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Computations of Floer Homology and Gauge Theoretic
Invariants of Montesinos Twins

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Doctoral degree in Mathematics

Ronald A. Fintushel
Major Professor's Signature

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COMPUTATIONS OF FLOER HOMOLOGY AND GAUGE THEORETIC INVARIANTS FOR MONTESINOS TWINS

By

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ABSTRACT

COMPUTATIONS OF FLOER HOMOLOGY AND GAUGE THEORETIC INVARIANTS FOR MONTESINOS TWINS

By

Adam C. Knapp

I compute the Lagrangian Floer cohomology groups of certain tori in closed simply connected symplectic 4-manifolds arising from Fintushel - Stern knot/link surgery. These manifolds are usually not symplectically aspherical. As a result of the computation we observe examples where $HF(L_0) \cong HF(L_1)$ and L_0 and L_1 are smoothly and Lagrangian isotopic but L_0, L_1 are not symplectically isotopic.

and

In [Gil82], C. Giller proposed an invariant of ribbon 2-knots in S^4 based on a "Conway calculus" of crossing changes in a projection to $\mathbb{R}^3$. In certain cases, this invariant computes the Alexander polynomial. Giller's invariant is, however, a symmetric polynomial - which the Alexander polynomial of a 2-knot need not be. After modifying a 2-knot into a Montesinos twin in a natural way, we show that Giller's invariant is actually the Seiberg-Witten invariant of the exterior of the twin, glued to the complement of a fiber in $E(2)$.

To

Melissa

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I would like to thank my advisor, Ronald Fintushel.

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1.1 Introduction

1.1.1 Symplectic Manifolds

An even dimensional manifold M^{2n} together with a two form ω is said to be *symplectic* if ω is a nondegenerate and closed. The condition of non-degeneracy is equivalent to ω^n being a volume form on M. In particular, M must be oriented. The prototype of a symplectic manifold is $\mathbb{R}^{2n}$ given coordinates (x_i, y_i) and symplectic form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$.

For example, all orientable 2-manifolds are symplectic with symplectic forms equal to their volume forms. In fact, Moser's stability theorem gives us that, up to diffeomorphism, symplectic forms ω on closed 2-manifolds Σ are determined entirely by $\int_\Sigma \omega$.

Theorem 1.1.1 (Moser). Let M be a closed manifold with ω_t a smooth family of cohomologous symplectic forms. Then there is a family of diffeomorphisms $\phi_t : M \to M$ so that $\phi_0 = \text{id}$ and $\phi_t^*(\omega_t) = \omega_0$.

From a manifold topologist's point of view, symplectic manifolds are interesting due to the fact that the symplectic structure has no local invariants. Hence, when we obtain invariants of symplectic manifolds, we get information about the global topology of the manifold. The statement of "no local invariants" is more precisely Darboux's theorem:

Theorem 1.1.2 (Darboux). Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.

A chart of such local diffeomorphisms is called a Darboux chart.

Not all manifolds admit a symplectic form. Consider a closed symplectic manifold M. As ω^n is a volume form, it orients M and gives a non-zero class in the top dimensional cohomology group $H^{2n}(M;\mathbb{R}) \cong \mathbb{R}$. Thus ω must represent a non-zero class in $H^2(M;\mathbb{R})$ in order for $[\omega^n] = [\omega]^n$ to be non-zero. In particular, even dimensional spheres other than S^2 cannot be symplectic.

On a closed 4-dimensional manifold X, once we pick an orientation $[X] \in H_4(X;\mathbb{Z}) \subset H_4(X;\mathbb{R})$ we get a symmetric inner product Q_X on $H^2(X;\mathbb{R})$ by $\alpha \cdot \beta = (\alpha \wedge \beta)([X])$. Let b_+ and b_- be the dimensions of the maximal positive and negative definite subspaces for this inner product, respectively. If X is to admit a symplectic form with this orientation, we must have $b_+ \geq 1$. Now, $\overline{\mathbb{C}P}^2$ (the complex projective space $\mathbb{C}P^2$ with the opposite orientation), has $Q_{\overline{\mathbb{C}P}^2} = (-1)$. Hence $\overline{\mathbb{C}P}^2$ does not admit a symplectic form with its given orientation.

On the other hand, $\mathbb{C}P^n$ with its complex orientation, admits a symplectic form τ_0, the Fubini-Study form. In the local coordinates $[z_0 : z_1 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n]$ we can write

$$\tau_0 = \frac{i}{2}\partial\overline{\partial}\log\left(\sum_{j=0}^{n}\overline{z}_j z_j\right)$$

In particular, $\mathbb{C}P^2$ is symplectic.

An *almost complex structure* on M^{2n} is a bundle map $J : TM \to TM$ with $J^2 = -I$. If M is symplectic, we say that ω *tames* J if $\omega(X, JX) > 0$ for any nonzero tangent vector X. A tame almost complex structure is said to be *compatible* with ω if $\omega(JX, JY) = \omega(X, Y)$. If J is compatible with ω, the symmetric 2-tensor $g(X, Y) = \omega(X, JY)$ is a Riemannian metric on M. The space of compatible almost complex structures is contractable and so one can make sense of invariants that depend on an almost complex structure such as the Chern classes $c_i(X)$ of a symplectic manifold (X, ω).

A Kähler manifold is a symplectic manifold (X, ω) which admits an integrable almost complex structure. That is, X has a set of complex charts with holomorphic transition functions so that J corresponds to multiplication by $i = \sqrt{-1}$. Complex projective spaces are Kähler as are all complex submanifolds of a Kähler manifold. In particular, smooth projective subvarities of $\mathbb{C}P^n$ are Kähler.

A technique which can be applied to a Kähler or symplectic manifold (X, ω) is the blow up at a point. This procedure replaces a point $p \in X$ with the projectivized tangent space to p. The procedure results in another smooth manifold and can also be done in the algebraic setting. In the 4-dimensional case, this is smoothly equivalent to connect summing X with $\overline{\mathbb{C}P}^2$.

Suppose that we take a generic pencil of cubics on $\mathbb{C}P^2$. Then there are 9 points common to each cubic. If we blow up at each of these 9 points, we get the manifold $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ which admits a holomorphic map $E(1) \to \mathbb{C}P^1$ where the fibers are the cubics in $\mathbb{C}P^2$. Genericity of the pencil ensures that a generic fiber of this map will be a smooth elliptic curve. Now, take two of these generic fibers, F_1, F_2. The n-fold branch cover of $E(1)$ along F_1, F_2 is the manifold $E(n)$. The manifolds $E(n)$ also admit holomorphic maps to $\mathbb{C}P^1$ where the generic fiber is an elliptic curve and are examples of elliptic surfaces.

There are other prototypes of symplectic manifolds: Suppose that we have a manifold Σ^n with cotangent bundle $T^*\Sigma$. Take a coordinate chart $x_i : U \to \mathbb{R}$ on $U \subset \Sigma$. Over each point x in U, the differentials $(dx_i)_x$ form a basis of $T^*_x\Sigma$ and give coordinate charts $(x_i, y_i) : T^*U \to \mathbb{R}^2$. We can then form the canonical 1-form

$$\alpha_{\text{can}} = \sum_{i=1}^{n} y_i dx_i$$

which is independent of the coordinates chosen. In this case the form $\omega_{\text{can}} = -d\alpha_{\text{can}} = \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic form on $T^*\Sigma$. These manifolds occur in Hamiltonian mechanics where the base is interpreted to be a configuration space and the fibers as momentum coordinates.

Suppose that we are given two compact symplectic manifolds $(X_1, \omega_1), (X_2, \omega_2)$ of the same dimension each with a codimension 2 symplectic embedding of (Q, ω_Q). Assume that $Q_i \subset X_i$ has trivial normal bundle for each i. Then for each embedding of Q, we have the symplectic neighborhood theorem:

Theorem 1.1.3 (Symplectic Neighborhood). Suppose that there is an isomorphism $\Phi : \nu(Q_1) \to \nu(Q_2)$ of symplectic normal bundles which covers a symplectomorphism $\phi : Q_1 \to Q_2$. Then ϕ extends to a symplectomorphism $\psi : N(Q_1) \to N(Q_2)$ of tubular neighborhoods so that $d\psi = \Phi$ along Q_1.

As Q is codimension 2 in each X_i, the isomorphism type of the symplectic normal bundle is determined by its first Chern class. We will be interested in the case where the dimension of X_i is 4, where the Chern number is the same as the self intersection number of the surface Q.

For each i, there are symplectic embeddings $f_i : Q \times B^2(\epsilon_i) \to X_i$ so that $f_i^*(\omega_i) = \omega_Q + dx \wedge dy$. Suppose $0 < \epsilon_1 < \epsilon_2$ and let $A(\epsilon_1, \epsilon_2) = B^2(\epsilon_2) \setminus B^2(\epsilon_1)$. There is an area and orientation preserving map ϕ which exchanges the two components of $\partial A(\epsilon_1, \epsilon_2)$. Then we form the *fiber connected sum*:

$$X_1 \underset{Q}{\#} X_2 = (X_1 \setminus Q) \underset{\phi}{\cup} (X_1 \setminus Q)$$

There are more general versions of this important construction in [Gom95] and in [MW94]. Our principal interest will be in gluing symplectic 4-manifolds along square zero symplectic tori such as we find in elliptic surfaces.

Having constructed a number of symplectic manifolds with the above procedures, we arrive at the question: How flexible is a symplectic structure ω on a fixed manifold X? Or rather, what does the space $\mathrm{Symp}(X, \omega)$ of diffeomorphisms of X which preserve ω look like?

Because it is non-degenerate, the symplectic form gives a bijection $V \to \iota(V)\omega$ between vector fields and 1-forms on X. Suppose that V_t is a time-dependant vector field on a symplectic manifold (X, ω). Cartan's formula and $d\omega = 0$ gives $\mathcal{L}_{V_t}\omega = \iota(X)d\omega + d\left(\iota(V_t)\omega\right) = d\left(\iota(V_t)\omega\right)$. We say that V_t is a *symplectic vector field* if $d(\iota(V_t)\omega) = 0$. i.e. if the corresponding 1-forms are closed. The flows of such time-dependant vector fields give elements in the identity component of $\mathrm{Symp}(X, \omega)$.

A time dependant *Hamiltonian vector field* is a vector field V_{H_t} such that $\iota(V_{H_t})\omega = dH_t$ for functions $H_t : X \to \mathbb{R}$. i.e. the corresponding 1-forms are exact. The flows of such vector fields are a subset of the symplectic diffeomorphisms and are called *Hamiltonian* *symplectomorphisms* or $\mathrm{Ham}(X, \omega)$. If $b_1(X) = 0$, then $\mathrm{Ham}(X, \omega)$ is the same as the identity component of $\mathrm{Symp}(X, \omega)$ as all closed forms are also exact.

One of the natural, early conjectures concerning symplectic or Hamiltonian diffeomorphisms was that of Arnold, who conjectured that the number of fixed points of a Hamiltonian diffeomorphism on X was always greater or equal to the rank of the singular homology of X. If we restrict to the Hamiltonian diffeomorphisms generated by generic time-*in*dependant vector fields this fact is comparatively easy. Hamiltonian vector fields V_H correspond to smooth functions $H : X \to \mathbb{R}$, modulo constants. Generic smooth functions are Morse and the Morse inequalities give that the number of critical points of H is at least as large as the rank of the homology of X. Then since $\iota(V_H)\omega = dH$, V_H must be zero exactly when H has a critical point. The general case is, of course, not this easy and was proved for symplectic manifolds with $\pi_2 = 0$ by A. Floer in [Flo87]. This result has been extended several times by a number of authors.

1.1.2 Lagrangian submanifolds

A *Lagrangian submanifold* L of a symplectic manifold (M^{2n}, ω) is a n-dimensional submanifold of M such that $\omega|_L \equiv 0$. Some examples include $\mathbb{R}P^n \subset \mathbb{C}P^n$ and the graphs of closed 1-forms within cotangent bundles with their canonical symplectic form. The closed condition which defines Lagrangians suggests that they should be fairly rigid objects, and this indeed seems to be the case. The question we seek to answer is: How rigid are they?

First, similar to symplectic submanifolds, Lagrangian submanifolds have "nice" neighborhoods:

Theorem 1.1.4. Let (M, ω) be a symplectic manifold and $L \subset M$ a compact Lagrangian submanifold. Then there exists a neighborhood $N(L_0) \subset T^*L$ of the zero section, a neighborhood $N(L) \subset M$ of L, and a diffeomorphism $\phi : N(L_0) \to N(L)$ such that $\phi^*(\omega) = -d\alpha_{\mathrm{can}}$ (where α is the canonical 1-form) and $\phi|_{L_0} \equiv \mathrm{id}$.

That is, Lagrangian submanifolds have neighborhoods which are symplectomorphic to the zero sections of their cotangent bundles. Therefore, when we look to dimension 4, a Lagrangian L is 2-dimensional and has self-intersection equal to $-\chi(L)$. This has immediate consequences, for example the only closed orientable Lagrangian submanifolds of $\mathbb{R}^4$ are tori.

We will assume that we are in 4-dimensional symplectic manifolds unless otherwise noted. We also require that Lagrangian submanifolds be orientable so that they represent homology classes.

For a pair L_0, L_1 of Lagrangian submanifolds in a symplectic 4-manifold, there are several types of isotopy that we may consider:

- Smooth isotopy.
- Lagrangian isotopy. L_0 and L_1 are Lagrangian isotopic in X if there is an smooth isotopy of L_0 to L_1 through Lagrangians.
- Symplectic/Hamiltonian isotopy. These are Lagrangian isotopies of L_0 to L_1 which extend to Symplectic/Hamiltonian isotopies of the ambient manifold X. These concepts coincide when $b_1(X) = 0$.

Let us consider the case of Lagrangian tori in $\mathbb{R}^4$. Consider $\mathbb{R}^4 \cong \mathbb{C}^2$ and the unit circle $S^1 \subset \mathbb{C}$. Then it is easy to verify that $S^1 \times S^1 \subset \mathbb{C}^2$ is a Lagrangian torus. This can be reinterpreted as a "spun knotted torus". i.e. Take a knot K lying away from the boundary in the half space $\mathbb{R}^{3+}$ and form $S^1 \times K \subset (S^1 \times \mathbb{R}^{3+})/\sim$ where $(\theta_1, x_1) \sim (\theta_2, x_2)$ iff $x_1 = x_2 \in \partial\mathbb{R}^{3+}$. Now, $(S^1 \times \mathbb{R}^{3+})/\sim \cong \mathbb{R}^4$ and $\pi_1(\mathbb{R}^4 \backslash S^1 \times K) = \pi_1(\mathbb{R}^{3+} \backslash K) = \pi_1(S^3 \backslash K)$. Therefore, distinct knots K determine distinct isotopy types of "spun knotted tori" in $\mathbb{R}^4$.

In [Lut95], K. Luttinger showed that, out of these isotopy classes, only the spun unknot can have a Lagrangian representative. His theorem is based on a kind of surgery which can be performed on Lagrangian tori which we describe here. We know that a Lagrangian torus L will have a trivial normal bundle since $\chi(T^2) = 0$. Pick a trivialization of the normal bundle, $L \times D^2$, so that $L \times \{pt\}$ is Lagrangian for all $pt \in D^2$. Such a trivialization is canonical.

In fact, since these tori are nullhomologous, there is another trivialization of the normal bundle coming from a push-off of the Lagrangian into the Seifert 3-manifold that it bounds. For Lagrangian tori in $\mathbb{R}^4$, Luttinger shows that the framing coming from Lagrangian push-offs and the nullhomologous framings coincide.

Suppose that L is a Lagrangian representative of a the isotopy class of the spin of a knot K. Give $L \times D^2$ coordinates $x, y \in \mathbb{R}^2/\mathbb{Z}^2$ on L and polar coordinates (r, θ) on D^2. We can then write $\omega = d(r(\cos(2\pi\theta)dx + \sin(2\pi\theta)dy))$. The map $F_{m,n}(x, y, r, \theta) = (x + m\theta, y + n\theta, r, \theta)$ preserves the restriction of ω to $\partial(L \times D^2)$. Therefore, we can excise $L \times D^2$ and glue it back in using $F_{m,n}$ for any m, n. Whenever the knot K is not the unknot, nontrivial surgeries will result in manifolds with π_1 nontrivial. However, there is a theorem due to Gromov with a refinement due to McDuff, which says

Theorem 1.1.5 (Gromov, McDuff). If (X, ω) is a symplectic 4-manifold which is symplectomorphic to $\mathbb{R}^4$ at infinity, then (X, ω) is symplectomorphic to $\mathbb{R}^4$ blown up finitely many times.

Since Luttinger's surgery only affects a compact region, this theorem holds. Therefore, K could not have been nontrivial so there are no Lagrangian representatives of the isotopy classes of spins of nontrivial knots.

Further restricting the isotopy classes of Lagrangians, Eliashberg and Polterovich showed in [EP96] that a Lagrangian $\mathbb{R}^2$ in $\mathbb{R}^4$ which is asymptotic to a Lagrangian plane at infinity is Hamiltonian isotopic to the planar embedding. This rules out the idea that you could have potentially chosen a Darboux chart centered at a point in the Lagrangian and performed a local knotting operation. i.e. a connect sum of a 2-knot in S^4.

Another result from Eliashberg and Polterovich ([EP93]) shows that if L is a Lagrangian sphere or torus embedded in T^*L, if L is homotopic to the zero section then it is smoothly isotopic to the zero section.

The previous three results each speak of topological unknottedness of Lagrangians. Things become more subtle when the symplectic invariants come into play. For example, in [EP97] Eliashberg and Polterovich gave examples of Lagrangian tori in $\mathbb{R}^4$ which are Lagrangian isotopic but not Hamiltonian isotopic. These examples require the introduction of the Maslov class. Suppose that we have a loop γ on a Lagrangian $L \subset X$ with a trivialization of TX over γ. If X is simply connected (or if $[\gamma] = 1 \in \pi_1(X)$), such a trialization may be given by fixing a disc D which γ bounds and trivializing the pullback of TX over D. Then the Maslov class $\mu(\gamma)$ is a winding number of the Lagrangian planes TL around

γ relative to this trivialization. For a general manifold, this may only be well defined up to adding of spheres S to D which changes the Maslov class by $c_1(X) \cdot S$. However, $\mathbb{R}^4$ is contractable so this issue is immaterial here. It is elementary to see that μ gives a homomorphism $H_1(L) \to \mathbb{Z}$ and so can be thought of as an element of $H^1(L;\mathbb{Z})$. It is also easy to see that μ is a Lagrangian isotopy invariant.

Now, thinking of $\mathbb{R}^4$ as $T^*\mathbb{R}^2$, we have the canonical 1-form α. Let $f : T^2 \to \mathbb{R}^4$ give the embedding of L. We call $f^*\alpha \in H^1(T^2)$ the symplectic area class of L. The torus L is monotone if $f^*\alpha = \lambda\mu$ for $\lambda \in \mathbb{R}^+$. Consider the following Lagrangian tori in $\mathbb{R}^4 \cong \mathbb{C}^2$:

$$L_{a,b} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = \frac{2a}{\pi}, |z_2| = \frac{2b}{\pi} \right\}$$

The $L_{a,a}$ are monotone while $L_{a,b}$ with $a \neq b$ are not. Clearly any $L_{a,b}$ and $L_{c,d}$ are Lagrangian isotopic and so have the same Maslov classes. They must have different symplectic area classes then — which are invariants of Hamiltonian isotopy. Thus these Lagrangian tori are smoothly isotopic through Lagrangians, but are not isotopic by Hamiltonian diffeomorphisms.

We can also look at Lagrangian tori in closed symplectic 4-manifolds and manifolds with nontrivial topology. The construction which we use for our theorem is originally due to Vidussi in [Vid06]. It will be described more throughly later. Roughly, however the construction starts with the Fintushel-Stern knot surgery manifolds. That is, a fibered knot K in S^3 is chosen and zero surgery on that knot is performed to get $S^3_0(K)$. Then with m a meridian,

$$E(2)_K = E(2) \mathop{\#}_{F=S^1\times m} S^1 \times S^3_0(K)$$

is a symplectic manifold. Vidussi pointed out that loops on the fiber of K cross S^1 correspond to Lagrangian tori. He showed that there are an infinite number of these Lagrangian tori which are smoothly nonisotopic inside $E(2)_K$ by computing relative Seiberg-Witten invariants and showing that the degrees go to infinity.

In the related paper [FS04], Fintushel and Stern defined an integer valued smooth invariant (extracted from the relative Seiberg-Witten invariants) of the same sort of Lagrangian tori, distinguishing the smooth isotopy class of an infinite family. In fact, the tori in question are nullhomologous and hence have a Lagrangian push-off and Seifert pushoff framing.

Unlike in $\mathbb{R}^4$, where Luttinger proved that the two coincide, Fintushel-Stern's integer invariant detects the difference between the two and is shown to take an infinite number of values.

The previous two results can be interpreted as stating that Lagrangians may be smoothly knotted - without giving any symplectic results. To get such results, a new tool is needed. This is Lagrangian Floer Homology, introduced by Floer in [Flo88]. The original purpose of Lagrangian Floer homology was to answer a variant of Arnold's conjecture. The variant of the conjecture in question is this: Suppose that $L \subset X$ is a Lagrangian submanifold of the symplectic manifold (X, ω). Then given a Hamiltonian diffeomorphism ϕ on X so that L and $\phi(L)$ are transverse, $|L \cap \phi(L)|$ is at least as large as the rank of the singular homology of L.

