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MIXED VOLUME AND TOTAL DEGREE

By

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ABSTRACT

MIXED VOLUME AND TOTAL DEGREE

By

Ying Zhang

This thesis focuses on the study of solving several extensible benchmark polynomial systems by homotopy continuation methods. By establishing the relationship between their mixed volume and total degree, we find that for most of those systems the difference between their mixed volume and total degree is very minimal. Consequently, those systems should be solved by the classical linear homotopy method rather than the polyhedral homotopy method, although in general the polyhedral homotopy method is the typical choice for solving sparse systems. Furthermore, by restricting to the classical linear homotopy on solving those systems, we may take the special structure of the systems into account for solving the systems efficiently. This precious aspect of the classical linear homotopy does not seem to exist in the polyhedral homotopy method.

To my sons, Ray and Ryan.

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Introduction

Polynomial systems arise very frequently in many fields of science and engineering [7], such as formula construction, geometric intersection problems, inverse kinematics, power flow problems with PQ-specified bases, computation of equilibrium states, etc. In 1977, Garcia and Zangwill [10] and Drexler [9] independently presented theorems, which suggested that homotopy continuation methods could be used to find the full set of isolated zeros of a polynomial system numerically. During the last three decades these methods have been developed into a reliable and efficient numerical algorithm for approximating all isolated zeros of polynomial systems.

Let $P(x) = 0$ be a system of n polynomial equations with n unknowns. Denoting $P = (p_1, \dots, p_n)$ and $x = (x_1, \dots, x_n)$, we want to find all isolated solutions of

$$P(x) = \begin{cases} p_1(x_1, \dots, x_n) = 0 \\ \qquad\vdots \\ p_n(x_1, \dots, x_n) = 0. \end{cases}$$

The classical homotopy continuation method [2, 3] for solving $P(x) = 0$ is to define a trivial system $Q(x) = (q_1(x), \dots, q_n(x))$ and then follow the solution curves in the real variable t from $t = 0$ to $t = 1$, which make up the solution set of

$$H(x,t) = (1-t)rQ(x) + tP(x) = 0 \qquad \text{with generic} \quad r \in \mathbb{C} \setminus \{0\}.$$

More precisely, all the isolated solutions of $P(x) = 0$ can be found if the system $Q(x) = 0$, known as the start system, is chosen properly to satisfy the following three properties:

- Property 0. The solutions of the start system $Q(x) = 0$ are known;
- Property 1. The solution set of $H(x,t) = 0$ for $0 \leqslant t \leqslant 1$ consists of a finite number of smooth paths, and each of them can be parameterized by t in [0,1);
- Property 2. Every isolated solution of $H(x,1) = P(x) = 0$ can be reached by some path originating at $t = 0$, that is, the path starts from a solution of the start system $H(x,0) = Q(x) = 0$.

A typical choice of a start system $Q(x) = 0$ satisfying Property 0–2 is

$$Q(x) = \begin{cases} q_1(x_1, \ldots, x_n) = a_1 x_1^{d_1} - b_1 \\ \qquad\qquad \vdots \\ q_n(x_1, \ldots, x_n) = a_n x_n^{d_n} - b_n \end{cases}$$

where $d_1, \ldots, d_n$ are the degrees of polynomials $p_1(x), \ldots, p_n(x)$, respectively, and a_j, b_j, $j = 1, \ldots, n$ are random complex numbers [6, 18, 23, 29, 37, 38]. The solutions of such a start system $Q(x) = 0$ can be explicitly obtained and the total number of solutions is $d = d_1 \times \cdots \times d_n$, which is known as the *total degree* or the Bézout number of the original polynomial system $P(x) = 0$ [33]. We may then find all the isolated solutions of $P(x) = 0$ by following the total degree number of paths originating from solutions of the start system $Q(x) = 0$. But, a great majority of the polynomial systems arising in applications have fewer than, and in some cases only a small fraction of $d = d_1 \times \cdots \times d_n$ isolated zeros. We call such a system *deficient.* In this case, many of the $d_1 \times \cdots \times d_n$ paths will diverge to infinity as $t \to 1$, and those paths become extraneous, causing highly wasteful computation.

In the middle of 1990's, a major computational breakthrough emerged in solving deficient polynomial systems efficiently by the homotopy continuation method. The new method, called the *polyhedral homotopy* method [13], takes a great advantage of the combinatorial root count in the Bernshteín's theory [4], which generally provides a much tighter bound, called *mixed volume*, for the number of isolated zeros of a polynomial system in the algebraic tori $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. When the polyhedral homotopy method is employed to solve a polynomial system, the number of homotopy paths that need to be traced agrees with twice of the mixed volume of the polynomial system. As an important consequence, when the mixed volume of a polynomial system is far less than its total degree, then solving the systems by the polyhedral homotopy method will greatly reduce the extraneous paths and thereby considerably limit the wasteful computations.

However, in the core of the polyhedral homotopy continuation method, there is a sometimes costly computation, namely the *mixed cell* computation, which provides the critically important start system that one can handle for the polyhedral homotopy. Indeed, this mixed cell computation can become very costly for large polynomial systems. Therefore, before the polyhedral homotopy is used to solve the polynomial system at the expense of the sometimes costly mixed cell computations, a *prior* knowledge on the comparison of the total degree of the system and its mixed volume is highly desirable. If a substantial difference between these two numbers of the system is absent, then, of course, the system should be solved by the classical linear homotopy rather than the polyhedral homotopy.

While it has been largely admitted for grant that for most of the sparse polynomial systems their mixed volume is far less than their total degree, in this thesis we analyze several extensible benchmark polynomial systems in the opposite. By establishing the relationship between their mixed volume and total degree, we find that for most of those systems the difference between their mixed volume and total degree is very minimal. Consequently, those systems should be solved by the classical linear homotopy method. Furthermore, by restricting to the classical linear homotopy on solving those systems, we may take the special structure of the systems into account for solving the systems efficiently. This precious aspect of the classical linear homotopy does not seem to exist in the polyhedral homotopy method.

The thesis is organized as follows. In Chapter 1, the Bernshteín theorem is introduced along with its application on the polyhedral homotopy continuation method for solving polynomial systems, including mixed volume and mixed cell computations. In Chapter 2, 3 and 4, we study the extensible benchmark systems Katsura-n [35], Reimer-n [35] and Noon-n [31] respectively. It is shown that for each of those systems the difference between the mixed volume of the system and its total degree is very slim, if not zero. Therefore the polyhedral homotopy continuation method, widely considered the state of the art, is inappropriate for solving those systems. They should be solved by the classical linear homotopy method. Furthermore, when the classical linear homotopy is used to solve the Noon-n systems in Chapter 4, particularly illuminating is the marvelous speed-ups by choosing proper start system to recognize the symmetry structure of the system.

In Chapter 5, we study the generalized eigenvalue problem $Ax = \lambda Bx$, where A and B are $n \times n$ matrices. When this problem is considered as a polynomial system, it contains n equations in $n+1$ variables. With an appended linear equation, the mixed volume of the resulting system is shown to be n, that is far less than its total degree 2^n. Nonetheless, from the obvious m-homogeneous structure of the system, proper start systems with n isolated solutions are always available in the classical linear homotopy. In this situation, the employment of the polyhedral homotopy method for solving this system is still unnecessary so that the mixed cell computations can be avoided.

It is commonly known that very efficient algorithms for matrix eigenvalue problems, QR algorithm for $Ax = \lambda x$ and QZ algorithm for $Ax = \lambda Bx$, have been implemented and asserted in the software package LAPACK [16]. However, as the size of the matrix becomes larger, more computing resources are required. And a natural way to allocate extra computing resources efficiently is to perform independent tasks simultaneously in parallel. Since each isolated zero of a polynomial system is computed independently of all the others in the homotopy continuation method, it provides a natural environment for the parallelization. In this regard, solving very large algebraic eigenvalue problems by the homotopy continuation method in parallel offers a great perspective in contrast to highly serial QR or QZ algorithms.

CHAPTER 1

The polyhedral homotopy

1.1 The Bernshteín theory

Let $P(x) = (p_1(x), \dots, p_n(x)) \in \mathbb{C}[x]$ be a given polynomial system, where $x = (x_1, ..., x_n)$. Denoting $x^a = x_1^{a_1} \dots x_n^{a_n}$ with $a = (a_1, \dots, a_n)$, write

$$P(x) = \begin{cases} p_1(x) = \sum_{a \in S_1} c^*_{1,a} x^a \\ \vdots \\ p_n(x) = \sum_{a \in S_n} c^*_{n,a} x^a \end{cases} \tag{1.1.1}$$

where $S_1, \dots, S_n$ are fixed subsets of $\mathbb{N}^n$ with cardinalities $k_j = \# S_j$ and $c^*_{j,a} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for $a \in S_j, j = 1, \dots, n$. Here S_j is called the *support* of $p_j(x)$ and denoted by $\mathrm{supp}(p_j)$. Its convex hull $Q_j = \mathrm{conv}(S_j)$ in $\mathbb{R}^n$ is called the *Newton polytope* of p_j and $S = (S_1, \dots, S_n)$ is the *support* of $P(x)$, denoted by $\mathrm{supp}(P)$. For nonnegative variables $\lambda_1, \dots, \lambda_n$ and the Newton polytopes Q_j of p_j for $j = 1, \dots, n$, let $\lambda_1 Q_1 + \cdots + \lambda_n Q_n$ be the *Minkowski* sum of $\lambda_1 Q_1, \dots, \lambda_n Q_n$, i.e.,

$$\lambda_1 Q_1 + \cdots + \lambda_n Q_n = \{\lambda_1 r_1 + \cdots + \lambda_n r_n : r_j \in Q_j, j = 1, \dots, n\}.$$

It can be shown that the n-dimensional volume, denoted Vol_n, of the polytope $\lambda_1 Q_1 + \cdots + \lambda_n Q_n$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$, and the coefficient of the term $\lambda_1 \times \cdots \times \lambda_n$ in this homogeneous polynomial is called the *mixed volume* of the polytopes $Q_1, \ldots, Q_n$, denoted $\mathcal{M}(Q_1, \ldots, Q_n)$, or the mixed volume of the supports $S_1, \ldots, S_n$, denoted $\mathcal{M}(S_1, \ldots, S_n)$. Sometimes, it is called the mixed volume of $P(x)$ when no ambiguities exist.

The system (1.1.1) can be embedded into the system $P(c, x) = (p_1(c, x), \ldots, p_n(c, x))$, where

$$P(c,x) = \begin{cases} p_1(c,x) = \sum_{a \in S_1} c_{1,a} x^a \\ \quad\vdots \\ p_n(c,x) = \sum_{a \in S_n} c_{n,a} x^a \end{cases} \tag{1.1.2}$$

and the coefficients $c_{j,a}$ with $a \in S_j$, $j = 1, \ldots, n$, are taken to be a set of $M = k_1 + \cdots + k_n$ variables. Namely, the system $P(x)$ in (1.1.1) is considered as a system in (1.1.2) corresponding to a set of specified values of coefficients $c^* = (c^*_{j,a})$ or $P(x) = P(c^*, x)$.

Lemma 1.1.1. [12] *For polynomial systems $P(c, x)$ in (1.1.2), there exists a polynomial system $G(c) = (g_1(c), \ldots, g_n(c))$ in the variables $c = (c_{j,a})$ for $a \in S_j$ and $j = 1, \ldots, n$ such that for those coefficients $\bar{c} = (c^*_{j,a})$ for which $G(\bar{c}) \neq 0$, the root count in $(\mathbb{C}^*)^n$ of the corresponding polynomial systems in (1.1.2) is a fixed number, and the root count in $(\mathbb{C}^*)^n$ of any other polynomial systems in (1.1.2) is bounded above by this number.*

Remark 1.1.1. *Since the zeros of the polynomial system $G(c)$ in Lemma 1.1.1 form an algebraic set with dimension smaller than M, its complement is open and dense with full measure in $\mathbb{C}^M$. Therefore, with probability one, $G(c^*) \neq 0$ for randomly chosen coefficients $c^* = (c^*_{j,a}) \in \mathbb{C}^M$. Hence, polynomial systems $P(c^*, x)$ in (1.1.2) with $G(c^*) \neq$*

0 *are said to be* " in general position".

Theorem 1.1.1. *(Bernshteín)* [4] *The number of isolated zeros in* $(\mathbb{C}^*)^n$*, counting multiplicities, of a polynomial system* $P(x) = (p_1(x), \ldots, p_n(x))$ *with support* $S = (S_1, \ldots, S_n)$ *is bounded above by the mixed volume* $\mathcal{M}(S_1, \ldots, S_n)$*. When* $P(x)$ *is in general position, it has exactly* $\mathcal{M}(S_1, \ldots, S_n)$ *isolated zeros in* $(\mathbb{C}^*)^n$.

In [5], this root count was nicknamed as BKK bound after its inventors, Bernshteín [4], Kushnirenko [15] and Khovanskii [14]. In general, it provides a much tighter bound compared to variant Bézout bounds [30, 33]. An apparent limitation of the theorem is that it only counts the isolated zeros of polynomial system in $(\mathbb{C}^*)^n$ rather than all the isolated zeros in the affine space $\mathbb{C}^n$. For the purpose of finding all the isolated zeros of a polynomial system in $\mathbb{C}^n$, a generalized version of the assertion in the theorem which counts the roots in $\mathbb{C}^n$ is strongly desirable. This problem was first attempted in [32] by introducing the notion of the *shadowed* sets and a bound for the root count in $\mathbb{C}^n$ was obtained. Later, a significantly much tighter bound was discovered in the following theorem.

