 1]$ (see [3] or [13]).

Now consider the initial value problem,

$$\begin{cases} y^{(\nu)}(t) = \hat{u}(t) + k^{(\nu)} * \hat{u}(t), \\ y(0) = y'(0) = \cdots = y^{(\nu-1)}(0) = 0, \end{cases} \tag{2.3}$$

where equality is understood to be pointwise. If $y(t) = \int_0^t k(t-s)\hat{u}(s)ds$ for all $t \in [0, 1]$, then it it easy to see that y satisfies the initial value problem (2.3). However, $y = f$ also solves (2.3), so by uniqueness of solutions, it follows that

$$f(t) = \int_0^t k(t-s)\hat{u}(s)ds,$$

for all $t \in [0, 1]$. Therefore given $f \in G_{\nu,0}$, we have found $\hat{u} \in C[0, 1]$ for which $A\hat{u} = f$. This proves $G_{\nu,0} \subseteq R(A)$. Therefore

$$R(A) = G_{\nu,0}.$$

One can also verify that the solution to the ν-smoothing problem defined in (2.2) is unique. Suppose that

$$\int_0^t k(t-s)u(s)ds = 0,$$

for all $t \in [0,1]$. As before, we may differentiate ν times. Using that k is ν-smoothing, we obtain the second-kind Volterra equation

$$u(t) + \int_0^t k^{(\nu)}(t-s)u(s)ds = 0,$$

for all $t \in [0,1]$, which has a unique solution that may be expressed using the variation of constants formula, [3] or [13] (see also equation (2.20) below). With this, we conclude that $u(t) = 0$ for all $t \in [0,1]$ and $\mathcal{N}(A) = \{0\}$.

Since $k \in C^{\nu}[0,1]$, it follows that A is compact (Theorem 2.5 in [13].) We showed that A is injective with $\dim R(A) = \infty$, however $R(A)$ is not closed in $C[0,1]$ and so A^{-1} fails to be continuous. Our problem is ill-posed due to a lack of continuous dependence of solutions on data.

In this chapter, we add to the theory of local regularization for solving the linear ν-smoothing problem, by introducing an *a posteriori* parameter selection strategy. We begin by deriving the second kind Volterra equation associated with local regularization for the underlying data space $C[0,1]$. We outline the sufficient conditions developed in [22] under which uniform convergence of regularized approximations to the true solution was achieved for exact data $f \in C[0,1]$ and for perturbed data $f^{\delta} \in C[0,1]$ for appropriate *a priori* parameter choices. While doing so, we normalize the approximating equation and alter one of the conditions therein slightly, defining the conditions here to be [A0] and [A1]. We make explicit a sufficient condition on the relationship between the kernel k and length of the interval $(0, R]$ for which the approximating equation is well-posed and the resolvent of the approximating equation is uniformly bounded for all $r \in (0, R]$. We restrict the choice of regularization parameter to an interval $(0, R]$ over which this condition is satisfied so that regularized approximations are uniformly bounded on $(0, R]$. This result sheds light on the condition in [22] and [36] that so long as k and R are sufficiently small, approximations are bounded uniformly.

Once applying the normalized local regularization scheme satisfying [A0] and [A1] to the ν-smoothing problem in equation (2.2), we define the additional assumption [A2] in preparation of introducing our *a posteriori* parameter selection strategy. Our strategy consists of a newly defined discrepancy principle to assist a user in selecting a constant value of the regularization parameter r. We establish that conditions [A0]-[A2] are sufficient to conclude that local regularization paired with our discrepancy principle is a convergent method. We do so by showing that approximations, constructed from noisy data $f^{\delta} \in C[0, 1]$ in a local regularization scheme satisfying [A0]-[A2] and the new discrepancy principle to select the regularization parameter, converge uniformly to the true solution as the noise level shrinks to zero. We also provide a rate of convergence under a suitable source condition. One will find that the discrepancy principle we introduce is a natural complement to the theory established in [22].

2.1 Local Regularization in $C[0,1]$

Let $\bar{u}$ denote the "true" solution given "exact" data f and f^{δ} the "noisy" version of f. Our first assumption concerns the availability of the data.

[A0] Let $0 < \bar{R} << 1$ be such that $\bar{u} \in C[0, 1+\bar{R}]$ and $f \in R(A) \subseteq C[0, 1+\bar{R}]$ so that $A\bar{u} = f$ for all t on the interval $[0, 1+\bar{R}]$. Given $\delta > 0$, the data $f^{\delta}(t)$ is available for all $t \in [0, 1+\bar{R}]$ and $f^{\delta} \in C[0, 1+\bar{R}]$ satisfies

$$\left\| f - f^{\delta} \right\|_{C[0,\,1\,+\,\bar{R}]} \leq \delta.$$

If additional data is unavailable, we suffice with approximating $\bar{u}$ on the slightly smaller interval $\left[0, 1-\bar{R}\right]$.

2.1.1 The Approximating Equation

Proceeding as in [22], it follows from [A0] that $\bar{u}$ satisfies

$$\int_0^{t\,+\,\rho} k(t+\rho-s)u(s)ds = f(t+\rho),$$

for all $t \in [0,1]$ and $\rho \in [0,r]$ for any $r \in (0,\bar{R}]$. Splitting the integral and using a change of variables we obtain

$$\int_0^{\rho} k(\rho-s)u(t+s)ds + \int_0^{t} k(t+\rho-s)u(s)ds = f(t+\rho), \qquad (2.4)$$

for all $t \in [0,1]$, $\rho \in [0,r]$ and for any $r \in (0,\bar{R}]$.

For each $r \in (0, \bar{R}]$, consider the space of all bounded linear functionals on $C[0, r]$. Recall that the continuous dual space of $C[0, r]$ can be identified with the space of all regular Borel measures on $\mathbb{R}$. Then for any $\Omega_r \in [C[0, r]]^*$, there exists a signed measure η_r defined on the σ-algebra of Borel sets of $\mathbb{R}$ [37] such that

$$\Omega_r(g) = \int_0^r g(\rho) d\eta_r(\rho),$$

for any $g \in C[0, r]$. Applying a functional Ω_r, we integrate both sides of equation (2.4) with respect to η_r and obtain

$$\int_0^r \int_0^\rho k(\rho - s) u(t+s) ds d\eta_r(\rho) + \int_0^t \int_0^r k(t+\rho-s) d\eta_r(\rho) u(s) ds = \int_0^r f(t+\rho) d\eta_r(\rho), \tag{2.5}$$

an equation that $\bar{u}$ still satisfies for each $0 < r \leq \bar{R}$ and all $t \in [0, 1]$.

Beck's original idea in [2] for solving the discretized inverse heat conduction problem involved stabilizing the problem by assuming u to be constant locally. Extending this idea to the abstract setting formulated above, i.e. taking u constant on the small local interval $[t, t + \rho]$, $\rho \in [0, r]$, leads to consideration of the *second-kind Volterra equation,*

$$u(t) \int_0^r \int_0^\rho k(\rho - s) ds d\eta_r(\rho) + \int_0^t \int_0^r k(t + \rho - s) d\eta_r(\rho) u(s) ds = \int_0^r f(t + \rho) d\eta_r(\rho), \tag{2.6}$$

for $t \in [0, 1]$.

2.1.2 Properties of η_r

In [22], hypotheses (H1)-(H3) were given indicating how to select a family of measures $\eta_r, r \in (0, \bar{R}]$, in the approximating equation (2.6). We state them here as assumption [A1], with any changes made to (H1)-(H3) mentioned in the remark that follows.

[A1] The measure η_r is chosen to satisfy the following hypotheses.

(H1) For $i = 0, 1, ..., \nu$, there is some $\sigma \in \mathbb{R}$ and $c_i = c_i(\nu) \in \mathbb{R}$ independent of r such that

$$\int_0^r \rho^i d\eta_r(\rho) = r^{i+\sigma} c_i,$$

with $c_\nu \neq 0$. Without loss of generality, we may assume that η_r is scaled so that $c_\nu = \nu!$.

(H2) The parameters c_i, $i = 0, 1, ..., \nu$, satisfy the condition that the roots of the polynomial $p_\nu(\lambda)$, defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + ... + \frac{c_1}{1!}\lambda + \frac{c_0}{0!},$$

have negative real part.

(H3) There exists $\tilde{C} > 0$ independent of r such that

$$\left| \int_0^r h(\rho) d\eta_r(\rho) \right| \leq \|h\|_{C[0,r]} \tilde{C} r^\sigma,$$

for each $h \in C[0, r]$ and any $0 < r \leq \bar{R}$.

Remark 2.1 *In [22], a generalization of (H1) was given which allowed for a larger class of η_r's. It allowed for c_i to be replaced by $c_i+O(r)$ as $r \to 0$ in (H1). In fact, the theory we develop in Chapters 2 and 3 carries over to this more general case without much conceptual difficulty; however, the increase in some technical detail needed to consider this more general case did not seem to be worth the slight additional benefit, especially since the measures η_r used in practice do not require this $O(r)$ term.*

We recall two classes of measures (constructed in Lemma 2.2 and 2.3 of [22]) that satisfy assumption [A1].

Lemma 2.1 *[22]*

1. *Let $\nu = 1, 2, \cdots,$ be arbitrary and let $\psi \in L^1[0,1]$ be such that $\int_0^1 \rho^\nu \psi(\rho)d\rho = \nu!$. Then for $r \in (0, \bar{R}]$, the measure η_r defined by*

$$\int_0^r g(\rho)d\eta_r(\rho) = \int_0^r g(\rho)\psi_r(\rho)d\rho, \quad g \in C[0,r],$$

where $\psi_r \in L^1[0,r]$ is given by

$$\psi_r(\rho) = \psi\left(\frac{\rho}{r}\right), \quad a.e.\ \rho \in [0,r],$$

satisfies condition (H1) (with $c_\nu = \int_0^1 \rho^\nu \psi(\rho)d\rho$ and $\sigma = 1$) and condition (H3) (with $\tilde{C} = \|\psi\|_{L^1[0,1]}$.) Further, for all $\nu = 1, 2, \cdots,$ and given arbitrary positive $m_1, m_2, \cdots,$ and m_ν, there is a unique monic polynomial ψ of degree ν so that the resulting family $\{\eta_r\}$ satisfies (H1) with $c_\nu = \nu!$ and $\sigma = 1$, (H2) *with the roots of the polynomial* p_ν *in (H2) given by* $(-m_i), i = 1, \cdots, \nu$ *and (H3).*

2. *Let* $\nu = 1, 2, \cdots,$ *be arbitrary and let* $\beta_\ell, \tau_\ell \in \mathbb{R}, \ell = 0, 1, \cdots L,$ *be fixed so that*

$$0 \leq \tau_0 < \tau_1 < \cdots < \tau_{\ell - 1} < \tau_\ell \leq 1, \tag{2.7}$$

and

$$\sum_{\ell = 0}^{L} \beta_\ell \tau_\ell^\nu = \nu!. \tag{2.8}$$

Then for $r \in (0, \bar{R}]$, *the discrete measure* η_r *defined via*

$$\int_0^r g(\rho) d\eta_r(\rho) = \sum_{\ell = 0}^{L} \beta_\ell g\left(\tau_\ell r\right), \quad g \in C[0, r],$$

satisfies condition (H1) (with $c_\nu = \sum_{\ell = 0}^{L} \beta_\ell \tau_\ell^\nu$ *and* $\sigma = 0$*) and condition (H3) (with* $\tilde{C} = \sum_{\ell = 0}^{L} |\beta_\ell|$ *.) Further, for all* $\nu = 1, 2, \cdots,$ *and given arbitrary positive* $m_1, m_2, \cdots,$ *and* m_ν, *and for* $L = \nu$ *there is a unique choice of* $\beta_0, \beta_1, \cdots, \beta_\nu$ *satisfying* (2.8) *(for each given collection of* τ_ℓ *satisfying* (2.7)*) and such that the resulting discrete measure* η_r *satisfies (H1) with* $c_\nu = \nu!$ *and* $\sigma = 0$, *(H2) with the roots of the polynomial* p_ν *in (H2) given by* $(-m_i), i = 1, \cdots, \nu$ *and (H3).*

For $r \in (0, \bar{R}]$, assume η_r is any measure satisfying [A1] and define

$$\gamma_r := \int_0^r d\eta_r(\rho). \tag{2.9}$$

If we let $(-m_i), i = 1, \cdots, \nu$, represent the roots of the polynomial p_ν in (H2) such that $m_i \in \mathbb{C}, Re(m_i) > 0$, then $p_\nu(\lambda) = \prod_{i=1}^{\nu} (\lambda + m_i)$ taking into account the scaling of η_r by assigning $c_\nu = \nu!$. Further, with $c_i \in \mathbb{R}, i = 0, \cdots, \nu$, then

$$c_0 = \prod_{i=1}^{\nu} m_i > 0. \tag{2.10}$$

It follows that

$$\gamma_r = r^\sigma c_0 > 0, \tag{2.11}$$

for all $r > 0$. Then define for any $u \in C[0, 1]$ and $v \in C[0, 1 + \bar{R}]$,

$$a_r := \frac{\int_0^r \int_0^\rho k(\rho - s) ds \; d\eta_r(\rho)}{\gamma_r}, \tag{2.12}$$

$$k_r(t) := \frac{\int_0^r k(t + \rho) d\eta_r(\rho)}{\gamma_r}, \tag{2.13}$$

$$A_r u(t) := \int_0^t k_r(t - s) \; u(s) ds, \tag{2.14}$$

$$f_r(t) := \frac{\int_0^r f(t + \rho) \; d\eta_r(\rho)}{\gamma_r}, \tag{2.15}$$

$$D_r v(t) := \frac{\int_0^r \int_0^\rho k(\rho - s) v(t + s) ds d\eta_r(\rho)}{\gamma_r}, \tag{2.16}$$

for all $t \in [0, 1]$ and each $r \in (0, \bar{R}]$. With this notation and recalling that $\bar{u}$ satisfies equation (2.5), then equations (2.5) and (2.6) may be written equivalently as

$$D_r\bar{u}(t) + A_r\bar{u}(t) = f_r(t), \qquad \text{for all} \qquad t \in [0, 1], \tag{2.17}$$

and

$$a_r u_r(t) + A_r u_r(t) = f_r(t), \qquad \text{for all} \qquad t \in [0, 1], \tag{2.18}$$

respectively.

Remark 2.2 *Equation* (2.18) *is a normalized version of the regularized equation considered in [22]. However, this is an important difference and will later allow for our discrepancy functional to be appropriately normalized.*

Provided $a_r \neq 0$, equation (2.18) is well-posed and there exists a unique solution $u_r \in C[0, 1]$ that depends continuously on $f \in C[0, 1]$ [13]. For each $r \in (0, \bar{R}]$ for which $a_r \neq 0$, $(a_r I + A_r)^{-1}$ is a bounded linear operator on $C[0, 1]$ [13], so that we may represent the solution as

$$u_r = (a_r I + A_r)^{-1} f_r. \tag{2.19}$$

We may also express u_r using the *variation of constants formula* in [3],

$$u_r(t) = \frac{f_r(t)}{a_r} - \int_0^t \mathcal{X}_r(t-s)\frac{f_r(s)}{a_r}ds, \tag{2.20}$$

for all $t \in [0, 1]$ and $r \in (0, \bar{R}]$ for which $a_r \neq 0$. For each such $r > 0$, there exists a unique function $\mathcal{X}_r \in C[0, 1]$ [13], called the *resolvent kernel*, that satisfies the *integral equation of the resolvent kernel* (corresponding to equation (2.18)) [4] given by

$$\mathcal{X}_r(t) + \int_0^t \frac{k_r(t-s)}{a_r} \mathcal{X}_r(s) ds = \frac{k_r(t)}{a_r}, \tag{2.21}$$

for all $t \in [0, 1]$.

Therefore to guarantee well-posedness of equation (2.18) for all $r \in (0, R]$, for some $0 < R \leq \bar{R}$, we must ensure that a_r does not vanish on the interval $(0, R]$. The following lemma gives a condition under which this is true. Based on assumptions [A0] and [A1], we obtain the following results that will be used frequently later sections. To simplify notation, define

$$\bar{C} := \frac{\tilde{C}}{c_0}.$$

Lemma 2.2 *Assume [A0] and [A1] are satisfied.*

1. *Let a_r be as defined in* (2.12). *If R and k satisfy*

$$R \left\| k^{(\nu)} \right\|_{C[0, R]} \leq \frac{(\nu+1)!}{\bar{C}\kappa}, \tag{2.22}$$

for some $\kappa > 1$ and $0 < R \leq \bar{R}$. then

$$\frac{\kappa - 1}{\kappa c_0} \cdot r^{\nu} \leq a_r \leq \frac{\kappa + 1}{\kappa c_0} \cdot r^{\nu}, \tag{2.23}$$

for all $r \in (0, R]$.

2. *Let $h \in C[0, 1 + \bar{R}]$ and define*

$$h_r(t) := \frac{\int_0^r h(t+\rho)d\eta_r(\rho)}{\gamma_r},$$

for all $t \in [0, 1]$. Then

$$\lim_{r \to 0} \|h_r - h\|_{C[0,1]} = 0. \tag{2.24}$$

3. *Let A_r be as defined in* (2.14) *for $r \in (0, \bar{R}]$. Then*

$$\lim_{r \to 0} \|A_r - A\| = 0, \tag{2.25}$$

where $\|\cdot\|$ is the operator norm on $\mathcal{B}(C[0, 1])$.

Proof:

1. Consider the general ν-smoothing kernel which may be expressed in terms of its Taylor expansion about 0. Then $k \in C^{\nu}[0, 1 + \bar{R}]$ is of the form

$$k(t) = \frac{t^{\nu - 1}}{(\nu - 1)!} + k^{(\nu)}(\zeta_t)\frac{t^{\nu}}{\nu!}, \tag{2.26}$$

for some $\zeta_t \in [0, t]$ and $t \in [0, 1 + \bar{R}]$. By [A1], for any $r \in (0, \bar{R}]$

$$
\begin{aligned}
a_r &= \frac{\int_0^r \left(\int_0^\rho k(\rho - s)ds\right) d\eta_r(\rho)}{\gamma_r} \\
&= \frac{\int_0^r \left(\int_0^\rho k(s)ds\right) d\eta_r(\rho)}{\gamma_r} \\
&= \frac{1}{\gamma_r}\left[\int_0^r \left(\int_0^\rho \frac{s^{\nu - 1}}{(\nu - 1)!}ds\right) d\eta_r(\rho) + \int_0^r \left(\int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!}ds\right) d\eta_r(\rho)\right] \\
&= \frac{1}{r^\sigma c_0}\left[\int_0^r \frac{\rho^\nu}{\nu!}d\eta_r(\rho) + \int_0^r \left(\int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!}ds\right) d\eta_r(\rho)\right] \\
&= \frac{1}{r^\sigma c_0}\left[r^{(\nu + \sigma)}\frac{c_\nu}{\nu!} + \int_0^r \left(\int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!}ds\right) d\eta_r(\rho)\right].
\end{aligned}
$$

If R and k satisfy

$$
R\left\|k^{(\nu)}\right\|_{C[0,\,R]} \le \frac{(\nu+1)!}{\tilde{C}\kappa},
$$

for some $\kappa > 1$ and $0 < R \le \bar{R}$, then since $r\left\|k^{(\nu)}\right\|_{C[0,\,r]}$ is a continuous, monotonically non-decreasing function of r that has value zero at $r = 0$, it follows that for all $r \in (0, R]$,

$$
r\left\|k^{(\nu)}\right\|_{C[0,\,r]} \le R\left\|k^{(\nu)}\right\|_{C[0,\,R]} \le \frac{(\nu+1)!}{\tilde{C}\kappa},
$$

and so

$$\begin{aligned}
a_r &\geq \frac{r^\nu}{c_0} - \frac{1}{r^\sigma c_0} \left\| k^{(\nu)} \right\|_{C[0,r]} \frac{r^{\nu+1}}{(\nu+1)!} \tilde{C} \cdot r^\sigma \\
&\geq \frac{r^\nu}{c_0} \left(1 - \frac{\tilde{C} r \left\| k^{(\nu)} \right\|_{C[0,r]}}{(\nu+1)!} \right) \\
&\geq \frac{r^\nu}{c_0} \left(1 - \frac{1}{\kappa} \right) \\
&= \frac{\kappa - 1}{\kappa c_0} \cdot r^\nu .
\end{aligned}$$

Similarly,

$$\begin{aligned}
a_r &\leq \frac{r^\nu}{c_0} + \frac{1}{r^\sigma c_0} \left\| k^{(\nu)} \right\|_{C[0,r]} \frac{r^{\nu+1}}{(\nu+1)!} \tilde{C} \cdot r^\sigma \\
&\leq \frac{r^\nu}{c_0} \left(1 + \frac{\tilde{C} r \left\| k^{(\nu)} \right\|_{C[0,r]}}{(\nu+1)!} \right) \\
&\leq \frac{r^\nu}{c_0} \left(1 + \frac{1}{\kappa} \right) \\
&= \frac{\kappa + 1}{\kappa c_0} \cdot r^\nu .
\end{aligned}$$

2. For all $t \in [0, 1]$, (H1) and (H3) imply

$$\begin{aligned}
|h_r(t) - h(t)| &= \left| \frac{\int_0^r h(t+\rho) d\eta_r(\rho)}{\gamma r} - h(t) \right| \\
&= \left| \frac{\int_0^r (h(t+\rho) - h(t))\, d\eta_r(\rho)}{r^\sigma c_0} \right| \\
&\leq \bar{C} \left\| h(t+\cdot) - h(t) \right\|_{C[0,r]} \\
&= \bar{C} \sup_{\rho \in [0,r]} |h(t+\rho) - h(t)| .
\end{aligned}$$

Since h is uniformly continuous on $[0, 1+\bar{R}]$ and by properties of suprema,

$$\lim_{r \to 0} \|h_r(t) - h(t)\|_{C[0,1]} \leq \bar{C} \lim_{r \to 0} \sup_{t \in [0,1]} \sup_{\rho \in [0,r]} |h(t+\rho) - h(t)| = 0.$$

3. Let $x \in C[0,1]$. Recalling the notation in (2.13) and (2.14), for any $r \in (0, \bar{R}]$,

$$\begin{aligned} \|A_r - A\| &= \sup_{\|x\|_{C[0,1]} = 1} \|(A_r - A)\,x\|_{C[0,1]} \\ &= \sup_{\|x\|_{C[0,1]} = 1} \sup_{t \in [0,1]} \left| \int_0^t (k_r(t-s) - k(t-s))\,x(s)ds \right| \\ &\leq \|k_r - k\|_{C[0,1]}. \end{aligned}$$

From the previous result, it follows that

$$\lim_{r \to 0} \|A_r - A\| \leq \lim_{r \to 0} \|k_r - k\|_{C[0,1]} = 0.$$

□

2.1.3 Properties of $\mathcal{X}_r$

We now turn to the establishment of an estimate on the size of $\left\|(a_r I + A_r)^{-1}\right\|$. This information can be obtained by examining the size of $\|\mathcal{X}_r\|$.

Ring and Prix established in [36] that for any $\nu-$smoothing k and *positive* measures η_r satisfying versions of (H1) and (H2) as given in [25], there exists a constant $\hat{D} = \hat{D}(\nu, c_0, c_1, \cdots, c_\nu)$ independent of r, such that if $\left\|k^{(\nu)}\right\|_{L^1[0,1]} \leq \hat{D}$ then the norm of the resolvent was bounded uniformly, i.e. there exists $0 < M = M(k, \nu, c_0, c_1, \cdots, c_\nu)$ for which $\|\mathcal{X}_r\|_{L^1[0,1]} \leq M$ for all r sufficiently small. A detailed proof of this result can be found in Lemma 1 of [36].

By introducing the use of *signed* measures η_r satisfying the versions of (H1) and (H2) in addition to the condition (H3) as given in [22], Lamm later proved the existence of a constant $\hat{C} = \hat{C}(\nu, c_0, c_1, \cdots, c_\nu)$ independent of r, such that if $\left\|k^{(\nu)}\right\|_{L^\infty[0,1+R]} \leq \hat{C}$ for $R > 0$ sufficiently small, then the resolvent $\|\mathcal{X}_r\|_{L^1[0,1]}$ is still bounded uniformly for all $r \in (0, R]$.

In Lemma 2.2, we proved that a sufficient condition for $a_r \neq 0$ for all $r \in (0, R]$ is that

$$R\left\|k^{(\nu)}\right\|_{C[0,R]} \leq \frac{(\nu+1)!}{\tilde{C}\kappa}, \tag{2.27}$$

for some $\kappa > 1$ and $0 < R \leq \bar{R}$. Therefore we have existence of a unique solution to equation (2.18) for all $r \in (0, R]$. In the two lemmas that follow, we take the analysis of $\|\mathcal{X}_r\|_{L^1[0,1]}$ in [36] and [22] further by clearly illustrating the dependence of the length of the interval $(0, R]$ on which $\|\mathcal{X}_r\|_{L^1[0,1]}$ is bounded for all $r \in (0, R]$ on the size of $R\left\|k^{(\nu)}\right\|_{C[0,R]}$.

In the first of two lemmas below, we establish a condition on the size of κ that determines the length of the interval $(0, R]$ to which we restrict our choice of r in relation to the selection of the roots of the polynomial p_ν in (H2) of [A1]. In the second lemma, we conclude that the resolvent $\|\mathcal{X}_r\|_{L^1[0,1]}$ is uniformly bounded for all $r \in (0, R]$, provided $\left\|k^{(\nu)}\right\|_{C[0,1+\bar{R}]}$ satisfies a particular bound. The proofs of these two lemmas are quite technical and serve as an update of the proof found in Lemma 1 of [36] with modifications found in [22]. Therefore many of the arguments follow closely or are identical to the proofs in the references mentioned.

Lemma 2.3 *Assume [A1] holds and that for some* $0 < R \leq \bar{R}$*,* R *and* k *satisfy*

$$R\left\|k^{(\nu)}\right\|_{C[0,\,R]} \leq \frac{(\nu+1)!}{\tilde{C}\kappa}, \tag{2.28}$$

for some $\kappa \geq \bar{\kappa}$ *where* $\bar{\kappa} > 1$ *is sufficiently large. Then the eigenvalues of* $A_r := A + M_r$ *have negative real part for all* $r \in (0, R]$*, where*

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_0/0! & -c_1/1! & \cdots & \cdots & -c_{\nu-1}/(\nu-1)! \end{pmatrix},$$

$$M_r = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{m}_{0,r} & -\bar{m}_{1,r} & \cdots & \cdots & -\bar{m}_{\nu-1,r} \end{pmatrix},$$

and

$$\bar{m}_{j,r} = \frac{1}{a_r\gamma_r}\left[r^{\nu-j}\int_0^r k^{(\nu)}(\hat{\zeta}_j)\frac{\rho^{j+1}}{(j+1)!}d\eta_r(\rho) - \frac{c_j}{j!}\int_0^r\int_0^\rho k^{(\nu)}(\zeta)\frac{s^\nu}{\nu!}ds\,d\eta_r(\rho)\right].$$

for $j = 0, 1, \cdots, \nu - 1$.

Further, for the matrix

$$B_r \equiv \int_0^\infty (\exp(A_r t))^T \exp(A_r) dt,$$

there exist positive constants L, K, *and* S *so that*

$$\begin{aligned} |x| &\geq 2L\left(x^T B_r x\right)^{\frac{1}{2}} \\ |B_r x| &\leq K\left(x^T B_r x\right)^{\frac{1}{2}} \\ |x| &\leq S\left(x^T B_r x\right)^{\frac{1}{2}} \end{aligned} \tag{2.29}$$

for all $x \in \mathbb{R}^\nu$ *and all* $r \in (0, R]$, *where* $|\cdot|$ *denotes the usual norm on* $\mathbb{R}^\nu$.

