DISCOURSE ANALYSIS AS A TOOL TO INVESTIGATE THE RELATIONSHIP BETWEEN WRITTEN AND ENACTED CURRICULA: THE CASE OF FRACTION MULTIPLICATION IN A MIDDLE SCHOOL *STANDARDS*-BASED CURRICULUM

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ABSTRACT

DISCOURSE ANALYSIS AS A TOOL TO INVESTIGATE THE RELATIONSHIP BETWEEN WRITTEN AND ENACTED CURRICULA: THE CASE OF FRACTION MULTIPLICATION IN A MIDDLE SCHOOL *STANDARDS*-BASED CURRICULUM

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In the 1990s, the National Science Foundation (NSF) funded the development of curricula based on the approach to mathematics proposed in *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989). Controversy over the effectiveness of these curricula and the soundness of the standards on which they were based, often labeled the "math wars," prompted a plethora of evaluative and comparative curricular studies. Critics of these studies called for mathematics education researchers to document the implementation of these curricula (e.g., National Research Council, 2004; Senk & Thompson, 2003) because "one cannot say that a curriculum is or is not associated with a learning outcome unless one can be reasonably certain that it was implemented as intended by the curriculum developers" (Stein, Remillard, & Smith, 2007, p. 337). Curriculum researchers have used a variety of methods for documenting curricular implementation, including table-of-content implementation records, teacher and student textbook use diaries, teacher and student interviews, and classroom observations. These methods record teacher and student beliefs, extent of content coverage, in-class and out-of-class textbook use, and classroom participation structures, but do little to compare the mathematics presented in the written curriculum (the student and teacher textbooks) and the way in which this mathematics plays out in the enacted curriculum (that which happens in classrooms).

In order to compare the mathematical features in the written and enacted curricula, I utilized Sfard's Commognition framework (most recently and fully described in *Thinking as Communicating: Human Development, the Growth of discourses, and Mathematizing* published in 2008). That is, I compared the mathematical words, visual mediators, endorsed narratives, and mathematical routines in the written and enacted curricula. Each of these mathematical features provided a different perspective on the mathematics present in the curricula. The written curriculum in this study was represented by Investigation 3 (*Multiplying with Fractions)* included in *Bits and Pieces II: Using Fraction Operations* in *Connected Mathematics 2* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). Videotapes of this same Investigation recorded in a sixth grade classroom in a small, rural town in the Midwest were used as the enacted curricula for this case.

The study revealed many similarities and differences between the written and enacted curricula; however, most prominent were the findings regarding objectification in the curricula. Sfard defines objectification as "a process in which a noun begins to be used as if it signifies an extradiscursive, self-sustained entity (object), independent of human agency" (Sfard, 2008, p. 412). She proposes that objectifying is an important process for students' discursive development and that it serves them particularly well in the study of advanced mathematics. Both objectification itself and the opportunities present for objectification were more prevalent in the written curriculum than in the enacted curriculum.

To my wonderful family and friends – how lucky I am

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TABLE OF CONTENTS

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Go
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CHAPTER
MATHEM
DIRECT
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CHAPTE
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LIST OF TABLES

LIST OF FIGURES

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W is used
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Examples f
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Examples i
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NOTES ON THE TEXT

"Unit," "Investigation," "Problem," and "Question" with upper-case first letters reference the particular components of *Connected Mathematics*. When these words appear with lower-case first letters, they are being used colloquially.

WC is used to designate "Written Curriculum" in examples
EC is used to designate "Enacted Curriculum" in examples

Examples from the written curriculum are cited using the following format (TG, p. 23) or (SG, p. 32) where TG indicates "Teacher's Guide" and SG indicates "Student's Guide"

Examples from the enacted curriculum are cited by day (e.g., "Day 4"). In addition, the following abbreviations are used to indicate the speaker:

T	Teacher
S	Student (used only in cases in which one identified student speaks)
S1, S2, etc.	Student (used in cases in which at least two identified students speak)
S?	Student (used in cases in which the speaking student can not be identified because he/she does not appear on the videotape and the voice is not recognizable)
Ss	Students (used in cases in which multiple students are speaking, at least one of which does not appear on the videotape and the voices are not recognizable)

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CHAPTER 1: INTRODUCTION

The mathematics education research community continues to grapple with the elusive question: Which mathematics curricula are the most effective for promoting student learning? While this question has long been of interest nationally, debates around and intensive study of this question were most recently prompted by a series of three events beginning in the 1980s: (1) the release of national reports (e.g., *A Nation at Risk* [National Commission for Excellence in Education, 1983]) which claimed that students in the United States were not adequately prepared to compete with their counterparts around the world in mathematics; (2) the publication of *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995) by the National Council of Teachers of Mathematics (NCTM); and (3) the National Science Foundation's (NSF) funding of the development of 15 elementary, middle, and high school mathematics curricula which were stipulated to use NCTM's standards documents as the foundation for their approach to mathematics teaching and learning in the early 1990s (hereafter referred to as "*standards*-based curricula").

The standards documents and the *standards*-based curricula promoted a new vision of what it meant to teach and learn mathematics. That is, NCTM advocated for a more participatory form of mathematics education that valued a wider range of mathematical skills, including not only traditional computational procedures but also the communication, problem solving, and reasoning required to engage collaboratively with other students on rich mathematical tasks. These curricula and the debates surrounding them have stimulated research studies from professional organizations (e.g., American resolution

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Association for the Advancement of Science, 2000) as well as individual researchers (e.g., Webb & Dowling, 1997; Wu, 2000). One of the most significant challenges in the attempt to study these new curricula has been measuring the "quality" and "quantity" of their implementation in classrooms. The question is whether and to what extent the materials are actually being used in a particular classroom that is part of a study. Are the materials to some degree faithful to the intent of the authors of the particular curriculum in question? Can a curriculum be assigned credit or blame for student achievement without a careful investigation of the ways in which the curriculum is enacted in the classroom?

This phenomenon prompted the call for the study of the implementation of *standards*-based curricula as a precursor to making claims regarding the effectiveness of these curricula (e.g., Senk & Thompson, 2003; National Research Council, 2004). In other words, learning outcomes of students in classrooms in which *standards*-based curricula are "used" cannot be attributed to the particular curriculum in use unless some degree of fidelity to the curriculum has been established. Studies of *standards*-based curricula have documented their implementation in many ways, including stating that "the teacher used the curriculum," interviewing teachers regarding their use of curricular materials, asking teachers to keep curriculum logs (i.e., recording when they use the curriculum), and conducting classroom observations detailing features of the environment called for in NCTM's standards documents (e.g., cooperative learning, questioning techniques) (e.g., Post et al., 2008; Tarr et al., 2008). These methods provide important information; however, they do not examine the details of the mathematics as it is presented in *standards*-based curricula or classrooms using these curricula. This study,

which compares the mathematical features in the written and enacted curricula, is a step in this direction.

Given the discursive nature of both the written and enacted curricula (i.e., both involve "talk" in different modes), discourse analysis in the form of the Commognition framework described in *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing* (Sfard, 2008) is utilized to investigate the relationship between the written and the enacted curricula.[1] Commognition requires a careful examination of key mathematical features, including (1) use of *mathematical words*, (2) use of uniquely mathematical *visual mediators* in the form of symbolic artifacts that have been created specifically for the purpose of communicating about quantities, (3) special *mathematical routines* with which the participants implement well-defined types of tasks, and (4) *endorsed narratives*, such as definitions, postulates, and theorems produced throughout the discursive activity. The Commognition framework provides a new lens through which to compare the relationship between the mathematical features of the written and enacted curricula.

In Chapter 2, I position this project among studies addressing curriculum in mathematics education in general and studies that examine the relationship between the written and enacted curricula in particular. In addition, I describe the Commognition framework in detail including a description of each of its four key mathematical features and important phenomena highlighted in this framework, including objectification.

[1] Although this framework has been described and used in other publications (e.g., Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005; Sfard, 2007). Sfard (2008) will be used as the primary reference throughout this study because it represents Sfard's most elaborated rationale for and detailed description of Commognition.

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Chapter 3 describes the methodology of the study, including depictions of the written and enacted curricula, fraction multiplication, and an outline of the design of the study.

Chapters 4-8 contain the results of the analyses conducted in this study along with interpretations of the findings. In Chapter 4, the stated goals of the written curriculum are examined in both the written and enacted curricula in an effort to compare the opportunities associated with each goal in the curricula. The key mathematical features associated with Commognition (i.e., mathematical words, visual mediators, endorsed narratives, and mathematical routines) are compared in the written and enacted curricula in Chapters 5-8 respectively. These analyses (Chapter 4-8) provide a detailed picture of the relationship between the written and enacted curricula using a Commognitive lens.

Finally, Chapter 9 provides a discussion of the findings across the five analyses highlighting the threads that run throughout, and Chapter 10 summarizes the study's contributions to the field, its limitations, and suggestions for future action.

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CHAPTER 2: THEORETICAL BACKGROUND

Review of Relevant Literature

The Development of Standards-based Curricula

During the 1980s, several national reports (e.g., *A Nation at Risk* [National Commission for Excellence in Education, 1983], *Educating Americans for the Twenty-first Century* [National Science Board Commission on Pre-College Education in Mathematics, Science, and Technology, 1983]) and results from the Second International Mathematics Study (SIMS) suggested that students from the United States were lagging behind students from other developed countries in mathematics. These reports indicated that this deficit was a threat, not only to the everyday functioning of U.S. schools and businesses, but to our national security as well. This was certainly not the first mathematics education crisis in the United States. In fact, Fey and Graeber (2003) characterize the direction of curricula and teaching in elementary and secondary school mathematics as having a "predictable rhythm of crisis-reform-reaction episodes" (p. 521). However, it was the "crisis-reform-reaction episode" catalyzed by these reports in the 1980s that is of particular interest here. These reports can be said to have communicated the "crisis" of the episode.

The episode's "reform" was largely taken up by the National Council of Teachers of Mathematics (NCTM) beginning with the publication of *An Agenda for Action: Recommendations for School Mathematics of the 1980s* in 1980 that outlined eight recommendations for improvement in school mathematics, including a call to emphasize problem solving, expand the meaning of "basic skills," and encourage the use of calculators and computers at all grade levels. As both a response to the reports mentioned

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mentioned earlier and in an attempt to expand the recommendations proposed in *An Agenda for Action*, NCTM initiated the development of a set of standards for mathematics education in 1986. A series of landmark publications resulted: *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995).[2]

A new approach to teaching school mathematics was put forth in these standards publications. This approach challenged the transmission model of the learning-teaching process which was thought to be prevalent at the time in mathematics education and suggested a more participatory form of learning. The standards emphasized that "the discourse of a classroom - the ways of representing, thinking, talking, agreeing and disagreeing - is central to what students learn about mathematics as a domain of human inquiry with characteristic ways of knowing" (NCTM, 1991, p. 34). This new conceptualization of mathematics education offered multiple discursive entry points for students which addressed a key goal of the standards: to make mathematics meaningful for *all* students. That is, mathematics education was expanded to value more than the speed and accuracy with which a student could state basic arithmetic facts or manipulate algebraic symbols. This new approach also valued mathematical skills such as communication, reasoning, and problem solving.

The national reports of the 1980s and NCTM's publication of *Curriculum and Evaluation Standards for School Mathematics* (1989) prompted the National Science Foundation (NSF) to fund the development of 15 mathematics curricula with the goal of

[2] A more recent standards document, *Principles and Standards for School Mathematics*, was published by NCTM in 2000. This newer version advocates for many of the same tenets found in the original standards.

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alignment with NCTM's standards documents.[3] These "*standards*-based" mathematics curricula included development at the elementary, middle, and high school levels. Following the development of these curricula, the problem of proving curricular effectiveness was brought to the fore in mathematics education. This demand for proof of the curricular effectiveness of *standards*-based curricula can be conceptualized as the "reaction" in this crisis-reform-reaction episode.

It is not surprising that the burden of proof for curricular effectiveness seemed to rest on those involved in the development of *standards*-based curricula because it is common for the developers of a new product to have the responsibility of convincing the public that their new product is better than the older, more familiar one.[4] Resources were granted to the authors to carry out small formative and summative evaluations. These evaluations were promising, but not convincing to critics. It became clear that providing the necessary evidence to justify a change to new and significantly different mathematics curricula was not going to be without its challenges (Kilpatrick, 2003; Romberg, 1992). In fact, the very nature of *standards*-based curricula complexifies the process of proving their effectiveness (Stein, Remillard, & Smith, 2007).

The Phases of Curricula

The factors contributing to the complex problem of proving curricular effectiveness of *standards*-based curricula are most easily conveyed through the use of a model of curricular phases. Figure 1 provides one such model.[5] Clearly, this

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3 NSF has a history of funding innovative mathematics and science curricula.
4 To clarify, "*standards*-based" mathematics curricula are those funded by NSF following the publication of NCTM's 1989 standards and "traditional" curricula are those with editions that were on the market prior to the development of NCTM's 1989 standards. These categorizations are used in Stein, Remillard, and Smith (2007) except that they use "conventional" instead of "traditional."
5 It should be noted that there are many versions of this model; this was used because it seemed to best suit the purposes here.

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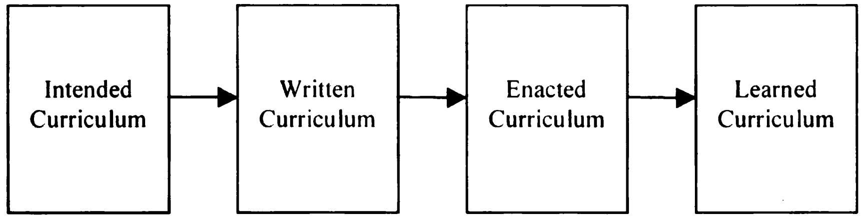

Figure 1. Model of four curricular phases.

representation is a simplified version of the phases of curricula. However, it provides ways to talk about curriculum as it is translated from the minds of the authors (the intended curriculum) to that which is learned by students and teachers (the learned curriculum).[6] Here, the written curriculum is represented by the textbooks (both teacher and student editions) and any ancillary materials provided by the publisher. The enacted curriculum is the curriculum as it plays out in classrooms with teachers and students.[7]

This model can be used to frame the problem of proving curricular effectiveness of *standards*-based curricula. The goal of the author(s) of any curriculum, regardless of whether it is traditional, *standards*-based, or a hybrid of these approaches, is that the intended curriculum (i.e., their vision) becomes the learned curriculum.[8] However, the intended curriculum, in most cases, is only accessible to the users of the curriculum by proxy of the written curriculum. Therefore, the intended curriculum will not be discussed further here. This decision, although it makes the model even less representative of the complex curricular process, leaves a version of the model focused on the key elements for this discussion. Figure 2 provides a representation of this revised model.

[6] Often the learned curriculum is reserved for student learning; however, given the breadth of research devoted to teacher learning through curriculum use it seems appropriate to include both students and teachers in this definition.

[7] Often the enacted curriculum is used to describe the work of teachers; however, this study conceptualizes the enacted curriculum as being co-constructed by teachers and students in classrooms.

[8] I have included the notion of a hybrid of the two approaches because since the publication of the standards in 1989 there have been efforts on the part of some curriculum developers to balance these approaches in their curricula.

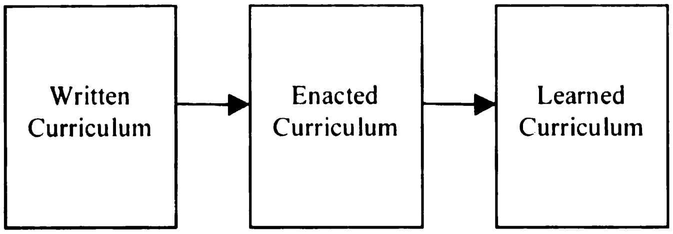

Figure 2. Model of three curricular phases.

What is it that is represented in the model that makes proving curricular effectiveness of *standards*-based curricula so complex? The authors' goal in this new revised framework would be that the written curriculum becomes the learned curriculum. This leaves the enacted curriculum, along with the transitions between these forms of curricula (represented by the arrows in the model which are arguably at least bi-directional) as the paths or the obstacles to the authors' goal. The complex nature of these three components (i.e., the transition between the written and the enacted curricula, the enacted curriculum itself, and the transition between the enacted and the learned curricula) will be described briefly here. Two arguments will be made: (1) These components are complex in nature regardless of which curricula are studied and (2) The nature of *standards*-based curricula makes these components even more complex.

The Complex Path between the Written and the Learned Curricula

First, the transition between the written and enacted curricula is most often addressed in the literature as the ways in which teachers make sense of the written curriculum. That is, the teacher is seen as the agent who facilitates this transition. Remillard (2005), in a review of the literature addressing teachers' use of curriculum, categorized those studies according to the way in which the researchers framed teachers' curricular use: (1) Curriculum use as following or subverting the text, (2) Curriculum use as drawing on the text, (3) Curriculum use as interpretation of the text, and (4) urriculu

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Curriculum use as participation with the text. The multiple perspectives on the ways in which teachers' use of curriculum is framed in research indicate the complex nature of this transition in the process. In addition, if the notion of this meaning making is extended to students because they also produce their own meaning from the written instructional materials, the transition becomes even more complex. This transition is complex regardless of the type of curriculum involved; however, Remillard (2005) stresses the increased complexity with *standards*-based curricula given that the new written materials are "foreign in form" to many teachers and students (p. 212).

Second, the enacted curriculum, defined here as the curriculum co-constructed by teachers and students in classrooms, is equally complex in nature. Lampert (2001) developed an "elaborated model of teaching practice" in which she began with the interaction between the teacher, an individual student, and the mathematical content, and expanded that three-dimensional model over a class of 20-30 students. She added the teacher's interaction with small groups and the whole class, student-student interactions, and located all of these interactions within mathematical content. In addition, she pointed out how both the temporal and social characteristics inherent in the classroom context further increase the complexity because the relationships between students, teachers, and content "have a history and project into future encounters" (p. 425). This "elaborated model" provides a compelling representation of the complex nature of the enacted curriculum. In *standards*-based enacted curricula, this model is potentially more elaborate given the different roles of both teachers and students proposed in NCTM's standards. Remillard (2005) substantiates this saying, "They [*standards*-based curricula] require the teacher to play a substantially different role in the mathematics classroom

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than has been typical among teachers in the United States" (p. 211). Similarly, the role of students in classrooms using *standards*-based curricula requires a shift in the students' role from that of passive recipient to active participant (NCTM, 1989, 2000). Sfard (2007) reinforces this message regarding the complexity of *standards*-based enacted curricula, saying "Our helplessness as researchers is aggravated by the fact that the current reform, promoting the pedagogy of talking classrooms and of communities of inquiry, makes learning processes not only more visible, but also much more intricate and messy" (p. 568).

Finally, the transition between the enacted curriculum and the learned curriculum is knotty as well, particularly because this project frames both of these phases of curricula in terms of teachers and students alike. That is, teachers and students co-construct the enacted curriculum, and both teachers and students are learners in the curricular process. The ways in which the transition from enacted to learned curriculum is conceptualized depends largely upon the learning theory adopted by the individual responsible for the conceptualization. Sfard (1998) problematizes the dichotomous relationship often assigned to the acquisitionist and participationist theories of learning and discusses the affordances and limitations of each theory as well as the dangers in developing too great a devotion to one of them. This leads to the conclusion that the variety of available learning theories and the relationships among these theories contributes to the complex nature of this final transition in the model in Figure 2.

In summary, the complex nature of these three components of the curricular model contributes to the difficulty of evaluating curricular effectiveness of *standards*-based curricula. Although all three of these components are of great importance, this

project focuses primarily on the first two (i.e., the transition from the written to the enacted curriculum and the enacted curriculum itself). The study of the final transition (i.e., the transition from the enacted curriculum to the learned curriculum) is particularly tenuous currently because the development of outcome measures for the foci of *standards*-based curricula (e.g., mathematical reasoning, communication) are not well developed in the field.

The Importance of Implementation in Establishing Curricular Effectiveness

Curricular implementation, as it will be used here, combines the two components addressed previously.[9] It includes both the ways in which teachers and students make sense of the written curriculum as well as the ways in which they enact these curricula in mathematics classrooms. An alternate conceptualization is that implementation "turns on" the relationship between the written and enacted curricula. The importance of the documentation of this implementation in curricular studies is well documented in recent literature.

Senk and Thompson (2003) edited a collection of studies that addressed the effectiveness of *standards*-based curricula. They were interested in the simple question, "How well do these *standards*-based instructional materials work?" (p. x). Early in the volume, the editors state the requirement that the researchers address the "fidelity of treatment" which they define as the "extent to which a curriculum is used in the way it was intended" (p. 20). That is, in order to associate achievement with a particular curriculum there must be a positive relationship between the intended curriculum (by

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[9] I don't actually use "implementation" in this study; rather I use "the relationship between the written and enacted curricula." However, I use "implementation" in the literature review because it is the term most often used in the field. Again, my choice not to use "implementation" comes from the fact that it has been used most often to refer to the teacher's agency (teacher as implementer), whereas I am conceptualizing both teachers and students as agents in curricular implementation in the classroom.

proxy of the written curriculum) and the enacted curriculum. The book contains sections committed to evaluations of *standards*-based elementary, middle, and high school curriculum projects. The need to document curricular implementation is echoed in each of these sections as well as in the volume's final conclusion.

First, in his summary of the elementary curriculum projects, Putnam (2003) states that "It is important to attend to such implementation issues, because without information about how curricula are being implemented, it is difficult to know what is being compared in student outcome studies" (p. 170). Chappell (2003), in her reaction to the middle school curriculum projects in the same volume reiterated this concern saying, "One shortcoming pertains to monitoring how the curriculum is actually implemented in the classrooms" (p. 292). Finally, Swafford (2003), in her reflections on the high school curriculum projects added that, "Future research on the impact under routine implementation also has to take into account the degree or faithfulness of the implementation and the relationship of the degree to which the materials are implemented to student outcomes" (p. 467). Kilpatrick (2003), in the final conclusion of this volume, summarized the findings of the evaluations:

> Students studying from *standards*-based curricula do as well as students studying from traditional curricula on standardized mathematics tests and other measures of traditional content. They score higher than those who have studied from traditional curricula on tests of newer content and processes highlighted in the Standards document. (p. 472)

He called the evidence "promising and substantial," but went on to point to the difficulties in "evaluating something as complex as curriculum" (p. 472). He emphasized that "two classrooms in which the same curriculum is supposedly being 'implemented' may look very different" (p. 473).

Similar conclusions regarding the importance of considering curricular implementation were drawn in the National Research Council's 2004 report, *On Evaluating Curricular Effectiveness: Judging the Quality of K-12 Mathematics Evaluations*. The committee's charge was twofold:

> First we aim to examine evidence currently available from the evaluation of effectiveness of mathematics curricula. Second, we will suggest ways to improve the evaluation process that will enhance the quality and usefulness of evaluations and help guide curriculum developers and evaluators in conducting better studies. (p. 15)

The major finding reported was that due to the number of and nature of the studies for any one given curriculum it was impossible to determine the effectiveness of individual programs. The report's major contributions to the mathematics education community included a detailed outline of the required evaluations needed to scientifically establish the effectiveness of a curriculum program and three essential components to be addressed in the evaluations. The second of these components is of primary importance here: "Evaluations should present evidence that provides reliable and valid indicators of the extent, quality, and type of the implementation of the materials" (p. 194).

These two documents call for careful consideration of curricular implementation for "one cannot say that a curriculum is or is not associated with a learning outcome unless one can be reasonably certain that it was implemented as intended by the curriculum developers" (Stein, Remillard, & Smith, 2007, p. 337).

Documenting the Implementation of Curricula

The studies described in Senk and Thompson (2003) account for curricular implementation in a variety of ways and to varying degrees. Interestingly, five of the 12 chapters (descriptions of studies of individual curriculum projects) say only that the curriculum was being "used" or that the students were "taking the course" that utilized the curriculum. They went no further in accounting for the implementation of the curriculum in the classrooms.[10] Two of the remaining seven chapters also mentioned that the teachers in the study received training in the use of the curriculum.

The remaining five chapters include a more thorough description of the extent and nature of the implementation. Of these five chapters, three include some measure of the content coverage. That is, they document (through surveys or interviews) the specific chapters which the teacher covered during the course. Four of the five chapters include classroom observations in their data collection methods. The purposes of these observations were "to establish that instruction was aligned with the program being taught" (p. 123) and "to provide a systematic basis for comparisons across classrooms" (p. 363). In the former case, the chapter does not provide a description of what it means to be "aligned" with the program. In the latter case, an observation instrument which provided a rating system was included as an appendix (pp. 372-373). The instrument addressed five major classroom components: (1) Structure, (2) Cooperative Student Learning, (3) Types of Thinking, (4) Gender Interaction, and (5) Other Striking Features.

Similar methods of documenting curricular implementation have continued to be used more recently. For example, Tarr, Chavez, Reys, and Reys (2006), in their study of

[10] It is possible that the original studies addressed implementation more thoroughly and that this information was not included in the book due to space constraints.

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the impact of middle school teachers on students' opportunities to learn, reported use of teacher surveys (asking questions about education, teaching experience, beliefs about teaching and learning mathematics, and practices regarding textbooks), textbook-use diaries (teachers' reports of their use of the textbooks and other curricular materials), classroom observations (documenting curricular use), teacher interviews (focusing on what and how to teach mathematics and the extent to which their textbook influenced these decisions), and table-of-content implementation records (recording the amount of the textbook "covered" during the school year). Using similar methodologies, Tarr et al., (2008) investigated textbook fidelity and its relationship to student outcomes by documenting content coverage and the amount of time spent on specific topics. In addition, they used classroom observations to describe the extent to which the classroom activities reflected a "standards-based learning environment" (SBLE).

Although these methods for documenting curricular implementation provide useful information, one is left wondering about the mathematics taking place in the classrooms and the alignment between the mathematics highlighted in the written curriculum (both student and teacher textbooks) and the mathematics enacted in the classrooms. In this study, I utilize a discursive framework to document the curricular implementation by carefully exploring the mathematical features of both the written and enacted curricula and describing the relationship between them.

The Discursive Nature of the Written and Enacted Curricula

Discourse has been defined in many ways, ranging from notions as simple as "language" or "talk" to much broader conceptualizations. For example, Fairclough (1992) wrote, "Discourse is, for me, more than just language use: it is language use, rather sp

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whether speech or writing, seen as a type of social practice" (p. 28). Similarly, Jaworski and Coupland (1999) propose an expanded notion of discourse as, "language use relative to social, political and cultural formations – it is language reflecting social order but also language shaping social order, and shaping individuals' interaction with society" (p. 3). For each of these definitions, the method of discourse analysis takes a different form. For example, Fairclough, based on his broad conceptualization of discourse developed a three-dimensional framework for studying discourse which mapped three separate forms of analysis onto one another. These forms of analysis included: (1) analysis of language texts, (2) analysis of discourse practice, and (3) analysis of discursive events as instances of social practice. Using this framework, he considers three levels of "language in use": details of the language itself, language patterns, and the implications of language for broader social contexts.

Regardless of whether the simple definition of discourse as "talk" or a more expanded notion such as those of Fairclough or Jaworski and Coupland is adopted, both the written and enacted curricula can be conceptualized as discursive constructs. The written curriculum contains "talk" in the form of written text as well as messages regarding social practices. For example, if a textbook always uses the pronoun *he* and never *she*, this may carry a message to the reader regarding who can do mathematics. Similarly, the enacted curriculum (that which occurs in classrooms) certainly contains "talk" as well as carries messages regarding social order. An example of this may come from something as simple as which students are chosen to answer the most complex problems. Additionally, the enacted curriculum can shape an individual's interaction

with society either by reinforcing the messages that students are receiving from outside the classroom or by providing alternative messages for them to carry into the world.

Treating the enactment of mathematics curriculum as discursive is certainly not without precedent. Halliday (1978) introduced the notion of the "mathematical register" to highlight the ways in which some new and some everyday words are used in particular ways in mathematical discourse. It is through the use of these words in conventional ways that one is said to be participating in mathematical discourse. In related work, Pimm (1987) proposed that mathematics might be construed as a language. He stated that "part of learning mathematics is learning to speak like a mathematician, that is, acquiring control over the mathematics register" (p. 76). He suggested three purposes for increasing student communication in mathematics classrooms:

> Within the educational context of a mathematics classroom there are two main reasons for pupils talking, namely talking to communicate with others and talking for themselves. There is also a further justification, namely for the teacher to gain access to and insight into the ways of thinking of the pupils. (p. 23)

In addition, Pierre van Hiele, in his model of geometric thinking proposed in 1957, posited that each level of geometric thinking has its own language and symbols (i.e., its own discourse). For example, a "square" may or may not also be a "rectangle" depending on a students' level of geometric thinking. Finally, although the research of Lemke (1990) was conducted in the context of science education, he suggests that it is applicable to all scientific and technical subjects, and arguably therefore to mathematics. He analyzed science classroom discourse and identified discursive patterns used to

communicate scientific content. Halliday, Pimm, and van Hiele all talk about the difficulties inherent in mathematical communication (and Lemke the difficulties inherent in communication in science classrooms) as justification for the need for its careful study.