Theorem 1.1.2. [27] *The root count in* $\mathbb{C}^n$ *of a polynomial system* $P(x) = (p_1(x), \ldots, p_n(x))$ *with supports* $S_j = \mathrm{supp}(p_j)$, $j = 1, \ldots, n$, *is bounded above by the mixed volume* $\mathcal{M}(S_1 \cup \{0\}, \ldots, S_n \cup \{0\})$.

Corollary 1.1.1. *For polynomial system* $P(x) = (p_1(x), \ldots, p_n(x))$ *in (1.1.1), assume all* $p_j(x)$*s have constant term, then the number of isolated zeros of* $P(x)$ *in* $\mathbb{C}^n$ *is bounded above by the mixed volume* $\mathcal{M}(S_1, \ldots, S_n)$ *of its supports* $S = (S_1, \ldots, S_n)$*. When* $P(x)$ *is in general position, all zeros of* $P(x)$ *in* $\mathbb{C}^n$ *are isolated and its total number is exactly equal to* $\mathcal{M}(S_1, \ldots, S_n)$.

For a polynomial $p(x) = p(x_1, \ldots, x_n)$ of degree d, denote *the associated homogeneous* *polynomial* by

$$\tilde{p}(x_0, x_1, \ldots, x_n) = x_0^d p(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$

The solutions of $p(x) = 0$ *at infinity* are those zeros of $\tilde{p}$ in projective space

$$\mathbb{P}^n = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \setminus (0, \ldots, 0)\}/\sim$$

with $x_0 = 0$ where the equivalent relation $\sim$ is given by $x \sim y$ if $x = cy$ for some nonzero $c \in \mathbb{C}$. On the other hand, zeros of $p(x)$ in $\mathbb{C}^n$ can be identified with zeros of $\tilde{p}$ in $\mathbb{P}^n$ with $x_0 = 1$.

When the system $P(x) = (p_1(x), \ldots, p_n(x))$ in (1.1.1) is viewed in $\mathbb{P}^n$, namely we consider $\tilde{P}(x_0, x_1, \ldots, x_n) = (\tilde{p}_1(x_0, x_1, \ldots, x_n), \ldots, \tilde{p}_n(x_0, x_1, \ldots, x_n))$, then

Theorem 1.1.3. *(Bézout) If all the zeros of $\tilde{P}(x_0, x_1, \ldots, x_n)$ in $\mathbb{P}^n$ are isolated, then the number of those isolated zeros, counting multiplicities, equals to its total degree.*

Together with Corollary 1.1.1, we conclude with the following proposition.

Proposition 1.1.1. *For polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in general position in which all $p_j(x)$s have constant term, assume the zeros of $P(x)$ at infinity are all isolated, then*

$$\textit{Total degree of } P(x) = \textit{Mixed volume of } P(x) + \textit{number of isolated zeros of } P(x) \textit{ at infinity.}$$

1.2 The polyhedral homotopy

In light of Theorem 1.1.2 given in the above section, to find all isolated zeros of polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in $\mathbb{C}^n$ with support $S = (S_1, \ldots, S_n)$, we first add the monomial $x^0 (= 1)$ to those p_is which do not have constant terms. Followed by choosing coefficients of all monomials at random, a new polynomial system $Q(x) = (q_1(x), \ldots, q_n(x))$ with support $S' = (S_1', \ldots, S_n')$, where $S_j' = S_j \cup \{0\}$ for $j = 1, \ldots, n$, is obtained:

$$Q(x) = \begin{cases} q_1(x) = \sum_{a \in S_1'} \bar{c}_{1,a} x^a \\ \vdots \\ q_n(x) = \sum_{a \in S_n'} \bar{c}_{n,a} x^a. \end{cases}$$

We call such a system an *augmented* system of $P(x)$. Since all those coefficients $\bar{c}_{j,a}$ for $a \in S_j'$ and $j = 1, \ldots, n$ are randomly chosen, this system may be regarded as a system in general position. We want to solve this system in the first place. Afterwards, this system will be used as the start system to solve $P(x) = 0$ via linear homotopy.

Let t denote a new complex variable and consider the polynomial system $\hat{Q}(x,t) = (\hat{q}_1(x,t), \ldots, \hat{q}_n(x,t))$ in the $n+1$ variables (x,t), where

$$\begin{aligned} \hat{q}_1(x,t) &= \sum_{a \in S_1'} \bar{c}_{1,a} x^a t^{w_1(a)} \\ &\vdots \\ \hat{q}_n(x,t) &= \sum_{a \in S_n'} \bar{c}_{n,a} x^a t^{w_n(a)} \end{aligned} \tag{1.2.1}$$

and the images of each $w_j : S_j' \to R$ for $j = 1, \ldots, n$ are chosen generically. For a fixed

t_0, system $\hat{Q}(x,t) = (\hat{q}_1(x,t), \ldots, \hat{q}_n(x,t))$ can be written as

$$\hat{Q}(x,t_0) = \begin{cases} \hat{q}_1(x,t_0) = \sum_{a \in S_1'} (\bar{c}_{1,a} t_0^{w_1(a)}) x^a \\ \vdots \\ \hat{q}_n(x,t_0) = \sum_{a \in S_n'} (\bar{c}_{n,a} t_0^{w_n(a)}) x^a. \end{cases} \tag{1.2.2}$$

Remark 1.2.1. [17] *System (1.2.2) is in general position.*

Now we regard $\hat{Q}(x,t) = 0$ as a homotopy, known as the *polyhedral homotopy*, defined on $(\mathbb{C}^*)^n \times [0,1]$ with target system $\hat{Q}(x,1) = Q(x)$. The zero set of this homotopy is made up of k homotopy paths $x^{(1)}(t), \ldots, x^{(k)}(t)$. Since each $\hat{q}_j(x,t)$ has nonzero constant term for all $j = 1, \ldots, n$, it follows from a standard application of generalized Sard's Theorem [1] that all those homotopy paths are smooth with no bifurcations. Therefore, both Property 1 and Property 2 hold for this homotopy. However, as for Property 0, at $t = 0$, $\hat{Q}(x,0) \equiv 0$, so those homotopy paths can not get started because the paths originating from $t = 0$ can not be identified. We deal with this problem with the following design.

The function $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_j : S_j' \to \mathbb{R}, j = 1, \ldots, n$, may be considered as a *generic lifting* on the support $S' = (S_1', \ldots, S_n')$ of $Q(x)$, which lifts S_j' to its graph

$$\hat{S}_j' = \{\hat{a} = (a, \omega_j(a)) \mid a \in S_j'\}, \quad j = 1, \ldots, n.$$

Let $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ satisfy the following condition:

Condition A: *There exists a collection of pairs* $\{a_1, a_1'\} \subset S_1', \ldots, \{a_n, a_n'\} \subset S_n'$, *where*

$\{a_1 - a_1', \ldots, a_n - a_n'\}$ *is linearly independent and for* $j = 1, \ldots, n$

$$\langle \hat{a}_j, \hat{\alpha} \rangle = \langle \hat{a}_j', \hat{\alpha} \rangle,$$

$$\langle \hat{a}, \hat{\alpha} \rangle > \langle \hat{a}_j', \hat{\alpha} \rangle, a \in S_j' \setminus \{a_j, a_j'\}.$$

Here $\langle .,. \rangle$ stands for the usual inner product in the Euclidean space. For such $\hat{\alpha} = (\alpha, 1)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, let

$$\begin{cases} y_1 = t^{-\alpha_1} x_1 \\ \quad \vdots \\ y_n = t^{-\alpha_n} x_n. \end{cases}$$

In short, we write $y = t^{-\alpha} x$ with $y = (y_1, \ldots, y_n)$ or $x = y t^{\alpha}$. By this transformation and $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we have

$$\begin{aligned} x^a &= x_1^{a_1} \cdots x_n^{a_n} = (y_1 t^{\alpha_1})^{a_1} \cdots (y_n t^{\alpha_n})^{a_n} \\ &= y_1^{a_1} \cdots y_n^{a_n} t^{\alpha_1 a_1 + \cdots + \alpha_n a_n} = y^a t^{\langle a, \alpha \rangle}. \end{aligned}$$

Consequently, $\hat{q}_j(x, t)$, for $j = 1, \ldots, n$, of $\hat{Q}(x, t)$ in (1.2.1) becomes

$$\hat{q}_j(yt^{\alpha}, t) = \sum_{a \in S_j'} \bar{c}_{j,a} y^a t^{\langle a, \alpha \rangle} t^{\omega_j(a)} = \sum_{a \in S_j'} \bar{c}_{j,a} y^a t^{\langle (a, \omega_j(a)), (\alpha, 1) \rangle} = \sum_{a \in S_j'} \bar{c}_{j,a} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle}$$

where $\hat{a} = (a, \omega_j(a))$ and $\hat{\alpha} = (\alpha, 1)$.

For $j = 1, \ldots, n$, let

$$\beta_j = \min_{a \in S_j'} \langle \hat{a}, \hat{\alpha} \rangle$$

and consider the homotopy $H^{\alpha}(y,t) = (h_1^{\alpha}(y,t),\ldots,h_n^{\alpha}(y,t)) = 0$ on $(\mathbb{C}^*)^n \times [0,1]$, where

$$h_j^{\alpha}(y,t) = t^{-\beta_j}\hat{q}_j(yt^{\alpha},t) = \sum_{a\in S_j'} \bar{c}_{j,a}y^a t^{\langle \hat{a},\hat{\alpha}\rangle-\beta_j}$$

$$= \sum_{\substack{a\in S_j' \\ \langle \hat{a},\hat{\alpha}\rangle=\beta_j}} \bar{c}_{j,a}y^a + \sum_{\substack{a\in S_j' \\ \langle \hat{a},\hat{\alpha}\rangle>\beta_j}} \bar{c}_{j,a}y^a t^{\langle \hat{a},\hat{\alpha}\rangle-\beta_j}.$$

This homotopy retains most of the properties of the homotopy $\hat{Q}(x,t) = 0$; in particular, both Property 1 and Property 2 remain valid and

$$H^{\alpha}(y,1) = \hat{Q}(y,1) = Q(y) = Q(x).$$

Moreover, with Condition A, for each $j = 1,\ldots,n$, $\langle \hat{a}_j,\hat{\alpha}\rangle = \langle \hat{a}_j',\hat{\alpha}\rangle = \beta_j$ and $\langle \hat{a},\hat{\alpha}\rangle > \beta_j$ for $a \in S_j' \setminus \{a_j, a_j'\}$. Thus when $t = 0$

$$H^{\alpha}(y,0) = \begin{cases} h_1^{\alpha}(y,0) = \sum_{a\in S_1'} \bar{c}_{1,a}y^a = \bar{c}_{1,a_1}y^{a_1} + \bar{c}_{1,a_1'}y^{a_1'} = 0 \\ \vdots \\ h_n^{\alpha}(y,0) = \sum_{a\in S_n'} \bar{c}_{n,a}y^a = \bar{c}_{n,a_n}y^{a_n} + \bar{c}_{n,a_n'}y^{a_n'} = 0 \end{cases}$$

which is known as the *binomial system.*

Proposition 1.2.1. *[12] The binomial system*

$$\begin{cases} \bar{c}_{1,a_1}y^{a_1} + \bar{c}_{1,a_1'}y^{a_1'} = 0, \\ \vdots \\ \bar{c}_{n,a_n}y^{a_n} + \bar{c}_{n,a_n'}y^{a_n'} = 0 \end{cases}$$

has

$$k_\alpha := \left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right|$$

nonsingular isolated solutions in $(\mathbb{C}^*)^n$.

Proposition 1.2.2. *[12] Different* $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ *that satisfy Condition A will induce different homotopies* $H^\alpha(y,t) = 0$. *Those different homotopies will reach different sets of isolated zeros of* $H^\alpha(y,1) = Q(y) = Q(x)$. *Moreover, those different sets of isolated zeros of* $Q(x)$ *are disjoint from each other.*

Proposition 1.2.3. *[12] The root count of* $Q(x) = 0$ *in* $(\mathbb{C}^*)^n$ *or the mixed volume of the augmented system of* $P(x)$ *is*

$$\sum_\alpha k_\alpha = \sum_\alpha \left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right|.$$

A key step in solving system $Q(x) = 0$ by the polyhedral homotopy method described above is the search of all those vectors $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ as well as their associated collections of pairs $C^\alpha = (\{a_1, a_1'\}, \ldots, \{a_n, a_n'\})$ which we call *mixed cells* with *inner normal* α that satisfy Condition A. This is one of the most time consuming parts of the polyhedral homotopy method and well developed algorithms for finding those mixed cells can be found in [11, 19, 20, 22].

After all isolated solutions of $Q(x) = 0$ are attained, the linear homotopy

$$H(x,t) = (1-t)rQ(x) + tP(x) = 0 \qquad \text{with generic} \quad r \in \mathbb{C}^*$$

will be used to solve the target system $P(x) = 0$ because this homotopy now satisfies all the three properties [24].

We now summarize the polyhedral homotopy procedure for solving polynomial systems.

Given polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with support $S = (S_1, \ldots, S_n)$, let $S' = (S'_1, \ldots, S'_n)$ with $S'_j = S_j \cup \{0\}$ for $j = 1, \ldots, n$.

- Step 0: Initialization.