Proof: The eigenvalues of A are exactly the roots of the polynomial $p_\nu(\lambda)$ in (H2) of [A1] which are chosen to have negative real part. To estimate the size of $\bar{m}_{j,r}$, $j = 0, 1, \cdots, \nu - 1$, we use the lower bound on a_r to write

$$\begin{aligned} m_{j,r} &= \\ &\frac{1}{a_r \gamma_r}\left[r^{\nu - j}\int_0^r k^{(\nu)}(\hat{\zeta}_j)\frac{\rho^{j+1}}{(j+1)!}d\eta_r(\rho) - \frac{c_j}{j!}\int_0^r\int_0^\rho k^{(\nu)}(\zeta)\frac{s^\nu}{\nu!}ds d\eta_r(\rho)\right] \\ \leq\ & \frac{\kappa}{(\kappa - 1)r^{\nu+\sigma}}\left[\left\|k^{(\nu)}\right\|_{C[0,r]}\frac{r^{\nu+1}}{(j+1)!}\tilde{C}r^\sigma + \left\|k^{(\nu)}\right\|_{C[0,r]}\frac{\left|c_j\right|}{j!}\frac{r^{\nu+1}}{(\nu+1)!}\tilde{C}r^\sigma\right] \\ \leq\ & \frac{\kappa}{(\kappa - 1)r^{\nu+\sigma}}\left\|k^{(\nu)}\right\|_{C[0,r]}\tilde{C}r^{\nu+1+\sigma}\left[\frac{1}{(j+1)!} + \frac{\left|c_j\right|}{j!(\nu+1)!}\right] \\ \leq\ & r\left\|k^{(\nu)}\right\|_{C[0,r]}\frac{\kappa}{\kappa - 1}\tilde{C}\left[\frac{1}{(j+1)!} + \frac{\left|c_j\right|}{j!(\nu+1)!}\right]. \end{aligned}$$

Since

$$r\left\|k^{(\nu}\right\|_{C[0,r]} \leq R\left\|k^{(\nu}\right\|_{C[0,R]},$$

for all $r \in (0, R]$, then assuming (2.28) gives

$$m_{j,r} \leq \frac{(\nu+1)!}{\kappa - 1}\left[\frac{1}{(j+1)!} + \frac{|c_j|}{j!(\nu+1)!}\right],$$

for all $r \in (0, R]$. Then for $\bar{\kappa}_1 > 1$ sufficiently large, if $\kappa \geq \bar{\kappa}_1$, then the eigenvalues of $A + M_r$ have negative real part. It follows that if R and k satisfy (2.28) with $\kappa \geq \bar{\kappa}_1$, then for $A_r = A + M_r$, the matrix

$$B_r \equiv \int_0^{\infty} (\exp(A_r t))^T \exp(A_r) dt$$

is well-defined, symmetric, positive definite and such that

$$A_r^T B_r + B_r A_r = -I$$

for all $r \in [0, R]$ [3]. Further, since $B_r \to B_0$ (as $r\left\|k^{(\nu}\right\|_{C[0.r]} \to 0$) as $r \to 0$, there exists a $\bar{\kappa} > \bar{\kappa}_1$ sufficiently large so that if R and k satisfy (2.28) with $\kappa \geq \bar{\kappa}$, then there exist positive constants L, K, and S so that (2.29) holds for all $x \in \mathbb{R}^{\nu}$ and any $r \in (0. R]$ [36]. □

Lemma 2.4 *Assume the hypotheses of Lemma 2.3 hold. Then there exist constants $\hat{C} > 0$ and $M > 0$, independent of r (but dependent on $k, \nu, c_0, c_1, \cdots, c_\nu$), such that* *if*

$$\left\|k^{(\nu)}\right\|_{C[0,\,1+\bar{R}]} \leq \hat{C},$$

then we have

$$\|\mathcal{X}_r\|_{L^1[0,\,1]} \leq M,$$

for all $r \in (0, R]$, *where* $\mathcal{X}_r$ *is the resolvent defined in equation* (2.21).

Proof: Let $\mathcal{X}_r$ denote the resolvent defined in equation (2.21). Then for $r \in (0, R]$ and $t \in \left[0, \frac{1}{r}\right]$, $\mathcal{X}_r$ satisfies

$$\mathcal{X}_r(rt) + \int_0^{rt} \frac{k_r(rt-s)}{a_r}\mathcal{X}_r(s)ds = \frac{k_r(rt)}{a_r},$$

and so making a change of variables, we obtain

$$\mathcal{X}_r(rt) + \int_0^{t} r\frac{k_r(r(t-s))}{a_r}\mathcal{X}_r(rs)ds = \frac{k_r(rt)}{a_r},$$

for all $t \in \left[0, \frac{1}{r}\right]$. Define

$$\hat{\mathcal{X}}_r(t) := r\mathcal{X}_r(rt).$$

Then

$$\hat{\mathcal{X}}_r(t) + \int_0^{t} r\frac{k_r(r(t-s))}{a_r}\hat{\mathcal{X}}_r(s)ds = r\frac{k_r(rt)}{a_r}, \tag{2.30}$$

for all $t \in \left[0, \frac{1}{r}\right]$. In order to bound $\|\mathcal{X}_r\|_{L^1[0,1]}$, we see that

$$\begin{aligned}
\|\mathcal{X}_r\|_{L^1[0,1]} &= \int_0^1 |\mathcal{X}_r(t)|\,dt \\
&= \int_0^1 \frac{1}{r}\left|\hat{\mathcal{X}}_r\left(\frac{t}{r}\right)\right| dt \\
&= \int_0^{1/r} \left|\hat{\mathcal{X}}_r(t)\right| dt \\
&= \left\|\hat{\mathcal{X}}_r\right\|_{L^1[0,1/r]},
\end{aligned}$$

and so it suffices to show that $\left\|\hat{\mathcal{X}}_r\right\|_{L^1[0,1/r]} \leq M$ for all $r \in (0, R]$ under the conditions of the lemma.

Proceeding as in [22] and [36], and since $k \in C^{\nu}[0,1]$, we may differentiate equation (2.30) $j = 1, \cdots, \nu$ times to obtain

$$\begin{aligned}
\hat{\mathcal{X}}_r^{(j)}(t) &= -\sum_{\ell=0}^{j-1} \frac{r^{\ell+1}}{a_r} k_r^{(\ell)}(0)\hat{\mathcal{X}}_r^{(j-1-\ell)}(t) \\
&\quad - \int_0^t \frac{r^{j+1}}{a_r} k_r^{(j)}(r(t-s))\hat{\mathcal{X}}_r(s)ds + \frac{r^{j+1}}{a_r} k_r^{(j)}(rt), \qquad (2.31)
\end{aligned}$$

for all $t \in \left[0, \frac{1}{r}\right]$. We focus on the ν-th differentiation

$$\begin{aligned}
\hat{\mathcal{X}}_r^{(\nu)}(t) &= -\sum_{\ell=0}^{\nu-1} \frac{r^{\ell+1}}{a_r} k_r^{(\ell)}(0)\hat{\mathcal{X}}_r^{(\nu-1-\ell)}(t) \\
&\quad - \int_0^t \frac{r^{\nu+1}}{a_r} k_r^{(\nu)}(r(t-s))\hat{\mathcal{X}}_r(s)ds + \frac{r^{\nu+1}}{a_r} k_r^{(\nu)}(rt), \qquad (2.32)
\end{aligned}$$

for all $t \in \left[0, \frac{1}{r}\right]$.

Recalling the definition of k_r in (2.13) and a_r in (2.12), using the Taylor expansion of k at 0 and [A1], we have

$$
\begin{aligned}
k_r^{(\ell)}(0) &= \frac{1}{\gamma_r}\int_0^r k^{(\ell)}(\rho)d\eta_r(\rho) \\
&= \frac{1}{\gamma_r}\left[\int_0^r \frac{\rho^{\nu-1-\ell}}{(\nu-1-\ell)!}d\eta_r(\rho) + \int_0^r k^{(\nu)}(\zeta_{\nu-1-\ell})\frac{\rho^{\nu-\ell}}{(\nu-\ell)!}d\eta_r(\rho)\right] \\
&= \frac{1}{\gamma_r}\left[\frac{c_{\nu-1-\ell}}{(\nu-1-\ell)!}r^{\nu-1-\ell+\sigma} + \int_0^r k^{(\nu)}(\zeta_{\nu-1-\ell})\frac{\rho^{\nu-\ell}}{(\nu-\ell)!}d\eta_r(\rho)\right],
\end{aligned}
$$

for each $\ell = 0, 1, \cdots, \nu-1$ and

$$
\begin{aligned}
a_r &= \frac{1}{\gamma_r}\left[\int_0^r \frac{\rho^\nu}{\nu!}d\eta_r(\rho) + \int_0^r\int_0^\rho k^{(\nu)}(\zeta)\frac{s^\nu}{\nu!}ds\,d\eta_r(\rho)\right] \\
&= \frac{1}{\gamma_r}\left[r^{\nu+\sigma} + \int_0^r\int_0^\rho k^{(\nu)}(\zeta)\frac{s^\nu}{\nu!}ds\,d\eta_r(\rho)\right].
\end{aligned}
$$

Then dividing we obtain

$$
\begin{aligned}
\frac{r^{\ell+1}k_r^{(\ell)}(0)}{a_r} &= \frac{c_{\nu-1-\ell}}{(\nu-1-\ell)!} \\
&+ \frac{1}{a_r\gamma_r}\left[r^{\ell+1}\int_0^r k^{(\nu)}(\zeta_{\nu-1-\ell})\frac{\rho^{\nu-\ell}}{(\nu-\ell)!}d\eta_r(\rho)\right. \\
&\left. - \frac{c_{\nu-1-\ell}}{(\nu-1-\ell)!}\int_0^r\int_0^\rho k^{(\nu)}(\zeta)\frac{s^\nu}{\nu!}ds\,d\eta_r(\rho)\right],
\end{aligned}
$$

for each $\ell = 0, 1, \cdots, \nu-1$.

Define $\bar{m}_{j,r}$ as in Lemma 2.3. Then substituting into equation (2.32), we arrive at

$$\hat{\mathcal{X}}_r^{(\nu)}(t) = -\sum_{\ell=0}^{\nu-1} \frac{c_{\nu-1-\ell}}{(\nu-1-\ell)!}\hat{\mathcal{X}}_r^{(\nu-1-\ell)}(t)$$
$$- \sum_{\ell=0}^{\nu-1} \bar{m}_{\nu-1-\ell, r}\hat{\mathcal{X}}_r^{(\nu-1-\ell)}(t)$$
$$- \int_0^t \frac{r^{\nu+1}}{a_r} k_r^{(\nu)}(r(t-s))\hat{\mathcal{X}}_r(s)ds + \frac{r^{\nu+1}}{a_r} k_r^{(\nu)}(rt),$$

for all $t \in \left[0, \frac{1}{r}\right]$.

As in [22] and [36], we rewrite the ν-th order integro-differential equation as the system

$$\mathcal{R}_r'(t) = A\mathcal{R}_r(t) + M_r\mathcal{R}_r(t) + \int_0^t D_r(t-s)\mathcal{R}_r(s)ds + g_r(t) \qquad (2.33)$$

where A and M_r are given in the statement of Lemma 2.3, and

$$\mathcal{R}_r(t) = \begin{pmatrix} \hat{\mathcal{X}}_r^{(0)}(t) \\ \vdots \\ \hat{\mathcal{X}}_r^{(\nu-1)}(t) \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r^{\nu+1}k_r^{(\nu)}(rt)/a_r & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$g_r(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r^{\nu+1}k_r^{(\nu)}(rt)/a_r \end{pmatrix}.$$

Now B_r defined in the statement of Lemma 2.3 is well-defined and satisfies (2.29) for

$r \in (0, R]$, and positive constants L, K, and S independent of r. Following closely the proof of Lemma 1 in [36], we choose a constant $\bar{K} > 0$ independent of $r \in (0, R]$ and define the Lyapunov functional

$$V_r(t, x(\cdot)) = \left(x^T(t) B_r x(t)\right)^{\frac{1}{2}} + \bar{K} \int_0^t \int_t^{1/r} \|D_r(\zeta - s)\| \, d\zeta x(s) ds, \qquad (2.34)$$

for suitable $x : \left[0, \frac{1}{r}\right] \to \mathbb{R}^\nu$, where $\|\cdot\|$ is the matrix norm on $\mathbb{R}^{\nu \times \nu}$ induced from the Euclidean norm $|\cdot|$ on $\mathbb{R}^\nu$.

Let $Z(\cdot)$ denote the $\nu \times \nu$ matrix whose columns $z_j(\cdot)$, $j = 1, \cdots \nu$, are solutions of the homogeneous part of (2.33) (taking $g_r \equiv 0$) such that $Z(0) = I$. Following the arguments in [36], we differentiate V_r along $z_j(\cdot)$ to obtain

$$\begin{aligned} &\frac{d}{dt}\left(V_r(t, z_j(t))\right) \\ &\leq -\left(L - \bar{K} \int_t^{1/r} \|D_r(\zeta - t)\| \, d\zeta\right) \left|z_j(t)\right| - (\bar{K} - K) \int_0^t \|D_r(t - s)\| \left|z_j(s)\right| ds. \end{aligned}$$

As in [36], our goal is choose $\bar{K} > 0$ independent of $r \in (0, R]$ such that

$$\frac{d}{dt}\left(V_r(t, z_j(t))\right) \leq -\bar{L} \left|z_j(t)\right|,$$

for $t \in \left[0, \frac{1}{r}\right]$ and some $\bar{L} > 0$ independent of $r \in (0, R]$.

If this can be done, then exactly as in [36] we may integrate to obtain

$$V_r(t, z_j(t)) \leq V_r(0, z_j(0)) - \bar{L} \int_0^t \left|z_j(s)\right| ds.$$

Then from (2.29), we have

$$\begin{aligned}
\frac{1}{S}\left|z_j(t)\right| &\leq \left(z_j^T(t)B_r z_j(t)\right)^{\frac{1}{2}} \\
&\leq V_r(t, z_j(t)) \\
&\leq V_r(0, z_j(0)) - \bar{L}\int_0^t \left|z_j(s)\right| ds \\
&= \left(z_j^T(0)B_r z_j(0)\right)^{\frac{1}{2}} - \bar{L}\int_0^t \left|z_j(s)\right| ds \\
&\leq \frac{1}{2L}\left|z_j(0)\right| - \bar{L}\int_0^t \left|z_j(s)\right| ds
\end{aligned}$$

for all $t \in \left[0, \frac{1}{r}\right]$. Exactly as in [36],

$$\left\|z_j\right\|_{L^1[0,1/r]} \leq \frac{1}{2L\bar{L}},$$

and $Z(t)$ satisfies

$$\|Z(t)\| \leq \frac{\nu+1}{2L} S \exp(-S\bar{L}t), \tag{2.35}$$

for $t \in \left[0, \frac{1}{r}\right]$ and so

$$\|Z\|_{L^1[0,1/r]} \leq \frac{\nu+1}{2L\bar{L}}. \tag{2.36}$$

Bounding $g_r(t)$ we have

$$
\begin{aligned}
|g_r(t)| &= \frac{r^{\nu+1}}{a_r}\left|k_r^{(\nu)}(rt)\right| \\
&\leq r\frac{\bar{\kappa}c_0}{\bar{\kappa}-1}\frac{1}{\gamma_r}\left|\int_0^r k^{(\nu)}(rt+\rho)d\eta_r(\rho)\right| \\
&\leq r\frac{\bar{\kappa}c_0}{\bar{\kappa}-1}\bar{C}\left\|k^{(\nu)}\right\|_{C[0,1+\bar{R}]} \\
&= \tilde{m}r\left\|k^{(\nu)}\right\|_{C[0,1+\bar{R}]} \qquad (2.37)
\end{aligned}
$$

for some constant $\tilde{m} > 0$ independent of $r \in (0, R]$.

Further, from (2.31)

$$
\hat{\mathcal{X}}_r(0) = \frac{r^{\ell+1}k_r^{(\ell)}(0)}{a_r}|_{\ell=0} = \frac{rk_r(0)}{a_r},
$$

and for $\ell = 0, \cdots, \nu-1$,

$$
\frac{r^{\ell+1}k_r^{(\ell)}(0)}{a_r} = \frac{c_{\nu-1-\ell}}{(\nu-1-\ell)!} + \bar{m}_{\nu-1-\ell,r},
$$

where

$$
\left|\bar{m}_{\nu-1-\ell,r}\right| \leq \frac{(\nu+1)!}{\kappa-1}\left[\frac{1}{(\nu-\ell)!} + \frac{\left|c_{\nu-1-\ell}\right|}{(\nu-1-\ell)!(\nu+1)!}\right] \leq \tilde{M}(\bar{\kappa}) < \infty.
$$

for all $r \in (0, R]$ using the bound in the proof of Lemma 2.3. Therefore

$$
\left|\hat{\mathcal{X}}_r(0)\right| \leq d_0(\bar{\kappa}),
$$

for some $0 \leq d_0(\bar{\kappa}) < \infty$. Evaluating (2.31) at $t = 0$, we have

$$\hat{\mathcal{X}}_r^{(j)}(0) = -\sum_{\ell=0}^{j-1} \frac{r^{\ell+1}}{a_r} k_r^{(\ell)}(0)\hat{\mathcal{X}}_r^{(j-1-\ell)}(0) + \frac{r^{j+1}}{a_r} k_r^{(j)}(0), \tag{2.38}$$

and we may argue by induction that

$$\left|\hat{\mathcal{X}}_r^{(j)}(0)\right| \leq d_j(\bar{\kappa}),$$

with $0 \leq d_j(\bar{\kappa}) < \infty$, for $j = 0, \cdots, \nu - 1$, and all $r \in (0, R]$. Therefore

$$|\mathcal{R}_r(0)| \leq d(\bar{\kappa}) < \infty, \tag{2.39}$$

for some $d(\bar{\kappa}) \geq 0$, independent of $r \in (0, R]$.

Using the variation of constants formula in [3] for the non-homogeneous equation, we have

$$\mathcal{R}_r(t) = Z(t)\mathcal{R}_r(0) + \int_0^t Z_r(t-s)g_r(s)ds,$$

for $t \in \left[0, \frac{1}{r}\right]$. From (2.35), (2.36), (2.37), and (2.39), we have

$$|\mathcal{R}_r(t)| \leq \frac{\nu+1}{2L} S \exp(-S\bar{L}t)d(\bar{\kappa}) + \frac{\nu+1}{2L\bar{L}} \tilde{m}r \left\|k^{(\nu)}\right\|_{C[0,1+\bar{R}]},$$

for all $t \in \left[0, \frac{1}{r}\right]$ and $r \in (0, R]$. Therefore,

$$\begin{aligned}
\left\|\hat{\mathcal{X}}_r\right\|_{L^1[0,1/r]} &\leq \|\mathcal{R}_r\|_{L^1[0,1/r]} \\
&= \left\|Z(\cdot)\mathcal{R}_r(0) + \int_0^{\cdot} Z_r(\cdot - s)g_r(s)ds\right\|_{L^1[0,1/r]} \\
&\leq \frac{\nu+1}{2L}Sd(\bar{\kappa})\int_0^{1/r} \exp(-S\bar{L}t)dt + \frac{1}{r}\frac{\nu+1}{2L\bar{L}}\tilde{m}r\left\|k^{(\nu)}\right\|_{C[0,1+\bar{R}]} \\
&\leq \frac{\nu+1}{2L}Sd(\bar{\kappa})\frac{1}{S\bar{L}} + \frac{\nu+1}{2L\bar{L}}\tilde{m}\hat{C} \\
&= M < \infty,
\end{aligned}$$

for all $r \in (0, R]$. Therefore the resolvent $\mathcal{X}_r$ is uniformly bounded for all $r \in (0, R]$.

It remains to show that we can choose $\bar{K} > 0$ independent of $r \in (0, R]$ such that

$$\frac{d}{dt}\left(V_r(t, z_j(t))\right) \leq -\bar{L}\left|z_j(t)\right|,$$

for $t \in \left[0, \frac{1}{r}\right]$ and some $\bar{L} > 0$. Returning to (2.35), if we take $\bar{K} > K > 0$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(V_r(t, z_j(t))\right) \\
&\leq -\left(L - \bar{K}\int_t^{1/r} \|D_r(\zeta - t)\|\, d\zeta\right)\left|z_j(t)\right|
\end{aligned}$$

for $t \in \left[0, \frac{1}{r}\right]$. And so we need $\bar{K} > 0$ and $\bar{L} > 0$ so that

$$L - \bar{K}\int_t^{1/r} \|D_r(\zeta - t)\|\, d\zeta \geq \bar{L}, \tag{2.40}$$

for all $t \in \left[0, \frac{1}{r}\right]$. Consider

$$
\begin{aligned}
\int_t^{1/r} \|D_r(\zeta - t)\| \, d\zeta &\leq \int_t^{1/r} \frac{r^{\nu+1}}{a_r} k_r^{(\nu)}(r(\zeta - t)) d\zeta \\
&= \frac{r^{\nu+1}}{a_r} \frac{1}{r} \int_0^{1-rt} \left| k_r^{(\nu)}(s) \right| ds \\
&\leq \frac{\bar{\kappa} c_0}{\bar{\kappa} - 1} \frac{1}{\gamma_r} \int_0^{1-rt} \left| \int_0^r k^{(\nu)}(s+\rho) d\eta_r(\rho) \right| ds \\
&\leq \frac{\bar{\kappa} c_0}{\bar{\kappa} - 1} \tilde{C} \left\| k^{(\nu)} \right\|_{C[0, 1+\bar{R}]} \\
&= \frac{\bar{\kappa}}{\bar{\kappa} - 1} \tilde{C} \hat{C},
\end{aligned}
$$

for all $t \in \left[0, \frac{1}{r}\right]$, then returning to (2.40), we have that

$$
\begin{aligned}
L - \bar{K} \int_t^{1/r} \|D_r(\zeta - t)\| \, d\zeta &\geq L - \bar{K} \frac{\bar{\kappa}}{\bar{\kappa} - 1} \tilde{C} \hat{C} \\
&= L - \bar{K} \frac{\bar{\kappa}}{\bar{\kappa} - 1} \tilde{C} \hat{C},
\end{aligned}
$$

which means we need $\bar{K} > 0$ and $\bar{L} > 0$ so that

$$
L - \bar{K} \frac{\bar{\kappa}}{\bar{\kappa} - 1} \tilde{C} \hat{C} > \bar{L},
$$

or so that

$$
\bar{K} \leq \frac{L - \bar{L}}{\hat{C}} \cdot \frac{\bar{\kappa} - 1}{\tilde{C} \bar{\kappa}},
$$

as well as $\bar{K} > K > 0$. But such a $\bar{K}$ can be found for any $\bar{L} \in (0, L)$ if

$$
\frac{\hat{C}}{L - \bar{L}} \cdot \frac{\tilde{C} \bar{\kappa}}{\bar{\kappa} - 1} K < 1.
$$

or equivalently, for any $\hat{C}$ satisfying

$$\hat{C} \leq \frac{L - \bar{L}}{K} \cdot \frac{\bar{\kappa} - 1}{\tilde{C}\bar{\kappa}}.$$

This proves existence of a

$$\bar{K} \in \left(K, \frac{L - \bar{L}}{\hat{C}} \cdot \frac{\bar{\kappa} - 1}{\tilde{C}\bar{\kappa}} \right),$$

for which (2.40) is satisfied for all $r \in (0, R]$. □

Corollary 2.1 *Assume that R and k are such that Lemmas 2.3 and 2.4 are satisfied.*

Define

$$M = \inf \left\{ \mu \mid \|\mathcal{X}_r\|_{L^1[0,1]} \leq \mu \quad \text{for all} \quad r in (0, R] \right\}. \tag{2.41}$$

For each $r \in (0, R]$,

$$\left\| (a_r I + A_r)^{-1} \right\| \leq \frac{1 + M}{a_r}, \tag{2.42}$$

where $\|\cdot\|$ is the operator norm on $\mathcal{B}(C[0,1])$.

Proof: Representing u_r using (2.20), we have for each $r \in (0, R]$ [13],

$$\begin{aligned}
\|u_r\|_{C[0,1]} &= \left\| \frac{f_r}{a_r} - \mathcal{X}_r * \frac{f_r}{a_r} \right\|_{C[0,1]} \\
&\leq \left\| \frac{f_r}{a_r} \right\|_{C[0,1]} + \|\mathcal{X}_r\|_{L^1[0,1]} \left\| \frac{f_r}{a_r} \right\|_{C[0,1]} \\
&\leq (1+M) \left\| \frac{f_r}{a_r} \right\|_{C[0,1]}.
\end{aligned}$$

Therefore representing u_r using (2.19),

$$\left\| (a_r I + A_r)^{-1} f_r \right\|_{C[0,1]} = \|u_r\|_{C[0,1]} \leq \frac{1+M}{a_r} \|f_r\|_{C[0,1]},$$

and thus

$$\left\| (a_r I + A_r)^{-1} \right\| \leq \frac{1+M}{a_r},$$

for all $r \in (0, R]$. □

2.1.4 Uniform Convergence with *A Priori* Parameter Selection

In Theorem 3.1 of [22], under condition [A0] and the original (H1)-(H3) in condition [A1] described in the remark, uniform convergence of solutions of (2.6) to $\bar{u}$ was proved given exact data $f \in \dot{C}[0, 1 + \bar{R}]$. Replacing f by f^δ in equation (2.6), an *a priori* rule was given to guarantee uniform convergence for the case when the given data $f^\delta \in C[0, 1 + \bar{R}]$ contained noise. For purposes of obtaining a rate of convergence, it was assumed that $\bar{u}$ is uniformly Hölder continuous with power $\alpha \in (0, 1]$ and Hölder constant $L_{\bar{u}}$, i.e. for any $x, y \in [0, 1 + \bar{R}]$,

$$|\bar{u}(x) - \bar{u}(y)| \leq L_{\bar{u}} |x - y|^{\alpha}. \tag{2.43}$$

We state the results of Theorem 3.1 of [22] using the hypotheses [A0] and [A1], as well as the conditions on R and k as given here. We instead consider solutions of the normalized approximating equation and give a proof that will be referenced in sections that follow.

Theorem 2.1 *[22] Assume that [A0] and [A1] hold and k satisfies Lemma* 2.4.

1. *Let u_r denote the solution of equation* (2.18) *for $r \in (0, R]$, for $R > 0$ sufficiently small. Then*

$$\|u_r - \bar{u}\|_{C[0,1]} \to 0 \quad as \quad r \to 0.$$

Moreover, if the true solution $\bar{u}$ satisfies (2.43), *then*

$$\|u_r - \bar{u}\|_{C[0,1]} = O\left(r^{\alpha}\right) \quad as \quad r \to 0.$$

2. *Let u_r^{δ} denote the solution to equation* (2.18) *with f_r replaced by f_r^{δ} for $r \in (0, R]$, then*

$$\left\|u_r^{\delta} - \bar{u}\right\|_{C[0,1]} \leq C_1 \frac{\delta}{r^{\nu}} + \|u_r - \bar{u}\|_{C[0,1]},$$

for some $C_1 \geq 0$, so that a choice of $r(\delta)$ satisfying *i)* $r(\delta) \to 0$ *as* $\delta \to 0$, *and*

ii) $\delta \left[r(\delta)\right]^{-\nu} \to 0$ *as* $\delta \to 0$,

ensures

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{C[0,\,1]} \to 0 \quad as \quad \delta \to 0.$$

If $\bar{u}$ *satisfies* (2.43) , *then*

$$\left\| u^{\delta}_{r} - \bar{u} \right\|_{C[0,\,1]} \leq C_1 \frac{\delta}{r^{\nu}} + C_2 r^{\alpha},$$

for some $C_1, C_2 \geq 0$, *and so for any* $K > 0$, *the choice*

$$r = r(\delta) = K\delta^{1/(\alpha+\nu)},$$

gives

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{C[0,\,1]} = O\left(\delta^{\alpha/(\alpha+\nu)}\right) \quad as \quad \delta \to 0.$$

Proof: Let $\|\cdot\| = \|\cdot\|_{C[0,\,1]}$.