Similarly, one need not look far in order to access examples in which mathematics curricula in written form has been treated as a discursive construct. A recent example is found in Herbel-Eisenmann's 2007 article, *From Intended Curriculum to Written Curriculum: Examining the "Voice" of a Mathematics Textbook.* Here, she investigated the roles of the authors and readers of a mathematics curriculum by examining particular language forms and concluded that shifting authority toward students' reasoning is difficult given the predominant tradition of authoritative "voice" in mathematics textbooks. Danielson (2005) also analyzed written curriculum materials as discursive constructs. He compared the communicational characteristics of two middle school mathematics textbooks, highlighting the distinctive features of each and finds, among other things, that tables, graphs, and equations serve one another in one curriculum whereas tables and graphs are in service of equations in the other curriculum. In a final example, Morgan (2005) investigated the notion of definition in secondary mathematics textbooks using systemic functional linguistic tools and found that "the ways in which definitions appear in school mathematics texts vary significantly with the type of mathematics involved and with the age of the intended student-readers" (p. 111).

Therefore, for the purposes of this project, the written and enacted curricula are conceptualized and analyzed as discursive constructs. Sfard's (2008) Commognition framework, with its focus on mathematical features, is utilized to examine the discursive

relationship between the mathematical features of the written and enacted *standards*-based curricula.[11]

Conceptual Framework

Commognition

Commognition (created by merging *communication* and *cognition*) treats communication (interpersonal exchange) and cognition (intrapersonal exchange) as two forms of the same phenomenon. It was developed to emphasize the close relationship between these two processes. In the spirit of Lave and Wenger (1991), commognition recognizes learning as occurring through legitimate peripheral participation. That is, learning is participatory in nature and requires social interaction, particularly with "oldtimers" (i.e., those with full membership) in the discourse community. The framework recognizes a broad definition of discourse (e.g., includes non-verbal gestures); however, it identifies linguistic commognition as the primary source of human uniqueness. When using the commognition frame, careful attention is paid to defining terms; it is seen as a matter of conceptual accountability as well as an important step to creating common language to talk about learning. In that spirit, the Commognition framework proposes the following: (1) *mathematics* is the discourse about mathematical objects and (2) *learning mathematics* is a change in participation in mathematical discourse.[12]

Commognition suggests a detailed analysis (i.e., a search for patterns) of the use of discursive features of mathematics, including (1) Mathematical Words, (2) Visual

[11] In this project, "curricula" is used to refer to the written curriculum and the enacted curriculum.
[12] It is noteworthy that mathematical objects (e.g., numbers, shapes) are discursive objects themselves and therefore part of the discourse.

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Mediators, (3) Endorsed Narratives, and (4) Mathematical Routines.[13] First, the words of interest in an analysis of mathematical discourse are primarily those that signify quantities and shapes (e.g., number, triangle) and those that highlight relationships between these quantities and shapes. Of particular interest are the ways in which the words are used. The Commognition framework proposes four phases in the development of word use, including (1) passive use, (2) routine-driven use, (3) phrase-driven use, and (4) object-driven use in which the word is used as a noun as if it has a life of its own.

Second, visual mediators are artifacts created for the primary purpose of mathematical communication, including but not limited to algebraic symbols, diagrams, and graphical representations. These visual mediators may be conventional or individually designed. They may be drawn on a chalkboard, built from toothpicks, or operated on in the mind. The power of these objects, however, does not reside in the objects themselves. Rather, their power is created as they are used in classrooms and discursively linked to other related mathematical experiences. Sfard (2008) categorizes visual mediators into concrete (e.g., fraction strips, geoboards), iconic (e.g., pictures, graphs), and symbolic (e.g., arithmetic number sentences, algebraic expressions).

Third, narratives include any text, spoken or written, which is framed as a description of objects, of relations between objects or processes with or by objects, and which is subject to endorsement or rejection (i.e., being labeled true or false). Definitions, axioms, theorems, and proofs are commonly endorsed narratives in mathematics. Endorsed narratives hold a special place in mathematics because "the overall goal of mathematizing is to produce narratives that can be endorsed, labeled as

[13] The mathematical features of commognition are described briefly here and more fully in Chapters 5-8 respectively.

true, and become known as mathematical facts" (Sfard, 2008, p. 289). Sfard argues that much of mathematical discourse consists of *recalling*, *constructing*, and *substantiating* narratives. *Recalling* entails bringing a previously-constructed narrative to mind, *constructing* results in a new narrative, and *substantiating* is the process that leads to the decision whether or not to endorse a narrative.

Finally, routines are repetitive characteristics of mathematical discourse. Both the *how* and *when* of routines are important in mathematics. For example, knowing *when* to find a common denominator is every bit as important as knowing *how* to find one. Sfard (2008) argues that the emphasis in school mathematics is too often on the *how* to the virtual exclusion of the *when* of routines. Closely related to the *when* is the *why* of routines. That is, understanding why a routine works is fundamental to assessing a situation in order to decide whether or not the routine is appropriate in a particular context. Mathematical routines are divided into three categories: (1) explorations, (2) deeds, and (3) rituals. *Explorations* are those routines whose goal is the creation of endorsed narratives about mathematical objects. *Rituals* and *deeds* are developmental predecessors to explorations. Sfard (2008) stresses that, "As long as school teaching focuses on the issue of *how* routines should be performed to almost total neglect of the question of *when* this performance would be most appropriate, it is more likely to result in the discourse of rituals than of explorations" (p. 289).

It should be noted that the four categories of mathematical features (mathematical words, visual mediators, endorsed narratives, and mathematical routines) are intricately related to one another. That is, a *visual mediator* (e.g., area model) has particular *words* associated with it (e.g., part, piece, fraction), is used in *routine* ways (e.g., partitioned), rican

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and can be described by a *narrative* (e.g., "$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$"). A detailed comparison of these four mathematical features in the written and enacted curricula provides new ways to talk about curricular implementation that highlight the mathematics in *standards*-based classrooms.

Objectification

One phenomenon that is particularly sensitive to this type of analysis is objectification, which will be discussed here in some detail because it is a major factor in the analysis of these curricula. Objectification is defined as "a process in which a noun begins to be used as if it signifies an extradiscursive, self-sustained entity (object), independent of human agency" (Sfard, 2008, p. 412). The process consists of two closely related sub-processes: reification and alienation. Reification and alienation are defined as follows:

- Reification: Replacement of talk about processes with talk about objects
- Alienation: Using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings

These two processes, when taken together indicate that what was previously something to "do" (*processual*) becomes a discursive "object" (*structural*). Mathematical and scientific discourses are particularly dependent upon objectification for their successful evolution.

The case of whole numbers illustrates this importance. Children first approach whole numbers processually; whole numbers are the result of counting sets of objects. If a young child is asked how many marbles are in a bag, he will probably count them and his answer will be the last number he states in his counting. Later, he may use numbers

as adjectives; he may state that he has "four" marbles. It becomes more probable that he has "objectified" whole numbers when he responds "four" to the same question. In this case, it is possible that he is using "four" as a noun, an object in its own right.[14] This process is important because it is only once a number is used as a noun (i.e., is objectified) that operating on it outside of contextual situations makes sense. Such objectification facilitates an individual's mathematical communication, and therefore, their learning of mathematics as defined here.

It is not easy to detect objectification in mathematics, but several discursive clues provide evidence. First is the use of "is" instead of "are" (i.e., a singular verb instead of a plural verb). For example, "four is greater than three" would indicate that "four" is being used as the noun (and therefore objectified). In contrast, "four are more than three" may indicate that the child is still imagining four objects ("four as an adjective"). In this case it is likely that the number has not been *encapsulated.* Encapsulation is a process associated with objectification in which objects previously seen as separate entities are encapsulated into one object. Here, what used to be four objects (i.e., marbles) for the child are now encapsulated into a single object, the number four. Again, this is evidenced by a change in verb use from plural to singular. Objectification was investigated in association with each of the four mathematical features in this study.

A Step Forward

This investigation does not take the position that *standards*-based curricula are more effective than traditional curricula. Rather, a *standards*-based curriculum will be used in the study because as noted earlier, the construct of curricular implementation is

[14] It is important to note here that this use of "object" is not Platonic in nature. That is, the implication is not that the child is discovering a previously existing object. Rather that the student speaks about the number as if it is an object. Therefore, mathematical objects are theoretical constructs and nothing more.

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particularly problematic for researchers who seek to study the effectiveness of these curricula. The question of which curricula provide the best learning opportunities for students remains unanswered, although many in the mathematics education community and beyond have strong feelings in one direction or the other (i.e., *standards*-based curricula or traditional curricula). This study provides a step in the process of answering that question by examining a new approach to investigate curricular implementation using a framework that aligns with the theoretical assumptions of the *standards*-based curricula themselves; that is, mathematics learning is participatory and therefore discursive in nature. This framework is promising as a means to study the implementation of *standards*-based curricula because its foundations are consistent with that of NCTM's standards (1989, 2000). It allows us to move beyond the structural features of classrooms and the content coverage described in teacher surveys used traditionally to describe curricular implementation to using mathematically discursive features (i.e., mathematical words, visual mediators, endorsed narratives, and mathematical routines) of both the written and enacted curricula to provide a detailed look at the ways in which the mathematics is addressed both in the written text materials and in the classroom.

This leads to the question guiding this study: What does an investigation of the key features of mathematical discourse, using the Commognition framework, in the written and enacted curricula reveal?

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CHAPTER 3: METHODOLOGY

The purpose of this project is to explore discourse analysis as a tool to describe the relationship between written and enacted *standards*-based mathematics curricula. To this end, I made use of two primary data sources, (1) the written version of a *standards*-based mathematics curriculum and (2) an enacted version of the same *standards*-based mathematics curriculum. I describe these sources in greater detail here followed by the methods used to analyze these data sources.

Written Curriculum

The Connected Mathematics Project (hereafter referred to as *Connected Mathematics*) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) was selected as the *standards*-based curriculum in this project because in the late 1990s it was named as one of five "exemplary" mathematics curricula by the U.S. Department of Education and was ranked first in a curricular evaluation of 13 middle school curricula by the American Association for the Advancement of Science (AAAS). The curriculum has also achieved substantial penetration of the middle school textbook market nationally. *Connected Mathematics* provides 24 Units for grades 6-8 (8 units at each grade level).[15] Each Unit is composed of a series of Investigations, each Investigation contains a sequence of Problems, and each Problem consists of a set of Questions.

Investigation 3, *Multiplying with Fractions*, included in *Bits and Pieces II: Using Fraction Operations* (the second of three Units in grade 6 that address fractions) was utilized in this study. This Investigation (pp. 32-47 in the Student's Guide and pp. 59-88

[15] Note that "Unit" here has an upper-case "U" so as not to be confused with the colloquial use of unit (this will appear with a lower case "u"). The same treatment will be given to "Investigation," "Problem," and "Question." That is, when they are used in reference to particular components of the written curriculum, they will be capitalized. When their use is colloquial, they will not be capitalized.

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in the Teacher's Guide) contains a sequence of five Problems and has a "suggested pacing" of 5.5 days.[16] The *Connected Mathematics* instructional model consists of three major components, (1) Launch, (2) Explore, and (3) Summary. The authors propose that each lesson begins with a problem that introduces the topic to the whole class (i.e., "Launch"), followed by time for students to work on a series of problems individually or in small groups (i.e., "Explore"), and concludes with a whole class discussion in which the teacher guides the students to reach the mathematical goals of the problem and connect their new understanding to prior mathematical experiences and knowledge (i.e., "Summary"). In addition, "Applications," "Connections," and "Extensions" (ACE) are included at the end of each Investigation. For the purposes of this study, only the five Problems in the Investigation, Problems 3.1-3.5, were included in the analysis (i.e., ACE was not considered for this analysis).

Enacted Curriculum

I utilized videotapes of this classroom enactment of *Investigation 3* in a sixth grade classroom recorded on five consecutive days in October 2006. This particular class, consisting of 14 girls and 9 boys, is heterogeneous in mathematical ability (i.e., the students are not tracked).[17] It is located in a middle school (grades 6-8) in a small rural town (population 3,800) in the Midwest approximately 30 miles from the state capitol. The district serves nearly 2,000 students, of whom 97% are Caucasian and 15% receive free or reduced lunch. In 2005, the school made Adequate Yearly Progress (AYP) and 79% of the students were proficient in mathematics compared to 69% in the state. The

[16] The last 0.5 day in the written curriculum is designated for a reflection on the work completed in the Investigation.

[17] I will use present tense verbs in reference to both the written and enacted curricula throughout this investigation as I am conceptualizing them both as texts with eternal form.

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teacher of this particular class is a veteran *Connected Mathematics* teacher. She has taught in a middle school for 20 years, the last 13 of those using *Connected Mathematics* at this particular middle school. She has attended and conducted professional development for *Connected Mathematics* and verbally endorses the curriculum. She represents a good starting point for this work due to her familiarity with the curriculum. That is, I expected to find a relationship between the written curriculum and the enacted curriculum in her classroom. This provided an opportunity to focus on the description of this relationship.

In this classroom, the *Connected Mathematics* instructional model varies depending on the day. The sequence of the components, Launch-Explore-Summary, is consistent; however, some variation exists in the time allotted to each component. For example, Day 4 consists only of a Launch-Explore followed on Day 5 by an Explore-Summary. The Explore on Day 5 is a continuation of the Explore on Day 4. In contrast, Day 3 contains two Launch-Explore-Summary sequences. Day 1 is the only day that contains one Launch-Explore-Summary sequence. The written curriculum recommends that the Launches and Summaries are whole class discussions and the Explorations are carried out either individually or in pairs or small groups. The teacher leads the Launch and students answer posed questions, working at the whiteboard upon request. During exploration, the teacher walks around interacting with individuals and groups. Finally, students present their work on posters or overhead transparences for teacher-facilitated whole class discussion during the Summary.

The analysis of the enacted curriculum required transcribing the spoken language from the classroom into written language for the purposes of comparison. The goal in produ

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producing these transcriptions was to represent the classroom discourse as completely and accurately as possible. Within the sections of transcripts included in the analysis chapters, I used "T" to indicate that the teacher is speaking and "S" to indicate that a student is speaking. "S1", "S2," "S3," etc. are used to designate specific students when more than one identified student has a turn in a particular conversation. Each section of transcript is treated separately (i.e., "S1" may represent different students in different sections of transcript). "S?" is used in situations in which the speaking student is off-camera and therefore unidentifiable. Finally, "Ss" is used when more than one student is speaking simultaneously. These are often situations in which the speaking students are off-camera as well. In situations in which the teacher or a student refers to another student by name, a pseudonym is used.

In the same way that the union of the discourse of the Teacher's Guide and the Student's Guide are considered the "written curriculum" for the purposes of this study, the discourse of the classroom (i.e., the union of the words and actions of the teacher and the students) is considered the "enacted curriculum" for the purposes of this study. Sfard (2008) supports this, saying "Discursive rules of mathematics classrooms, rather than being implicitly dictated by the teacher through her discursive actions, are an evolving product of teacher's and students' collaborative efforts" (p. 262).

Fraction Multiplication

I selected fractions as the topic for study because it has traditionally been a difficult topic to teach and learn. Both the first and second handbooks of research on mathematics teaching and learning (published in 1992 and 2007 respectively) support this statement, encouraging research addressing rational numbers. In the first handbook, Betw

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Behr, Harel, Post, and Lesh (1992) state that, "There is a great deal of agreement that learning rational number concepts remains a serious obstacle in the mathematical development of children" (p. 296). Fifteen years later, in the *Second Handbook of Research on Mathematics Teaching and Learning*, Lamon (2007) makes an even stronger statement:

> Of all the topics in the school curriculum, fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites. (p. 629)

She goes on to comment on the state of research addressing this topic:

> In the last decade, researchers have made little progress in unraveling the complexities of teaching and learning these topics [fractions, ratios, and proportions]. Worse yet, the number of scholars pursuing long-term research agendas in the field of rational numbers is disproportionate to the mathematical richness of the domain. (p. 629)

These statements regarding the problematic nature of the domain of fractions combined with research that emphasizes the difficulties associated with fraction operations (e.g., Erlwanger, 1973; Graeber, 1993; Ma, 1999) served to justify the selection of fraction multiplication as an important topic for study.

The use of "fraction" in this study was a conscious choice; however, I could also have argued for the use of "rational number." This distinction is a contentious one with the possibility of unfavorable consequences for mathematics students. A rational number is a n

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is a number that can be expressed as a ratio in the form of $\frac{a}{b}$ where a and b are integers and b does not equal zero, whereas a given fraction is one of infinitely many possible representations of a rational number. "Fraction," perhaps because of its colloquial use as a part, is often used in school mathematics when speaking about the number represented by the fraction. This discursive confusion between mathematical discourse communities may contribute to the difficulty of the transition to rational numbers later in the school mathematics curriculum. I chose to use "fraction" because it is the word used both in the written and enacted curricula. That is, "rational number" was not present in either curriculum on the five days under investigation in this study.

Regardless of which word is used, objectification of rational numbers (called "fractions" throughout this study) is important. If we extend the earlier argument regarding the objectification of whole numbers to include fractions, several of the same principles apply. First, objectification of fractions involves "encapsulation." For example, to objectify fractions, two whole numbers (and later integers) (e.g., "2" and "3") need to be "encapsulated" into a fraction ("$\frac{2}{3}$"). Second, similar types of evidence can be utilized to detect objectification of fractions. If a student says, "two thirds are…," the chances are that she has not objectified the fraction. That is, for her, "two" is an adjective describing the number of "thirds." However, if she says, "two thirds is…," then "two thirds" (actually "two-thirds" in this case) has been objectified because it is being treated as a singular noun (evidenced by a corresponding singular verb). Often, it is only after such objectification has occurred that operating on fractions as numbers makes sense to students. The objectification of fractions is important for the study of advanced mathe

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mathematics because as students are expected to operate within other number systems (e.g., rational numbers, integers, real numbers), the objects representing numbers become more complex (e.g., "$\sqrt{-3}$"). In later years, students will be expected to use "$\sqrt{-3}$," not as something to "do," but rather as something that "is" (i.e., a number in its own right). In sum, the objectification of fractions is but one such objectification expected in students' ever expanding domain of numbers.

The objectification of fractions is related to the various interpretations (or constructs or personalities) of fractions described in the literature (e.g., Kieran, 1976, 1980, 1988; Behr, Lesh, Post, and Silver, 1983; Nesher, 1985; Ohlsson, 1988). Lamon (2007) summarizes these descriptions, saying "When these analyses were reconciled, *measure*, *quotient*, *ratio*, and *operator* were recognized as distinct subconstructs … and there was some disagreement about whether *part-whole* was distinct from the *measure* sub-construct" (p. 630). She goes on to say that, "Whatever the number of subconstructs, the overriding issue was that current instruction, which addresses only one of them, is inadequate" (p. 630). Here, she challenges the overuse of the *part-whole* or *measure* subconstruct in school mathematics. The overuse of the *part-whole* subconstruct is relevant to this study because it may not promote the objectification of fractions.

Design of the Study

In the tradition of case study, I observed one curriculum in one classroom with one teacher. This choice was particularly appropriate for several reasons. First, utilizing commognition to describe the relationship between written and enacted mathematics curricula represents a new application for its use. Therefore, an in-depth analysis of one classroom's mathematical features provides a useful starting point. Second, it is importa

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important to first establish the usefulness of the Commognition framework for this particular purpose before using it to compare cases or to generalize to a broader set of curricula or classrooms. Stake (2005) would classify this study as an "instrumental case study" because "the case is of secondary interest, it plays a supportive role, and it facilitates our understanding of something else" (p. 445). In this case, that "something else" is the implementation of *standards*-based curricula and the usefulness of the Commognition framework to investigate this implementation.

I can imagine many further applications of the Commognition framework: (1) to compare the mathematical features of several curricula (either *standards*-based curricula with traditional curricula or several *standards*-based curricula), (2) to describe the mathematical features in classrooms of teachers with varying amounts of teaching experience, (3) to compare the ways in which the mathematical features of a particular curricula play out in various contexts (e.g., urban, rural) and with particular student populations. However, given the complex nature of both the written and enacted curricula and the novel use of this framework to examine these forms, a case study such as the one described here provides the first step toward these further investigations. Of course the use of a case study methodology will limit my ability to make claims about either the written or enacted curricula as a whole.

My data analysis entails, broadly speaking, applying Sfard's Commognition framework to both the written and enacted curricula, and using the results of these analyses to investigate both the relationship between the two curricula as well as the usefulness of the framework to describe the results. Here, I outline the steps of the analyses in greater detail. First, I compared the treatment of the *Investigation 3* goals stated

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(stated in the written curriculum) in the written and enacted curricula to determine the amount of emphasis placed on each goal in each curriculum (see the description of both the particular methods utilized and the results of this analysis in Chapter 4). Second, I identified the mathematical features in the written curriculum in *Investigation 3* (both the Student's Guide and the Teacher's Guide) and the enacted curriculum during the five days of videotaping using the four categories in the framework: (1) Mathematical words, (2) Visual mediators, (3) Endorsed narratives, and (4) Mathematical routines (see the descriptions of both the particular methods utilized and the results of these analyses in Chapters 5-8 respectively).

The Venn diagram in Figure 3 provides one way to represent the expected results. Mathematical features found in the written curriculum but not found in the enacted curriculum are located in Area A. Conversely, Mathematical features located in Area C are found in the enacted curriculum but not in the written curriculum. Finally, features located in Area B are found in both the written and enacted curricula. In a classroom using a particular curriculum, many features are expected to be located in Area B although it would be surprising to not also find features in Areas A and C. In addition, detailed descriptions of the use of the mathematical features in the written and enacted curricula are included in each analysis.

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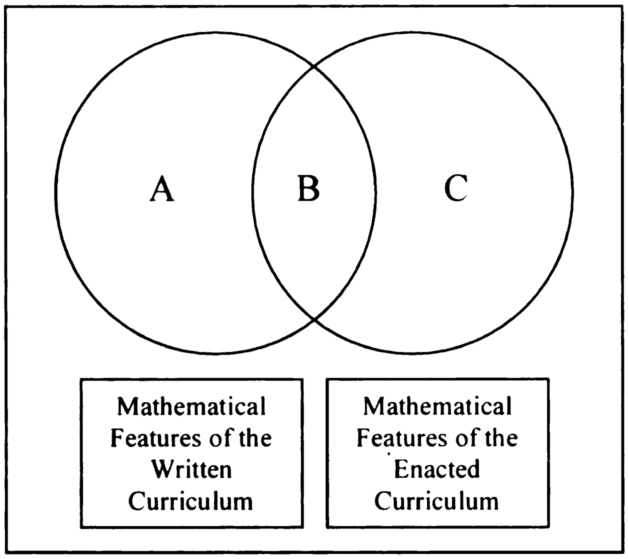

Figure 3. Model for the analysis of the relationship between the written and enacted curricula.

The descriptions of the goals and mathematical features serve two purposes: (1) to describe the relationship between the written and enacted curricula, and therefore curricular implementation in this particular case, and (2) to describe the use of the Commognition framework as a way to investigate this relationship.

The comparison of written and enacted forms of curricula brings with it inherent challenges since these two forms of curriculum differ considerably in nature. For example, the written curriculum is in written mode while the enacted curriculum contains both written (e.g., posters) and spoken modes (e.g., classroom dialogue). At times, I made specific analytic decisions in an attempt to address these differences; these are outlined in the chapters describing particular analyses. In addition, I conjecture several times in the analyses when it seems that the difference in curricular form impacts the results. This said, it is impossible to know precisely to what extent the difference in curricular form has influenced the findings of the analyses presented here.

Because this case study investigates the relationship between two forms of curricula, written and enacted, the descriptions include comparative discourse such as "similar" and "different." Due to the qualitative nature of this study, I have not determined the statistical significance of these differences. Rather, I provide here some guidelines for my use of comparative discourse. First, in all cases of comparisons of counts of various mathematical features, either the raw numbers and/or the relative frequencies have been provided in the text. Second, I considered less than 10% difference in relative frequency as "similar" and less than 5% difference in relative frequency "very" or "quite" similar. Finally, I used differed "greatly" or "substantially" in cases in which there was at least a 20% difference in relative frequencies.

Throughout this process, I have acted as a Commognitive researcher. This label carries with it important assumptions and responsibilities. First, I acknowledge that research is a form of communication. That is, I communicate my perception of the analysis to the reader. Therefore, my research is as much about how I see the world as it is about the world itself because I bring my own perspective and biases with me to this study.

Second, my principal object of attention in this project is discourse. I am guided by the principle of "utmost verbal fidelity" in my transcriptions while recognizing that "perfect" transcription is an impossible feat. I have used the voices of my participants whenever possible to allow readers the opportunity to draw their own conclusions. Third, I recognize that interpersonal communication is observable, while intrapersonal communication is not, therefore, I do not have access to the whole story since I do not have direct access to individual's intrapersonal communication. Finally, I consciously fluctuated between an insider's and an outsider's point of view, recognizing the benefits and limitations of each and noting the particular difficulties of taking on the eyes of an outsider to mathematical discourse because I have been involved, as a student and/or a teacher, in mathematics education for nearly forty years.

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CHAPTER 4: GOALS OF THE WRITTEN AND ENACTED CURRICULA

The primary goal of this chapter is to describe the relationship between the goals stated in the written curriculum and their enactment in the classroom on the five days analyzed in this study. This study was limited to five days as this is the "suggested pacing" for *Investigation 3: Multiplying Fractions* in the written curriculum. That is, what happens before and after these five days is unknown and not considered in this analysis. Table 1 includes the title of each Problem in *Investigation 3* in the written curriculum along with the goals for each Problem as stated in the written curriculum.[18]

Table 1.

Goals of Problems 3.1-3.5 in "Investigation 3: Multiplying Fractions" (as stated in the Written Curriculum)

Problem	Title	Goal
3.1	*How Much of the Pan Have We Sold? (A Model for Multiplication)*	• Estimate products of fractions • Use models to represent the product of two fractions • Understand that finding a fraction *of* a number means multiplication

[18] "Investigation," "Problem," and "Question" will begin with upper case letters when they refer to components of the written curriculum. They will begin with lower case letters when they are used colloquially.

Tab

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Table 1 (cont'd)

Problem	Title	Goal
3.2	*Finding a Part of a Part (Another Model for Multiplication)*	• Estimate products of fractions • Use models to represent the product of two fractions • Understand that finding a fraction *of* a number means multiplication
3.3	*Modeling More Multiplication Situations*	Estimate products of fractions • Develop and use strategies and models for multiplying combinations of fractions, whole numbers, and mixed numbers to solve problems • Determine when multiplication is an appropriate operation
3.4	*Changing Forms (Multiplication with Mixed Numbers)*	• Explore the relationships between two numbers and their product • Develop and use algorithms for multiplying combinations of fractions, whole numbers, and mixed numbers

Table 1 (cont'd)

Problem	Title	Goal
3.5	*Writing a Multiplication Algorithm*	• Develop and use an efficient algorithm to solve any fraction multiplication problem

Note. The information included in this table was collected from pp. 60, 65, 71, 75, and 81 in the Teacher's Guide for Investigation 3.

Each problem is intended to be completed during a single day; therefore, Problem 3.1 will be associated with Day 1, Problem 3.2 with Day 2, and so on in the written curriculum.[19] Three of the eight goals are included on more than one day (e.g., "Estimate Products of Fractions" is included on Days 1-3). In addition, multiple goals are included in the written curriculum for all but Day 5. Table 2 summarizes the Questions included in the written curriculum and those enacted in the classroom for each day.[20]

Table 2.

Investigation 3 Problems and Questions in the Written and Enacted Curricula

	Written Curriculum		Enacted Curriculum	
Day	Problem	Question(s)	Problem	Question(s)
1	3.1	A - D	3.1	A & B
2	3.2	A - D	3.1	C & D
3	3.3	A - D	3.2	A2, A3, B, C

[19] This is not exactly true as the written curriculum explicitly states that Problem 3.2 may be summarized on the following day; however, the curriculum's pacing allows for one day per problem in this Investigation.
[20] Recall that only the problems completed during class are included in these analyses (i.e., not ACE problems or other problems given for homework).