  Choose polynomial system $Q(x) = (q_1(x), \ldots, q_n(x))$ with support $S' = (S'_1, \ldots, S'_n)$ and generically chosen coefficients $c_{j,a}$, for $a \in S'_j$ and $j = 1, \ldots, n$. That is,

$$q_j(x) = \sum_{a \in S'_j} c_{j,a} x^a, \quad j = 1, \ldots, n.$$

- Step 1: Solve $Q(x) = 0$.

  - Step 1.1: Choose a set of real valued functions $\omega_j : S'_j \to \mathbb{R}$, $j = 1, \ldots, n$, their images are generic numbers.

  - Step 1.2: Find all the cells $C^\alpha = (\{a_1, a'_1\}, \ldots, \{a_n, a'_n\})$, $j = 1, \ldots, n$ of $S' = (S'_1, \ldots, S'_n)$ induced by $\omega = (\omega_1, \ldots, \omega_n)$ with $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ being the inner normal of $(\{\hat{a}_1, \hat{a}'_1\}, \ldots, \{\hat{a}_n, \hat{a}'_n\})$ in $S' = (S'_1, \ldots, S'_n)$.

  - Step 1.3: For each $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$ and its associated cell C^α obtained in Step 1.2.

◦ Step 1.3.1 Solve the binomial system

$$\begin{cases} c_{1,a_1} y^{a_1} + \bar{c}_{1,a_1'} y^{a_1'} = 0 \\ \vdots \\ c_{n,a_n} y^{a_n} + \bar{c}_{n,a_n'} y^{a_n'} = 0 \end{cases}$$

in $(\mathbb{C}^*)^n$, let the solution set be X_α^*.

◦ Step 1.3.2: Let $\beta_j = \langle \hat{a}_j, \hat{\alpha} \rangle$ for $j = 1, \ldots, n$. Follow homotopy paths $y(t)$ of the polyhedral homotopy

$$H^\alpha(y,t) = \begin{cases} h_1^\alpha(y,t) = \sum_{a \in S_1'} c_{1,a} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle - \beta_1} \\ \vdots \\ h_n^\alpha(y,t) = \sum_{a \in S_n'} c_{n,a} y^a t^{\langle \hat{a}, \hat{\alpha} \rangle - \beta_n} \end{cases}$$

starting from the solutions in X_α^*. Collect all the solutions of $y(1)$ as a subset of isolated zeros of $Q(x)$.

- Step 2: Solve $P(x) = 0$.

Follow the homotopy paths of the linear homotopy

$$H(x,t) = (1-t)rQ(x) + tP(x) = 0 \qquad \text{with generic} \quad r \in \mathbb{C}^*$$

starting from the solutions of $Q(x) = 0$ obtained in Step 1 to get all the isolated solutions of $P(x) = 0$ at $t = 1$.

Remark 1.2.2. *To find all isolated zeros of $P(x)$ in $\mathbb{C}^n$, $k = \mathcal{M}(S_1', \ldots, S_n')$ homotopy paths need to be traced in both Step 1.3 and Step 2, making it $2k$ homotopy paths in total.*

CHAPTER 2

Katsura-n system

In this Chapter, we consider the Katsura-n system [35]. For this polynomial system, we shall prove that the mixed volume of its augmented system is equal to its total degree. Therefore, for finding all the isolated zeros of this system in $\mathbb{C}^n$, the polyhedral homotopy method offers no advantages as regard to minimizing the number of homotopy paths one needs to trace.

2.1 Katsura-n system

Katsura-n system actually contains $n+1$, rather than n, variables $x_1, \ldots, x_{n+1}$. It has the following forms:

- if n is odd, the polynomials are

$$\begin{cases} 2x_{n+1} + 2x_n + \cdots + 2x_2 + x_1 - 1 \\ 2x_{n+1}^2 + 2x_n^2 + \cdots + 2x_2^2 + x_1^2 - x_1 \\ 2x_n x_{n+1} + 2x_{n-1}x_n + \cdots + 2x_1x_2 - x_2 \\ 2x_{n-1}x_{n+1} + 2x_{n-2}x_n + \cdots + 2x_1x_3 + x_2^2 - x_3 \\ \vdots \\ 2x_2x_{n+1} + 2x_1x_n + 2x_2x_{n-1} + \cdots + x_{\frac{n+1}{2}}^2 - x_n; \end{cases}$$

- if n is even, the polynomials are

$$\begin{cases} 2x_{n+1} + 2x_n + \cdots + 2x_2 + x_1 - 1 \\ 2x_{n+1}^2 + 2x_n^2 + \cdots + 2x_2^2 + x_1^2 - x_1 \\ 2x_n x_{n+1} + 2x_{n-1}x_n + \cdots + 2x_1x_2 - x_2 \\ 2x_{n-1}x_{n+1} + 2x_{n-2}x_n + \cdots + 2x_1x_3 + x_2^2 - x_3 \\ \vdots \\ 2x_2x_{n+1} + 2x_1x_n + 2x_2x_{n-1} + \cdots + 2x_{\frac{n}{2}}x_{\frac{n+2}{2}} - x_n. \end{cases}$$

Appending constant terms to those polynomials without them and choosing all the coefficients randomly yield the following augmented Katsura-n system

- if n is odd,

$$\begin{cases} c_{1,1}x_{n+1} + c_{1,2}x_n + \cdots + c_{1,n}x_2 + c_{1,n+1}x_1 + c_{1,n+2} \\ c_{2,1}x_{n+1}^2 + c_{2,2}x_n^2 + \cdots + c_{2,n}x_2^2 + c_{2,n+1}x_1^2 + c_{2,n+2}x_1 + c_{2,n+3} \\ c_{3,1}x_n x_{n+1} + c_{3,2}x_{n-1}x_n + \cdots + c_{3,n}x_1x_2 + c_{3,n+1}x_2 + c_{3,n+2} \\ c_{4,1}x_{n-1}x_{n+1} + c_{4,2}x_{n-2}x_n + \cdots + c_{4,n-1}x_1x_3 + c_{4,n}x_2^2 + c_{4,n+1}x_3 + c_{4,n+2} \\ \vdots \\ c_{n,1}x_2x_{n+1} + c_{n,2}x_1x_n + c_{n,3}x_2x_{n-1} + \cdots + c_{n,\frac{n+3}{2}}x_{\frac{n+1}{2}}^2 + c_{n,\frac{n+5}{2}}x_n + c_{n,\frac{n+7}{2}}; \end{cases}$$

- if n is even,

$$\begin{cases} c_{1,1}x_{n+1} + c_{1,2}x_n + \cdots + c_{1,n}x_2 + c_{1,n+1}x_1 + c_{1,n+2} \\ c_{2,1}x_{n+1}^2 + c_{2,2}x_n^2 + \cdots + c_{2,n}x_2^2 + c_{2,n+1}x_1^2 + c_{2,n+2}x_1 + c_{2,n+3} \\ c_{3,1}x_n x_{n+1} + c_{3,2}x_{n-1}x_n + \cdots + c_{3,n}x_1x_2 + c_{3,n+1}x_2 + c_{3,n+2} \\ c_{4,1}x_{n-1}x_{n+1} + c_{4,2}x_{n-2}x_n + \cdots + c_{4,n-1}x_1x_3 + c_{4,n}x_2^2 + c_{4,n+1}x_3 + c_{4,n+2} \\ \vdots \\ c_{n,1}x_2x_{n+1} + c_{n,2}x_1x_n + c_{n,3}x_2x_{n-1} + \cdots + c_{n,\frac{n+2}{2}}x_{\frac{n}{2}}x_{\frac{n+2}{2}} + c_{n,\frac{n+4}{2}}x_n + c_{n,\frac{n+6}{2}}. \end{cases}$$

As elaborated in the previous chapter, for the purpose of finding all isolated zeros of polynomial system $P(x)$ in $\mathbb{C}^n$, rather than in $(\mathbb{C}^*)^n$, the number of paths that need to be traced in the polyhedral homotopy method is twice the mixed volume of the *augmented* system of $P(x)$. Therefore the major difference, in terms of the number of paths one needs to trace, in employing the classical linear homotopy or the polyhedral homotopy for solving $P(x) = 0$ in $\mathbb{C}^n$ lies in the comparison of the mixed volume of the augmented system of $P(x)$ and its total degree. The following proposition shows that for a Katsura-n system these two numbers are actually the same.

Proposition 2.1.1. *Mixed volume of augmented Katsura-n system is equal to its total degree* 2^n.

Proof. By Proposition 1.1.1, the mixed volume of an augmented polynomial system and its total degree will be the same if the polynomial system has no zeros at infinity. We will therefore prove the assertion of the proposition by showing there is no zeros at infinity for the augmented Katsura-n system by induction on n.

First of all, when $n = 1$, the augmented Katsura-1 system is reduced to

$$\begin{cases} c_{11}x_2 + c_{12}x_1 + c_{13} \\ c_{21}x_2^2 + c_{22}x_1^2 + c_{23}x_1 + c_{24}. \end{cases} \tag{2.1.1}$$

The zeros of this system at infinity are the zeros of its associated homogeneous polynomial

$$\begin{cases} c_{11}x_2 + c_{12}x_1 + c_{13}x_0 \\ c_{21}x_2^2 + c_{22}x_1^2 + c_{23}x_1x_0 + c_{24}x_0^2 \end{cases}$$

in $\mathbb{P}^2$ with $x_0 = 0$. However, solving system

$$\begin{cases} c_{11}x_2 + c_{12}x_1 = 0 \\ c_{21}x_2^2 + c_{22}x_1^2 = 0 \end{cases}$$

will result in $x_1 = 0$ and $x_2 = 0$. Hence, system (2.1.1) has no zero at infinity.

For $n = 2$, augmented Katsura-2 system becomes

$$\begin{cases} c_{11}x_3 + c_{12}x_2 + c_{13}x_1 + c_{14} \\ c_{21}x_3^2 + c_{22}x_2^2 + c_{23}x_1^2 + c_{24}x_1 + c_{25} \\ c_{31}x_2x_3 + c_{32}x_1x_2 + c_{33}x_2 + c_{34} \end{cases} \tag{2.1.2}$$

and its associated homogeneous polynomial is

$$\begin{cases} c_{11}x_3 + c_{12}x_2 + c_{13}x_1 + c_{14}x_0 \\ c_{21}x_3^2 + c_{22}x_2^2 + c_{23}x_1^2 + c_{24}x_1x_0 + c_{25}x_0^2 \\ c_{31}x_2x_3 + c_{32}x_1x_2 + c_{33}x_2x_0 + c_{34}x_0^2. \end{cases}$$

Hence, zeros at infinity of system (2.1.2) is the nonzero solutions of system

$$\begin{cases} c_{11}x_3 + c_{12}x_2 + c_{13}x_1 = 0 \\ c_{21}x_3^2 + c_{22}x_2^2 + c_{23}x_1^2 = 0 \\ c_{31}x_2x_3 + c_{32}x_1x_2 = 0. \end{cases} \tag{2.1.3}$$

If $x_3 = 0$, system (2.1.3) becomes

$$\begin{cases} c_{12}x_2 + c_{13}x_1 = 0 \\ c_{22}x_2^2 + c_{23}x_1^2 = 0 \\ c_{32}x_1x_2 = 0 \end{cases}$$

and the only solution of which is $x_1 = 0$ and $x_2 = 0$.

When $x_3 \neq 0$, let $x_3 = 1$ in system (2.1.3), we have

$$\begin{cases} pp_1(x) = c_{11} + c_{12}x_2 + c_{13}x_1 = 0 \\ pp_2(x) = c_{21} + c_{22}x_2^2 + c_{23}x_1^2 = 0 \\ pp_3(x) = c_{31}x_2 + c_{32}x_1x_2 = 0. \end{cases} \tag{2.1.4}$$

It is clear that this system has no solutions because the isolated zeros of the first two equations $pp_1(x) = 0$ and $pp_2(x) = 0$ are all in $(\mathbb{C}^*)^2$ and those generically chosen coefficients c_{31} and c_{32} in $pp_3(x)$ will not subject to the nonzero constraint that $pp_3(x) =$

0 imposed.

So the proposition is true for $n = 1$ and $n = 2$.