1.1.3 Lagrangian Floer Homology

Floer's solution to Arnold's (Lagrangian) conjecture is an invariant of Lagrangian submanifolds, now called Lagrangian Floer Homology, which is well defined up to Hamiltonian isotopy. We will now sketch a basic "User's Guide" to the construction.

Suppose that (X, ω, J) is a symplectic manifold together with an almost complex structure and that $L \subset X$ is a compact Lagrangian submanifold of X. Let ϕ be a Hamiltonian diffeomorphism of X so that L and $\phi(L)$ intersect transversely. Then L and $\phi(L)$ will intersect in a finite number of points. Define the chain group

$$CF(L, \phi(L)) = \bigoplus_{x \in L \cap \phi(L)} \mathbb{Z}_2 x$$

With this definition in mind, we can see that any homology built out of this chain group will have at least as many generators as there are points of $L \cap \phi(L)$. Thus, once a set of invariant homology groups are built, its rank will give a lower bound on $|L \cap \phi(L)|$.

Consider the moduli space $\mathcal{M}(x, y)$ of maps $\tilde{u}$ of the infinite strip $S = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq 1\}$ to X such that

- $\lim_{\Im(z) \to \infty} \tilde{u}(z) = x$ and $\lim_{\Im(z) \to -\infty} \tilde{u}(z) = y$,
- $\tilde{u}(0 + iy) \subset L$ and $\tilde{u}(1 + iy) \subset \phi(L)$, and

- $\tilde{u}$ is J-holomorphic. i.e. with J_0 the standard complex structure on $\mathbb{C}$, $J \circ d\tilde{u} = d\tilde{u} \circ J_0$

The first two conditions define what we refer to as a "topological Floer disc" connecting x and y. Together with the third we define a "Floer disc" connecting x and y. There is a free $\mathbb{R}$-action on $\mathcal{M}(x,y)$ given by precomposing with a reparameterization of the infinite strip $r : x + iy \to x + i(y + r)$. Also, we can consider maps u from the unit ball around zero $D^2 \subset \mathbb{C}$ by a conformal map $S \to D^2 \setminus \{\pm i\}$ which sends $(\partial D^2 \cap \{\Re(z) \leq 0\}) \setminus \{\pm i\}$ to $\{0 + iy | y \in \mathbb{R}\}$ and $(\partial D^2 \cap \{\Re(z) \geq 0\}) \setminus \{\pm i\}$ to $\{1 + iy | y \in \mathbb{R}\}$.

Homotopy classes of topological Floer discs form a space $\pi_2(X, L)$ that has an action of $\pi_2(X)$ on it. These split $\mathcal{M}(x,y)$ into sets of connected components divided by their homotopy class. For a moment, assume that $\pi_2(X) = 0$. Then for a generic choice of J, $\mathcal{M}(x,y)$ is a smooth orientable manifold of dimension $\mu(x,y)$ where $\mu(x,y)$ a Maslov class defined as follows: Take a topological disc in the (unique) homotopy class connecting x and y. Trivialize the (pullback of the) tangent bundle of X over this disc. Then $\mu(x,y)$ is the winding number of T^*L and $T^*\phi(L)$ with respect to this trivialization. A more precise definition can be found in [RS93].

When $\pi_2(X, L) \neq 0$, we must specify the homotopy class in order to define $\mu(x,y)$. If D_1, D_2 are two such homotopy classes with $[\partial D_1] = [\partial D_2] \in \pi_1(L)$, then $D_1 - D_2$ is a element of $\pi_2(X)$. It is elementary to compute that $\mu_{D_1}(x,y) - \mu_{D_2}(x,y) = c_1(X)[D_1 - D_2]$.

Define

$$\partial x = \sum_{\substack{y \in L \cap \phi(L) \\ \mu(x,y) = 1}} \#\left(\mathcal{M}(x,y)/\mathbb{R}\right) y$$

where $\#\left(\mathcal{M}(x,y)/\mathbb{R}\right)$ is the point count modulo 2 of the 0-dimensional space. Compactness arguments show that this sum is finite.

Gromov showed in [Gro85] that when bounds on symplectic energy (integral of the symplectic form over the surface) are satisfied, the spaces $\mathcal{M}(x,y)$ can be compactified by adding in configurations of nodal curves to the boundary. There are three types of degeneration, which in a high dimensional moduli space may each happen together and/or more than once:

- Breaking into trajectories. i.e. A Floer disc from x to z may degenerate into a Floer disc from x to y and one from y to z.

- Bubble sphere off interior. i.e. A Floer disc D_1 from x to y may degenerate into another Floer disc D_2 from x to y plus a J-holomorphic sphere S. In this case, $\mu_{D_1}(x, y) = \mu_{D_2} + c_1(X)[S]$.
- Bubble disc off boundary. i.e. A Floer disc D_1 from x to y may degenerate into another Floer disc D_2 from x to y plus a J-holomorphic disc D_3 with boundary entirely on L or $\phi(L)$.

As J-holomorphic curves are necessarily symplectic, any non-constant J-holomorphic spheres S or J-holomorphic discs D with boundary on L have $\int_S \omega, \int_D \omega > 0$. Therefore, when $\pi_2(X, L) = 0$ or $\int \omega|_{\pi_2(X,L)} \equiv 0$ there are no non-constant spheres or discs with boundaries entirely on one Lagrangian. For a moment assume that $\int \omega|_{\pi_2(X,L)} \equiv 0$.

Now, for ∂ to be a differential, we must have $\partial^2 \equiv 0$. With the assumption that $\int \omega|_{\pi_2(X,L)} \equiv 0$, the boundary of a compactified 1-dimensional moduli space $\mathcal{M}(x, z)/\mathbb{R}$ of Floer discs consists only of products of zero dimensional moduli spaces $\mathcal{M}(x, y)/\mathbb{R} \times \mathcal{M}(y, z)/\mathbb{R}$ for $y \in L \cap \phi(L)$. Then we compute,

$$\begin{aligned}
\partial^2 x &= \sum_{\substack{z \in L \cap \phi(L) \\ \mu(y,z)=1}} \ \sum_{\substack{y \in L \cap \phi(L) \\ \mu(x,y)=1}} \# (\mathcal{M}(x, y)/\mathbb{R}) \, \# (\mathcal{M}(y, z)/\mathbb{R}) \, z \\
&= \sum_{\substack{z \in L \cap \phi(L) \\ \mu(x,z)=2}} \# (\partial(\mathcal{M}(x, z)/\mathbb{R})) \, z
\end{aligned}$$

which is zero as $\mathcal{M}(x, z)/\mathbb{R}$ is a 1-dimensional space and so has an even number of boundary components. Thus $\partial^2 = 0$.

Floer showed that his homology is a Hamiltonian isotopy invariant and independent of almost complex structure by continuation. Suppose that we have two generic almost complex structures J_0, J_1. Then the chain group $CF(L, \phi(L))$ has two differentials ∂_0, ∂_1 each defined as above but counting J_0- and J_1-holomorphic Floer discs respectively. As the space of almost complex structures is contractable, there is a path J_t connecting J_0 and J_1.

Now define

$$\Phi_{01}(x) = \sum_{\substack{y \in L \cap \phi(L) \\ \mu(x,y)=0}} \hat{\mathcal{M}}(x,y)y,$$

a map from $(CF(L,\phi(L)),\partial_1)$ to $(CF(L,\phi(L)),\partial_0)$. Here, $\hat{\mathcal{M}}(x,y)$ is a zero dimensional moduli space of Floer discs where the almost complex structures vary. That is, maps $\hat{u}$ of the infinite strip $S = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq 1\}$ to X such that

- $\lim_{\Im(z)\to\infty} \hat{u}(z) = x$ and $\lim_{\Im(z)\to-\infty} \hat{u}(z) = y$,
- $\hat{u}(0+iy) \subset L$ and $\tilde{u}(1+iy) \subset \phi(L)$, and
- $\hat{u}$ has
  - $J_0 \circ d\hat{u} = d\hat{u} \circ J_{std}$ over points z with $\Im(z) \leq 0$,
  - $J_1 \circ d\hat{u} = d\hat{u} \circ J_{std}$ over points z with $\Im(z) \geq 1$, and
  - $J_t \circ d\hat{u} = d\hat{u} \circ J_{std}$ over points z with $\Im(z) = t \in [0,1]$,

  where J_{std} is the standard almost complex structure on $\mathbb{C}$.

Note that there is no $\mathbb{R}$-action on $\hat{\mathcal{M}}(x,y)$ defined this way so when $\hat{\mathcal{M}}(x,y)$ is 0-dimensional, it need not be empty.

To prove that Φ_{01} is a chain map, and thus induces a map on homology, we must show that $\partial_0 \circ \Phi_{01} - \Phi_{01} \circ \partial_1 = 0$. This is accomplished by describing the boundary of the compactification $\overline{\hat{\mathcal{M}}(x,z)}$ of $\hat{\mathcal{M}}(x,z)$ when $\mu(x,z) = 1$. In this case $\overline{\hat{\mathcal{M}}(x,z)}$ is compact, one dimensional and so modulo two the count of points in its boundary is zero.

In this setup, Gromov compactness means that elements of $\hat{\mathcal{M}}(x,z)$ will degenerate into two types of curves in the limit:

- J_0-holomorphic Floer discs in $\mathcal{M}(x,y)$ with $\mu(x,y) = 1$ and elements of $\mathcal{M}(y,z)$ with $\mu(y,z) = 0$
- Elements of $\mathcal{M}(x,y)$ with $\mu(x,y) = 0$ and J_1-holomorphic Floer discs in $\mathcal{M}(y,z)$ with $\mu(y,z) = 1$.

These configurations are exactly those counted by $\partial_0 \circ \Phi_{01}$ and $\Phi_{01} \circ \partial_1$ respectively. Therefore, as $\left|\partial\overline{\hat{\mathcal{M}}(x,z)}\right| = 0 \mod 2$, $\partial_0 \circ \Phi_{01} - \Phi_{01} \circ \partial_1 = 0 \mod 2$. So Φ_{01} is a chain map.

Now we describe why Φ_{01} is well defined (on homology) independent of the path J_t. Suppose that we had two paths J_t, J_t' interpolating between J_0, J_1. We then get two map Φ_{01}, Φ_{01}'. We wish to show that they are chain homotopic. That is, there is a map H on $CF(L, \phi(L))$ for which $\Phi_{01} - \Phi_{01}' = -H\,\partial_0 - \partial_1\, H$. Take a homotopy $J_{s,t}$ so that $J_{0,t} = J_t$ and $J_{1,t} = J_t'$ and define

$$H(x) = \sum_{\substack{y \in L \cap \phi(L) \\ \mu(x,y) = -1}} \check{\mathcal{M}}(x,y)y$$

where $\check{\mathcal{M}}(x,y)$ is the space of maps $\check{u}$ from $[0,1] \times i\mathbb{R} \subset \mathbb{C}$ with

- $\lim_{\Im(z)\to\infty} \check{u}(z) = x$ and $\lim_{\Im(z)\to-\infty} \check{u}(z) = y$,
- $\check{u}(0 + iy) \subset L$ and $\check{u}(1 + iy) \subset \phi(L)$, and
- $\check{u}$ has
  - $J_0 \circ d\check{u} = d\check{u} \circ J_{std}$ over points z with $\Im(z) \leq 0$,
  - $J_1 \circ d\check{u} = d\check{u} \circ J_{std}$ over points z with $\Im(z) \geq 1$, and
  - $J_{s,t} \circ d\check{u} = d\check{u} \circ J_{std}$ over points $z = s + it$ with $\Im(z) = t \in [0,1]$,

  where J_{std} is the standard almost complex structure on $\mathbb{C}$.

With $\mu(x,y) = -1$, $\check{\mathcal{M}}(x,y)$ is a compact manifold of dimension 0.

The statement $\Phi_{01} - \Phi_{01}' = -H\,\partial_0 - \partial_1\, H$ then follows by viewing $\Phi_{01} - \Phi_{01}' + H\,\partial_0 + \partial_1\, H$ as counting elements of the boundary of the 1-dimensional space $\check{\mathcal{M}}(x,z)$, $\mu(x,z) = 0$.

Now Φ_{01} is an isomorphism on homology as a result of the invariance result we have just discussed. In particular, consider the map $\Phi_{10} \circ \Phi_{01}$. This can be thought of as being given by a homotopy of J_t to itself. We have invariance under such moves so $\Phi_{10} \circ \Phi_{01} = I$ and the converse. Hence, $\Phi_{01}^{-1} = \Phi_{10}$ so Φ_{10} and Φ_{01} are isomorphisms.

A set of similar arguments can be made to show that Lagrangian Floer homology is independent of Hamiltonian isotopy. The reader should note that the assumptions on $\int \omega$

on $\pi_2(X, L)$ made for the purposes of the discussion are in some places necessary for the sketch as stated. This is because, without said assumption, the boundaries of moduli spaces considered may contain other configurations of curves. While some boundary configurations, like that of bubbled-off spheres S with non-negative $c_1(X) \cdot S$, can be dealt with via largely algebraic constructions, (see [HS95] for the relevant idea of Novikov homology done in the symplectic Floer homology setting) bubbles on the boundary pose more fundamental problems. For our own theorem, we will show non-existence of these problem configurations.

The property of Lagrangian Floer homology which proves the Lagrangian Arnold conjecture is that, on small time flows of 1-parameter Hamiltonians, there is a bijection between Floer discs and gradient flows of a Morse function on the Lagrangian. This means that Lagrangian Floer homology is isomorphic to the standard Morse homology. Invariance of Lagrangian Floer homology then gives us the desired result for any Hamiltonian flow. (Remember that all of this is only the case when $\int \omega|_{\pi_2(X,L)} \equiv 0$)

We say L_0, L_1 have clean intersection if their intersection is a embedded submanifold and $T(L_0 \cap L_1) = TL_0 \cap TL_1$. The result of Poźniak in [Poź94] was a chain homotopy between the Morse complex of the clean intersection and the Lagrangian Floer complex with differential restricted to counting intersections and Floer discs in a neighborhood of the clean intersection. i.e. the local Floer homology $HF(L, \phi(L); U)$ where U is a neighborhood of the clean intersection. Poźniak's result is used in key places in both our own work and the work of Seidel described below.

When $\int \omega$ and c_1 restricted to $\pi_2(X, L)$ are nontrivial, we may have problems as in the boundary, moduli spaces may have different curve configurations than we have specified previously. This can be remedied by restricting to Lagrangians and spaces which are "nice" in the sense that all the above constructions work.

When Lagrangian Floer homology is defined but $\int \omega$ is nontrivial, we have the phenomena of so called "quantum corrections", i.e. pseudoholomorphic discs which contribute to ∂ and have large energy even for small time Hamiltonian isotopies. In the presence of quantum corrections, $HF(L, \phi(L))$ may not be isomorphic to $H_*(L; \mathbb{Z}_2)$. In fact, quantum corrections give higher order differentials computing $HF(L, \phi(L))$ from a chain complex starting at the sum of local Floer homologies, $\bigoplus_{S \subset L \cap \phi(L)} HF(L, \phi(L); N(S))$ where S is a connected component of $L \cap \phi(L)$ and $N(S)$ is a tubular neighborhood of S.

We now have an invariant sensitive enough to distinguish non-Hamiltonianly isotopic Lagrangian submanifolds. In [Sei99], Seidel uses Lagrangian Floer homology to distinguish an infinite family of smoothly isotopic but symplectically nonisotopic Lagrangian spheres in an exact symplectic 4-manifold. He extended this result in [Sei00] to certain embeddings of these examples into $K3$ and Enriques surfaces. Seidel's computation uses the "Morse-Bott" spectral sequence for clean intersections of Lagrangians from [Poź94] to compute these Lagrangian Floer homologies. We now describe his construction.

Suppose there is a Lagrangian 2-sphere, L, in X.

$$T^*S^2 = \left\{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle u, v \rangle = 0, ||v|| = 1\right\}$$

with $\omega = \Sigma_{i=1}^3 du_i \wedge dv_i$ (restricted to T^*S^2. The function $h(u, v) = ||u||$ induces a Hamiltonian circle action on $T^*S^2 \setminus S^2$

$$\sigma_t(u, v) = \left(\cos(t)u - \sin(t)||u||v, \cos(t)v + \sin(t)\frac{u}{||u||}\right)$$

Then σ_π extends to the antipodal map while for $t \in (0, 2\pi) \setminus \pi$, the map does not extend to the zero section. Seidel describes this as the normalized geodesic flow for the round metric on S^2 as it transports each (co)tangent vector along the corresponding geodesic at unit speed.

Letting $r : \mathbb{R} \to \mathbb{R}$ be such that $r(-t) = r(t) - t$ and when $t \gg 0$, $r(t) = 0$. Then letting $H = r \circ h$, we get a Hamiltonian flow $\phi_t = \sigma_{r'(||u||}(u, v)$. Then as $r'(0) = \frac{1}{2}$, the 2π map can be extended over the zero section as the antipodal map. We then get the symplectic diffeomorphism

$$\tau(u, v) = \begin{cases} \sigma_{2\pi r'(||u||)}(u, v) & \text{if } u \neq 0 \\ (0, -v) & \text{if } u = 0 \end{cases}$$

which is compactly supported and called a Dehn twist. As τ is compactly supported, it extends to a symplectomorphism τ_L of X which is the identity outside a tubular neighborhood of L. The action of τ_L on $H_2(X)$ is

$$(\tau_L)_*(x) = x + (x \cdot L)L$$

and since $L^2 = -2$, $(\tau_L)_*$ has trivial square. In fact, as a smooth map τ_L^2 is isotopic to the identity.

Now let X be a linear plumbing of three T^*S^2s, L_1, L_2, L_3 so that $L_1 \cap L_2 = pt_1$, $L_2 \cap L_3 = pt_2$, with pt_1, pt_2 antipodal, and $L_1 \cap L_3 = \emptyset$. Then $c_1(X)$ and ω_X are both nullhomologous. Consider the sequence of Lagrangian S^2s $L_1^n = \tau_{L_2}^{2n}(L_1)$. As τ^2 is smoothly isotopic to the identity, each L_1^n is smoothly isotopic.

In the symplectic world, consider the intersections $L_1^n \cap L_3$. By hypothesis, $L_1 \cap L_3 = \emptyset$ so we must have $HF(L_1, L_3) \cong 0$. As X is a plumbing, L_1 and L_2 intersect so that, in the tubular neighborhood $N(L_2)$ in which the support of $\tau_{L_2}^2$ lies, $L_1 \cap N(L_2)$ is a normal disc. By the description previously, we can then see that the image $\tau_{L_2}^2(L_1 \cap N(L_2))$ projects to a double cover $N(L_2) \to L_2$. As pt_1, pt_2 are antipodal, they are each other's conjugate locus with the round metric. Thus $\tau_{L_2}^2(L_1 \cap N(L_2))$ intersects the fiber over pt_2 in an S^1 so $L_1^1 \cap L_3 \cong S^1$. Moreover, this is a clean intersection - $TL_1^1 \cap TL_3 = T(L_1^1 \cap L_3)$.

By similar argument, $L_1^n \cap L_3$ is a clean intersection that consists of n copies of S^1. Pozniak's result then gives us the E_2 page of a spectral sequence converging to $HF(L_1^n, L_3)$: $E_2 \cong H_*(S^1; \mathbb{Z}_2)^{\otimes n}$. Seidel then proves that

Theorem 1.1.6 (Seidel). $HF(L_1, L_3) \cong 0$ and $HF(L_1^n, L_3) \neq 0$

by showing that, due to large (≥ 2) differences in Maslov class of discs connecting the components of $L_1^n \cap L_3$, at least one term of the spectral sequence does not vanish. Thus L and L_1^n are Hamiltonianly isotopic. In fact, all L_1^r and L_1^s are Hamiltonianly non-isotopic for $r > s$, as we can pull back via $\tau_{L_2}^{2s}$ and see that L_1 and L_1^{r-s} are Hamiltonianly non-isotopic.

1.2 Construction

Vidussi's symplectic version of the Fintushel-Stern link surgery can be described as follows: Consider M_L, a 3 manifold obtained from zero surgery on a nontrivial fibered m component link L in S^3 with a fibration $\pi : M_L \to S^1$. Choose metrics on M_L and S^1 appropriately so that the fibration map π is harmonic. (Without loss of generality assume that the metric on S^1 gives it volume 1.) Then $S^1 \times M_L$ has a symplectic form $\omega = d\theta \wedge d\pi + *_3 d\pi$. Here θ is a coordinate on S^1 and $*_3 d\pi$ indicates the pullback of $*d\pi$ in M_L via the projection

$S^1 \times M_L \to M_L$. The form ω is closed since π is harmonic and nondegenerate since L is fibered. If m_i are meridians to the components K_i of L, then $S^1 \times m$ is a symplectic torus of square zero. Let X_i be symplectic 4-manifolds each with a symplectic torus F_i of square zero and tubular neighborhood $N(F_i)$. Suppose that $\pi_1(X_i \setminus N(F_i)) = 0$, then the symplectic fiber sum

$$X_L = S^1 \times M_L \underset{\substack{i=1,\dots,m \\ F_i = S^1 \times m_i}}{\#} X_i$$

is simply connected and symplectic. Symplectic and Lagrangian submanifolds of each X_i and of $S^1 \times M_L$ which do not intersect the $F_i = S^1 \times m_i$ remain so under this process. (We also note that on each link component the choice of meridian m_i does not matter since isotopies of m_i induce deformation equivalences of symplectic structures on X_L.)