1. Let u_r denote the solution to equation (2.18) for $r \in (0, R]$. We bound the error

due to regularization using (2.17), (2.18), (2.19), and (2.23) to obtain

$$\begin{aligned}
\|u_r - \bar{u}\| &= \left\|(a_r I + A_r)^{-1}(f_r - A_r\bar{u} - a_r\bar{u})\right\| \\
&= \left\|(a_r I + A_r)^{-1}(D_r\bar{u} - a_r\bar{u})\right\| \\
&\leq \frac{(1+M)}{a_r}\|D_r\bar{u} - a_r\bar{u}\| \\
&\leq \frac{(1+M)\,\kappa c_0}{r^{\nu}(\kappa - 1)}\|D_r\bar{u} - a_r\bar{u}\|. \qquad (2.44)
\end{aligned}$$

Recall the respective definitions of a_r and D_r in (2.12) and (2.16). Then using the Taylor expansion of k in (2.26), we have

$$\begin{aligned}
\|D_r\bar{u} - a_r\bar{u}\| &= \left\|\frac{\int_0^r \int_0^{\rho} k(\rho - s)\,(\bar{u}(\cdot + s) - \bar{u}(\cdot))\,ds\,d\eta_r(\rho)}{\gamma_r}\right\| \\
&\leq \bar{C}\left\|\int_0^{\cdot} k(\cdot - s)ds\right\|_{C[0,r]} \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]} \\
&\leq \bar{C}r\,\|k\|_{C[0,r]} \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]} \\
&\leq \bar{C}r\frac{r^{\nu-1}}{(\nu-1)!} \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]} \\
&\quad + \bar{C}r\frac{r^{\nu}\left\|k^{(\nu)}\right\|_{C[0,r]}}{\nu!} \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]} \\
&\leq \bar{C}r^{\nu}\frac{2}{(\nu-1)!} \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]}, \qquad (2.45)
\end{aligned}$$

for all $r \in (0, R]$, for $R > 0$ sufficiently small. Substituting into (2.44), we have

$$\begin{aligned}
\|u_r - \bar{u}\| &\leq \frac{(1+M)\,\kappa c_0}{r^{\nu}(\kappa - 1)}\bar{C}r^{\nu}\frac{2}{(\nu-1)!} \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]} \\
&= D_1 \sup_{t\in[0,1]} \|\bar{u}(t+\cdot) - \bar{u}(t)\|_{C[0,r]}, \qquad (2.46)
\end{aligned}$$

for $D_1 > 0$ constant. Since $\bar{u} \in C[0, 1 + \bar{R}]$ it follows that

$$\lim_{r \to 0} \|u_r - \bar{u}\| \le D_1 \lim_{r \to 0} \sup_{t \in [0,1]} \|\bar{u}(t + \cdot) - \bar{u}(t)\|_{C[0,r]} = 0.$$

If the true solution $\bar{u}$ satisfies (2.43), it follows that

$$\begin{aligned} \sup_{t \in [0,1]} \|\bar{u}(t + \cdot) - \bar{u}(t)\|_{C[0,r]} &\le \sup_{t \in [0,1]} \|L_{\bar{u}}(\cdot)^{\alpha}\|_{C[0,r]} \\ &\le L_{\bar{u}} r^{\alpha}. \end{aligned}$$

Returning to (2.46), we have

$$\begin{aligned} \|u_r - \bar{u}\| &\le D_1 L_{\bar{u}} r^{\alpha} \\ &= C_2 r^{\alpha}, \end{aligned}$$

for $C_2 > 0$ constant. Therefore

$$\|u_r - \bar{u}\| = O\left(r^{\alpha}\right) \to 0 \quad \text{as} \quad r \to 0.$$

2. Let u_r^{δ} denote the solution to equation (2.18) with f_r replaced by f_r^{δ} for $r \in (0, R]$. We bound the error due to regularization and noise in the data using

(2.19), Corollary 2.1, and (2.23) to obtain

$$
\begin{aligned}
\left\| u_r^\delta - u_r \right\| &= \left\| (a_r I + A_r)^{-1} \left(f_r^\delta - f_r \right) \right\| \\
&\leq \frac{(1+M)}{a_r} \left\| f_r^\delta - f_r \right\| \\
&\leq \frac{(1+M)\,\kappa c_0}{r^\nu (\kappa - 1)} \bar{C} \delta. \\
&= C_1 \frac{\delta}{r^\nu}, \qquad (2.47)
\end{aligned}
$$

for $C_1 > 0$ constant. This establishes the bound on the total error for $\bar{u} \in C[0,1]$ to be

$$
\begin{aligned}
\left\| u_r^\delta - \bar{u} \right\| &\leq \left\| u_r^\delta - u_r \right\| + \| u_r - \bar{u} \| \\
&\leq C_1 \frac{\delta}{r^\nu} + \| u_r - \bar{u} \| . \qquad (2.48)
\end{aligned}
$$

If $\bar{u}$ satisfies (2.43), then

$$
\left\| u_r^\delta - \bar{u} \right\| \leq C_1 \frac{\delta}{r^\nu} + C_2 r^\alpha,
$$

and so for the choice $r = r(\delta) = K\delta^{1/(\alpha+\nu)}$ for some $K > 0$, we have

$$
\begin{aligned}
\left\| u_{r(\delta)}^\delta - \bar{u} \right\| &\leq C_1 \frac{\delta}{[r(\delta)]^\nu} + C_2 [r(\delta)]^\alpha \\
&= C_1 \delta (K\delta^{1/(\alpha+\nu)})^{-\nu} + C_2 \left(K\delta^{1/(\alpha+\nu)} \right)^\alpha \\
&= \tilde{C}_1 \delta^{\alpha/(\alpha+\nu)} + \tilde{C}_2 \delta^{\alpha/(\alpha+\nu)},
\end{aligned}
$$

for $\tilde{C}_1, \tilde{C}_2 > 0$ constants, and so

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{\alpha/(\alpha+\nu)}\right) \quad \text{as} \quad \delta \to 0.$$

□

2.2 A Discrepancy Principle for Local Regularization Given $f^{\delta} \in C[0,1]$

2.2.1 Preliminaries

For the purpose of defining our discrepancy principle, we make our final assumption that the choice of measures η_r satisfies the following continuity property on $(0, \bar{R})$.

[A2] The measure η_r is chosen so that for any $g \in C[0, \bar{R}]$,

$$\left| \int_0^{r+h} g(\rho) d\eta_{r+h}(\rho) - \int_0^{r} g(\rho) d\eta_r(\rho) \right| \to 0 \quad \text{as} \quad h \to 0,$$

where the convergence is uniform in bounded equicontinuous sets of $g \in C[0, \bar{R}]$.

Our assumption implies the following.

Lemma 2.5 *Assume [A0] - [A2] are satisfied. For any $g \in C[0, 1+\bar{R}]$, define*

$$g_r(t) := \frac{\int_0^r g(t+\rho) d\eta_r(\rho)}{\gamma_r},$$

for all $t \in [0,1]$. *Then the mapping* $r \mapsto g_r$ *is continuous in* $C[0,1]$, *for all* $r \in (0, \bar{R})$.

Proof: Let $\|\cdot\| = \|\cdot\|_{C[0,1]}$ and $r \in (0, \bar{R})$ be fixed. Let h be such that $r + h \in (0, \bar{R})$. By (2.11), we have $\dfrac{1}{\gamma_r} = \dfrac{1}{r^\sigma c_0}$. Then

$$\begin{aligned}
\left\|g_{r+h} - g_r\right\| &= \left\| \frac{\int_0^{r+h} g(\cdot + \rho) d\eta_{r+h}(\rho)}{\gamma_{r+h}} - \frac{\int_0^r g(\cdot + \rho) d\eta_r(\rho)}{\gamma_r} \right\| \\
&= \left\| \frac{\int_0^{r+h} g(\cdot + \rho) d\eta_{r+h}(\rho)}{(r+h)^\sigma c_0} - \frac{\int_0^r g(\cdot + \rho) d\eta_r(\rho)}{r^\sigma c_0} \right\| \\
&\leq \left\| \frac{\int_0^{r+h} g(\cdot + \rho) d\eta_{r+h}(\rho) - \int_0^r g(\cdot + \rho) d\eta_r(\rho)}{(r+h)^\sigma c_0} \right\| \\
&\quad + \left\| \frac{\int_0^r g(\cdot + \rho) d\eta_r(\rho)}{(r+h)^\sigma c_0} - \frac{\int_0^r g(\cdot + \rho) d\eta_r(\rho)}{r^\sigma c_0} \right\|.
\end{aligned}$$

Since $\gamma_r > 0$ for all $r > 0$, the mapping $r \mapsto \dfrac{1}{\gamma_r}$ is continuous on $(0, \bar{R})$. Then

$$\lim_{h \to 0} \left\| \frac{\int_0^r g(\cdot + \rho) d\eta_r(\rho)}{(r+h)^\sigma c_0} - \frac{\int_0^r g(\cdot + \rho) d\eta_r(\rho)}{r^\sigma c_0} \right\| = 0,$$

and

$$\begin{aligned}
&\lim_{h \to 0} \left\| \frac{\int_0^{r+h} g(\cdot + \rho) d\eta_{r+h}(\rho) - \int_0^r g(\cdot + \rho) d\eta_r(\rho)}{(r+h)^\sigma c_0} \right\| \\
&= \frac{1}{r^\sigma c_0} \lim_{h \to 0} \sup_{t \in [0,1]} \left| \int_0^{r+h} g(t + \rho) d\eta_{r+h}(\rho) - \int_0^r g(t + \rho) d\eta_r(\rho) \right|.
\end{aligned}$$

Define

$$L(h) = \sup_{t \in [0,1]} \left| \int_0^{r+h} g(t+\rho) d\eta_{r+h}(\rho) - \int_0^r g(t+\rho) d\eta_r(\rho) \right|,$$

and let $\epsilon > 0$. By properties of suprema, there exists $t_0 \in [0.1]$ for which

$$L(h) < \left| \int_0^{r+h} g(t_0+\rho) d\eta_{r+h}(\rho) - \int_0^r g(t_0+\rho) d\eta_r(\rho) \right| + \frac{\epsilon}{2}.$$

From the continuity property of η_r in [A2] and the uniform continuity of g on $[0, 1+\bar{R}]$,

$$\lim_{h \to 0} \left| \int_0^{r+h} g(t+\rho) d\eta_{r+h}(\rho) - \int_0^r g(t+\rho) d\eta_r(\rho) \right| = 0,$$

uniformly for $t \in [0,1]$, and so in particular, there exists $\delta(\epsilon) > 0$ such that for all $|h| < \delta(\epsilon)$,

$$\left| \int_0^{r+h} g(t+\rho) d\eta_{r+h}(\rho) - \int_0^r g(t+\rho) d\eta_r(\rho) \right| < \frac{\epsilon}{2},$$

for all $t \in [0,1]$. Let $|h| < \delta(\epsilon)$. Then

$$L(h) < \left| \int_0^{r+h} g(t_0+\rho) d\eta_{r+h}(\rho) - \int_0^r g(t_0+\rho) d\eta_r(\rho) \right| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore

$$\lim_{h \to 0} \left\| \frac{\int_0^{r+h} g(\cdot+\rho) d\eta_{r+h}(\rho) - \int_0^r g(\cdot+\rho) d\eta_r(\rho)}{(r+h)^\sigma c_0} \right\| = 0.$$

$\square$

We see that measures standardly used in local regularization also satisfy this additional assumption.

Lemma 2.6 *The measures given in Lemma* 2.1 *satisfy [A2].*

Proof:

1. Recall the continuous measure defined in part 1 of Lemma 2.1. Namely, for $\nu = 1, 2, \cdots$, arbitrary, let $\psi \in L^1[0,1]$ and for $r \in (0, \bar{R}]$, let the measure η_r be defined by

$$\int_0^r g(\rho) d\eta_r(\rho) = \int_0^r g(\rho)\psi_r(\rho) d\rho, \qquad g \in C[0,r],$$

where $\psi_r \in L^1[0,r]$ is given by

$$\psi_r(\rho) = \psi\left(\frac{\rho}{r}\right), \quad \text{a.e. } \rho \in [0,r].$$

Let $r \in (0, \bar{R})$ be fixed. Then for all $g \in C[0, \bar{R}]$,

$$\begin{aligned}
&\lim_{h \to 0} \left| \int_0^{r+h} g(\rho) d\eta_{r+h}(\rho) - \int_0^r g(\rho) d\eta_r(\rho) \right| \\
= &\lim_{h \to 0} \left| \int_0^{r+h} g(\rho)\psi\left(\frac{\rho}{r+h}\right) d\rho - \int_0^r g(\rho)\psi\left(\frac{\rho}{r}\right) d\rho \right| \\
= &\lim_{h \to 0} \left| (r+h) \int_0^1 g((r+h)\rho)\psi(\rho)\, d\rho - r \int_0^1 g(r\rho)\psi(\rho)\, d\rho \right| \\
\leq &\lim_{h \to 0} \int_0^1 |(r+h)g((r+h)\rho) - rg(r\rho)|\, |\psi(\rho)|\, d\rho \\
\leq &\lim_{h \to 0} \|(r+h)g((r+h)\cdot) - rg(r\cdot)\|_{C[0,1]} \|\psi\|_{L^1[0,1]} \\
\leq &\|\psi\|_{L^1[0,1]} \lim_{h \to 0} \left[r \|g((r+h)\cdot) - g(r\cdot)\|_{C[0,1]} + h \|g((r+h)\cdot)\|_{C[0,1]} \right] \\
= &\ 0,
\end{aligned}$$

where the rate is uniform in bounded equicontinuous sets of g in $C[0, \bar{R}]$.

2. Recall the discrete measure defined in part 2 of Lemma 2.1. Namely, for $\nu = 1, 2, \cdots$, arbitrary, let $\beta_\ell, \tau_\ell \in \mathbb{R}$, $\ell = 0, 1, \cdots L$, be fixed so that

$$0 \leq \tau_0 < \tau_1 < \cdots < \tau_{\ell-1} < \tau_\ell \leq 1,$$

and let the discrete measure η_r be defined by

$$\int_0^r g(\rho) d\eta_r(\rho) = \sum_{\ell=0}^{L} \beta_\ell g(\tau_\ell r), \quad g \in C[0, r].$$

Let $r \in (0, \bar{R})$ be fixed. Then for all $g \in C[0, \bar{R}]$,

$$\begin{aligned}
&\lim_{h \to 0} \left| \int_0^{r+h} g(\rho) d\eta_{r+h}(\rho) - \int_0^{r} g(\rho) d\eta_r(\rho) \right| \\
&= \lim_{h \to 0} \left| \sum_{\ell=0}^{L} \beta_\ell \, g((r+h)\tau_\ell) - \sum_{\ell=0}^{L} \beta_\ell \, g(r\tau_\ell) \right| \\
&\leq \lim_{h \to 0} \sum_{\ell=0}^{L} |\beta_\ell| \, |g((r+h)\tau_\ell) - g(r\tau_\ell)| \\
&\leq \lim_{h \to 0} \sup_{\ell = 0, \cdots, L} |g((r+h)\tau_\ell) - g(r\tau_\ell)| \sum_{\ell=0}^{L} |\beta_\ell| \\
&= 0,
\end{aligned}$$

where the rate is uniform in equicontinuous sets of g in $C[0, \bar{R}]$.

□

2.2.2 Definition and Properties

We are now ready to define our discrepancy principle for selecting the regularization parameter in local regularization on $C[0, 1]$. We restate our assumptions and modify [A0] to be:

[A0] Let $0 < \bar{R} << 1$ be such that $\bar{u} \in C[0, 1 + \bar{R}]$ and $f \in R(A) \subseteq C[0, 1 + \bar{R}]$ so that $A\bar{u} = f$ for all t on the interval $[0, 1+\bar{R}]$. Assume that $\|f'\|_{C[0, 1+\bar{R}]} \neq 0$. Given $\delta > 0$, the data $f^\delta(t)$ is available for all $t \in [0, 1+\bar{R}]$ and $f^\delta \in C[0, 1+\bar{R}]$

satisfies

$$\left\|f - f^\delta\right\|_{C[0,\,1\,+\,\bar{R}]} \leq \delta \quad \text{and} \quad \left\|f^\delta\right\|_{L^2[0,\,1]} > (\tau + 1)\delta,$$

with $\tau \in (1, 2)$ fixed for all δ. If additional data is unavailable, we suffice with approximating $\bar{u}$ on the slightly smaller interval $\left[0, 1 - \bar{R}\right]$.

[A1] The measure η_r satisfies hypotheses (H1)-(H3) for all $r \in (0, \bar{R}]$.

[A2] The measure η_r is chosen so that for any $g \in C[0, \bar{R}]$,

$$\left| \int_0^{r\,+\,h} g(\rho) d\eta_{r\,+\,h}(\rho) - \int_0^r g(\rho) d\eta_r(\rho) \right| \to 0, \quad \text{as} \quad h \to 0,$$

where the convergence is uniform in bounded equicontinuous sets of $g \in C[0, \bar{R}]$.

Remark 2.3 *A specification of the form* $\left\|f^\delta\right\|_{L^2[0,\,1]} > (\tau + 1)\delta$ *is a classical assumption when working with a posteriori parameter selection rules (see [7], [17], and [43].) The specification* $\|f'\|_{C[0,\,1\,+\,\bar{R}]} \neq 0$ *simply implies that our true data is sufficiently "rich".*

Henceforth we will assume that R and k satisfy the conditions of Lemmas 2.3 and 2.4.

Definition 2.1 Discrepancy Principle for Local Regularization in C[0,1] *Assume that the conditions [A0]-[A2] are satisfied. Let* $d : (0, R) \to [0, \infty)$ *be the discrepancy functional defined by*

$$d(r) := a_r^m \left\|A_r u_r^\delta - f_r^\delta\right\|_{L^2[0,\,1]}, \tag{2.49}$$

for $m \in (0, 1]$ fixed. Choose the regularization parameter r so that

$$a_r^m \left\| A_r u_r^\delta - f_r^\delta \right\|_{L^2[0,1]} = \tau\delta. \tag{2.50}$$

By Lemma 2.2, we observe that

$$A_r \to A \quad \text{and} \quad f_r^\delta \to f^\delta$$

in $\mathcal{B}(C[0, 1])$ and $C[0, 1]$ respectively as $r \to 0$ (and therefore also in $\mathcal{B}\left(L^2[0, 1]\right)$ and $L^2[0, 1]$ respectively.) Also, by Lemma 2.2 we have that $a_r \sim r^\nu$. Thus for R small, our rule leads one to select the parameter $r = r(\delta)$ such that

$$\left\| A u_r^\delta - f^\delta \right\|_{L^2[0,1]} \approx \left\| A_r u_r^\delta - f_r^\delta \right\|_{L^2[0,1]} = \frac{\tau\delta}{a_r^m} \approx \frac{\tau\delta}{r^{\nu m}}$$

which corresponds roughly to the class of modified discrepancy principles defined in Chapter 1.

We will show existence of an $r \in (0, R)$ for which $d(r) = \tau\delta$ once establishing properties of d. To do so, we use the bounds on $\|u_r\|_{L^2[0,1]}$ in following lemma.

Lemma 2.7 *Assume [A0] and [A1] are satisfied. For any $r \in (0, R]$,*

$$\frac{\|f_r\|_{L^2[0,1]}}{a_r + \bar{C}\,\|k\|_{C[0,1+R]}} \le \|u_r\|_{L^2[0,1]} \le (1 + M) \left\| \frac{f_r}{a_r} \right\|_{L^2[0,1]}, \tag{2.51}$$

where u_r is the solution of (2.18).

Proof: Let $\|\cdot\| = \|\cdot\|_{L^2[0,1]}$. Representing u_r using (2.19), for any $r \in (0, R]$,

$$\begin{aligned} \|f_r\| &= \|(a_r I + A_r)u_r\| \\ &\leq (a_r + \|A_r\|)\,\|u_r\|\,. \end{aligned}$$

Let $x \in L^2[0,1]$. Recalling the notation in (2.13),

$$\begin{aligned} \|A_r\| &= \sup_{\|x\|=1} \|A_r x\| \\ &= \sup_{\|x\|=1} \left\| \int_0^{\cdot} k_r(\cdot - s)x(s)ds \right\| \\ &\leq \sup_{\|x\|=1} \|k_r\|\,\|x\| \\ &= \left\| \frac{\int_0^r k(\cdot + \rho)d\eta_r(\rho)}{\gamma_r} \right\| \\ &\leq \bar{C}\,\|k\|_{C[0,1+\bar{R}]}\,. \end{aligned}$$

Therefore

$$\|f_r\| \leq \left(a_r + \bar{C}\,\|k\|_{C[0,1+\bar{R}]}\right)\|u_r\|,$$

or

$$\frac{\|f_r\|}{a_r + \bar{C}\,\|k\|_{C[0,1+\bar{R}]}} \leq \|u_r\|\,.$$

To establish the upper bound, we represent u_r using (2.19) and use the bound on the operator norm in Corollary 2.1 to obtain

$$\begin{aligned} \|u_r\| &= \left\|(a_r I + A_r)^{-1} f_r\right\| \\ &\leq \frac{1+M}{a_r} \|f_r\|, \end{aligned}$$

for all $r \in (0, R]$. $\square$

Lemma 2.8 *Assume that the conditions [A0]-[A2] are satisfied. The function $d : (0, R) \to (0, \infty)$ defined as in (2.49) has the following properties:*

i) The mapping $r \mapsto d(r)$ is continuous on $(0, R)$.

ii) $\lim_{r \to 0} d(r) = 0$.

iii) There exists an $\tilde{R} = \tilde{R}(f, R)$, $\bar{\delta} = \bar{\delta}(f, k, R)$, and $\gamma_S = \gamma_S(f, k, R)$, such that if $\delta \in (0, \bar{\delta})$ or $\left\|f^\delta\right\|_{L^2[0,1]} > \gamma_S \delta$, then $\lim_{r \to \tilde{R}} d(r) > \tau\delta$.

Therefore, for δ sufficiently small or $\dfrac{\left\|f^\delta\right\|_{L^2[0,1]}}{\delta}$ sufficiently large, there exists $r \in (0, R)$ such that $d(r) = \tau\delta$.

Proof: Let $\|\cdot\| = \|\cdot\|_{L^2[0,1]}$.

i) Fix $r \in (0, R)$ and let h be such that $r + h \in (0, R)$. From the definition of u_r, we have

$$d(r) = a_r^{1+m} \left\|u_r^\delta\right\|,$$

where Lemma 2.2 gives $a_r > 0$ for all $r \in (0, R]$. Thus

$$\begin{aligned} |d(r+h) - d(r)| &= \left| a_{r+h}^{1+m} \left\| u_{r+h}^{\delta} \right\| - a_r^{1+m} \left\| u_r^{\delta} \right\| \right| \\ &\leq a_{r+h}^{1+m} \left\| u_{r+h}^{\delta} - u_r^{\delta} \right\| + \left| a_{r+h}^{1+m} - a_r^{1+m} \right| \left\| u_r^{\delta} \right\| . \end{aligned}$$

Since r is fixed, we may use (2.51) to bound $\left\| u_r^{\delta} \right\|$ and obtain

$$|d(r+h) - d(r)| \leq a_{r+h}^{1+m} \left\| u_{r+h}^{\delta} - u_r^{\delta} \right\| + \left| a_{r+h}^{1+m} - a_r^{1+m} \right| (1+M) \left\| \frac{f_r^{\delta}}{a_r} \right\| .$$

Since $\rho \mapsto \int_0^{\rho} k(s)ds$ is continuous on $(0, r)$ for all $r \in (0, R]$, it follows from [A2] that $r \mapsto a_r$ is continuous on $(0, R)$. Therefore

$$\left| a_{r+h}^{1+m} - a_r^{1+m} \right| \to 0 \quad \text{as} \quad h \to 0,$$

and thus

$$\lim_{h \to 0} |d(r+h) - d(r)| \leq a_r^{1+m} \limsup_{h \to 0} \left\| u_{r+h}^{\delta} - u_r^{\delta} \right\| . \tag{2.52}$$

From the representation of u_r^{δ} in (2.20), we have

$$\left\| u_{r+h}^{\delta} - u_r^{\delta} \right\| = \left\| \frac{f_{r+h}^{\delta}}{a_{r+h}} - \frac{f_r^{\delta}}{a_r} - \mathcal{X}_{r+h} * \frac{f_{r+h}^{\delta}}{a_{r+h}} + \mathcal{X}_r * \frac{f_r^{\delta}}{a_r} \right\| .$$

Define for all $t \in [0, 1]$,

$$\tilde{f}_h(t) := \frac{f^\delta_{r+h}(t)}{a_{r+h}} - \frac{f^\delta_r(t)}{a_r}$$

$$\tilde{\mathcal{X}}_h(t) := \mathcal{X}_{r+h}(t) - \mathcal{X}_r(t)$$

$$\tilde{k}_h(t) := \frac{k_{r+h}(t)}{a_{r+h}} - \frac{k_r(t)}{a_r}.$$

Using the above notation and adding and subtracting $\mathcal{X}_r * \dfrac{f^\delta_{r+h}}{a_{r+h}}$, we have

$$\begin{aligned}
\left\|u^\delta_{r+h} - u^\delta_r\right\| &\leq \left\|\tilde{f}_h\right\| + \left\|\tilde{\mathcal{X}}_h * \frac{f^\delta_{r+h}}{a_{r+h}}\right\| + \left\|\mathcal{X}_r * \tilde{f}_h\right\| \\
&\leq \left\|\tilde{f}_h\right\| + \left\|\tilde{\mathcal{X}}_h\right\|_{L^1[0,1]} \left\|\frac{f^\delta_{r+h}}{a_{r+h}}\right\| + \|\mathcal{X}_r\|_{L^1[0,1]} \left\|\tilde{f}_h\right\| \\
&\leq (1+M)\left\|\tilde{f}_h\right\| + \left\|\tilde{\mathcal{X}}_h\right\|_{L^1[0,1]} \left\|\frac{f^\delta_{r+h}}{a_{r+h}}\right\|.
\end{aligned}$$

Using the fact that $r \mapsto a_r$ is continuous on $(0, R)$ and Lemma 2.5, we have

$$\left\|\tilde{f}_h\right\| \leq \frac{1}{a_{r+h}} \left\|f^\delta_{r+h} - f^\delta_r\right\| + \left|\frac{1}{a_{r+h}} - \frac{1}{a_r}\right| \left\|f^\delta_r\right\| \to 0 \quad mboxas \quad h \to 0,$$

and

$$\left|\left\|\frac{f^\delta_{r+h}}{a_{r+h}}\right\| - \left\|\frac{f^\delta_r}{a_r}\right\|\right| \leq \left\|\tilde{f}_h\right\| \to 0 \quad \text{as} \quad h \to 0.$$

Therefore

$$\limsup_{h \to 0} \left\| u^{\delta}_{r+h} - u^{\delta}_{r} \right\| \leq \left\| \frac{f^{\delta}_{r}}{a_r} \right\| \limsup_{h \to 0} \left\| \tilde{\mathcal{X}}_h \right\|_{L^1[0,1]}$$

and

$$\lim_{h \to 0} |d(r+h) - d(r)| \leq a_r^{1+m} \left\| \frac{f^{\delta}_{r}}{a_r} \right\| \limsup_{h \to 0} \left\| \tilde{\mathcal{X}}_h \right\|_{L^1[0,1]}.$$

It remains to prove that $\limsup_{h \to 0} \left\| \tilde{\mathcal{X}}_h \right\|_{L^1[0,1]} = 0$. Using equation (2.21), we have

$$\mathcal{X}_{r+h}(t) - \mathcal{X}_r(t) + \frac{k_{r+h}}{a_{r+h}} * \mathcal{X}_{r+h}(t) - \frac{k_r}{a_r} * \mathcal{X}_r(t) = \frac{k_{r+h}(t)}{a_{r+h}} - \frac{k_r(t)}{a_r}.$$

for all $t \in [0,1]$. Adding and subtracting $\frac{k_{r+h}}{a_{r+h}} * \mathcal{X}_r$, we obtain

$$\tilde{\mathcal{X}}_h(t) + \frac{k_{r+h}}{a_{r+h}} * \tilde{\mathcal{X}}_h(t) + \left(\frac{k_{r+h}}{a_{r+h}} - \frac{k_r}{a_r} \right) * \mathcal{X}_r(t) = \frac{k_{r+h}(t)}{a_{r+h}} - \frac{k_r(t)}{a_r}.$$

Substituting the notation $\tilde{k}_h$,

$$\tilde{\mathcal{X}}_h(t) = -\int_0^t \frac{k_{r+h}(t-s)}{a_{r+h}} \tilde{\mathcal{X}}_h(s) ds - \tilde{k}_h * \mathcal{X}_r(t) + \tilde{k}_h(t),$$

so that for all $t \in [0,1]$,

$$\begin{aligned}
\left|\tilde{\mathcal{X}}_h(t)\right| &\leq \int_0^t \left|\frac{k_{r+h}(t-s)}{a_{r+h}}\right| \left|\tilde{\mathcal{X}}_h(s)\right| ds + \left|\tilde{k}_h * \mathcal{X}_r(t)\right| + \left|\tilde{k}_h(t)\right| \\
&\leq \left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]} \int_0^t \left|\tilde{\mathcal{X}}_h(s)\right| ds + \left\|\tilde{k}_h * \mathcal{X}_r\right\|_{C[0,1]} + \left\|\tilde{k}_h\right\|_{C[0,1]} \\
&\leq \left(\left\|\tilde{k}_h * \mathcal{X}_r\right\|_{C[0,1]} + \left\|\tilde{k}_h\right\|_{C[0,1]}\right) \exp\left(\left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]}\right)
\end{aligned}$$

taking the supremum over $t \in [0,1]$ of the last two terms and then using Gronwall's Inequality. It follows from [13] that

$$\begin{aligned}
\left\|\tilde{\mathcal{X}}_h\right\|_{L^1[0,1]} &\leq \left(\left\|\tilde{k}_h * \mathcal{X}_r\right\|_{C[0,1]} + \left\|\tilde{k}_h\right\|_{C[0,1]}\right) \exp\left(\left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]}\right) \\
&\leq \left(\left\|\tilde{k}_h\right\|_{C[0,1]} \|\mathcal{X}_r\|_{L^1[0,1]}\right) \exp\left(\left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]}\right) \\
&\quad + \left\|\tilde{k}_h\right\|_{C[0,1]} \exp\left(\left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]}\right) \\
&\leq (1+M) \left\|\tilde{k}_h\right\|_{C[0,1]} \exp\left(\left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]}\right).
\end{aligned}$$