Now suppose the augmented Katsura-$(k-1)$ system has no zeros at infinity. We assume k is even. (The proof is the same for odd k.) Then the augmented Katsura-$(k-1)$ system takes the form

$$\begin{cases} c_{1,1}x_k + c_{1,2}x_{k-1} + \cdots + c_{1,k-1}x_2 + c_{1,k}x_1 + c_{1,k+1} \\ c_{2,1}x_k^2 + c_{2,2}x_{k-1}^2 + \cdots + c_{2,k-1}x_2^2 + c_{2,k}x_1^2 + c_{2,k+1}x_1 + c_{2,k+2} \\ c_{3,1}x_{k-1}x_k + c_{3,2}x_{k-2}x_{k-1} + \cdots + c_{3,k-1}x_1x_2 + c_{3,k}x_2 + c_{3,k+1} \\ c_{4,1}x_{k-2}x_k + c_{4,2}x_{k-3}x_{k-1} + \cdots + c_{4,k-2}x_1x_3 + c_{4,k-1}x_2^2 + c_{4,k}x_3 + c_{4,k+1} \\ \vdots \\ c_{k,1}x_2x_k + c_{k,2}x_1x_{k-1} + c_{k,3}x_2x_{k-2} + \cdots + c_{k,\frac{k+2}{2}}x_{\frac{k}{2}}^2 + c_{k,\frac{k+4}{2}}x_{k-1} + c_{k,\frac{k+6}{2}}. \end{cases} \tag{2.1.5}$$

This system has no zeros at infinity, so the system

$$\begin{cases} c_{1,1}x_k + c_{1,2}x_{k-1} + \cdots + c_{1,k-1}x_2 + c_{1,k}x_1 \\ c_{2,1}x_k^2 + c_{2,2}x_{k-1}^2 + \cdots + c_{2,k-1}x_2^2 + c_{2,k}x_1^2 \\ c_{3,1}x_{k-1}x_k + c_{3,2}x_{k-2}x_{k-1} + \cdots + c_{3,k-1}x_1x_2 \\ c_{4,1}x_{k-2}x_k + c_{4,2}x_{k-3}x_{k-1} + \cdots + c_{4,k-2}x_1x_3 + c_{4,k-1}x_2^2 \\ \vdots \\ c_{k,1}x_2x_k + c_{k,2}x_1x_{k-1} + c_{k,3}x_2x_{k-2} + \cdots + c_{k,\frac{k+2}{2}}x_{\frac{k}{2}}^2 \end{cases} \tag{2.1.6}$$

has no nontrivial zeros. For $n = k$, the zeros at infinity of the augmented Katsura-k system

$$\begin{cases} c_{1,1}x_{k+1} + c_{1,2}x_k + \cdots + c_{1,k}x_2 + c_{1,k+1}x_1 + c_{1,k+2} \\ c_{2,1}x_{k+1}^2 + c_{2,2}x_k^2 + \cdots + c_{2,k}x_2^2 + c_{2,k+1}x_1^2 + c_{2,k+2}x_1 + c_{2,k+3} \\ c_{3,1}x_kx_{k+1} + c_{3,2}x_{k-1}x_k + \cdots + c_{3,k}x_1x_2 + c_{3,k+1}x_2 + c_{3,k+2} \\ c_{4,1}x_{k-1}x_{k+1} + c_{4,2}x_{k-2}x_k + \cdots + c_{4,k-1}x_1x_3 + c_{4,k}x_2^2 + c_{4,k+1}x_3 + c_{4,k+2} \\ \qquad\qquad\vdots \\ c_{k,1}x_3x_{k+1} + c_{k,2}x_2x_k + c_{k,3}x_1x_{k-1} + \cdots + c_{k,\frac{k+4}{2}}x_{\frac{k}{2}}^2 + c_{k,\frac{k+6}{2}}x_{k-1} + c_{k,\frac{k+8}{2}} \\ c_{k+1,1}x_2x_{k+1} + c_{k+1,2}x_1x_k + c_{k+1,3}x_2x_{k-1} + \cdots + \\ \qquad\qquad c_{k+1,\frac{k+2}{2}}x_{\frac{k}{2}}x_{\frac{k+2}{2}} + c_{k+1,\frac{k+4}{2}}x_k + c_{k+1,\frac{k+6}{2}} \end{cases} \tag{2.1.7}$$

are nontrivial zeros of

$$\begin{cases} c_{1,1}x_{k+1} + c_{1,2}x_k + \cdots + c_{1,k}x_2 + c_{1,k+1}x_1 \\ c_{2,1}x_{k+1}^2 + c_{2,2}x_k^2 + \cdots + c_{2,k}x_2^2 + c_{2,k+1}x_1^2 \\ c_{3,1}x_kx_{k+1} + c_{3,2}x_{k-1}x_k + \cdots + c_{3,k}x_1x_2 \\ c_{4,1}x_{k-1}x_{k+1} + c_{4,2}x_{k-2}x_k + \cdots + c_{4,k-1}x_1x_3 + c_{4,k}x_2^2 \\ \qquad\qquad\vdots \\ c_{k,1}x_3x_{k+1} + c_{k,2}x_2x_k + c_{k,3}x_1x_{k-1} + \cdots + c_{k,\frac{k+4}{2}}x_{\frac{k}{2}}^2 \\ c_{k+1,1}x_2x_{k+1} + c_{k+1,2}x_1x_k + c_{k+1,3}x_2x_{k-1} + \cdots + c_{k+1,\frac{k+2}{2}}x_{\frac{k}{2}}x_{\frac{k+2}{2}}. \end{cases} \tag{2.1.8}$$

For $x_{k+1} = 0$, the above system becomes

$$\begin{cases} c_{1,2}x_k + \cdots + c_{1,k}x_2 + c_{1,k+1}x_1 \\ c_{2,2}x_k^2 + \cdots + c_{2,k}x_2^2 + c_{2,k+1}x_1^2 \\ c_{3,2}x_{k-1}x_k + \cdots + c_{3,k}x_1x_2 \\ c_{4,2}x_{k-2}x_k + \cdots + c_{4,k-1}x_1x_3 + c_{4,k}x_2^2 \\ \qquad\qquad\vdots \\ c_{k,2}x_2x_k + c_{k,3}x_1x_{k-1} + \cdots + c_{k,\frac{k+4}{2}}x_{\frac{k}{2}}^2 \\ c_{k+1,2}x_1x_k + c_{k+1,3}x_2x_{k-1} + \cdots + c_{k+1,\frac{k+2}{2}}x_{\frac{k}{2}}x_{\frac{k+2}{2}} \end{cases} \tag{2.1.9}$$

and the first k polynomials of system (2.1.9) have the same form as the system in (2.1.6) with randomly chosen coefficients, which has no nontrivial zeros. Consequently, system (2.1.9) itself can not have nontrivial zeros, i.e., system (2.1.7) has no zeros at infinity when $x_{k+1} = 0$.

For $x_{k+1} \neq 0$, let $x_{k+1} = 1$ in system (2.1.8) and consider

$$\begin{cases} pp_1(x) = c_{1,1} + c_{1,2}x_k + \cdots + c_{1,k}x_2 + c_{1,k+1}x_1 \\ pp_2(x) = c_{2,1} + c_{2,2}x_k^2 + \cdots + c_{2,k}x_2^2 + c_{2,k+1}x_1^2 \\ pp_3(x) = c_{3,1}x_k + c_{3,2}x_{k-1}x_k + \cdots + c_{3,k}x_1x_2 \\ pp_4(x) = c_{4,1}x_{k-1} + c_{4,2}x_{k-2}x_k + \cdots + c_{4,k-1}x_1x_3 + c_{4,k}x_2^2 \\ \qquad\qquad\vdots \\ pp_k(x) = c_{k,1}x_3 + c_{k,2}x_2x_k + c_{k,3}x_1x_{k-1} + \cdots + c_{k,\frac{k+4}{2}}x_{\frac{k}{2}}^2 \\ pp_{k+1}(x) = c_{k+1,1}x_2 + c_{k+1,2}x_1x_k + c_{k+1,3}x_2x_{k-1} + \cdots + c_{k+1,\frac{k+2}{2}}x_{\frac{k}{2}}x_{\frac{k+2}{2}}. \end{cases} \tag{2.1.10}$$

First of all, no zeros $x^0 = (x_1^0, x_2^0, \ldots, x_k^0)$ of the above system can have all $x_2^0 = x_3^0 = \cdots = x_k^0 = 0$. Otherwise it would lead to a contradiction to $pp_1(x^0) = 0$ together with $pp_2(x^0) = 0$. So, without loss of generality, we suppose $x_2^0 \neq 0$. Now consider x^0 an isolated solution of k equations

$$
\begin{aligned}
pp_1(x_1, \ldots, x_k) &= 0 \\
pp_2(x_1, \ldots, x_k) &= 0 \\
&\vdots \\
pp_k(x_1, \ldots, x_k) &= 0
\end{aligned}
$$

in k variables whose coefficients are randomly chosen but then fixed (and therefore all the solutions are isolated). However, when we substitute $(x_1^0, x_2^0, \ldots, x_k^0)$ into $pp_{k+1}(x) = 0$, it imposes a nonzero constraint for the coefficients $c_{k+1,1}, \ldots, c_{k+1,\frac{k+2}{2}}$ of $pp_{k+1}(x)$ since $x_2^0 \neq 0$. This can't occur since those coefficients are arbitrarily chosen, they do not subject to any particular constraints. Therefore system (2.1.10) has no zeros, i.e., system (2.1.7) has no zeros at infinity when $x_{k+1} \neq 0$. Thus the assertion of the proposition is valid for $n = k$. This completes the proof. $\square$

2.2 Numerical results

As a comparison, we solve the Katsura-n system numerically by both the classical linear homotopy and the polyhedral homotopy, and results are listed in Table 2.1. All the computations here as well as in the following chapters were carried out on a Dell PC with a Pentium 4 CPU of 2.2GHz, 1GB of memory, and results presented are restricted to the systems that can be solved within 12 hours of CPU time. Recall that we use the typical start system $Q(x) = (q_1(x), \ldots, q_n(x))$ where

$$q_1(x) = a_1 x_1^{d_1} - b_1,$$
$$\vdots$$
$$q_n(x) = a_n x_n^{d_n} - b_n$$

with randomly chosen complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ in the classical linear homotopy

$$H(x,t) = (1-t)Q(x) + tP(x) = 0.$$

The speed-up ratio is the ratio of the CPU time of solving the system by the polyhedral homotopy to that by the classical linear homotopy with the typical start system. Apparently the table shows the classical linear homotopy works much better for finding all the solutions of Katsura-n systems [21]. For instance, when $n = 16$, the polyhedral homotopy takes more than 12 hours to find all isolated zeros of the system whereas the classical linear homotopy only takes 16 minutes and 25 seconds.

System	Total Degree	CPU time		Speed-up ratio
		Linear	**Polyhedral**	
Katsura-11	2,048	11s	23s	2.09
Katsura-12	4,096	26s	1m22s	3.15
Katsura-13	8,192	1m06s	5m32s	5.03
Katsura-14	16,384	2m38s	22m14s	8.44
Katsura-15	32,768	7m03s	1h50m26s	15.66
Katsura-16	65,536	16m25s	-	-
Katsura-17	131,072	40m48s	-	-
Katsura-18	262,144	1h35m47s	-	-
Katsura-19	524,288	3h50m48s	-	-
Katsura-20	1,048,576	8h58m00s	-	-

Table 2.1. Comparison of the classical linear homotopy with the typical start system and the polyhedral homotopy in solving Katsura-n systems.

CHAPTER 3

Reimer-n system

Reimer-n system [35] is a polynomial system whose mixed volume is equal to its total degree.

3.1 Reimer-n system

The general form of Reimer-n system is

$$\begin{cases} 2x_1^2 - 2x_2^2 + \cdots + (-1)^{n+1}2x_n^2 - 1 \\ 2x_1^3 - 2x_2^3 + \cdots + (-1)^{n+1}2x_n^3 - 1 \\ \vdots \\ 2x_1^{n+1} - 2x_2^{n+1} + \cdots + (-1)^{n+1}2x_n^{n+1} - 1. \end{cases} \tag{3.1.1}$$

Since polynomials in the system all have constant term, its augmented system will consist of the same monomials with generically chosen coefficients:

$$\begin{cases} c_{11}x_1^2 + c_{12}x_2^2 + \cdots + c_{1n}x_n^2 + c_{1n+1} \\ c_{21}x_1^3 + c_{22}x_2^3 + \cdots + c_{2n}x_n^3 + c_{2n+1} \\ \vdots \\ c_{n1}x_1^{n+1} + c_{n2}x_2^{n+1} + \cdots + c_{nn}x_n^{n+1} + c_{nn+1}. \end{cases} \tag{3.1.2}$$

Proposition 3.1.1. *For Reimer-n system, mixed volume = total degree* $= 2 \times 3 \times \cdots \times (n+1) = (n+1)!$.

Proof. While exactly the same argument that was used in the proof of Proposition 2.1.1 can also be applied here, we shall provide a different proof for this proposition due to the special structure of the system.

For the Reimer-n system given in (3.1.1), the supports of polynomials in the system are

$$\begin{aligned} S_1 &= \{(2,0,\ldots,0),(0,2,\ldots,0),\ldots,(0,0,\ldots,2)\} = \{2e_1, 2e_2, \ldots, 2e_n\} \\ S_2 &= \{(3,0,\ldots,0),(0,3,\ldots,0),\ldots,(0,0,\ldots,3)\} = \{3e_1, 3e_2, \ldots, 3e_n\} \\ &\vdots \\ S_n &= \{(n+1,0,\ldots,0),(0,n+1,\ldots,0),\ldots,(0,0,\ldots,n+1)\} \\ &= \{(n+1)e_1, (n+1)e_2, \ldots, (n+1)e_n\}, \end{aligned}$$

where for $i = 1, \ldots, n$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the ith unit vector with its ith component 1 and all other components zero.