Let γ be a loop on a fiber of π. Then with the specified symplectic form, $L_\gamma = S^1 \times \gamma$ is a Lagrangian torus in $S^1 \times M_K$. When γ and the m_i are disjoint, L_γ is also naturally a Lagrangian torus in X_L. It is this class of Lagrangian tori which we will be considering here.

There is one more observation of note. In the 3-manifold M_L there is a natural construction of a vector field μ, namely the vector field uniquely determined by $\iota(\mu)(*d\pi) \equiv 0$ and $\pi^*(dvol_{S^1})(\mu) \equiv 1$. By construction, the time t flow of μ preserves the fibers of π, moving them in the forward monodromy direction. Thus the time 1 flow of μ on M_L gives the monodromy map when restricted to a fiber of π. If we extend μ to a vector field on $S^1 \times M_L$ which we also call μ, then we note that $\iota(\mu)\omega$ is a closed, but not exact 1-form. Thus we get a 1-parameter family of symplectomorphisms ϕ_t on $S^1 \times M_L$ which are not Hamiltonian.

Consider the action of ϕ_t on our Lagrangian torus L_γ. When $t \notin \mathbb{Z}$, $\phi_t(L_\gamma) \cap L_\gamma = \emptyset$ as γ is moved to a disjoint fiber. However, when $t \in \mathbb{Z}$ it is possible that $\phi_t(L_\gamma)$ and L_γ intersect. Further, when the monodromy is of finite order, we can find a good choice of meridian m so that the symplectic isotopies of L_γ to its iterates under the monodromy stay away from the $S^1 \times m_i$. These symplectic isotopies survive as Lagrangian isotopies in X_L.

1.3 Calculation of Floer Cohomology

We use the variant of Lagrangian Floer cohomology over the universal Novikov ring Λ in [FOOO]. In this theory, the construction of the Floer cohomology groups is defined when certain obstruction classes vanish. These classes count pseudoholomorphic discs with boundary on the Lagrangians. The following lemmas show that we are in the situation where these classes vanish and serve to compute the homology.

Lemma 1.3.1. Suppose $(S^1 \times M_L, \omega)$ is as above and that the link L is nontrivially fibered in the sense that the genus of the fiber is at least 1. Then $S^1 \times M_L$ contains no pseudoholomorphic spheres. Further, suppose that γ_i, $i = 0, 1$, are loops on a fiber of π which meet transversely in exactly one point and let $L_i = L_{\gamma_i}$. Then all pseudoholomorphic discs in $S^1 \times M_L$ with boundary on L_0 or on L_1 are constant and there are no nonconstant Floer discs for L_0, L_1.

Note that we cannot extend this lemma to say that there are no pseudoholomorphic representatives of $\pi_2(S^1 \times M_L, L_0 \cup L_1)$. We see such a counterexample in Section 1.4.

Proof of Lemma 1.3.1. As was assumed, M_L is a fibration over S^1 with fiber Σ_g of genus $g \geq 1$ and projection π. As $[\gamma_0] \cdot [\gamma_1] = \pm 1$ in the homology of the fiber no nonzero multiples of the two may be homologous. Thus they represent distinct infinite order elements of $\pi_1(\Sigma_g, \gamma_0 \cap \gamma_1)$ for which no powers $i, j \neq 0$ give $[\gamma_0]^i = [\gamma_1]^j$. By considering the universal abelian cover $\Sigma_g \times \mathbb{R}$, we see that $\pi_1(\Sigma_g)$ injects into $\pi_1(M_L)$ by the inclusion of a fiber. Then the subgroups generated by γ_0, γ_1 intersect trivially in $\pi_1(M_L, \gamma_0 \cap \gamma_1)$.

Since the genus of the fiber $g \geq 1$, $\pi_2(M_L) = 0$ and thus $\pi_2(S^1 \times M_L) = 0$. Now consider the exact sequence

$$0 = \pi_2(S^1 \times M_L) \longrightarrow \pi_2(S^1 \times M_L, L_i) \longrightarrow \pi_1(S^1 \times \gamma_i) \xrightarrow{i} \pi_1(S^1 \times M_L)$$

It follows that $\pi_2(S^1 \times M_L, L_i) = \ker(i) = 0$ by our assumptions on γ_i. Therefore, the homomorphism $\int \omega$ on $\pi_2(S^1 \times M_L, L_i)$ given by choosing a representative and integrating the pullback of ω over the disc is the zero homomorphism.

Let $\Omega(L_0, L_1)$ denote the space of paths $\delta : ([0,1], 0, 1) \to (S^1 \times M_L, L_0, L_1)$ and $\Omega_0(L_0, L_1)$ be that subset whose members are homotopic to a point. As $L_0 \cap L_1$ is connected, $\Omega_0(L_0, L_1)$ is also connected. Let i_0, i_1 be the inclusions of L_0 and L_1 into $S^1 \times M_L$ respectively. There is an evaluation map $p : \Omega_0(L_0, L_1) \to L_0 \times L_1$, $p(\delta) = (\delta(0), \delta(1))$. This is a Serre fibration whose fiber is homotopy equivalent to $\Omega_0(S^1 \times M_L, x)$. Thus $\pi_k(p^{-1}(\delta_0, \delta_1)) \cong \pi_{k+1}(S^1 \times M_L)$ and there is the exact sequence

$$\begin{tikzcd}
\pi_2(S^1 \times M_L) \arrow[r] \arrow[d, "\cong"] & \pi_1(\Omega_0(L_0, L_1)) \arrow[r, "p_*"] \arrow[d, "="] & \pi_1(L_0) \times \pi_1(L_1) \arrow[r, "i_{0*} \cdot i_{1*}^{-1}"] \arrow[d, "\cong"] & \pi_1(S^1 \times M_L) \arrow[d, "\cong"] \\
0 \arrow[r] & \pi_1(\Omega_0(L_0, L_1)) \arrow[r] & \mathbb{Z}^2 \times \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \times \pi_1(M_L)
\end{tikzcd}$$

For the last map, the sequence is exact in the sense that

$$\mathrm{im}(p_*) = \ker(i_{0*} \cdot i_{1*}^{-1}) = \{(a, b) \in \pi_1(L_0) \times \pi_1(L_1) \mid i_{0*}(a) \cdot (i_{1*}(b))^{-1} = e\}.$$

Then $\pi_1(\Omega_0(L_0, L_1)) \cong \ker\left(i_{0*} \cdot i_{1*}^{-1}\right)$. Since the γ_i are nontrivial and nontorsion each i_{0*}, i_{1*} is individually injective on π_1. So the kernel of $i_{0*} \cdot i_{1*}^{-1}$ depends only on the intersection of the images of i_{0*} and i_{1*} in $\mathbb{Z} \times \pi_1(M_L)$. Then as we know the subgroups generated by γ_0, γ_1 in $\pi_1(M_L)$ intersect trivially, we see that $\pi_1(\Omega_0(L_0, L_1)) \cong \mathbb{Z}$.

Consider D^2 as the unit disc in $\mathbb{C}$ and let $\partial_+ = \partial D^2 \cap \{z \in \mathbb{C} \mid \Re z \geq 0\}$ and $\partial_- = \partial D^2 \cap \{z \in \mathbb{C} \mid \Re z \leq 0\}$. Consider $\mathbb{A}$ as the annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ with boundary components $\partial_{|z|=1}$ and $\partial_{|z|=2}$. We may represent elements of $\pi_1(\Omega_0(L_0, L_1))$ as maps of the annulus $(\mathbb{A}, \partial_{|z|=2}, \partial_{|z|=1})$ into $(S^1 \times M_L, L_0, L_1)$. As with $\pi_2(S^1 \times M_L, L_i)$ there is a homomorphism on $\pi_1(\Omega_0(L_0, L_1))$ which we shall call $\int \omega$ which is given by integrating the pullback of ω over (in this case) the annulus. We now show $\int \omega$ on $\pi_1(\Omega_0(L_0, L_1))$ is the zero homomorphism.

To see this, consider a certain generator for $\pi_1(\Omega_0(L_0, L_1)) \cong \mathbb{Z}$ represented by a map $u : (\mathbb{A}, \partial_{|z|=2}, \partial_{|z|=1}) \to (S^1 \times M_L, L_0, L_1)$ for which $u(\mathbb{A}) \subset L_0 \cap L_1$ and $\partial_{|z|=1}, \partial_{|z|=2}$ both map to $\pm$ the generator of $\pi_1(L_0 \cap L_1) = \mathbb{Z}$. Clearly $\int_{\mathbb{A}} u^*\omega = 0$. Then $\int \omega \equiv 0$ on $\pi_1(\Omega(L_0, L_1))$.

If we have topological Floer disc, that is, a map

$$u : (D^2, \partial_-, \partial_+) \to (S^1 \times M_L, L_0, L_1),$$

then the images of $\pm i$ lie in $L_0 \cap L_1$. As $L_0 \cap L_1 \cong S^1$ is connected, we can connect the images of $\pm i$ by an arc $\gamma : ([-\frac{\pi}{2}, \frac{\pi}{2}], \{-\frac{\pi}{2}\}, \{\frac{\pi}{2}\}) \to (L_0 \cap L_1, u(-i), u(i))$. There are, of course, two ways of doing this but that will be immaterial. Then define $\tilde{u} : (\mathbb{A}, \partial_{|z|=2}, \partial_{|z|=1}) \to (S^1 \times M_L, L_0, L_1)$ by

$$\tilde{u}(z) = \begin{cases} u \circ \phi & \text{if } \Re z < 0 \\ \gamma(\theta) & \text{if } \Re z \geq 0, z = re^{i\theta}, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{cases}$$

Where ϕ is a diffeomorphism from the interior of $D = \mathbb{A} \cap \{z \in \mathbb{C} \mid \Re z \leq 0\}$ to the interior of D^2 whose extension to the boundary takes $D \cap \partial_{|z|=2}$ to ∂_-, $D \cap \partial_{|z|=1}$ to ∂_+, the part of D lying on the positive imaginary axis to i and the part of D lying on the negative imaginary axis to $-i$.

Then $\int_{\mathbf{A}} \tilde{u}^*\omega = \int_{D^2} u^*\omega$. Now, $\tilde{u}$ defines an element of $\pi_1(\Omega(L_0, L_1))$ so $\int_{D^2} u^*\omega = \int_{\mathbf{A}} \tilde{u}^*\omega = 0$. Thus as all nonconstant pseudoholomorphic curves have positive symplectic area, there are no nonconstant Floer discs.

□

Lemma 1.3.2. Suppose that the Lagrangian tori $L_i = S^1 \times \gamma_i$ meet in an S^1 in $(S^1 \times M_L, \omega)$ as in Lemma 1.3.1. Let m_i be meridians to each component K_i of L each away from the γ_i and (X_i, ω_{X_i}) be a collection of symplectic 4-manifolds each containing an embedded symplectic torus F_i of square zero. Then, given any bound on energy E, there exists an almost complex structure on the fiber sum manifold X_L (which on each side of the fiber sum is sufficiently close to one for which the F_i and $S^1 \times m_i$ are pseudoholomorphic) for which all (perturbed) pseudoholomorphic discs in X_L with boundary on L_0 or on L_1 and all Floer discs for L_0, L_1 have energy greater than E.

Note that if J_t is a loop of almost complex structures, starting at a J as described, which is contained within a small neighborhood of J, then J_t-pseudoholomorphic strips (Floer discs) can be considered as solutions to the perturbed pseudoholomorphic curve equations.

Proof of Lemma 1.3.2. In [IP04], Ionel and Parker construct a 6 dimensional symplectic manifold Z with a map to D^2 so that over $\lambda \in D^2 \setminus \{0\}$ the fiber is the symplectic sum X_L and over 0 the fiber is the singular manifold $S^1 \times M_L \bigcup_{\substack{i=1,\dots,m \\ F_i = S^1 \times m_i}} X_i$. Each fiber $X_{L,\lambda}$ is canonically symplectomorphic to $S^1 \times M_L \setminus N(S^1 \times m_i) \bigcup_{i=1}^{m} X_i \setminus N(F_i)$ away from the fiber sum region.

Further, the almost-complex structure J_Z on Z is chosen so that in the singular fiber $X_{L,0}$, $S^1 \times m_i$ is a pseudoholomorphic torus and so that the restriction of J_Z to each fiber is an almost complex structure. (The singular fiber is pseudoholomorphic in the sense that each inclusion of X and $S^1 \times M_L$ is pseudoholomorphic.)

Suppose that γ is a smooth embedded path in D^2 which passes through zero and that $\{\lambda_n\}_{n=0}^{\infty} \to 0$ with $\lambda_n \in \gamma \setminus \{0\}$. Since the L_i are disjoint from the fiber sum region, there are Lagrangian submanifolds $\tilde{L}_i$ in Z for which in every fiber X_{L,λ_n} above λ_n, $\tilde{L}_i \cap X_{L,\lambda_n} = L_{i,n}$ which is mapped to L_i under the canonical identification.

Suppose that we have a family of pseudoholomorphic discs Σ_n in X_{L,λ_n} that have boundary in $\tilde{L}_i$ for each n and have energy bounded by E. Then by Gromov compactness for pseudoholomorphic curves with Lagrangian boundary conditions, this sequence has a subsequence which converges to a pseudoholomorphic curve $\hat{u}$ with image in the singular fiber $X_{L,0} \cong S^1 \times M_L \bigcup_{\substack{i=1,\dots,m \\ F_i = S^1 \times m_i}} X_i$ and boundary on $L_{i,0} \cong L_i$. The domain C of $\hat{u}$ is then a collection of spheres and discs.

We next state Lemma 3.4 of [IP03] within our context. Note that here, ν is a perturbation of the $\overline{\partial}_J$ operator. (We suppress the perturbation elsewhere.)

Lemma 1 (Ionel,Parker). Suppose that C is a smooth connected curve and $f : C \to S^1 \times M_L$ is a (J, ν)-holomorphic map that intersects $S^1 \times m$ at a point $p = f(z_0) \in S^1 \times m$. Then either

1. $f(C) \subset S^1 \times m_i$ for some i or
2. there is an integer $d > 0$ and a nonzero $a_0 \in \mathbb{C}$ so that in local holomorphic coordinates centered at p
$$f(z, \bar{z}) = (p^i + O(|z|), a_0 z^d + O(|z|^{d+1}))$$
where $O(|z|^k)$ denotes a function which vanishes to order k at $z = 0$.

We note that no irreducible component of $\hat{u}$ is mapped entirely into any $S^1 \times m_i$ as there are no nonconstant holomorphic maps of spheres into tori and all discs have boundary on

L_i away from each $S^1 \times m_i$. Part (2) of the lemma shows us that on each component of the domain, $\hat{u}$ meets the F_i as algebraic curves do. Thus we know that $\hat{u}$ intersects each $S^1 \times m_i$ at a finite number of points and that the image of each spherical or disc component of the domain of $\hat{u}$ will lie entirely in one of the X_i or in $S^1 \times M_L$.

Then our map $\hat{u}$ splits into two pseudoholomorphic maps $\hat{u}_1 : (C_1, \partial C_1) \to (S^1 \times M_L, L_i)$ and $\hat{u}_2 : C_2 \to \coprod_i X_i$. Here $C = C_1 \cup C_2$ with C_1 a collection of spheres and discs and C_2 a (possibly empty) collection of spheres so that $C_1 \cap C_2 \subset \hat{u}^{-1}\left(\bigcup_{i=1}^{m} S^1 \times m_i\right)$ are nodes of C.

Thus we have obtained $\hat{u}_1$, a pseudoholomorphic curve in $S^1 \times M_L$ with boundary on L_i. By our Lemma 1.3.1 this map must be constant. Thus the image of $\hat{u}_1$ is in L_i disjoint from the fiber sum region. Then since the images of $\hat{u}_2$ and $\hat{u}_1$ are connected, $\hat{u}_2$ must have empty domain. Therefore, $\hat{u}$ is constant.

This implies that for λ sufficiently small, all of the discs Σ_n must have been contained in $S^1 \times M_L$ disjoint from the fiber sum region. Therefore by Lemma 1.3.1, they were constant in the nonsingular fiber sum. Let λ' be one such value. Then picking J equal to the fiber-wise almost complex structure $J_{\lambda'}$ on $X_{L,\lambda'}$, we have the desired result.

The proof follows identically if we consider pseudoholomorphic Floer discs for L_0, L_1.

$\square$

Theorem 1.3.3. Let Lagrangian tori $L_i = S^1 \times \gamma_i$ in $(S^1 \times M_L, \omega)$ meet in a S^1 as in Lemma 1.3.1. Then

$$HF_{S^1 \times M_L}(L_0, L_0) \cong HF_{S^1 \times M_L}(L_1, L_1) \cong H^*(T^2) \otimes \Lambda$$

and

$$HF_{S^1 \times M_L}(L_0, L_1) \cong H^*(S^1) \otimes \Lambda.$$

Corollary 1.3.4. The Lagrangian tori $L_i = S^1 \times \gamma_i$ are not Hamiltonian isotopic in $(S^1 \times M_L, \omega)$.

Proof of Theorem 1.3.3. We begin by computing $HF(L_0, L_0)$. Since we have found in Lemma 1.3.1 that $\pi_2(S^1 \times M_L, L_0) = 0$, we have that

$$HF(L_0, L_0) \cong H^*(T^2) \otimes \Lambda$$

as in Floer's original work [Flo88]. Similarly, $HF(L_1, L_1) = H^*(T^2) \otimes \Lambda$.

Now we consider $HF(L_0, L_1)$. As L_0, L_1 intersect cleanly, Proposition 3.4.6 of [Poź94] implies that in some neighborhood $N(L_0 \cap L_1)$ of $L_0 \cap L_1$ the Floer complex and Morse complex for some Morse function $f : L_0 \cap L_1 \to \mathbb{R}$ coincide. This allows us to consider a slight modification of the action spectral sequence of [FOOO].

The universal Novikov ring Λ can be written as the ring of formal sums $\sum_i a_i T^{\lambda_i} e^{n_i}$ with

1. $\lambda_i \in \mathbb{R}$ and $n_i \in \mathbb{Z}$
2. for each $\lambda^* \in \mathbb{R}$, $\#\{i | \lambda_i \leq \lambda^*\} < \infty$

Here the T^λ parameter will be used to keep track of the action of a pseudoholomorphic disc. The formula $\deg T^\lambda e^n = 2n$ determines a grading on Λ and we denote by Λ^k the homogeneous degree k part.

An $\mathbb{R}^+$ filtration on Λ is given by

$$F^\lambda \Lambda = \left\{ \sum_i a^i T^{\lambda_i} e^{n_i} \mid \lambda_i \geq \lambda \right\}$$

We can then get a $\mathbb{Z}$ filtration by picking some $\lambda^* \in \mathbb{R}^+$ and setting $\mathcal{F}^q \Lambda = F^{q\lambda^*}\Lambda$. The homogeneous elements of level q are then $\mathrm{gr}_q(\mathcal{F}\Lambda) = \mathcal{F}^q \Lambda / \mathcal{F}^{q+1}\Lambda$.

Then as in Theorem 6.13 of [FOOO] we have a spectral sequence $E_r^{p,q}$ with

$$E_2^{p,q} = \bigoplus_k H^k(L_0 \cap L_1; \mathbb{Q}) \otimes \mathrm{gr}_q(\Lambda^{(p-k)})$$

Thus $E_2 \cong H^*(S^1) \otimes \Lambda$. We now want to see that all higher order differentials vanish. Recall that in Lemma 1.3.1 we showed that $\omega|_{\pi_2(S^1 \times M_L, L_0 \cup L_1)} \equiv 0$. Since the Hamiltonian perturbation may be chosen to be very small so that the local curves are of area less than λ^*, Lemma 1.3.1 shows that we have found all of the discs to be counted. Therefore, there are no higher order differentials and

$$HF(L_0, L_1) = H^*(S^1) \otimes \Lambda$$

$\square$

In [Che98], Chekanov showed that when the Hofer energy of a Hamiltonian symplectomorphism is less than the minimum symplectic area of pseudoholomorphic sphere or disc with boundary on a Lagrangian, a modified Lagrangian Floer homology can be computed which has rank equal to the rank of the singular homology. The Hofer norm of a Hamiltonian function $H : [0,1] \times P \to \mathbb{R}$ on a symplectic manifold P is defined as

$$||H|| = \int_0^1 \left(\max_{r \in P} H(s,r) - \min_{r \in P} H(s,r) \right) ds.$$

This extends to a definition of Hofer energy of a Hamiltonian symplectomorphism ϕ by

$$E(\phi) = \inf \{ ||H|| \;\; | \, \phi \text{ is the time 1 flow of } H \}$$

Theorem 1.3.5. For all $E > 0$, let X_L be the fiber sum of $S^1 \times M_L$ and the X_i, L_i the image of $S^1 \times \gamma_i$ in the fiber sum with γ as in 1.3.1, J as found in Lemma 1.3.2, and ϕ a Hamiltonian symplectomorphism which has Hofer energy less than E. Then, supposing that for each L_i, the Maslov class $\mu : \pi_2(X_L, L_i) \to \mathbb{Z}$ takes only even values, in the fiber sum manifold X_L,

$$HF_{X_L}(L_0, L_0; J, \phi) \cong HF_{X_L}(L_1, L_1; J, \phi) \cong H^*(T^2) \otimes \Lambda_E$$

where Λ_E is the truncated Novikov ring $\Lambda / F^E \Lambda$.

Note that this version of Lagrangian Floer homology is not invariant under change of ϕ to a Hamiltonian with larger Hofer energy.