Table 2 (cont'd)

	Written Curriculum		Enacted Curriculum	
Day	Problem	Question(s)	Problem	Question(s)
4	3.4	A - D	3.3	A – D
5	3.5	A – C	3.3	A – D

Table 2 indicates that three of the five Problems (Problems 3.1-3.3) are enacted in the classroom during the five days included in this analysis. That is, Days 1 and 2 in the classroom are used to complete Problem 3.1 and Days 4 and 5 in the classroom are used to complete Problem 3.3. The only Problem that is enacted in a single day is Problem 3.2 on Day 3 in the classroom. The Questions in the written curriculum for Problems 3.4 and 3.5 are not enacted in the classroom during these five days. In the remainder of this chapter, I compare the experiences associated with each goal in the written and enacted curricula followed by a categorization of the experiences associated with each goal as "similar," displaying a "low level of discrepancy," or "displaying a higher level of discrepancy" in the written and enacted curricula. In each case, I explain how the goal was categorized. Finally, I discuss briefly the possible importance of these differences through the lens of objectification.

Goal 1: "Estimate products of fractions"

Written Curriculum

The goal, "Estimate products of fractions," is included in Problems 3.1, 3.2, and 3.3 (Days 1, 2, and 3) of the written curriculum in *Investigation 3*. In particular, Question

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D in Problem 3.1 (see Example 1) and Question A-1 in Problem 3.2 (see Example 2) explicitly address estimation:

Example 1 (WC).

> **D.** Use estimation to decide if each product is greater than or less than 1. To help, use the "of" interpretation for multiplication. For example, in part (1), think "$\frac{5}{6}$ of $\frac{1}{2}$."
>
> **1.** $\frac{5}{6} \times \frac{1}{2}$ **2.** $\frac{5}{6} \times 1$ **3.** $\frac{5}{6} \times 2$ **4.** $\frac{3}{7} \times 2$
>
> **5.** $\frac{3}{4} \times \frac{3}{4}$ **6.** $\frac{1}{2} \times \frac{9}{3}$ **7.** $\frac{1}{2} \times \frac{10}{7}$ **8.** $\frac{9}{10} \times \frac{10}{7}$

Example 2 (WC).

> **A. 1.** For parts (a)–(d), use estimation to decide if the product is greater than or less than $\frac{1}{2}$.
>
> **a.** $\frac{1}{3} \times \frac{1}{2}$ **b.** $\frac{2}{3} \times \frac{1}{2}$ **c.** $\frac{1}{8} \times \frac{4}{5}$ **d.** $\frac{5}{6} \times \frac{3}{4}$

These Questions instruct students to use estimation to decide if a product of two fractions is greater or less than a given value (e.g., "$\frac{1}{2}$"). In Problem 3.3 (see Example 3), the written curriculum contains two experiences with estimation. First, four non-contextual mixed number multiplication problems are included in the "Getting Ready" (SG, p. 36)[21]:

Example 3 (WC).

> **Getting Ready for Problem 3.3**
>
> Estimate each product to the nearest whole number (1, 2, 3, . . .).
>
> $\frac{1}{2} \times 2\frac{9}{10}$ $1\frac{1}{2} \times 2\frac{9}{10}$ $2\frac{1}{2} \times \frac{4}{7}$ $3\frac{1}{4} \times 2\frac{11}{12}$
>
> Will the actual product be greater than or less than your whole number estimate?

[21] Recall that Problems 3.2, 3.3, and 3.4 begin with a "Getting Ready" problem. Excerpts from the Teacher's Guide (TG) and Student Guide (SG) of Investigation 3 will be cited in the form presented here.

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The expectation in Example 3 is slightly different than in Examples 1 and 2. Instead of being asked to determine if the product will be greater than or less than a particular number, here students are asked to both estimate the product "to the nearest whole number" and decide whether this will be an overestimate or an underestimate. Example 4 presents Questions A-D from Problem 3.3 in which the class is asked to estimate answers to four contextual fraction and mixed number multiplication problems:

Example 4 (WC).

> **For each question:**
>
> - **Estimate the answer.**
> - **Create a model or a diagram to find the exact answer.**
> - **Write a number sentence.**
>
> **A.** **The sixth-graders have a fundraiser. They raise enough money to reach $\frac{7}{8}$ of their goal. Nikki raises $\frac{3}{4}$ of this money. What fraction of the goal does Nikki raise?**
>
> **B.** **A recipe calls for $\frac{2}{3}$ of a 16-ounce bag of chocolate chips. How many ounces are needed?**
>
> **C.** **Mr. Flansburgh buys a $2\frac{1}{2}$-pound wheel of cheese. His family eats $\frac{1}{3}$ of the wheel. How much cheese have thev eaten?**
>
> **D.** **Peter and Erin run the corn harvester for Mr. McGreggor. They harvest about $2\frac{1}{3}$ acres each day. They have only $10\frac{1}{2}$ days to harvest the corn. How many acres of corn can they harvest for Mr. McGreggor?**

The experience with estimation provided in Example 4 is different yet from the previous three examples. In the Questions in Example 4, students are expected to estimate answers to contextual problems including different combinations of fractions, whole numbers, and mixed numbers. (i.e., Question A includes two fractions, Question B

includes a fraction and a whole number, Question C includes a fraction and a mixed number, and Question D includes two mixed numbers).

Enacted Curriculum

Question D from Problem 3.1 (Example 1 on p. 42) is enacted on Day 2. The teacher asks the students to follow the instructions given with the problem, "Use estimation to decide if each product is greater than or less than 1." The second problem of this type (i.e., where students are expected to estimate the magnitude of fraction products), Question A-1 from Problem 3.2 (Example 2 on p. 42), is not completed by the class. Instead, they complete Question A-2 from Problem 3.2 in which they "solve" the problem rather than estimate the answer. In the enactment of Problem 3.3 on Day 4, the class completes the "Getting Ready" (Example 3 on p. 43) which contains, as mentioned previously, four non-contextual mixed number multiplication problems. Much emphasis on estimation is present during the discussion around these four problems. Example 5 provides an excerpt from that discussion.[22] The bolded words indicate language associated with estimation (i.e., "estimate," "round") used by both the teacher and students. More will be said about the use of these words in *Chapter 5: Mathematical Words.*

Example 5 (EC):

(1) *T: Could we just get an* ***estimate*** *[points to* $3\frac{1}{4} \times 2\frac{11}{12}$ *on the overhead]?*

(2) *S: Yeah.*

[22] Recall that "T" is used to indicate that the teacher is speaking. "S" is used to indicate that a student is speaking – numbers will be used to distinguish between students (e.g., S1, S2). The same number is associated with a particular student within an example. "S?" is used when the identity of the student is unknown (i.e., they are out of the camera's view). "Ss" is used when several students in the class answer simultaneously. For emphasis, bolding will be added to particular words in examples throughout the chapter.

(3) *T: Could we figure out an **estimate**? Is there anything I could **round up** a bit to make it easier to work with? Steven, what should we do?*

(4) *S: **Round** the eleven twelfths up to twelve twelfths so the two would be three.*

(5) *T: Would be a three. Would that help us?*

(6) *S: Yeah, yes, six.*

(7) *T: So, what, what, what if I just got rid of this too [points to "$\frac{1}{4}$"]? Could you guys find three groups of three?*

(8) *S: It's nine.*

(9) *T: It's nine? So, would nine be a halfway fair **estimate**?*

(10) *S: Yes.*

Example 5 illustrates a discussion that took place on Day 4 in the enacted curriculum in which estimation was the focus in the context of multiplying mixed numbers.

The second opportunity for estimation in Problem 3.3 involves four contextual Questions (see Example 4) that ask students to first "estimate the answer," then "create a model" and finally to "write a number sentence." Example 6 indicates the way in which the problem is presented to the class:

Example 6 (EC):

*T: "Okay. You need to do all three of those things. So you need to **estimate** what you think the answer should be, **about how much**, you need to create some sort of model or diagram, so that you can show how you solved it, and then when you're done try to write a number sentence. Okay? So you need to **estimate**, model, write a number sentence for A, do all three for B, C and D."*

During the Summary, however, no mention is made of the estimates; only the models and number sentences are discussed.

Given the information presented here, I argue that there are more experiences with estimation present in the written curriculum than in the enacted curriculum in

Investigation 3. I also argue that opportunities for estimating fraction products contribute to the objectification of fractions because estimation has the potential to encourage students to make statements such as "$\frac{5}{6}$ ***is** greater than* $\frac{1}{2}$." As mentioned previously, the use of "is" instead of "are" may indicate that students have encapsulated the "5" and the "6" or the "5" and the "sixth" into a fraction "$\frac{5}{6}$" as an entity in and of itself. Recall that encapsulation involves taking mathematical objects that were previously seen as separate (e.g., "5," "6") and seeing them as together forming a new mathematical object (e.g., "$\frac{5}{6}$"). In addition to encapsulation, which represents a form of reification, the use of "is" also works to fulfill the other requirement for objectification, alienation. Recall that alienation is depersonalization of the mathematics. Here, the use of "is" alienates the mathematics from the action of doing it. That is, "$\frac{5}{6}$ ***is** less than* $\frac{1}{2}$" is stated as a mathematical fact rather than as an action carried out by an individual. Much more will be said about the role of estimation in objectification throughout this analysis.

Goal 2: "Use models to represent the product of two fractions"

Written Curriculum

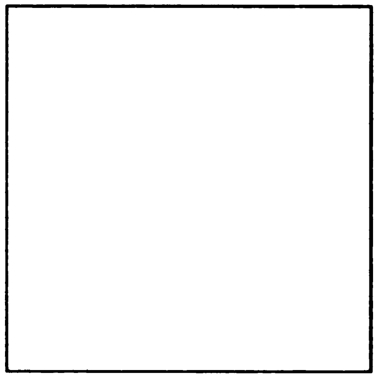
The goal, "Use models to represent the product of two fractions," appears in Problems 3.1 and 3.2 (Days 1 and 2) in the written curriculum. Students are expected to construct iconic mediators to represent the products of two fractions in Problems 3.1 ("Getting Ready," A-1, A-2, B-1, B-2, C-1, C-2, C-3, and C-4) and 3.2 ("Getting Ready," A-2, B-1, B-2, and B-3).[23] Example 7 presents Question A-1 in Problem 3.1 which contains explicit directions regarding the construction of an iconic mediator to represent fraction multiplication:

Example 7 (WC).

A. **Mr. Williams asks to buy $\frac{1}{2}$ of a pan that is $\frac{2}{3}$ full.**

1. **Use a copy of the brownie pan model shown at the right. Draw a picture to show how the brownie pan might look before Mr. Williams buys his brownies.**
2. **Use a different colored pencil to show the part of the brownies that Mr. Williams buys. Note that Mr. Williams buys *a part of a part* of the brownie pan.**

Model of a Brownie Pan



Note the level of detail in the directions in Example 7 regarding the construction of an iconic mediator representing fraction multiplication. This same level of detail is not present in later problems. In Example 8, Question A-2 in Problem 3.2 simply instructs students to "use the brownie-pan model or the number-line model":

Example 8 (WC).

2. Solve parts (a)–(d) above. Use the brownie-pan model or the number-line model.

The brevity of the directions for the use of iconic mediators in Example 8 seems to indicate an expectation that by this point in the Investigation students have developed

[23] In an effort to use language consistently throughout this project, I will use "iconic mediator" (see description on p. 130) in the text instead of "model" because in all cases except one in the enacted curriculum, "model" refers to this type of visual mediator. However, examples lifted directly from the written and enacted curricula will maintain their original wording.

skills for constructing iconic mediators for fraction multiplication without detailed instructions.

Enacted Curriculum

All of the aforementioned Questions that address the construction of iconic mediators in the written curriculum are enacted in the classroom. That is, the students are expected to construct iconic mediators for all of these Questions and therefore are exposed to the intended opportunities for this goal. One difference is that these Questions are completed over the course of three days in the enacted curriculum instead of the two days suggested in the written curriculum. In addition, the amount of time taken to describe the method of iconic mediator construction lessens over the course of the days in ways similar to the written curriculum. For example, if we consider the enactment of the same two Questions mentioned above from the written curriculum (Examples 7 and 8), the difference in time devoted to discussing the construction of the iconic mediators is striking. In the enacted curriculum, Question A-1 in Problem 3.1 (Example 7) results in an eight-minute discussion of iconic mediator construction on Day 1. The discussion presented in Example 9 follows students' previous interactions with iconic mediators that represent fraction multiplication. The brief excerpt is associated with Question A-2 in Problem 3.2.[24]:

Example 9 (EC).

(1) T: Okay. Alright now, you're going to really have to listen 'cause each group of, each couple of tables only had to do one of these, so they're going to be a couple up here that you haven't looked at yet. So be sure that you really listen, okay? Could, the first one was one third of one half. So, whoever did that one, could you come over and talk about that one, please?

[24] The student in this example is referring to an iconic mediator that was constructed on the whiteboard and not saved. Therefore, it is not presented here.

(2) *S1: I filled in half because I knew it was going to be half of a bar, and then I made three parts of that half and I filled in one half, er, one part of that half, and I got one sixth, because if you still had this half, there'd be these three pieces.*

(3) *T: What do you guys think?*

(4) *Ss: Yeah.*

In both the written and enacted curricula, descriptions of the process of constructing iconic mediators decreased in length over the course of the week except in cases where a new type of iconic mediator was being introduced.

Given the fact that all Questions in Problems 3.1 and 3.2 addressing this goal are enacted in the classroom, I argue that similar emphasis is given to representing fraction multiplication in iconic mediators in the written and enacted curricula. In terms of objectification, most of the talk surrounding the construction of iconic mediators in both curricula is personalized. This is evidenced in Example 9 in which the student says "I filled in half, "I made three parts," and "I got one sixth." In addition, the use of models, although potentially serving students well for making sense of fraction multiplication, does not necessarily encourage the reification of fractions. For example, in the statement, "Mr. Williams asks to buy $\frac{1}{2}$ of a pan that is $\frac{2}{3}$ full." (see example 7), $\frac{1}{2}$ and $\frac{2}{3}$ are used as descriptors of "a pan" and "full" rather than as reified objects in their own right. A much more in-depth comparison of models and their use in the written and enacted curricula can be found in *Chapter 6: Visual Mediators.*

Goal 3: "Understand that finding a fraction *of* a number means multiplication"

Written Curriculum

The goal, "Understand that finding a fraction 'of' a number means multiplication," is stated for Problems 3.1 and 3.2 (Days 1 and 2) in the written curriculum. Again, I was interested in whether or not the enacted curriculum emphasizes this goal in ways similar to the expectations in the written curriculum. In Problems 3.1 and 3.2 in the written curriculum, the notion that "of" means multiplication is addressed implicitly and explicitly in several places. Explicit statements like those included in Examples 10-12 are found in the Investigation (specifically in Problems 3.1 and 3.2).

> ***Example 10 (WC).***
> "When you multiply a fraction by a fraction, you are finding 'a part **of** a part'." (SG, p. 33)
>
> ***Example 11 (WC).***
> "Understanding that 'part **of** a part' means 'X' is raised in Question C." (TG, p. 60)
>
> ***Example 12 (WC).***
> "To help, use the **'of'** interpretation for multiplication." (SG, p. 33)

Examples 10-12 include explicit reference to the fact that "of" often indicates the need for fraction multiplication or that fraction multiplication can be thought of as a "part *of* a part." Implicitly, Questions in the written curriculum in which students are writing number sentences using "X" when confronted with contextual problems containing "of" provide opportunities to make sense of the connection between "of" and fraction multiplication. Example 13 contains a Question (SG, p. 35) that exemplifies a case in which "of" in the contextual problem can be interpreted as multiplication to aid in writing the number sentence.

Example 13 (WC).

B. Solve the following problems. Write a number sentence for each.
1. Seth runs $\frac{1}{4}$ of a $\frac{1}{2}$-mile relay race. How far does he run?

The number sentence stated for this Question in the written curriculum is "$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$" (TG, p. 70). The opportunities for considering the relationship between "of" and fraction multiplication in the written curriculum occur in Problems 3.1 ("Getting Ready," A, B, C, D) and 3.2 ("Getting Ready" and B).

Enacted Curriculum

Both explicit and implicit opportunities to examine the relationship between "of" and fraction multiplication are also found in the enacted curriculum. For example, the interaction in Example 14 containing an explicit reference to this goal occurs on Day 1 in the classroom:

Example 14 (EC).

*(1) S: Does the **"of"** mean like times?*

*(2) T: It can mean "times," yes. I worry about saying that **"of"** is every single possible time always multiplication, but yeah.*

Although, the teacher seems hesitant to suggest that "of" always implies multiplication, she answers the student in the affirmative. Also explicit in nature, the statements included in Examples 15 and 16 are made by the teacher on Day 2:

Example 15 (EC).

*T: "Okay. We're going to come back and look at all these [a series of mathematical statements of the form A of B = C, where A, B, and C are proper fractions], 'cause you know what I'm going to do? I'm going to change that word **"of"**[points to "of" in "$\frac{9}{10}$ of $\frac{1}{6} = \frac{9}{60}$ "] to a mathematic symbol. What you guys have done here. I love the way that you're all looking at me. I can tell you're listening. Thank you very much. When I take a part [points to*

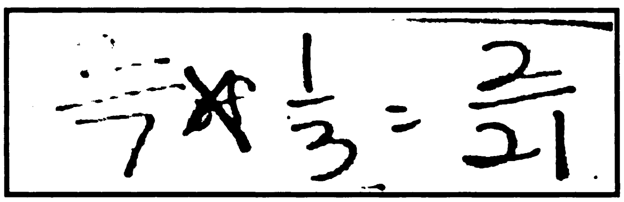
"$\frac{2}{7}$" in "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"] of another part [points to "$\frac{1}{3}$" in "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"] . I can write that as a multiplication problem [writes "X" over "of" in "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"]."

Example 16 (EC).

*T: " Instead of saying nine tenths **of** a sixth, I can also say nine tenths **times** a sixth. Go ahead, Trevor. "*

In Examples 15 and 16, the teacher refers to six problems that included "of" in the number sentence (e.g., "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"). She writes "X" on top of "of" in each example, producing a new number sentence (e.g., "$\frac{2}{7} x \frac{1}{3} = \frac{2}{21}$"). Example 17 illustrates a sample result of this process:

Example 17 (EC – Student Work).



In addition to explicit statements regarding the relationship between "of" and fraction multiplication, students complete the same contextual problems included in the written curriculum in which "of" is embedded in the context and students are expected to write number sentences using "X."

Given the explicit and implicit references to this relationship provided in both curricula and the fact that students enact all of the Questions that provide opportunities to develop this relationship, I conclude that the emphasis on this particular goal, "*Understand that finding a fraction of a number means multiplication*" is similar in the written and enacted curricula. In terms of objectification, this goal provides a combination of personalization and alienation in both the written and enacted curricula. This goal from the written curriculum states that "a fraction of a number means multiplication." This statement is void of human agency which indicates alienation to some extent; however, other statements, such as "When mathematicians write $\frac{1}{2}$ of $\frac{1}{4}$, they mean the operation of multiplication, or $\frac{1}{2} \times \frac{1}{4}$." (SG, p. 33) include human agency, albeit the agency of professional mathematicians. This goal seems to be providing opportunities for students to reify fractions because the move from "$\frac{1}{2}$ of $\frac{1}{4}$" to "$\frac{1}{2} \times \frac{1}{4}$" can serve to facilitate reification given the encapsulation of "1" and "2" into "$\frac{1}{2}$" and "1" and "4" into "$\frac{1}{4}$" as entities themselves (i.e., numbers) that can be multiplied.

Goal 4: "Develop and use strategies and models for multiplying combinations of fractions, whole numbers, and mixed numbers to solve problems"

Written Curriculum

The goal stated in the written curriculum, "Develop and use strategies and models for multiplying combinations of fractions, whole numbers, and mixed numbers to solve problems," extends fraction multiplication, which has thus far been largely limited to fractions (with the exception of Question D in Problem 3.1 in the written curriculum which includes whole numbers), to include combinations of fractions, whole numbers, and mixed numbers. This goal is stated only for Problem 3.3 (Day 3) in the written curriculum. I also argue that several activities in the written curriculum in Problem 3.4 (Day 4) are closely related to this goal, including opportunities for discussion of the distributive property and changing mixed numbers to improper fractions in order to facilitate multiplication. On Day 3, Questions A, B, C, and D in Problem 3.3 ask students to "estimate the answer," "create a model or a diagram to find the exact answer," and "write a number sentence" for four contextual problems involving mixed numbers and whole numbers (see Example 4 on p. 44). These problems require students to develop strategies for multiplying combinations of fractions, whole numbers, and mixed numbers and to construct iconic mediators. In addition, sample "Suggested Questions" from Day 3 in the written curriculum that solicit responses regarding strategies are provided in Examples 18 and 19:

Example 18 (WC).
"How do you find $\frac{2}{3}$ of something?" (TG, p. 72)

Example 19 (WC).
"Does anyone have a different way to think about the problem?" (TG, p. 72)

The questions included in Examples 18 and 19 are "suggested" in the written curriculum to encourage students to share their strategies for multiplying combinations of fractions, whole numbers, and mixed numbers.

As mentioned previously, Problem 3.4 (Day 4) also provides opportunities directly related to this goal. The "Getting Ready" (SG, p. 37) and Question C (SG, p. 38) are presented in Examples 20 and 21 respectively. They introduce two important strategies for multiplication involving mixed numbers: (1) changing a mixed number to an improper fraction and (2) using the distributive property.

Example 20 (WC).

Yuri and Paula are trying to find the following product.

$$2\frac{2}{3} \times \frac{1}{4}$$

Yuri says that if he rewrites $2\frac{2}{3}$, he can use what he knows about multiplying fractions. He writes:

$$\frac{8}{3} \times \frac{1}{4}$$

Paula asks, "Can you do that? Are those two problems the same?"

What do you think about Yuri's idea? Are the two multiplication problems equivalent?

Example 21 (WC).

C. Takoda answers Question A part (1) by doing the following:

$$\left(2 \times 1\frac{1}{6}\right) + \left(\frac{1}{2} \times 1\frac{1}{6}\right)$$

1. Do you think Takoda's strategy works? Explain.
2. Try Takoda's strategy on parts (2) and (5) in Question A. Does his strategy work? Why or why not?

Examples 20 and 21 illustrate opportunities for students to discuss strategies for multiplying combinations of fractions, whole numbers, and mixed numbers using two particular strategies proposed by fictitious students in the written curriculum.

Questions A and B in Problem 3.4 also address this goal. These Questions are provided in Example 22.

Example 22 (WC).

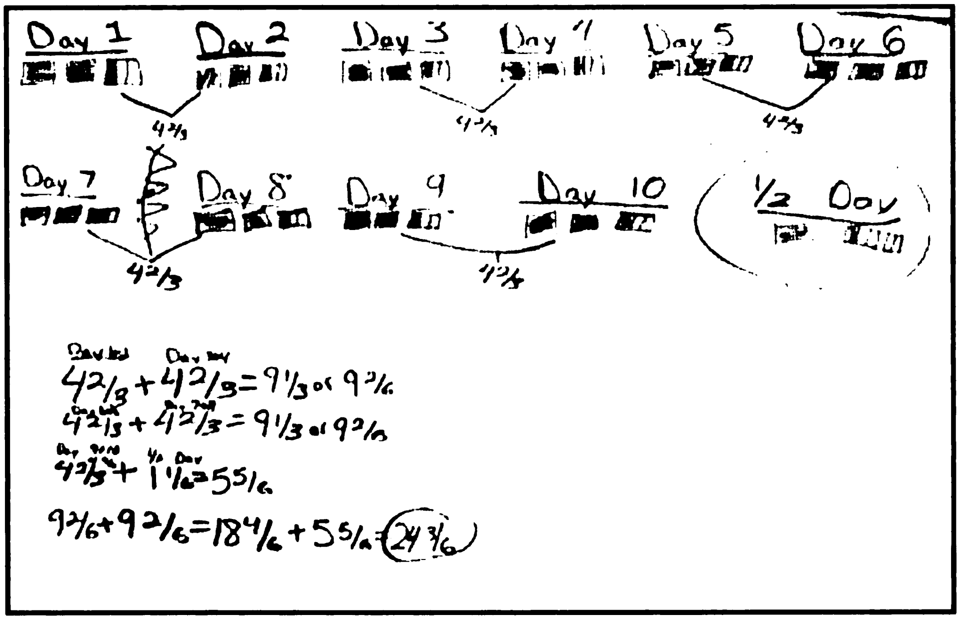
A. Use what you know about equivalence and multiplying fractions to first estimate, and then determine, the following products. **1.** $2\frac{1}{2} \times 1\frac{1}{6}$ **2.** $3\frac{4}{5} \times \frac{1}{4}$ **3.** $\frac{3}{4} \times 16$ **4.** $\frac{5}{3} \times 2$ **5.** $1\frac{1}{3} \times 3\frac{6}{7}$ **6.** $\frac{1}{4} \times \frac{9}{4}$ **B.** Choose two problems from Question A. Draw a picture to prove that your calculations make sense.

Question A instructs students to multiply six combinations of fractions, mixed numbers, and whole numbers using "what you know about equivalence and multiplying fractions." This requires the use of "strategies" as mentioned in Goal 4. Question B expects students to draw pictures for two of the problems from Question A to "prove that your calculations make sense." This Question emphasizes the use of "models" as mentioned in Goal 4. Therefore, I argue that Day 3 and Day 4 in the written curriculum emphasize this goal even though it is only explicitly stated for Day 3.

Enacted Curriculum

In the enacted curriculum, this goal is addressed on Days 4 and 5. After some introduction, the students are asked to complete Questions A-D in Problem 3.3. As discussed previously, these four problems directly address both the "strategies" and "models" components of this goal (see Example 4 on p. 44). Student work included in Example 23 illustrates a possible strategy for solving Question D regarding the harvesting of $2\frac{1}{3}$ acres of corn on each of $10\frac{1}{2}$ days.

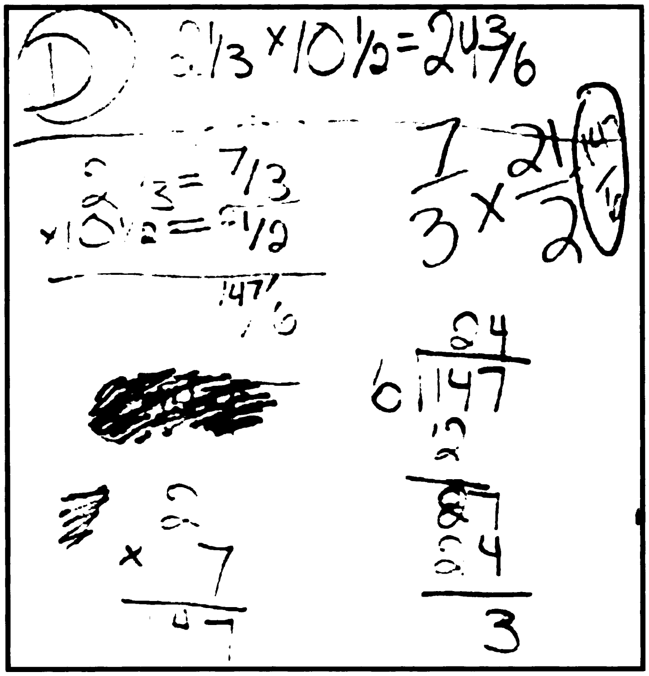
Example 23 (EC – Student Work).



This student work illustrates both the use of a "strategy" and a "model" for multiplying mixed numbers as called for in Goal 4.

As noted previously, the Questions in Problem 3.4 (Day 4 in the written curriculum) are not enacted in the classroom during these five days. Therefore, the Questions in Problem 3.4 related to this goal (i.e., "Getting Ready" and Questions A, B, and C) are not included in the enacted curriculum. It is interesting to note that although these specific exercises are not included, important points made in these exercises emerge in the enacted curriculum. For example, in the "Getting Ready" from Problem 3.4 (see Example 3 on p. 43), Yuri, a fictitious student, suggests the strategy of converting a mixed number to an improper fraction in order to facilitate fraction multiplication. A second fictitious student, Paula, questions this strategy asking, "Can you do that?" Although there is no discussion of "Yuri" and "Paula" in the enacted curriculum, a similar strategy emerges from a student in the classroom (see Example 24):

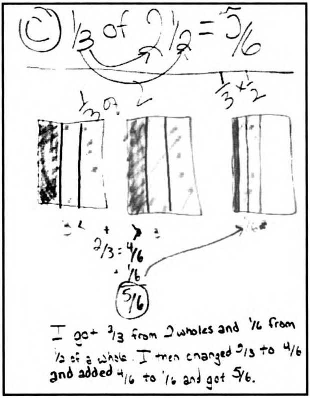
Example 24 (EC – Student Work).