Recall that for mixed cell $\{a_1 - a_1', \ldots, a_n - a_n'\}$ induced by the lifted support $\hat{S} = (\hat{S}_1, \ldots, \hat{S}_n)$ with inner normal $\hat{\alpha} = (\alpha, 1) \in \mathbb{R}^{n+1}$, we have $\{a_1, a_1'\} \subset S_1, \ldots, \{a_n, a_n'\} \subset$

S_n and by Proposition 1.2.3,

$$\text{Mixed volume of the system} = \sum_{\alpha} \left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right| .$$

Now, for each $i = 1, \ldots, n$, $\{a_i, a_i'\} \subset S_i = \{(i+1)e_1, \ldots, (i+1)e_n\}$ implies $a_i - a_i' = (i+1)(e_1^i - e_2^i)$ where $e_1^i \neq e_2^i$ and they are both in $\{e_1, \ldots, e_n\}$. Let

$$A := \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} = \begin{pmatrix} 2(e_1^1 - e_2^1) \\ \vdots \\ (n+1)(e_1^n - e_2^n) \end{pmatrix} .$$

Then

$$\det A = 2 \times \cdots \times (n+1) \det \begin{pmatrix} e_1^1 - e_2^1 \\ \vdots \\ e_1^n - e_2^n \end{pmatrix} = 2 \times \cdots \times (n+1) \det B$$

where

$$B = \begin{pmatrix} e_1^1 - e_2^1 \\ \vdots \\ e_1^n - e_2^n \end{pmatrix}$$

is a matrix with all arrays being either -1, 0, or 1. When $\det B \neq 0$, then $|\det B| \geq 1$ and consequently $|\det A| \geq 2 \times \cdots \times (n+1)$. Thus the mixed volume of the Reimer-n system, $\sum_\alpha |\det A|$, is greater than or equal to $2 \times \cdots \times (n+1)$.

On the other hand, the mixed volume of any system is less than or equal to its total degree. Here the total degree of the Reimer-n system is $2\times\cdots\times(n+1)$. It follows that the mixed volume of the Reimer-n system agrees with its total degree $2\times\cdots\times(n+1) = (n+1)!$.

$\square$

Corollary 3.1.1. *For the Reimer-n system, there is one and only one mixed cell regardless of what sort of liftings being applied to the support* $(S_1, \ldots, S_n)$.

Proof. For mixed cell $\{a_1 - a_1', \ldots, a_n - a_n'\}$ of Reimer-n system, where $\{a_1, a_1'\} \subset S_1, \ldots, \{a_n, a_n'\} \subset S_n$, by Proposition 1.2.3,

$$\text{Mixed volume of Reimer-}n\text{ system} = \sum_{\alpha} \left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right| = (n+1)!.$$

However, from the proof in Proposition 3.1.1, for any mixed cell $\{a_1 - a_1', \ldots, a_n - a_n'\}$

$$\left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right| \geq (n+1)!.$$

Therefore, there can be at most one mixed cell $\{a_1 - a_1', \ldots, a_n - a_n'\}$ with

$$\left| \det \begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_n - a_n' \end{pmatrix} \right| = (n+1)!.$$

$\square$

3.2 Numerical results

Listed in Table 3.1 is the numerical result for solving Reimer-n system by the classical linear homotopy with the typical start system $Q(x) = (q_1(x), \ldots, q_n(x))$ where

$$q_1(x) = a_1 x_1^{d_1} - b_1,$$
$$\vdots$$
$$q_n(x) = a_n x_n^{d_n} - b_n$$

with randomly chosen complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ and the polyhedral homotopy continuation method [21]. Apparently the speed-ups of the classical linear homotopy with the typical start system over the polyhedral homotopy in solving these systems shown in the table are not as dramatic as Table 2.1 shows for Katsura-n systems. A major reason is, as indicated in Corollary 3.1.1, finding only one mixed cell may not be as costly when the system is solved by the polyhedral homotopy method.

System	Total Degree	CPU time		Speed-up ratio
		Linear	**Polyhedral**	
Reimer-4	120	0.08s	0.1s	1.25
Reimer-5	720	0.7s	0.99s	1.41
Reimer-6	5,040	9.2s	12.8s	1.39
Reimer-7	40,320	1m58s	2m49s	1.43
Reimer-8	362,880	30m43s	36m43s	1.20
Reimer-9	3,628,800	7h52m40s	8h47m42s	1.12

Table 3.1. Comparison of the classical linear homotopy with the typical start system and the polyhedral homotopy in solving Reimer-n systems.

CHAPTER 4

Noon-n system

4.1 Noon-n system

In this Chapter, we discuss the Noon-n system [31],

$$P(x) = \begin{cases} x_1(x_2^2 + x_3^2 + \cdots + x_n^2 - 1.1) + 1 \\ x_2(x_1^2 + x_3^2 + \cdots + x_n^2 - 1.1) + 1 \\ \vdots \\ x_n(x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - 1.1) + 1. \end{cases}$$

Since polynomials in the system all have constant terms, generically choosing its coefficients yields the augmented system:

$$\begin{cases} x_1(c_{12}x_2^2 + c_{13}x_3^2 + \cdots + c_{1n}x_n^2 + c_{10}) + d_1 \\ x_2(c_{21}x_1^2 + c_{23}x_3^2 + \cdots + c_{2n}x_n^2 + c_{20}) + d_2 \\ \vdots \\ x_n(c_{n1}x_1^2 + c_{n2}x_2^2 + \cdots + c_{nn-1}x_{n-1}^2 + c_{n0}) + d_n. \end{cases} \tag{4.1.1}$$

Before relating total degree of the system to its mixed volume, we first recall certain properties concerning the multiplicity of an isolated zero of a polynomial system. Simply denoting the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ by $\mathcal{P}^n$ and treating it as a vector space, we use $(\mathcal{P}^n)^*$ to represent its dual space, consisting of all the linear functionals on $\mathcal{P}^n$.

Definition 4.1.1. *For a given polynomial ideal $\mathcal{I} \subset \mathcal{P}^n$ with quotient ring $\mathcal{P}^n/\mathcal{I}$, the dual space $\mathcal{D}[\mathcal{I}]$ of the ideal $\mathcal{I}$ is the set of linear functionals in the dual space $(\mathcal{P}^n/\mathcal{I})^*$ with their domain extended to $\mathcal{P}^n$. Namely, for $l \in (\mathcal{P}^n/\mathcal{I})^*$ and $p \in \mathcal{P}^n$*

$$l(p) := l(r) \quad \textit{where} \quad p \in r + \mathcal{I}.$$

An immediate consequence of this definition is,

Proposition 4.1.1. *For an ideal $\mathcal{I} \subset \mathcal{P}^n$ and $l \in (\mathcal{P}^n)^*$,*

$$l \in \mathcal{D}[\mathcal{I}] \Longleftrightarrow l(p) = 0 \quad \forall p \in \mathcal{I}.$$

Definition 4.1.2. *A subset $\mathcal{D}$ of the dual space $(\mathcal{P}^n)^*$ is closed iff*

$$l \in \mathcal{D} \Rightarrow \quad l \cdot q \in \mathcal{D} \quad \forall q \in \mathcal{P}^n$$

where linear functional $l \cdot q \in (\mathcal{P}^n)^*$ is defined by

$$(l \cdot q)p := l(qp) \quad for \quad p \in \mathcal{P}^n.$$

Definition 4.1.3. *For $j = (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ with $|j| := \sum_\alpha j_\alpha$, and for $z \in \mathbb{C}^n$, the* *differential functional* $\partial_j[z] \in (\mathcal{P}^n)^*$ *with evaluation at* z *is defined by*

$$\partial_j[z](p) := \frac{1}{j_1! \dots j_n!}\left(\frac{\partial^{|j|}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} p\right)(z) \quad \forall p \in \mathcal{P}^n.$$

An ideal $\mathcal{I} \subset \mathcal{P}^n$ is called 0-dimensional if all zeros of $\mathcal{I}$ are isolated.

Definition 4.1.4. *A zero* z_0 *of a 0-dimensional ideal* $\mathcal{I} \subset \mathcal{P}^n$ *is an m-fold zero of* $\mathcal{I}$ *if there exists a closed set of* m*, but no more than* m*, linearly independent differentiation functionals* $dl = \sum_j \beta_{ij}\partial_j[z_0]$ *with evaluation at* z_0 *in the dual space* $\mathcal{D}[\mathcal{I}]$.

Definition 4.1.5. *For* $\sigma = 1, \dots, n$ *the anti-differentiation operators* s_σ *is defined by*

$$s_\sigma \partial_j[z] := \begin{cases} \partial_{j-e_\sigma}[z] & \text{if } j_\sigma > 0 \\ 0 & \text{if } j_\sigma = 0 \end{cases}$$

where e_σ *is the* σ*th unit vector with its* σ*th component 1 and all other components zero and*

$$s_\sigma\Big(\sum_j \gamma_j \partial_j[z_0]\Big) := \sum_j \gamma_j s_\sigma \partial_j[z_0].$$

Theorem 4.1.1. *[34] In* $(\mathcal{P}^n)^*$*, a subset* $\mathcal{D}(z_0)$ *of differential functionals with evaluation at* z_0 *is closed iff*

$$dl \in \mathcal{D}(z_0) \Longrightarrow s_\sigma dl \in \mathcal{D}(z_0) \quad \text{for all } \sigma = 1, \dots, n.$$

We now establish the relation between the mixed volume and the total degree of the Noon-n system.

Proposition 4.1.2. *For the Noon-n system,*

$$\textit{Mixed volume} \ = \ \textit{Total degree} \ -2n \ = \ 3^n - 2n.$$

Proof. From Proposition 1.1.1, the above equality holds if the system in (4.1.1) has $2n$ zeros at infinity. Consider its associated homogeneous polynomial system

$$\tilde{P}(x_0, x_1, \dots, x_n) = \begin{cases} x_1(c_{12}x_2^2 + c_{13}x_3^2 + \cdots + c_{1n}x_n^2 + c_{10}x_0^2) + d_1 x_0^3 \\ x_2(c_{21}x_1^2 + c_{23}x_3^2 + \cdots + c_{2n}x_n^2 + c_{20}x_0^2) + d_2 x_0^3 \\ \vdots \\ x_n(c_{n1}x_1^2 + c_{n2}x_2^2 + \cdots + c_{nn-1}x_{n-1}^2 + c_{n0}x_0^2) + d_n x_0^3. \end{cases} \tag{4.1.2}$$

The zeros at infinity of system (4.1.1) are the nontrivial zeros of system (4.1.2) in $\mathbb{P}^n$ with $x_0 = 0$, i.e., nontrivial zeros of the system

$$\begin{cases} x_1(c_{12}x_2^2 + c_{13}x_3^2 + \cdots + c_{1n}x_n^2) \\ x_2(c_{21}x_1^2 + c_{23}x_3^2 + \cdots + c_{2n}x_n^2) \\ \vdots \\ x_n(c_{n1}x_1^2 + c_{n2}x_2^2 + \cdots + c_{nn-1}x_{n-1}^2). \end{cases}$$

It is clear that $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ are n isolated zeros of this system. For the multiplicity of each of those solutions, we add one more polynomial

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + c_{n+1}$$

to system (4.1.2), where c_i, $i = 1, \dots, n+1$ are randomly chosen complex numbers, resulting in a system of $n+1$ equations in $n+1$ variables

$$\tilde{P}(x_0, x_1, \ldots, x_n) = \begin{cases} \tilde{p}_1(x_0, x_1, \ldots, x_n) = 0 \\ \vdots \\ \tilde{p}_n(x_0, x_1, \ldots, x_n) = 0 \\ \tilde{p}_{n+1}(x_0, x_1, \ldots, x_n) = 0 \end{cases}$$

where

$$\begin{aligned} \tilde{p}_1(x_0, x_1, \ldots, x_n) &= x_1(c_{12}x_2^2 + c_{13}x_3^2 + \cdots + c_{1n}x_n^2 + c_{10}x_0^2) + d_1 x_0^3 \\ &\vdots \\ \tilde{p}_n(x_0, x_1, \ldots, x_n) &= x_n(c_{n1}x_1^2 + c_{n2}x_2^2 + \cdots + c_{nn-1}x_{n-1}^2 + c_{n0}x_0^2) + d_n x_0^3 \\ \tilde{p}_{n+1}(x_0, x_1, \ldots, x_n) &= c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_{n+1}. \end{aligned} \tag{4.1.3}$$

For $x_0 = 0$, the solutions are $z_1 = (0, -\frac{c_{n+1}}{c_1}, 0, \ldots, 0)$, $z_2 = (0, 0, -\frac{c_{n+1}}{c_2}, 0, \ldots, 0)$, $\ldots$, $z_n = (0, 0, \ldots, 0, -\frac{c_{n+1}}{c_n})$. In projective space $\mathbb{P}^n$, these solutions are in the same equivalence class as $(0,1,0,\ldots,0), (0,0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$.