Corollary 1.3.6. Under the conditions of Theorem 1.3.5, the Lagrangian tori $L_i = S^1 \times \gamma_i$ are not Hamiltonianly isotopic in X_L.

In section 1.4, We give examples which are Lagrangian isotopic. As these examples are in simply connected X_L, we may strengthen the corollary to exclude symplectic isotopies.

Proof of Corollary 1.3.6. From Theorem 1.3.5 we see that for any energy bound E on ϕ, $HF(L_i, L_i); J, \phi)$ has 4 generators so $\# L_i \cap \phi(L_i) \geq 4$ for any Hamiltonian symplectomorphism of energy at most E. Also, there is a Hamiltonian isotopy which is given by smoothly extending a perfect Morse function on $S^1 = L_0 \cap L_1$ to be constant outside a neighborhood of $L_0 \cap L_1$. Letting one of the L_i flow by this Hamiltonian, we get a pair of Lagrangians with two intersection points. Therefore, L_0 and L_1 cannot be Hamiltonian isotopic. □

Proof of Theorem 1.3.5. This is proved using the same methods as in the previous theorem. Lemma 1.3.2 ensures that the obstructions to defining a small perturbation Lagrangian Floer cohomology on the $L_i \subset X_L$ vanish and we make make the following local (in J and H) computation: Using the action spectral sequence for $HF(L_0, L_0)$ we see that $E_2 = H^*(T^2) \otimes \Lambda_E$. By Lemma 1.3.2, we see that all the higher order differentials vanish for a small perturbation as all flow lines either have already been counted via d_1 or are of energy higher than E and thus die in Λ_E. (This can be seen by Lemma 6 of [Che98]) In this case, the calculation of $HF(L_0, L_0)$ goes through as in the case of Theorem 1.3.3 and we find that $HF(L_0, L_0; J, \phi) \cong H^*(T^2) \otimes \Lambda_E$. Similarly, we get $HF(L_1, L_1; Jm\phi) \cong H^*(T^2) \otimes \Lambda_E$.

Now we show that the computed groups are invariant under the Hofer energy hypothesis. Invariance is guaranteed as long as in 1-parameter families, there are no discs with boundary on L_0, L_1 of index -1 which bubble off.Our restriction on the Maslov class ensures that there are no index -1 discs which appear in the boundary of 1-parameter families. Thus there is a continuation isomorphism and we have a well defined invariant of (energy bounded) Hamiltonian isotopy in X_L.

□

1.4 Example

If we let L be the union of the right-hand trefoil knot K_1 and one of its meridians K_2, the loops $\gamma_1, \gamma_2, \gamma_3$ in Figure 1.1 are all *freely* smoothly isotopic in M_L and meet transversely pairwise.[1] We select the fibration $\pi : M_L \to S^1$ so that the Seifert surface containing γ_i shown in Figure 1.1 is the fiber. The smooth isotopies of γ_1 and γ_2 to a common curve are shown in Figures 1.4 and 1.5. In each of these figures going from (1) to (2) involves sliding over the 0-surgery on the meridian, going from (2) to (3) is an isotopy. To relate

[1]Though they are freely isotopic, the proof of Lemma 1.3.1 shows that they are not equal in π_1 i.e. fixing a basepoint.

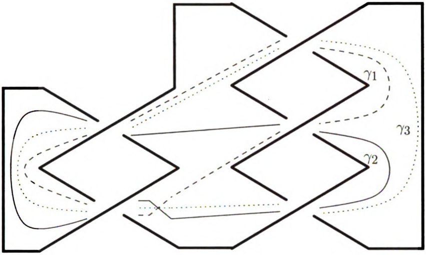

Figure 1.1. γ_i, $i = 1, 2, 3$ intersecting pairwise transversely (within the fiber) in single points

the end results we have the additional move of "twisting up the corkscrew" which takes the curves each (3) to the other. Note that this smooth isotopy is different than the Lagrangian isotopy which we will mention later.

Since the γ_i are smoothly isotopic, the Lagrangian tori $L_i = S^1 \times \gamma_i$ are smoothly isotopic. As Lagrangians do not have canonical orientations we neglect the orientations of the loops here.

Addressing the comment made after Lemma 1.3.1, we note that we can choose the almost complex structure J so that the Seifert surface Σ (a T^2) is pseudoholomorphic. Then choosing any pair γ_i, γ_j ($i \neq j$), $\Sigma \setminus (\gamma_i \cup \gamma_j)$ is a disc. This is however, not a Floer disc as it does not satisfy the correct boundary conditions.

For the left and right handed trefoils, the monodromy is of order 6 and in the basis A, B shown in Figure 1.2 is given by the matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

On the (positive/negative) Hopf link the monodromy is a (positive/negative) Dehn twist about a curve parallel to the components. The connect sum of fibered links is fibered with monodromy which splits around the connect sum region.

From this computation of monodromy, we see that γ_2 is $\pm$ the 2nd and 5th image of

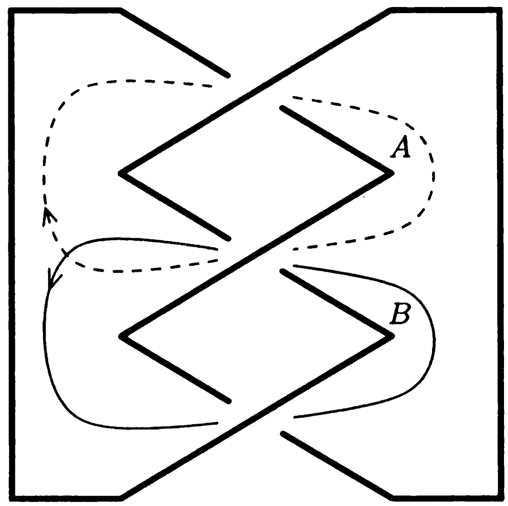

Figure 1.2. Basis for monodromy on trefoil

γ_1 under the monodromy map and that γ_3 is $\pm$ the 1st and 4th image of γ_1. Finally, γ_1 is sent to $-\gamma_1$ under the third iteration of the monodromy. Thus, as the monodromy gives a symplectic isotopy (c.f. Section 1.2) in $S^1 \times M_L$ the tori L_1, L_2, L_3 are all symplectically isotopic. In X_L they are all Lagrangian isotopic as the fiber sum is taken away from the isotopy. Despite the existence of these symplectic isotopies in $S^1 \times M_L$, Theorem 1.3.3 shows that the $L_i = S^1 \times \gamma_i$ are not Hamiltonian isotopic there.

Now we consider how Theorem 1.3.5 applies. That all the hypotheses of the theorem are satisfied, except that on the Maslov class, is clear. With the following lemma we see that the remaining condition is satisfied. Then Theorem 1.3.5 shows that the L_i are not symplectically isotopic in X_L.

Lemma 1.4.1. For the Lagrangian tori L_i, we may choose $X_i = E(1)$, and the particular identification of F_i and $S^1 \times m_i$ so that $\mu_{L_i} : \pi_2(X_L, L_i) \to \mathbb{Z}$ is even.

Proof. As L is a two component link with odd linking number, X_L is a homotopy $E(2)$ and thus is spin. See [FS98]. In fact $E(2)_L$ is $E(2)_{\text{Trefoil}}$. We shall write $E(2)_L := X_L$. For such a 4-manifold, the first Chern class is an even multiple of the fiber.

Note that μ_{L_i} factors through $\pi_2(E(2)_L, L_i) \to H_2(E(2)_L, L_i)$. As $H_1(E(2)_L) = 0$, the Meyer-Vietoris sequence gives that the group $H_2(E(2)_L, L_i)$ is generated by elements of $H_2(E(2)_L)/\langle L_i \rangle$ and relative classes with boundary spanning $H_1(L_i)$. The Maslov index of a class β, $\mu_{L_i}(\beta)$, $\beta \in H_2(E(2)_K, L_i)$, will change by an even amount whenever an element of $H_2(E(2)_L)/\langle L_i \rangle$ is added as $c_1(E(2)_L)$ is divisible by 2. Thus if we can find a pair of relative discs whose boundaries generate $H_1(L_i)$ and whose Maslov indices are even, we have shown that μ is even.

For $i = 1, 2$, we will choose the identification of F_i and $S^1 \times m_i$ so that

1. pt $\times\, m_i$ is identified with a vanishing cycle on F_i and

2. $S^1 \times$ pt is identified with the sum of two vanishing cycles on F_i whose boundaries meet once, transversely, in F_i.

Because $\pi_1(E(1) \setminus F_i) = 1$, we may select an elliptic fibration on $E(1)$ with nodal fibers having vanishing cycles a and b where a, b generate $\pi_1(F_i)$. With the decomposition $F_i = a \times b$, identify a with pt $\times\, m$ and $a + b$ with $S^1 \times$ pt. This gives us the desired identification of F_i and $S^1 \times m_i$.

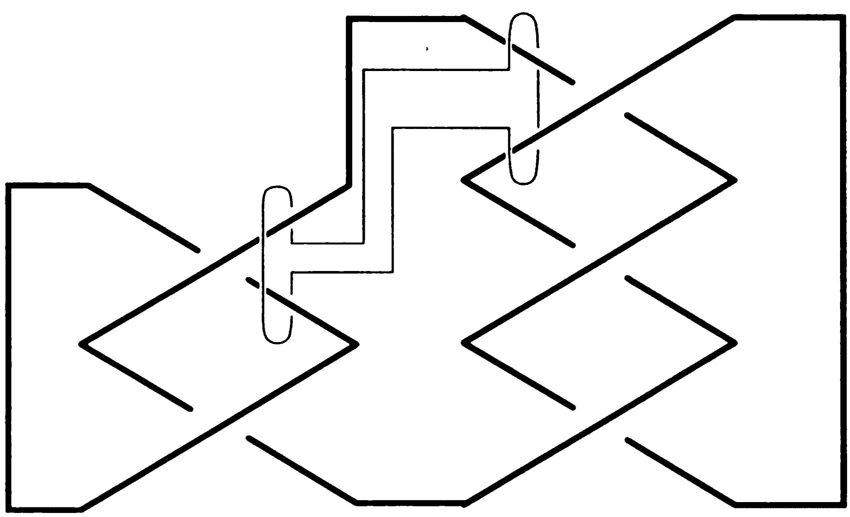

Figure 1.3. An isotope of γ_1 bounding meridians to K_1 and K_2

Each of the γ_i bounds a four times punctured disc D_{0,γ_i} in M_L where three punctures are meridians of K_1 and one is a meridian to K_2. See Figure 1.3. By our choice of fiber sum gluing, each meridian of K_1 bounds a vanishing disc on its $E(1)$ side of the fiber sum. Similarly, the meridian to K_2 bounds a vanishing disc on its $E(1)$ side of the fiber sum. Take three copies of the vanishing disc D_1, D_2, D_3 from the $E(1)$ fiber summed to

$S^1 \times m_1$ and one vanishing disc D_4 of from the $E(1)$ fiber summed to $S^1 \times m_2$ and form $D_{\gamma_i} = D_{0,\gamma_i} \cup D_1 \cup D_2 \cup D_3 \cup D_4$.

Each γ_i's Lagrangian framing relative to that induced by trivializing over D_{γ_i} is -2 and is given by a pushoff in the Seifert surface. See [FS04]. The framing coming from this disc is then $-1-1-1-1-(-2) = -2$ and gives us $\mu_{\gamma_i}(D_{\gamma_i}) = -2$.

By our choice of gluing, the $S^1 \times \mathrm{pt} \subset L_i$ is bounded by a pair of vanishing discs. This pair of vanishing discs intersects at one point on F and so can be smoothed to a disc $D_{S^1 \times \mathrm{pt}}$ with relative framing -2. For this loop the framing defect from the torus is 0 (given by pushoff in the monodromy direction) and so $\mu_{L_i}(D_{S^1 \times \mathrm{pt}}) = -2$.

Thus we have found a basis on which μ is even, hence μ is even.

□

This example generalizes to similar links with $K_1 = T_{2,2n+1}$ where we find many Lagrangian but not symplectically isotopic tori.

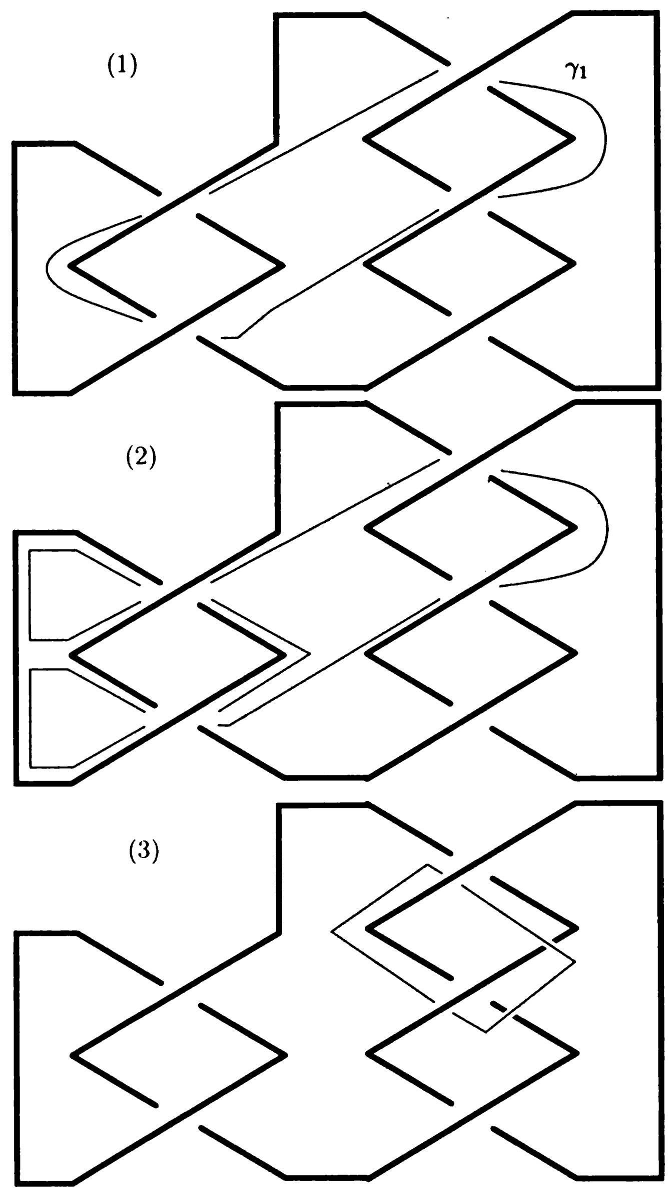

Figure 1.4. Isotopy of γ_1

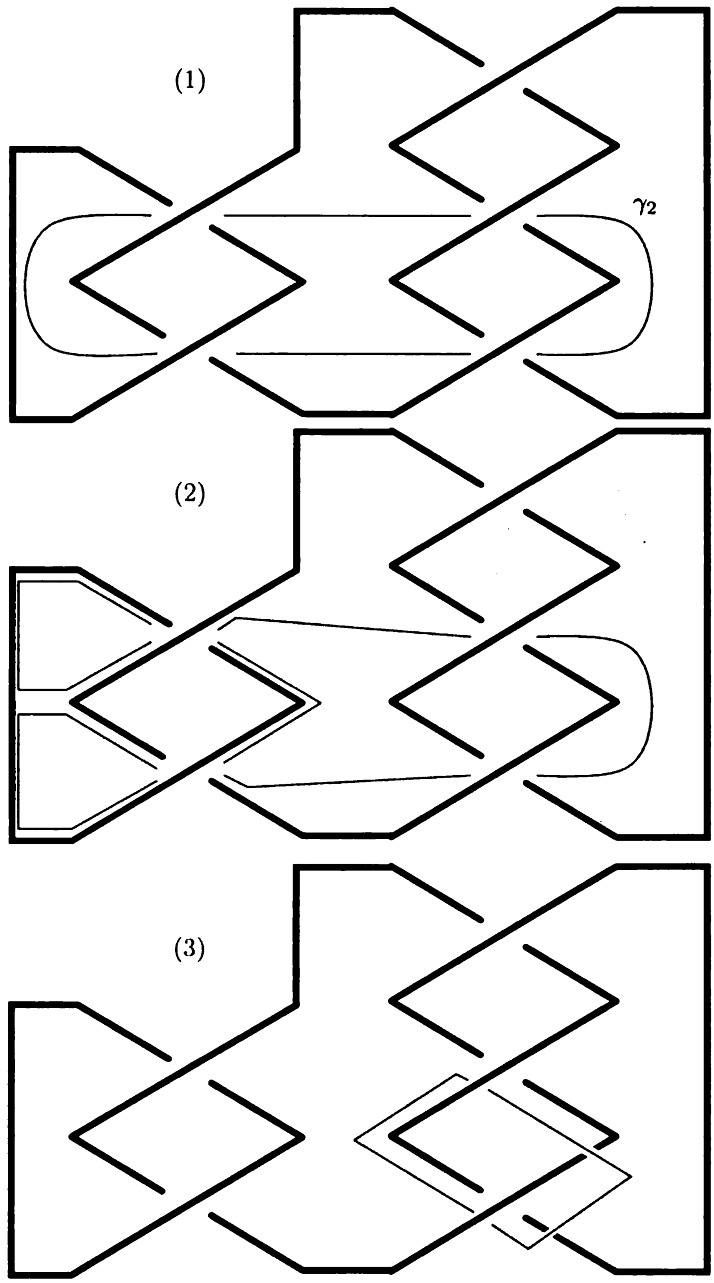

Figure 1.5. Isotopy of γ_2

CHAPTER 2

Gauge-theoretic Invariants of Twins

2.1 Introduction

A 2-knot K is an embedded S^2 in S^4. We say that K is unknotted when it bounds a D^3. In contrast to the classical case of embedded S^1s in S^3, 2-knots display many pathologies; including that their exteriors need not be $K(\pi_1, 1)$s and that the homeomorphism type of the knot exterior does not determine the knot's isotopy class. For instance, see [AC59]) and [Gor76]. From a gauge theoretical perspective, 2-knots are uncomfortable to deal with as their exteriors are homology $S^1 \times D^3$s and even after the surgery which replaces $N(K) = S^2 \times D^2$ with $D^3 \times S^1$, we obtain homology $S^1 \times S^3$s. With $b_2 = 0$ for these manifolds, current tools for smooth 4-manifolds offer little help.

On the other hand, the natural generalizations of gadgets like the Alexander ideal/polynomial can be computed for 2-knots. In [Gil82], C. Giller proposed a definition and method of computation for an invariant we will call Δ_G. The proposed invariant is derived from the projection of 2-knots to $\mathbb{R}^3$ and for a certain class of 2-knots is known to compute the Alexander polynomial. This is a result of the proposed invariant obeying a relation similar to Conway's for computing the Alexander polynomial for classical knots. The relation, which we discuss more thoroughly in section 2.1.8, always results in a symmetric polynomial. However, the Alexander polynomial of a 2-knot need not be symmetrizable. For example, the example given in Figure 2.7 has Alexander polynomial $\Delta_A = 1 - 2t$ while Giller's polynomial is $\Delta_G = t^{-2} - 1 + t^2$.

Further complicating matters is that Giller's polynomial is not actually known to be an invariant. If instead of 2-knots, we look at "twins" in S^4, we have objects to which standard gauge-theoretic methods can be applied. It is in this context we see that Giller's polynomial computes, in the relevant cases, the Seiberg-Witten invariant of the exterior of the twin.

2.1.1 Twins

In [Mon83] and [Mon84], José María Montesinos introduced the concept of a twin in S^4. Such an object consists of two embedded S^2s which meet exactly twice, transversely. As the second homology of S^4 is trivial, these intersection points have opposite orientations. By "standard twins", I will mean that both S^2s bound embedded B^3s and that the exterior of the pair is $T^2 \times D^2$. In general, the exterior of such a twin is only a homology $T^2 \times D^2$ with boundary T^3.

Standard twins may be described as follows: Take a 3-ball B^3 in $\mathbb{R}^3$. Form the space $(S^1 \times B^3)/\sim$ with $(\theta_0, x_0) \sim (\theta_1, x_1)$ iff $x_0 = x_1 \in \partial B^3$. This space is S^4. Then consider the image of $\{1\} \times \partial B^3$ and the $S^1 \times z$-axis under $\sim$. Call these S_1, S_2 respectively. S_1 bounds $\{1\} \times B^3$ and if we take S^1 times the half plane $\{y = 0, x \geq 0\}$ then after quotienting by $\sim$ we get a B^3 which S_2 bounds. Also we can see that the exterior of these twins is S^1 times the exterior of the z-axis in B^3 — i.e. $S^1 \times (S^1 \times D^2) = T^2 \times D^2$. See Figure 2.1.

In fact, all twins in S^4 have as their exterior a homology $T^2 \times D^2$ with T^3 boundary.

As $\pi_2(SO(2))$ is trivial, an orientation on an $S^2 \subset S^4$ determines a trivialization $N(S^2) \cong S^2 \times D^2$. Thus, for a $S^2 \subset S^4$, all framings are equivalent to the Seifert framing. Fix orientations on the 2-spheres, K_1, K_2, in a twin $\mathbb{T}\mathrm{w}$ and consider $\partial N(\mathbb{T}\mathrm{w}) \cong T^3$. Take a simple closed curve γ on the twin passing through the intersection points of K_1, K_2 and lift it to $\partial N(\mathbb{T}\mathrm{w})$ using the framings. All such γ are isotopic on the twin and as each K_i has a single framing, this lift is canonical. This decomposes $\partial N(\mathbb{T}\mathrm{w}) = \gamma \times T^2$ The boundaries of normal discs D_1^2, D_2^2 to K_1, K_2 complete the decomposition to give $\partial N(\mathbb{T}\mathrm{w}) = \gamma \times \partial D_1^2 \times \partial D_2^2$.