With $r \in (0, R)$ fixed, the maps $r \mapsto k_r$ and $r \mapsto a_r$ continuous on $(0, R)$ by Lemma 2.5 and [A2], and $a_r > 0$, it follows that

$$\left\|\tilde{k}_h\right\|_{C[0,1]} \leq \left|\frac{1}{a_{r+h}}\right| \left\|k_{r+h} - k_r\right\|_{C[0,1]} + \left|\frac{1}{a_{r+h}} - \frac{1}{a_r}\right| \|k_r\|_{C[0,1]} \to 0,$$

as $h \to 0$, and

$$\exp\left(\left\|\frac{k_{r+h}}{a_{r+h}}\right\|_{C[0,1]}\right) \to \exp\left(\left\|\frac{k_r}{a_r}\right\|_{C[0,1]}\right) \qquad \text{as } h \to 0.$$

Therefore

$$\limsup_{h \to 0} \left\|\tilde{\mathcal{X}}_h\right\|_{L^1[0,1]} = 0,$$

and so $\lim_{h \to 0} |d(r+h) - d(r)| = 0$ proving that $r \mapsto d(r)$ is continuous on $(0, R)$.

ii) Using that $a_r > 0$, (H3) to bound $\left\|f_r^\delta\right\|$, and the upper bounds on a_r in Lemma 2.2 and $\left\|u_r^\delta\right\|$ in Lemma 2.7,

$$\begin{aligned} d(r) &= a_r^m \left\|A_r u_r^\delta - f_r^\delta\right\| \\ &= a_r^{1+m} \left\|u_r^\delta\right\| \\ &\leq a_r^{1+m} (1+M) \frac{\left\|f_r^\delta\right\|}{a_r} \\ &= a_r^m (1+M) \left\|f_r^\delta\right\| \\ &\leq r^{\nu m}(1+M)\left(\frac{\kappa+1}{\kappa c_0}\right)^m \bar{C} \left\|f^\delta\right\|_{C[0,1+R]}. \end{aligned} \tag{2.53}$$

Since $m \in (0,1]$, it follows that $\lim_{r \to 0} d(r) = 0$.

iii) Define

$$\tilde{R} := \min\left\{R,\ \frac{c_0}{2\tilde{C}} \cdot \frac{\|f\|_{L^2[0,1]}}{\|f'\|_{C[0,1+\bar{R}]}}\right\}. \tag{2.54}$$

Then for all $r \in (0, \tilde{R}]$,

$$\begin{aligned}
|f_r(t) - f(t)| &= \left| \frac{\int_0^r f(t+\rho) - f(t)\, d\eta_r(\rho)}{\gamma r} \right| \\
&\leq \bar{C}\, \|f'\|_{C[0,\,1+r]}\, \tilde{R} \\
&\leq \frac{\|f\|}{2} \\
&\leq \frac{1}{2}\left[\left\|f^\delta - f\right\| + \left\|f^\delta\right\|\right] \\
&\leq \frac{1}{2}\left[\delta + \left\|f^\delta\right\|\right]. \qquad (2.55)
\end{aligned}$$

For all $r \in (0, R]$, define

$$B(r) := \frac{a_r^{1+m}}{a_r + \bar{C}\, \|k\|_{C[0,\,1+R]}}. \qquad (2.56)$$

Then using the lower bound on $\left\|u_r^\delta\right\|$ in Lemma 2.7 and (2.55), we have for all $r \in (0, R)$

$$\begin{aligned}
d(r) &= a_r^{1+m} \left\|u_r^\delta\right\| \\
&\geq B(r) \left\|f_r^\delta\right\| \\
&\geq B(r) \left[\left\|f^\delta\right\| - \left\|f_r^\delta - f_r\right\| - \left\|f^\delta - f\right\| - \|f_r - f\|\right] \\
&\geq B(r) \left[\left\|f^\delta\right\| - \bar{C}\delta - \delta - \frac{\delta}{2} - \frac{\left\|f^\delta\right\|}{2}\right] \\
&= B(r) \left[\frac{\left\|f^\delta\right\|}{2} - \left(\bar{C} + \frac{3}{2}\right)\delta\right].
\end{aligned}$$

(2.57)

For all $r \in (0, R]$, define

$$F(r) := \frac{B(r)}{2\left[\tau + B(r)\left(\bar{C} + 3/2\right)\right]}. \tag{2.58}$$

Define

$$\bar{\delta} := \|f\| \frac{F(\tilde{R})}{1 + F(\tilde{R})}, \tag{2.59}$$

and

$$\gamma_S := \frac{1}{F(\tilde{R})}. \tag{2.60}$$

By [A0],

$$\|f\| \leq \left\|f^{\delta}\right\| + \left\|f - f^{\delta}\right\| \leq \left\|f^{\delta}\right\| + \delta,$$

and

$$(\tau + 1)\delta < \left\|f^{\delta}\right\| \leq \|f\| + \left\|f - f^{\delta}\right\| \leq \|f\| + \delta.$$

imply

$$\left\|f^{\delta}\right\| \geq \|f\| - \delta > 0. \tag{2.61}$$

If $\delta \in (0, \bar{\delta})$,

$$\begin{aligned}
\frac{\delta}{\left\|f^{\delta}\right\|} &\leq \frac{\delta}{\|f\| - \delta} \\
&< \frac{\|f\| F(\tilde{R})}{\|f\| (1 + F(\tilde{R})) - \|f\| F(\tilde{R})} \\
&= F(\tilde{R}).
\end{aligned}$$

Equivalently,

$$\left\| f^{\delta} \right\| > \frac{\delta}{F(\tilde{R})}. \tag{2.62}$$

Alternatively, if $\delta \notin (0, \bar{\delta})$, the assumption that $\left\| f^{\delta} \right\|_{L^2[0,\,1]} > \gamma_S \delta$ implies that (2.62) still holds. Then substituting (2.58) and (2.62) into the lower bound on d in (2.57) and taking the limit as r approaches $\tilde{R}$, we have that

$$\begin{aligned}
\lim_{r \to \tilde{R}} d(r) &\geq B(\tilde{R}) \left[\frac{\left\| f^{\delta} \right\|}{2} - \left(\bar{C} + \frac{3}{2} \right) \delta \right] \\
&> \frac{B(\tilde{R})}{2} \left[\frac{\delta}{F(\tilde{R})} - \left(2\bar{C} + 3 \right) \delta \right] \\
&= \frac{B(\tilde{R})}{2} \left[\frac{1}{F(\tilde{R})} - \left(2\bar{C} + 3 \right) \right] \delta \\
&= \frac{B(\tilde{R})}{2} \left[2 \cdot \frac{\tau + B(\tilde{R}) \left(\bar{C} + 3/2 \right)}{B(\tilde{R})} - \left(2\bar{C} + 3 \right) \right] \delta \\
&= \left[\tau + B(\tilde{R}) \left(\bar{C} + \frac{3}{2} \right) - B(\tilde{R}) \left(\bar{C} + \frac{3}{2} \right) \right] \delta \\
&= \tau \delta.
\end{aligned}$$

□

We observe that the choice $r(\delta)$ of the regularization parameter given by the discrepancy principle in (2.50) is bounded away from zero by $r^*(\delta) > 0$, where $r^*(\delta)$ is defined in the next lemma.

Lemma 2.9 *Let* $\delta \in \left(0, \bar{\delta}\right)$ *or* $\left\| f^{\delta} \right\| > \gamma_S \delta$, *where* $\bar{\delta}, \gamma_S > 0$ *are as given in Lemma* 2.8. *Let* $r = r(\delta)$ *be defined by*

$$r(\delta) = \min \left\{ r \in (0, R) \mid d(r) = \tau \delta \right\}. \tag{2.63}$$

There exists an $r^* = r^*(\delta) > 0$ *such that* $r(\delta) \geq r^* > 0$, *where* $r^* \in (0, R)$ *is given by*

$$r^* := \left(\frac{\tau\delta}{\epsilon}\right)^{1/(\nu m)},$$

with $\epsilon := \left(\frac{\kappa+1}{\kappa c_0}\right)^m (1+M)\bar{C} \left\|f^\delta\right\|_{C[0,\,1+R]} > 0.$

Proof: We first observe that the set $\{r \in (0, R) \mid d(r) = \tau\delta\}$ is compact and thus has a minimum value $r(\delta)$. Note that

$$\begin{aligned} d(r) &\leq a_r^m (1+M) \left\|f_r^\delta\right\|_{L^2[0,\,1]} \\ &\leq r^{\nu m} \left(\frac{\kappa+1}{\kappa c_0}\right)^m (1+M)\bar{C} \left\|f^\delta\right\|_{C[0,\,1+R]} \\ &= \epsilon r^{\nu m}, \end{aligned}$$

by Lemma 2.2 and Corollary 2.1.

Since $r \mapsto \epsilon\, r^{\nu m}$ is a continuous, strictly increasing function that bounds d from above for all $r \in (0, \tilde{R})$,

$$\lim_{r \to \tilde{R}} \epsilon r^{\nu m} \geq \lim_{r \to \tilde{R}} d(r) > \tau\delta.$$

Therefore there exists a unique $r^*(\delta) \in (0, \tilde{R}]$ for which

$$\epsilon \left(r^*\right)^{\nu m} = \tau\delta,$$

and so for $r(\delta) \in (0, \tilde{R})$ for which $d(r(\delta)) = \tau\delta$, we have necessarily that $r(\delta) \geq r^*(\delta) > 0$. Further,

$$r^* = \left(\frac{\tau\delta}{\epsilon}\right)^{1/(\nu m)} = \left(\tau\delta \frac{c_0^{1+m}\kappa^m}{(\kappa+1)^m(1+M)\tilde{C}\left\|f^\delta\right\|_{C[0,1+R]}}\right)^{1/(\nu m)}.$$

□

2.2.3 Uniform Convergence

We now make more definite the choice of the regularization parameter r given by our discrepancy principle.

Definition 2.2 Discrepancy Principle for Local Regularization

Let $d : (0, R) \to [0, \infty)$ be the discrepancy functional defined by

$$d(r) := a_r^m \left\|A_r u_r^\delta - f_r^\delta\right\|_{L^2[0,1]}, \tag{2.64}$$

for $m \in (0, 1]$ fixed. Choose the regularization parameter $r = r(\delta)$ to be the smallest $r \in (0, R)$ so that

$$a_r^m \left\|A_r u_r^\delta - f_r^\delta\right\|_{L^2[0,1]} = \tau\delta. \tag{2.65}$$

Remark 2.4 *Any $r \in (0, \tilde{R})$ satisfying* (2.65) *would be acceptable.*

We now prove that local regularization with the discrepancy principle defined via equation (2.65) is a convergent regularization method for $f^\delta \in C[0, 1]$. For purposes of obtaining a rate of convergence, we make the additional smoothness assumption that $\bar{u}$ is uniformly Hölder continuous with power $\alpha \in (0, 1]$ and Hölder constant $L_{\bar{u}}$. Recall the definitions of $\tilde{R}$, $B(r)$, and $F(r)$ in (2.54), (2.56), and (2.58) respectively.

Theorem 2.2 *Assume that [A0]-[A2] hold and let $\bar{\delta}, \gamma_S > 0$ be given as in Lemma* 2.8. *For $\delta \in (0, \bar{\delta})$ or $\left\|f^\delta\right\| > \gamma_S\delta$, let u_r^δ denote the solution to equation* (2.18) *with f_r replaced by f_r^δ. Then for $r(\delta)$ selected according to the discrepancy principle in definition* 2.2, *we have*

1. $r(\delta) \to 0$ *as* $\delta \to 0$.
2. $\left\|u^\delta_{r(\delta)} - \bar{u}\right\|_{C[0,1]} \to 0$ *as* $\delta \to 0$.
3. *If $\bar{u}$ satisfies the condition* (2.43), *then*

$$\left\|u^\delta_{r(\delta)} - \bar{u}\right\|_{C[0,1]} = O\left(\delta^{m/m+1}\right) + O\left(\delta^{\alpha/(\nu(1+m))}\right) \quad \text{as } \delta \to 0,$$

and so the rate of convergence is determined by $\min\{\alpha, \nu m\}$.

If $\omega = \min\{\alpha, \nu m\}$, *then*

$$\left\|u^\delta_{r(\delta)} - \bar{u}\right\|_{C[0,1]} = O\left(\delta^{\omega/(\nu(1+m))}\right) \quad \text{as } \delta \to 0.$$

Moreover, if the choice of m is such that $m = \dfrac{\alpha}{\nu}$, *then*

$$\left\|u^\delta_{r(\delta)} - \bar{u}\right\|_{C[0,1]} = O\left(\delta^{\alpha/(\alpha+\nu)}\right) \quad \text{as } \delta \to 0,$$

which is identical to the rate of convergence obtained in Theorem 2.1 *using the a priori rule to select the parameter* $r = r(\delta)$.

Proof:

1. Let $\{\delta_n\}_{n \geq 1}$ be a positive sequence for which $\delta_n \to 0$ as $n \to \infty$, with $\delta_n \in (0, \bar{\delta})$ or $\left\| f^{\delta_n} \right\| > \gamma_s \delta_n$ for each n, and $\left\| f - f^{\delta_n} \right\|_{C[0,1+\bar{R}]} \leq \delta_n$ for each n. Let $\{r_n\}_{n \geq 1}$ be the corresponding sequence of regularization parameter values selected according to the discrepancy principle for local regularization given in definition 2.2, namely for each n,

$$r_n = r(\delta_n) = \min\left\{r \in (0, R) \mid d(r) = \tau\delta_n\right\}.$$

Using the lower bound on d in (2.57), we have

$$\begin{aligned}
\tau\delta_n &= d(r_n) \\
&\geq B(r_n)\left[\frac{\left\|f^{\delta_n}\right\|}{2} - \left(\bar{C} + \frac{3}{2}\right)\delta_n\right] \\
&\geq B(r_n)\left[\frac{\|f\|}{2} - \frac{\left\|f - f^{\delta_n}\right\|}{2} - \left(\bar{C} + \frac{3}{2}\right)\delta_n\right] \\
&\geq B(r_n)\left[\frac{\|f\|}{2} - (\bar{C} + 2)\,\delta_n\right],
\end{aligned} \tag{2.66}$$

and so

$$\left[\tau + B(r_n)\left(2 + \bar{C}\right)\right]\delta_n \geq B(r_n)\frac{\|f\|}{2}.$$

Then

$$0 = \lim_{n \to \infty} \delta_n \geq \liminf_{n \to \infty} \frac{B(r_n)}{\tau + B(r_n)\left(2 + \bar{C}\right)} \frac{\|f\|}{2} \geq 0.$$

We now claim that

$$\lim_{n \to \infty} r_n = 0.$$

By (2.61), $\|f\| > 0$, so this can only be true if

$$\lim_{n \to \infty} B(r_n) = 0.$$

Recalling the definition of $B(r_n)$ in (2.56), it must be that $\lim_{n \to \infty} a_{r_n} = 0$. Using the lower bound on a_r in Lemma 2.2, we have

$$0 = \lim_{n \to \infty} a_{r_n} \geq \frac{(\kappa - 1)}{\kappa c_0} r_n^{\nu} \geq 0$$

Therefore

$$\lim_{n \to \infty} r_n = 0.$$

2. Let r be chosen according to the discrepancy principle for local regularization given in definition 2.2, i.e.

$$r = r(\delta) = \min\left\{r \in (0, R) | \ d(r) = \tau\delta\right\}.$$

Let $\|\cdot\| = \|\cdot\|_{L^2[0,1]}$. Using (2.53), our choice of $r = r(\delta)$ is such that

$$a_{r(\delta)}^{1+m} \left\| u_{r(\delta)}^{\delta} \right\| = \tau\delta.$$

and so representing $u_{r(\delta)}^{\delta}$ as in (2.19), we obtain by substituting into (2.47),

$$\begin{aligned}
\left\| u_{r(\delta)}^{\delta} \right\| - \left\| u_{r(\delta)} \right\| &\leq \left\| u_{r(\delta)}^{\delta} - u_{r(\delta)} \right\| \\
&\leq C_1 \frac{\delta}{[r(\delta)]^{\nu}} \\
&= \frac{C_1}{[r(\delta)]^{\nu}} \cdot \frac{1}{\tau} a_{r(\delta)}^{1+m} \left\| u_{r(\delta)}^{\delta} \right\| \\
&\leq \frac{C_1}{[r(\delta)]^{\nu}} \left(\frac{\kappa+1}{\kappa c_0} \right)^{+m} \frac{1}{\tau} [r(\delta)]^{\nu(1+m)} \left\| u_{r(\delta)}^{\delta} \right\| \\
&= \hat{C}_1 [r(\delta)]^{\nu m} \left\| u_{r(\delta)}^{\delta} \right\|,
\end{aligned}$$

for $\hat{C}_1 > 0$ constant. Using Theorem 2.1 and the first part of the theorem where we proved that $\lim_{\delta \to 0} r(\delta) = 0$, we have

$$\begin{aligned}
\left\| u_{r(\delta)} \right\| &\leq \left\| u_{r(\delta)} - \bar{u} \right\| + \|\bar{u}\| \\
&\leq \frac{\|\bar{u}\|}{2} + \|\bar{u}\|.
\end{aligned}$$

for $\delta > 0$ sufficiently small. Therefore

$$\left\| u_{r(\delta)}^{\delta} \right\| \leq \hat{C}_1 [r(\delta)]^{\nu m} \left\| u_{r(\delta)}^{\delta} \right\| + \frac{3}{2} \|\bar{u}\|.$$

and thus

$$\limsup_{\delta \to 0} \left\| u^{\delta}_{r(\delta)} \right\| \leq \frac{3}{2} \|\bar{u}\|. \tag{2.67}$$

Substituting the principle in for δ into (2.48), we have

$$\begin{aligned} \left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{C[0,1]} &\leq C_1 \frac{\delta}{[r(\delta)]^{\nu}} + \left\| u_{r(\delta)} - \bar{u} \right\| \\ &\leq \hat{C}_1 [r(\delta)]^{\nu m} \left\| u^{\delta}_{r(\delta)} \right\| + \left\| u_{r(\delta)} - \bar{u} \right\|. \end{aligned}$$

Then by (2.67) and part 1 of the theorem, it follows that

$$\lim_{\delta \to 0} \left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{C[0,1]} \leq \lim_{\delta \to 0} \left(\hat{C}_1 [r(\delta)]^{\nu m} \left\| u^{\delta}_{r(\delta)} \right\| + \left\| u_{r(\delta)} - \bar{u} \right\| \right) = 0,$$

proving $u^{\delta}_{r(\delta)}$ converges uniformly to $\bar{u}$ on $[0, 1]$ as $\delta \to 0$.

3. Returning to (2.48), we have

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{C[0,1]} \leq C_1 \frac{\delta}{[r(\delta)]^{\nu}} + C_2 [r(\delta)]^{\alpha}.$$

To obtain a rate of convergence, it remains to bound $\dfrac{1}{[r(\delta)]^{\nu}}$ and $[r(\delta)]^{\alpha}$ in terms of δ using our rule. First we bound $[r(\delta)]^{\alpha}$ in terms of δ.

Using the upper bound on a_r in Lemma 2.2 to bound $B(r)$ in (2.56), we have for all $r \in (0, R]$

$$B(r) \leq \left(\frac{\kappa + 1}{\kappa c_0} \right)^m r^{\nu m},$$

and so it follows from the lower bound on d in (2.66),

$$\begin{aligned}
\tau\delta &= d(r(\delta)) \\
&\geq B(r(\delta))\left\|f^{\delta}_{r(\delta)}\right\| \\
&\geq B(r(\delta))\frac{\|f\|}{2} - B(r(\delta))\left(2+\bar{C}\right)\delta \\
&\geq B(r(\delta))\frac{\|f\|}{2} - \left(\frac{\kappa+1}{\kappa c_0}\right)^{m}\tilde{R}^{\nu}\left(2+\bar{C}\right)\delta.
\end{aligned} \tag{2.68}$$

Thus,

$$(\tau + E_1)\delta \geq B(r(\delta))\frac{\|f\|}{2}, \tag{2.69}$$

for the constant $E_1 > 0$. Now using both bounds on a_r in Lemma 2.2 to bound $B(r)$ in (2.56), we obtain for all $r \in (0, R)$,

$$B(r) \geq \left(\frac{\kappa-1}{\kappa c_0}\right)^{1+m} r^{\nu(1+m)}\left(\frac{\kappa+1}{\kappa c_0}\tilde{R}^{\nu} + \bar{C}\,\|k\|_{C[0,1+R]}\right)^{-1}.$$

Therefore,

$$B(r(\delta)) \;\geq\; E_2\,[r(\delta)]^{\nu(1+m)}, \tag{2.70}$$

for the constant $E_2 > 0$. Combining (2.69) and (2.70), we have

$$\frac{2(\tau+E_1)}{E_2\,\|f\|}\delta \;\geq\; [r(\delta)]^{\nu(1+m)},$$

and so raising both sides to the $\dfrac{\alpha}{\nu(1+m)}$ power,

$$[r(\delta)]^{\alpha} \leq \tilde{E}\delta^{\alpha/(\nu(1+m))}, \tag{2.71}$$

for $\tilde{E} > 0$ constant.

Next we obtain the bound for $\dfrac{1}{[r(\delta)]^{\nu}}$ in terms of δ. Bounding d above using the inequality in (2.53), Lemma (2.2), and [A0], we have

$$\begin{aligned}
\tau\delta &= a_{r(\delta)}^{1+m}\left\|u_{r(\delta)}^{\delta}\right\| \\
&\leq a_{r(\delta)}^{1+m}\left[\left\|u_{r(\delta)}^{\delta} - u_{r(\delta)}\right\| + \left\|u_{r(\delta)} - \bar{u}\right\| + \|\bar{u}\|\right] \\
&\leq a_{r(\delta)}^{m}(1+M)\bar{C}\delta + a_{r(\delta)}^{1+m}\left[C_2\,[r(\delta)]^{\alpha} + \|\bar{u}\|\right] \\
&\leq \left(\frac{(\kappa+1)\,[r(\delta)]^{\nu}}{\kappa c_0}\right)^{m}(1+M)\bar{C}\delta \\
&\quad + \left(\frac{(\kappa+1)\,[r(\delta)]^{\nu}}{\kappa c_0}\right)^{1+m}\left(C_2\hat{R}^{\alpha} + \|\bar{u}\|\right) \\
&= G_1\,[r(\delta)]^{\nu m}\,\delta + G_2\,[r(\delta)]^{\nu(1+m)},
\end{aligned}$$

for $G_1 > 0$ and $G_2 > 0$ constants. Since $\tau - G_1\,[r(\delta)]^{\nu m} > \tau - G_1\hat{R}^{\nu m} > 0$ for $r(\delta) \in (0, \hat{R}]$ for some $\hat{R}$ sufficiently small, we have that

$$\frac{\tau - G_1\hat{R}^{\nu m}}{G_2}\delta \;\leq\; [r(\delta)]^{\nu(1+m)},$$

for $\delta >$ sufficiently small so that $r(\delta) \in (0, \tilde{R})$. Then

$$\frac{1}{[r(\delta)]^{\nu}} \leq \tilde{G}\delta^{-1/(1+m)}, \tag{2.72}$$

for $\tilde{G} > 0$ constant and for $\delta > 0$ sufficiently small.

Substituting (2.71) and (2.72) into (2.48), we have

$$\begin{aligned} \left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{C[0,1]} &\leq C_1 \frac{\delta}{[r(\delta)]^{\nu}} + C_2 [r(\delta)]^{\alpha} \\ &\leq C_1 \delta \tilde{G} \delta^{-1/(1+m)} + C_2 \tilde{E} \delta^{\alpha/(\nu(1+m))}, \end{aligned}$$

and so

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{m/(m+1)}\right) + O\left(\delta^{\alpha/(\nu(1+m))}\right) \text{ as } \delta \to 0.$$

If $\omega = \min\{\alpha. \nu m\}$, then

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{\omega/(\nu(1+m))}\right) \quad \text{as} \quad \delta \to 0.$$

Then taking $m = \dfrac{\alpha}{\nu}$, it follows that

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{\alpha/(\alpha+\nu)}\right) \quad \text{as} \quad \delta \to 0.$$

□

CHAPTER 3

EXTENSIONS OF THE THEORY OF LOCAL REGULARIZATION

The purpose of this chapter is to extend the theory of local regularization to solving the ν-smoothing problem in the case when the true solution is no longer continuous, but instead contained in the space $L^p[0,1]$, for some $1 < p < \infty$.

In this chapter, we take the operator $A : L^p[0,1] \to L^p[0,1]$ to be defined by

$$Au(t) := \int_0^t k(t-s)u(s)ds, \qquad \text{a.e.} \qquad t \in [0,1], \tag{3.1}$$

where the kernel $k \in C^\nu[0,1]$ is ν-smoothing, $\nu \geq 1$, and it is assumed without loss of generality that $k^{(\nu-1)}(0) = 1$. We would now like to solve $Au = f$ for $u \in L^p[0,1]$ with $f \in R(A) \subseteq L^p[0,1]$. An argument may be made similar to the one in Chapter 2 to show that

$$R(A) = G_{\nu,p} := \left\{ g \in W^{\nu,p}[0,1] | g(0) = g'(0) = \cdots = g^{(\nu-1)}(0) = 0 \right\},$$

and that the $N(A) = \{0\}$. Since $\dim R(A) = \infty$ and A is compact, it follows that

A^{-1} is unbounded and the problem is ill-posed.

We begin our extension of the ideas in [22] by rederiving the second kind equation associated with local regularization for the case when the underlying space is $L^p[0, 1]$, $1 < p < \infty$. Modifying the conditions in [22], assumptions [A0] and [A1] are redefined to be [A0] and [A1-p]. We again restrict the choice of regularization parameter to an interval $(0, R]$ over which $R\|k^{(\nu)}\|_{C[0, R]}$ is sufficiently small so that the resolvent of the approximating equation remains uniformly bounded for all $r \in (0, R]$. We prove that these conditions are sufficient to guarantee convergence in $L^p[0, 1]$, $1 < p < \infty$, of regularized approximations to the true solution $\bar{u} \in L^p[0, 1]$ in the noise free-case. Given noisy data $f^\delta \in L^p[0, 1]$, we provide an *a priori* parameter selection strategy and determine a rate of L^p-convergence of approximations to $\bar{u}$ satisfying the source condition of uniform Hölder continuity. In doing so, we complete our generalization of [22].

Next, we once again apply the normalized local regularization scheme satisfying [A0] and [A1-p] to our problem, define the additional assumption [A2], redefine our discrepancy principle in this new context and establish its properties. We show that approximations, constructed from noisy data $f^\delta \in L^p[0, 1]$, $1 < p < \infty$ using a local regularization scheme satisfying [A0], [A1-p], and [A2], and the new discrepancy principle to select the regularization parameter, converge in $L^p[0, 1]$, $1 < p < \infty$ to the true solution $\bar{u} \in L^p[0, 1]$ as the noise level shrinks to zero. We also give a rate of convergence for $\bar{u}$ uniformly Hölder continuous. We conclude that local regularization paired with the redefined discrepancy principle is a convergent regularization method.

3.1 Extensions to $L^p[0,1], 1 < p < \infty$

Again, let $\bar{u}$ denote the "true" solution given "exact" data f and let f^δ denote the "noisy" version of f. We begin with an assumption on the data.

[A0] Let $0 < \bar{R} << 1$ be such that $\bar{u} \in L^p[0, 1 + \bar{R}]$ and $f \in R(A) \subseteq L^p[0, 1 + \bar{R}]$ so that $A\bar{u} = f$ for a.e. t on the interval $[0, 1 + \bar{R}]$. Given $\delta > 0$, the data $f^\delta(t)$ is available for a.e. $t \in [0, 1 + \bar{R}]$ and $f^\delta \in L^p[0, 1 + \bar{R}]$ satisfies

$$\left\| f - f^\delta \right\|_{L^p[0,\,1+\bar{R}]} \leq \delta.$$

If additional data is unavailable, we suffice with approximating $\bar{u}$ on the slightly smaller interval $\left[0, 1 - \bar{R}\right]$.

3.1.1 The Approximating Equation

As in [22], it follows from [A0] that $\bar{u}$ satisfies

$$\int_0^\rho k(\rho - s)u(t + s)ds + \int_0^t k(t + \rho - s)u(s)ds = f(t + \rho), \tag{3.2}$$

for a.e. $t \in [0, 1]$, $\rho \in [0, r]$ and for any $r \in (0, \bar{R}]$.