Let $\mathcal{I} =< \hat{P}(x_0, x_1, \ldots, x_n) >$ be the ideal in $\mathcal{P}^{n+1} = \mathbb{C}[x_0, x_1, \ldots, x_n]$ generated by the polynomials in (4.1.3). At solution $z_i = (0, \ldots, 0, -\frac{c_{n+1}}{c_i}, 0, \ldots, 0)$ for $i = 1, \ldots, n$, we assert that the following two linearly independent differentiation functionals

$$dl_i^1 = \partial_{00\ldots0}[z_i](p) = p(z_i)$$

and

$$dl_i^2 = \partial_{10\ldots0}[z_i](p) = \frac{\partial}{\partial x_0} p(z_i)$$

constitute a closed subset of the dual space $\mathcal{D}[\mathcal{I}]$ with maximal number of differential functionals with evaluation at z_i. Obviously $\hat{P}(z_i) = 0$, so by Proposition 4.1.1, $dl_i^1 \in \mathcal{D}[\mathcal{I}]$. Further, dl_i^2 is also in $\mathcal{D}[\mathcal{I}]$ because

$$\frac{\partial}{\partial x_0}\tilde{p}_j(z_i) = (2c_{j0}x_jx_0 + 3d_jx_0^2)(z_i) = 0 \quad \text{for } j = 1, \ldots, n$$

and

$$\frac{\partial}{\partial x_0}\tilde{p}_{n+1}(z_i) = 0.$$

Moreover,

$$\partial_{20\ldots0}[z_i]\,(\tilde{p}_i) = \frac{1}{2}\frac{\partial^2}{\partial x_0^2}\,\tilde{p}_i(z_i) = \frac{1}{2}(2c_{i0}x_i + 6d_ix_0)(z_i) = -\frac{c_{i0}c_{n+1}}{c_i} \neq 0.$$

So, by Proposition 4.1.1, $\partial_{20\ldots0}(z_i) \notin \mathcal{D}[\mathcal{I}]$. On the other hand, for $j = 1, \ldots, n$,

$$\partial_{0\ldots010\ldots0}[z_i]\,(\tilde{p}_{n+1}) = \frac{\partial}{\partial x_j}\tilde{p}_{n+1}(z_i) = c_j \neq 0,$$

hence, $\frac{\partial}{\partial x_j}[z_i] \notin \mathcal{D}[\mathcal{I}]$ for all $j = 1, \ldots, n$. Consequently, by Theorem 4.1.1, dl_i^1 and dl_i^2 are the only two linearly independent differential functionals with evaluation at z_i that form a closed subset of $\mathcal{D}[\mathcal{I}]$. Therefore, for each $i = 1, \ldots, n$, the multiplicity of z_i is two. All together, they account for $2n$ solutions at infinity for the system in (4.1.1). $\square$

4.2 A special start system for Noon-n system

As Proposition 4.1.2 indicates, for the Noon-n system, the difference between the total degree and its mixed volume is $2n$, which, compared with its total degree 3^n, becomes very slim even when n is of moderate size. Such slim difference will certainly make it difficult to enjoy the benefits that are commonly provided by using the polyhedral homotopy method in solving them. Thus, the classical linear homotopy

$$H(x,t) = (1-t)Q(x) + tP(x) = 0 \tag{4.2.4}$$

with the typical start system $Q(x) = (q_1(x), \ldots, q_n(x))$ where

$$\begin{aligned} q_1(x) &= a_1 x_1^{d_1} - b_1, \\ &\vdots \\ q_n(x) &= a_n x_n^{d_n} - b_n \end{aligned}$$

with randomly chosen complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ appears to be the proper choice for solving those systems. On the other hand, beyond the reach of the polyhedral homotopy method, the classical linear homotopy can take a huge advantage on the strong symmetry structure existed in the Noon-n system.

In the first place, the multi-homogeneous structure [36] of the system allows the choice of the start system

$$Q(x) = \begin{cases} c_1(x_1+\alpha)(x_2+x_3+\cdots+x_n+\beta)(x_2+x_3+\cdots+x_n+\gamma) \\ c_2(x_2+\alpha)(x_1+x_3+\cdots+x_n+\beta)(x_1+x_3+\cdots+x_n+\gamma) \\ \vdots \\ c_n(x_n+\alpha)(x_1+x_2+\cdots+x_{n-1}+\beta)(x_1+x_2+\cdots+x_{n-1}+\gamma) \end{cases} \tag{4.2.5}$$

where c_1, c_2, $\ldots$, c_n, α, β, and γ are randomly chosen complex numbers. Clearly, with this choice, the symmetry in the Noon-n system is retained and with the invariance of this symmetry in the homotopy, much fewer paths need to be traced for generating the whole solution set.

For instance, for Noon-3 system

$$P(x) = \begin{cases} x_1(x_2^2 + x_3^2 - 1.1) + 1 \\ x_2(x_1^2 + x_3^2 - 1.1) + 1 \\ x_3(x_1^2 + x_2^2 - 1.1) + 1, \end{cases}$$

the start system for the linear homotopy in (4.2.4) is

$$Q(x) = \begin{cases} c_1(x_1 + \alpha)(x_2 + x_3 + \beta)(x_2 + x_3 + \gamma) \\ c_2(x_2 + \alpha)(x_1 + x_3 + \beta)(x_1 + x_3 + \gamma) \\ c_3(x_3 + \alpha)(x_1 + x_2 + \beta)(x_1 + x_2 + \gamma). \end{cases}$$

To attain the whole solution set of $Q(x) = 0$, we only need to permute the variables x_1, x_2, x_3 on a subset of solutions of $Q(x) = 0$. For example, the solutions of the following three linear subsystems

$$\begin{cases} x_1 + \alpha = 0 \\ x_1 + x_3 + \beta = 0 \\ x_1 + x_2 + \beta = 0 \end{cases} \qquad \begin{cases} x_2 + x_3 + \beta = 0 \\ x_2 + \alpha = 0 \\ x_1 + x_2 + \beta = 0 \end{cases} \qquad \begin{cases} x_2 + x_3 + \beta = 0 \\ x_1 + x_3 + \beta = 0 \\ x_3 + \alpha = 0 \end{cases}$$

of $Q(x) = 0$ are $z_1 = (-\alpha, \alpha-\beta, \alpha-\beta)$, $z_2 = (\alpha-\beta, -\alpha, \alpha-\beta)$, and $z_3 = (\alpha-\beta, \alpha-\beta, -\alpha)$ respectively. Obviously, solution z_3 may be attained by permuting x_1 and x_3 in z_1 or permuting x_2 and x_3 in z_2. In fact, any one of z_1, z_2, and z_3 can generate all the others by permutations. This property is actually retained on the homotopy paths initiated from those three solutions of $Q(x) = 0$. So we may just follow one of these homotopy paths to reach a solution of $P(x) = 0$ and generate the other two solutions from this solution by permutations.

It is clear that the above three linear subsystems of $Q(x) = 0$ share one thing in common: they all have one α and two βs as constant terms. Let us represent this set of linear subsystems of $Q(x) = 0$ by $(\alpha, 2\beta)$. In general, denoting $(m_1\alpha, m_2\beta, m_3\gamma)$ for the set of linear subsystems of $Q(x)$ with m_1 αs, m_2 βs, and m_3 γs as constants terms where $m_1 + m_2 + m_3 = 3$, it is easy to see that solution of any one linear subsystem in the set can generate solutions of all other linear subsystems in the set by proper permutations. Among all the possible divisions of the linear subsystems of $Q(x) = 0$ in this manner, there are two sets of singular linear subsystems, hence no solutions for such systems, they are $(2\alpha, \beta)$ and $(2\alpha, \gamma)$. The rest of the 8 groups and the number of linear subsystems in the same group are listed in Table 4.1.

constant terms	number of solutions
$(3\alpha) = (\alpha, \alpha, \alpha)$	1
$(3\beta) = (\beta, \beta, \beta)$	1
$(3\gamma) = (\gamma, \gamma, \gamma)$	1
$(\alpha, 2\beta) = (\alpha, \beta, \beta)$	3
$(\alpha, 2\gamma) = (\alpha, \gamma, \gamma)$	3
$(2\beta, \gamma) = (\beta, \beta, \gamma)$	3
$(\beta, 2\gamma) = (\beta, \gamma, \gamma)$	3
$(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma)$	6

Table 4.1. Number of linear subsystems for different grouping of constant terms.

Exhibited in Table 4.1, there are, in total, 21 solutions of $Q(x) = 0$, and all those solutions can be generated by just solving one linear subsystem from each of those 8 groups. Note that $21 = 3^3 - 2 \times 3$, which agrees with the mixed volume of Noon-3 system given in Proposition 4.1.2.

For the start system $Q(x) = (q_1(x), \ldots, q_n(x))$ in (4.2.5) where

$$q_1(x) = c_1(x_1 + \alpha)(x_2 + x_3 + \cdots + x_n + \beta)(x_2 + x_3 + \cdots + x_n + \gamma)$$

$$q_2(x) = c_2(x_2 + \alpha)(x_1 + x_3 + \cdots + x_n + \beta)(x_1 + x_3 + \cdots + x_n + \gamma)$$

$$\vdots$$

$$q_n(x) = c_n(x_n + \alpha)(x_1 + x_2 + \cdots + x_{n-1} + \beta)(x_1 + x_2 + \cdots + x_{n-1} + \gamma),$$

let $(l_1(x), \ldots, l_n(x)) = 0$ be linear subsystems of $Q(x) = 0$, where, for each $i = 1, \ldots, n$, $l_i(x)$ is a linear factor of $q_i(x)$. We divide those linear subsystems as follows: let $(m_1\alpha,\ m_2\beta,\ m_3\gamma)$ be the set of those linear subsystems in which m_1, m_2 and m_3 of the equations in $(l_1(x), \ldots, l_n(x)) = 0$ have α, β and γ as constant terms respectively. Of course, $m_1 + m_2 + m_3 = n$. It is clear that for fixed m_1, m_2 and m_3 there are $C_n^{m_1} \cdot C_{n-m_1}^{m_2}$ linear subsystems of $Q(x) = 0$ in $(m_1\alpha,\ m_2\beta,\ m_3\gamma)$ and except for $m_1 = n - 1$, all those linear subsystems are nonsingular.

To count the total number of possible different combinations of m_1, m_2 and m_3 for which $(m_1\alpha,\ m_2\beta,\ m_3\gamma)$ provide nonsingular subsystems, note that for fixed $m_1 \in \{0, 1, \ldots, n-2, n\}$, there are $n + 1 - m_1$ choices for m_2, and when m_1 and m_2 are determined, $m_3 = n - (m_1 + m_2)$. Therefore, in total, there are

$$(n+1) + n + (n-1) + \cdots + 3 + 1 = \frac{n^2 + 3n - 2}{2}$$

different such combinations.

One of the solutions for linear subsystems $(m_1\alpha,\ m_2\beta,\ m_3\gamma)$ where $m_1 \neq n - 1$ is

$$(\overbrace{\alpha_1, \ldots, \alpha_1}^{m_1}, \overbrace{\alpha_2, \ldots, \alpha_2}^{m_2}, \overbrace{\alpha_3, \ldots, \alpha_3}^{m_3}) \qquad (4.2.6)$$

for certain α_1, α_2, and α_3. This solution comes from the subsystem whose first m_1 equations have constant term α, followed by the next m_2 equations having constant term β and all the remaining equations having constant term γ. We may use this solution to generate the solution set of all linear subsystems in $(m_1\alpha,\ m_2\beta,\ m_3\gamma)$ by permutations. For simplicity in the description, we shall replace α_1, α_2 and α_3 by 1, 2, and 3 respectively, so the solution in (4.2.6) becomes $(\overbrace{1,\ldots,1}^{m_1},\overbrace{2,\ldots,2}^{m_2},\overbrace{3,\ldots,3}^{m_3})$. We generate all the permutations of $(\overbrace{1,\ldots,1}^{m_1},\overbrace{2,\ldots,2}^{m_2},\overbrace{3,\ldots,3}^{m_3})$ by the following steps:

Step 1: Let $sq_1 = (\overbrace{1,\ldots,1}^{m_1},\overbrace{2,\ldots,2}^{m_2+m_3})$ and $sq_2 = (\overbrace{2,\ldots,2}^{m_2},\overbrace{3,\ldots,3}^{m_3})$.

Step 2: For each sq_1 and sq_2, search from left for the first smaller-larger pair and interchange them. If an interchange is made, check if there are some smaller numbers to the left of the interchange. If not, save the resulting permutation. Otherwise, move all the smaller numbers to the left of the interchange to the most left end and save the permutation. For the currently saved permutation, repeat the procedure until no smaller-larger pairs exist. From this step, $C_n^{m_1}$ permutations will be produced from sq_1 and $C_{n-m_1}^{m_2}$ permutations from sq_2.

Step 3: For each permutation produced from sq_1, replace all the numbers labeled 2 by the numbers in the permutations from sq_2 and save all the possible permutations. For instance, if one of the permutations generated from sq_1 is $(1,2,1,2)$ and all the permutations produced by sq_2 are $(2,3)$ and $(3,2)$, then the resulting permutations are $(1,2,1,3)$ and $(1,3,1,2)$. Namely we replace $2,2$ in $(1,2,1,2)$ by $(2,3)$ and $(3,2)$.