With this decomposition in mind, we define a standard surgery a twin $\mathbb{T}\mathrm{w}$ in S^4: form

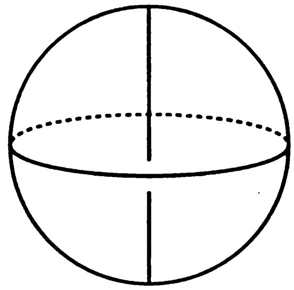

Figure 2.1. Spin to get standard twins

$S^4(\mathbb{T}\mathrm{w}) = (S^4 \setminus \mathbb{T}\mathrm{w}) \underset{\phi}{\cup} (T^2 \times D^2)$ where $\phi : T^3 \to T^3$ identifies ∂D^2 and γ. Up to isotopy, any two such ϕ differ by a diffeomorphism $T^2 \times \mathrm{pt} \to T^2 \times \mathrm{pt}$. Since any such map extends over $T^2 \times D^2$, the resulting manifold only depends on the embedding of the twin. Also, $H_*(S^4(\mathbb{T}\mathrm{w}); \mathbb{Z}) \cong H_*(T^2 \times S^2; \mathbb{Z})$ so that the core T^2 of the surgery is identified with $T^2 \times pt \subset T^2 \times S^2$ as a homology class.

2.1.2 Definition of the invariant for twins

Let F be a fiber in an elliptic fibration of the $K3$ surface, $E(2)$. Then $N(F)$ comes with a trivialization $N(F) = D^2 \times T^2$ from the fibration map which induces $\partial N(F) = \partial D^2 \times F$. Taking a twin $\mathbb{T}\mathrm{w} \subset S^4$, we also have a decomposition $\partial N(\mathbb{T}\mathrm{w}) = \gamma \times T^2$. Fix an identification of this T^2 and F and using an orientation reversing diffeomorphism between ∂D^2 and γ we obtain an identification ϕ of $\partial(E(2) \setminus N(F))$ and $\partial(S^4 \setminus \mathbb{T}\mathrm{w})$. Form

$$E(2)_{\mathbb{T}\mathrm{w}} := (E(2) \setminus N(F)) \underset{\phi}{\cup} (S^4 \setminus \mathbb{T}\mathrm{w}).$$

Although the resultant manifold may depend on the precise choice of ϕ, we will omit the distinction. As $\pi_1(E(2) \setminus N(F)) = 0$ and the image of $\pi_1(T^3)$ normally generates $\pi_1(S^4 \setminus \mathbb{T}\mathrm{w})$, $E(2)_{\mathbb{T}\mathrm{w}}$ is simply connected.

This procedure may also be thought of as the generalized fiber sum of $E(2)$ and $S^4(\mathbb{T}\mathrm{w})$ along F and the core T^2 of the surgered twin. As such, we will sometimes write $E(2)_{\mathbb{T}\mathrm{w}}$ as $E(2) \underset{F=T^2}{\#} S^4(\mathbb{T}\mathrm{w})$.

The invariant of the twin $\mathbb{T}\mathrm{w}$ which we will consider will be the Seiberg-Witten invariant of $E(2)_{\mathbb{T}\mathrm{w}}$. i.e.

Definition 2.1.1. $I(\mathbb{T}\mathrm{w}) := SW(E(2)_{\mathbb{T}\mathrm{w}})$ thought of as an element of the group ring $\mathbb{Z}[H_2(E(2);\mathbb{Z})]$.

Precisely, $SW(E(2)_{\mathbb{T}\mathrm{w}})$ lives in $\mathbb{Z}[H_2(E(2)_{\mathbb{T}\mathrm{w}};\mathbb{Z})]$ and we identify $H_2(E(2)_{\mathbb{T}\mathrm{w}})$ with $H_2(E(2))$ by a homomorphism which extends the identity map on $H_2(E(2)_{\mathbb{T}\mathrm{w}} \setminus (S^4 \setminus \mathbb{T}\mathrm{w})) \to H_2(E(2) \setminus F)$. While $E(2)_{\mathbb{T}\mathrm{w}}$ may depend on the choice of the gluing map ϕ, the Seiberg-Witten invariant does not. This is due to the gluing formulas in [MMS97]. The invariant is well defined up to a sign which depends on a homology orientation of $H_0(E(2)) \oplus \det H_2^+(E(2)) \oplus H_1(E(2))$

As standard twins $\mathbb{T}\mathrm{w}_{std}$ have exterior equal to $T^2 \times D^2$, the gluing map ϕ extends over the interior so $E(2)_{\mathbb{T}\mathrm{w}_{std}} \cong E(2)$ and

$$I(\mathbb{T}\mathrm{w}_{std}) = SW(E(2)) = 1. \tag{2.1}$$

Now suppose that we have a twin $\mathbb{T}\mathrm{w}$ and a disjoint torus T in S^4. T has a canonical framing given by pushing off into its Seifert manifold. Thus we can form the self fiber sum $S^4(\mathbb{T}\mathrm{w}) \underset{T_{\mathbb{T}\mathrm{w}}=T}{\#}$, where $T_{\mathbb{T}\mathrm{w}}$ is the core of the standard surgery on $\mathbb{T}\mathrm{w}$ with its default framing. This manifold is well defined with a choice of an identification of $T_{\mathbb{T}\mathrm{w}}$ and T. Again, any choice of identification will have the same Seiberg-Witten polynomial. Then form $E(2)_{\mathbb{T}\mathrm{w},T} = E(2) \underset{F=T'_{\mathbb{T}\mathrm{w}}}{\#} \left(S^4(\mathbb{T}\mathrm{w}) \underset{T_{\mathbb{T}\mathrm{w}}=T}{\#} \right)$ with $T'_{\mathbb{T}\mathrm{w}}$ a pushoff of $T_{\mathbb{T}\mathrm{w}}$. Now, $E(2)_{\mathbb{T}\mathrm{w},T}$ is a homology $E(2) \underset{F=F'}{\#}$ with F, F' elliptic fibers and $SW(E(2) \underset{F=F'}{\#}) = (t - t^{-1})^2$ with $t = \exp([F])$. This is a result of the gluing theorems of [MMS97]. We define

Definition 2.1.2. $I(\mathbb{T}\mathrm{w}, T) := (t - t^{-1})^{-1} SW(E(2)_{\mathbb{T}\mathrm{w},T})$ thought of as a element of the group ring $\mathbb{Z}[H_2(E(2);\mathbb{Z})]$.

2.1.3 Construction of Knotted twins

We will, of course, want to consider twins other than the standard ones. There are several techniques we will consider.

Construction 2.1.3 (Connect sum). Take K_0, a knotted S^2, and a twin $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ in S^4. Then, selecting one of the spheres K_1 in the twin, we form the connected sum

$(S^4 \# S^4, K_0 \# K_1)$ at some point away from the 2 double points of the twins. (This construction is not independent of the choice of K_1, K_2 in $\mathbb{T}\mathrm{w}$ in general.)

If we take $\mathbb{T}\mathrm{w}$ to be the standard twins, then this construction *is* independent of the choice of S^2s and provides a handy method for studying 2-knots via twins. This independence is due to the existence of a orientation preserving diffeomorphism ρ of S^4 which interchanges K_1, K_2 in standard twins. The diffeomorphism ρ is constructed as follows: View S^4 as $N(\mathbb{T}\mathrm{w}_{std}) \cup T^2 \times D^2$. Define ρ on $N(\mathbb{T}\mathrm{w}_{std})$ to be the obvious map which interchanges K_1 and K_2. Then ρ induces the map

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on $\partial N(\mathbb{T}\mathrm{w}_{std}) = T^2 \times \partial D^2 = T^3$ under the basis for $H_1(T^3)$ given by $\partial D^2_{K_1}, \partial D^2_{K_2}$, and $\partial D^2_{T^2}$ where ∂D^2_S is the boundary of the normal bundle to S. This map then extends over $T^2 \times D^2$ giving ρ on S^3.

Construction 2.1.4 (Artin Spin). In the construction in the introduction, if we replace the z-axis with a knotted arc, $\dot{K}$, meeting ∂B^3 at the north and south poles, the procedure gives us a twin $\mathbb{T}\mathrm{w}_K$ whose complement is $S^1 \times (S^3 \setminus K)$ with

$$K = \dot{K} \cup \{\text{the international date line of } \partial B^3\}$$

and consists of a unknotted sphere (the image of ∂B^3) and an Artin spun knot (the image of $\dot{K}$). This spinning construction is originally due to Artin in [Art26].

When the standard twin surgery is performed on twins formed by Artin spinning K, the result is the manifold $S^1 \times S^3_0(K)$. (Where $S^3_0(K)$ is the result of zero surgery on $K \subset S^3$.) This case is identical to the knot surgery considered by Fintushel and Stern in [FS98] and so we have

Theorem 2.1.5 (Fintushel,Stern). $I(\mathbb{T}\mathrm{w}_K) = \Delta_K(t^2)$ where $\Delta_K(t^2)$ is the symmetrized Alexander polynomial of K and $t = \exp([F])$.

Construction 2.1.6 (Twist Spin). As before, take a knotted arc $\dot{K}_1$ in B^3 with boundary on the north and south poles. For each $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ we take the image of $\dot{K}$ rotated by $n\theta$

radians. Then in $S^1 \times B^3$ we have an annulus formed by the rotated $\dot{K}_1$s. This descends to a knotted S^2, K_2, in $S^4 = S^1 \times B^3/\sim$ which together with the image of $\{\theta\} \times \partial B^3$, forms a twin. We call K_2 the n-twist spin of K_1 and write $K_2 = \tau^n K_1$ and $\mathbb{T}\mathrm{w}_{\tau^n K_1}$ for the associated twin.

This comes from Zeeman, who in [Zee65], showed that the n-twist spin of a knot was not isotopic to any 2-knot obtained by Artin spinning when $n > 1$. The $n = 1$ case is interesting in that the 1-twist spin, K_2 is unknotted independently of choice of K_1.

Construction 2.1.7 (Roll Spin). Similar to the twist spin, this construction involves a deformation of a knotted arc $\dot{K}_1$ in B^3 fixing the north and south poles which returns the arc to the starting point. Take a international date line of ∂B^3 union $\dot{K}_1$ and push it into $B^3 \setminus \dot{K}_1$ so that it is null homologous. Call this $\tilde{K}_1$. Then consider the 1-parameter family of diffeomorphisms given by pushing a base point x n times along $\tilde{K}_1$. This gives us a diffeomorphism of the quadruple $(B^3, \partial B^3, \dot{K}_1, x)$ which is the identity on all but the first component. Proceed as before and quotient $S^1 \times B^3$ by $\sim$ to get K_2 to be the image of the $\dot{K}_1$ in each $\{\theta\} \times B^3$. We call K_2 the n-roll spin of K_1 and write $K_2 = \rho^n K_1$ and $\mathbb{T}\mathrm{w}_{\rho^n K_1}$ for the associated twin.

This idea is due to Fox who, in [Fox66], showed that, for $\dot{K}_1 = \dot{4}_1$, the knotted 2-sphere K_2 coming from the deformed arc is not isotopic to any n-twist spun knot. In this case, the 1-roll spin had a corresponding visualization of the motion of $\dot{K}_1$ in B^3 which explains why "roll" was chosen to describe this. I duplicate the rolling move in Figure 2.2.

Note that both twist and roll spinning can be described in terms of certain diffeomorphisms of B^3 which keep ∂B^3 and $\dot{K}_1$ fixed identically. With this in mind, we now consider their mutual generalization, Deform spinning:

Construction 2.1.8 (Deform Spin). Let g be a self diffeomorphism of B^3 keeping $\partial B^3, \dot{K}$ and a base point x fixed identically. Then the mapping torus of g is $S^1 \times B^3 \cong ([0,1] \times B^3)/\langle (0,x) \tilde{\sim} (1, g(x)) \rangle$ with an embedded annulus $\dot{K}_2$ which is the quotient of $I \times \dot{K}_1$. Then after quotienting by $\sim$ as before, we have a knotted 2-sphere K_2, the image of $\dot{K}_2$, which together with the image of $\{\theta\} \times \partial B^3$, forms a twin. We write $K_2 = gK_1$ and $\mathbb{T}\mathrm{w}_{gK_1}$ for the twin pair. The isotopy class of K_2 and of $\mathbb{T}\mathrm{w}_{gK_1}$ is determined solely by the isotopy

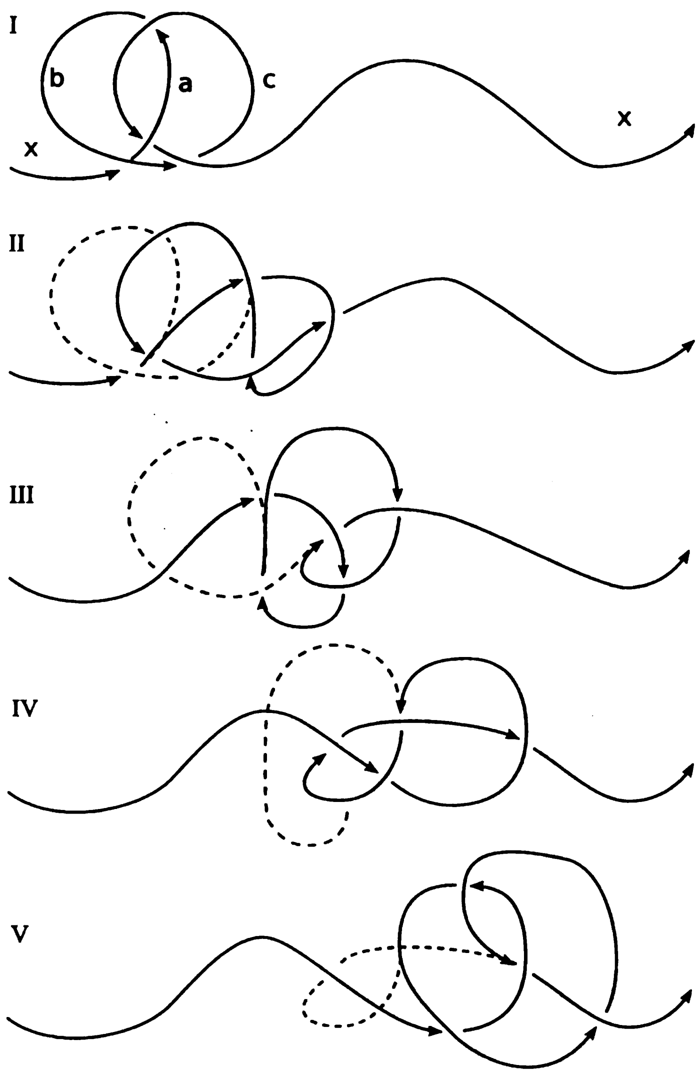

Figure 2.2. Fox's Roll Spin

class of g.

This construction was introduced by Litherland in [Lit79]. Diffeomorphisms such as g are called "deformations" and form a group $\mathcal{D}(K_1)$ of deformations modulo isotopy. $\mathcal{D}(K_1)$ is isomorphic to the group $\mathrm{Aut}_\partial(\pi_1(S^3 \setminus K_1))$ of automorphisms of $\pi_1(S^3 \setminus K_1)$ preserving some fixed peripheral subgroup. In this setup we find that τ corresponds to conjugation by the meridian of K_1 and ρ corresponds to conjugation by the longitude of K_1. Then

Lemma 2.1.9 (Litherland). If K_1

- is not a torus knot, $\mathcal{D}(K_1) \cong \mathbb{Z}_\tau \oplus \mathbb{Z}_\rho$.
- is a torus knot, $\mathcal{D}(K_1) \cong \mathbb{Z}_\tau$ and $\tau^{pq}\rho = \mathrm{id}$.

2.1.4 Ribbon Knots and Twins

We say that a 2-knot K is ribbon if it is formed by the following construction: Let $D = \amalg D^3$ (bases), $B = \amalg D^2 \times I$ (bands) each be embedded in $\mathbb{R}^4$ with $(\partial D) \cap B = \amalg D^2 \times \pm 1$. If a band intersects a base elsewhere, $\left(D^2 \times (-1,1)\right) \cap D^3 = D^2 \times t$, $t \in (-1,1)$ and $\left(D^2 \times (-1,1)\right) \cap (\partial D^3) = \emptyset$. The second type of intersection is called a ribbon intersection of K or ribbon singularity of $D \cup B$. Then

$$K = (\partial D \setminus \underbrace{\amalg D^2 \times \pm 1}_{\text{in } \partial B}) \cup (\amalg(\partial D^2) \times I)$$

is a ribbon knot with ribbon presentation given by $D \cup B$. We can define ribbon surfaces of arbitrary genus in the same manner.

Suppose that we have a twin $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ for which the K_i are ribbon. Then for each K_i we have a set of bases D_i and bands B_i. We will say that $\mathbb{T}\mathrm{w}$ is ribbon if

- $B_1 \cap B_2 = \emptyset$
- $D_1 \cap B_2 = \amalg D^2 \times t$, $t \in (-1,1)$ with $(\partial D_1) \cap D^2 \times (-1,1) = \emptyset$ for each band in B_2 (ribbon intersection)
- $D_2 \cap B_1 = \amalg D^2 \times t$, $t \in (-1,1)$ with $(\partial D_2) \cap D^2 \times (-1,1) = \emptyset$ for each band in B_1 (ribbon intersection)

- $D_1 \cap D_2 = 2D^2$. More specifically, D_1 and D_2 meet only in two balls D_i', D_i'' from each and at these intersections (corresponding to the intersection points $K_1 \cap K_2$), we have the following local model:

$$D_1' = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re z_2 \leq 1, \Im z_2 = 0\} \subset \mathbb{C}^2,$$

$$D_2' = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re z_1 \leq 1, \Im z_1 = 0\} \subset \mathbb{C}^2,$$

  So that taking the boundary of each gives us the cone of the positive Hopf link. The D_1'', D_2'' case is the same but with orientations reversed on D_2'', giving us a negative Hopf link cone boundary.

In this case we say that $\mathbb{T}\mathrm{w}$ is a *ribbon twin.*

It is known that of the deform spun knots, only the Artin and 1-twist spun knots are ribbon. Artin spun twins are also ribbon. It is not known to the author if 1-twist spun twins are ribbon (and suspected not to be the case). All other deform spun twins are not ribbon.

There are several operations which will be useful to perform on ribbon presentations. Addition of a trivial base/band pair, sliding the disc to which a band attaches (band slide), and moving a ribbon intersection along a base/band sequence (band pass) are shown in Figure 2.3 and together with isotopy generate *stable equivalence* of ribbon presentations. Clearly stable equivalence of ribbon presentations generates isotopies of the corresponding ribbon knot but the converse also holds — isotopic ribbon knots have stably equivalent ribbon presentations. For a proof of this, see [Mar92].

It will occasionally be easier to deal with simplified ribbon presentations. Let $\Gamma = \Gamma(D, B)$ be the graph which has vertices corresponding to bases and edges given by bands, connected in the natural way. It is clear that $b_1(\Gamma)$ (thought of as a cell complex) is the genus of the ribbon surface specified by the ribbon presentation. Restrict ourselves to the case where the ribbon surface is a sphere or torus. Suppose that Γ has a vertex x of valence 3 or greater. Then one of the outgoing edges of x has a path, never returning to x, which ends at a vertex y which has only one incoming edge. Perform the band slide corresponding to this path to get a new ribbon presentation Γ' with the same set of bases. In Γ', the

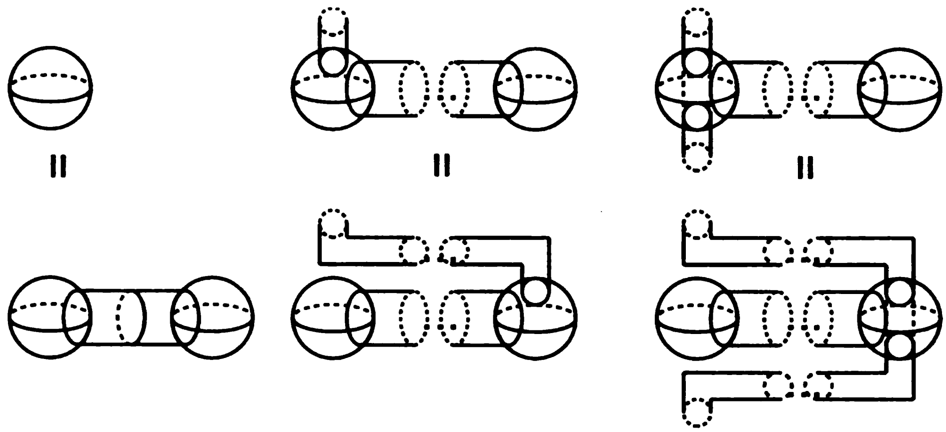

Figure 2.3. A) Trivial addition/deletion B) Band slide C) Band pass

valence of x has decreased by 1 and the valence of y is now 2. Continue this procedure until we arrive at a graph $\tilde{\Gamma}$ (and corresponding ribbon presentation) for which each vertex has valence at most 2. Then, as cell complexes, $\tilde{\Gamma}$ is either an interval or an S^1 as the ribbon surface is a S^2 or a torus. We will call such ribbon presentations *linear.*

Consider the connect sum of a ribbon 2-knot K_0 with standard twins $\mathbb{T}\mathrm{w} = K_1 \cup K_2$. Standard twins have a simple ribbon presentation given by two bases and a band each. (Each base is for one of the twin intersection points.) Stabilize the band in K_1 by switching it for two bands and a base. Then the connect sum $K_0 \# K_1$ is formed by taking K_0 and K_1 each in a D^3 and adding a band from an endpoint base D' of K_0 to the "middle" base of K_1. Then sliding the band from the "middle" base of K_0 to the $\ominus$ along K_0's linear ribbon presentation to the other endpoint base, we get a linear ribbon presentation of $K_0 \# K_1$ which has the twin intersection bases as its endpoints.