This student uses the same strategy as Yuri and Paula (i.e., converting a mixed number to an improper fraction for the purposes of fraction multiplication); therefore, this strategy is discussed in the enacted curriculum.[25] Likewise, the distributive property suggested by "Takoda" in Question C of Problem 3.4 (see Example 21 on p. 56) emerges in the enacted curriculum. In the written curriculum, Takoda suggests decomposing the "$2\frac{1}{2}$" in "$2\frac{1}{2}$ x $1\frac{1}{6}$" to facilitate multiplication of mixed numbers. Example 25 presents student work in which a similar strategy emerges, albeit with different numbers:

[25] This conversation will be further discussed later in the chapter.

Example 25 (EC - Student Work).



In this case, the student decomposed the "$2\frac{1}{2}$" in "$\frac{1}{3}$ of $2\frac{1}{2}$" to facilitate multiplication. Therefore, I conclude that the primary strategies and "models" for multiplication of combinations of fractions, whole numbers, and mixed numbers addressed in the written curriculum also emerge in the enacted curriculum. That being said, the additional discursive opportunities provided by Question A in Problem 3.4 that are not enacted in the classroom could potentially further develop the use of these and other strategies and "models" for the multiplication of combinations of fractions, whole numbers, and mixed numbers. Whether or not objectification (i.e., alienation and reification) is promoted depends on both which "strategies" and "models" are used as well as how they are used. For example, the use of fictitious students to present the strategy of converting mixed numbers to improper fractions for the purpose of fraction multiplication (see Examples

20 and 21 on p. 56) promotes the personalization of mathematics; however, the use of this strategy and the equivalence employed in it promotes the reification of "$2\frac{2}{3}$" and "$\frac{8}{3}$" because "equivalence" is used in numerical discourse.

Goal 5: "Determine when multiplication is an appropriate operation"

The goal, "Determine when multiplication is an appropriate operation," is included in Problem 3.3 (Day 3) of the written curriculum. There are at least three possible interpretations of this goal: (1) the goal is closely related to Goal 3, "*Understand that finding a fraction of a number means multiplication.*" (2) the goal expects students to decide whether to use fraction multiplication in a particular situation (versus other fraction operations), and/or (3) the goal expects students to make sense of the transition from multiplication of fractions to multiplication of combinations of fractions, whole numbers, and mixed numbers.

The first possibility was addressed earlier in the discussion of Goal 3 in which the argument was made that the written and enacted curricula give similar emphases to the goal. The second possibility seems unlikely because all contextual problems in Investigation 3 are appropriate situations for multiplication (versus other operations). It is difficult for me to imagine how students are to gain understanding of when multiplication is the appropriate operation without a series of problems in which they must decide on the appropriate operation for the situation. The third possibility is addressed briefly in this section.

Written Curriculum

Because Problem 3.3 addresses modeling situations involving combinations of fractions, whole numbers, and mixed numbers, it is possible that this goal is implying that students will need to move past the notion of "part of part" given these new combinations. If this is the case then "suggested questions" in the written curriculum, such as "**What does it mean** when it says '$\frac{2}{3}$ of a 16-ounce bag'?" (TG, p. 72) may provide support for this transition.

Enacted Curriculum

A related discussion in the enacted curriculum (see Example 26) occurs when students are presented with "$3\frac{1}{4} \times 2\frac{11}{12}$":

Example 26 (EC).

(1) *T: You told us how that you think you might solve that [$3\frac{1}{4} \times 2\frac{11}{12}$], but I'm not sure we even understand what this is asking. If it were just three times two, **what would it mean**?*

(2) *S1: Three groups of two.*

(3) *T: **It would mean** three groups of two?*

(4) *S1: Yeah.*

(5) *T: So, **what does it mean now**? I need to get some more of you involved. Chloe, **what does it mean now**?*

(6) *S2: Three and one fourth groups of two and eleven twelfths.*

(7) *S?: So maybe you have to*

(8) *T: Say that again, Chloe. **What does it mean**?*

(9) *S2: Three and one fourth groups of two and eleven twelfths.*

(10) *T: Let's look - hold on for just a second. Let's look at this for a minute.* ***Does this just mean*** *I have more than three groups of more than two?*

(11) *Ss: Yeah.*

This discussion, like the "suggested question" in the written curriculum addresses the "meaning" of the multiplication of these new combinations of fractions, whole numbers, and mixed numbers. Therefore, if the goal expects students to make sense of multiplying combinations of fractions, whole numbers, and mixed numbers, I argue that the emphasis on this goal is similar in the written and enacted curriculum because all Questions in Problem 3.3 are enacted in the classroom.

Goal 6: "Explore the relationships between two numbers and their product"

Written Curriculum

The goal, "Explore the relationships between two numbers and their product," is included in Problem 3.4 (Day 4) in the written curriculum. There are several Questions in the Investigation that address the magnitude of the product of combinations of fractions, whole numbers, and mixed numbers. Fewer Questions, however, explicitly address the relationship between the magnitude of the product and the magnitude of the factors as this goal seems to demand. In Problem 3.1, Question D instructs students to estimate the size of the product of eight fraction multiplication problems; however, the Teacher's Guide suggests that teachers "be sure to focus the conversation on how students decided if each product was greater than or less than 1 and greater than or less than each of the factors" (TG, p. 62). Question A in Problem 3.2 asks students to estimate the magnitude of products of fraction multiplication problems. In this case, however, the Teacher's Guide makes no suggestion regarding a comparison to the magnitude of the factors. Example 27 presents Question D in Problem 3.2. This

Question represents the first time in the Student's Guide that a Question explicitly references the comparison between the product and factors:

Example 27 (WC):

D. **Ian says, "When you multiply, the product is greater than each of the two numbers you are multiplying: $3 \times 5 = 15$, and 15 is greater than 3 and 5." Libby disagrees. She says, "When you multiply a fraction by a fraction, the product is less than each of the two fractions you multiplied." Who is correct and why?**

Note that in Example 27 from the written curriculum, students are expected to discuss the relationship between the magnitude of the factors and the magnitude of the product. Example 28 illustrates the Teacher's Guide that suggests several questions for discussion:

Example 28 (WC):

For Question D, have students present their reactions to Ian and Libby's conversation.

- ***Does multiplying two numbers always lead to a greater product? Why?***
- ***Can you give an example that shows what Libby is talking about?***
- ***Does multiplication with fractions always lead to a product that is less than each factor?***

The Teacher's Guide goes on to state, "Do not expect students to resolve this last question. They will explore this idea in the problems that follow." (TG, p. 67). Indeed the written curriculum does come back to the issue on Day 4. The introduction to Problem 3.4 in the Student's Guide reminds students to consider the magnitude of the product, saying "Before you begin a problem, you should always ask yourself: 'About how large will the product be?'" (SG, p. 37) Finally, the last two parts of Question D in Problem 3.4 (see Example 29) ask students to:

Example 29 (WC):

> **4.** Describe when a product will be less than each of the two factors.
> **5.** Describe when a product will be greater than each of the two factors.

The Teacher's Guide emphasizes the need to address the facts that the product will be less than both factors when the factors are both less than one and that the product will be greater than both factors when the factors are both greater than one. Therefore, even though the written curriculum states this goal only for Day 4, the relationship between the magnitude of the factors and product is also addressed relatively substantially on Day 2 in the written curriculum.

Enacted Curriculum

In the enacted curriculum, the issue of product magnitude is addressed several times. For example, Question D in Problem 3.1 is enacted in the classroom and discussions addressing the magnitude of the product occur in conjunction with that Question. Examples 30-32 contain statements from the enacted curriculum referencing the magnitude of the products of fractions are included here:

Example 30 (EC).
*T: " It was five sixths of one half, and we're trying to figure out, **is five sixths of one half going to be more or less than the whole thing.**" (Day 2)*

Example 31 (EC).

*T: You know what. I'm noticing that every time. I'm noticing, that, ooh, I got a little two sixths, or a third. I got three eighths, I got two twenty oneths. **My answer got so little. Why? Why am I getting a little answer? Yeah?***

S: Because the denominator is getting larger and that makes the pieces smaller? (Day 1)

Example 32 (EC).
T: " Can anybody think of a situation where I would multiply and ***it wouldn't get bigger?"*** *(Day 1)*

Example 30 asks students to compare the product to "the whole thing" and therefore addresses the magnitude of the product, but does not relate its magnitude to the magnitude of the factors. Examples 31 and 32 address the relationship between the factors and the product using comparative language (e.g., "larger," "bigger"). A generalized discussion regarding when the product is less than or greater than the factors, as described in the written curriculum, does not occur in the enacted curriculum. In addition, the majority of the problems in the written curriculum that address this issue (i.e., Questions A and D in Problem 3.2, and Question D in Problem 3.4) are not enacted in the classroom. Therefore, I conclude that there is less emphasis on this goal in the enacted curriculum than in the written curriculum.

This goal, as several others, presents mathematics as a combination of personalized and alienated. For example, fictitious students, Ian and Libby, are again used to present mathematical conjectures. In this case, the conjectures address the relationship between the magnitude of the factors and products. The use of these students may personalize the mathematics. However, Question D-4 provides an example of a question that alienates the mathematics, "Describe when a product will be less than each of the two factors." (see Example 29 on p. 65). In this example, the use of "a product will be" implies that there is a mathematical fact that can be used to determine the answer to this Question. The ways in which this goal is presented in the written curriculum provide opportunities for reification of fractions. That is, if students are asked about the relationship between the magnitude of the factors and the product, statements such as "$\frac{1}{3}$

is less than $\frac{1}{2}$ and $\frac{2}{3}$" may occur. Statements that use language such as "is less than" possibly indicate reification of fractions because this type of language is used with numbers. If reification has not taken place, individuals are more likely to say "$\frac{1}{3}$ is smaller than $\frac{1}{2}$ and $\frac{2}{3}$." Because this goal is less emphasized in the enacted curriculum than in the written curriculum and it provides opportunities for objectification, it again appears (as with Goal 1) that opportunities for objectification (and reification more specifically) are less prominent in the enacted curriculum.

Goal 7: "Develop and use algorithms for multiplying combinations of fractions, whole numbers, and mixed numbers"

Written Curriculum

"Develop and use algorithms for multiplying combinations of fractions, whole numbers, and mixed numbers," is stated as a goal for Problem 3.4 (Day 4) in the written curriculum. "Algorithm" is defined in the Student Guide as "a reliable mathematical procedure" (p. 38). The line between "strategy" and "algorithm" seems somewhat blurred here. As described earlier, Problems 3.3 and 3.4 address strategies associated with the multiplication of combinations of fractions, whole numbers, and mixed numbers. These strategies include: (1) converting mixed numbers or whole numbers to improper fractions and then applying the traditional algorithm for fraction multiplication and (2) using some method of decomposing the mixed numbers and applying the distributive property. Are these strategies or algorithms, or both? It is interesting that this goal addressing algorithms is included in Day 4 and "algorithm(s)" is entirely absent in the written curriculum for this day. The Teacher's Guide discusses both of the above-mentioned strategies (p. 75) and goes on to discuss the "case-specific" nature of these strategies (p. 77). The notion that particular strategies serve some situations and not others leads into Goal 8, which stresses the development of an "efficient" algorithm.

Enacted Curriculum

When the class begins their discussion of multiplication of combinations of fractions, whole numbers, and mixed numbers in the enacted curriculum, the teacher stresses that she is only interested in strategies, not algorithms, for the time being (see Example 33):

> ***Example 33 (EC)***
> *T: "**I'm not so interested in an algorithm today. I'm interested in how can we literally solve through a drawing or something a problem like this.** How can we do it? And it looks like some of you have some good ideas of ways to start. Maybe you're going to have a picture of something and be fractioning off parts or whatever. **We're not on some mission to find an algorithm. We're on a mission to make sense of what is this really saying, and how can we do that."** (Day 4)*

The enacted curriculum contains discussions of the same two primary strategies for multiplying combinations of fractions, whole numbers, and mixed numbers found in the written curriculum. In the context of these problems, "algorithm" comes up in reference to the traditional algorithm for use after whole numbers and mixed numbers have been converted to improper fractions. Examples 34 and 35 are two such statements from Day 5:

> ***Example 34 (EC).***
> *T: "Oh my gosh! Now wouldn't that be nice if we could multiply any fraction by a whole number just by doing that [giving it a denominator of 1], and we could still use our **algorithm**?"*

Example 35 (EC).

T: "She didn't like the idea of mixed numbers, so she switched them over [converted them to improper fractions] so now she has two fractions. And then did she just used our ***algorithm****, multiply the numerators and multiply the denominators?"*

Because algorithms for use with multiplication of combinations of fractions, whole numbers, and mixed numbers are largely discussed as strategies on Day 4 in both the written and enacted curriculum, and in fact, the same two strategies are emphasized, I conclude that the emphasis on this goal is similar in the written and enacted curricula.

Because both the written and enacted curricula expect students to develop their own strategies/algorithms for multiplying combinations of fractions, whole numbers, and mixed numbers, I argue that the mathematics associated with this goal is largely personalized in both curricula. However, the movement toward algorithms for operating on fractions seems to provide an opportunity for reification because algorithms are largely performed on numbers.

Goal 8: "Develop and use an efficient algorithm to solve any fraction multiplication problem"

Written Curriculum

The goal, "Develop and use an efficient algorithm to solve any fraction multiplication problem" is included in Problem 3.5 (Day 5) of the written curriculum. The focus on this goal in Problem 3.5 is evidenced by the title of the problem, "Writing a Multiplication Algorithm" as well as an additional 23 appearances of "algorithm(s)" throughout the Problem. Examples 36-38 illustrate several uses:

Example 36 (WC).

"Test your ***algorithm*** *on the problems in the table. If necessary, change your* ***algorithm*** *until you think it will work all the time." (SG, p. 39)*

Example 37 (WC).
*"Several students may offer the traditional **algorithm** of multiplying the numerators together and multiplying the denominators together. Be sure they talk about how this is done when one or both of the factors is a mixed number or whole number." (TG, p. 81)*

Example 38 (WC).
*"Evaluating whether each **algorithm** is useable and helpful, and how it compares with other **algorithms**, will further students' understanding of multiplication of fractions, mixed numbers, and whole numbers." (TG, p. 82)*

In addition, all of the Questions (i.e., A, B, and C) in Problem 3.5 address algorithm development, albeit in slightly different ways. Question A instructs students to "write," "test," and, if necessary, "change" an algorithm. Questions B and C provides opportunities in which to "use" the developed algorithm; Question C contains the additional task of noting patterns in the products obtained from algorithm use.

Enacted Curriculum

In the enacted curriculum, the final day of the week (i.e., Day 5) ends with the discussion included in Example 39:

Example 39 (EC).

*(1) T: Okay, now. This is what I see, and tell me if you see this. I see a couple of different ways to think about this. I see this way [points to an example in which a mixed number was changed to an improper fraction and then the traditional algorithm was utilized] where we take any mixed number or any whole number and write it as a fraction, as an improper fraction, and then just use our **algorithm** - hold on a minute - and then I see another way. I see kind of picking it apart [points to an example in which numbers are decomposed, multiplied, and then put back together] and saying "Ten and a half twos, ten and a half of thirds," or really picking it apart. I see this sort of as one way [points to example where mixed number was changed to improper fraction], and this as another way [points to example in which mixed number is decomposed]. What are you thinking about those two strategies at this point? What are you thinking about them? Is there one that you're like, "Whoa, I like that one. I'm going to use that one." Or are you kind of like, "Hmmm, I still want to think about it." Tell me where you're at.*

(2) *S1: I think the first one is more **efficient.***

(3) *Ss: Yeah, yes.*

(4) *T: This one [points to example in which the mixed number is changed to an improper fraction]? **More efficient?***

(5) *Ss: Yeah.*

(6) *S2: I like the ten and a half -*

(7) *T: You like this one [points to example in which mixed number is decomposed]? Okay?*

In this discussion, two "different ways to think about" multiplication of combinations of fractions, whole numbers, and mixed numbers are compared. Students develop these strategies/algorithms and a student proposes "efficient" when explaining why he prefers one strategy/algorithm to the other. Therefore, I argue that the enacted curriculum addresses this goal in spite of the fact that Problem 3.4 is not enacted in the classroom. I would also argue that the development of the goal is more extensive in the written curriculum. In terms of objectification, the same comments from Goal 7 apply also to Goal 8 because both address algorithm development.

Summary

The goal of this chapter is to provide evidence to address two main questions: (1) What does an investigation of the goals in the written and enacted curricula allow us to see? (2) What do we know now about the relationship between the written and enacted curricula that we did not know before? This investigation allows us to see whether or not the goals stated in the written curriculum receive similar emphasis in the enacted curriculum. What we know now that we did not know before is that the mathematics in some of the goals receive similar treatment in the written and enacted curricula while the mathematics in other goals receive less emphasis in the enacted curriculum than in the written curriculum.

The statements in this summary and throughout this analysis are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days in the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or the enacted curriculum as a whole. Rather, my statements highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.[26]

Of the eight goals included in the written curriculum for this Investigation, four receive "similar" emphases in the written and enacted curricula. That is, students are given opportunities to interact with these goals in ways closely aligned with the suggestions in the written curriculum:

Goal 2: Use models to represent the product of two fractions
Goal 3: Understand that finding a fraction 'of' a number means multiplication
Goal 5: Determine when multiplication is an appropriate operation
Goal 7: Develop and use algorithms for multiplying combinations of fractions, whole numbers, and mixed numbers

[26] These caveats will be included as a footnote in each of the chapter summaries and the final discussion (Chapter 9) to emphasize the importance of these statements.

The remaining four goals receive greater attention in the written curriculum than in the enacted curriculum. The degree of discrepancy varies among these four goals. One of these goals displays a "low level of discrepancy" between the written and enacted curricula. That is, qualitatively the goal is addressed in similar ways; however, the amount of time spent on the goal is slightly less in the enacted curriculum than in the written curriculum:

Goal 4: Develop and use strategies and models for multiplying combinations of fractions, whole numbers, and mixed numbers to solve problems

Finally, three goals display a higher level of discrepancy between the written and enacted curriculum. That is, several experiences that address these goals in the written curriculum are not enacted in the classroom on the days examined in this study:

Goal 1: Estimate products of fractions
Goal 6: Explore the relationships between two numbers and their product
Goal 8: Develop and use an efficient algorithm to solve any fraction multiplication problem

The differential treatment of these goals may provide different opportunities for learning mathematics than was intended by the authors of the written curriculum.[27] In particular, the lack of emphasis on Goals 1 and 6 in the enacted curriculum, which address estimating products of fractions and the relationship between the magnitude of the factors and the product of fraction multiplication, may limit students' access to opportunities for the objectification of fractions. That is, both of these goals encourage the use of fractions as numbers and promote discourse associated with numbers. As mentioned earlier in this chapter, objectification of fractions (i.e., their use as numbers) provides an important step in the process of expanding the domain of numbers in which students are able to function.

[27] Recall that "learning mathematics" here is conceptualized as a change in participation in mathematical discourse.

This investigation of the goals has set the stage for a more detailed examination of the discursive practices present in the written and enacted curricula. Subsequent chapters address the relationships between four key features of mathematical discourse in the written and enacted curricula: (1) Mathematical words, (2) Visual mediators, (3) Endorsed narratives, and (4) Mathematical routines.

CHAPTER 5: MATHEMATICAL WORDS IN THE WRITTEN AND ENACTED CURRICULA

Mathematical words, as described previously, are largely words that signify quantities and shapes (e.g., "number," "triangle"). These words describe the mathematical objects or the products of mathematical discourse. In addition to these mathematical "product" words, here I also include words that signify mathematical processes (e.g., "multiply," "estimate") for consideration in this analysis. Because "learning mathematics" is conceptualized here as changes in participation in mathematical discourse, a close look at the uses of mathematical words seems essential for this analysis. Figure 4 provides one way to represent the process of word development:

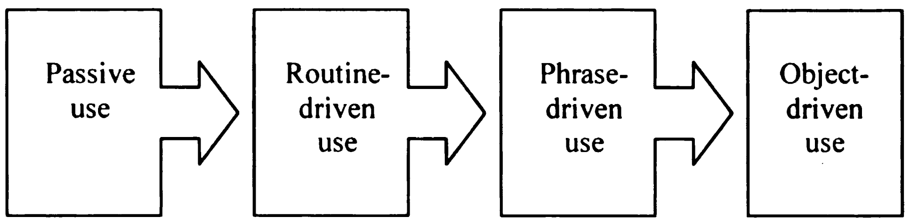


Figure 4. Four-stage model of the development of word use. (Sfard, 2008, p. 236)

An individual engages in "passive word use" when she can respond to questions about the word, but does not actually utter the word herself. The final three stages are characterized by "active word use." That is, the individual uses the word in her own utterances. She may begin by using the word within very particular discursive sequences – "routine-driven use." As her experience with the word increases, she may begin to use it more flexibly, but still primarily in a limited number of phrases (i.e., "phrase-driven use"). Finally, she will use the word as if it has a life of its own (object-driven use). This is the primary goal of word development as described here, so more must be said about this stage.

The "object-driven use" stage of this model is characterized by the objectification of the word in question (i.e., using it as a noun). Recall that objectification is defined as a process in which a word begins to be used as if it signifies an extra-discursive, self-sustained entity (object), independent of human agency. The process of objectification consists of two closely related, but not inseparable sub-processes: reification and alienation. Reification involves the replacement of talk about processes with talk about objects. Alienation is the use of discursive forms that present phenomena in an impersonal way, as if they are occurring of themselves, without the participation of human beings. That is, objectification, the final stage of word use, includes both talking about products rather than processes and describing the mathematical objects void of human agency. For example, "I multiplied the numerators," is a processual, personalized use of "multiply." It describes something that the student is doing. In contrast, "Multiplication is the opposite of division," uses "multiply in a structural (i.e., objectified) way because multiplication is a noun and the sentence is void of human agency.

Throughout this analysis of mathematical words, it is important to keep in mind that written and enacted forms of curricula are being compared. This may impact the results of this analysis. For example, the words in the written curriculum are solely in written form while the words in the enacted curriculum are primarily spoken. The words used in written text may differ from those in spoken text due to the nature of the mode; it is impossible to predict the impact of this difference on the findings presented here. In addition, the voice of the written text is largely that of the textbook authors whereas the classroom voice includes that of the teacher and students. It might be expected that these voices would differ in their word use (e.g., formal word use may be more common in textbooks than in classrooms).

Because it would be impossible to examine each and every mathematical word in the written and enacted curricula, I analyzed only the mathematical words that seem "central" to this Investigation. The title of the Investigation, "Multiplying with Fractions," provided two important words: "multiplying" and "fractions." I investigated all derivatives of "multiplying" (e.g., "multiplication") and other words that signify multiplication (e.g., "times") that were present in either the written or enacted curricula. For "fractions," I also investigated the singular version as well as any specific case of a fraction (e.g., "two thirds," "$\frac{2}{3}$") and words such as "fifths," "parts," and "pieces." I used the goals of the Investigation as stated in the written curriculum to make further decisions regarding the key mathematical words.[28] Table 3 presents these goals by Problem. Words signifying multiplication and fractions have been underlined to illustrate their prevalence.

[28] This process of choosing key words using the goals of the Investigation is but one possible method for word selection. I argue, however, that this process provides a set of important words in the Investigation. I do not argue that my key word list is comprehensive.

Table 3.

Goals of Problems 3.1-3.5 in "Investigation 3: Multiplying Fractions" (as Stated in the Written Curriculum) with "Multiplication" and "Fractions" and Their Derivatives Underlined

Problem	Goal
3.1	• Estimate <u>products</u> of <u>fractions</u> • Use models to represent the <u>product</u> of two <u>fractions</u> • Understand that finding a <u>fraction</u> <u>*of*</u> a number means <u>multiplication</u>
3.2	• Estimate <u>products</u> of <u>fractions</u> • Use models to represent the <u>product</u> of two <u>fractions</u> • Understand that finding a <u>fraction</u> <u>*of*</u> a number means <u>multiplication</u>
3.3	• Estimate <u>products</u> of <u>fractions</u> • Develop and use strategies and models for <u>multiplying</u> combinations of <u>fractions</u>, whole numbers, and mixed numbers to solve problems • Determine when <u>multiplication</u> is an appropriate operation
3.4	• Explore the relationships between two numbers and their <u>product</u> • Develop and use algorithms for <u>multiplying</u> combinations of <u>fractions</u>, whole numbers, and mixed numbers
3.5	• Develop and use an efficient algorithm to solve any <u>fraction</u> <u>multiplication</u> problem

Words signifying "multiplication" (appearing 14 times in the goals) and "fraction(s)" (appearing 10 times in the goals) figured prominently throughout the goals. In addition, "estimate" appears as the leading word in the first goal in Lessons 3.1, 3.2, and 3.3, "models" appears in the goals of Lessons 3.1, 3.2, and 3.3, "number" appears nine times within the goals, and "algorithm" appears in the goals for Lessons 3.4 and 3.5. Therefore, these words were also selected for analysis. Table 4 illustrates the goals again, this time with these additional key mathematical words underlined.

Table 4.

Goals of Problems 3.1-3.5 in Investigation 3: Multiplying Fractions (as Stated in the Written Curriculum) with Key Mathematical Words and Their Derivatives Underlined

Lesson	Goal
3.1	• <u>Estimate</u> <u>products</u> of <u>fractions</u>
	• Use <u>models</u> to represent the <u>product</u> of two <u>fractions</u>
	• Understand that finding a <u>fraction</u> <u>*of*</u> a <u>number</u> means <u>multiplication</u>
3.2	• <u>Estimate</u> <u>products</u> of <u>fractions</u>
	• Use <u>models</u> to represent the <u>product</u> of two <u>fractions</u>
	• Understand that finding a <u>fraction</u> <u>*of*</u> a <u>number</u> means <u>multiplication</u>

Table 4 (cont'd)

Lesson	Goal
3.3	• Estimate products of fractions
	• Develop and use strategies and models for multiplying combinations of fractions, whole numbers, and mixed numbers to solve problems
	• Determine when multiplication is an appropriate operation
3.4	• Explore the relationships between two numbers and their product
	• Develop and use algorithms for multiplying combinations of fractions, whole numbers, and mixed numbers
3.5	• Develop and use an efficient algorithm to solve any fraction multiplication problem

In this chapter, I will address these words in increasing order of frequency in the goals (i.e., from "algorithm," "estimate," "number," "fraction," to "multiplication"). "Number" is included in *Words Signifying Fractions* because most of the numbers in this analysis are fractions. In addition, specific fractions (e.g., "$\frac{2}{3}$") will be discussed in the same section. "Models" will be addressed in the next chapter, *Visual Mediators in the Written and Enacted Curriculum,* when both references to models and the use of models will be analyzed.

Words Signifying Algorithm

"Algorithm(s)" appears 30 times in the written curriculum and 24 times in the enacted curriculum.[29] Twenty-six of the occurrences of "algorithm(s)" in the written curriculum (87%) are in Lesson 3.5 (Day 5). In contrast, uses of "algorithm(s)" are spread fairly evenly across Days 3-5 (8, 7, and 8 occurrences respectively) in the enacted curriculum. The three most common actions associated with "algorithm(s)" in both the written and enacted curriculum were *writing/developing*, *using*, and *testing/checking*. Figure 5 indicates the relative frequencies of each action in the respective curriculum.

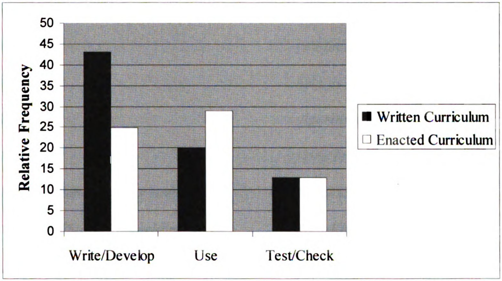


Figure 5. Relative frequencies of uses of "algorithm(s)" in the written and enacted curricula.

Figure 5 indicates that "writing/developing" algorithms is the most prevalent action in the written curriculum whereas "writing/developing" and "using" algorithms occur in nearly

[29] It is important to remember that the word use described here reflects only the five days under consideration in these analyses and cannot be generalized to statements regarding the overall word use in the written or enacted curriculum. That is, it is not possible to know whether this word use is "typical" for the textbooks or the classroom outside of these five days.

equal proportions in the enacted curriculum. Table 5 provides examples to demonstrate the use of "algorithm(s)" in these three categories in each curriculum.

Table 5.