In total, there will be $C_n^{m_1} \cdot C_{n-m_1}^{m_2}$ permutations for $(\overbrace{1,\ldots,1}^{m_1},\overbrace{2,\ldots,2}^{m_2},\overbrace{3,\ldots,3}^{m_3})$. For instance, when $n = 4$, let $m_1 = 2$, $m_2 = 1$, $m_3 = 1$ for the start system

$$Q(x) = \begin{cases} q_1(x) = c_1(x_1 + \alpha)(x_2 + x_3 + x_4 + \beta)(x_2 + x_3 + x_4 + \gamma) \\ q_2(x) = c_2(x_2 + \alpha)(x_1 + x_3 + x_4 + \beta)(x_1 + x_3 + x_4 + \gamma) \\ q_3(x) = c_3(x_3 + \alpha)(x_1 + x_2 + x_4 + \beta)(x_1 + x_2 + x_4 + \gamma) \\ q_4(x) = c_4(x_4 + \alpha)(x_1 + x_2 + x_3 + \beta)(x_1 + x_2 + x_3 + \gamma), \end{cases}$$

there are $C_n^{m_1} \cdot C_{n-m_1}^{m_2} = C_4^2 \cdot C_2^1 = 12$ linear subsystems $(l_1(x), \ldots, l_n(x)) = 0$ of $Q(x) = 0$ in which $(l_1(x), \ldots, l_n(x)) = 0$ have 2 α and 1 β and 1 γ as constant terms. One of them is

$$L_1(x) = \begin{cases} l_1(x) = x_1 + \alpha = 0 \\ l_2(x) = x_2 + \alpha = 0 \\ l_3(x) = x_1 + x_2 + x_4 + \beta = 0 \\ l_4(x) = x_1 + x_2 + x_3 + \gamma = 0 \end{cases}$$

and the solution for $L_1(x) = 0$ is $(-\alpha, -\alpha, 2\alpha - \gamma, 2\alpha - \beta)$. For simplicity, write the solution $(1, 1, 2, 3)$. For the permutations of $(1, 1, 2, 3)$, we follow the above steps. Let $sq_1 = (1, 1, 2, 2)$ and $sq_2 = (2, 3)$. All the permutations generated from $sq_1 = (1, 1, 2, 2)$ by following Step 2 are $(1, 2, 1, 2)$, $(2, 1, 1, 2)$, $(1, 2, 2, 1)$, $(2, 1, 2, 1)$ and $(2, 2, 1, 1)$, and the permutation from $sq_2 = (2, 3)$ is $(3, 2)$. Then by Step 3, replace all the number 2s in all the permutations generated from sq_1 by the numbers in each of the permutations produced by sq_2 and reach all the permutations for $(1, 1, 2, 3)$. They are $(1, 1, 3, 2)$, $(1, 2, 1, 3)$, $(1, 3, 1, 2)$, $(2, 1, 1, 3)$, $(3, 1, 1, 2)$, $(1, 2, 3, 1)$, $(1, 3, 2, 1)$, $(2, 1, 3, 1)$, $(3, 1, 2, 1)$, $(2, 3, 1, 1)$ and $(3, 2, 1, 1)$. By replacing 1, 2 and 3 back to $-\alpha$, $2\alpha - \gamma$, and $2\alpha - \beta$ respectively, solutions $(-\alpha, -\alpha, 2\alpha - \beta, 2\alpha - \gamma)$, $(-\alpha, 2\alpha - \gamma, -\alpha, 2\alpha - \beta)$, $(-\alpha, 2\alpha - \beta, -\alpha, 2\alpha - \gamma)$, $(2\alpha - \gamma, -\alpha, -\alpha, 2\alpha - \beta)$, $(2\alpha - \beta, -\alpha, -\alpha, 2\alpha - \gamma)$, $(-\alpha, 2\alpha - \gamma, 2\alpha -$

$\beta,\ -\alpha)$, $(-\alpha,\ 2\alpha-\beta,\ 2\alpha-\gamma,\ -\alpha)$, $(2\alpha-\gamma, -\alpha,\ 2\alpha-\beta,\ -\alpha)$,$(2\alpha-\beta, -\alpha,\ 2\alpha-\gamma,\ -\alpha)$, $(2\alpha-\gamma,\ 2\alpha-\beta,\ -\alpha,\ -\alpha,\ )$, and $(2\alpha-\beta,\ 2\alpha-\gamma,\ -\alpha,\ -\alpha,\ )$ are attained. They are exactly the solutions of the following remaining 11 linear subsystems in $(2\alpha\,,1\beta\,,1\gamma)$:

$$L_2(x)=\begin{cases} l_1(x)=x_1+\alpha \\ l_2(x)=x_2+\alpha \\ l_3(x)=x_1+x_2+x_4+\gamma \\ l_4(x)=x_1+x_2+x_3+\beta \end{cases} \qquad L_3(x)=\begin{cases} l_1(x)=x_1+\alpha \\ l_2(x)=x_1+x_3+x_4+\beta \\ l_3(x)=x_3+\alpha \\ l_4(x)=x_1+x_2+x_3+\gamma \end{cases}$$

$$L_4(x)=\begin{cases} l_1(x)=x_1+\alpha \\ l_2(x)=x_1+x_3+x_4+\gamma \\ l_3(x)=x_3+\alpha \\ l_4(x)=x_1+x_2+x_3+\beta \end{cases} \qquad L_5(x)=\begin{cases} l_1(x)=x_2+x_3+x_4+\beta \\ l_2(x)=x_2+\alpha \\ l_3(x)=x_3+\alpha \\ l_4(x)=x_1+x_2+x_3+\gamma \end{cases}$$

$$L_6(x)=\begin{cases} l_1(x)=x_2+x_3+x_4+\gamma \\ l_2(x)=x_2+\alpha \\ l_3(x)=x_3+\alpha \\ l_4(x)=x_1+x_2+x_3+\beta \end{cases} \qquad L_7(x)=\begin{cases} l_1(x)=x_1+\alpha \\ l_2(x)=x_1+x_3+x_4+\beta \\ l_3(x)=x_1+x_2+x_4+\gamma \\ l_4(x)=x_4+\alpha \end{cases}$$

$$L_8(x)=\begin{cases} l_1(x)=x_1+\alpha \\ l_2(x)=x_1+x_3+x_4+\gamma \\ l_3(x)=x_1+x_2+x_4+\beta \\ l_4(x)=x_4+\alpha \end{cases} \qquad L_9(x)=\begin{cases} l_1(x)=x_2+x_3+x_4+\beta \\ l_2(x)=x_2+\alpha \\ l_3(x)=x_1+x_2+x_4+\gamma \\ l_4(x)=x_4+\alpha \end{cases}$$

$$L_{10}(x) = \begin{cases} l_1(x) = x_2 + x_3 + x_4 + \gamma \\ l_2(x) = x_2 + \alpha \\ l_3(x) = x_1 + x_2 + x_4 + \beta \\ l_4(x) = x_4 + \alpha \end{cases} \qquad L_{11}(x) = \begin{cases} l_1(x) = x_2 + x_3 + x_4 + \beta \\ l_2(x) = x_1 + x_3 + x_4 + \gamma \\ l_3(x) = x_3 + \alpha \\ l_4(x) = x_4 + \alpha \end{cases}$$

and

$$L_{12}(x) = \begin{cases} l_1(x) = x_2 + x_3 + x_4 + \gamma \\ l_2(x) = x_1 + x_3 + x_4 + \beta \\ l_3(x) = x_3 + \alpha \\ l_4(x) = x_4 + \alpha. \end{cases}$$

Therefore, tracing one of the homotopy paths of the classical linear homotopy in (4.2.4) emanated from one of the linear subsystems of $Q(x) = 0$ in $(m_1\alpha,\ m_2\beta,\ m_3\gamma)$ is sufficient to generate all the corresponding solutions of $P(x) = 0$ in this group.

4.3 Numerical results

Numerical results on solving Noon-n systems by different homotopies are listed in Table 4.2. Recall that the difference between the total degree and the mixed volume of Noon-n systems is $2n$. Total degree, 3^n, paths need to be traced when the typical start system $Q(x) = (q_1(x), \ldots, q_n(x))$ where

$$\begin{aligned} q_1(x) &= a_1 x_1^{d_1} - b_1, \\ &\vdots \\ q_n(x) &= a_n x_n^{d_n} - b_n \end{aligned}$$

with randomly chosen complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ is used in the classical linear homotopy

$$H(x,t) = (1-t)Q(x) + tP(x) = 0.$$

And twice of the mixed volume, $2 \times (3^n - 2n)$, paths need to be traced when the polyhedral homotopy is applied. The table clearly shows that the classical linear homotopy with the typical start system works better than polyhedral homotopy because much more paths need to be followed as n gets larger for the polyhedral homotopy method. Moreover, due to the symmetry of Noon-n systems, only $\dfrac{n^2+3n-2}{2}$ homotopy paths need to be traced when a proper start system as in (4.2.5) is assigned for the classical linear homotopy. This number is shown in the 3rd column on the table, it is much smaller than the mixed volume. Apparently, solving Noon-n systems by choosing start system $Q(x)$ in (4.2.5) leads in speed by a huge margin in finding all the isolated zeros of Noon-n systems.

System	Mixed Volume	# of paths	CPU time		
			Polyhedral	**Typical $Q(x)$**	**Special $Q(x)$**
noon-10	59,029	64	5m12s	1m27s	1.1s
noon-11	177,125	76	23m27s	5m32s	1.4s
noon-12	531,417	89	1h28m00s	27m29s	2.2s
noon-13	1,594,297	103	7h02m10s	3h7m10s	3.1s
noon-19	1,162,261,429	208	-	-	18.5s
noon-29	6.8630377E13	463	-	-	2m10s
noon-39	4.0525551E18	818	-	-	16m48s
noon-49	2.3929933E23	1,273	-	-	29m58s
noon-59	1.4130386E28	1,828	-	-	1h11m39s
noon-69	8.3438517E32	2,483	-	-	2h16m59s
noon-79	4.9269609E37	3,238	-	-	4h31m03s
noon-89	2.9093212E42	4,093	-	-	10h17m23s

Table 4.2. Comparison of the polyhedral homotopy and the classical linear homotopy with different start systems in solving Noon-n systems.

CHAPTER 5

Generalized eigenvalue problem

By and large, polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ whose mixed volume is much less than its total degree should be solved by the polyhedral homotopy method. However, when certain m-homogeneous structure of the system is apparent, we may still choose appropriate start systems $Q(x) = 0$ in the classical linear homotopy so that the number of homotopy paths that need to be traced matches the mixed volume of the system. In such situations, the polyhedral homotopy method can no longer be beneficial in reducing wasteful computations.

5.1 Generalized eigenvalue problem

Consider the generalized eigenvalue problem

$$Ax = \lambda Bx$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

are $n \times n$ matrices. This problem is actually an n polynomial equations in $n+1$ variables $\lambda, x_1, \dots, x_n$:

$$\begin{cases} \lambda(b_{11}x_1 + \cdots + b_{1n}x_n) - (a_{11}x_1 + \cdots + a_{1n}x_n) = 0, \\ \vdots \\ \lambda(b_{n1}x_1 + \cdots + b_{nn}x_n) - (a_{n1}x_1 + \cdots + a_{nn}x_n) = 0. \end{cases}$$

We augment the system with a linear equation

$$c_1x_1 + \cdots + c_nx_n + c_{n+1} = 0$$

where $c_1, \ldots, c_{n+1}$ are randomly chosen complex numbers, yielding a polynomial system of $n+1$ equations in $n+1$ variables

$$\begin{cases} \lambda(b_{11}x_1 + \cdots + b_{1n}x_n) - (a_{11}x_1 + \cdots + a_{1n}x_n) \\ \qquad\qquad\vdots \\ \lambda(b_{n1}x_1 + \cdots + b_{nn}x_n) - (a_{n1}x_1 + \cdots + a_{nn}x_n) \\ c_1x_1 + \cdots + c_nx_n + c_{n+1}. \end{cases} \tag{5.1.1}$$

Proposition 5.1.1. *The mixed volume of system (5.1.1) is* n.

To prove this proposition, we need to introduce the following fundamental lemmas.

Lemma 5.1.1. *If* $z_0, z_1, \ldots, z_n \in \mathbb{R}^n$ *are affinely independent, i.e.,* $z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0$ *are linearly independent, then the volume of the convex hull of those points, denoted by* $\mathrm{Vol}_n(\mathrm{conv}(z_0, z_1, \ldots, z_n))$, *is*

$$\frac{1}{n!}\left|\begin{pmatrix} z_1 - z_0 \\ z_2 - z_0 \\ \vdots \\ z_n - z_0 \end{pmatrix}\right|.$$

Lemma 5.1.2. *For a polynomial system* $P(x) = (p_1(x), \ldots, p_n(x))$, *let* $S_1, \ldots, S_n$ *be the support of* $p_1(x), \ldots, p_n(x)$ *respectively. If* $S_1 = S_2 = \cdots = S_n = S$, *then*

$$M(S, \ldots, S) = n!\,\mathrm{Vol}_n(\mathrm{conv}(S)).$$

Proof. The mixed volume of $P(x)$ is the coefficient of the term $\lambda_1 \times \cdots \times \lambda_n$ in the homogeneous polynomial $\mathrm{Vol}_n(\lambda_1\mathrm{conv}(S_1) + \cdots + \lambda_n\mathrm{conv}(S_n))$ of degree n in $\lambda_1, \ldots, \lambda_n$.