For each $r \in (0, \bar{R}]$, consider the space of all bounded linear functionals on $L^p[0, r]$, $1 < p < \infty$. Recall that the continuous dual space of $L^p[0, r]$ can be identified with the space $L^q[0, r]$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\Omega_r \in [L^p[0, r]]^*$, there exists

$\psi_r \in L^q[0, r]$ such that

$$\Omega_r(g) = \int_0^r g(\rho)\psi_r(\rho)d\rho,$$

for any $g \in L^p[0, r]$. We define the measure η_r by

$$\int_0^r g(\rho)d\eta_r(\rho) := \int_0^r g(\rho)\psi_r(\rho)d\rho, \tag{3.3}$$

for $g \in L^p[0, r]$, where $d\rho$ denotes Lebesgue measure.

Applying a functional Ω_r, we integrate both sides of equation (3.2) with respect to the measure η_r and obtain

$$\int_0^r \int_0^\rho k(\rho-s)u(t+s)dsd\eta_r(\rho) + \int_0^t \int_0^r k(t+\rho-s)d\eta_r(\rho)u(s)ds = \int_0^r f(t+\rho)d\eta_r(\rho), \tag{3.4}$$

which $\bar{u}$ satisfies for a.e. $0 < r \leq \bar{R}$ and a.e. $t \in [0, 1]$.

With the idea of holding u constant locally, we consider the *second-kind Volterra equation*

$$u(t)\int_0^r \int_0^\rho k(\rho - s)dsd\eta_r(\rho) + \int_0^t \int_0^r k(t + \rho - s)d\eta_r(\rho)u(s)ds = \int_0^r f(t + \rho)d\eta_r(\rho), \tag{3.5}$$

for a.e. $t \in [0, 1]$ and $r \in (0, \bar{R}]$.

3.1.2 Properties of η_r

We again specify how to select a family of measures η_r, $r \in (0, \bar{R}]$, in the approximating equation (3.5), redefining [A1] to be [A1-p].

[A1-p] The measure η_r is chosen to satisfy the following hypotheses.

(H1) For $i = 0, 1, ..., \nu$, there is some $\sigma \in \mathbb{R}$ and $c_i = c_i(\nu) \in \mathbb{R}$ independent of r such that

$$\int_0^r \rho^i d\eta_r(\rho) = r^{i+\sigma} c_i,$$

with $c_\nu \neq 0$. Without loss of generality, we may assume that η_r is scaled so that $c_\nu = \nu!$.

(H2) The parameters c_i, $i = 0, 1, ..., \nu$, satisfy the condition that the roots of the polynomial $p_\nu(\lambda)$ defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + ... + \frac{c_1}{1!}\lambda + \frac{c_0}{0!}$$

have negative real part.

(H3) There exists $\tilde{C} > 0$ independent of r such that

$$\left| \int_0^r h(\rho) d\eta_r(\rho) \right| \leq \|h\|_{C[0,r]} \tilde{C} r^\sigma,$$

for each $h \in C[0, r]$ and any $r \in (0, \bar{R}]$, and there exists $\tau_p < 1$, independent of r, such that

$$\left| \int_0^r h(\rho) d\eta_r(\rho) \right| \leq \|h\|_{L^p[0,r]} \tilde{C} r^{\sigma - \tau_p},$$

for each $h \in L^p[0, r]$, $1 < p < \infty$ and any $r \in (0, \bar{R}]$.

Remark 3.1 *If the function to which the measure η_r is applied is in $C[0,r]$, then (H3) of [A1-p] coincides with (H3) of [A1] in Chapter 2. However, if the measure η_r is applied to a function that is only contained in $L^p[0,r]$, then the bound in (H3) of [A1-p] requires the factor $r^{-\tau_p}$.*

We give an example of a class of measures (similar to the one defined in Lemma 2.2 of [22]) that can be constructed to satisfy assumption [A1-p].

Lemma 3.1 *Let $\nu = 1, 2, \cdots$, be arbitrary and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, for each $1 < p < \infty$. Let $\psi \in L^q[0,1]$ be such that $\int_0^1 \rho^\nu \psi(\rho)d\rho = \nu!$. Then for $r \in (0, \bar{R}]$, the measure η_r defined by*

$$\int_0^r g(\rho)d\eta_r(\rho) = \int_0^r g(\rho)\psi_r(\rho)d\rho, \quad g \in L^p[0,r],$$

where $\psi_r \in L^q[0,r]$ is given by

$$\psi_r(\rho) = \psi\left(\frac{\rho}{r}\right), \quad a.e.\ \rho \in [0,r],$$

satisfies condition (H1) (with $c_\nu = \int_0^1 \rho^\nu \psi(\rho)d\rho$ and $\sigma = 1$) and condition (H3) (with $\tilde{C} = \|\psi\|_{L^q[0,1]}$ and $\tau_p = \frac{1}{p}$). Further, for all $\nu = 1, 2, \cdots$, and given arbitrary positive $m_1, m_2, \cdots$, and m_ν, there is a unique monic polynomial ψ of degree ν so that the resulting family $\{\eta_r\}$ satisfies (H1) with $c_\nu = \nu!$ and $\sigma = 1$, (H2) with the roots of the polynomial p_ν in (H2) given by $(-m_i), i = 1, \cdots, \nu$ and (H3).

Proof: The proper construction of the measure η_r satisfying (H1) and (H2) can be handled as in Lemma 2.2 of [22], namely for $i = 0, 1, \ldots, \nu$,

$$\begin{aligned}
\int_0^r \rho^i d\eta_r(\rho) &= \int_0^r \rho^i \psi\left(\frac{\rho}{r}\right) d\rho \\
&= r \int_0^1 (r\rho)^i \psi(\rho)\, d\rho \\
&= r^{i+1} \int_0^1 \rho^i \psi(\rho)\, d\rho \\
&= r^{i+1} c_i.
\end{aligned}$$

Then (H1) holds with $\int_0^1 \rho^i \psi(\rho)\, d\rho = c_i \in \mathbb{R}$, independent of $r \in (0, R]$, and $\sigma = 1$. In order to satisfy (H2), one can always construct a νth degree polynomial ψ for which $\int_0^1 \rho^i \psi(\rho)\, d\rho = c_i$, $i = 0, 1, \ldots, \nu$, where the c_i are determined by the choice of roots of $p_\nu(\lambda)$ and $\displaystyle\int_0^1 \rho^\nu \psi(\rho) d\rho = c_\nu = \nu!$.

To show that (H3) is satisfied for such $\psi \in L^q[0,1]$, consider $g \in C[0,r]$. Then

$$\begin{aligned}
\left|\int_0^r g(\rho) d\eta_r(\rho)\right| &= \left|\int_0^r g(\rho)\psi\left(\frac{\rho}{r}\right) d\rho\right| \\
&\leq \|g\|_{C[0,r]} \int_0^r \left|\psi\left(\frac{\rho}{r}\right)\right| d\rho \\
&= \|g\|_{C[0,r]}\, r \int_0^1 |\psi(\rho)|\, d\rho \\
&= \|g\|_{C[0,r]} \|\psi\|_{L^1[0,1]}\, r \\
&\leq \|g\|_{C[0,r]} \|\psi\|_{L^q[0,1]}\, r.
\end{aligned}$$

If $g \in L^p[0, r]$, then applying Hölder's inequality,

$$\begin{aligned}
\int_0^r g(\rho)d\eta_r(\rho) &= \int_0^r g(\rho)\psi\left(\frac{\rho}{r}\right) d\rho \\
&\leq \left(\int_0^r |g(\rho)|^p \, d\rho\right)^{1/p} \left(\int_0^r \left|\psi\left(\frac{\rho}{r}\right)\right|^q d\rho\right)^{1/q} \\
&= \|g\|_{L^p[0, r]} \left(r\int_0^1 |\psi(\rho)|^q \, d\rho\right)^{1/q} \\
&= \|g\|_{L^p[0, r]} \|\psi\|_{L^q[0, 1]} \, r^{1/q} \\
&= \|g\|_{L^p[0, r]} \|\psi\|_{L^q[0, 1]} \, r^{1 - 1/p}.
\end{aligned}$$

For $\sigma = 1$, (H3) holds with $\tilde{C} = \|\psi\|_{L^q[0, 1]}$ and $\tau_p = \dfrac{1}{p}$. □

For $r \in (0, \bar{R}]$, assume η_r is any measure satisfying [A1-p] and define

$$\gamma_r := \int_0^r d\eta_r(\rho). \tag{3.6}$$

As described in Section 2.1.1, it follows from (H1) and (H2) that

$$\gamma_r = r^{\sigma} c_0 > 0. \tag{3.7}$$

for all $r > 0$. Then we may define for any $u \in L^p[0, 1]$ and $v \in L^p[0, 1 + \bar{R}]$,

$$a_r := \frac{\int_0^r \int_0^\rho k(\rho - s)ds \; d\eta_r(\rho)}{\gamma_r}, \tag{3.8}$$

$$k_r(t) := \frac{\int_0^r k(t + \rho)d\eta_r(\rho)}{\gamma_r}. \tag{3.9}$$

$$A_r u(t) := \int_0^t k_r(t-s)\; u(s)ds, \tag{3.10}$$

$$f_r(t) := \frac{\int_0^r f(t+\rho)\; d\eta_r(\rho)}{\gamma_r}, \tag{3.11}$$

$$D_r v(t) := \frac{\int_0^r \int_0^\rho k(\rho-s)v(t+s)dsd\eta_r(\rho)}{\gamma_r}, \tag{3.12}$$

for a.e. $t \in [0,1]$ and each $r \in (0, \bar{R}]$. With this notation and recalling that $\bar{u}$ satisfies (3.4), equations (3.4) and (3.5) may be written equivalently as the normalized equations

$$D_r\bar{u}(t) + A_r\bar{u}(t) = f_r(t), \qquad \text{a.e. } t \in [0,1], \tag{3.13}$$

and

$$a_r u_r(t) + A_r u_r(t) = f_r(t), \qquad \text{a.e. } t \in [0,1], \tag{3.14}$$

respectively.

Provided $a_r \neq 0$, equation (3.14) is well-posed and there exists a unique solution $u_r \in L^p[0,1], 1 < p < \infty$, that depends continuously on $f \in L^p[0,1]$ [13]. For each $r \in (0, \bar{R}]$ for which $a_r \neq 0$, $(a_r I + A_r)^{-1}$ is a bounded linear operator on $L^p[0,1]$ [13], so that we may represent the solution as

$$u_r = (a_r I + A_r)^{-1} f_r. \tag{3.15}$$

We may also express u_r using the *variation of constants formula* in [3],

$$u_r(t) = \frac{f_r(t)}{a_r} - \int_0^t \mathcal{X}_r(t-s)\frac{f_r(s)}{a_r}ds, \tag{3.16}$$

for a.e. $t \in [0, 1]$ and $r \in (0, \bar{R}]$ for which $a_r \neq 0$. Exactly as in Chapter 2, for each such $r > 0$, the resolvent kernel $\mathcal{X}_r \in L^1[0, 1]$ uniquely satisfies

$$\mathcal{X}_r(t) + \int_0^t \frac{k_r(t-s)}{a_r} \mathcal{X}_r(s) ds = \frac{k_r(t)}{a_r}, \tag{3.17}$$

for a.e. $t \in [0, 1]$.

As in the $C[0, 1]$ case, to guarantee well-posedness of equation (3.14) for all $r \in (0, R]$, for some $0 < R \leq \bar{R}$, we must ensure that a_r does not vanish on the interval $(0, R]$. The following lemma gives a condition under which this is true based on assumptions [A0] and [A1-p].

Lemma 3.2 *Assume [A0] and [A1-p] are satisfied.*

1. *Let a_r be as defined in* (3.8)*. If R and k satisfy*

$$R \left\| k^{(\nu)} \right\|_{C[0,\, R]} \leq \frac{(\nu + 1)!}{\tilde{C}\kappa}, \tag{3.18}$$

for some $\kappa > 1$ and $0 < R \leq \bar{R}$, then

$$\frac{\kappa - 1}{\kappa c_0} \cdot r^{\nu} \leq a_r \leq \frac{\kappa + 1}{\kappa c_0} \cdot r^{\nu}. \tag{3.19}$$

for all $r \in (0, R]$.

2. *Let $h \in C[0, 1 + \bar{R}]$ and define*

$$h_r(t) := \frac{\int_0^r h(t + \rho) d\eta_r(\rho)}{\gamma_r},$$

for all $t \in [0, 1]$. Then

$$\lim_{r \to 0} \|h_r - h\|_{L^p[0,1]} = 0. \tag{3.20}$$

3. *Let A_r be as defined in* (3.10) *for $r \in (0, \bar{R}]$. Then*

$$\lim_{r \to 0} \|A_r - A\| = 0, \tag{3.21}$$

where $\|\cdot\|$ is the operator norm on $\mathcal{B}(L^p[0,1])$, $1 < p < \infty$.

Proof:

1. It was established in the proof of Lemma 2.2 that

$$a_r = \frac{1}{r^{\sigma} c_0}\left[\frac{c_\nu}{\nu!} r^{\nu+\sigma} + \int_0^r \int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!} ds d\eta_r(\rho)\right].$$

Since

$$\rho \mapsto \int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!} ds \in C[0, r],$$

it follows from (H3) of [A1-p] and the hypothesis of the lemma that

$$\int_0^r \int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!} ds d\eta_r(\rho) \leq \frac{r^{\nu+1}}{(\nu+1)!}\left\|k^{(\nu)}\right\|_{C[0,r]} \tilde{C} \cdot r^{\sigma} \leq \frac{r^{\nu+\sigma}}{\kappa},$$

therefore the bounds on a_r may be established exactly as in the proof of Lemma 2.2.

2. Since h is continuous, it follows from (H3) that for each $t \in [0, 1]$,

$$\left| \int_0^r [h(t+\rho) - h(t)]\, d\eta_r(\rho) \right| \leq \|h(t+\cdot) - h(t)\|_{C[0,\, r]}\, \tilde{C} r^{\sigma}.$$

Therefore,

$$\begin{aligned}
\|h_r - h\|_{L^p[0,\, 1]} &= \left(\int_0^1 \left| \frac{\int_0^r h(t+\rho) d\eta_r(\rho)}{\gamma_r} - h(t) \right|^p dt \right)^{1/p} \\
&= \left(\int_0^1 \left| \frac{\int_0^r (h(t+\rho) - h(t))\, d\eta_r(\rho)}{r^{\sigma} c_0} \right|^p dt \right)^{1/p} \\
&\leq \bar{C} \left(\int_0^1 \|h(t+\cdot) - h(t)\|_{C[0,\, r]}^p\, dt \right)^{1/p} \\
&\leq \bar{C} \sup_{t \in [0,\, 1]} \|h(t+\cdot) - h(t)\|_{C[0,\, r]}, \qquad (3.22)
\end{aligned}$$

and so

$$\lim_{r \to 0} \|h_r - h\|_{L^p[0,\, 1]} \leq \bar{C} \lim_{r \to 0} \sup_{t \in [0,\, 1]} \|h(t+\cdot) - h(t)\|_{C[0,\, r]} = 0.$$

3. Let $x \in L^p[0, 1]$. Recalling the notation in (3.9),

$$\begin{aligned}
\|A_r - A\| &= \sup_{\|x\|_{L^p[0,\, 1]} = 1} \|(A_r - A)x\|_{L^p[0,\, 1]} \\
&= \sup_{\|x\|_{L^p[0,\, 1]} = 1} \left\| \int_0^{\cdot} (k_r(\cdot - s) - k(\cdot - s))\, x(s) ds \right\|_{L^p[0,\, 1]} \\
&\leq \sup_{\|x\|_{L^p[0,\, 1]} = 1} \|k_r - k\|_{L^q[0,\, 1]} \|x\|_{L^p[0,\, 1]} \\
&= \|k_r - k\|_{L^q[0,\, 1]}.
\end{aligned}$$

Since $k \in C[0, 1 + \bar{R}]$ and $q \in (1, \infty)$, as in (3.22) in the previous part of the lemma, we have that

$$\|k_r - k\|_{L^q[0,1]} \leq \bar{C} \lim_{r \to 0} \sup_{t \in [0,1]} \|k(t + \cdot) - k(t)\|_{C[0,r]},$$

and so

$$\lim_{r \to 0} \|A_r - A\| = 0.$$

$\square$

In Lemma 3.2, we established again that

$$R \left\| k^{(\nu)} \right\|_{C[0,R]} \leq \frac{(\nu + 1)!}{\bar{C}_K}$$

is a sufficient condition so that $a_r \neq 0$ for all $r \in (0, R]$, for some $R \in (0, \bar{R}]$. We obtained the same bounds on a_r for the L^p case as in the continuous case; a direct result of (H3) in [A1-p] and the fact that the functions being integrated with respect to the measure η_r, namely $\int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!}ds$, were continuous. The implication then is that all of the estimates used to prove Lemmas 2.3 and 2.4 require η_r to be applied to a continuous function, and so these estimate will be exactly the same for *any* measure η_r now satisfying [A1-p]. Therefore we restate Lemmas 2.3 and 2.4 replacing condition [A1] with [A1-p] for completeness, with proofs the same as those for Lemmas 2.3 and 2.4.

Lemma 3.3 *Assume [A1-p] holds and that for some $0 < R \leq \bar{R}$, R and k satisfy*

$$R \left\| k^{(\nu)} \right\|_{C[0,\, R]} \leq \frac{(\nu + 1)!}{\tilde{C}\kappa},$$

for some $\kappa \geq \bar{\kappa}$ where $\bar{\kappa} > 1$ is sufficiently large. Then the eigenvalues of $A_r := A + M_r$ have negative real part for all $r \in (0, R]$, where

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_0/0! & -c_1/1! & \cdots & \cdots & -c_{\nu - 1}/(\nu - 1)! \end{pmatrix},$$

$$M_r = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{m}_{0,\, r} & -\bar{m}_{1,\, r} & \cdots & \cdots & -\bar{m}_{\nu - 1,\, r} \end{pmatrix},$$

and

$$\bar{m}_{j,\, r} = \frac{1}{a_r \gamma_r} \left[r^{\nu - j} \int_0^r k^{(\nu)}(\hat{\zeta}_j) \frac{\rho^{j + 1}}{(j + 1)!} d\eta_r(\rho) - \frac{c_j}{j!} \int_0^r \int_0^\rho k^{(\nu)}(\zeta) \frac{s^\nu}{\nu!} ds d\eta_r(\rho) \right],$$

for $j = 0, 1, \cdots, \nu - 1$.

Further, for the matrix

$$B_r \equiv \int_0^\infty (\exp(A_r t))^T \exp(A_r))dt,$$

there exist positive constants L, K, and S so that

$$|x| \geq 2L\left(x^T B_r x\right)^{\frac{1}{2}}$$
$$|B_r x| \leq K\left(x^T B_r x\right)^{\frac{1}{2}} \tag{3.23}$$
$$|x| \leq S\left(x^T B_r x\right)^{\frac{1}{2}},$$

for all $x \in \mathbb{R}^\nu$ and all $r \in (0, R]$, where $|\cdot|$ denotes the usual norm on $\mathbb{R}^\nu$.

Lemma 3.4 *Assume the hypotheses of Lemma 3.3 hold. Then there exist constants $\hat{C} > 0$ and $M > 0$, independent of r (but dependent on $k, \nu, c_0, c_1, \cdots, c_\nu$), such that if*

$$\left\|k^{(\nu)}\right\|_{C[0,\, 1+\bar{R}]} \leq \hat{C},$$

then we have

$$\|\mathcal{X}_r\|_{L^1[0,\, 1]} \leq M.$$

for all $r \in (0, R]$, where $\mathcal{X}_r$ is the resolvent defined in equation (3.17).

We have the following estimate on the size of $\left\|(a_r I + A_r)^{-1}\right\|$.

Corollary 3.1 *Assume that R and k are such that Lemmas 3.3 and 3.4 are satisfied. Define*

$$M = \inf\left\{\mu \mid \|\mathcal{X}_r\|_{L^1[0,\, 1]} \leq \mu \quad \text{for all} \quad r \in (0, R]\right\}. \tag{3.24}$$

For each $r \in (0, R]$,

$$\left\|(a_r I + A_r)^{-1}\right\| \leq \frac{1+M}{a_r}. \tag{3.25}$$

where $\|\cdot\|$ *is the operator norm on* $\mathcal{B}(L^p[0,1])$, $1 < p < \infty$.

Proof: Representing u_r using (3.16), we have for each $r \in (0, R]$ [13],

$$
\begin{aligned}
\|u_r\|_{L^p[0,1]} &= \left\| \frac{f_r}{a_r} - \mathcal{X}_r * \frac{f_r}{a_r} \right\|_{L^p[0,1]} \\
&\le \left\| \frac{f_r}{a_r} \right\|_{L^p[0,1]} + \|\mathcal{X}_r\|_{L^1[0,1]} \left\| \frac{f_r}{a_r} \right\|_{L^p[0,1]} \\
&\le (1+M) \left\| \frac{f_r}{a_r} \right\|_{L^p[0,1]}.
\end{aligned}
$$

Therefore representing u_r using (3.15),

$$
\left\| (a_r I + A_r)^{-1} f_r \right\|_{L^p[0,1]} = \|u_r\|_{L^p[0,1]} \le \frac{1+M}{a_r} \|f_r\|_{L^p[0,1]}.
$$

and thus

$$
\left\| (a_r I + A_r)^{-1} \right\| \le \frac{1+M}{a_r},
$$

for all $r \in (0, R]$. □

3.1.3 L^p-Convergence with *A Priori* Parameter Selection

We obtain the following convergence results in the case when $\bar{u} \in L^p[0, 1+\bar{R}]$. Given noisy data $f^\delta \in L^p[0, 1+\bar{R}]$, $1 < p < \infty$, we provide an *a priori* rule for which approximations converge to the true solution in $L^p[0,1]$. For purposes of obtaining a rate, we take as our source condition that $\bar{u}$ is uniformly Hölder continuous with power $\alpha \in (0, 1]$ and Hölder constant $L_{\bar{u}}$.

Theorem 3.1 *Assume that [A0] and [A1-p] hold and k satisfies Lemma* 3.4.

1. *Let u_r denote the solution of equation* (3.14) *for $r \in (0, R]$, for R sufficiently small. If the true solution $\bar{u} \in L^p[0, 1 + \bar{R}]$, then*

$$\|u_r - \bar{u}\|_{L^p[0,1]} \to 0 \quad as \quad r \to 0.$$

Moreover, if the true solution $\bar{u}$ satisfies (2.43) , *then*

$$\|u_r - \bar{u}\|_{L^p[0,1]} = O\left(r^{\alpha}\right) \quad as \quad r \to 0.$$

2. *Let u_r^{δ} denote the solution to equation* (3.14) *with f_r replaced by f_r^{δ} for $r \in (0, R]$. If the true solution $\bar{u} \in L^p[0, 1 + \bar{R}]$, then with τ_p given in (H3),*

$$\left\|u_{r(\delta)}^{\delta} - \bar{u}\right\|_{L^p[0,1]} \le C_1 \frac{\delta}{r^{\nu + \tau_p}} + \|u_r - \bar{u}\|_{L^p[0,1]},$$

for some $C_1 \ge 0$, so that a choice of $r(\delta)$ satisfying

i) $r(\delta) \to 0$ *as* $\delta \to 0$, *and*

ii) $\delta\left[r(\delta)\right]^{-(\nu + \tau_p)} \to 0$ *as* $\delta \to 0$,

ensures

$$\left\|u_{r(\delta)}^{\delta} - \bar{u}\right\|_{L^p[0,1]} \to 0 \quad as \quad \delta \to 0.$$

If $\bar{u}$ satisfies (2.43) , *then*

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{L^p[0,\,1]} \leq C_1 \frac{\delta}{r^{\nu + \tau p}} + C_2 r^{\alpha},$$

for some $C_1, C_2 \geq 0$, and so for any $K > 0$, the choice

$$r = r(\delta) = K \delta^{1/(\alpha + \nu + \tau p)},$$

gives

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{L^p[0,\,1]} = O\left(\delta^{\alpha/(\nu + \alpha + \tau p)}\right) \quad as \quad \delta \to 0. \tag{3.26}$$

Proof: Let $\|\cdot\| = \|\cdot\|_{L^p[0,\,1]}$.