Examples of the Use of "Algorithm" in the Written and Enacted Curricula

Action	Written Curriculum	Enacted Curriculum
Write/ Develop	*"Write an **algorithm** that will work for multiplying any two fractions, including mixed numbers." (SG, p. 39)*	*T: "We're going to try to write an **algorithm**, an **algorithm** for multiplying, but we're going to kind of do this one as a big class. So, if you finish B early, come on over and help us get started to write an **algorithm** for what we think so far." (Day 3)*
Use	*"Some students may use the traditional **algorithm** but also develop special rules for special situations." (TG, p. 83).*	*T: "Well, if we, if we use our **algorithm**, we said if I have any two fractions, I can multiply across the numerator and multiply across the denominator, so that would be twenty halves, right?" (Day 5)*
Test/ Check	*"Test your **algorithm** on the problems in the table." (SG, p. 39)*	*T: "Can, can I just make sure that our **algorithm** worked here? Did, did you guys check that? Did our **algorithm** work?" (Day 5)*

In addition to the explicit use of "algorithm(s)," both the written and enacted curricula contain references to algorithms without including the word itself. For example, both curricula include descriptions of the traditional (and other) algorithms for multiplying fractions. The examples included here illustrate such references:

Example 40 (WC).
"Double the denominator because you are making pieces one-half as large, or two times smaller." (TG, p. 67)

Example 41 (EC).
T: "Do you think that these are the steps that we should tape, take? Multiply the numerators and multiply the denominators, then you put them together as a fraction." (Day 3)

Comparing or evaluating algorithms is evidenced in both curricula as well. In fact, the goal of Problem 3.5 in the written curriculum states *"Develop and use an efficient algorithm to solve any fraction multiplication problem"* (TG, p. 81). The use of "efficient" here seems to indicate something beyond developing an algorithm that works. Rather it alludes to the need for an algorithm that has specific characteristics. The following example from the written curriculum indicates that the authors see comparing algorithms as a useful enterprise:

Example 42 (WC).
"Evaluating whether each algorithm is useable and helpful, and how it compares with other algorithms, will further students' understanding of multiplication of fractions, mixed numbers, and whole numbers." (TG, p. 82)

Here, the written curriculum advocates for the evaluation of each algorithm and comparing algorithms. A discussion on Day 5 in the enacted curricula compares two possible algorithms:

Example 43 (EC).

(1) T: Okay, now. This is what I see, and tell me if you see this. I see a couple of different ways to think about this. I see this way [points to an example in which a mixed number was changed to an improper fraction and then the traditional algorithm was utilized] where we take any mixed number or any whole number and write it as a fraction, as an improper fraction, and then just use our algorithm - hold on a minute - and then I see another way. I see kind of picking it apart [points to an example in which numbers are decomposed, multiplied, and then put back together] and saying "Ten and a half twos, ten and a half of thirds," or really picking it apart. I see this sort of as one way [points to example where mixed number was changed to improper fraction], and this as another way [points to example in which mixed number is decomposed]. What are you thinking about those two strategies at this point? What are you thinking about them? Is there one that you're like, "Whoa, I like that one. I'm going to use that one." Or are you kind of like, "Hmmm, I still want to think about it." Tell me where you're at.

(2) S1: I think the first one is more efficient.

(3) Ss: Yeah, yes.

(4) T: This one [points to example in which mixed number is changed to an improper fraction]? More efficient?

(5) Ss: Yeah.

(6) S2: I like the ten and a half -

(7) T: You like this one [points to example in which mixed number is decomposed]? Okay?

As in the written curriculum, "efficient" is used in the enacted curriculum as a way to describe one of the possible algorithms. Neither the written nor the enacted curricula are explicit about what makes an algorithm "efficient" in the days I considered in this analysis.

On Day 2 in the enacted curriculum after the class has completed several fraction multiplication number sentences using models, Trevor notices the pattern that leads him to suggest the traditional algorithm:

Example 44 (EC).
S: "I think it's times the numerator and numerator and then times the denominator and denominator."

From this moment forward, Trevor's name is often attached to this algorithm in the enacted curriculum. In contrast to the written curriculum which uses "algorithm" or simply describes the process without naming it, the traditional algorithm takes on several other labels in the enacted curriculum:

Example 45 (EC).
T: "Okay, guys. Okay. Listen up, please, okay. What we're going to do now is we're going to take a peek at these, keeping in mind ***Trevor's idea*** *that he brought up earlier about "Hmmm, I wonder if we can just multiply and multiply." (Day 2)*

Example 46 (EC).
S: "I know that ***Trevor's way*** *works." (Day 3)*

Example 47 (EC).
S: "Yeah, but you can't, ***Trevor's method*** *doesn't apply with wholes." (Day 2)*

In Examples 45-47, the algorithm is referred to as "Trevor's idea," "Trevor's way," and "Trevor's method." This is one of many examples in the enacted curriculum in which mathematics is portrayed as personal and happening in the here and now. This personal nature of mathematics seems to take precedence over the historical nature of mathematics (i.e., that someone discovered this algorithm prior to Trevor's observation). As mentioned previously, this personalization of mathematics is also found in the written curriculum when fictitious students suggest strategies. One such example is presented here:

Example 48 (WC).

> **C.** Takoda answers Question A part (1) by doing the following:
>
> $$\left(2 \times 1\tfrac{1}{6}\right) + \left(\tfrac{1}{2} \times 1\tfrac{1}{6}\right)$$
>
> **1.** Do you think Takoda's strategy works? Explain.
>
> **2.** Try Takoda's strategy on parts (2) and (5) in Question A. Does his strategy work? Why or why not?

This reference to the distributive property is presented as a students' proposed strategy rather than as a historically established mathematical property. This is in sharp contrast to the way in which the distributive property might be presented in a more traditional mathematics textbook. Recall that depersonalization is a requirement for objectification. Therefore, this personalization of the traditional algorithm and the distributive property, although no doubt used for particular purposes, might be seen as unsupportive of the process of objectification. As mentioned in Chapter 4, the movement toward algorithms for operating on fractions seems to provide an opportunity for reification because algorithms are largely performed on numbers. Therefore, the use of "algorithm" in both curricula is personalized and offers opportunities for reification; the former does not necessarily support objectification but the latter may.

Words Signifying Estimation

"Estimate" and its derivatives (e.g., "estimation") are present in both the written and enacted curricula, appearing 50 and 23 times respectively, more than twice as often in the written curriculum as in the enacted curriculum. Table 6 includes the derivatives of "estimate", the parts of speech they represent, and examples of their use in each curriculum.

Table 6.

Examples of Uses of "Estimate" and its Derivatives in the Written and Enacted Curricula

Word	Part of Speech	Written Curriculum	Enacted Curriculum
Estimate	Noun	*"Remind students that they can use their* ***estimate*** *as a guide for a reasonable answer when they compute." (TG, p. 66)*	*S: "Yeah. That's what I got, too. I got sixteen for my* ***estimate****." (Day 4)*
	Verb	*"****Estimate*** *each product to the nearest whole number (1, 2, 3, . . .)." (SG, p. 36)*	*S: "How did they* ***estimate*** *to get the same answer?" (Day 4)*
Under-estimate	Noun	*"It would be an* ***underestimate*** *because I rounded* $1\frac{1}{6}$ *down to 1." (TG, p. 76)*	-----
Over-estimate	Noun	*"Would the estimate of* $2\frac{1}{2}$ *be an underestimate or an* ***overestimate****?"(TG, p. 76)*	-----

Table 6 (cont'd)

Word	Part of Speech	Written Curriculum	Enacted Curriculum
Estimated	Verb	*"Who can explain how they **estimated** $1\frac{1}{2}$ x $2\frac{9}{10}$?" (TG, p. 72)*	*T: "We **estimated** how much it would be if we put a couple of fractions or a couple of decimals together." (Day 1)*
Estimating	Verb	-----	*T: "Graham, how could we think about **estimating** that answer?" (Day 4)*
Estimation	Noun	*"For parts (a)–(d), use **estimation** to decide if the product is greater than or less than $\frac{1}{2}$." (SG, p. 35)*	-----
	Adjective	*Finish up by having students share their **estimation** strategies in Question D. (TG, p. 62)*	-----

Note. Dashes (i.e., "-----") indicate that the word does not appear as this part of speech in the designated curriculum.

It is interesting to note that only four of the eight forms of "estimate" present in the written curriculum appear in the enacted curriculum. That is, there are no explicit references to "underestimate," "overestimate," or "estimation" (as a noun or an adjective) in the enacted curriculum. It is also notable that, although not captured in Table 6, 19 of the 23 instances of "estimate" and its derivatives (83%) in the enacted curriculum occur on Day 4, whereas occurrences in the written curriculum occur over the first four days (Problems 3.1-3.4) in relatively consistent numbers (ranging from 8 to 14). Recall that "object-driven" word use, the final stage in the word development process, is found only when words are used as nouns (i.e., "object-like"). Figure 6 summarizes the relative frequencies of the parts of speech presented in Table 6.

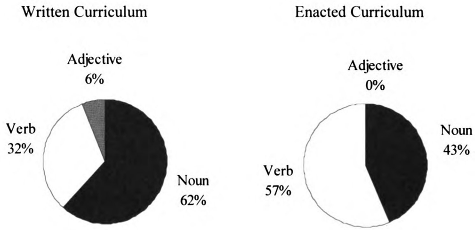


Figure 6. Relative frequencies of the parts of speech of "estimate" and its derivatives in the written and enacted curricula.

Figure 6 indicates that objectification of "estimate" (i.e., nouns) occurs more often in the written curriculum than in the enacted curriculum, representing 62% and 43% in each curriculum respectively. This seems to indicate that "estimate" is objectified more often in the written curriculum than in the enacted curriculum. That is, for the class, "estimation" is largely something you "do" and not an object to be discussed in its own right.

The enacted curriculum contains other less formal words and phrases associated with estimation. This is not surprising since, as mentioned earlier, it is more common to use formal language in spoken than in written mode. Examples of this other language (i.e., "*about*," "*almost*," "*round up*") are provided here:

Example 49 (EC).
*T: "So you would call this **about** three?" (Day 4)*

Example 50 (EC).
*S: "Well, two thirds is **pretty much almost** half." (Day 1)*

Example 51 (EC).
*S: "**Round** the eleven twelfths up to twelve twelfths so the two would be three." (Day 4)*

Closely related to "estimation" and its derivatives are situations found in the written curriculum in which questions are posed regarding whether, for example, the product of a fraction multiplication problem is greater or less than 1. In fact, the written curriculum recommends that students "use estimation to solve" such problems (SG, pp. 33, 35). An example of this type of discussion is included here from Problem 3.1 (Day 1) of the written curriculum along with a related discussion from Day 2 of the enacted curricula:

Example 52 (WC).

"What is $\frac{5}{6}$ of 1 whole? (part of a whole, or $\frac{5}{6}$)

Does this help you estimate whether $\frac{5}{6}$ x 1 is greater than or less than 1? (Yes, it is less than 1)

Now think about 1 x $\frac{1}{2}$. Is this greater than 1? (no)

Is it greater than $\frac{1}{2}$? (no)

So is $\frac{5}{6}$ x $\frac{1}{2}$ greater than $\frac{1}{2}$? (No. You have just $\frac{5}{6}$ of $\frac{1}{2}$, which is less than $\frac{1}{2}$.)"[30] *(TG, pp. 60-61)*

Example 53 (EC).

(1) T: Let me give you an example. I have, mmmm, a half of a submarine sandwich left, and I eat a third of it. I eat one third of one half [writes "$\frac{1}{3}x\frac{1}{2}$"]. Okay. Did I eat more or less than a whole submarine sandwich [spreads hands apart to indicate the length of a whole submarine sandwich]?

(2) Ss: Less.

(3) T: Okay. Now I need you to explain why. How did you know I ate less? Connor, how did you know it was less than a whole?

(4) S1: Um -

(5) T: Not sure about that one? How do we know? Maria?

(6) S2: Because, um, when you times the denominators you get kind of like a big, um, well

(7) T: Without multiplying it can I know? Or do I have to multiply it? Do I have to do Trevor's idea and solve it, or can I know just by looking at it?

(8) Ss: Just by looking at it.

(9) T: Just by looking at it?

(10) S3: Just the whole, you only had half left in the first place.

(11) T: Tell Maria. What do you think about that?

(12) S3: She said out of a whole, but then there was half gone, there was only half left out of the whole, and if you're, if you have, if you're taking

(13) S4: Even if you don't eat any of it, it would still be less than a whole anyway because there was only a half left.

---

[30] The parenthetical components in this example are possible question responses provided in the written curriculum.

(14) *T: Does that make sense? I started with a half of a sub, and I'm eating a part of that part. Am I ever going to get the whole sub?*

(15) *S4: No, because you have half already, and, uh, then you have to, uh, take away more, so you'd have less than half.*

(16) *T: Okay. Ashley, did you want to add something to Grace's?*

(17) *S5: You're just taking the half and cutting it into thirds and taking one out of the three thirds.*

(18) *T: Exactly.*

(19) *S5: You're taking less than*

(20) *T: Yeah. I'm taking a part of what's already a part.*

Example 52 and 53 illustrate the use of estimation strategies for determining the magnitude of the product of two fractions. Again, differences are present in terms of objectification. Example 52 (from the written curriculum) uses primarily language associated with numbers (i.e., reified fractions), such as "greater than 1" and "less than 1." Example 53 (from the enacted curriculum) uses language not associated with reification, such as "more than a whole, "less than a whole." The uses in the written curriculum treat fractions as numbers by comparing them with numbers. The uses in the enacted curriculum treat fractions as pieces or parts. This supports our finding earlier in this section that objectification of "estimation" is more common in the written curriculum than in the enacted curriculum.

Words Signifying Fractions

As discussed previously (see pp. 30-31), "fraction" has been historically problematic because it is used in many different ways.[31] For example, it is used when referring to a rational number itself (i.e., as a number) or as one of many representations

[31] When fraction appears within quotation marks (i.e., "fraction"), I am referring to the word, when it appears without quotation marks, I am referring to the fractions themselves.

of a rational number (i.e., as a numeral). The glossary of the Student's Guide of *Bits and Pieces I* (the first unit addressing fractions in *Connected Mathematics*) provides several definitions explaining how "fraction" is used in the written curriculum. The glossary defines a "fraction" as "a number (quantity) of the form $\frac{a}{b}$ where a and b are whole numbers," an "improper fraction" as "a fraction in which the numerator is larger than the denominator" and as "a fraction that is greater than 1," and a "mixed number" as "a number that is written with both a whole number and a fraction" (p. 74-75). By defining "fraction" as "a number (quantity) of the form $\frac{a}{b}$" and defining improper fractions and mixed numbers in terms of this definition, it is not clear whether "fraction" is used as a number or a particular "form" of a number (i.e., a numeral). Since there is no mention in either the written or enacted curriculum of alternative "forms" or representations of these numbers (e.g., decimals) nor is there any mention of "rational number," it seems that both curricula are treating fractions as numbers rather than as numerals. For the purposes of this study, I will extend the definitions provided in the written curriculum to include fractions equivalent to one (e.g., $\frac{3}{3}$) as improper fractions and use "proper fraction" only when referring to fractions less than 1.

The analysis of words signifying fractions begins with an examination of the words "fraction," "piece," "part," and "number" and their derivatives in the written and enacted curricula. "Number" is included here for several reasons: (1) The numbers in question in this Investigation are primarily fractions or parts of fractions, (2) The written curriculum defines a fraction as a number, and (3) The fraction literature (e.g., Kieran, 1976; Lamon, 1999) characterizes fractions in several ways including as parts of a whole and as numbers. The final analysis in this section will examine the use of proper and improper fractions and mixed numbers in the written and enacted curricula.

Uses of "fraction" in the Curricula

Four common categories of the use of "fraction" and its derivatives (e.g., "fractioning") emerged from the written and enacted curricula, including fraction as a number, a part, an adjective, and a verb. Table 7 provides examples of each category of use in both curricula.

Table 7.

Examples of Categories of "Fraction" Use in the Written and Enacted Curricula

Category	Written Curriculum	Enacted Curriculum
Number	*"The strategy introduced in the Getting Ready involves changing the form of mixed numbers and whole numbers so students can operate in the same way as when both factors are **fractions**." (TG, p. 75)*	*S: "One thing I don't get is, you know how when we said eight times seven and we showed like the picture. Like, that wouldn't work with **fractions**, would it?" (Day 2)*

Table 7 (cont'd)

Category	Written Curriculum	Enacted Curriculum
Part	*"What* ***fraction*** *of the goal does Nikki raise?" (SG, p. 36)*	*S: "Um, well, there, we knew that she, there was, it was half full, so we split it into half, and then we, she bought two thirds of it, so we shaded, we made thirds first, and then we shaded in two, and then, um, she, it said what* ***fraction*** *of the pan she bought, and she would buy two sixths of the pan because if you split all these into thirds you'll get two sixths." (Day 1)*
Adjective	*"Sometimes, they have to find a* ***fractional*** *part of another fraction." (SG, p. 32)*	*T: "You can buy any* ***fractional*** *part of a pan of brownies and pay that fraction of twelve dollars." (Day 1)*
Verb	-----	*T: "Maybe you're going to have a picture of something and be* ***fractioning*** *off parts or whatever." (Day 4)*

Note. Dashes (i.e., "-----") indicate that the category of use of "fraction" is not present in the designated curriculum.

I distinguished between the "number" and "part" categories by determining which word (i.e., "number" or "part") could replace "fraction" in each sentence. Even though "fraction" is used as a noun in both of these categories, the "number" category is further along in the word development process (see Figure 4 on p. 75) because when "fraction" is used as "part," it is used only in specific types of phrases (e.g., "**fraction of** the whole pan," "**fraction of** a fundraising goal"). Therefore, it is only when "fraction" is used as a number that it is objectified and discussed as if it has a life of its own. The fraction as number example from the written curriculum refers to fractions as factors and certainly factors are objects in their own right. In the fraction as number example from the enacted curriculum, the student is comparing fraction multiplication to whole number multiplication which reifies "fraction." In contrast, the other examples use "fraction" and its derivatives as processes or descriptors which does not fulfill the reification requirement of objectification. Figure 7 summarizes the relative frequency of each category in the curricula.

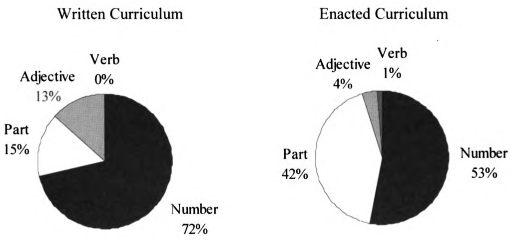


Figure 7. Relative frequencies of categories of "fraction" use in the written and enacted curricula.

Figure 7 indicates that "fraction" as number is the most common use of "fraction" in both the written and enacted curricula; however, its use as number represents nearly three-fourths of the uses in the written curriculum compared to slightly more than half in the enacted curriculum. "Fraction" as part is more common in the enacted curriculum than in the written curriculum (42% compared to 15%) which indicates that "fraction" and its derivatives are objectified more often in the written curriculum than in the enacted curriculum.

Uses of "piece" and "part" in the Curricula

Both "part(s)" and "piece(s)" are present in the written and enacted curricula.[32] "Part(s)" was much more common than "piece(s)" in the written curriculum (72 appearances compared to 11 appearances). The opposite was true in the enacted curriculum; "Piece(s)" was nearly twice as prevalent as "part(s)" (238 appearances compared to 120 appearances). Both "part(s)" and "piece(s)" are included in the following excerpts from the written and enacted curricula:

> ***Example 54 (WC).***
> *"The $\frac{2}{3}$-sized section of the thermometer is divided into four equal **parts**. When the rest of the thermometer is sectioned into **pieces** of the same size, there are six **pieces**. Then $\frac{1}{4}$ of $\frac{2}{3}$ on the thermometer is $\frac{1}{6}$ of the distance from 0 to 1." (TG, p. 65)*

[32] Only instances of "part(s)" and "piece(s)" in mathematical context were included in this analysis.

Example 55 (EC).

*S: "What we did is, it was in one sixths, so we split it up into six **pieces**, and then they had just this **part**, and then all this **part** was gone, and they had, we split this up into ten **pieces**, and then they had nine of them, they bought nine of them, and then there was this one, and then we figured, to figure out how many, um, we would have for the denominator, we had this in ten **pieces**, so we split each of the other six into ten **pieces**, so we got sixty, and then we had nine in here and then all the rest wasn't there so we got nine sixtieths." (Day 2)*

The significance of this distinction is not clear; however, it provides another example of differences in fractional discourse between the written and enacted curricula. It seems that "piece(s)," more prevalent in the enacted curriculum, is less obviously "fraction" talk than "part(s)." That is, "part" might more often be used with a question such as "What part of the cake was eaten?" which elicits a fraction answer; whereas, "piece" is more likely to be used in a question such as "How many pieces?" which elicits a whole number answer. If this conjecture is true, then the fractional discourse is more advanced in the written curriculum than in the enacted curriculum because "pieces" is more common in the enacted curriculum than "parts." Closely related to pieces and parts are instances of words such as "thirds," "fifteenths," etc. The written curriculum contains 7 distinct words of this type with 30 total appearances. The enacted curriculum included 14 words of this type with 217 total appearances.[33] Examples are included here from both curricula:

Example 56 (WC).

*"The alternative approach below leads to **twelfths** as the unit rather than **sixths.** In this approach, you first partition a thermometer or number line into **thirds** and label the **thirds.**" (TG, p. 65)*

[33] The only instances counted here were those with no number (e.g., "two," or "2") in front of the word (e.g., "thirds").

Example 57 (EC).
*S: "I think they're all in **sixths** because you had them into **fifths** and then you cut them into **thirds** so now they're all in **sixths** so you have, let's, er, **fifteenths**." (Day 1)*

The prevalence of these types of words in the enacted curriculum seems consistent with the prevalence of "piece(s)" in the enacted curriculum because these words indicate a particular fractional piece.

Uses of "number" in the Curricula

Uses of "number(s)" in the written and enacted curricula occur in the word phrases "number line," "number sentence," "whole number," and "mixed number," as well as several instances of "number(s)" standing alone. When "number(s)" is used outside of the word phrases it refers to numbers more generally. Examples from the written and enacted curricula are included here:

Example 58 (WC).
*"Question C provides additional multiplication practice and looks at the result of multiplying a **number** by its reciprocal." (TG, p. 81)*

Example 59 (EC).
*T: "Is it? Is multiplication always commutative, or just with some **numbers**?" (Day 4)*

In Example 58, the "numbers" include fractions, whole numbers, and mixed numbers. In Example 59, the teacher is asking about how broadly the commutative property of multiplication can be applied. The Question under consideration when this question is asked is "$2\frac{1}{2} \times \frac{4}{7}$." Statements/Questions such as these may serve to facilitate the objectification of fractions because they are included here under the broad heading of "numbers." Figure 8 summarizes the relative frequencies of these uses in the written and enacted curricula.

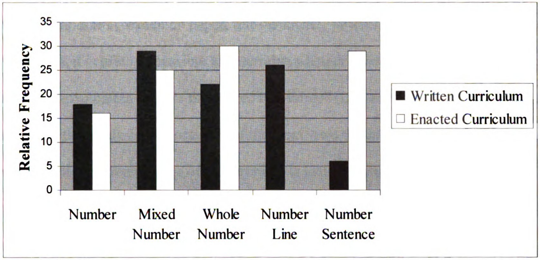


Figure 8. Relative frequencies of uses of "number(s)" in the written and enacted curricula.

Figure 8 indicates several similarities and differences in the use of "number(s)" in the written and enacted curricula. "Mixed number" and "whole number" occur in similar proportions in the curricula (ranging from 22% to 30%). In contrast, "number line" and "number sentence" are present in quite different proportions in the curricula (ranging from 0% to 29%).[34] In the case of "number(s)" standing alone, which as described above may facilitate the objectification of fractions, the relative frequencies in the written and enacted curricula are similar.

Fractions in the Curricula

The final analysis in this section examines the use of fractions (e.g., "$\frac{3}{4}$," "$\frac{4}{3}$," "$3\frac{1}{4}$") in the written and enacted curricula. Examples of proper fractions, improper fractions, and mixed numbers are present in both curricula. Nearly all cases of fractions

[34] These differences will be expanded upon in Chapter 6: *Visual Mediators in the Written and Enacted Curricula.*.

in the written curriculum are in symbolic form, whereas the enacted curriculum includes both symbolic form (written on paper, transparencies, posters, etc.) and spoken form. In spoken form, it is often impossible to determine whether "two thirds" refers to "two thirds" (i.e., two pieces that are thirds) or "two-thirds" (i.e., a rational number). For the purposes of this analysis, all instances of the form "two thirds" are included as fractions; however, a closer investigation of this problem is included later in this section. I recorded each distinct proper and improper fraction and mixed number and noted the number of times they appeared in each curriculum. Figure 9 indicates the number of distinct fractions in each of the three categories included in the curricula.[35]

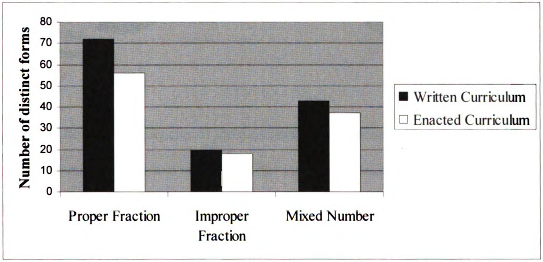


Figure 9. Frequencies of distinct proper and improper fractions and mixed numbers in the written and enacted curricula.

Figure 9 indicates that in both the written and enacted curricula the number of distinct proper fractions is the greatest, followed by the number of distinct mixed numbers, and finally the number of distinct improper fractions. The largest discrepancy between the

[35] Equivalent fractions (e.g., $\frac{2}{3}, \frac{6}{9}$) are considered distinct forms in this analysis.

written and enacted curricula is in the number of distinct proper fractions, 72 in the written curriculum compared to 56 in the enacted curriculum. Figure 10 summarizes the relative frequencies of the total number of appearances of these forms in the written and enacted curricula.

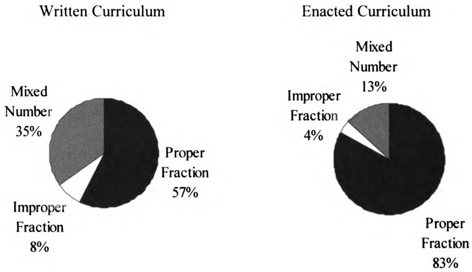


Figure 10. Relative frequencies of the total number of appearances of proper and improper fractions and mixed numbers in the written and enacted curricula.

Figure 10 indicates that the relative frequency of the appearance of each type varies substantially. For example, proper fractions represent 57% of all fractions present in the written curriculum compared to 83% in the enacted curriculum, whereas mixed numbers represent 35% of all fractions present in the written curriculum compared to just 13% in the enacted curriculum. This may indicate that in this Investigation, students have more experience with proper fractions and less experience with improper fractions and mixed numbers than intended by the authors of the written curriculum. In terms of objectification, the extension from multiplication of fractions to include improper fractions and mixed numbers may facilitate reification (and therefore objectification) by problematizing the "part of part" discursive pattern. For example, it is possible to speak about "$\frac{1}{6} x \frac{3}{4}$" as "$\frac{1}{6}$ of $\frac{3}{4}$" (i.e., part of a part) indefinitely. Therefore, if only proper fractions are provided as examples, the reification of fractions may never happen. However, it is much more problematic to speak about "$\frac{7}{6} x \frac{3}{4}$" as "$\frac{7}{6}$ of $\frac{3}{4}$." This problematizing is important because it may facilitate the reification of fractions.

Table 8 summarizes particular fractions in the three categories which represent at least 5% of the total number of fractions in each curriculum.

Table 8.

Fractions Representing at Least 5% of the Total Number of Fractions in the Written and Enacted Curricula

	Written Curriculum		Enacted Curriculum	
Category	Fraction	Relative Frequency	Fraction	Relative Frequency
Proper Fractions	$\frac{1}{2}$	14%	$\frac{1}{2}$	27%
	$\frac{2}{3}$	10%	$\frac{2}{3}$	17%
	$\frac{1}{3}$	7%	$\frac{1}{3}$	14%
	$\frac{1}{4}$	6%	$\frac{1}{4}$	6%
	$\frac{3}{4}$	5%	$\frac{3}{4}$	6%

Table 8 (cont'd)

Category	Written Curriculum Fraction	Written Curriculum Relative Frequency	Enacted Curriculum Fraction	Enacted Curriculum Relative Frequency
Improper Fractions	No improper fraction represented more than 1% of the total number of fractions in either the written or enacted curriculum.			
Mixed Numbers	$1\frac{1}{2}$	6%	$2\frac{1}{3}$	5%
	$2\frac{1}{2}$	6%		

Table 8 indicates that the most common proper fractions in the written and enacted curricula are the same. In fact, even their order of prevalence is the same. It is notable; however, that the three most common proper fractions are approximately twice as prevalent in the enacted curriculum as in the written curriculum. When taken as a whole, these common fractions make up 42% of all proper fractions that appear in the written curriculum and 70% of all proper fractions included in the enacted curriculum. This statistic seems to indicate that students are having less exposure to a variety of proper fractions than the authors intended. Particular improper fractions make up a very small percentage (less than 1%) of the total number of fractions in either curriculum. Particular mixed numbers are also not very prevalent and those that appear most often in the written and enacted curricula are different.