Since $S_1 = S_2 = \cdots = S_n = S$, we have

$$\begin{aligned}
&\mathrm{Vol}_n(\lambda_1 \mathrm{conv}(S_1) + \cdots + \lambda_n \mathrm{conv}(S_n)) \\
=&\mathrm{Vol}_n((\lambda_1 + \cdots + \lambda_n)\mathrm{conv}(S)) \\
=&(\lambda_1 + \cdots + \lambda_n)^n \mathrm{Vol}_n(\mathrm{conv}(S)) \\
=&n!\, \mathrm{Vol}_n(\mathrm{conv}(S))\lambda_1 \times \cdots \times \lambda_n + \text{other terms.}
\end{aligned}$$

Therefore,

$$M(S, \ldots, S) = n!\, \mathrm{Vol}_n(\mathrm{conv}(S)).$$

□

Proof of the proposition. Consider the system

$$\begin{cases}
\lambda(\bar{b}_{11}x_1 + \cdots + \bar{b}_{1n}x_n) - (\bar{a}_{11}x_1 + \cdots + \bar{a}_{1n}x_n) + \bar{a}_{1n+1} \\
\qquad\qquad\vdots \\
\lambda(\bar{b}_{n1}x_1 + \cdots + \bar{b}_{nn}x_n) - (\bar{a}_{n1}x_1 + \cdots + \bar{a}_{nn}x_n) + \bar{a}_{nn+1} \\
\lambda(\bar{b}_{n+11}x_1 + \cdots + \bar{b}_{n+1n}x_n) + c_1x_1 + \cdots + c_nx_n + c_{n+1}
\end{cases} \tag{5.1.2}$$

where $\bar{a}_{ij}$, $\bar{b}_{ji}$, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$ are randomly chosen complex numbers. The supports of all the polynomials of this system are the same. Taking $n = 2$ for instance, the above system becomes

$$\begin{cases}
\lambda(\bar{b}_{11}x_1 + \bar{b}_{12}x_2) - (\bar{a}_{11}x_1 + \bar{a}_{12}x_2) + \bar{a}_{13} \\
\lambda(\bar{b}_{21}x_1 + \bar{b}_{22}x_2) - (\bar{a}_{21}x_1 + \bar{a}_{22}x_2) + \bar{a}_{23} \\
\lambda(\bar{b}_{31}x_1 + \bar{b}_{32}x_2) + c_1x_1 + c_2x_2 + c_3
\end{cases} \tag{5.1.3}$$

with variables λ, x_1, and x_2. We list the points in the support in descending lexicographic

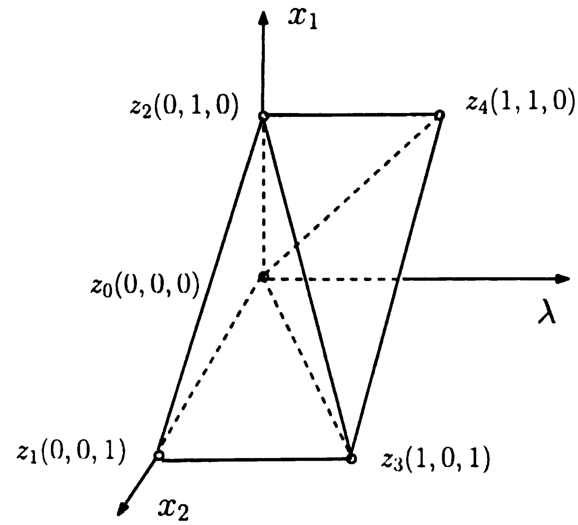

Figure 5.1. The convex hull of S for $n = 3$.

order, that is $z_1 >_{lex} z_0$ for z_0, $z_1 \in \mathbb{Z}^n_{\geq 0}$ if the left-most nonzero entry of the vector $z_1 - z_0$ is positive. Then $S = S_1 = S_2 = S_3 = \{(0,0,0),(0,0,1),(0,1,0),(1,0,1),(1,1,0)\} = \{z_0, z_1, z_2, z_3, z_4\}$. The graph of the convex hull of S is shown in Figure 5.1.

From Lemma 5.1.2, the mixed volume of system (5.1.3) is $M(S,S,S) = 3!\,\mathrm{Vol}_3(\mathrm{conv}(S))$. To compute the volume of the convex hull of S, we divide $\mathrm{conv}_3(S)$ into two simplices: $Q_1 = \mathrm{conv}\{z_0, z_1, z_2, z_3\}$, and $Q_2 = \mathrm{conv}\{z_0, z_2, z_3, z_4\}$. Graphically it is clear from Figure 5.1,

$$\mathrm{Vol}_3(\mathrm{conv}(S)) = \mathrm{Vol}_3(Q_1) + \mathrm{Vol}_3(Q_2).$$

For a proof of this equality, first note that Q_1 and Q_2 have three points z_0, z_2, z_3 in common and $\mathrm{conv}\{z_0, z_2, z_3\}$ is a face for each. A normal of the face is $v = (1, 0, -1)$ since

$$\begin{pmatrix} i & j & k \\ & z_2 - z_0 & \\ & z_3 - z_0 & \end{pmatrix} = \begin{pmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1i + 0j + (-1)k.$$

Apparently, z_1 and z_4 lie on different sides of the face because

$$\langle z_1, v\rangle \times \langle z_4, v\rangle = \langle (0,0,1),(1,0,-1)\rangle \times \langle (1,1,0),(1,0,-1)\rangle$$
$$= -1 \times 1 = -1 < 0.$$

Therefore, $Q_1 \bigcap Q_2 = \varnothing$ and $\text{conv}(S) = Q_1 \bigcup Q_2$, it follows that

$$\text{Vol3}(\text{conv}(S)) = \text{Vol3}(Q_1) + \text{Vol3}(Q_2).$$

On the other hand, z_0, z_1, z_2, z_3 are affinely independent. By Lemma 5.1.1

$$\text{Vol3}(Q_1) = \frac{1}{3!}\left|\begin{pmatrix} z_1 - z_0 \\ z_2 - z_0 \\ z_3 - z_0 \end{pmatrix}\right| = \frac{1}{3!}\left|\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right| = \frac{1}{3!},$$

$$\text{Vol3}(Q_2) = \frac{1}{3!}\left|\begin{pmatrix} z_2 - z_0 \\ z_3 - z_0 \\ z_4 - z_0 \end{pmatrix}\right| = \frac{1}{3!}\left|\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right| = \frac{1}{3!},$$

thus,

$$\text{Vol3}(\text{conv}(S)) = \frac{2}{3!}.$$

And

$$M(S,S,S) = 3!\,\text{Vol3}(\text{conv}(S)) = 3! \times \frac{2}{3!} = 2.$$

For the general system in (5.1.2), the variables are $\lambda, x_1, x_2, \ldots, x_n$. Listing the points in the support in descending lexicographic order yields

$$
\begin{aligned}
S = S_1 = S_2 = \cdots = S_{n+1} \\
&= \{(0,0,\ldots,0), e_{n+1}, e_n, \ldots, e_2, e_1 + e_{n+1}, e_1 + e_n, \ldots, e_1 + e_2\} \\
&= \{z_0, z_1, z_2, \ldots, z_{2n}\}
\end{aligned}
$$

where $z_0 = (0,0,\ldots,0)$, and

$$
z_i = \begin{cases} e_{n+2-i} & \text{for} \quad i = 1, \ldots, n, \\ e_1 + e_{2(n+1)-i} & \text{for} \quad i = n+1, \ldots, 2n. \end{cases}
$$

Similarly, $\mathrm{conv}(S)$ can be divided into n simplicies:

$$
\begin{aligned}
Q_1 &= \mathrm{conv}\{z_0, z_1, \ldots, z_{n+1}\} \\
Q_2 &= \mathrm{conv}\{z_0, z_2, \ldots, z_{n+2}\} \\
&\vdots \\
Q_n &= \mathrm{conv}\{z_0, z_n, \ldots, z_{2n}\}.
\end{aligned}
$$

These simplicies can intersect only at their faces and for each $i = 1, \ldots, n$,

$$\mathrm{Vol}_{n+1}(Q_i) = \frac{1}{(n+1)!}\left|\begin{pmatrix} z_i - z_0 \\ z_{i+1} - z_0 \\ \vdots \\ z_{i+n} - z_0 \end{pmatrix}\right| = \frac{1}{(n+1)!}\left|\begin{pmatrix} z_i \\ z_{i+1} \\ \vdots \\ z_{i+n} \end{pmatrix}\right|$$

$$= \frac{1}{(n+1)!}\left|\begin{pmatrix} e_{n+2-i} \\ \vdots \\ e_2 \\ e_1 + e_{n+1} \\ \vdots \\ e_1 + e_{n+2-i} \end{pmatrix}\right| = \frac{1}{(n+1)!}\left|\begin{pmatrix} e_{n+2-i} \\ \vdots \\ e_2 \\ e_{n+1} \\ \vdots \\ e_{n+3-i} \\ e_1 \end{pmatrix}\right|$$

$$= \frac{1}{(n+1)!}\left|\begin{pmatrix} e_1 \\ \vdots \\ e_{n+1} \end{pmatrix}\right| = \frac{1}{(n+1)!}.$$

Hence,

$$\mathrm{Vol}_{n+1}(\mathrm{conv}(S)) = n \times \frac{1}{(n+1)!} = \frac{n}{(n+1)!},$$

and the mixed volume of system (5.1.2) is

$$M(S, S, \ldots, S) = (n+1)!\,\mathrm{Vol}_{n+1}(\mathrm{conv}(S)) = (n+1)! \times \frac{n}{(n+1)!} = n.$$

Now, support $\bar{S} = (\bar{S}_1, \ldots, \bar{S}_n)$ of system (5.1.1) is a subset of the support $S = (S, \ldots, S)$ of system (5.1.2), that is, $\bar{S}_i \subseteq S$ for all $i = 1, \ldots, n$. Thus

$$M(\bar{S}_1, \ldots, \bar{S}_n) \leqslant M(S, \ldots, S) = n.$$

On the other hand, consider the system

$$\begin{cases} Ax - \lambda Bx = 0 \\ x_1 + x_2 + \cdots + x_n - n - 1 = 0 \end{cases} \tag{5.1.4}$$

where

$$A = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{bmatrix}^{-1}$$

and $B = I$, the $n \times n$ identity matrix. For these specific matrices A and B, the zeros of system (5.1.4) are $(\lambda, x_1, x_2, \ldots, x_n) = (1, 2, 1, 1, \ldots, 1)$, $(2, 1, 2, 1, \ldots, 1)$, $\ldots$, $(n, 1, 1, 1, \ldots, 2)$, that is, for $i = 1, \ldots, n$, the first component of the ith zero is i, the $(i+1)$th component is 2 and all the others are 1. So the number of isolated zeros of system (5.1.4) in $(\mathbb{C}^*)^n$ is n, and therefore

$$M(\bar{S}_1, \ldots, \bar{S}_n) \geq n.$$

So,

$$M(\bar{S}_1, \ldots, \bar{S}_n) = n.$$

□

5.2 A special start system for generalized eigenvalue problem

The total degree of system (5.1.1) is 2^n, but the system has at most n isolated zeros. So the system is deficient. If the classical linear homotopy is used to solve the problem, at least $2^n - n$ paths will be extraneous, representing huge wasteful computations. If the polyhedral homotopy is applied, $2n$ paths need to be followed. But when n becomes larger, finding all the mixed cells requires a big amount of computations . To alleviate this problem, Li et al. [17] suggested the random product homotopy and proposed a more efficient choice for the start system $Q(x) = 0$:

$$Q(x) = \begin{cases} q_1(x) = (\lambda + c_{11})(x_1 + c_{12}) \\ q_2(x) = (\lambda + c_{21})(x_2 + c_{22}) \\ \qquad \vdots \\ q_n(x) = (\lambda + c_{n1})(x_n + c_{n2}) \\ q_{n+1}(x) = c_1 x_1 + \cdots + c_n x_n + c_{n+1} \end{cases} \tag{5.2.5}$$

where c_{ij}'s $i = 1, \ldots, n$, $j = 1, 2$ and c_k, $k = 1, \ldots, n+1$ are randomly chosen complex numbers. It is clear that $Q(x) = 0$ has exactly n isolated solutions. It is proved in [17] that for this choice of $Q(x) = 0$, properties 0-2 hold for the linear homotopy

$$H(x, t) = (1 - t)Q(x) + tP(x) = 0.$$

Thus all the isolated solutions for the generalized eigenvalue problem can be found by following n paths emanating from the solutions of $Q(x) = 0$.

The generalized eigenvalue problem

$$Ax = \lambda Bx,$$

in particular the eigenvalue problem $Ax = \lambda x$, for $B = I$, has important applications in many scientific areas. It is widely known that very efficient algorithms for matrix eigenvalue problems, QR algorithm for $Ax = \lambda x$ and QZ algorithm for $Ax = \lambda Bx$, have been implemented and asserted in the software package LAPACK [16]. However, as the size of the matrix becomes larger, more computing resources are required. And a natural way to allocate extra computing resources efficiently is to perform independent tasks simultaneously in parallel. Since each isolated zero of a polynomial system is computed independently of all the others in the homotopy continuation method, it provides a natural environment for the parallelization. We have tested the parallel version of our homotopy algorithms on generalized eigenvalue problem

$$Ax = \lambda Bx$$

with augmented linear equation

$$c_1x_1 + \cdots + c_nx_n + c_{n+1} = 0$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

are randomly chosen $n \times n$ matrices and $c_1, \ldots, c_{n+1}$ are randomly chosen complex numbers for $n = 100$ and $n = 200$. We also tested the parallelization for eigenvalue problem, i.e. $B = I$, the $n \times n$ identity matrix and A is a randomly chosen matrix. Our preliminary numerical results are listed in Table 5.1. The speed-up ratio on the table represents the ratio of the CPU time of a single processor to that of multiple processors. Almost perfect speed-ups in the table illustrate the great potential in solving very large algebraic generalized eigenvalue problems in parallel by the homotopy continuation method in contrast to highly serial QZ or QR algorithms.

n	# of processors	CPU time		Speed-up ratio	
		$Ax = \lambda Bx$	$Ax = \lambda x$	$Ax = \lambda Bx$	$Ax = \lambda x$
100	1	86.5	54.6	1	1
	2	43.7	27.5	1.99	1.99
	3	28.9	18.2	2.99	3
	5	17.9	11.2	4.83	4.88
	7	12.6	7.9	6.86	6.91
200	1	1795	1146	1	1
	2	898	573	2	2
	3	598	384	3	2.98
	5	366	233	4.9	4.92
	7	260	166	6.9	6.9

Table 5.1. The parallel speed-up for solving generalized eigenvalue problem $Ax = \lambda Bx$ and eigenvalue problem $Ax = \lambda x$ with $n = 100$ and $n = 200$.

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