1. We bound the error due to regularization using (3.13), (3.14), (3.15), and (3.19) to obtain

$$\begin{aligned} \|u_r - \bar{u}\| &= \left\| (a_r I + A_r)^{-1} \left[f_r - a_r \bar{u} - A_r \bar{u} \right] \right\| \\ &\leq \frac{(1 + M)}{a_r} \left\| D_r \bar{u} - a_r \bar{u} \right\| \\ &\leq \frac{\kappa c_0 (1 + M)}{(\kappa - 1) r^{\nu}} \left\| D_r \bar{u} - a_r \bar{u} \right\|. \end{aligned} \tag{3.27}$$

Notice that

$$\rho \mapsto \int_0^{\rho} k(\rho - s) \left[\bar{u}(t + s) - \bar{u}(t) \right] ds \in C[0, r],$$

and recall the respective definitions of a_r and D_r in (3.8) and (3.12). Then by

(H3), we have

$$
\begin{aligned}
\|D_r\bar{u} - a_r\bar{u}\| &= \left\| \frac{\int_0^r \int_0^\rho k(\rho - s)\left(\bar{u}(\cdot + s) - \bar{u}(\cdot)\right) ds d\eta_r(\rho)}{\gamma r} \right\| \\
&= \frac{1}{\gamma r}\left(\int_0^1 \left|\int_0^r \int_0^\rho k(\rho - s)\left(\bar{u}(t + s) - \bar{u}(t)\right) ds d\eta_r(\rho)\right|^p dt\right)^{1/p} \\
&\le \frac{1}{\gamma r}\left(\int_0^1 \left\|\int_0^\cdot k(\cdot - s)\left[\bar{u}(t + s) - \bar{u}(t)\right] ds\right\|^p_{C[0,r]} \tilde{C}^p r^{\sigma p} dt\right)^{1/p} \\
&= \bar{C}\left(\int_0^1 \sup_{\rho \in [0,r]} \left|\int_0^\rho k(\rho - s)\left[\bar{u}(t + s) - \bar{u}(t)\right] ds\right|^p dt\right)^{1/p} \\
&\le \bar{C}\left(\int_0^1 \sup_{\rho \in [0,r]} \|k\|^p_{L^q[0,\rho]} \|\bar{u}(t + \cdot) - \bar{u}(t)\|^p_{L^p[0,\rho]} dt\right)^{1/p} \\
&\le \bar{C} r^{1/q} \|k\|_{C[0,r]}\left(\int_0^1 \int_0^r |\bar{u}(t + s) - \bar{u}(t)|^p ds dt\right)^{1/p} \\
&\le \bar{C} r^{1 - 1/p} \|k\|_{C[0,r]}\left(\int_0^1 \int_0^r |\bar{u}(t + s) - \bar{u}(t)|^p ds dt\right)^{1/p} \\
&\le \bar{C} r \|k\|_{C[0,r]}\left(\int_0^1 \frac{1}{r}\int_0^r |\bar{u}(t + s) - \bar{u}(t)|^p ds dt\right)^{1/p}.
\end{aligned}
$$

Now using the Taylor expansion of k in (2.26) and the assumption that k and

R satisfy (3.18), we obtain

$$\begin{aligned}
&\|D_r\bar{u} - a_r\bar{u}\| \\
&\leq \bar{C}r\,\|k\|_{C[0,r]} \left(\int_0^1 \frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, dsdt\right)^{1/p} \\
&\leq \bar{C}r \left(\frac{r^{\nu} - 1}{(\nu - 1)!} + \frac{r^{\nu}\left\|k^{(\nu)}\right\|_{C[0,r]}}{\nu!}\right) \left(\int_0^1 \frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, dsdt\right)^{1/p} \\
&\leq \bar{C}r^{\nu}\frac{2}{(\nu - 1)!} \left(\int_0^1 \frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, dsdt\right)^{1/p}, \qquad (3.28)
\end{aligned}$$

for all $r \in (0, R]$, for $R > 0$ sufficiently small. Substituting into (3.27), we have

$$\begin{aligned}
\|u_r - \bar{u}\| &\leq \frac{\kappa c_0(1 + M)}{(\kappa - 1)r^{\nu}}\bar{C}r^{\nu}\frac{2}{(\nu - 1)!} \left(\int_0^1 \frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, dsdt\right)^{1/p} \\
&= D_1 \left(\int_0^1 \frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, dsdt\right)^{1/p}
\end{aligned}$$

for $D_1 > 0$ constant. By Lebesgue differentiation theorem [10],

$$\lim_{r \to 0} \frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, ds = 0,$$

for a.e. $t \in [0, 1]$, but then for r sufficiently small, it follows that for a.e. $t \in [0, 1]$,

$$\frac{1}{r}\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, ds \leq 2^p \, |\bar{u}(t)| \, .$$

Then by Lebesgue Dominated Convergence,

$$\lim_{r \to 0} \|u_r - \bar{u}\| \leq \lim_{r \to 0} D_1 \left(\int_0^1 \frac{1}{r} \int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, ds dt \right)^{1/p} = 0. \tag{3.29}$$

If the true solution $\bar{u}$ satisfies (2.43) , then

$$\int_0^r |\bar{u}(t+s) - \bar{u}(t)|^p \, ds \leq \frac{L_{\bar{u}}^p r^{1+\alpha p}}{1+\alpha p},$$

and so

$$\|u_r - \bar{u}\| \;\leq\; D_1 \left(\int_0^1 \frac{L_{\bar{u}}^p r^{\alpha p}}{1+\alpha p} dt \right)^{1/p} = O\left(r^{\alpha}\right) \quad \text{as} \quad r \to 0. \tag{3.30}$$

2. Let u_r^{δ} denote the solution to equation (3.14) with f_r replaced by f_r^{δ} for $r \in (0, R]$. We bound the error due to regularization and noise in the data using

(3.15), Corollary 3.1. (3.19), and (H3) to obtain

$$\begin{aligned}
\left\|u_r^\delta - u_r\right\| &= \left\|(a_r I + A_r)^{-1}\left(f_r^\delta - f_r\right)\right\| \\
&\leq \frac{(1+M)}{a_r}\left\|f_r^\delta - f_r\right\| \\
&= \frac{(1+M)}{a_r}\left(\int_0^1 \left|f_r^\delta(t) - f_r(t)\right|^p dt\right)^{1/p} \\
&= \frac{(1+M)}{a_r}\frac{1}{\gamma_r}\left(\int_0^1 \left|\int_0^r f^\delta(t+\rho) - f(t+\rho) d\eta_r(\rho)\right|^p dt\right)^{1/p} \\
&\leq \frac{(1+M)}{a_r}\frac{\bar{C}r^{\sigma - \tau_p}}{\gamma_r}\left(\int_0^1 \left\|f^\delta(t+\cdot) - f(t+\cdot)\right\|_{L^p[0,r]}^p dt\right)^{1/p} \\
&\leq \frac{(1+M)}{a_r}\bar{C}r^{-\tau_p}\left\|f^\delta - f\right\|_{L^p[0,1+\bar{R}]} \\
&\leq \frac{(1+M)\kappa c_0}{r^\nu(\kappa - 1)}\bar{C}r^{-\tau_p}\delta \\
&= C_1\frac{\delta}{r^{\nu + \tau_p}}, \qquad (3.31)
\end{aligned}$$

for $C_1 > 0$ constant. This establishes the bound on the total error for $\bar{u} \in L^p[0,1]$ to be

$$\begin{aligned}
\left\|u_r^\delta - \bar{u}\right\| &\leq \left\|u_r^\delta - u_r\right\| + \left\|u_r - \bar{u}\right\| \\
&\leq C_1\frac{\delta}{r^{\nu + \tau_p}} + \left\|u_r - \bar{u}\right\|. \qquad (3.32)
\end{aligned}$$

If the true solution $\bar{u}$ satisfies (2.43), then

$$\left\|u_{r(\delta)}^\delta - \bar{u}\right\| \leq C_1\frac{\delta}{[r(\delta)]^{\nu + \tau_p}} + C_2\,[r(\delta)]^\alpha,$$

and so for the choice $r = r(\delta) = K\delta^{1/(\alpha + \nu + \tau p)}$ for some $K > 0$, we have

$$\begin{aligned}
\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| &\le C_1 \frac{\delta}{[r(\delta)]^{\nu + \tau p}} \\
&\quad + C_2 [r(\delta)]^{\alpha} \\
&= C_1 \delta (K\delta^{1/(\alpha + \nu + \tau p)})^{-(\nu + \tau p)} \\
&\quad + C_2 (K\delta^{1/(\alpha + \nu + \tau p)})^{\alpha} \\
&= \tilde{C}_1 \delta^{\alpha/(\alpha + \nu + \tau p)} + \tilde{C}_2 \delta^{\alpha/(\alpha + \nu + \tau p)},
\end{aligned}$$

for $\tilde{C}_1, \tilde{C}_2 > 0$ constants, and so

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{\alpha/(\alpha + \nu + \tau p)}\right) \quad \text{as} \quad \delta \to 0.$$

□

3.2 A Discrepancy Principle for Local Regularization Given $f^{\delta} \in L^p[0, 1], 1 < p < \infty$

3.2.1 Preliminaries

Before redefining our discrepancy principle, we assume that the choice of measures η_r satisfies the following continuity property on $(0, \bar{R})$. Recall that there exists

$\psi_r \in L^q[0, r]$ for which

$$\int_0^r g(\rho)d\eta_r(\rho) = \int_0^r g(\rho)\psi_r(\rho)d\rho,$$

for all $g \in L^p[0, r]$ and any $r \in (0, \bar{R}]$. We first embed $L^q[0, r]$ into $L^q[0, \bar{R}]$ via the zero extension. For all $r \in (0, \bar{R}]$ and $\psi_r \in L^q[0, r]$, define the function $\tilde{\psi}_r \in L^q[0, \bar{R}]$ such that

$$\tilde{\psi}_r(\rho) := \begin{cases} \psi_r(\rho) & \text{a.e.} \quad \rho \in [0, r], \\ 0 & \text{a.e.} \quad \rho \in (r, \bar{R}]. \end{cases} \tag{3.33}$$

We make our final assumption.

[A2] The measure η_r (and thus $\psi_r \in L^q[0, r]$) is chosen so that

$$\left\| \tilde{\psi}_{r+h} - \tilde{\psi}_r \right\|_{L^q[0, \bar{R}]} \to 0 \quad \text{as} \quad h \to 0,$$

for all $r \in (0, \bar{R})$.

Our assumption implies the following.

Lemma 3.5 *Assume [A0], [A1-p], and [A2] are satisfied. For any $g \in L^p[0, 1 + \bar{R}]$, define*

$$g_r(t) := \frac{\int_0^r g(t+\rho)d\eta_r(\rho)}{\gamma_r},$$

for all $t \in [0, 1]$. Then the mapping $r \mapsto g_r$ is continuous in $L^p[0, 1]$, for all $r \in (0, \bar{R})$.

Proof: Let $\|\cdot\| = \|\cdot\|_{L^p[0,1]}$ and $r \in (0, \bar{R})$ be fixed. Let h be such that $r + h \in (0, \bar{R})$. Without loss of generality, let $h > 0$. Recall that by (3.7), we have

$\dfrac{1}{\gamma_r} = \dfrac{1}{r^\sigma c_0}$. Then

$$\left\| g_{r+h} - g_r \right\| = \left\| \frac{\int_0^{r+h} g(\cdot+\rho)d\eta_{r+h}(\rho)}{\gamma_{r+h}} - \frac{\int_0^{r} g(\cdot+\rho)d\eta_{r}(\rho)}{\gamma_{r}} \right\|$$

$$= \left\| \frac{\int_0^{r+h} g(\cdot+\rho)\psi_{r+h}(\rho)d\rho}{(r+h)^\sigma c_0} - \frac{\int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{r^\sigma c_0} \right\|$$

$$\leq \left\| \frac{\int_0^{r+h} g(\cdot+\rho)\psi_{r+h}(\rho)d\rho - \int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{(r+h)^\sigma c_0} \right\|$$

$$+ \left\| \frac{\int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{(r+h)^\sigma c_0} - \frac{\int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{r^\sigma c_0} \right\|.$$

Since $\gamma_r > 0$ for all $r > 0$, the mapping $r \mapsto \dfrac{1}{\gamma_r}$ is continuous on $(0, \bar{R})$. Then

$$\lim_{h \to 0} \left\| \frac{\int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{(r+h)^\sigma c_0} - \frac{\int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{r^\sigma c_0} \right\| = 0,$$

and

$$\lim_{h \to 0} \left\| \frac{\int_0^{r+h} g(\cdot+\rho)\psi_{r+h}(\rho)d\rho - \int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho}{(r+h)^\sigma c_0} \right\|$$

$$= \frac{1}{r^\sigma c_0} \lim_{h \to 0} \left\| \int_0^{r+h} g(\cdot+\rho)\psi_{r+h}(\rho)d\rho - \int_0^{r} g(\cdot+\rho)\psi_{r}(\rho)d\rho \right\|.$$

Then for a.e. $t \in [0, 1]$,

$$\begin{aligned}
&\left| \int_0^{r+h} g(t+\rho)\psi_{r+h}(\rho)d\rho - \int_0^r g(t+\rho)\psi_r(\rho)d\rho \right| \\
&= \left| \int_0^{r+h} g(t+\rho)\tilde{\psi}_{r+h}(\rho)d\rho - \int_0^{r+h} g(t+\rho)\tilde{\psi}_r(\rho)d\rho \right| \\
&\leq \int_0^{r+h} |g(t+\rho)| \left| \tilde{\psi}_{r+h}(\rho) - \tilde{\psi}_r(\rho) \right| d\rho \\
&\leq \|g\|_{L^p[0,1+\bar{R}]} \left\| \tilde{\psi}_{r+h} - \tilde{\psi}_r \right\|_{L^q[0,\bar{R}]}.
\end{aligned}$$

Therefore by assumption [A2],

$$\begin{aligned}
&\lim_{h \to 0} \left\| \frac{\int_0^{r+h} g(\cdot+\rho)\psi_{r+h}(\rho)d\rho - \int_0^r g(\cdot+\rho)\psi_r(\rho)d\rho}{(r+h)^\sigma c_0} \right\| \\
&\leq \frac{1}{r^\sigma c_0} \|g\|_{L_p[0,1+\bar{R}]} \lim_{h \to 0} \left\| \tilde{\psi}_{r+h} - \tilde{\psi}_r \right\|_{L^q[0,\bar{R}]} \\
&= 0,
\end{aligned}$$

and so

$$\lim_{h \to 0} \left\| g_{r+h} - g_r \right\|_{L^p[0,1]} = 0,$$

for all $g \in L^p[0, 1 + \bar{R}]$. □

Lemma 3.6 *Let $\psi \in C[0,1]$ be such that $\psi_r \in C[0,r]$ is given by*

$$\psi_r(\rho) = \psi\left(\frac{\rho}{r}\right),$$

for all $\rho \in [0, r]$ and $r \in (0, \bar{R}]$. Then for any $g \in L^p[0, r]$ and $r \in (0, \bar{R}]$, the measure

η_r *defined by*

$$\int_0^r g(\rho)d\eta_r(\rho) = \int_0^r g(\rho)\psi_r(\rho)d\rho,$$

satisfies assumption [A2].

Proof: Let $r \in (0, \bar{R})$ be fixed, $h \in (0, \bar{R})$ such that $r + h \in (0, \bar{R})$. Without loss of generality, let $h > 0$. Then for all $\rho \in [0, \bar{R}]$,

$$\tilde{\psi}_{r+h}(\rho) - \tilde{\psi}_r(\rho) = \begin{cases} \psi_{r+h}(\rho) - \psi_r(\rho), & \text{if } \rho \in [0, r], \\ \psi_{r+h}(\rho), & \text{if } \rho \in (r, r+h], \\ 0, & \text{if } \rho \in (r+h, \bar{R}]. \end{cases}$$

Then

$$\begin{aligned}
&\lim_{h \to 0} \left\| \tilde{\psi}_{r+h} - \tilde{\psi}_r \right\|_{L^q[0, \bar{R}]} \\
&= \lim_{h \to 0} \left(\int_0^{\bar{R}} \left| \tilde{\psi}_{r+h}(\rho) - \tilde{\psi}_r(\rho) \right|^q d\rho \right)^{1/q} \\
&= \lim_{h \to 0} \left(\int_0^r \left| \psi\left(\frac{\rho}{r+h}\right) - \psi\left(\frac{\rho}{r}\right) \right|^q d\rho + \int_r^{r+h} \left| \psi\left(\frac{\rho}{r+h}\right) \right|^q d\rho \right)^{1/q} \\
&= \lim_{h \to 0} \left(\int_0^r \left| \psi\left(\frac{\rho}{r+h}\right) - \psi\left(\frac{\rho}{r}\right) \right|^q d\rho + (r+h) \int_{r/(r+h)}^1 |\psi(\rho)|^q d\rho \right)^{1/q} \\
&= 0,
\end{aligned}$$

which follows from the continuity of ψ. Therefore

$$\lim_{h \to 0} \left\| \tilde{\psi}_{r+h} - \tilde{\psi}_r \right\|_{L^q[0, \bar{R}]} = 0.$$

$\square$

3.2.2 Definition and Properties

We assume that we are given data $f^\delta \in L^p[0, 1 + R]$ that is a version of the true data $f \in L^p[0, 1 + R]$ that contains noise. We restate our assumptions and modify [A0] to be:

[A0] Let $0 < \bar{R} << 1$ be such that $\bar{u} \in L^p[0, 1 + \bar{R}]$ and $f \in R(A) \subseteq L^p[0, 1 + \bar{R}]$ so that $A\bar{u} = f$ for a.e. t on the interval $[0, 1 + \bar{R}]$. Assume that $\|f'\|_{L^p[0,\,1+\bar{R}]} \neq 0$. Given $\delta > 0$, the data $f^\delta(t)$ is available for a.e. $t \in [0, 1 + \bar{R}]$ and $f^\delta \in L^p[0, 1 + \bar{R}]$ satisfies

$$\|f - f^\delta\|_{L^p[0,\,1+\bar{R}]} \leq \delta \quad \text{and} \quad \|f^\delta\|_{L^p[0,\,1]} > (\tau + 1)\delta,$$

with $\tau \in (1, 2)$ fixed for all δ. If additional data is unavailable, we suffice with approximating $\bar{u}$ on the slightly smaller interval $\left[0, 1 - \bar{R}\right]$.

[A1-p] The measure η_r satisfies hypotheses (H1)-(H3) with $\tau_p > 0$ for all $r \in (0, \bar{R}]$.

[A2] The measure η_r (and thus $\psi_r \in L^q[0, r]$) is chosen so that

$$\left\|\tilde{\psi}_{r+h} - \tilde{\psi}_r\right\|_{L^q[0,\,\bar{R}]} \to 0 \quad \text{as} \quad h \to 0,$$

for all $r \in (0, \bar{R})$. for $\tilde{\psi}$ defined in (3.33).

Henceforth we will assume that R and k satisfy the conditions of Lemmas 3.3 and 3.4.

Definition 3.1 Discrepancy Principle for Local Regularization in $L^p[0,1]$
Assume that the conditions [A0], [A1-p], and [A2] are satisfied. Let $d : (0, R) \to [0, \infty)$ be the discrepancy functional defined by

$$d(r) := a_r^m \left\| A_r u_r^\delta - f_r^\delta \right\|_{L^p[0,\,1]}, \tag{3.34}$$

for $m \in (0, 1]$ fixed. Choose the regularization parameter r so that

$$a_r^m \left\| A_r u_r^\delta - f_r^\delta \right\|_{L^p[0,\,1]} = \tau\delta. \tag{3.35}$$

We now show existence of an $r \in (0, R)$ for which $d(r) = \tau\delta$ once establishing properties of d. To do so, we use the bounds on $\|u_r\|_{L^p[0,\,1]}$ in following lemma.

Lemma 3.7 *Assume [A0] and [A1-p] are satisfied. For any $r \in (0, R]$,*

$$\frac{\|f_r\|_{L^p[0,\,1]}}{a_r + \bar{C}\,\|k\|_{C[0,\,1+R]}} \le \|u_r\|_{L^p[0,\,1]} \le (1 + M)\left\|\frac{f_r}{a_r}\right\|_{L^p[0,\,1]}, \tag{3.36}$$

where u_r is the solution of (3.14).

Proof: Representing u_r using (3.15), for any $r \in (0, R]$

$$\begin{aligned}\|f_r\|_{L^p[0,\,1]} &= \left\|(a_r I + A_r)(a_r I + A_r)^{-1} f_r\right\|_{L^p[0,\,1]} \\ &\le (a_r + \|A_r\|)\,\|u_r\|_{L^p[0,\,1]}.\end{aligned}$$

Let $x \in L^p[0,1]$, $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Recalling the notation in (3.9)

$$\begin{aligned}
\|A_r\| &= \sup_{\|x\|_{L^p[0,1]}=1} \|A_r\|_{L^p[0,1]} \\
&= \sup_{\|x\|_{L^p[0,1]}=1} \left\| \int_0^{\cdot} k_r(\cdot - s)x(s)ds \right\|_{L^p[0,1]} \\
&\leq \sup_{\|x\|_{L^p[0,1]}=1} \|k_r\|_{L^q[0,1]} \|x\|_{L^p[0,1]} \\
&= \left\| \frac{\int_0^r k(\cdot + \rho)d\eta_r(\rho)}{\gamma r} \right\|_{L^q[0,1]} \\
&\leq \bar{C} \|k\|_{C[0,1+R]} .
\end{aligned} \tag{3.37}$$

Therefore

$$\|f_r\|_{L^p[0,1]} \leq \left(a_r + \bar{C}\|k\|_{C[0,1+R]}\right) \|u_r\|_{L^p[0,1]},$$

or

$$\frac{\|f_r\|_{L^p[0,1]}}{a_r + \bar{C}\|k\|_{C[0,1+R]}} \leq \|u_r\|_{L^p[0,1]} .$$

To establish the upper bound, we represent u_r using (3.15) and the bound on the operator norm in Corollary 3.1 to obtain

$$\begin{aligned}
\|u_r\|_{L^p[0,1]} &= \left\|(a_r I + A_r)^{-1} f_r\right\|_{L^p[0,1]} \\
&\leq \frac{1+M}{a_r} \|f_r\|_{L^p[0,1]},
\end{aligned}$$

for all $r \in (0, R]$. $\square$

Lemma 3.8 *Assume that [A0],[A1-p], and [A2] hold with $0 < \tau_p < \nu m$ in hypothesis (H3). The function $d : (0, R) \to (0, \infty)$ defined as in (3.34) has the following properties:*

i) The mapping $r \mapsto d(r)$ is continuous on $(0, R)$.

ii) $\lim_{r \to 0} d(r) = 0$.

iii) There exists an $\tilde{R} = \tilde{R}(f, R)$, $\bar{\delta} = \bar{\delta}(f, k, R)$, and $\gamma_S = \gamma_S(f, k, R)$, such that if $\delta \in (0, \bar{\delta})$ or $\left\|f^\delta\right\|_{L^p[0,1]} > \gamma_S \delta$, then $\lim_{r \to \tilde{R}} d(r) > \tau\delta$.

Therefore, for δ sufficiently small or $\dfrac{\left\|f^\delta\right\|_{L^p[0,1]}}{\delta}$ sufficiently large, there exists $r \in (0, R)$ such that $d(r) = \tau\delta$.

Proof: Let $\|\cdot\| = \|\cdot\|_{L^p[0,1]}$.

i) Fix $r \in (0, R)$ and let h be such that $r + h \in (0, R)$. From the definition of u_r, we have

$$d(r) = a_r^{1+m} \left\|u_r^\delta\right\|,$$

and by Lemma 3.2, we again have $a_r > 0$ for all $r \in (0, R]$ and so

$$\begin{aligned} |d(r+h) - d(r)| &= \left|a_{r+h}^{1+m} \left\|u_{r+h}^\delta\right\| - a_r^{1+m} \left\|u_r^\delta\right\|\right| \\ &\leq a_{r+h}^{1+m} \left\|u_{r+h}^\delta - u_r^\delta\right\| + \left|a_{r+h}^{1+m} - a_r^{1+m}\right| \left\|u_r^\delta\right\|. \end{aligned}$$

Since r is fixed, we may use (3.36) to bound $\left\| u_r^\delta \right\|$ and obtain

$$|d(r+h) - d(r)| \leq a_{r+h}^{1+m} \left\| u_{r+h}^\delta - u_r^\delta \right\| + \left| a_{r+h}^{1+m} - a_r^{1+m} \right| (1+M) \left\| \frac{f_r^\delta}{a_r} \right\| .$$

To show that $r \mapsto a_r$ is continuous on $(0, R)$, we note that

$$a_{r+h} - a_r = \int_0^{r+h} \left(\int_0^\rho k(s) ds \right) \left(\tilde{\psi}_{r+h}(\rho) - \tilde{\psi}_r(\rho) \right) d\rho,$$

(where we assume without loss of generality that $h > 0$), and use Hölder's inequality and [A2]. Therefore

$$\left| a_{r+h}^{1+m} - a_r^{1+m} \right| \to 0 \quad \text{as} \quad h \to 0,$$

and thus

$$\lim_{h \to 0} |d(r+h) - d(r)| \leq a_r^{1+m} \limsup_{h \to 0} \left\| u_{r+h}^\delta - u_r^\delta \right\| . \tag{3.38}$$

From (3.16), we have

$$\left\| u_{r+h}^\delta - u_r^\delta \right\| = \left\| \frac{f_{r+h}^\delta}{a_{r+h}} - \frac{f_r^\delta}{a_r} - \mathcal{X}_{r+h} * \frac{f_{r+h}^\delta}{a_{r+h}} + \mathcal{X}_r * \frac{f_r^\delta}{a_r} \right\| .$$

Define for all $t \in [0, 1]$,

$$\tilde{f}_h(t) := \frac{f_{r+h}^\delta(t)}{a_{r+h}} - \frac{f_r^\delta(t)}{a_r}$$

$$\tilde{\mathcal{X}_h}(t) := \mathcal{X}_{r+h}(t) - \mathcal{X}_r(t)$$

$$\tilde{k_h}(t) := \frac{k_{r+h}(t)}{a_{r+h}} - \frac{k_r(t)}{a_r}.$$

Using the the representation of u_r^δ in (3.16) and the above notation exactly as in Lemma 2.8, we have

$$\begin{aligned}
\left\| u_{r+h}^\delta - u_r^\delta \right\| &\leq \left\| \tilde{f}_h \right\| + \left\| \tilde{\mathcal{X}_h} * \frac{f_{r+h}^\delta}{a_{r+h}} \right\| + \left\| \mathcal{X}_r * \tilde{f}_h \right\| \\
&\leq \left\| \tilde{f}_h \right\| + \left\| \tilde{\mathcal{X}_h} \right\|_{L^1[0,1]} \left\| \frac{f_{r+h}^\delta}{a_{r+h}} \right\| + \left\| \mathcal{X}_r \right\|_{L^1[0,1]} \left\| \tilde{f}_h \right\| \\
&\leq (1+M) \left\| \tilde{f}_h \right\| + \left\| \tilde{\mathcal{X}_h} \right\|_{L^1[0,1]} \left\| \frac{f_{r+h}^\delta}{a_{r+h}} \right\|.
\end{aligned}$$

Using the fact that $r \mapsto a_r$ is continuous on $(0, R)$ and Lemma 3.5, we have

$$\left\| \tilde{f}_h \right\| \leq \frac{1}{a_{r+h}} \left\| f_{r+h}^\delta - f_r^\delta \right\| + \left| \frac{1}{a_{r+h}} - \frac{1}{a_r} \right| \left\| f_r^\delta \right\| \to 0 \quad \text{as} \quad h \to 0,$$

and

$$\left| \left\| \frac{f_{r+h}^\delta}{a_{r+h}} \right\| - \left\| \frac{f_r^\delta}{a_r} \right\| \right| \leq \left\| \tilde{f}_h \right\| \to 0 \quad \text{as} \quad h \to 0.$$

Therefore

$$\limsup_{h \to 0} \left\| u_{r+h}^\delta - u_r^\delta \right\| \leq \left\| \frac{f_r^\delta}{a_r} \right\| \limsup_{h \to 0} \left\| \tilde{\mathcal{X}_h} \right\|_{L^1[0,1]}$$

and

$$\lim_{h \to 0} |d(r+h) - d(r)| \leq a_r^{1+m} \left\| \frac{f_r^\delta}{a_r} \right\| \limsup_{h \to 0} \left\| \tilde{\mathcal{X}_h} \right\|_{L^1[0,1]}.$$

It remains to prove that $\limsup_{h \to 0} \left\| \tilde{\mathcal{X}}_h \right\|_{L^1[0,1]} = 0$, though this was already shown in Lemma 2.8 based on the continuity of the maps $r \mapsto k_r$ and $r \mapsto a_r$, and $a_r > 0$, to argue that $\left\| \tilde{k}_h \right\|_{C[0,1]} \to 0$ as $h \to 0$. However this still holds under the assumptions of this Lemma. Therefore $\lim_{h \to 0} |d(r+h) - d(r)| = 0$ proving that $r \mapsto d(r)$ is continuous on $(0, R)$.

ii) Using the fact that $a_r > 0$, (H3) to bound $\left\| f_r^\delta \right\|$, and the upper bounds on a_r in Lemma 3.2 and $\left\| u_r^\delta \right\|$ in Lemma 3.7,

$$\begin{aligned} d(r) &= a_r^m \left\| A_r u_r^\delta - f_r^\delta \right\| \\ &= a_r^{1+m} \left\| u_r^\delta \right\| \\ &\leq a_r^{1+m} (1+M) \frac{\left\| f_r^\delta \right\|}{a_r} \\ &= a_r^m (1+M) \left\| f_r^\delta \right\| \\ &\leq r^{\nu m - \tau_p} (1+M) \left(\frac{\kappa+1}{\kappa c_0} \right)^m \bar{C} \left\| f^\delta \right\|_{L^p[0,1+R]}. \end{aligned} \tag{3.39}$$

Since $m \in (0, 1]$, and $\nu m > \tau_p > 0$, it follows that $\lim_{r \to 0} d(r) = 0$.

iii) Define

$$\tilde{R} := \min \left\{ R, \left(\frac{1}{2\bar{C}} \cdot \frac{\|f\|_{L^p[0,1]}}{\|f'\|_{L^p[0,1+\bar{R}]}} \right)^{1/(1-\tau_p)} \right\}. \tag{3.40}$$

Then for all $r \in (0, \tilde{R}]$,

$$
\begin{aligned}
|f_r(t) - f(t)| &= \left| \frac{\int_0^r f(t+\rho) - f(t)\, d\eta_r(\rho)}{\gamma r} \right| \\
&\leq \bar{C} \, \|f'\|_{L^p[0,\,1+r]} \, \tilde{R}^{1-\tau p} \\
&\leq \frac{\|f\|}{2} \\
&\leq \frac{1}{2}\left[\left\| f^\delta - f \right\| + \left\| f^\delta \right\| \right] \\
&\leq \frac{1}{2}\left[\delta + \left\| f^\delta \right\| \right]. \qquad (3.41)
\end{aligned}
$$

For all $r \in (0, R]$, define

$$
B(r) := \frac{a_r^{1+m}}{a_r + \bar{C}\,\|k\|_{C[0,\,1+R]}}. \qquad (3.42)
$$

Then using the lower bound on $\left\| u_r^\delta \right\|$ in Lemma 3.7 and (3.41), we have for all $r \in (0, R]$,

$$
\begin{aligned}
d(r) &= a_r^{1+m} \left\| u_r^\delta \right\| \\
&\geq B(r) \left\| f_r^\delta \right\| \\
&\geq B(r) \left[\left\| f^\delta \right\| - \left\| f_r^\delta - f_r \right\| - \left\| f^\delta - f \right\| - \|f_r - f\| \right] \\
&\geq B(r) \left[\left\| f^\delta \right\| - \bar{C} r^{-\tau p} \delta - \delta - \frac{\delta}{2} - \frac{\left\| f^\delta \right\|}{2} \right] \\
&= B(r) \left[\frac{\left\| f^\delta \right\|}{2} - \left(\bar{C} r^{-\tau p} + \frac{3}{2} \right) \delta \right]. \qquad (3.43)
\end{aligned}
$$

For all $r \in (0, R]$, define

$$F(r) := \frac{B(r)}{2\left[\tau + B(r)\left(\bar{C}r^{-\tau p} + 3/2\right)\right]}. \tag{3.44}$$

Define

$$\bar{\delta} := \|f\| \frac{F(\tilde{R})}{1 + F(\tilde{R})}. \tag{3.45}$$

and

$$\gamma_S := \frac{1}{F(\tilde{R})}. \tag{3.46}$$

By [A0],

$$\|f\| \leq \left\|f^\delta\right\| + \left\|f - f^\delta\right\| \leq \left\|f^\delta\right\| + \delta,$$

and

$$(\tau + 1)\delta < \left\|f^\delta\right\| \leq \|f\| + \left\|f - f^\delta\right\| \leq \|f\| + \delta.$$

imply

$$\left\|f^\delta\right\| \geq \|f\| - \delta > 0. \tag{3.47}$$

If $\delta \in (0. \bar{\delta})$,

$$\begin{aligned}
\frac{\delta}{\left\|f^\delta\right\|} &\leq \frac{\delta}{\|f\| - \delta} \\
&< \frac{\|f\| F(\tilde{R})}{\|f\|(1 + F(\tilde{R})) - \|f\| F(\tilde{R})} \\
&= F(\tilde{R}).
\end{aligned}$$

Equivalently,

$$\left\|f^{\delta}\right\| > \frac{\delta}{F(\tilde{R})}. \tag{3.48}$$

Alternatively, if $\delta \notin (0, \bar{\delta})$, the assumption that $\left\|f^{\delta}\right\|_{L^p[0,\,1]} > \gamma_S \delta$ implies that (3.48) still holds. Then substituting (3.44) and (3.48) into the lower bound on d in (3.43) and taking the limit as r approaches $\tilde{R}$, we have that

$$\begin{aligned}
\lim_{r \to \tilde{R}} d(r) &\geq B(\tilde{R}) \left[\frac{\left\|f^{\delta}\right\|}{2} - \left(\bar{C} \tilde{R}^{-\tau p} + \frac{3}{2} \right) \delta \right] \\
&> \frac{B(\tilde{R})}{2} \left[\frac{\delta}{F(\tilde{R})} - \left(2\bar{C} \tilde{R}^{-\tau p} + 3 \right) \delta \right] \\
&= \frac{B(\tilde{R})}{2} \left[\frac{1}{F(\tilde{R})} - \left(2\bar{C} \tilde{R}^{-\tau p} + 3 \right) \right] \delta \\
&= \frac{B(\tilde{R})}{2} \left[2 \cdot \frac{\tau + B(\tilde{R}) \left(\bar{C} \tilde{R}^{-\tau p} + 3/2 \right)}{B(\tilde{R})} - \left(2\bar{C} \tilde{R}^{-\tau p} + 3 \right) \right] \delta \\
&= \left[\tau + B(\tilde{R}) \left(\bar{C} \tilde{R}^{-\tau p} + \frac{3}{2} \right) - B(\tilde{R}) \left(\bar{C} \tilde{R}^{-\tau p} + \frac{3}{2} \right) \right] \delta \\
&= \tau \delta.
\end{aligned}$$

□

As before, we still have a lower bound on the choice of the regularization parameter as a function of δ.