2.1.5 Projections

In the study of classical knots in $\mathbb{R}^3$, their generic projections to $\mathbb{R}^2$ together with crossing information completely determine their isotopy type and have proved extremely useful. Projections are also quite useful for twins and surfaces in $\mathbb{R}^4 = S^4 \setminus \mathrm{pt}$.

Giller proves in [Gil82] that projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ with only double and triple points exist and are generic. In these generic projections, the double points either exist in families which are either simple closed curves or embedded open intervals whose closed endpoints are triple points. See Figure 2.4. In the same paper, he gives methods of decorating these projections with over/middle/under crossing information and a way of determining if an arbitrary set of crossing information gives a lift of such an immersion of a surface in $\mathbb{R}^3$ to an embedding in $\mathbb{R}^4$.

Figure 2.4. Local models for a family of double points and a triple point

We will only consider those knots and twins which admit a projection which contain no triple points. Not all twins or surfaces have such a projection and those that do are said to be *simply knotted*. First examples of simply knotted 2-knots include Artin spun knots and ribbon 2-knots. For Artin spun knots, we can get a projection with no triple points by doing the same spinning construction (one dimension down) to the projection (to $\mathbb{R}^2$) of the original, classical knot. This creates an S^1s worth of double points for each crossing in the classical knot's projection. We will call a twin $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ simply knotted if both K_i are simply knotted and pairwise have no triple points.

Ribbon knots have embedded projections away from the ribbon singularities — the intersections of the interiors of bands with interiors of the bases. (This is in contrast with ribbon 1-knots, for the which projections of bands may have crossings. The analogous situation here is an under/over crossing of the whole band — which does not result in a crossing in the projection.) Nearby the ribbon singularities, we have projections which appear as in Figure 2.5. It was proved in [Yaj64] that all simply knotted S^2s are ribbon.

When we have a S^1 family of double points, we have local neighborhoods around each which appear as in the first picture in Figure 2.4. This gives the neighborhood of the family the structure of an bundle over S^1, possibly with nontrivial monodromy. As the surfaces in $\mathbb{R}^4$ are orientable and the two preimages of the double points are separated, the monodromy must be trivial. This means that, local to the S^1 family of double points, the projection is that of a classical knot crossing times S^1.

Then, for a simply knotted projection of a (oriented) surface in $\mathbb{R}^4$, it is sufficient to label one of the surfaces as being over crossing at each family of double points. We will use

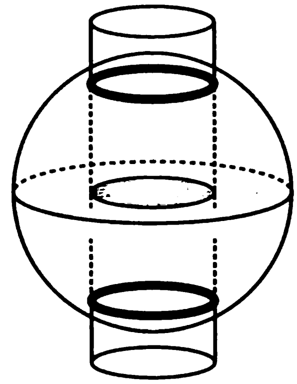

Figure 2.5. A projection of the neighborhood of a ribbon singularity (shaded disc) and the corresponding two S^1s of double points.

"+" to denote this.

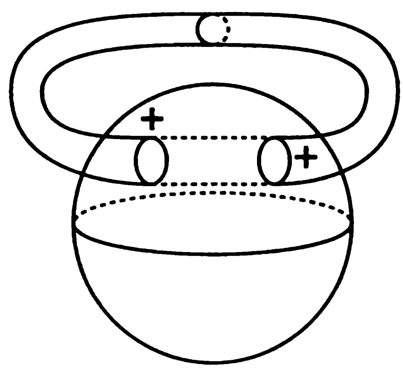

Figure 2.6. A sphere and a torus with crossing information.

For a twin, a few additional pieces of information are needed. We need to keep track of the two intersection points of the spheres. In S^4, the neighborhood of each is diffeomorphic to the cone on a positive or negative Hopf link. Then the (undecorated) projection of such a neighborhood appears as does a neighborhood of double points. We decorate the projection with a solid dot to indicate the intersection point of the S^2s and + signs to indicate over/under crossings on the double point arcs which emanate. We switch from over to under at the intersection of the spheres in twins. See Figure 2.8

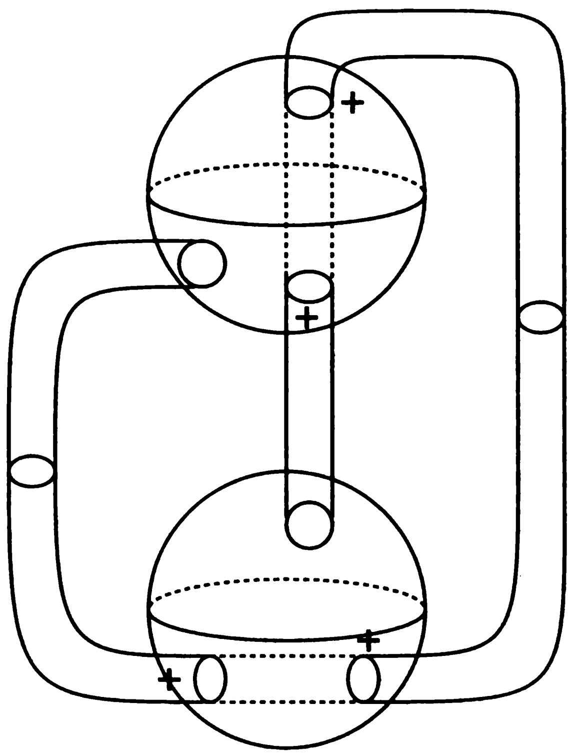

Figure 2.7. A 2-knot with Alexander polynomial $1 - 2t$ and Giller polynomial $\Delta_G = t^{-2} - 1 + t^2$

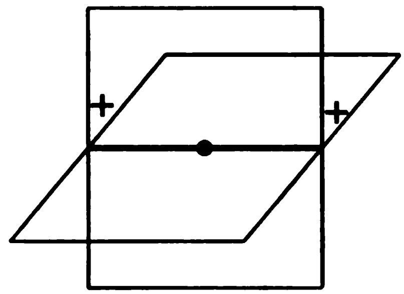

Figure 2.8. The projection of the neighborhood of the intersection of spheres in a twin. To the left, the vertical plane crosses above and to the right, the horizontal does.

2.1.6 Virtual knot presentation

In [Sat00], Satoh showed how to represent ribbon surfaces of genus 0 and 1 in $\mathbb{R}^4$ by means of virtual knots/links. For our purposes, a virtual knot (or link) is a diagram in $\mathbb{R}^2$ of embedded, oriented arcs which end either at "crossings" as in the (top) first two pictures in Figure 2.9 or at endpoints as in the (top) third picture. Each such diagram corresponds to a collection of immersed surfaces in $\mathbb{R}^3$ by replacing each of the crossings and endpoints in Figure 2.9 with the corresponding surfaces in $\mathbb{R}^3$ under them. These are then connected via tubes parallel to the embedded arcs. Thus, any virtual link corresponds to the projection, with crossing information, of a collection of ribbon 2-spheres and tori in $\mathbb{R}^4$.

Conversely, linear ribbon presentations of knots correspond to virtual knots. Take a projection of $K \subset \mathbb{R}^4 \to \mathbb{R}^3$ having only double points at ribbon singularities. For each band in the linear ribbon presentation, consider the image of its core in $\mathbb{R}^3$ extended to the center of the bases to which the band attaches. This gives an immersed (at ribbon singularities) arc $\hat{K}$ in $\mathbb{R}^3$. Taking a generic projection $\mathbb{R}^3 \to \mathbb{R}^2$ we get an arc $\check{K}$ immersed in $\mathbb{R}^2$ with two kinds of singularities:

- double points of the projection $\hat{K} \subset \mathbb{R}^3 \to \mathbb{R}^2 \supset \check{K}$ and
- projections of immersion points $\hat{K} \hookrightarrow \mathbb{R}^3$.

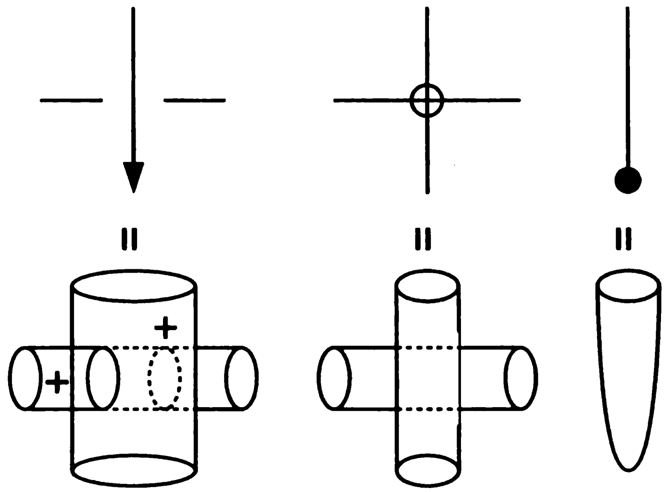

Figure 2.9. Correspondence of crossings in virtual knots to crossings in projections of surfaces to $\mathbb{R}^3$

Each of the first kind of double point corresponds to a virtual crossing. For the second kind of crossing we must first consider a diversion about orientations.

The endpoints of $\check{K}$ each correspond to a base with only one band attached — here, K locally consists of the discs D_1, D_2. With a fixed orientation on K, we orient the boundary of the D_i with the outward normal. We then say that the endpoint of $\check{K}$ is out/in as the boundary orientation on the D_i is counterclockwise or clockwise, respectively (when D_i is orientation-preserving identified with the unit complex disc.) This orients $\check{K}$.

Then, with $\check{K}$ oriented, we can check that the second type of immersion point corresponds to the ribbon intersection in Figure 2.9. If it does not (i.e. the two crossings have the opposite under/over information) then perform the isotopy in Figure 2.10. Once this has been done, we may use our correspondence from Figure 2.9 to label each of the immersion points of $\check{K}$ as virtual knot crossings.

In addition to the "classical" Reidemeister moves in Figure 2.11, associated to a virtual knot, we have the series of "virtual" Reidemeister moves in Figure 2.12 giving allowable isotopies. Notice that move D is one of the forbidden moves of the virtual knots of Kauffman. The type of virtual knot we consider here is sometimes referred to a being *weakly* virtual but

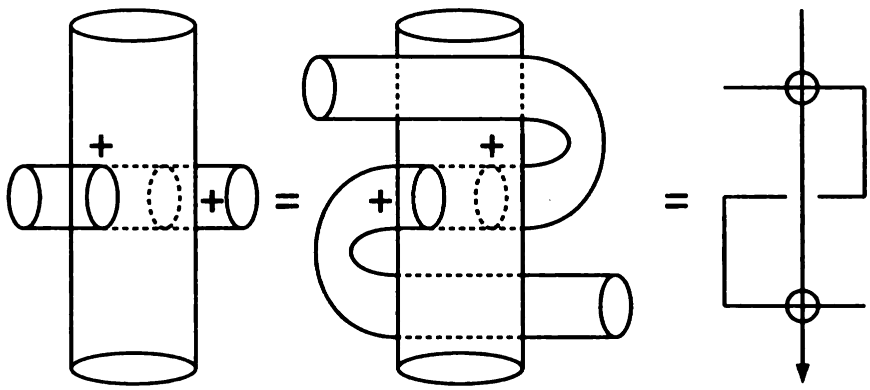

Figure 2.10. Fixing a "bad" ribbon crossing by an isotopy which creates two virtual crossings

in the spirit of brevity we will omit "weakly" in this paper. These concepts of virtual knots are inequivalent as there are virtual knots (in the sense of Kauffman) which are knotted (nonisotopic to a standard configuration) which, when move D is allowed, are unknotted. An example of this is given in [Sat00].

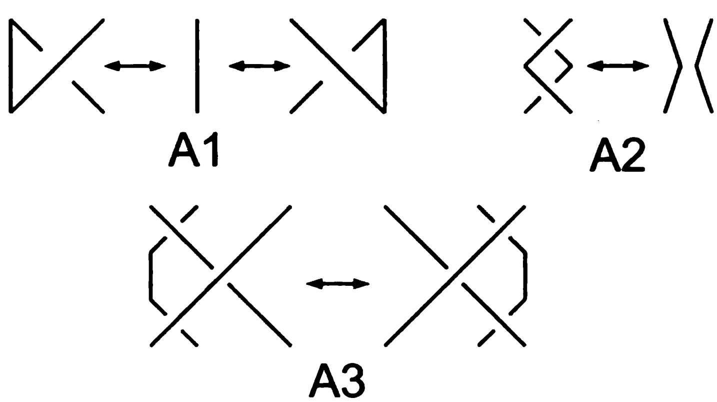

Figure 2.11. Reidemeister Moves for Classical knots

We will add two more items to these diagrams. For a ribbon twin $\mathbb{T}\mathrm{w} = K_1 \cup K_2$, there are two bases D_i', D_i'' in the ribbon representation of each K_i which correspond to the twin intersection points $K_1 \cap K_2$. Perform band slides until the ribbon presentations of the K_i are linear with endpoints D_i', D_i''. Then we have corresponding virtual knot representations

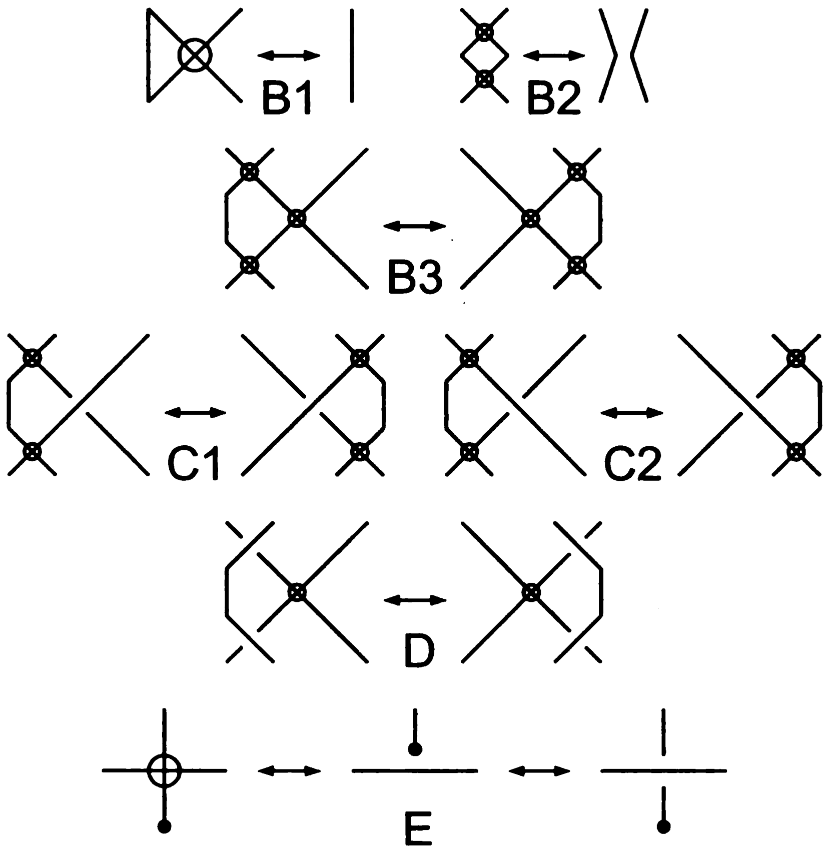

Figure 2.12. "Reidemeister" Moves for Virtual knots

of the K_i which have identical endpoints. We will use $\oplus, \ominus$ to mark each of these as they correspond to the cone on the positive and negative Hopf bands at the intersection points. With this in mind, we get the moves in Figure 2.13.

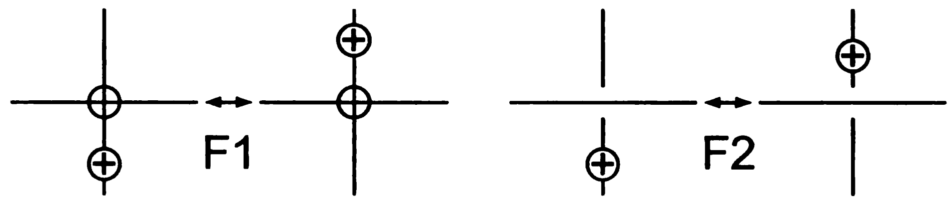

Figure 2.13. "Reidemeister" Moves $F1$,$F2$ for Twins, versions for $\ominus$ are identical

Now let us consider the connect sum of a ribbon 2-knot K_0 with standard twins $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ as described in Section 2.1.4. Notice that K_2 will not have any ribbon singularities with $K_0 \# K_1$. In fact, we can drag the bases of K_1 and K_2 corresponding to one of their twin intersections along together when we perform the band slide described previously. This creates only virtual crossings between the new diagrams for $K_0 \# K_1$ and K_2. By the "Reidemeister" moves $B1 - B3$ for virtual knots (Figure 2.12) or the standard Reidemeister moves $A1 - A3$, we can separate the arc for K_2 entirely from that for $K_0 \# K_1$, except for their common twin points. This means, in practice, we can obtain the virtual knot presentation of the connect sum of a ribbon 2-knot and ribbon twins by:

- using move E of Figure 2.12 to obtain a diagram where the endpoints are on the boundary of a D^2.
- connecting these points via an unknotted arc in the complement of the D^2 and changing the source point to a $\oplus$ and the sink point to a $\ominus$.

For example, see Figure 2.14

2.1.7 Surgery diagrams

As discussed earlier, a twin in S^4 has a canonical surgery associated to it. Since our decorated projections determine isotopy type, no additional information is needed to carry

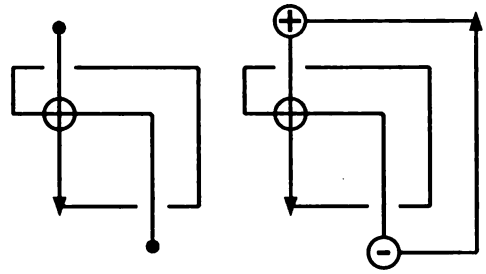

Figure 2.14. Giller's example (left, projection version in Figure 2.7) and twin version (right)

out surgery. For a T^2 in S^4, however, we will need additional information.

As any $T^2 \subset S^4$ is nullhomologous, it bounds a Seifert manifold[1] which, via its inward normal, gives a Seifert framing for the T^2. This gives us a decomposition, $\partial N(T^2) = T^2 \times S^1$. As surgery replacing $N(T^2)$ with $T^2 \times D^2$ is determined by the image of ∂D^2, we see that we can entirely describe surgery by specifying a curve on T^2 and an integer giving the winding about a meridian (boundary of normal disc) to T^2. See Figure 2.15.

When the T^2 is ribbon with a linear ribbon presentation and corresponding virtual knot diagram, we can decompose T^2 in the following manner: Let C be the core of the ribbon presentation, projected to $\mathbb{R}^3$. Let α be an essential loop on T^2 which, when projected to $\mathbb{R}^3$ is null-homologous in $\mathbb{R}^3 \setminus C$. Any such loop represents the same homology class on T^2. Let β be $\partial D^2 \times \{t\}$ in a band in the ribbon presentation. Orient α to coincide with the orientation of the virtual knot diagram. Then orient β so that $\alpha \cdot \beta = +1$ with respect to the orientation on T^2. So $T^2 \cong \alpha \times \beta$. Then, in the virtual knot diagram, labeling the knot corresponding to T^2 with $(\gamma, \beta/\alpha)$ where γ is the winding number of the attached ∂D^2 with respect to the Seifert framing and β/α is the slope of ∂D^2 projected to T^2.

We will write an S together with $*$s on the appropriate components when we wish to

[1] A Seifert manifold for a surface Σ in S^4 is a 3 manifold M with boundary diffeomorphic Σ, smoothly embedded in S^4 so that $\partial M = \Sigma$. As in [Rol76], Seifert manifolds exist because of the following: Consider the map $\Sigma \times \partial D^2 \to \partial D^2$ which is given by a trivialization of the normal bundle of Σ. Obstruction to extending this map over all of $S^4 \setminus N(\Sigma)$ vanish, giving us a map $S^4 \setminus N(\Sigma) \to S^1$. We can homotope this map to a smooth map which remains equal to the projection $\Sigma \times \partial D^2 \to \partial D^2$ on a tubular neighborhood of the boundary. Then as $S^4 \setminus N(\Sigma)$ is compact there are a finite number of critical points. Let M be the preimage of one of the regular points.

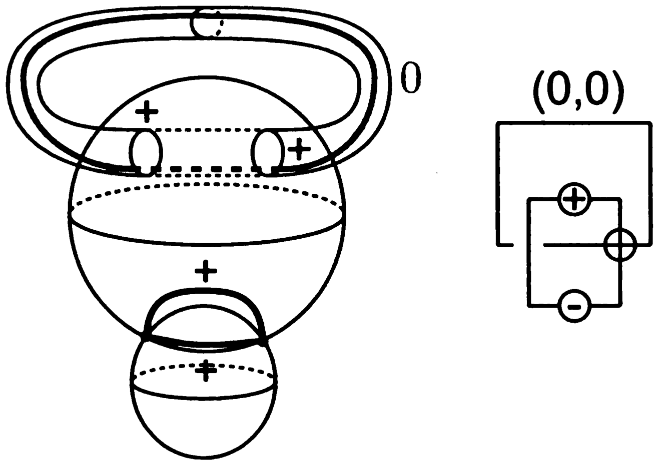

Figure 2.15. Projection and Virtual Knot surgery diagrams for a twin and torus.

denote this surgery.