As mentioned previously, reification of fractions is dependent upon the encapsulation of the two numbers in the fraction (e.g., "4" and "7") into one number (e.g., "$\frac{4}{7}$"). Evidence of such reification in association with proper and improper fractions and mixed numbers is rare in the enacted curriculum because nearly all fraction discourse is of the form "X of Y" where X is a fraction (e.g., two thirds) and Y is either another fraction or a contextual object (e.g., acre of land). Examples such as the ones presented here are ubiquitous throughout the five days:

Example 60 (EC).
*S: "Because he wants to buy **one half of the pan** that is two thirds full." (Day 1)*

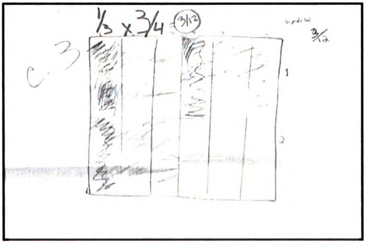
Example 61 (EC).
*T: "**One third of a half,** right? So what are you starting with, if you **have one third of a half,** what are you starting with?" (Day 3)*

Example 62 (EC).
*S: "And then, then we cut each of them into thirds, and colored in **two thirds of each sixteenths.**" (Day 5)*

This prevalence of "X of Y" is also found in the written curriculum in Problems 3.1-3.3; however, Problems 3.4 and 3.5 seem to contain less of this use. This may be partly attributed to the fact that the problems on the last two days are much less contextualized.

One feature of both the written and enacted curricula that might lead to the reification of fractions is the use of number sentences. That is, once the contextual situation (representing "X of Y") is translated into a number sentence the fractions can be treated primarily as numbers. However, this does not seem to be the case in Problems 3.1-3.3 in the written curriculum or in the enacted curriculum because the number sentences are so closely associated with the contexts and the iconic mediators. When the discussion in the enacted curriculum turns to the algorithm, the iconic mediators are used to demonstrate its reasonableness and this leads to additional references to fractions as "X of Y." The discussion of how the algorithm is represented in the iconic mediator for the number sentence "$\frac{1}{3} x \frac{3}{4} = \frac{3}{12}$" taken from Day 2 in the enacted curriculum provides a typical example of these discussions. The student work (including the iconic mediator and number sentence) and conversation associated with the Question are included in Example 63. Recall that "Trevor's idea" is the traditional algorithm for multiplying fractions.

Example 63 (EC - Student Work).



(1) *T: Does that make sense? Thanks, Grace. So if I think about Trevor's idea, in every one of the fourths [points to each of the fourths in the model], how many pieces did she split it into? How many pieces are in each?*

(2) *Ss: Three.*

(3) *T: Three. So could that explain the four groups of three in our denominator [points to the "12" in the number sentence]?*

(4) *Ss: Yeah.*

(5) *T: How about in each of those three fourths [points to the three one-fourth pieces in the model], how many did she color in?*

(6) *Ss: One.*

(7) *T: One here, one here, and one here [points to each of the one-fourth pieces in the model]. Would that be one times three?*

(8) *Ss: Yeah.*

Note that this discussion includes "pieces," "coloring," and "groups of" which are more indicative of the "X of Y" use of fractions. Further evidence that number sentences are closely linked to their associated iconic mediators is provided by statements in the written curriculum that advocate drawing connections between the number sentence, algorithm, and iconic mediator:

> ***Example 64 (WC).***
> *"If students should happen to notice that they can multiply the numerators and multiply the denominators, ask them to* ***use their drawings to show why they think this works.****" (TG, p. 60).*
>
> ***Example 65 (WC).***
> *"Students might point out that the numbers give the right answer. If students do,* ***direct them to consider both the numbers and the brownie pans.*** *" (TG, p. 61).*

It can certainly be argued that using iconic mediators to establish the traditional algorithm for fraction multiplication serves important purposes mathematically; however, it does not seem to facilitate the objectification of fractions in terms of reification or alienation because the discussions addressing algorithms are so closely linked to the actions of students.

In both the written and enacted curricula, estimation discussions seem to facilitate the objectification of fractions. For example, the following "Getting Ready" in Problem 3.3 from the written curriculum provides such opportunities:

Example 66 (WC).

Getting Ready for Problem 3

Estimate each product to the nearest whole number (1, 2, 3, . . .).

$\frac{1}{2} \times 2\frac{9}{10}$ $1\frac{1}{2} \times 2\frac{9}{10}$ $2\frac{1}{2} \times \frac{4}{7}$ $3\frac{1}{4} \times 2\frac{11}{12}$

Will the actual product be greater than or less than your whole number estimate?

Here, the question asks "Will the actual product be greater than or less than your whole number estimate?" Numbers are "greater than or less than," therefore, the use of these words promotes the objectification of fractions. This type of language is used in several places in the written curriculum in association with estimation. Another example is included here:

Example 67 (WC).
"*$\frac{9}{3}$ is 3 and $\frac{1}{2}$ of 3 is $1\frac{1}{2}$, which is greater than 1." (TG, p. 62)*

Again in Example 67, $1\frac{1}{2}$ "is greater than" 1 seems to imply the reification of "$1\frac{1}{2}$" as a mathematical object in its own right rather than as a whole number (i.e., 1) and a fraction (i.e., $\frac{1}{2}$). This is not the case for all estimation problems in the written curriculum; however, it is notable that such instances are common in the written curriculum in association with estimation.

Some language used in association with estimation in the enacted curriculum also indicates the possible reification of fractions. The following statement is made by a student that is rounding up $2\frac{9}{10}$ to 3:

Example 68 (EC).

S: "Because if I was going to round up the nine tenths like Graham did so" (Day 4)

"Rounding up" is language associated with numbers. That is, $2\frac{9}{10}$ (and $\frac{9}{10}$ itself) seems to be used here as a number. Another example comes from a student who is trying to multiply $4\frac{1}{2}$ by 1:

Example 69 (EC).

S: "Well I think you'd have to times it 'cause one times any number is always itself, so I think it'd be about four and a half because" (Day 4)

This student states explicitly that $4\frac{1}{2}$ is a number and therefore multiplying it by 1 would "be about four and a half." The next excerpt, taken from the same day indicates the complexity of determining whether or not reification has occurred. The Question being discussed is "$\frac{1}{2} \times 2\frac{9}{10}$":

Example 70 (EC).

(1) T: Could you help me with an estimate? If I said I was going to get one half of two and nine tenths, Graham, how could we think about estimating that answer?

(2) S1: Well, nine tenths, that's close to a whole, so -

(3) S2: One.

(4) S3: Yeah, so

(5) T: So what, Graham?

(6) S2: Nine tenths is close to a whole.

(7) T: Okay. So you could round that up into a whole and then two would be a three. So you would call this [points to $2\frac{9}{10}$] about three?

This excerpt is interesting because the student seems to be reifying "$\frac{9}{10}$" because he says "nine tenths is" in Line 6. This is in contrast to saying "nine tenths are" in which the plural verb is used because the nine is plural. In addition, the teacher uses "round that up" in Line 7 which also tends to be used in cases where a fraction is reified because numbers are rounded up. The complexity comes from these indications of reification combined with the use of "whole" in Lines 1, 5, and 6 by both the teacher and the students. Discourse indicative of reification would use "one" instead of "whole" to indicate that "$\frac{9}{10}$," because it is a number, should be compared to another number.

Words Signifying Multiplication

In the analysis of "multiplication," all versions of the words "multiplication" (e.g., "multiplying") were considered as were several other words that signified multiplication. These included "times," "product," "group," and "of." In all cases, only uses of each word directly related to multiplication were included in the analysis. Figure 11 presents a summary of the relative frequencies of the use of mathematical words that signify multiplication in the written and enacted curricula.

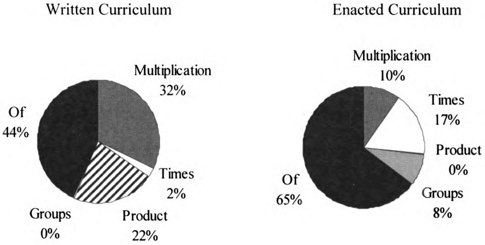


Figure 11. Relative frequencies of words signifying multiplication in the written and enacted curricula.

As Figure 11 indicates, "of" is the most common word signifying multiplication in both the written and the enacted curricula; however, proportionally it is more prevalent in the enacted curriculum. Forms of "multiplication" and "product" are common in the written curriculum (32% and 22% respectively), while forms of "multiplication," "times," and "groups" occur in substantial numbers in the enacted curriculum (10%, 17%, and 8% respectively).

Uses of "product" in the Curricula

"Product," while used actively in the written curriculum is used passively in the enacted curriculum (see Figure 4 on p. 75). That is, the class (both students and teacher) access statements using "product" from the written curriculum throughout the lessons; however, the only utterance of "product" in the enacted curriculum occurs when a student is reading aloud from the written curriculum. Also interesting is that "product," being a noun, is objectified. The uses of "product" can be characterized as both reified and alienated. For example, *"Use estimation to decide if the product is greater than or less* *than* $\frac{1}{2}$*."* occurs in Problem 3.2 in the Student Guide (p. 35). In this sentence, "product" is presented as an object and is void of human agency. Therefore, when comparing the written curriculum and the enacted curriculum, it is notable that "product" is used passively (the first stage in the word development model) in the enacted curriculum, while objectified (the final stage in the word development model) in the written curriculum.

Uses of "group" in the Curricula

"Group" and its derivatives are virtually absent in the written curriculum, occurring only once in the teachers guide, *"Other students group the 16 oz into sets of 3 and take two out of each set." (TG, p. 74)* In the enacted curriculum, "group" (also appearing as "groups" and "grouped") occurs 75 times. Of these, 73 instances (97%) occur in one of three primary contexts: (1) Drawing connections between multiplication of whole numbers and multiplication of fractions, (2) Confirming the algorithm for multiplying fractions, and (3) Discussing the multiplication of mixed numbers. In all three of these contexts, "group" is used in the following form: "X groups of Y" where X and Y are numbers. It is notable that it is always used in conjunction with "of." This is most likely a residual discursive pattern from whole number multiplication where it is commonly used (e.g., "four **groups of** six"). Table 9 shows the relative frequency of these instances along with an example of their use.

Table 9.

Examples of Uses of "group" in the Enacted Curriculum

Context	Relative Frequency	Example
Multiplication of whole numbers	23%	*S: "Because it's zero groups of six so you don't have any." (Day 1)*
Algorithm for multiplying fractions	15%	*T: "Where do I have ten groups of six?" (Day 2)*
Multiplication of mixed numbers	59%	*T: How could we find three and a quarter groups of two and eleven twelfths? (Day 4)*

These three categories appear at distinct times in the enacted curriculum. The instances related strictly to the multiplication of whole numbers occur during the Launch on the first day of the Investigation.[36] The second category occurs within a discussion of how the algorithm for multiplying fractions is represented in buying parts of brownie pans primarily on the second day of the Investigation. The instances in the final category occur primarily on Days 4 and 5 while multiplying combinations of fractions, whole numbers, and mixed numbers. The frequent use of "group" in the enacted curriculum and the lack of its use in the written curriculum indicate a difference in the ways in which multiplication is being talked about in the two curricula. Again, in this framework in

[36] Recall that the Launch is the first component of the Launch-Explore-Summary sequence of *Connected Mathematics* lessons.

which learning mathematics is defined as a change in participation in mathematical discourse, such differences may play a role in the change.

Uses of "times" in the Curricula

"Times" and its derivatives are much more prevalent in the enacted curriculum than in the written curriculum (17% compared to 2%). Four of the seven instances of "times" in the written curriculum are statements in which "times" is in between two objects being multiplied. Provided here are two examples:

> ***Example 71 (WC).***
> *"Yes. Takota broke the $2\frac{1}{2}$ into two parts and multiplied $1\frac{1}{6}$ separately by each part. First he multiplied $1\frac{1}{6}$ by 2, and then he multiplied $1\frac{1}{6}$ **times** $\frac{1}{2}$." (TG, p. 80)*
>
> ***Example 72 (WC).***
> *"Creating models for multiplication problems that involve combinations of fractions, whole numbers, and mixed numbers is a bit more complex than fraction **times** fraction situations." (TG, p. 71)*

In the other three instances, "times" follows a number. Two examples are provided here:

> ***Example 73 (WC):***
> *"It is four **times** as great as the answer to $\frac{1}{3}$ of $\frac{1}{2}$." (TG, p. 67)*
>
> ***Example 74 (WC):***
> *"You can repeatedly add $2\frac{1}{3}$ 10 **times** for the 10 days." (TG, p. 74).*

Both of these categories of use (Examples 71-74) seem to indicate a lack of objectification of fraction multiplication. In all four cases, the uses are operational (i.e., not structural) and personalized (i.e., not alienated). For instance, Example 74 describes a procedure (i.e., "repeatedly add") carried out by a person "You."

Of the 151 instances of "times" (including two instances of "timesed" and four instances of "timesing") in the enacted curriculum, 115 (76%) are of the first form mentioned above (i.e., an object "times" an object). These objects are present as number words (e.g., "two" or "two thirds") or names of numbers (e.g., "numerator," "mixed number"). Two examples are included here:

Example 75 (EC).
T: "Oh, I just set you up. It's like I paid you to say that. That's exactly what we're going to figure out tomorrow. What happens if I don't just have a fraction of a fraction, if I don't have a part of a part? What if I have a fraction of a whole number? What if I have a fraction of a mixed number? What if I have a mixed number ***times*** *a mixed number?" (Day 3)*

Example 76 (EC).
T: "Okay. I tell you what. Let's look at one more and let's see if we can just figure out what it's saying. Not necessarily how to solve it, but what is it saying if I have three and a fourth ***times*** *two and eleven twelfths? What does that mean? What does that mean?" (Day 4)*

The two most common types of objects being "timesed" in the enacted curriculum are whole numbers (74 instances) and fractions (18 instances).[37] The second type mentioned above (i.e., "times" following a number) occurs 26 times (17% of all instances of "times") in the enacted curriculum. Two examples are given here:

Example 77 (EC).
S: "It's like maybe we have to add two and eleven twelfths like, three ***times***, *and then another fourth time?" (Day 4)*

Example 78 (EC).
S: "Um, I knew I had to do two and a third ten and a half ***times***, *and I found that one half of two and one third was one and one sixth, so, um, I added all the wholes, which got me twenty-one, and then I added all the thirds, and three thirds is one whole, and I had one whole, two wholes, and then I had three wholes and a sixth, and a, one, a third and one sixth, and um, I can't add those so I changed the third to two sixths and I added those and it was three sixths, so altogether I got twenty-four and three sixths." (Day 5)*

[37] This whole number multiplication occurred either during the Launch which led into fraction multiplication or as a component of fraction multiplication (e.g., multiplying a whole number numerator times a whole number numerator).

In summary, the qualitative nature of the use of "times" in the written and enacted curricula is quite similar, primarily taking one of two forms, either between two numbers (e.g., three "times" four) or after a number (e.g., three four "times"). However, proportionally, "times" is much more common in the enacted curriculum than in the written curriculum (17% compared to 2%). Again, both of these forms are operational rather than structural and personal rather than alienated, and therefore are not likely to facilitate objectification of fraction multiplication.

Uses of "multiplication" in the Curricula

"Multiplication" and its derivatives are more common in the written curriculum than in the enacted curriculum (32% compared to 10%). Four forms appear in both curricula: "multiply," "multiplied," "multiplying," and "multiplication." Figure 12 summarizes their relative frequencies in each curriculum.

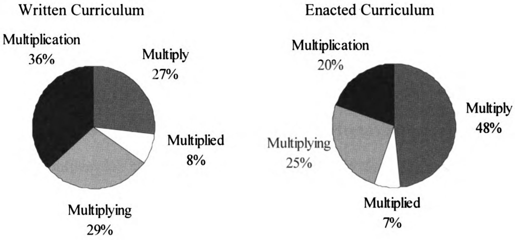


Figure 12. Relative frequencies of "multiplication" and its derivatives in the written and enacted curricula.

Figure 12 indicates that the operational forms of "multiplication" (i.e., "multiply," "multiplied," and "multiplying") are more prevalent than the structural form (i.e., "multiplication") in both curricula; however, the relative frequency of the structural forms varies, 36% and 20% in the written and enacted curriculum respectively. Recall from Figure 4 (p. 75) that object-driven word use is the final stage in word development and therefore the desired result. The fact that the written curriculum includes the objectified form of the word (i.e., "multiplication") more frequently than the enacted curriculum seems notable. Two examples of the use of "multiplication" from the written curriculum are included here:

> ***Example 79 (WC).***
> *"In this investigation, you will relate what you already know about **multiplication** to situations involving fractions" (SG, p. 32).*

> ***Example 80 (WC).***
> *"This helps students decide whether **multiplication** will help solve a problem in other situations" (TG, p. 60).*

In both instances, multiplication is not presented strictly as a process, rather as an entity in and of itself to be discussed. In these cases (and others like them in which multiplication is a noun), the word is objectified. Remember that the other requirement for objectification is alienation (i.e., no person needed). This requirement is also fulfilled here because multiplication is not presented as something to be "done" by a person in these instances. Similar examples of objectified use of "multiplication" occur in the enacted curricula:

> ***Example 81 (EC).***
> *S: "Um, division is like the opposite of **multiplication**" (Day 5)*

> ***Example 82 (EC).***
> *T: "Is **multiplication** always commutative, or just with some numbers?" (Day 4)*

A closer look at the word use in terms of the part of speech represented revealed statements in the written and enacted curricula in which "multiplication" served as an adjective rather than a noun. For example, the following statements in which multiplication functions as an adjective occur in the written and enacted curricula respectively:

> ***Example 83 (WC).***
> *"Question C provides additional **multiplication** practice and looks at the result of multiplying a number by its reciprocal" (TG, p. 81).*
>
> ***Example 84 (EC).***
> *T: "Just like I wrote three groups of eight, I wrote like this, groups of, I put a **multiplication** symbol in" (Day 1).*

Figure 13 quantifies the relative frequencies of the parts of speech (i.e., noun, verb, and adjective) of each use of "multiplication" and its derivatives in the written and enacted curricula.

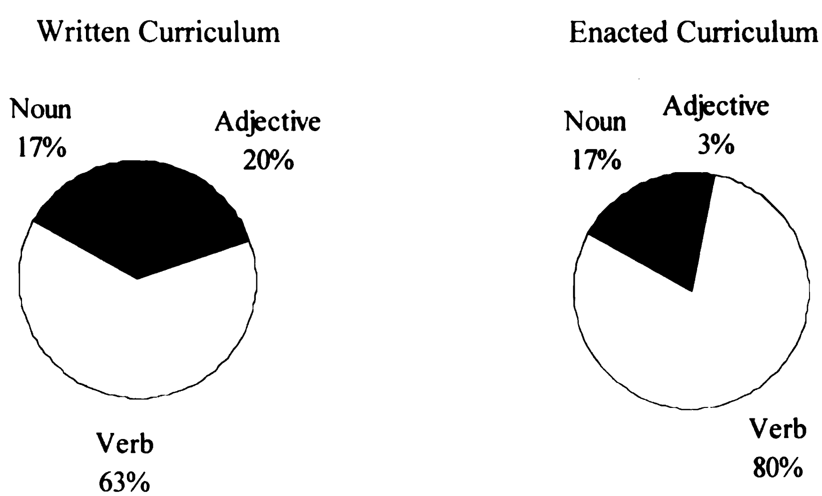


Figure 13. Relative frequencies of the parts of speech of "multiplication" and its derivatives in the written and enacted curricula.

Figure 13 indicates that multiplication is objectified (used as a noun) in equal frequency in the two curricula.

Uses of "of" in the Curricula

When all uses of "of" in the written and enacted curricula were counted, it was the third most common word in both curricula making up nearly 3% of all words in each text. For this analysis, however, only mathematical uses of "of" were considered. Even when considering only these instances, Figure 11 (p. 111) indicates that "of" was the most common word associated with multiplication in both the written and enacted curricula, 44% and 65% of all references to multiplication respectively. In fact, a stated goal in Problems 3.2 and 3.3 in the written curriculum is "Understand that finding a fraction *of* a number means multiplication.[38] Three statements explicitly addressing the meaning of "of" in fraction multiplication are present in the enacted curriculum. For example,

> ***Example 85 (EC).***
> *T: "Okay. We're going to come back and look at all these [a series of mathematical statements of the form A of B = C, where A, B, and C are proper fractions], 'cause you know what I'm going to do? I'm going to change that word* ***"of"****[points to "of" in "$\frac{9}{10}$ of $\frac{1}{6} = \frac{9}{60}$"] to a mathematic symbol. What you guys have done here. I love the way that you're all looking at me, I can tell you're listening. Thank you very much. When I take a part [points to "$\frac{2}{7}$" in "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"] of another part [points to "$\frac{1}{3}$" in "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"] , I can write that as a multiplication problem [writes "X" over "of" in "$\frac{2}{7}$ of $\frac{1}{3} = \frac{2}{21}$"]." (Day 2)*

In Example 85, "of" is replaced by "X" in a series of fraction multiplication number sentences to indicate the relationship between "of" and multiplication in these cases.

Approximately half of all instances of "of" in the written and enacted curricula are related to multiplication, 47% and 53% respectively. When examined for discursive

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[38] For a discussion of this goal in the written and enacted curricula, see pp. 50-54.

patterns, several categories of use emerged. These categories, along with their relative frequencies in the written and enacted curricula are provided in Figure 14.

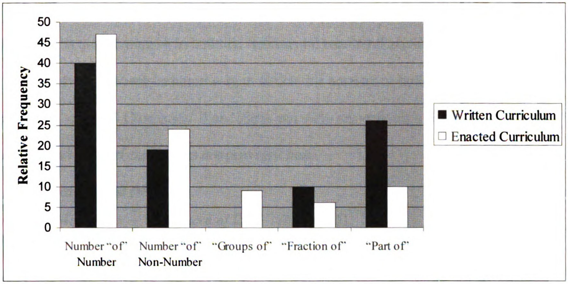

Figure 14. Relative frequencies of mathematical uses of "of" in the written and enacted curricula.

Figure 14 indicates *Number "of" Number* is the most common usage of "of" in both the written and enacted curricula, representing 40% and 47% of all mathematical uses of "of" respectively. The majority of these (65% in both curricula) are *Fraction "of" Fraction,* followed by *Fraction "of" Whole Number* (25% and 18% in the written and enacted curriculum respectively). Examples of these types from both curricula are provided here:

Example 86 (WC).
"How much is $\frac{1}{3}$ ***of*** *$\frac{2}{3}$?" (SG, p. 32)*

Example 87 (EC).
S: "Two sixths ***of*** *half." (Day 1)*

Example 88 (WC).
"Since $2\frac{9}{10}$ is almost 3, a reasonable estimate would be $\frac{1}{3}$ ***of*** *3 or $1\frac{1}{2}$." (TG, p. 71)*

Example 89 (EC).
T: "So what would one third ***of*** *sixteen be?" (Day 4)*

Figure 14 also indicates that the use of "of" in *Number "of" Non-Number* situations is quite common in both the written and enacted curricula, 19% and 24% respectively. The most common way in which this appears in both curricula is as a *Fraction "of a(n)" Contextual Word* (86% and 70% respectively). Examples from the written and enacted curriculum are provided here:

Example 90 (WC).
"Mali owns $\frac{4}{5}$ ***of*** *an acre of land." (SG, p. 35)*

Example 91 (EC).
S: "That is nine tenths ***of*** *a mile long." (Day 3)*

Two other categories of *Number "of" Non-Number* situations occur in reasonably substantial numbers in the enacted curriculum. Approximately 14% and 16% of the instances are *Fraction "of a part"* and *Fraction "of a whole"* respectively. Examples of these two types from the enacted curriculum are provided here:

Example 92 (EC).
T: "You want one third ***of just this orange part****. How can we think about doing that?" (Day 3)*

Example 93 (EC).
T: "Half ***of the whole thing****, right?"(Day 3)*

The *"Part of"* category is much more common in the written curriculum than the enacted curriculum (26% compared to 10%). The most common form of this category is *"part of a part"* in both the written and enacted curricula (33% and 44% respectively). Also common in the written curriculum are *"part of a whole"* (30%) and *"part of a(n)"*

Contextual Word (24%). Examples of these three types are provided here from the written curriculum:

Example 94 (WC).
"As you work on this problem, think about the size of the answer when you are finding a ***part of a part****." (TG, p. 60)*

Example 95 (WC).
"How many ***parts of the whole*** *are being bought?" (TG, p. 61)*

Example 96 (WC).
"Use a different colored pencil to show the ***part of the brownies*** *that Mr. Williams buys." (SG, p. 33)*

Besides *"part of a part,"* two other categories are present in more than 10% of the *"Part of"* category in the enacted curriculum: *"part of" fraction* (20%) and *"part of a whole"* (14%). Examples of all three types from the enacted curriculum are given here:

Example 97 (EC).
S: "They got ***part of a part****." (Day 2)*

Example 98 (EC).
S: "I filled in half because I knew it was going to be half of a bar, and then I made three ***parts of*** *that half and I filled in one half, er, one* ***part of*** *that half, and I got one sixth, because if you still had this half, there'd be these three pieces." (Day 3)*

Example 99 (EC).
T: "Did they take a ***part of the whole****?" (Day 2)*

"Fraction of," which is not very common in either curriculum (10% and 6% in the written and enacted curricula respectively) is nearly twice as prevalent in the written curriculum as in the enacted curriculum. The most common form of this type in the written curriculum is *"Fraction of a(n)" Contextual word*, representing 69% of the appearances in this category. In the enacted curriculum, the most common form (also representing 69% of the instances) is *"Fraction of a whole."* Examples of the most common types in each curriculum are included here:

Example 100 (WC).
*"What **fraction of** the goal did Nikki raise?" (SG, p. 36)"*

Example 101 (EC).
*T: "So then what **fraction of** this whole thing are they?" (Day 3)*

Interestingly, *"Groups of"* which represents 9% of the mathematical instances of "of" in the enacted curriculum is absent in the written curriculum. Examples of its use in the enacted curriculum are provided here:

Example 102 (EC).
*T: "We're not, Trevor, we're not subtracting. We're taking groups - I told you this was going to be tough today, right? Now we're not even having part of a part. Now we have **groups of** a part. Whooo. I told you we were going to have to think extra hard today. Elliot?" (Day 4)*

Example 103 (EC).
*T: "Three and one fourth **groups of** two and eleven twelfths." (Day 4)*

Overall, the most striking similarity between the use of "of" in the written and enacted curriculum is the sheer quantity of its use in both curricula. Beyond this, the categories of use of "of" in the curricula are also quite similar. It seems that the use of "of" does little to facilitate objectification of fraction multiplication except in that it has the potential to move students toward situations in which fractions become encapsulated as mathematical objects.

Summary

The question addressed in this summary is, "What does an investigation of the mathematical words in the written and enacted curricula allow us to see?" That is, "What do we know now about the relationship between the written and enacted curricula that we did not know before?"[39] Table 10 summarizes the findings from the investigation of mathematical words for the purposes of examination for overall conclusions.

Table 10.

Summary of "Mathematical Words" Analysis

Word	Written Curriculum	Enacted Curriculum
"Algorithm"	30 appearances	24 appearances
	87% on Day 5	Spread evenly across Days 3-5
	"Writing" algorithms most common (43%)	"Writing" (25%) and "Using" (29%) algorithms most common
	Both include references without use of "algorithm"	
	Both expect comparison or evaluation of algorithms (including "efficient" as an adjective)	
	Suggests strategies for facilitating the discovery of the traditional algorithm	Uses strategies suggested in the written curriculum and student notices the traditional algorithm
	Personalizes proposed strategies through the use of fictitious students	Personalizes traditional algorithm by connecting student's name to it

[39] The statements in this summary and throughout this analysis are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days in the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or the enacted curriculum as a whole. Rather, my statements highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.