Lemma 3.9 *Let* $\delta \in (0, \bar{\delta})$ *or* $\left\|f^{\delta}\right\| > \gamma_S \delta$, *where* $\bar{\delta}, \gamma_S > 0$ *are as given in Lemma* 3.8. *Let* $r = r(\delta)$ *be defined by*

$$r(\delta) = \min \left\{ r \in (0, R) \mid d(r) = \tau \delta \right\}. \tag{3.49}$$

If $\nu m > \tau p$, then there exists an $r^* = r^*(\delta) > 0$ such that $r(\delta) \geq r^* > 0$, where $r^* \in (0, R)$ is given by

$$r^* := \left(\frac{\tau\delta}{\epsilon}\right)^{1/(\nu m - \tau p)},$$

with $\epsilon := \left(\frac{\kappa + 1}{\kappa c_0}\right)^m (1 + M)\bar{C} \left\|f^\delta\right\|_{L^p[0,\,1 + R]} > 0.$

Proof: We first observe that $r(\delta)$ is well-defined since the set $\{r \in (0, R) \mid d(r) = \tau\delta\}$ is compact and thus has a minimum value. Note that

$$\begin{aligned} d(r) &\leq a_r^m (1 + M) \left\|f_r^\delta\right\|_{L^p[0,\,1]} \\ &\leq r^{\nu m} \left(\frac{\kappa + 1}{\kappa c_0}\right)^m (1 + M)\bar{C} r^{-\tau p} \left\|f^\delta\right\|_{L^p[0,\,1 + R]} \\ &= \epsilon\, r^{\nu m - \tau p} \end{aligned}$$

by Lemma 3.2 and Corollary 3.1. Since $r \mapsto \epsilon\, r^{\nu m - \tau p}$ is a continuous, strictly increasing function that bounds d from above for all $r \in (0, \tilde{R})$, then

$$\lim_{r \to \tilde{R}^-} \epsilon r^{\nu m - \tau p} \geq \lim_{r \to \tilde{R}^-} d(r) > \tau\delta.$$

Therefore there exists a unique $r^*(\delta) \in (0, \tilde{R}]$ for which

$$\epsilon \left(r^*\right)^{\nu m - \tau p} = \tau\delta,$$

and so for any $r(\delta) \in (0, \tilde{R}]$ for which $d(r(\delta)) = \tau\delta$, we have necessarily that $r(\delta) \geq r^* > 0$. Further,

$$r^* = \left(\frac{\tau\delta}{\epsilon}\right)^{1/(\nu m - \tau_p)} = \left(\tau\delta \frac{c_0^{1+m}\kappa^m}{(\kappa+1)^m(1+M)\tilde{C}\left\|f^\delta\right\|_{L^p[0,1+R]}}\right)^{1/(\nu m - \tau_p)}.$$

□

3.2.3 L^p-Convergence

We again make more definite the choice of the regularization parameter r given by our discrepancy principle for the case $f^\delta \in L^p[0,1], 1 < p < \infty$.

Definition 3.2 Discrepancy Principle for Local Regularization in $L^p[0,1]$
Let $d : (0, R) \to [0, \infty)$ be the discrepancy functional defined by

$$d(r) := a_r^m \left\|A_r u_r^\delta - f_r^\delta\right\|_{L^p[0,1]}, \tag{3.50}$$

for $m \in (0,1]$ fixed, with $\nu m > \tau_p > 0$. Choose the regularization parameter $r = r(\delta)$ to be the smallest $r \in (0, R)$ so that

$$a_r^m \left\|A_r u_r^\delta - f_r^\delta\right\|_{L^p[0,1]} = \tau\delta. \tag{3.51}$$

Remark 3.2 *Any $r \in (0, \tilde{R})$ satisfying (3.51) would be acceptable.*

We prove that local regularization with the discrepancy principle defined via equation (3.51) is a convergent regularization method for $f^\delta \in L^p[0,1]$. For purposes of obtaining a rate of convergence, we make the usual smoothness assumption that $\bar{u}$ is uniformly Hölder continuous with power $\alpha \in (0,1]$ and Hölder constant $L_{\bar{u}}$.

Theorem 3.2 *Assume that [A0],[A1-p], and [A2] hold and let $\bar{\delta}, \gamma_S > 0$ be given as in Lemma* 3.8. *For $\delta \in (0, \bar{\delta})$ or $\left\|f^\delta\right\| > \gamma_S \delta$, let u_r^δ denote the solution to equation* (3.14) *with f_r replaced by f_r^δ. Then for $r(\delta)$ selected according to the discrepancy principle in definition* 3.2, *we have*

1. $r(\delta) \to 0$ *as* $\delta \to 0$.

2. $\left\|u_{r(\delta)}^\delta - \bar{u}\right\|_{L^p[0,1]} \to 0$ *as* $\delta \to 0$.

3. *If $\bar{u}$ satisfies the condition* (2.43), *then for $\tau_p > 0$ as defined in (H3),*

$$\left\|u_r^\delta - \bar{u}\right\|_{L^p[0,1]} \le C_1 \frac{\delta}{r^{\nu + \tau_p}} + C_2 r^\alpha,$$

and hence,

$$\left\|u_{r(\delta)}^\delta - \bar{u}\right\|_{L^p[0,1]} = O\left(\delta^{(\nu m - \tau_p)/(\nu(1+m))}\right) + O\left(\delta^{\alpha/(\nu(1+m))}\right).$$

as $\delta \to 0$. Thus the rate of convergence is determined by $\min\left\{\alpha, \nu m - \tau_p\right\}$.

If $\omega = \min\left\{\alpha, \nu m - \tau_p\right\}$, *then*

$$\left\|u_{r(\delta)}^\delta - \bar{u}\right\|_{L^p[0,1]} = O\left(\delta^{\omega/(\nu(1+m))}\right) \text{ as } \delta \to 0.$$

Moreover, if the choice of m is such that $m = \dfrac{\alpha + \tau p}{\nu}$, then

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\|_{L^p[0,\,1]} = O\left(\delta^{\alpha/(\alpha + \nu + \tau p)}\right) \quad \textit{as } \delta \to 0,$$

which is same rate of convergence as obtained in Theorem 3.1 *using the a priori rule to select the parameter $r = r(\delta)$.*

Proof: Let $\|\cdot\| = \|\cdot\|_{L^p[0,\,1]}$.

1. Let $\{\delta_n\}_{n \geq 1}$ be a positive sequence for which $\delta_n \to 0$ as $n \to \infty$, with $\delta_n \in (0, \bar{\delta})$ or $\left\| f^{\delta_n} \right\| > \gamma_s \delta_n$ for each n, and $\left\| f - f^{\delta_n} \right\|_{L^p[0,\,1 + \bar{R}]} \leq \delta_n$ for each n. Let $\{r_n\}_{n \geq 1}$ be the corresponding sequence of regularization parameter values selected according to the discrepancy principle for local regularization given in definition 3.2, namely for each n,

$$r_n = r(\delta_n) = \min \left\{ r \in (0, R) \mid d(r) = \tau \delta_n \right\}.$$

Using the lower bound on d in (3.43), we have that

$$\begin{aligned} \tau \delta_n &= d(r_n) \\ &\geq B(r_n) \left[\frac{\left\| f^{\delta_n} \right\|}{2} - \left(\bar{C} r_n^{-\tau p} + \frac{3}{2} \right) \delta_n \right] \\ &\geq B(r_n) \left[\frac{\|f\|}{2} - \frac{\left\| f - f^{\delta_n} \right\|}{2} - \left(\bar{C} r_n^{-\tau p} + \frac{3}{2} \right) \delta_n \right] \\ &\geq B(r_n) \left[\frac{\|f\|}{2} - \left(\bar{C} r_n^{-\tau p} + 2 \right) \delta_n \right], \end{aligned} \tag{3.52}$$

and so

$$\left[\tau + B(r_n)\left(2 + \bar{C}r_n^{-\tau p}\right)\right]\delta_n \geq B(r_n)\frac{\|f\|}{2}.$$

Then

$$0 = \lim_{n \to \infty}\delta_n \geq \liminf_{n \to \infty}\frac{B(r_n)}{\tau + B(r_n)\left(2 + \bar{C}r_n^{-\tau p}\right)}\frac{\|f\|}{2},$$

and by (3.47), $\|f\| > 0$, therefore

$$\lim_{n \to \infty}\frac{B(r_n)}{\tau + B(r_n)\left(2 + \bar{C}r_n^{-\tau p}\right)} = 0. \tag{3.53}$$

Equation (3.53) can only be true if $\lim_{n \to \infty} B(r_n) = 0$ or if $\lim_{n \to \infty}\left(\bar{C}r_n^{-\tau p} + 2\right) = \infty$.

If $\lim_{n \to \infty} B(r_n) = 0$, then recalling the definition of $B(r)$ in (3.42) and using both bounds on a_r in Lemma 3.2, we have we have that

$$\begin{aligned}
0 &= \lim_{n \to \infty} B(r_n) \\
&\geq \lim_{n \to \infty}\left(\frac{\kappa - 1}{\kappa c_0}r_n^{\nu}\right)^{1+m}\left(\left(\frac{\kappa + 1}{\kappa c_0}\cdot R^{\nu}\right)^{m} + \bar{C}\,\|k\|_{C[0,1+R]}\right)^{-1} \\
&= \bar{D}r_n^{\nu(1+m)},
\end{aligned}$$

for $\bar{D} > 0$ constant. Then

$$0 = \lim_{n \to \infty}\bar{D}r_n^{\nu(1+m)} \geq 0.$$

If $\lim_{n \to \infty} \left(\bar{C} r_n^{-\tau p} + 2 \right) = \infty$, then

$$0 = \lim_{n \to \infty} r_n^{\tau p}.$$

Since $\tau p > 0$, we conclude that

$$\lim_{n \to \infty} r_n = 0.$$

2. Let r be chosen according to the discrepancy principle for local regularization given in definition 3.2, i.e.

$$r = r(\delta) = \min \{ r \in (0, R) | \; d(r) = \tau \delta \} .$$

Using (3.39), our choice of $r = r(\delta)$ is such that

$$a_r^{1+m} \left\| u_{r(\delta)}^{\delta} \right\| = \tau \delta.$$

and so representing $u^{\delta}_{r(\delta)}$ as in (3.15), we obtain by substituting into (3.31),

$$\begin{aligned}
\left\|u^{\delta}_{r(\delta)}\right\| - \left\|u_{r(\delta)}\right\| &\leq \left\|u^{\delta}_{r(\delta)} - u_{r(\delta)}\right\| \\
&\leq C_1 \frac{\delta}{[r(\delta)]^{\nu + \tau p}} \\
&= C_1 \frac{1}{[r(\delta)]^{\nu + \tau p}} \cdot \frac{a^{1+m}_{r(\delta)}}{\tau} \left\|u^{\delta}_{r(\delta)}\right\| \\
&\leq C_1 \left(\frac{\kappa + 1}{\kappa c_0}\right)^{1+m} \frac{[r(\delta)]^{\nu(1+m)}}{\tau\, [r(\delta)]^{\nu + \tau p}} \left\|u^{\delta}_{r(\delta)}\right\| \\
&= \hat{C}_1 \, [r(\delta)]^{\nu m - \tau p} \left\|u^{\delta}_{r(\delta)}\right\|,
\end{aligned}$$

for $\hat{C}_1 > 0$ constant. Using Theorem 3.1 and the first part of this theorem in which it was proved that $\lim_{\delta \to 0} r(\delta) = 0$, we have

$$\begin{aligned}
\left\|u_{r(\delta)}\right\| &\leq \left\|u_{r(\delta)} - \bar{u}\right\| + \|\bar{u}\| \\
&\leq \frac{\|\bar{u}\|}{2} + \|\bar{u}\|.
\end{aligned}$$

for $\delta > 0$ sufficiently small. Therefore

$$\left\|u^{\delta}_{r(\delta)}\right\| \leq \hat{C}_1 \, [r(\delta)]^{\nu m - \tau p} \left\|u^{\delta}_{r(\delta)}\right\| + \frac{3}{2} \|\bar{u}\|.$$

And since $\nu m > \tau p > 0$,

$$\limsup_{\delta \to 0} \left\|u^{\delta}_{r(\delta)}\right\| \leq \frac{3}{2} \|\bar{u}\|. \qquad (3.54)$$

Substituting the principle in for δ into (3.32), we have

$$\begin{aligned} \left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| &\le C_1 \frac{\delta}{[r(\delta)]^{\nu + \tau p}} + \left\| u_{r(\delta)} - \bar{u} \right\| \\ &\le \hat{C}_1 \left[r(\delta) \right]^{\nu m - \tau p} \left\| u^{\delta}_{r(\delta)} \right\| + \left\| u_{r(\delta)} - \bar{u} \right\|. \end{aligned}$$

Then by (3.54), part 1 of this theorem, and $\nu m > \tau p > 0$, it follows that

$$\begin{aligned} \lim_{\delta \to 0} \left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| &\le \lim_{\delta \to 0} \left(\hat{C}_1 \left[r(\delta) \right]^{\nu m - \tau p} \left\| u^{\delta}_{r(\delta)} \right\| + \left\| u_{r(\delta)} - \bar{u} \right\| \right) \\ &= 0, \end{aligned}$$

proving $u^{\delta}_{r(\delta)}$ converges to $\bar{u}$ in $L^p[0,1]$ as $\delta \to 0$.

3. Returning to the bound on the total error in (3.32), we have

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| \le C_1 \frac{\delta}{[r(\delta)]^{\nu + \tau p}} + C_2 \left[r(\delta) \right]^{\alpha}.$$

To obtain a rate of convergence, it remains to bound $\dfrac{1}{[r(\delta)]^{\nu + \tau p}}$ and $[r(\delta)]^{\alpha}$ in terms of δ using our rule. First we bound $[r(\delta)]^{\alpha}$ in terms of δ.

Bound $B(r)$ above using the upper bound on a_r to obtain

$$B(r) \le a_r^m \cdot 1 \le \left(\frac{\kappa + 1}{\kappa c_0} \right)^m r^{\nu m},$$

for all $r \in (0, R]$. Therefore using the lower bound on d in (3.52), we have

$$\begin{aligned}
\tau\delta &= d(r(\delta)) \\
&\geq B(r(\delta)) \left\| f^{\delta}_{r(\delta)} \right\| \\
&\geq B(r(\delta)) \frac{\|f\|}{2} - B(r(\delta)) \left(2 + \bar{C}\,[r(\delta)]^{-\tau p}\right) \delta \\
&\geq B(r(\delta)) \frac{\|f\|}{2} - \left(\frac{\kappa + 1}{\kappa c_0}\right)^{m} \tilde{R}^{\nu m} \left(2 + \bar{C}\tilde{R}^{-\tau p}\right) \delta,
\end{aligned}$$

since $\nu m > \tau p$. Thus

$$2(\tau + E_1)\delta \geq B(r(\delta)) \, \|f\| , \tag{3.55}$$

for the constant $E_1 > 0$. As before, using both bounds on a_r in Lemma 3.2 to bound $B(r)$ in (3.42), we obtain

$$\begin{aligned}
B(r(\delta)) &\geq \left(\frac{\kappa - 1}{\kappa c_0}\right)^{1+m} [r(\delta)]^{\nu(1+m)} \left(\frac{\kappa + 1}{\kappa c_0}\tilde{R}^{\nu} + \bar{C}\,\|k\|_{C[0,1+R]}\right)^{-1} \\
&= E_2\,[r(\delta)]^{\nu(1+m)}, \qquad (3.56)
\end{aligned}$$

for the constant $E_2 > 0$. Combining (3.55) and (3.56), we have

$$2\left(\frac{\tau + E_1}{E_2\,\|f\|}\right)\delta \;\geq\; [r(\delta)]^{\nu(1+m)},$$

and so raising both sides to the $\dfrac{\alpha}{\nu(1+m)}$ power,

$$[r(\delta)]^{\alpha} \leq \tilde{E}\delta^{\alpha/(\nu(1+m))}, \tag{3.57}$$

for $\tilde{E} > 0$ constant.

Next we obtain the bound for $\dfrac{1}{[r(\delta)]^{\nu + \tau p}}$ in terms of δ. Bounding d above using the inequality in (3.39), Lemma (3.2), and [A0], we have

$$\begin{aligned}
\tau\delta &= a_{r(\delta)}^{1+m}\left\|u_{r(\delta)}^{\delta}\right\| \\
&\leq a_{r(\delta)}^{1+m}\left[\left\|u_{r(\delta)}^{\delta} - u_{r(\delta)}\right\| + \left\|u_{r(\delta)} - \bar{u}\right\| + \|\bar{u}\|\right] \\
&\leq a_{r(\delta)}^{m}(1+M)\bar{C}\,[r(\delta)]^{-\tau p}\,\delta + a_{r(\delta)}^{1+m}\left[D_2\,[r(\delta)]^{\alpha} + \|\bar{u}\|\right] \\
&\leq \left(\frac{(\kappa+1)\,[r(\delta)]^{\nu}}{\kappa c_0}\right)^{m}(1+M)\bar{C}\,[r(\delta)]^{-\tau p}\,\delta \\
&\quad + \left(\frac{(\kappa+1)\,[r(\delta)]^{\nu}}{\kappa c_0}\right)^{1+m}\left(D_2\tilde{R}^{\alpha} + \|\bar{u}\|\right) \\
&= G_1\,[r(\delta)]^{\nu m - \tau p}\,\delta + G_2\,[r(\delta)]^{\nu(1+m)},
\end{aligned}$$

for $G_1 > 0$ and $G_2 > 0$ constants. Since

$$\tau - G_1\,[r(\delta)]^{\nu m - \tau p} > \tau - G_1\hat{R}^{\nu m - \tau p} > 0,$$

for $r \in (0, \hat{R}]$, for some $\hat{R}$ sufficiently small, we have that

$$\frac{\tau - G_1\hat{R}^{\nu m - \tau p}}{G_2}\delta \;\leq\; [r(\delta)]^{\nu(1+m)}.$$

for $\delta > 0$ sufficiently small. Then

$$\frac{1}{[r(\delta)]^{\nu + \tau p}} \leq \tilde{G}\delta^{-(\nu + \tau p)/(\nu(1+m))}, \tag{3.58}$$

for $\tilde{G} > 0$ constant.

Substituting (3.57) and (3.58) into (3.32), we have

$$\begin{aligned}\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| &\leq C_1 \frac{\delta}{[r(\delta)]^{\nu + \tau p}} + C_2 [r(\delta)]^{\alpha} \\ &\leq C_1 \delta \tilde{G} \delta^{-(\nu + \tau p)/(\nu(1+m))} + C_2 \tilde{E} \delta^{\alpha/(\nu(1+m))},\end{aligned}$$

and so

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{(\nu m - \tau p)/(\nu(1+m))}\right) + O\left(\delta^{\alpha/(\nu(1+m))}\right) \text{ as } \delta \to 0.$$

If $\omega = \min\left\{\alpha, \nu m - \tau p\right\}$, then

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{\omega/(\nu(1+m))}\right) \quad \text{as} \quad \delta \to 0.$$

Then taking $m = \dfrac{\alpha + \tau p}{\nu}$, it follows that

$$\left\| u^{\delta}_{r(\delta)} - \bar{u} \right\| = O\left(\delta^{\alpha/(\alpha + \nu + \tau p)}\right) \quad \text{as} \quad \delta \to 0.$$

$\square$

CHAPTER 4

Discretization and Numerical Results

We illustrate the practical use of the theory developed in Chapters 2 and 3 with a few numerical examples. In each the following cases, we plot the true $\bar{u}$ (dashed line) versus the approximation u_τ^δ (solid line). We selected the true solution $\bar{u}$ for each case and found the true data f by computing $A\bar{u}$ exactly. We then discretized the data and added uniformly distributed random error to generate the vector f^δ. We take $\tau = \sqrt{2}$ for our discrepancy principle and take the value of δ to be $\left\| f - f^\delta \right\|_{C[0,\,1\,+\,\bar{R}]}$ in Examples 4.1–4.3 and $\left\| f - f^\delta \right\|_{L^2[0,\,1\,+\,\bar{R}]}$ in Example 4.4

4.1 One-Smoothing Problem, Continuous Measure

Example 4.1 *Let* $k(t) = 1$, $\bar{u}(t) = 1+3t\,[\sin(10t) - \sin(t)]$, *and* η_r *a continuous measure as defined in Lemma* 2.1, *where* ψ *is a first-degree polynomial with* $p_\nu(\lambda) = \lambda + 5$. *For the discretization* $n = 600$, $\bar{R} = 0.583$ *and* $m = 0.001$ *in our discrepancy principle,* *we have the following results:*

$\dfrac{\delta}{\|f\|_{C[0,1+\bar{R}]}}$	δ	$r(\delta)$	$\left\|\bar{u}-u_r^\delta\right\|_{C[0,1]}$	$\dfrac{\left\|\bar{u}-u_{rc}^\delta\right\|_{C[0,1]}}{\left\|\bar{u}-u_{rp}^\delta\right\|_{C[0,1]}}$
0.0500	*0.0450*	*0.35*	*0.8040*	–
0.0250	*0.0233*	*0.25*	*0.3654*	*0.454*
0.0125	*0.0126*	*0.183*	*0.1954*	*0.535*
0.007	*0.0071*	*0.133*	*0.1021*	*0.523*

Table 4.1. Example 4.1 Error Analysis

Based on the values of $m, \alpha,$ *and* ν, *the ratios in the last column are predicted by Theorem* 2.2 *to be approximately* $\left(\dfrac{1}{2}\right)^{0.000999} = 0.999$. *See graphical illustrations in Figures* 4.1–4.4 *below.*

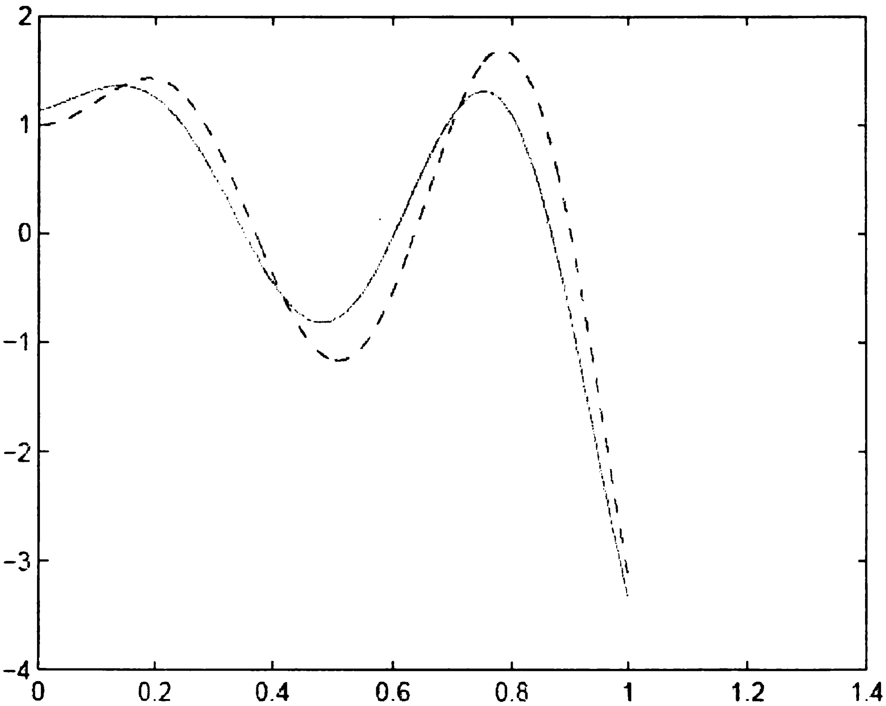

Figure 4.1. One-Smoothing Problem with Continuous Measure given in Example 4.1 with 5% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.35$

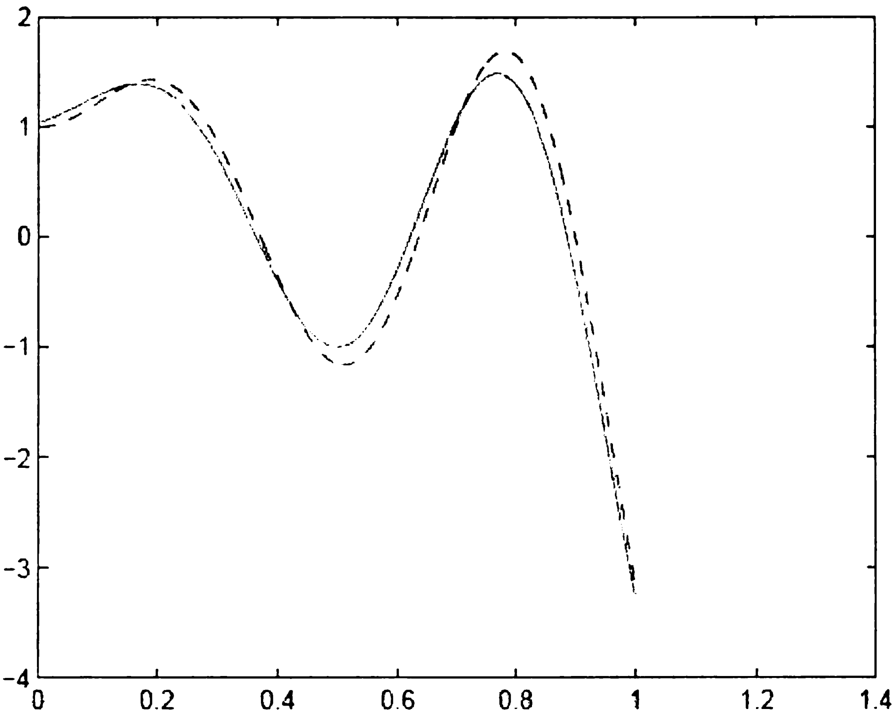

Figure 4.2. One-Smoothing Problem with Continuous Measure given in Example 4.1 with 2.5% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.25$