2.1.8 Giller's Polynomial

In [Gil82], C. Giller defines a polynomial $\Delta_G(t)$ of simply knotted S^2s in S^4. This supposed invariant obeys a "Conway calculus" relation similar to that of the Alexander polynomial for classical knots.

That is, consider a embedded circle of double points in a projection of a (collection of) oriented sphere or torus in $\mathbb{R}^4$. As mentioned before, we can trivialize the neighborhood of the double points so that we have the neighborhood of a classical knot crossing times S^1. All surfaces in question are oriented and so orient the double points of their projection — this orients both strands in the classical picture. We can then replace this neighborhood with S^1 times any of the 3 options in Figure 2.16, obtaining

The invariant is then defined by the relation:

$$\Delta_G(L_+) - \Delta_G(L_-) = (t^{1/2} - t^{-1/2})\Delta_G(L_0) \qquad (2.2)$$

together with

$$\Delta_G(\text{unknotted sphere}) = 1 \qquad (2.3)$$

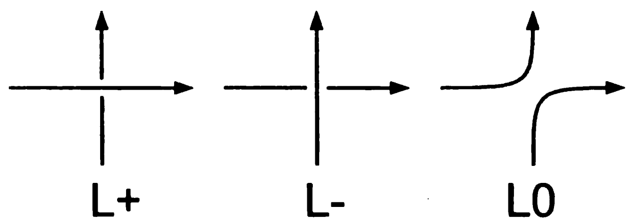

Figure 2.16. Resolution of a knot crossing

and

$$\Delta_G(\text{surfaces separated by an } S^3) = 0. \tag{2.4}$$

Giller also describes Δ_G in a manner similar to that of the Alexander polynomial. That is, letting M be a Seifert manifold for K, he forms the infinite cyclic cover X of $S^4 \setminus K$ and presents $H_1(X)$ as a $\mathbb{Q}[t, t^{-1}]$ module. Then Δ_G is defined by $T = \mathbb{Q}[t, t^{-1}]/(\Delta_G)$ where T is the $\mathbb{Q}[t, t^{-1}]$ torsion part of $H_1(X)$.

Whenever K is ribbon, we can choose M to be a punctured $nS^1 \times S^2$ given by the ribbon presentation. It is easy to verify by standard arguments that isotopies and band-stabilizations of the ribbon presentation yield the same Δ_G. Therefore, Δ_G is well-defined for ribbon knots.

For Artin spun knots, Giller's polynomial is the Alexander polynomial. In the case that we apply these computations to the projection of the knotted sphere in an Artin spun twin, Giller's polynomial is the Seiberg-Witten polynomial. (as shown in [FS98]).

Interestingly, 2-knots and twins need not have a symmetric Alexander polynomial. Giller's polynomial and the Seiberg-Witten polynomial, however, are symmetric. For example, see Figure 2.7, which is the spun right hand trefoil with crossing changes.

The natural questions to ask are then: Is Giller's polynomial an invariant of 2-knots? If so, is it equal to the Seiberg-Witten polynomial for the corresponding twin? For twins, what is the relationship between the Alexander polynomial and the Seiberg-Witten invariant? Our invariant provides suggestive evidence that the second question, at least, should be answered in the affirmative.

2.2 The 4-dimensional Macareña

2.2.1 3-dimensional Hoste Move

The main theorem of Fintushel and Stern in [FS98] gives a way of computing the Seiberg-Witten Invariants of classical-knot surgered 4-manifolds in terms of the symmetrized Alexander polynomial of the knot. The proof relies on a technique J. Hoste developed in [Hos84] which is a method for obtaining Kirby calculus diagrams for so called "sewn-up r-link exteriors" in S^3. (Like Fintushel and Stern, we will only consider the case where r-links are actually knots and links.) We discuss a simplified but sufficient version of the original move below so to demonstrate the ideas involved.

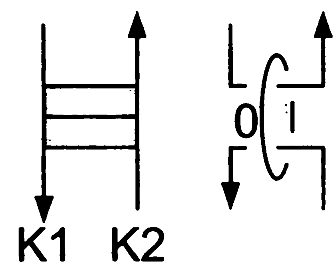

Figure 2.17. 3-dimensional Hoste move

A sewn up knot exterior is formed by taking either two oriented knots in one copy of S^3 or in two separate copies, excising a normal neighborhood of each knot, and gluing the resulting boundary T^2s by a diffeomorphism. For our purposes, we will let the diffeomorphism be the one which identifies oriented meridians and longitudes for the Seifert framings of each knot.

This procedure does two things, it removes two copies of $S^1 \times D^2$ with a chosen framing and orientation, and it replaces them with an $S^1 \times S^1 \times I$. Together, these are the boundary of $S^1 \times D^2 \times I$. Thus we may think of forming a sewn up link exterior as the result (on the boundary) of adding a round 4-dimensional 1-handle to B^4 so that the feet of the round 1-handle are the two knots, each with the proper framing.

Now, consider a projection of a link L in S^3 with oriented components K_1, K_2 and a small region in the projection where K_1 and K_2 run parallel but in opposite directions. We can then connect K_1 and K_2 via an arc. See Figure 2.17. Attach the round handle as above to form the sewn up link exterior for K_1 and K_2. Note that we can choose the attaching map of the round handle so that in the corresponding Morse-Bott function, the points p_1, p_2 on K_1, K_2 where the arc touches each knot are both connected to the same point p on the critical S^1 by gradient flow lines.

Take a perfect Morse function on the critical S^1 of the round handle so that the index zero critical point is p. This decomposes the round 1-handle into a 1-handle and 2-handle corresponding to the 0- and 1-handles of the Morse function on S^1.

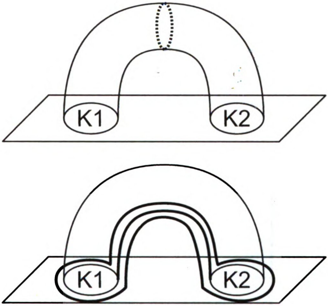

Figure 2.18. Round handle becomes a 1 and 2 handle

Attaching the 1-handle to S^3 results in self connect summing S^3 at the points p_1, p_2. By standard tricks, this is the same as zero surgery on the unknot around the arc in the second drawing in Figure 2.17.

The attaching circle of the 2-handle is the "band sum" K of K_1 and K_2 as shown in Figure 2.18 and the second picture in Figure 2.17. Attaching the 2-handle to the result (an

$S^1 \times S^2$) of the previous surgery is then merely zero surgery on K.

2.2.2 4-dimensional Hoste Move

In [FS98], the Hoste move shows up in 4-dimensions with an S^1 equivariance as we cross the 3-manifold with S^1. When that is done, the surgeries on knots show up as surgeries on square zero tori which, by using [MMS97], are amenable to computations of the Seiberg-Witten invariant. The 4-dimensional version of the Hoste move we will discuss here does not assume this S^1 equivariance, although local S^1 equivariances will occur.

Proposition 2.2.1. Consider two embedded, oriented square zero tori T_1, T_2 in a 4-manifold X. Suppose that T_1, T_2 are connected by an annulus $A = S^1 \times I$, embedded in X, so that ∂A consists of an essential curve on each torus. Let each T_i be framed so that $A \cap \partial N(T_i)$ is in the subspace of $H_1(\partial N(T_i)) = H_1(T^3)$ generated by the pushoffs of loops on T_i with respect to the framing. Let ϕ be a diffeomorphism $T_1 \to T_2$ which identifies the components of ∂A in T_1 and T_2. Then the self fiber sum, $X \underset{T_1 =_\phi T_2}{\#}$, is also the result of surgery on two tori: the "band sum" of the tori along A and torus given by the loop in Figure 2.19 in the neighborhood of $\theta \times I \subset A$ for each $\theta \in S^1$.

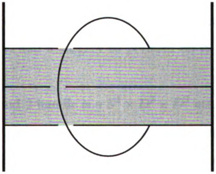

Figure 2.19. Band sum

Proof of Proposition 2.2.1. We can reinterpret the fiber sum as the result (on the boundary) of adding a 5-dimensional toric 1-handle (a $T^2 \times D^2 \times I$) to $X \times I$ so that the attaching region, $T^2 \times D^2 \times \pm 1$ is identified with the normal bundles to T_1 and T_2 with their chosen framing. This results in deleting the two $T^2 \times D^2$s and replacing them with a $T^2 \times \partial D^2 \times I$.

This identification is determined by choice of framings for T_1, T_2 and a diffeomorphism ϕ between them.

Choose a factorization $T_1 \cong_\phi T_2 \cong S^1_\alpha \times S^1_\beta$ so that the S^1_α factor is ∂A in both T_is. The critical T^2 for the 5-dimensional toric handle is identified with the T_is by the gradient flow. Pick a perfect Morse function on S^1_β and perturb the Morse-Bott function on the 5-dimensional handle by an extension of it. This gives us a reinterpretation of the 5-dimensional toric 1-handle as two round (S^1) handles — a round 2-handle and a round 1-handle — corresponding to the critical points of the Morse function on S^1_β.

Consider the round 1-handle first. Such a handle is a $S^1 \times D^3 \times I$ so that it attaches along $S^1 \times D^3 \times \pm 1$. By construction, the two $S^1 \times D^3$s are neighborhoods of the components of ∂A, with framing given by the inward normal along A, a vector field along ∂A parallel to T_i, and a third vector field defined by orthogonality to these and the tangent space to ∂A.

Consider a neighborhood of A which is S^1 equivariant, matching the S^1 equivariance of $A = I \times S^1$. When small, such a neighborhood is diffeomorphic to S^1 times the "H" in Figure 2.19. (The vertical lines are in T_i; the horizontal, slices of A.) Attaching the round 1-handle is the same as (equivariantly) self-connect summing at the places where the vertical lines intersect the horizontal core of the band. In each 3-manifold slice, this is equivalent to performing zero surgery on the loop linking the band in Figure 2.19. Then, in turn, this gives us a square zero torus L and a surgery to perform on it within the neighborhood of A.

Now, a 5-dimensional round 2-handle is a $S^1 \times D^2 \times D^2$ attached along $S^1 \times D^2 \times \partial D^2$. Outside of the neighborhood of A, the attaching torus is equal to the T_i. (two annuli) Inside the neighborhood of A, the attaching torus T is given by S^1 times the boundary of the band in Figure 2.19. (two more annuli) The framing of this torus is given by the framings of the T_i outside the neighborhood of A and by the inward normal to the band on (each slice of) the inside. This can be seen by band summing pushoffs of the T_i. (By hypothesis, A has zero winding with respect to our framing so this can be done by band summing in the S^1 trivialization.) T inherits a factorization $S^1_a \times S^1_b$ from the T_i by noting that the S^1_α factors of the T_i survive and that the S^1_β factors are themselves band summed. Attaching the round 2-handle then performs a surgery on this torus which sends ∂D^2 to the S^1_b factor. □

4-Dimensional Hoste Move for a Twin and Torus

Let us now examine how Proposition 2.2.1 can be described in terms of surgery information on projections. First we will look at the case when the tori come from surgeries on a twin and a torus.

Let X be the result of standard surgery on a ribbon twin $\mathbb{T}\mathrm{w}$ and $(0,0)$ surgery on a ribbon torus T in S^4, both specified by a virtual knot diagram as in sections 2.1.6 and 2.1.7. Note that the cores $T_{\mathbb{T}\mathrm{w}}, T_T$ of each surgery inherit preferred framings from their surgery description.

Suppose that $\mathbb{T}\mathrm{w}$ and T have a classical knot crossing in the virtual knot diagram and hence a ribbon intersection. Then, in a projection to $\mathbb{R}^3$, there is a neighborhood as in Figure 2.20. Consider the annuli shown in the figure. Each of these annuli are isotopic. This can be seen by the fact that on the left of each picture, the horizontal surface overcrosses the vertical surface so the annulus must lie completely under the horizontal surface to the left of the ribbon singularity. Thus we can isotope the annulus freely on the left of the ribbon singularity. Similarly, on the right the annulus lies completely above the horizontal surface and so we can isotope it freely on that side.

Now let B be the particular representation of the annulus corresponding to the particular orientations of the virtual knot crossing depicted below it. Now, B connects an equator $\gamma_{\mathbb{T}\mathrm{w}}$ to one of K_i in $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ to a essential curve γ_T on the torus. Since we have done $(0,0)$ surgery on the torus (in virtual knot notation), the surgery curve (in the projection notation) meets γ_T once.

The method of constructing B ensures that ∂B consists of essential curves on the cores of the twin and torus surgeries when B is extended by the projection to the surgered manifold. This means that we can apply proposition 2.2.1 once we have chosen the diffeomorphism ϕ between $T_{\mathbb{T}\mathrm{w}}, T_T$. We have already required that ϕ identify the components of ∂B. ϕ is then determined when we require that it identify the following:

- the projection to T_T of $\mathrm{pt}_1 \times \partial D^2$, where $\mathrm{pt}_1 \in T$, $T \times D^2 \subset S^4$ the normal bundle, and
- the projection to $T_{\mathbb{T}\mathrm{w}}$ of $\mathrm{pt}_2 \times \partial D^2$, where $\mathrm{pt}_2 \in \mathbb{T}\mathrm{w} \setminus (S^2$ intersection points) and

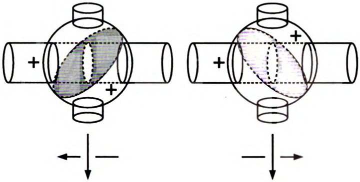

Figure 2.20. Annulus B used for smoothing

lies in the S^2 which does not contain the preimage of the double points. ∂D^2 is the boundary of the fiber of the normal bundle to this S^2.

With this data fixed, proposition 2.2.1 can be applied. The result is that

$$X \setminus (N(T_{\mathbb{T}\mathrm{w}}) \cup N(T_T)) = S^4 \setminus (N(\mathbb{T}\mathrm{w}) \cup N(T))$$

sewn up by ϕ is diffeomorphic to

$$S^4 \setminus (N(\mathbb{T}\mathrm{w} \underset{B}{\#} T) \cup \tau) \cup T^2 \times D^2 \cup T^2 \times D^2$$

where

- $\mathbb{T}\mathrm{w} \underset{B}{\#} T$ denotes the "band sum" of $\mathbb{T}\mathrm{w}$ and T along B,
- the first $T^2 \times D^2$ is glued to $\partial N(\mathbb{T}\mathrm{w} \underset{B}{\#} T)$ by the standard surgery on twins,
- τ is the torus given by the loop in Figure 2.19, and
- the second $T^2 \times D^2$ is glued to $\partial N(\tau)$ so that ∂D^2 goes to the nullhomologous pushoff of the loop in Figure 2.19.

Finally, we can isotopy this region in $\mathbb{T}\mathrm{w} \underset{B}{\#} T$ to be as shown in one of the pictures in the top row of Figure 2.21. The appropriate smoothing depends on the orientation of the horizontal surface and corresponds to the selection of band previously. In Figure 2.21, the correspondence of the orientation of the horizontal surface to the virtual knot diagrams is

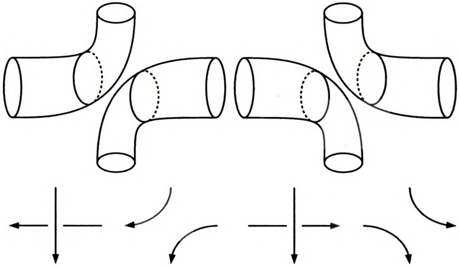

Figure 2.21. Two smoothings of the intersection on Figure 2.20 with corresponding virtual knot diagrams

show by the original diagrams to the lower left of each smoothing and the smoothed virtual knot diagrams to the lower right.

4-Dimensional Hoste Move for two Tori

We will require the Hoste Move between two tori in only one case. Suppose that both tori lie in a neighborhood diffeomorphic to $S^1 \times D^3$ so that in each $\theta \times D^3$ the tori are as shown in Figure 2.22. The we only need to describe one aspect of the 3-dimensional local picture — the gluing map for the sewn up exterior for the right hand side of Figure 2.22. This map is given by identifying meridians to each loop and the pushoffs along the obvious once punctured discs to each.

2.2.3 4-Dimensional Crossing Change

Consider a classical crossing in a virtual knot diagram for a twin and/or a torus and the corresponding annulus from Figure 2.23. Notice that the correspondence is reversed from that of the $4D$ Hoste move. As before, both bands shown are isotopic. Push the horizontal surface along the annulus in Figure 2.23 to get the configuration in Figure 2.24.

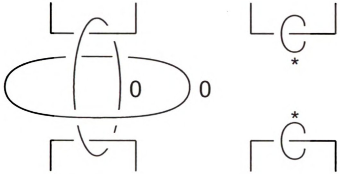

Figure 2.22. S^1 times the left is S^1 times the right with the two starred tori sewn up

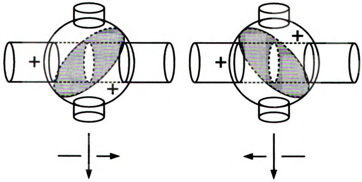

Figure 2.23. Annulus B used for crossing change

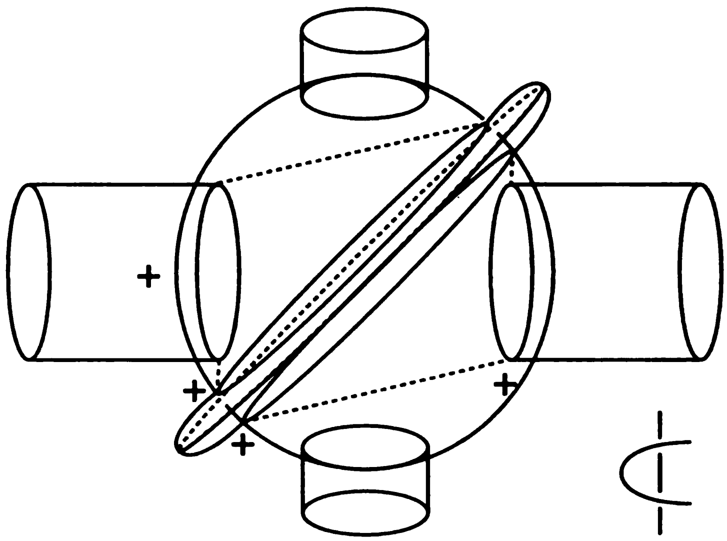

Figure 2.24. Diagram obtained from the first picture in Figure 2.23 by pushing horizontal tube along band, over crossing the vertical tube in the two new sets of double points. The new pair of crossings is locally modeled on S^1 times the diagram in the lower right. The diagram corresponding to the second picture in Figure 2.23 is similar.

Focus our attention to the lower of the two new loops of crossings in Figure 2.24 (or the corresponding picture for the other band.) Call this crossing C. Local to C, we have the model of S^1 times a 3-dimensional oriented knot crossing. Note that if we form an annulus by taking a path from the lower to the upper double point in each 3-manifold picture, we get an annulus A which is isotopic to the annulus B from before.

Perform one of the surgeries on a torus indicated by the 3-dimensional pictures in Figure 2.25 localized at C in Figure 2.24 (or the corresponding picture for the other band.) The appropriate surgery is the top for the first picture of Figure 2.23 and the lower for the second. This changes the crossing C from an overcrossing of the horizontal surface to an under crossing, resulting in Figures 2.26 and 2.27. Finally, we perform the isotopies indicated in these figures and see that our result has changed the crossing type from a classical $+$ to $-$ or from a classical $-$ to $+$ in the virtual knot diagram.

It is important to note that the torus which we have surgered is isotopic to the surgered torus τ of section 2.2.2. We identify the two via the isotopy of the band B.

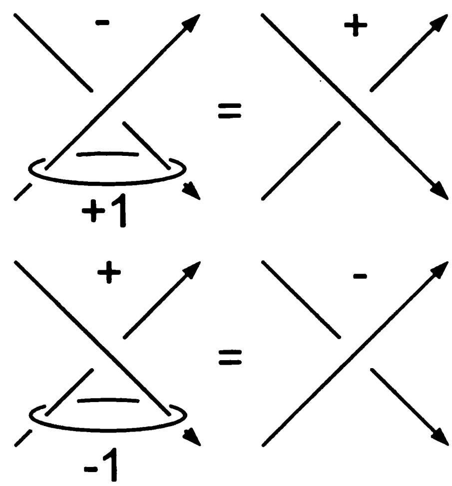

Figure 2.25. Isotopy in 3-dimensional picture

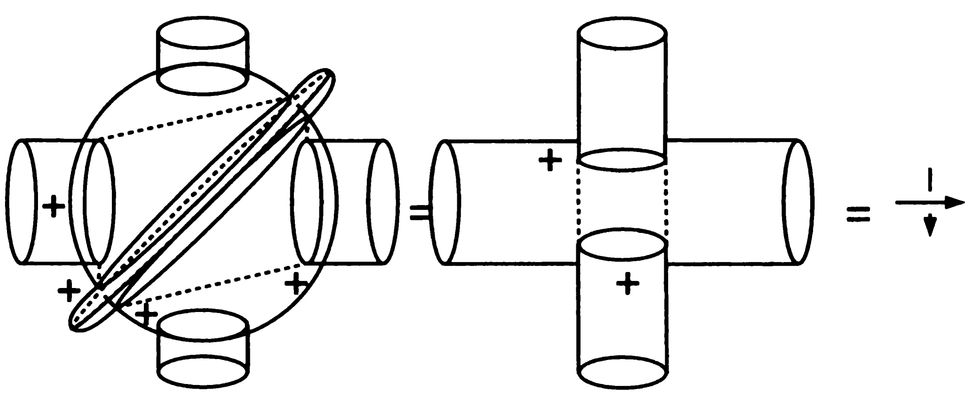

Figure 2.26. Isotopy of first picture in Figure 2.23 after surgery and the corresponding virtual knot crossing of result

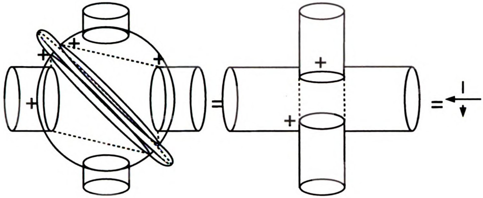

Figure 2.27. Isotopy of second picture in Figure 2.23 after surgery and the corresponding virtual knot crossing of result

2.3 Calculation of the Invariant for Certain Ribbon Twins

Consider a twin $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ given by a virtual knot presentation and the manifold $X(\mathbb{T}\mathrm{w}) = E(2)_{\mathbb{T}\mathrm{w}}$. Suppose that the virtual knot presentation for K_1 contains a classical crossing; so K_1 has a ribbon intersection with itself. Let $\mathbb{T}\mathrm{w}_+$, $\mathbb{T}\mathrm{w}_-$ and $\mathbb{T}\mathrm{w}_0$ be the results of replacing the crossing in the virtual knot diagram with the three options in Figure 2.28. Note that $\mathbb{T}\mathrm{w}_0$ will actually be a twin and a torus.