Table 10 (cont'd)

Word	Written Curriculum	Enacted Curriculum
"Estimation"	50 appearances	23 appearances
	Spread evenly across Days 1-4	83% on Day 4
	Eight forms of "estimate"	Four forms of "estimate"
	62% objectified	43% objectified
	When using estimation strategies, uses discourse that promotes fraction reification (e.g., "greater than")	When using estimation strategies, uses discourse supporting fraction as a descriptor (e.g., "whole")
	-----	Includes other discourse associated with estimation (e.g., "round up")
"Fraction"	Most often used as number (72%)	Most often used as number (53%) and part (42%)
"Piece"	72 appearances of "part"	11 appearances of "part"
"Part"	120 appearances of "piece"	238 appearances of "piece"
"Halves" "Thirds" "Fourths" etc.	7 distinct words with 30 appearances	14 distinct words with 217 appearances

Table 10 (cont'd)

Word	Written Curriculum	Enacted Curriculum
"Number"	Most common in "mixed number" (29%) and "number line" (26%)	Most common in "whole number (30%) and "number sentence" (29%)
	Similar frequency (18% compared to 16%) of use of "number" alone (i.e., not in phrase such as "number line")	
Fraction (e.g., "$\frac{2}{3}$")	72% of distinct fractions are proper fractions (remainder are improper fractions or mixed numbers)	56% of distinct fractions are proper fractions (remainder are improper fractions or mixed numbers)
	57% of all fractions are proper fractions (remainder are improper fractions or mixed numbers)	83% of all fractions are proper fractions (remainder are improper fractions or mixed numbers)
	Both include the same five most common proper fractions	
	Five most common fractions represent 42% of all proper fractions	Five most common fractions represent 70% of all proper fractions
	Prevalence of "X of Y" (X and Y are fractions) primarily on Days 1-3	Prevalence of "X of Y" (X and Y are fractions) throughout all five days

Table 10 (cont'd)

Word	Written Curriculum	Enacted Curriculum
"Product"	22% of words signifying multiplication	0% of words signifying multiplication
	Objectified use	Passive use
"Group"	0% of words signifying multiplication	8% of words signifying multiplication
	-----	All uses in "X groups of Y"
	-----	Most common use is with multiplication of mixed number (59%)
"Times"	17% of words signifying multiplication	2% of words signifying multiplication
	Both include uses of two types (i.e., "three times four" and "three four times")	
"Multiplication"	32% of words signifying multiplication	10% of words signifying multiplication
	Both include four forms of "multiplication"	
	Both include more operational use than structural (i.e., objectified use represents 17% in both curricula)	

Table 10 (cont'd)

Word	Written Curriculum	Enacted Curriculum
"Of"	Third most common word in the written and enacted curricula	
	44% of words signifying multiplication	65% of words signifying multiplication
	Both explicitly and implicitly relate "of" to multiplication	
	Both use "of" most often in "X of Y" where X and Y are numbers (40% and 47% respectively). In 65% of the cases of "X of Y" in both curricula, X and Y are fractions	

Note. Dashes (i.e., "-----") indicate that the category is not applicable to the designated curriculum.

Table 10 and the more detailed analysis included within the chapter highlight many similarities and differences in the uses of particular mathematical words between the written and enacted curricula. The problem with such a table is that some of the richness and interpretation present within the text of the chapter is lost. However, the summary provided within the table allows a look at the data as a whole.

With object-driven word use (see Figure 4 on p. 75) as a stated goal for the use of mathematical words, it seems that, according to the evidence presented in the chapter and summarized in Table 10, the written curriculum affords objectification in association with more words than the enacted curriculum. That is, although some words are used and objectified in similar ways in both curricula (e.g., "algorithm," "multiplication), other word use that affords objectification (e.g., "estimation," "product") are more common in the written curriculum whereas word use that seems to offer less opportunities for objectification (e.g., "group," "times") are more prevalent in the enacted curriculum.[40]

[40] These differences are highlighted in Table 10.

CHAPTER 6: VISUAL MEDIATORS IN THE WRITTEN AND ENACTED CURRICULA

In this chapter, I describe the use of visual mediators in the written and enacted curricula. Recall that visual mediators are defined as symbolic artifacts created for the purposes of mathematical communication. Given the ubiquitous nature of visual mediators in mathematics at all levels, it is critical to investigate their use as part of this analysis. Sfard (2008) recognizes three categories of visual mediators: (1) Concrete (e.g., blocks, fraction strips), (2) Iconic (e.g., pictures, diagrams), and (3) Symbolic (e.g., "3 x 4 = 12," "2x + y = 14"). These three types may be seen as a trajectory through which students move in school mathematics. That is, concrete mediators are much more common in early elementary school, whereas symbolic mediators are the norm in most high school mathematics courses. Iconic mediators are often present beginning in early elementary through high school, increasing in complexity over the years. The decision to use "visual mediator" instead of the more commonly used "representation" comes from the fact that "representation" implies that there is a "real" mathematical object which is being represented. In this analysis, I am conceptualizing mathematical objects strictly as discursive constructs. Therefore, the visual mediator is as much the mathematical object as anything else.

Several additional features of visual mediators are worth noting. First, the power of visual mediators lies in their use, rather than in the artifact itself. For example, marshmallows and toothpicks serve little mathematical purpose until they are put together in such a way that allows an individual to make sense of mathematical terminology such as "vertex" and "edge." Second, visual mediators are extraordinarily useful for facilitating discussions around particular mathematical topics. That is, examining a graph, table, and equation allows an individual to examine distinctive features of a function. Each of these visual mediators serves certain purposes of their own and making connections between them can be even more enlightening. For example, discussions regarding the domain and range of a function may be facilitated by relating their symbolic and graphical forms. Third, all visual mediators are not created equal. Some visual mediators are more or less useful for particular students with particular topics in particular situations. That is, the usefulness of visual mediators is context dependent. Finally, the use of visual mediators in curricula involves many questions leading to pedagogical decisions, both by curriculum developers and teachers, such as: (1) Which visual mediators should be used to introduce and develop a particular mathematical topic? (2) How should the specific visual mediators be used? (3) When should students be expected to transition between visual mediators (e.g., concrete mediators and symbolic mediators)?

Sfard (2008) introduces two important goals for visual mediation in mathematical discourse. The first goal for students is "mediational diversity." That is, the ability to use a variety of visual mediators flexibly. She argues that students who display "mediational diversity" are better prepared to learn mathematics. Metaphorically, the mediationally flexible student can be compared to a person building a house with an extensive toolkit rather just a hammer and nails. The second goal for visual mediation is to move students toward symbolic form (e.g., "$\frac{3}{4} \times 12 = 9$"). Other visual mediators (e.g., concrete objects) are used to make sense of mathematical topics, but the goal in school mathematics is for individuals to operate mathematically in symbolic notation as much as possible. This is a great advantage over carrying marshmallows and toothpicks or unifix cubes with them from place to place. That being said, concrete mediators remain the norm in many situations outside of formal mathematics.

In this chapter, I will examine both the words associated with visual mediators (an extension of the previous chapter) and the actual use of the visual mediators themselves in the written and enacted curricula.

Words Signifying Visual Mediators

Both the written and enacted curricula include references to visual mediators in general as well as references to specific types of visual mediators. Table 11 provides examples of each category from the curricula.

Table 11.

Examples of General and Specific References to Visual Mediators in the Written and Enacted Curricula

Category	Written Curriculum	Enacted Curriculum
General	*"Create a **model or diagram** to find the exact answer." (SG, p. 36)*	*T: "As - whatever **model** you feel comfortable with, that's fine." (Day 4)*
Specific	*"Draw a **number line** under the **thermometer**." (TG, p. 65)*	*S: "So can we show it like anything, like **brownie pans** or anything?" (Day 4)*

There are many cases in which the context provides information that indicates that the statement which appears to be a "general" reference when only the utterance is considered is actually a reference to a specific type of visual mediator. Such an example from the enacted curriculum is provided here:

Example 104 (EC).
T: "Are you showing that anywhere in your ***drawing****" (Day 1)*

In this particular excerpt the teacher is referring to a drawing of a brownie pan model presented by a student. In these instances, the reference is categorized as "specific." Figure 15 illustrates the relative frequencies of "general" and "specific" references to visual mediators in the written and enacted curricula.

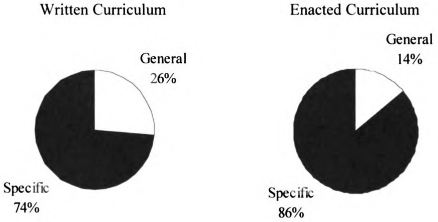

Figure 15. Relative frequencies of general and specific references to visual mediators in the written and enacted curricula.

Figure 15 indicates that both the written and enacted curricula have more references to specific visual mediators (e.g., number line) than visual mediators in general. The difference between the two types, however, is more pronounced in the enacted curriculum. It seems that more general references to visual mediators would facilitate the goal of mediational diversity especially when the general references mention the option of choosing an appropriate model.

As mentioned previously, Sfard (2008) describes three types of visual mediators, concrete, iconic, and symbolic. Figure 16 provides a summary of the relative frequencies of references to each of these types in the written and enacted curricula.[41]

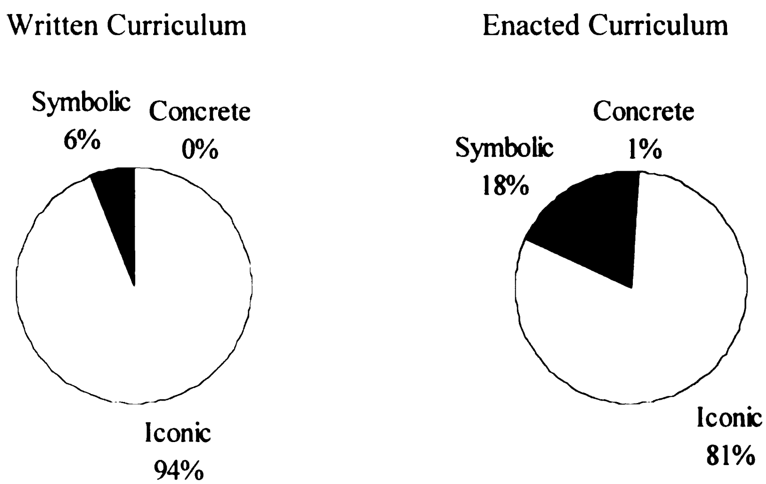

Figure 16. Relative frequencies of references to concrete, iconic, and symbolic visual mediators in the written and enacted curricula.

Figure 16 indicates that the vast majority of references to visual mediators in the written and enacted curricula refer to iconic mediators; however, this difference is more marked in the written curriculum than in the enacted. In fact, the written curriculum includes no mention of concrete mediators and only six references to symbolic mediators in this Investigation.[42] In the remainder of this section, I provide details of the analysis of references to each of the three categories of visual mediators and will end with a summary of the verb use associated with these visual mediators.

[41] Remember that these are word references and do not include the actual visual mediators themselves.
[42] Remember that the analysis of the enacted curriculum relied on videotapes of the lessons. As mentioned previously, in most cases the video camera followed the teacher. Therefore, it is unknown whether other groups used concrete mediators in their work. Here, as throughout this analysis, I relied completely on what was captured on the videotapes and in student work.

Words Signifying Concrete Mediators

Only one reference to a concrete mediator appears in the enacted curriculum (i.e., "blocks") and no such reference exists in the written curriculum. The reference on Day 4 of the enacted curriculum is included here:

Example 105 (EC).

> *T: "How could we do that? How could we get two thirds of all of these wholes? We have sixteen whole* ***blocks*** *here. How much, could you get a half of them?"*

Words Signifying Iconic Mediators

Ninety-four percent and 81% of the references to visual mediators in the written and enacted curricula respectively refer to iconic mediators (see Figure 16). The general words used to signify the iconic mediators include "diagram(s)," "drawing(s)," "model(s)," and "picture(s)." The written curriculum and enacted curriculum contain 85 and 95 appearances of these words respectively. Figure 17 summarizes the relative frequencies of these words in the written and enacted curricula.[43]

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[43] This figure only presents the uses of these words as nouns/objects. Verb use associated with visual mediators will be described later in the section.

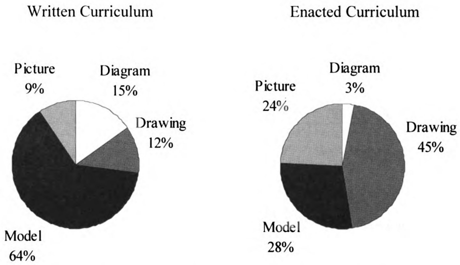


Figure 17. Words signifying iconic mediators in the written and enacted curricula.

Figure 17 indicates that the written curriculum uses “model(s)” most often to signify an iconic mediator, while the enacted curriculum most often uses “drawing(s).” The written curriculum uses the other words relatively infrequently; “diagram(s)” is the second most common representing 15% of the words signifying iconic mediators in the written curriculum. In contrast, two other words, “picture(s)” and “model(s),” each represent approximately 25% of the words signifying iconic mediators in the enacted curriculum. The significance of these findings is not obvious; however, because learning mathematics is conceptualized here as changes in mathematical discourse, the differences in the ways of referring to iconic mediators may impact such change. It is interesting to note that “representation(s),” a particularly common word used in mathematics to signify iconic mediators does not appear in either curriculum. Table 12 provides examples of the use of each of these words in the written and enacted curricula.

Table 12.

Examples of General Words referencing Iconic Mediators in the Written and Enacted Curricula

Word	Written Curriculum	Enacted Curriculum
Model(s)	*"Check to see if the context is one where it would make sense to use a number line **model**." (TG, p. 70)*	*T: "Now, what I want us to think about today is a fundraiser situation for sixth graders, so I want to share this with you and see what you think about a different kind of **model**." (Day 3)*
Drawing(s)	*"One student makes the **drawings** shown below." (SG, p. 34)*	*T: "Whoever did that **drawing**, can you explain your **drawing** real quick?" (Day 1)*
Diagram(s)	*"For Question A, how did you first mark your brownie pan? (into thirds, as in Part 1 of the **diagram** in the next column)." (TG, p. 61)*	*T: "What we want to do now is we want to take a look at what we did and see, is this what you did or different from what you did, did you get the same answer, maybe a different **diagram**, okay?" (Day 5)*

Table 12 (cont'd)

Word	Written Curriculum	Enacted Curriculum
Picture(s)	*"Draw a* ***picture*** *to show how the brownie pan might look before Mr. Williams buys his brownies." (SG, p. 33)*	*S: "One thing I don't get is, you know how when we said eight times seven and we showed like the* ***picture****. Like, that wouldn't work with fractions, would it?" (Day 2)*

Table 12 includes examples in which the word is used alone (e.g., "picture") as well as examples in which the word is used in conjunction with a specific type of iconic mediator (e.g., "number line model"). In addition to these two types, the curricula also include references to specific iconic mediators in which none of the general words mentioned above are present. An example from the written curriculum is provided here:

> ***Example 106 (WC).***
> *"These* ***thermometers*** *are like* ***number lines****." (SG, p. 34)*

Example 106 references two iconic mediators (i.e., thermometer, number line) without the use of the general words mentioned earlier. Instances of all three types (i.e., "model," "number line," and "number line model") were included in the remainder of the analysis of iconic mediators.

When all instances were examined, four primary categories of the functions of these references to iconic mediators emerged: (1) Constructing visual mediators (including both original construction and operating on the visual mediator), (2) Using visual mediators for various purposes, (3) Sharing visual mediators with the class, and (4) Mediational Diversity (including selecting, comparing, and translating between visual mediators). Table 13 summarizes the categories and provides examples of each type from the written and enacted curricula.

Table 13.

Examples of Functions of References to Iconic Mediators in the Written and Enacted Curricula

Category	Written Curriculum	Enacted Curriculum
Construct	*"**Draw a picture** to prove that your calculations make sense." (SG, p. 38)*	*T: "We're gonna use one brownie pan, two different colors. One color to show what's in the pan, and another color to show what we actually did." (Day 1)*
Use	*"If students should happen to notice that they can multiply the numerators and multiply the denominators, ask them to **use their drawings to show why they think this works**." (TG, p. 60)*	*T: "Kristen used just numbers and she **used a picture**." (Day 5)*
Share	*"When students **share their models** and exact answers, focus on their reasoning." (TG, p. 72)*	*T: "Okay. Are you guys **following his drawing**?" (Day 1)*

Table 13 (cont'd)

Category	Written Curriculum	Enacted Curriculum
Mediational Diversity	*"Both models are useful because each seems more natural in different situations." (TG, p. 65)*	*T: "Kelsey's trying it linear, like the longer model, and you try it the square and see if one is easier than another. " (Day 3)*

The analysis does not reveal any patterns regarding when particular words (e.g., drawing, diagram) are used. That is, particular words referencing general iconic mediator do not occur more often in one category than another. The words seem to be used interchangeably. Figure 18 illustrates the relative frequencies of each category of iconic mediator use in the written and enacted curricula.

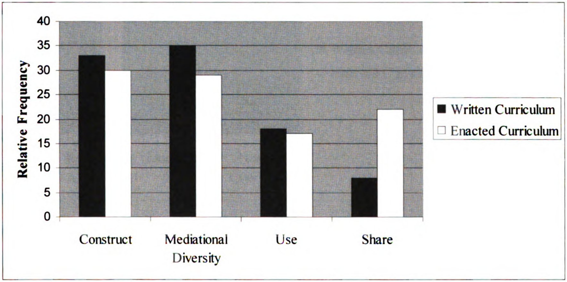


Figure 18. Relative frequencies of categories of the uses of iconic mediators in the written and enacted curricula.

Figure 18 shows that references involving constructing visual mediators and mediational diversity are the most common in both curricula. The largest discrepancy between the written and enacted curricula are the references to sharing iconic mediators; these references are much more common (more than three times as prevalent) in the enacted curriculum than in the written curriculum. This is not surprising given that the written curriculum may include one suggestion to encourage students to share their iconic mediators and this suggestion may be enacted multiple times in the classroom.

References to three primary types of iconic mediators are found in the written and enacted curriculum: rectangular area models (a partitioned rectangle of dimensions a x b where a and b are both greater than 1), linear area models (a partitioned rectangle of dimensions a x b where either a or b is 1), and number lines. Figure 19 summarizes the relative frequencies of references to these types of iconic mediators in the curricula.

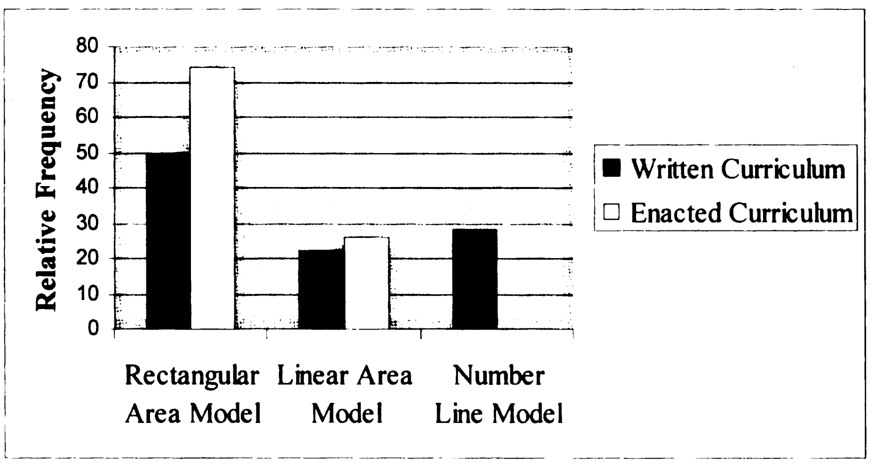


Figure 19. Relative frequencies of references to iconic visual mediators in the written and enacted curricula.

Figure 19 indicates that both the written and enacted curricula reference rectangular area models most often. These usually refer to models of brownie pans. Examples are provided here from each curriculum:

> ***Example 107 (WC).***
> *"Can someone share a way to mark the **brownie pan** so it is easy to see what part of the whole pan is bought?" (TG, p. 61)*
>
> ***Example 108 (EC).***
> *S: "But we put it all on one **brownie pan**?" (Day 1)*

Less common in both curricula are references to linear area models. These most often refer to fundraising "thermometers." Examples are provided here from each curriculum:

> ***Example 109 (WC).***
> *"To figure out the new length, the student divides the whole **thermometer** into pieces of the same size." (SG, p. 35)*
>
> ***Example 110 (EC).***
> *T: "I'm going to use a **long, skinny model**." (Day 3)*

Finally, references to number lines represent 28% of the references to iconic mediators in the written curriculum, but such references are absent in the enacted curriculum. An example from the written curriculum is provided here:

> ***Example 111 (WC).***
> *"If you partition the **number line** this way, there will be four sections in each third or 3 x 4 = 12 parts in the whole." (TG, p. 65)*

It is certainly notable that the enacted curriculum makes no reference to number lines. In terms of objectification, this is very important because number line models are more effective facilitators of the reification of fractions than area models. More will be said about this later in the chapter.

Words Signifying Symbolic Mediators

The only words signifying symbolic visual mediators in the written and enacted curricula come in the form of the phrase "number sentences(s)." This phrase appears in both curricula; however, it is more prevalent in the enacted curriculum than in the written curriculum (18% compared to 6% of all references to visual mediators). Examples are included here to illustrate its use:

> ***Example 112 (WC).***
> *"It is helpful to write the **number sentence** by the models." (TG, p.61)*
>
> ***Example 113 (EC).***
> *S: "That's what I was thinking, but I don't how I do, like do the **number sentence.**" (Day 4)*

In both the written and the enacted curricula, the vast majority (83% and 86% respectively) of the references to symbolic mediators address writing a number sentence. The remaining references involve either a comparison of two or more number sentences or a comparison of the result of a number sentence to an answer acquired using a different strategy.

Verbs Associated with Visual Mediators

Many of the verbs associated with constructing iconic mediators in this Investigation are derivatives of the nouns discussed earlier. Table 14 summarizes these verbs as they are found in the written and enacted curricula along with "write," a verb associated with the construction of symbolic mediators.

Table 14.

Verbs associated with Visual Mediators in the Written and Enacted Curricula

Verb	Written Curriculum	Enacted Curriculum
Model	Model(s)	Model
	Modeling	Modeled
Draw	Draw	Draw
	Drew	Drew
	Drawing	Drawing
	Drawn	
Represent	Represent	Represent(s)
	Representing	
Write	Write(s)	Write
	Writing	Writing

Table 14 indicates that virtually the same set of verbs associated with visual mediators is present in both curricula. It is notable that although "representation" does not appear in either curriculum as a noun, derivatives of "represent" as a verb are present in both the written and enacted curricula. Table 15 provides examples of the use of each family of verbs.

Table 15.

Examples of Uses of Verbs associated with Visual Mediators in the Written and Enacted Curricula

Verb	Written Curriculum	Enacted Curriculum
Model	*"You have developed some strategies for **modeling** multiplication and finding products involving fractions." (SG, p. 37)*	*T: "Can you **model** it?" (Day 4)*
Draw	*"In the last problem, we **drew** models to multiply with mixed numbers." (TG, p. 75)*	*S: "You have to **draw** ten of these. They did this and that." (Day 5)*
Represent	*"How would you **represent** $\frac{1}{4}$ x $\frac{2}{3}$ on a number line?" (SG, p. 34)*	*T: "Well, you need sixteen something. What do you want to **represent** your ounces?" (Day 4)*
Write	*"What number sentence could I **write** for Question A? ($\frac{1}{2} x \frac{2}{3} = \frac{2}{6}$)" (TG, p. 61)*	*T: "That's how much they ended up with. But help me **write** my number sentence." (Day 2)*

As mentioned previously, three of the four verb families presented in Tables 14 and 15 (i.e., "model," "draw," and "represent") are used in reference to iconic mediators whereas "Write" is used in reference to symbolic mediators. Figure 20 summarizes the relative frequencies of these verbs in the written and enacted curricula.

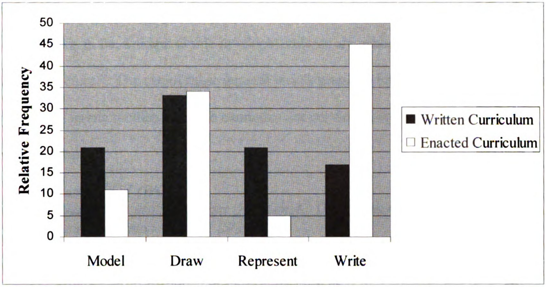


Figure 20. Relative frequencies of verbs associated with visual mediators in the written and enacted curricula.

Figure 20 indicates that 75% of the verbs associated with visual mediators in the written curriculum are used in conjunction with iconic mediators (i.e., "Model," "Draw," and "Represent") This frequency increases to 83% if the less common verbs are included because they are all associated with iconic mediators, leaving just 17% of the verbs associated with symbolic mediators in the written curriculum. In contrast, the corresponding frequencies in the enacted curriculum are 55% associated with iconic mediators and 45% associated with symbolic mediators. This makes sense given that references to "Number sentence(s)" are also more common in the enacted curriculum, 18% compared to 6% in the written curriculum. Figure 20 further indicates that "Draw" and its derivatives are the most common verbs associated with iconic mediators in both the written and enacted curricula (33% and 34% respectively).

Recall that one of the goals of visual mediator use is mediational flexibility (i.e., the ability to use multiple forms of visual mediators and to move flexibly between them).

Several types of references to this issue are found in the written curriculum. First, there are instances in the Student Guide in which students are asked to construct a "model" or draw a "picture." The use of these general words seems to imply acceptance of a variety of types of iconic mediators. Two examples are provided here to illustrate this phenomenon:

> ***Example 114 (WC).***
> *"Choose two problems from Question A. Draw a **picture** to prove that your calculations make sense." (SG, p. 39)*
>
> ***Example 115 (WC).***
> *"Your group is responsible for creating a **model** that you can use in the summary to talk about your assigned problem." (TG, p. 76)*

Second, the written curriculum suggests that the teacher encourage students to share a variety of models. Two examples illustrating such suggestions are included here:

> ***Example 116 (WC).***
> *"Lee used a brownie pan model for part (1a). Who can share a number line model?" (TG, p. 67)*
>
> ***Example 117 (WC).***
> *"Look for **various models** to use in the summary." (TG, p. 72)*

The enacted curriculum contains many statements in which the relationships and differences between visual mediators are highlighted. In addition, the teacher selects a variety of visual mediators to be presented by the class. Examples of statements addressing these issues in the enacted curriculum are included here:

> ***Example 118 (EC).***
> *T: "So you need to estimate what you think the answer should be, about how much, **you need to create some sort of model or diagram**, so that you can show how you solved it, and then when you're done try to write a number sentence." (Day 4)*

Example 119 (EC).
T: "Cool. Okay. Now we're all going to switch gears over there, and we have ***three different drawings. Maybe they're similar, maybe they're different. We're going to take a look at them and see.*** *" (Day 5)*

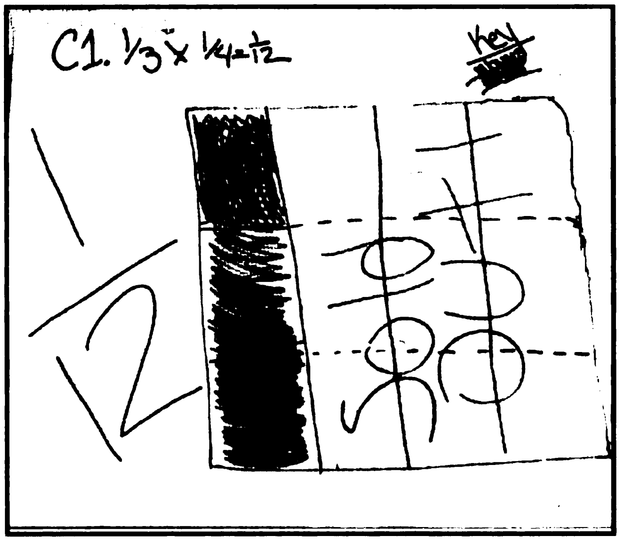
Example 120 (EC).
T: "Could, could hers be a picture of Katie's?" (Day 5)

In Example 118, the teacher is asking the students to create "some sort of model or diagram." Her language here indicates openness to several possible types of visual mediators. Examples 119 and 120 indicate drawing connections between iconic mediators presented by several students.

In addition to these references to mediational flexibility between iconic mediators, references to the relationship between iconic mediators and symbolic mediators are also present in both the written and enacted curricula. Examples of such statements from both curricula are provided here. Recall that "Trevor's way" is the traditional algorithm for fraction multiplication:

Example 121 (WC).
"It is helpful to write the number sentences by the models. By doing this, students typically notice that you can multiply the numerators and denominators to find the product." (TG, p. 61)

Example 122 (EC - Student Work).