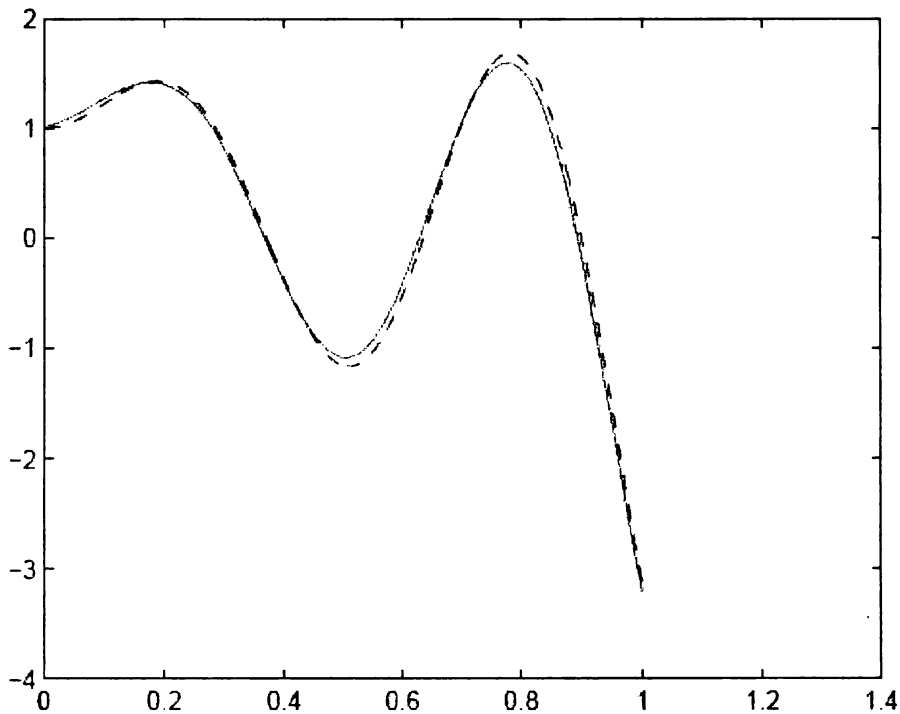

Figure 4.3. One-Smoothing Problem with Continuous Measure given in Example 4.1 with 1.25% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.183$

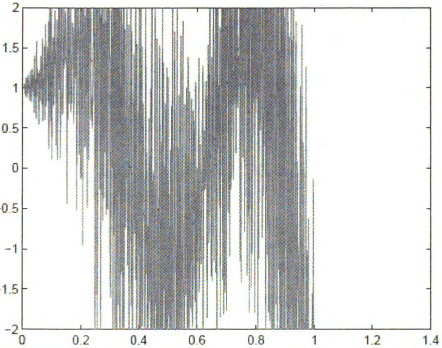

Figure 4.4. One-Smoothing Problem given in Example 4.1 with 1.25% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and the Solution with No Regularization

4.2 Four-smoothing Problem, Lebesgue Measure

Example 4.2 *Let* $k(t) = \frac{t^3}{6}$, $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$, *and* η_r *Lebesgue measure. For the discretization* $n = 100$, $\bar{R} = 1$ *and* $m = 0.001$ *in our discrepancy principle, we have the following results:*

$\frac{\delta}{\|f\|_{C[0,1+\bar{R}]}}$	δ	$r(\delta)$	$\left\|\bar{u} - u_r^\delta\right\|_{C[0,1]}$	$\frac{\left\|\bar{u} - u_{rc}^\delta\right\|_{C[0,1]}}{\left\|\bar{u} - u_{rp}^\delta\right\|_{C[0,1]}}$
0.00200	*4.1430e-004*	*0.50*	*2.8683*	–
0.00100	*2.0218e-004*	*0.42*	*3.0266*	*0.969*
0.00050	*9.8152e-005*	*0.45*	*3.3783*	*0.75*
0.00025	*5.4340e-005*	*0.39*	*2.4749*	*0.978*

Table 4.2. Example 4.2 Error Analysis

The ratios in the last column are predicted by Theorem 2.2 *to be approximately* $\left(\frac{1}{2}\right)^{0.000999} = 0.9993$. *The variation in the values obtained may in part be due the the fact that the ratio of the* δ *values is not exactly* $\frac{1}{2}$. *We note that the value of* $m = 0.001$ *means that the discrepancy principle is approximately one like Morozov's Discrepancy Principle; the slow rate is associated with* $m \approx 0$. *See graphical illustrations in Figures* 4.5–4.8 *below.*

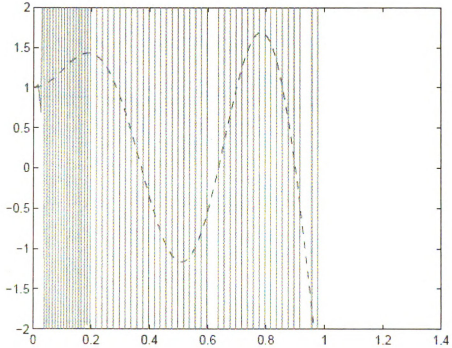

Figure 4.5. Four-smoothing Problem given in Examples 4.2 and 4.3 with 0.1% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and the Solution with No Regularization

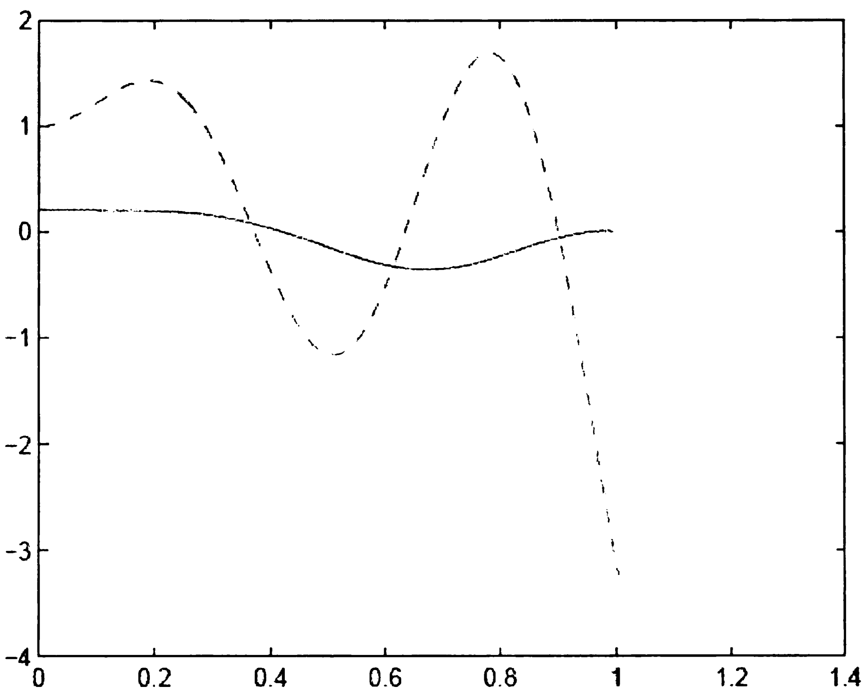

Figure 4.6. Four-smoothing Problem with Lebesgue Measure given in Example 4.2 with 0.1% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.69$

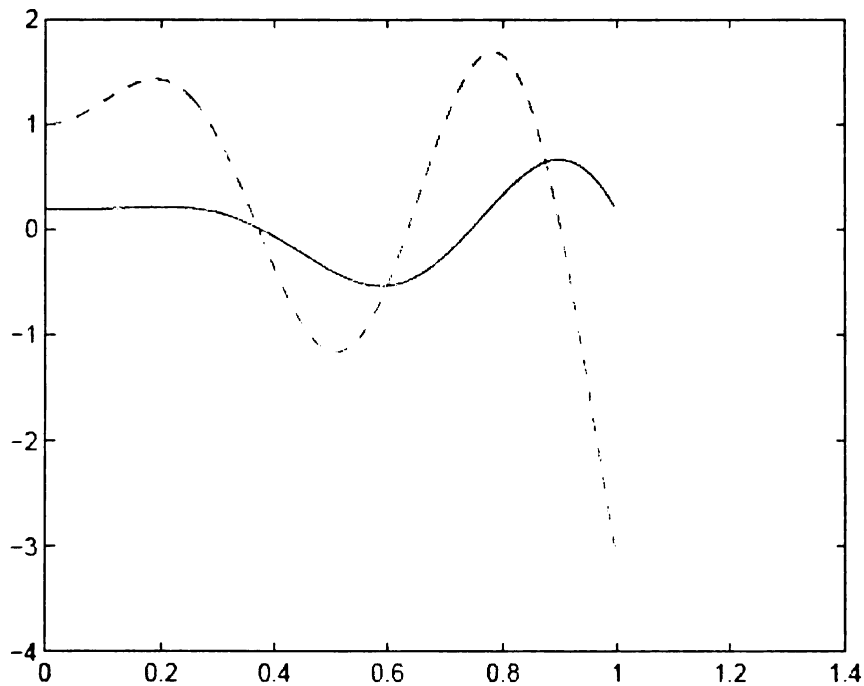

Figure 4.7. Four-smoothing Problem with Lebesgue Measure given in Example 4.2 with 0.05% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.58$

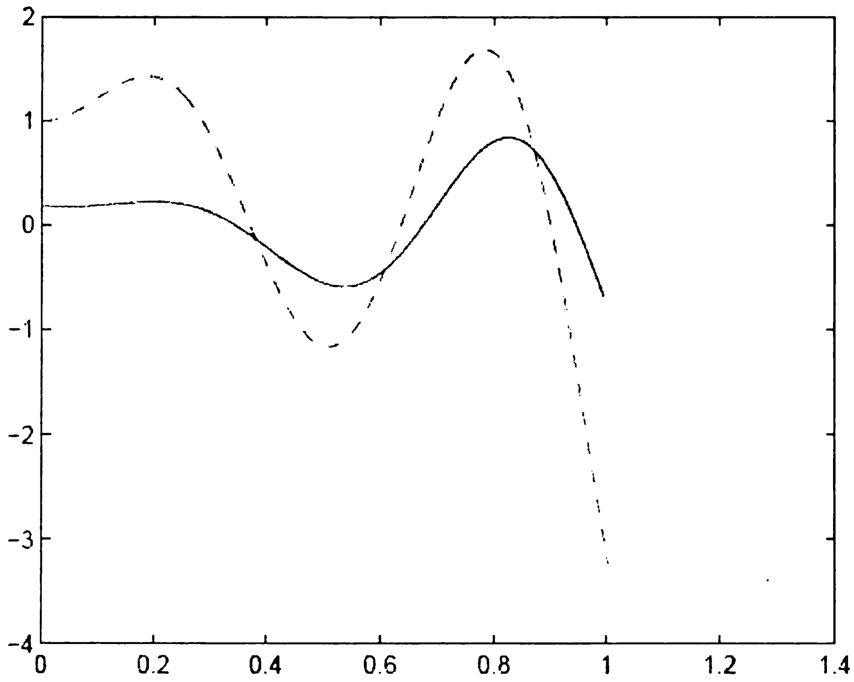

Figure 4.8. Four-smoothing Problem with Lebesgue Measure given in Example 4.2 with 0.025% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.51$

4.3 Four-smoothing Problem, Continuous Measure

Example 4.3 *Let* $k(t) = \frac{t^3}{6}$, $\bar{u}(t) = 1 + 3t\left[\sin(10t) - \sin(t)\right]$, *and* η_r *a continuous measure as defined in Lemma* 2.1, *where* ψ *is a fourth-degree polynomial with* $p_\nu(\lambda) = (\lambda + 5)^4$. *For the discretization* $n = 100$, $\bar{R} = 1$ *and* $m = 0.001$ *in our discrepancy principle, we have the following results:*

$\frac{\delta}{\Vert f\Vert_{C[0,1+\bar{R}]}}$	δ	$r(\delta)$	$\left\Vert \bar{u} - u_r^\delta \right\Vert_{C[0,1]}$	$\frac{\left\Vert \bar{u} - u_{rc}^\delta \right\Vert_{C[0,1]}}{\left\Vert \bar{u} - u_{rp}^\delta \right\Vert_{C[0,1]}}$
0.00200	*4.1784e-004*	*0.78*	*3.4179*	–
0.00100	*1.9812e-004*	*0.69*	*3.3356*	*0.976*
0.00050	*1.0249e-004*	*0.58*	*3.8640*	*1.158*
0.00025	*6.2523e-005*	*0.51*	*4.3731*	*1.11864*

Table 4.3. Example 4.3 Error Analysis I

$\frac{\delta}{\Vert f\Vert_{C[0,1+\bar{R}]}}$	δ	$r(\delta)$	$\left\Vert \bar{u} - u_r^\delta \right\Vert_{L^2[0,1]}$	$\frac{\left\Vert \bar{u} - u_{rc}^\delta \right\Vert_{L^2[0,1]}}{\left\Vert \bar{u} - u_{rp}^\delta \right\Vert_{L^2[0,1]}}$
0.0002	*4.1784e-004*	*0.78*	*0.75813*	–
0.00100	*1.9812e-004*	*0.69*	*0.68989*	*0.910*
0.00050	*1.0249e-004*	*0.58*	*0.68321*	*0.990*
0.00025	*6.2523e-005*	*0.51*	*0.69961*	*1.019*

Table 4.4. Example 4.3 Error Analysis II

The ratios in the last column are predicted by Theorem 2.2 *to be approximately* $\left(\frac{1}{2}\right)^{0.000999} = 0.9993$. *and should be the same as for Example* 4.2 *above. However, instability associated with finding the polynomial* ψ *for the measure* η_r *given in Lemma* 2.1 *leads to instability of the solution near* $t = 0$. *The results overall are better than in Example* 4.2, *but the growing error near* $t = 0$ *has a large negative effect on the ratios of* $C[0,1]$ *errors in the table above, compared to* $L^2[0,1]$ *errors as we expected to be smaller. This instability of the polynomial at* $t = 0$ *is a subject of future research. See graphical illustrations in Figure* 4.5 *above, and Figures* 4.9–4.11 *below.*

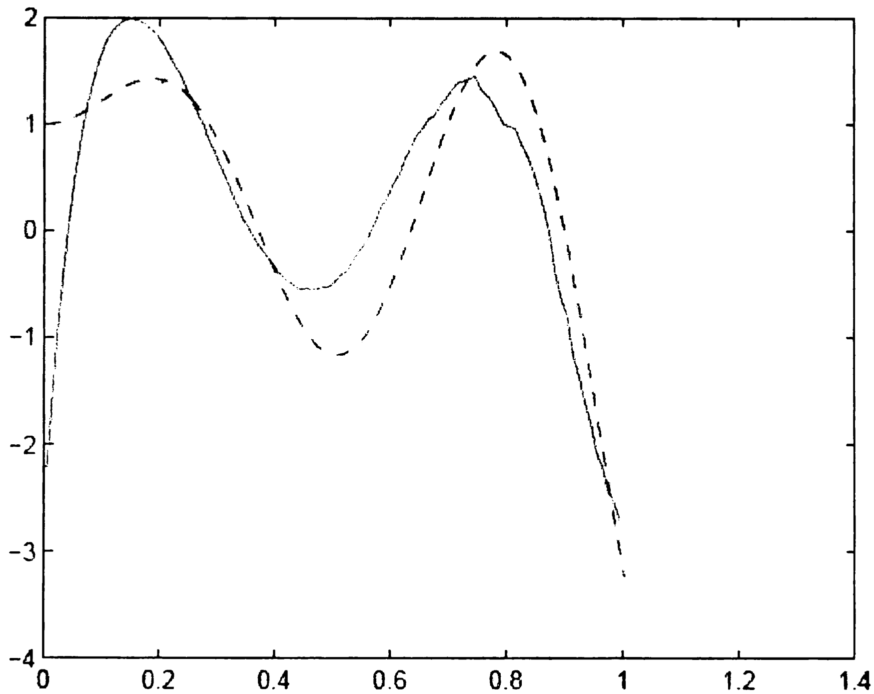

Figure 4.9. Four-smoothing Problem with Continuous Measure given in Example 4.3 with 0.1% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.69$

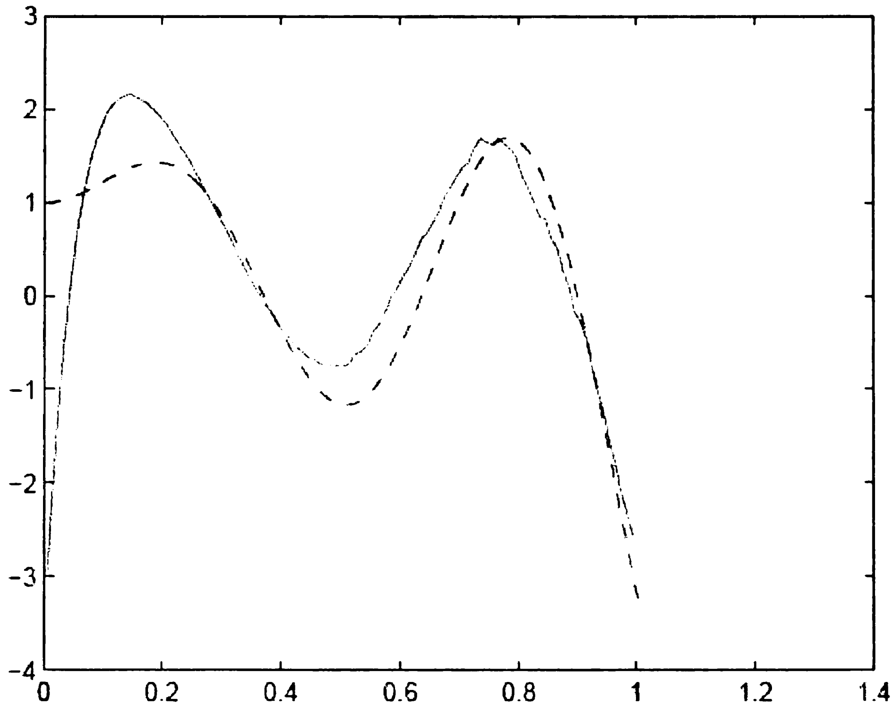

Figure 4.10. Four-smoothing Problem with Continuous Measure given in Example 4.3 with 0.05% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.58$

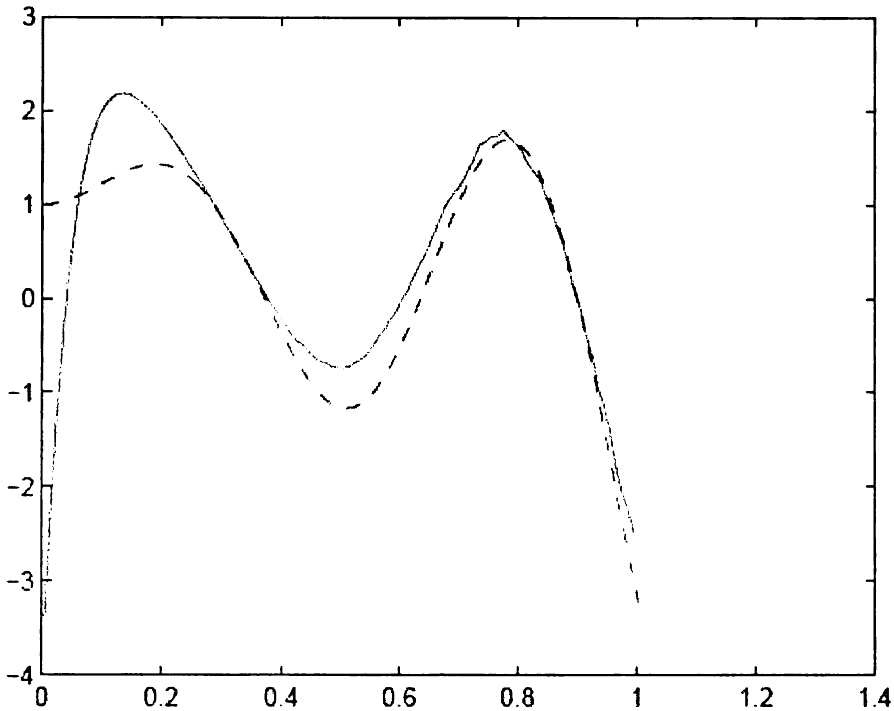

Figure 4.11. Four-smoothing Problem with Continuous Measure given in Example 4.3 with 0.025% Relative Error in the Data. Plots of $\bar{u}(t) = 1 + 3t\,[\sin(10t) - \sin(t)]$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.51$

4.4 Two-smoothing Problem, Discrete Measure

Example 4.4 *Let $k(t) = t$ and consider the true solution*

$$\bar{u}(t) = \begin{cases} 500, & \text{if } t \in [.25, .35) \cup [.6, .7), \\ 1, & \text{otherwise.} \end{cases}$$

We take η_r to be the discrete measure defined in Lemma 2.1 with β_ℓ, τ_ℓ corresponding to Beck's method as in [23]. For the discretization $n = 600$, $\bar{R} = 0.067$, and $m = 0.001$ in our discrepancy principle, we have the following results:

$\dfrac{\delta}{\lVert f \rVert_{L^2[0,1+\bar{R}]}}$	δ	$r(\delta)$	$\left\lVert \bar{u} - u_r^\delta \right\rVert_{L^2[0,1]}$	$\dfrac{\left\lVert \bar{u} - u_{rc}^\delta \right\rVert_{L^2[0,1]}}{\left\lVert \bar{u} - u_{rp}^\delta \right\rVert_{L^2[0,1]}}$
0.0210	*0.0737*	*0.0383*	*283.28*	–
0.0100	*0.0376*	*0.0233*	*366.56*	*1.294*
0.0050	*0.0189*	*0.015*	*483.08*	*1.318*
0.0024	*0.0071*	*0.010*	*561.55*	· *1.161*

Table 4.5. Example 4.4 Error Analysis

Because the true $\bar{u}$ is discontinuous, our theory does not predict a convergence rate. It is obvious that the general shape of the curve is found quite well by our method, however the anomalies at the jumps lead to large $L^2[0,1]$ error. The sharp spikes in the graphs of the approximations extend beyond the viewing window making it difficult to visualize the impact they have on the $L^2[0,1]$ error as shown in the table. See graphical illustrations in Figures 4.12–4.14 below.

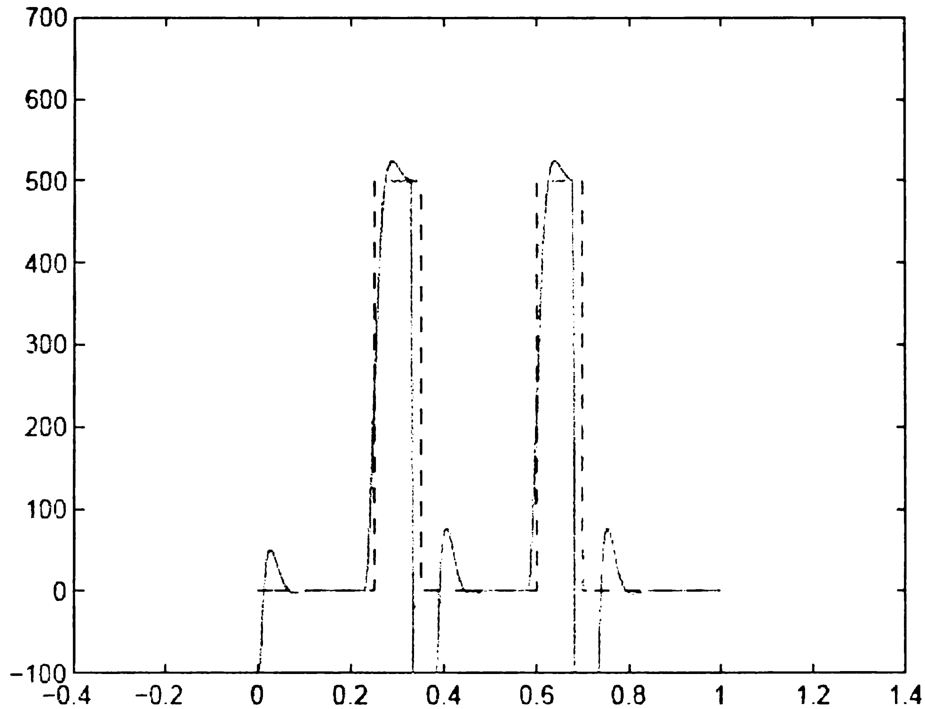

Figure 4.12. Two-smoothing Problem with Discrete Measure given in Example 4.4 with 1% Relative Error in the Data. Plots of the step function $\bar{u}$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.0233$

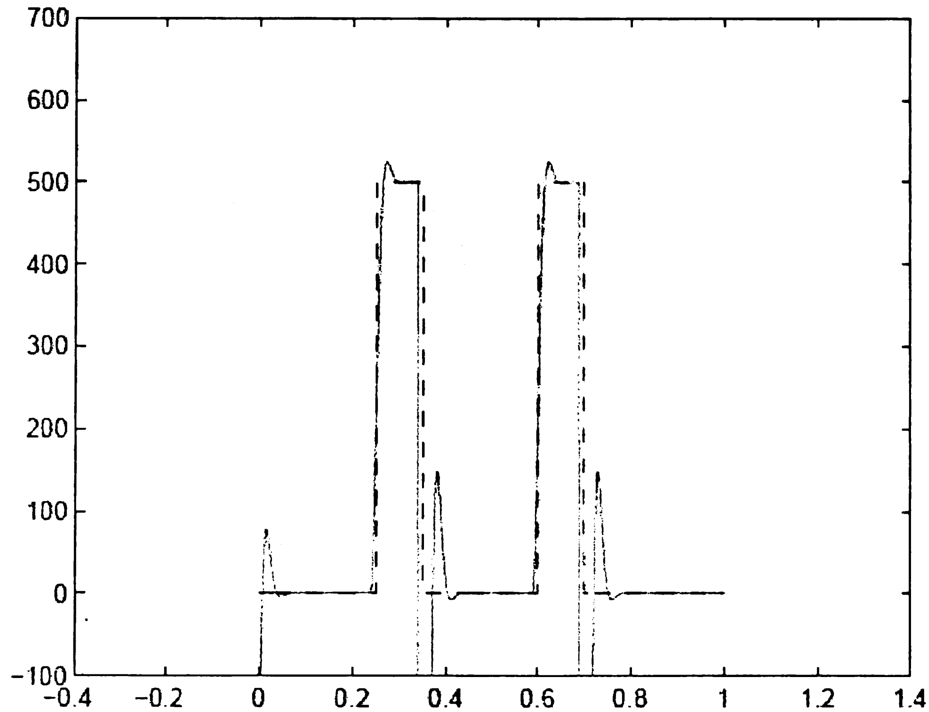

Figure 4.13. Two-smoothing Problem with Discrete Measure given in Example 4.4 with 0.5% Relative Error in the Data. Plots of the step function $\bar{u}$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.015$

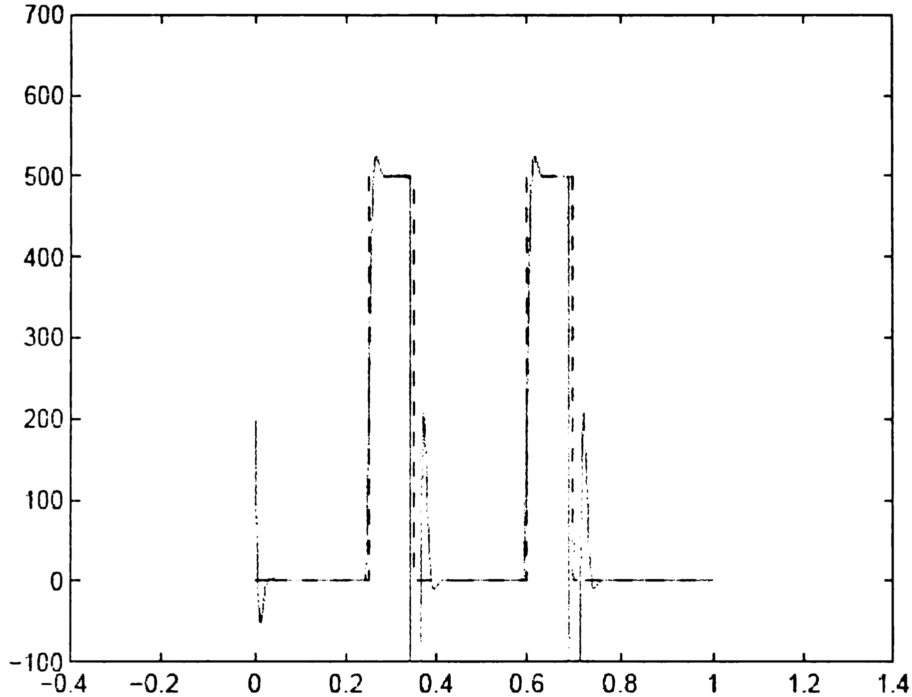

Figure 4.14. Two-smoothing Problem with Discrete Measure given in Example 4.4 with 0.25% Relative Error in the Data. Plots of the step function $\bar{u}$ and $u^{\delta}_{r(\delta)}$ with predicted value of $r(\delta) = 0.010$

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BIBLIOGRAPHY

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