Figure 2.28.

Consider the square zero torus τ from the previous sections. Now, $\partial N(\tau)$ admits a decomposition $\partial N(\tau) = S^1_\alpha \times S^1_\beta \times S^1_\gamma$ from it lying in the S^1 equivariant neighborhood of the annulus B. Namely, in each 3-manifold slice of the neighborhood, τ is given by a loop α linking the slice of B once. Then $S^1_\alpha \times S^1_\beta = \tau$ where β is pt $\times$ S^1 in the equivariant neighborhood of B. Finally $S^1_\gamma = \partial D^2$ finishes the decomposition.

Log transform surgery on τ is defined by removing $N(\tau) \cong T^2 \times D^2$ and replacing with another copy of $T^2 \times D^2$. Such a surgery is uniquely determined up to diffeomorphism by the image of ∂D^2 in $\partial N(\tau) \cong T^3$. Using our decomposition above, we can describe such a surgery by a triplet of integers (a, b, c) with no common factor. Such a triplet gives specifies the isotopy type of the curve to which we will glue ∂D^2. We will write $X_{\tau(a,b,c)}$ for the result of the (a, b, c) log transform surgery on X.

From section 2.2.3 we see that we can move from $\mathbb{T}\mathrm{w}_-$ to $\mathbb{T}\mathrm{w}_+$ by surgery on $\tau \subset S^4 \backslash \mathbb{T}\mathrm{w}_-$ which in each 3-manifold slice is $+1$ surgery on the loop which gives τ. Thus we can move from $\mathbb{T}\mathrm{w}_-$ to $\mathbb{T}\mathrm{w}_+$ by a $(0, 1, 1)$ log transform on τ. Note that $(0, 0, 1)$ surgery on τ is the identity.

Morgan, Mrowka, and Szabo's formula in [MMS97] then gives

$$SW(E(2)_{\mathbb{T}\mathrm{w}_+}) = SW(E(2)_{\mathbb{T}\mathrm{w}_-}) + SW(E(2)_{\mathbb{T}\mathrm{w}_-,\tau(0,1,0)})$$

Now by our description in section 2.2.2, $E(2)_{\mathbb{T}\mathrm{w}_-,\tau(0,1,0)}$ is the sewn up twin/torus exterior of $\mathbb{T}\mathrm{w}_0$ fiber summed to $E(2)$. Thus, by our definition of I, $SW(E(2)_{\mathbb{T}\mathrm{w}_-,\tau(0,1,0)} = (t - t^{-1})I(\mathbb{T}\mathrm{w}_0)$, while $I(\mathbb{T}\mathrm{w}_+) = SW(E(2)_{\mathbb{T}\mathrm{w}_+})$ and $I(\mathbb{T}\mathrm{w}_-) = SW(E(2)_{\mathbb{T}\mathrm{w}_-})$. Therefore,

$$I(\mathbb{T}\mathrm{w}_+) = I(\mathbb{T}\mathrm{w}_-) + (t - t^{-1})I(\mathbb{T}\mathrm{w}_0)$$

Now consider a twin $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ and torus $\mathbb{T}$ given by a virtual knot presentation. Suppose that the virtual knot presentation contains a classical crossing between K_1 and $\mathbb{T}$; so K_1 has a ribbon intersection with T. Let $\mathbb{T}\mathrm{w}_+$, $\mathbb{T}\mathrm{w}_-$ and $\mathbb{T}\mathrm{w}_0$ be the results of replacing the crossing in the virtual knot diagram with the three options in Figure 2.28. Note that $\mathbb{T}\mathrm{w}_\pm$ will each be a twin and a torus while $\mathbb{T}\mathrm{w}_0$ will be a single twin.

As before, we consider the torus $\tau \subset S^4 \setminus \mathbb{T}\mathrm{w}_-$. From section 2.2.3 we see that we can move from $\mathbb{T}\mathrm{w}_-$ to $\mathbb{T}\mathrm{w}_+$ by surgery on $\tau \subset S^4 \setminus \mathbb{T}\mathrm{w}_-$ which in each 3-manifold slice is $+1$ surgery on the loop which gives τ. Thus we can move from $\mathbb{T}\mathrm{w}_-$ to $\mathbb{T}\mathrm{w}_+$ by a $(0, 1, 1)$ log transform on τ. Note that $(0, 0, 1)$ surgery on τ is the identity.

Morgan, Mrowka, and Szabo's formula in [MMS97] then gives

$$SW(E(2)_{\mathbb{T}\mathrm{w}_+} = SW(E(2)_{\mathbb{T}\mathrm{w}_-}) + SW(E(2)_{\mathbb{T}\mathrm{w}_-,\tau(0,1,0)})$$

Now, since each $\mathbb{T}\text{w}_\pm$ are composed of a torus and a twin, the manifolds $E(2)_{\mathbb{T}\text{w}_\pm}$ are each sewn up twin/torus exteriors fiber summed to $E(2)$. Thus $SW(E(2)_{\mathbb{T}\text{w}_\pm}) = (t-t^{-1})I(\mathbb{T}\text{w}_\pm)$.

Nearby τ we have a local model for $\mathbb{T}\text{w}_-$ given in the left picture of Figure 2.29. If we perform the $4D$ Hoste move on $E(2)_{\mathbb{T}\text{w}_-,\tau(0,1,0)}$ at an annulus equal to S^1 times the horizontal line in this picture, we obtain the picture to the right in Figure 2.29.

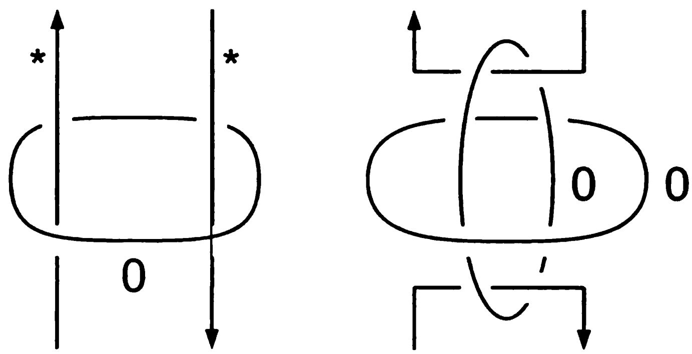

Figure 2.29. Hoste move nearby τ

Note that this picture is identical to that of Figure 2.22. Applying the version of the $4D$ Hoste move from section 2.2.2, we get a local picture equal to that in the right hand side of Figure 2.22. The tori in this picture are each isotopic to the torus $\partial D_1^2 \times \partial D_2^2$ – where ∂D_i^2 is the normal bundle to K_i, the knots which comprise $\mathbb{T}\text{w}_0$. This torus can also be described as one of the components of $\partial N(K_1) \cap \partial N(K_2)$. In the fiber sum manifold, $E(2)_{\mathbb{T}\text{w}_-,\tau(0,1,0)}$, each torus we have just described is isotopic to the fiber F. Thus, $E(2)_{\mathbb{T}\text{w}_-,\tau(0,1,0)}$ is diffeomorphic to $E(2)_{\mathbb{T}\text{w}_0} \underset{F=F'}{\#}$. Therefore, $SW(E(2)_{\mathbb{T}\text{w}_-,\tau(0,1,0)}) = (t-t^{-1})^2 SW(E(2)_{\mathbb{T}\text{w}_0})$ so

$$SW(E(2)_{\mathbb{T}\text{w}_+} = SW(E(2)_{\mathbb{T}\text{w}_-}) + (t-t^{-1})^2 SW(E(2)_{\mathbb{T}\text{w}_0})$$

and

$$(t-t^{-1})I(\mathbb{T}\text{w}_+) = (t-t^{-1})I(\mathbb{T}\text{w}_-) + (t-t^{-1})^2 I(\mathbb{T}\text{w}_0)$$

so

$$I(\mathbb{T}\text{w}_+) = I(\mathbb{T}\text{w}_-) + (t-t^{-1})I(\mathbb{T}\text{w}_0)$$

2.3.1 Relation to Giller's polynomial

Recall the definition of Giller's polynomial given by Equations (2.2), (2.3), and (2.4). These are the Conway-style relation, the value on the unknotted sphere, and the vanishing of the polynomial for split links, respectively. We now discuss similar results for our invariant.

That $I(\mathbb{T}\mathrm{w}_{std}) = 1$ was shown at Equation (2.1).

Suppose that $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ and T are ribbon with a virtual knot presentation which is split or only has pairwise virtual crossings. Then using the virtual Reidemeister moves of Figures 2.12 and 2.13, we can separate $\mathbb{T}\mathrm{w}$ and T. This means that the projections of $\mathbb{T}\mathrm{w}$ and T are separated by an S^3. So $\mathbb{T}\mathrm{w}$ and T are separated by an S^3. This means that the manifold formed by surgering $\mathbb{T}\mathrm{w}$ and T is a connected sum. Now, on the T side of the connected sum, the intersection form on H_2 will be a hyperbolic pair as will the intersection form on the side given by surgery on $\mathbb{T}\mathrm{w}$. Thus we are given a manifold which is the connected sum of two manifolds each with $b_+ > 0$. Therefore, the Seiberg-Witten invariant of this surgery vanishes. It follows that $SW(E(2)_{\mathbb{T}\mathrm{w},T}) = 0$ and that

$$I(\mathbb{T}\mathrm{w}, T) = 0 \text{ for } \mathbb{T}\mathrm{w}, T \text{ split.} \tag{2.5}$$

Now let us consider the Conway-style relation for Giller's polynomial we initially discussed in Section 2.1.8. This relation involves crossing changes and resolution at individual loops of double points. Now, in our crossing change surgery, we had a similar action of changing the lower crossing from Figure 2.24. In the diagrams for our $4D$ Hoste move, we took a different local projection to illustrate the appropriate surgery. However, smoothing the lower crossing from Figure 2.24 also yields a smoothing in the virtual knot diagram.

In other words, selecting a particular set of double points to apply the relation in Equation (2.2) to, $\Delta_G(\mathbb{T}\mathrm{w})$ computes $I(\mathbb{T}\mathrm{w})$. Therefore, $\Delta_G(\mathbb{T}\mathrm{w}) = I(\mathbb{T}\mathrm{w})$ for ribbon knots. We cannot make a stronger statement of equality however, as the relation from Equation (2.2) allows us to move into configurations of surfaces which are inaccessible to the invariant I.

2.3.2 The Class of Ribbon Twins

We now make some remarks on computations.

Now suppose that $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ is ribbon with a virtual knot presentation which only has virtual crossings. Then we can use the virtual Reidemeister moves B and F from Figures 2.12 and 2.13 to completely unknot the diagram for $\mathbb{T}\mathrm{w}$. Therefore, $\mathbb{T}\mathrm{w} = \mathbb{T}\mathrm{w}_{std}$ and so $I(\mathbb{T}\mathrm{w}) = I(\mathbb{T}\mathrm{w}_{std}) = 1$.

Suppose that $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ is a twin possibly with accompanying torus T with the configuration ribbon. Suppose that we reverse the orientation of one of the K_i or of T. This reverses the orientation of the torus $T_{\mathbb{T}\mathrm{w}}$ (or T_T) and induces a chance in homology orientation from the change of sign in pairing with $T_{\mathbb{T}\mathrm{w}}$. Then, if $\tilde{\mathbb{T}\mathrm{w}} = \overline{K_1} \cup K_2$, $I(\tilde{\mathbb{T}\mathrm{w}}) = -I(\mathbb{T}\mathrm{w})$. Similarly, $I(\mathbb{T}\mathrm{w}, \overline{T}) = -I(\mathbb{T}\mathrm{w}, T)$.

Currently, the author is unaware if crossings can be chosen so that the tree of terminates in standard twins and unlinked twin/torus pairs. The previous work of Fintushel and Stern guarantees that the process terminates when K_1 in $\mathbb{T}\mathrm{w} = K_1 \cup K_2$ is knotted with only classical crossings and K_2 is unknotted with no ribbon intersections with K_1. The presence of virtual crossings in the diagrams complicates the general case. Additionally, the author has yet to find a general method of dealing with ribbon intersections between the K_i.

However, there seem to be a fairly large number of new examples which we may compute using the current tools. The first we will compute is the twin version of the example from Giller's paper. Call this twin $\mathbb{T}\mathrm{w}_G$. This was encountered previously in Figure 2.14.

Follow the computation through Figures 2.30, 2.31, 2.32, and 2.33. In Figure 2.33, we arrive at configurations C, D, E, and F. Here C and F are isotopic to standard twins, so $I(C) = I(F) = 1$. The other configurations D, E differ by the orientation of the torus and so $I(D) = -I(E)$. Therefore,

$$\begin{aligned} I(\mathbb{T}\mathrm{w}_G) &= I(A) + (t - t^{-1})I(B) \\ &= I(C) + (t - t^{-1})I(D) + (t - t^{-1})I(E) + (t - t^{-1})^2 I(F) \\ &= 1 + 0 + (t - t^{-1})^2 \\ &= t^{-2} - 1 + t^2 \end{aligned}$$

Now let us look at the twin in Figure 2.34, a twin in which both 2-knots are unknots but which pairwise have ribbon intersections. Call this twin $\mathbb{T}\mathrm{w}_U$. Our current tools do not allow us to deal directly with pairwise ribbon intersections.

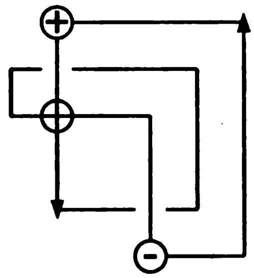

Figure 2.30. Giller Twin $\mathbb{T}\mathrm{w}_G$ with highlighted crossing

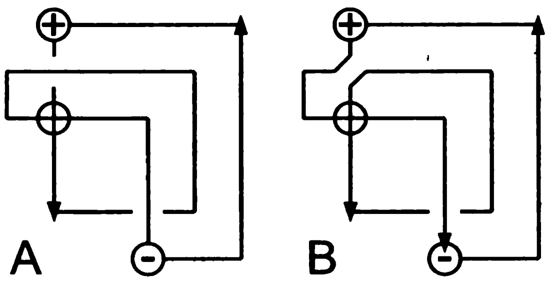

Figure 2.31. $I(\mathbb{T}\mathrm{w}_G) = I(A) + (t - t^{-1})I(B)$

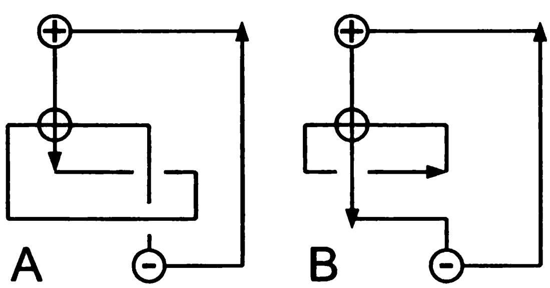

Figure 2.32. Isotopy of A and B from Figure 2.31 with highlighted crossings

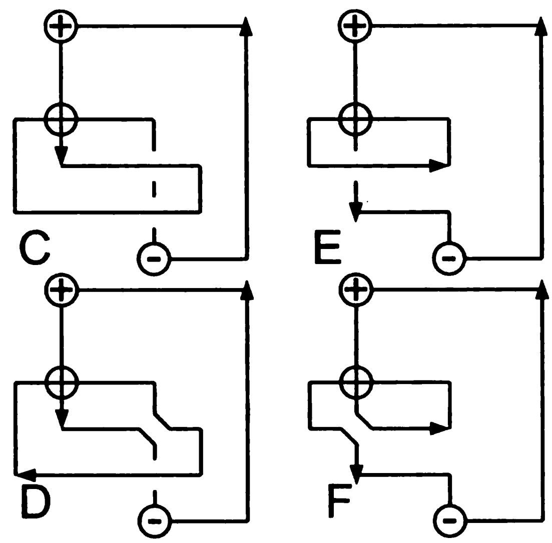

Figure 2.33. $I(\mathbb{T}\mathrm{w}_G) = I(C) + (t - t^{-1})I(D) + (t - t^{-1})I(E) + (t - t^{-1})^2 I(F)$

Follow the computation through Figures 2.35, 2.36, and 2.37. We arrive at configurations H, K, and L. Here H and L are isotopic to standard twins, so $I(H) = I(L) = 1$. The other configurations K contains a separated torus so $I(K) = 0$. Therefore,

$$\begin{aligned} I(\mathbb{T}\mathrm{w}_U) &= I(H) - (t - t^{-1})I(J) \\ &= I(H) - (t - t^{-1})I(K) + (t - t^{-1})^2 I(L) \\ &= 1 + 0 + (t - t^{-1})^2 \\ &= t^{-2} - 1 + t^2 \end{aligned}$$

Finally, we remark on uniqueness and related topics. In what is our Artin spun case, Fintushel and Stern have conjectured that their knot surgery construction yields nondiffeomorphic manifolds for "essentially different" knots. (Here, "essentially different" means that two knots are not isotopic nor are they mirror images of each other.) The Alexander polynomial does not completely distinguish knots however, so the Seiberg-Witten invariants in their current form do not shed any light on their conjecture. Similarly, it seems doubtful that the manifolds $E(2)_{\mathbb{T}\mathrm{w}_G}$ and $E(2)_{\mathbb{T}\mathrm{w}_U}$ are diffeomorphic, but with the Seiberg-Witten invariants being equal, we have no obvious way in which to distinguish them. In particular, it seems possible that $E(2)_{\mathbb{T}\mathrm{w}_U}$ is diffeomorphic to $E(2)_{\mathbb{T}\mathrm{w}_K}$ where $\mathbb{T}\mathrm{w}_K$ is the Artin spin of the left handed trefoil.

Also, results of C. Taubes in [Tau94] show that a manifold X with $b_+ > 2$ admits a symplectic form, the leading term in $SW(X)$ will have coefficient equal to one. The converse to this statement is known to be false by work of Fintushel and Stern in [FS97]. In the classical (or Artin spun) case, it is possible to construct a symplectic form on $E(2)_{\mathbb{T}\mathrm{w}}$ when the classical knot K from which $\mathbb{T}$w is constructed is a fibered knot. While it may be possible to rephrase this construction in terms of twins, it is unclear what topological conditions are required on $\mathbb{T}$w to achieve the same result. (A sufficient condition is that $S^4(\mathbb{T}\mathrm{w})$ fibers over T^2 or S^2.)

We then ask, do $E(2)_{\mathbb{T}\mathrm{w}_G}, E(2)_{\mathbb{T}\mathrm{w}_U}$ admit symplectic forms? What conditions on the exterior of the twin guarantee a symplectic form?

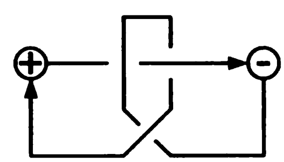

Figure 2.34. A twin in which both 2-knots are unknots

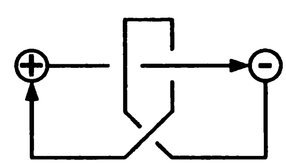

Figure 2.35. A negative crossing from Figure 2.34

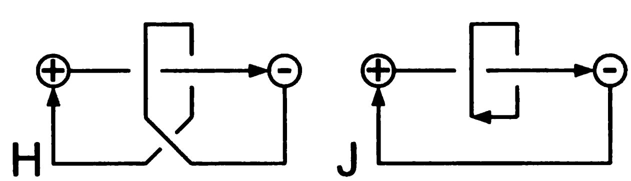

Figure 2.36. $I(\mathbb{T}\mathrm{w}_U) = I(H) - (t - t^{-1})I(J)$

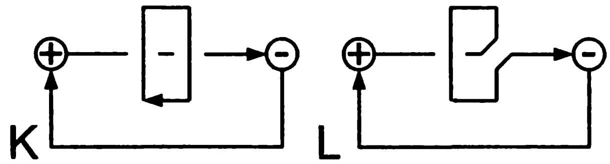

Figure 2.37. $I(J) = I(K) - (t - t^{-1})I(L)$

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