S: *"What we did is, we knew that one fourth meant there was four pieces, so we divided it up into four pieces [points to each fourth]. And then it says that we had one out of the four pieces, so we made that [points to shaded fourth], and then we had to divide that up into three pieces [indicates thirds], so we divided that up into three pieces, and we had one of those three pieces, which is there [points to shaded third]. And then we divided the rest of them up like if they were there [indicates dashed lines], we divided those up into three, and we ended up get, that gave us a total of twelve, and so we ended up getting one twelfth [points to double-shaded piece], which we also figured out like if you did it Trevor's way, too [points to number sentence "$\frac{1}{3}x\frac{1}{4}=\frac{1}{12}$"]."* *(Day 2)*

Finally, both curricula include references to the relationship between two or more symbolic mediators. Examples are included here:

Example 123 (WC).
"To raise the issue, list the number sentences solved so far on the board (e.g. $\frac{2}{3}x\frac{1}{2}=\frac{2}{6}$), then ask: What patterns do you see between the fraction numerators and denominators and their products?" (TG, p. 67)

Example 124 (EC).
T: *"They're the same or they're, you're saying they're opposite and you're saying they're the same thing, those two number sentences?" (Day 5)*

Example 123 from the written curriculum suggests listing a series of number sentences in hopes that students will recognize the pattern of the traditional algorithm for fraction multiplication. Example 124 also addresses the relationship between symbolic mediators, but in this case the teacher asks the students about the equivalence of two number sentences.

References to a variety of carefully selected visual mediators and discussions regarding the relationships between them in both the written and enacted curricula enhance the possibility of accomplishing both of the visual mediation goals put forth here (i.e., mediational flexibility, fluency in the use of symbolic mediators).

Use of Visual Mediators

To clarify, the difference between this section and the previous one is that the previous section highlighted the uses of words associated with visual mediators. For example, it answered questions such as "How many times does "number sentence" appear in each curriculum and how is the phrase used?" In contrast, this section addresses the actual use of visual mediators in the curricula. It answers questions such as, "How many number sentences are present in each curriculum and how are they used?"

The primary use of all visual mediators in this Investigation in both curricula is to construct narratives (i.e., mathematical statements) addressing fraction multiplication. This use of visual mediators will be detailed in the next chapter (*Endorsed Narratives in the Written and Enacted Curricula*); therefore, this section will address their use more generally. All three types of visual mediators (i.e., concrete, iconic, and symbolic) are present in the enacted curriculum, whereas the written curriculum does not contain concrete visual mediators. It may seem somewhat odd to expect a written curriculum to contain concrete mediators (e.g., cubes); however, some written curricula (e.g., *Bridges in Mathematics*, *Everyday Mathematics*) include a variety of such materials. In fact, *Connected Mathematics* provides some concrete mediators (e.g., fraction strips) for use in other Investigations. In any case, the written curriculum does not make reference to concrete visual mediators in this Investigation.[44] It should be noted; however, that many "general" references to visual mediators are included and it would certainly be possible for students to choose to "model" a problem using concrete objects. In fact, it is one of these "general" references that led to the one instance of concrete visual mediator use in the enacted curriculum. Figure 21 summarizes the relative frequencies of the three types of visual mediators in each curriculum.

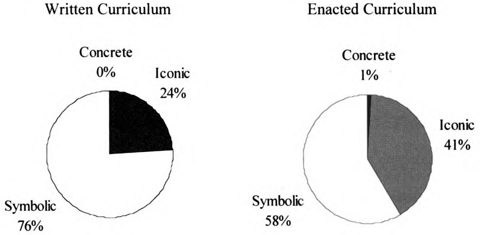


Figure 21. Relative frequencies of the types of visual mediators in the written and enacted curricula.

[44] The truth of this statement can be debated because the use of some of the iconic mediators prevalent in this Investigation (e.g., model of a brownie pan) is enhanced by students' possible experiences and/or imagining of the actual concrete object itself. Due to the fact that it is impossible to know whether students are operating on pictures of brownie pans or an image of the actual brownie pans, brownie pans and the like are not included as concrete mediators for the purposes of this investigation.

Figure 21 indicates that symbolic mediators (i.e., number sentences) are the most prevalent type of visual mediator in both the written and enacted curricula. Their prevalence, however, is more pronounced in the written curriculum than in the enacted curriculum. That is, the discrepancy between symbolic and iconic visual mediators is greater in the written curriculum than in the enacted curriculum, a difference of 51% and 17% respectively. As with many statistical discrepancies in this analysis, it is difficult to determine how much of this difference is attributable to the differences between the two forms of curricula, written and enacted. In this case, it seems likely that the consideration of the number of pages in the textbooks would influence the decision to include less iconic mediators because they take up more space in the text.

It is interesting to note that the relative frequencies of *references* to iconic and symbolic mediators in the written and enacted curricula differ substantially from the presence of the visual mediators themselves in the curricula (see Figures 16 and 19). The relative frequencies of references to iconic mediators in the written and enacted curricula are 94% and 81% respectively compared to 24% and 41% here. The relative frequencies of references to symbolic mediators in the written and enacted curricula are 6% and 18% respectively compared to 76% and 58% here. That is, it is more common to reference iconic visual mediators than to include them in the curricula, whereas it is more common to include symbolic visual mediators in the curricula than to reference them. Again, this may be related to space limits in written text materials.

The remainder of this section provides details of the categories of these types of visual mediators (i.e., concrete, iconic, and symbolic) and their use in the written and enacted curricula.

Concrete Mediators

Remember that concrete visual mediators never explicitly appear in this Investigation in the written curriculum and they are used only once on Day 4 in the enacted curriculum. Here, I present that use. The students were asked to complete the following problem from p. 36 in the Student Guide:

Example 125 (WC).

For each question:

- Estimate the answer.
- Create a model or a diagram to find the exact answer.
- Write a number sentence.

A. The sixth-graders have a fundraiser. They raise enough money to reach $\frac{7}{8}$ of their goal. Nikki raises $\frac{3}{4}$ of this money. What fraction of the goal does Nikki raise?

B. A recipe calls for $\frac{2}{3}$ of a 16-ounce bag of chocolate chips. How many ounces are needed?

C. Mr. Flansburgh buys a $2\frac{1}{2}$-pound wheel of cheese. His family eats $\frac{1}{3}$ of the wheel. How much cheese have they eaten?

D. Peter and Erin run the corn harvester for Mr. McGreggor. They harvest about $2\frac{1}{3}$ acres each day. They have only $10\frac{1}{2}$ days to harvest the corn. How many acres of corn can they harvest for Mr. McGreggor?

Four students are using 16 multi-colored wooden cubes to model Problem 3.3B when the teacher walks up to the group:

Example 126 (EC).

(1) *T: They're saying if I took, what you want to do is, you want to get two thirds of all of these [indicates the 16 cubes].*

(2) *S1: Okay.*

(3) T: How could we do that? How could we get two thirds of all of these wholes [indicates the 16 cubes]? We have sixteen whole blocks here [indicates the 16 cubes]. How much, could you get a half of them?

(4) S2: Yeah.

(5) T: Get a half. Show me a half.

(6) S1: It would be eight of them [moves the 16 cubes into two groups of 8].

(7) T: How did you know it was going to be eight of them?

(8) S2: 'Cause half of sixteen's eight.

(9) T: Okay [moves cubes back into one group of 16]. So now what if I wanted you to get a fourth of them? Could you do that?

(10) S1: Four [moves the 16 cubes into four groups of four].

(11) T: Why would it be four?

(12) S2: 'Cause four divided by sixteen is four.

(13) T: Sixteen divided by four is four, you mean?

(14) S2: Yeah.

(15) T: Okay [moves cubes back into one group of 16]. So, how could I think about getting thirds now?

(16) S2: Well, it would be an odd number, because you never, I mean it would have a remainder.

(17) T: Could we do that? Could we have parts of an ounce?

(18) S3: Yeah.

(19) T: Why not?

(20) S1: Yeah.

(21) T: Can we have parts of ounces?

(22) S1: Yeah.

(23) *T: So think about that [teacher is interrupted by another student – she turns her attention to him].*

In Example 126, the students use the 16 cubes to represent the 16 ounces in the problem. The teacher, perhaps thinking that finding two-thirds of the cubes is a difficult place to start, begins with "a half" which is a more familiar fraction. She then asks the same question about getting "a fourth of them" and then asks about finding a third. This is when one of the students realizes that unlike half and fourth, finding a third would give an "odd" number, adding that it would "have a remainder." The teacher interprets this to mean "parts of an ounce" and proceeds to ask if it is possible to have "parts of ounces." Recall that the problem actually asks for two-thirds, so even if the students figure out one-third of 16, their work is not finished. The teacher is interrupted by another student, but the group continues their discussion:

Example 127 (EC).

(24) *S2: Yeah. Or six or seven, so try over here. Let's try one. I don't think that you can divide sixteen by four, er, three.*

(25) *S1: It'd be harder.*

(26) *S2: Well, here. Let's try and divide it by three and see what the remainder is [Student 4 divides the cubes into five groups of 3 which leaves one block alone]. Let's see - yeah, 'cause if we split it - yeah, we have, well, what if we divided all of them into halves?*

(27) *S1: Divided them into five. So, it would be three wholes and*

(28) *S2: If we divided this one into fifths, every, everybody would have like, um, three and one fifth.*

(29) *S1: What?*

(30) *S2: I think I get it.*

In this excerpt, the students divide the cubes into five groups and see that one is left alone. After some discussion and looking at and touching the cubes, but not rearranging them, one of the students gives the answer of "three and one-fifth." It is unclear if the other students make sense of this answer, because the fourth student in the group recommends using an iconic mediator and the cubes are not used again by this or any other group.

Because we are examining the relationship between the written and enacted curricula, it is notable that this use of concrete mediators is a direct result of the request in the written curriculum to "Create a model or diagram to find the exact answer." (SG, p. 36). The students in the group illustrated here choose blocks to model this situation. Blocks (and other concrete objects) are available to students in the room at all times to use when they decide it is appropriate; however, most students (all but this group on Day 4 for this particular problem) choose not to use concrete mediators to solve the problems.[45] Iconic mediators are much more common.

Iconic Mediators

Types of Iconic Mediators

As mentioned previously, 24% and 41% of the visual mediators in the written and enacted curricula respectively are iconic mediators. These represent 34 iconic mediators in the written curriculum and 42 iconic mediators in the enacted curriculum. The four types of iconic mediators present in the curricula are (1) Contextual pictures, (2) Rectangular area models, (3) Linear area models, and (4) Number lines. Figure 22 summarizes the relative frequencies of these types of iconic mediators in the written and enacted curricula.

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[45] Again, this refers to groups which appear on the videotapes.

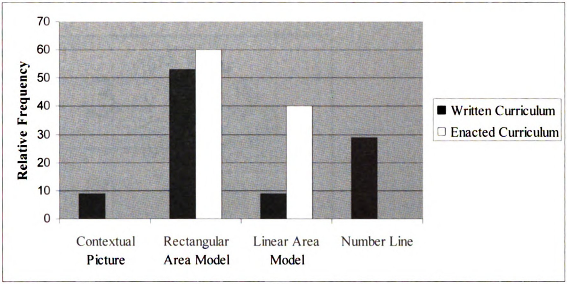
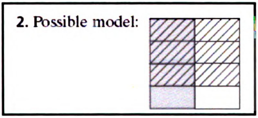


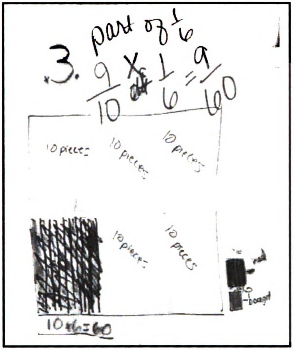
Figure 22. Relative frequencies of iconic mediators in the written and enacted curricula. Rectangular area models are the most common in both the written and enacted curricula, representing 53% and 60% respectively. Provided here are examples of rectangular area models from both the written and enacted curricula.

Example 128 (WC).



(TG, p. 64)

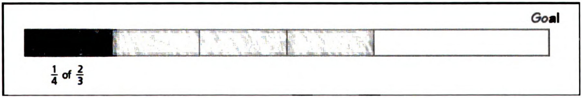
Example 129 (EC - Student Work).



(Day 2)

The other type of iconic mediator present in the enacted curriculum is the linear area model. This type is also present, although not common, in the written curriculum. Provided here is an example of a linear area model from the written curriculum.[46]

Example 130 (WC).



(SG, p. 34)

Finally, two additional types of iconic mediators are present in the written curriculum but absent in the enacted curriculum. The first of these is contextual pictures,

[46] All student work from the enacted curriculum containing linear area models was completed on the whiteboard and not saved, however as will be discussed later in the section, the linear area models in the enacted curriculum look very similar to the one from the written curriculum included here.

representing 9% of the iconic visual mediators in the written curriculum. An example is provided here:

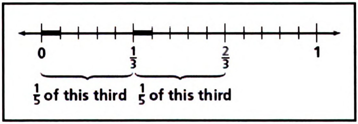
Example 131 (WC).

(SG, p. 35)

Example 131 accompanies a problem about painting two-thirds of a stripe that is nine-tenths of a mile long. It could be argued that this contextual picture (and the others) is accessed in the enacted curriculum because the students complete these problems.

The final type of iconic mediator is the number line model. It is absent in the enacted curriculum, but is represented by 29% of the iconic mediators in the written curriculum. An example from the written curriculum is provided here:

Example 132 (WC).



(TG, p. 66)

When comparing the relative frequencies of the actual use of iconic mediators to the relative frequencies of references to iconic mediators (see Figures 19 and 22), the prevalence of the actual and referenced iconic mediators are quite similar in most cases. For example, the references to the number line in the written curriculum represent 28% of all references to iconic mediators compared to representing 29% of the actual visual mediators. The largest discrepancies occur with linear area models. In the written curriculum, 22% of the references to iconic mediators referred to linear area models compared to only 9% of the actual iconic mediators. In the enacted curriculum, 26% of the references to iconic mediators referred to linear area models compared to 40% of the actual iconic mediators. That is, linear area models are referenced more than twice as often as they are included in the written curriculum, whereas they are included more than four times as often as they are referenced in the enacted curriculum.

Probably most notable in this section is the absence of number lines in the enacted curriculum. As mentioned previously, number lines may facilitate the reification of fractions as mathematical objects (i.e., numbers) in their own right in contrast to the "part of a whole" use of fractions because students need to make decisions regarding the location of a particular fraction (i.e., between which two numbers). The placement of a fraction or mixed number between whole numbers affords discussions regarding whether the fraction is greater than or less than particular whole numbers. This language (i.e., "greater than," "less than") is reserved for numbers and therefore may serve to facilitate the reification of fractions. NCTM (2000) made the following statement regarding the affordance of number lines in particular and multiple representations of fractions in general:

> Different representations often illuminate different aspects of a complex concept or relationship. For example, students usually learn to represent

> fractions as sectors of a circle or as pieces of a rectangle or some other figure. Sometimes they use physical displays of pattern blocks or fraction bars that convey the part-whole interpretation of fractions. Such displays can help students see fraction equivalence and the meaning of the addition of fractions, especially when the fractions have the same denominator and when their sum is less than 1. Yet this form of representation does not convey other interpretations of fraction, such as ratio, indicated division, or fraction as number. Other common representations for fractions, such as points on a number line or ratios of discrete elements in a set, convey some but not all aspects of the complex fraction concept. Thus, in order to become deeply knowledgeable about fractions—and many other concepts in school mathematics—students will need a variety of representations that support their understanding. (p. 69)

Here, NCTM promotes multiple representations and uses of fractions, mentioning number line as a possible alternative to area models to serve other "aspects of the complex fraction concept." In the next section, I describe the ways in which iconic mediators are used in the written and enacted curricula.

Use of Iconic Mediators

The written curriculum provides suggestions for the use of all types of iconic mediators except for the contextual pictures. It is likely that these pictures have been included in the written curriculum to assist students in understanding particular contexts. For example, the picture of the truck shown previously (see Example 131 on p. 159) may help students imagine how a truck paints a stripe on a road.

Several statements in the written curriculum provide guidance regarding the use of rectangular area models. For example:

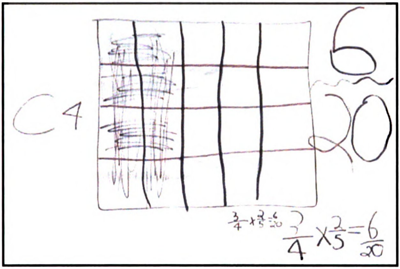
> ***Example 133 (WC).***
> *"Draw a picture to show how the brownie pan might look before Mr. Williams buys his brownies.* ***Use a different colored pencil*** *to show the part of the brownies that Mr. Williams buys." (SG, p. 33)*
>
> ***Example 134 (WC).***
> *"Can someone share a way to mark the brownie pan so it is easy to see what part of the whole pan is bought? (Here is an opportunity to suggest that* ***using horizontal and vertical lines*** *makes it clear what is happening." (TG, p. 61)*

Examples 133 and 134 refer to ways to use the brownie pan model to illustrate fraction multiplication, the first suggesting the use of two colors and the second suggesting the use of horizontal and vertical lines. Rectangular area models are often used in similar ways in the enacted curriculum. In fact, the enacted curriculum contains 158 references to coloring either a rectangular area model or a linear area model. Two examples are provided here:

> ***Example 135 (EC).***
> *T: "Okay. I used* ***red*** *to show how much is left in the pan. I'm gonna use a different* ***color*** *to show what we're gonna buy of it. Natalie, could you come up here on this, could you show us what three fourths of that would look like?" (Day 1)*
>
> ***Example 136 (EC).***
> *S: "Well, I knew we had one half of the bar left, so I had to* ***color*** *in one half, and then I had to divide each half into three pieces, and then I* ***colored*** *in two pieces of the one half, and I got two sixths." (Day 3)*

There is also discussion in the enacted curriculum regarding drawing lines on the rectangular and linear area models. In fact, 40 statements addressing drawing lines on these models are present in the enacted curriculum. In addition, 85 statements address cutting (i.e., using lines to "cut" the rectangular or linear area model into pieces) and 66 statements address dividing (i.e., using lines to "divide" the rectangular or linear area model into pieces). Examples of statements associated with rectangular area models that involve lines, cutting, and dividing are provided here with additional explanations and student work. Example 137 illustrates student work presented in association with Question C-4 from Problem 3.1:

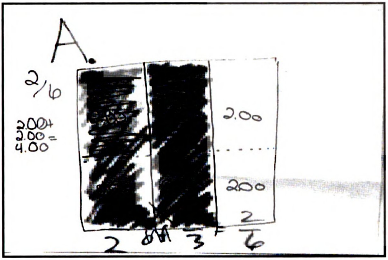
Example 137 (EC - Student Work).



S: "Do the rest of them, the ***vertical lines****." (Day 2)*

The statement included in Example 137 is made by a student encouraging the student constructing the rectangular area model to draw all of the vertical lines clearly. Originally, the student had only drawn the two leftmost vertical lines with marker. The rest were drawn lightly with pencil. This statement expresses the students' desire for all of the vertical lines to be drawn clearly for the group's presentation to the class. Example 138 illustrates student work associated with Question A from Problem 3.1:

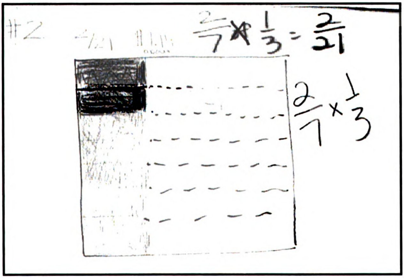
Example 138 (EC - Student Work).



> S: *"We drew two thirds and then we **cut** the two thirds in half to see what he would buy. And then, he would buy two sixths, because if you **cut** it in half he would get three, and then three which is six. And that's equivalent to one third. So he would either, he would get two sixths or one third." (Day 1)*

The student's explanation of the work provided in Example 138 emphasizes "cutting" the rectangular area model using vertical and horizontal lines. The final student work and associated student explanation illustrates the use of "dividing":

Example 139 (EC - Student Work).



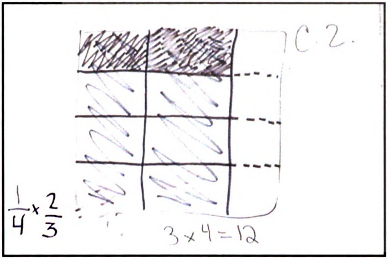
*S: "Well, um, he had one, there was one third of the pan left over, so we had to **divide** it into seven pieces, and he bought two sevenths, two sevenths of the one third, and if you **divide** all the rest into seven pieces, it would be twenty one pieces, so all together he bought two twenty oneths of the whole pan." (Day 1)*

The explanation included in Example 139 is similar to the previous one. Again a student is describing the use of lines except that here the student talks about "dividing" it into pieces rather than "cutting" it.

It is important to note here that in all of these cases and many others, the discussions of these models contained many hand movements in a vertical and horizontal direction as the students and teacher talked about the lines drawn on the models. Also notice that in these examples of student work, some of the lines have been extended in order to see the total number of pieces in the model. The students often make these line extensions dotted or dashed as illustrated by the student work in Examples 138 and 139.

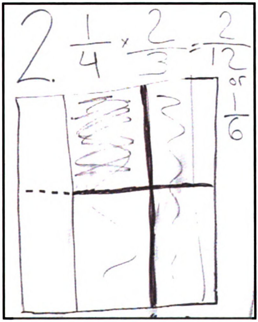
Although the way in which these three students draw lines and color is fairly typical in the enacted classroom (i.e., dividing the rectangle into (A x B) pieces where A and B are the denominators of the two fractions), this is not always the case. In fact, several long discussions take place in which students choose to use lines and coloring differently. An example of one such discussion is provided here (along with the corresponding poster of student work). The problem under consideration is "$\frac{1}{4} x \frac{2}{3}$". Emma and Katie have just explained their model which uses the strategy illustrated above and stated their answer as "$\frac{2}{12}$."

Example 140 (EC - Student Work).



Note that they have used the method described above (i.e., dividing the rectangle into (A x B) pieces where A and B are the denominators of the two fractions). Another group is asked to explain their model (included here in Example 141) of the same problem, and the discussion ensues.

Example 141 (EC - Student Work).



(1) T: Okay. How about this one? You've got a fourth of two thirds? Could you guys, somebody from this one and someone from this one, talk about what you did?

(2) S1: In this one, we had the two thirds [indicates the four shaded pieces], which is what we had to work with, and then we split it up into fourths [indicates the bold horizontal line and bold vertical line], and then we took one of them away [points at double-shaded piece], and then we made a line right here [indicates solid portion of horizontal line], and then that would, continuing right here [points at dashed portion of horizontal line], and then if you had a line down here [points at non-bold vertical line], then it would make it into sixths, so we've made it one sixth, because there was one piece taken out of these six.

(3) T: So, you guys disagreed with that at your table [addressing Emma and Katie who constructed the first model]?

(4) S2: Well, we didn't really get

(5) S3: It was a different way.

(6) S2: Yeah, we didn't really get how they got it, like how she kind of knew

(7) T: So ask her, ask her then what you didn't understand.

(8) S2: Because we didn't get, um, how you could just break this [points to shaded portion] up into fourths because that really doesn't make it into fourths, it kind of makes it into sixes, like we

(9) T: Hold on, hold on for just a second. Why doesn't it split that purple section into fourths [points to shaded portion]? 'Cause it looks like fourths to me.

(10) S3: 'Cause you don't look at this part [points to unshaded part].

(11) S2: I know, but I don't like it. We didn't get how they got it, like how they just knew to draw that line into fourths.

(12) T: So ask her, how did she just know?

(13) S2: How did you know?

(14) S1: Because in the problem it says, um, one fourth times two thirds, I think, so we have the two thirds right here [points to two shaded thirds], and then we had the one fourth so we split these two pieces [points to shaded two thirds] into fourths.

(15) *S2: Oh, okay.*

(16) *S1: And then we had the one.*

(17) *S4: What did you get for your answer, Emma?*

(18) *S2: We got, um, two twelfths.*

(19) *T: Is that a different answer than one sixth?*

(20) *S2: Yeah, but they're -*

(21) *T: Is it?*

(22) *S2: They're common, they're, they're equal to each other.*

(23) *T: So is your, did you get a different answer from them?*

(24) *S2: No.*

(25) *S3: Um, what -*

(26) *T: Go ahead.*

(27) *S3: Um, we have to split each third into four pieces and you only split each third into two pieces.*

(28) *T: But why do you have to do that, Katie?*

(29) *S3: Because, um, it only shows six pieces, and you have two thirds, like, if you had only one third you could do that, I think, but you need, you have two thirds, so you have to split each third into four pieces.*

(30) *S2: Because we didn't really get how they couldn't, 'cause we thought that you had to split them all into four pieces, and, 'cause you have three, which we got how they got the three, and then, um, how they just kind of like just split it.*

(31) *T: Okay. So if I look at yours, this right here is a fourth, right, and this is a fourth, and this is a fourth, and this is a fourth [indicates each of the fourths in the two-thirds]. If I look at theirs, isn't this a fourth [indicates one of the fourths in the two-thirds]? Didn't they split theirs into four pieces, too? They just split it up differently. But, but I think, Katie, correct me if I'm wrong, but what I hear you saying is you thought you had to cut it like this [gestures horizontally with hands]? Okay. Do I have to?*

She ended up with a sixth, and you said you got two twelfths. Can I write your, are those two answers different from each other?

(32) *S2: No.*

(33) *T: Are they the same amount?*

(34) *S3: Yeah.*

(35) *T: Does the same amount just look a little bit different?*

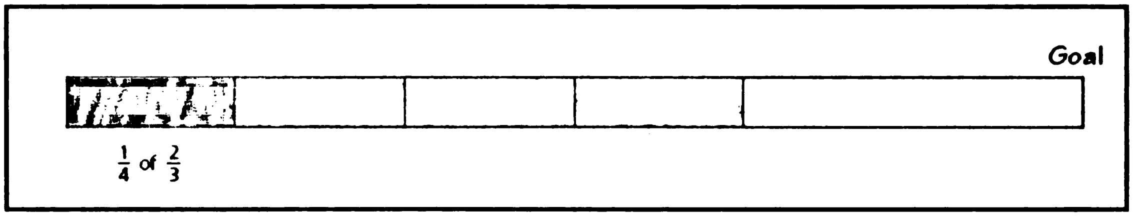
(36) *S3: Yeah.*

(37) *T: In every other problem we've done, we have cut it like this [points to Katie and Emma's poster]. I think what these guys saw was, gosh, maybe I'll take advantage of that cut I already have [indicates bold vertical line] and get my pieces that way. Is it okay if they both ended up with one sixth? It would just mean that maybe this notion of going across like this [gestures horizontally] we didn't really do. In her drawing she didn't take each third and split it into four pieces. They just did their drawing a little bit different.*

This excerpt exemplifies one of several discussions that took place over the course of the week in the enacted curriculum in which questions arose about the appropriate types and uses of iconic mediators.

There are similarities between the use of rectangular area models and the use of linear area models, but there are distinctions to be made as well. One distinction is that the partitioning suggested for the linear area model in the written and enacted curricula only uses vertical lines. That is, the linear area model is partitioned in ways similar to a number line. The use of two colors, one to denote each fraction so that the overlap of colors indicates the answer, is continued from the work with the rectangular area model. Similar language is also used (e.g., "lines," "cutting," "dividing"). Example 142 from the written curriculum includes both a sample linear area model and a statement addressing its use:

Example 142 (WC).



"The student above divides the fraction of the goal ($\frac{2}{3}$) that is met in four days into fourths to find the length equal to $\frac{1}{4}$ of $\frac{2}{3}$. To figure out the new length, the student divides the whole thermometer into pieces of the same size." (SG, p. 35)

Note the use of vertical lines and two colors.[47] The written curriculum also recommends that students use the linear area model with the number line. Example 143 illustrates this recommendation:

Example 143 (WC).
*"Draw a **number line** under the **thermometer**." (TG, p. 65)*

Students are encouraged to locate 0 and 1 on a number line drawn beneath a thermometer and to use it to assist in the partitioning of the thermometer (i.e., linear area model).

The primary discussion regarding the use of the linear area model in the enacted curriculum took place on Day 3. Example 144 includes the problem that was displayed using the overhead projector and the class discussion that ensued:[48]

Example 144 (EC).
"One sixth-grade class raises $\frac{2}{3}$ of their goal in four days. They wonder what fraction of the goal they raise each day on average. To figure this out, they find $\frac{1}{4}$ of $\frac{2}{3}$."

(1) T: So, let's say that a sixth grade class, maybe it's us, let's say that we raise two thirds of our goal. Instead of making our goal this big square

[47] Here the colors (originally red and blue) display as two shades of gray.
[48] It should be noted that this problem occurs verbatim in the written curriculum (SG, p. 34).

box like a brownie pan, I'm going to use this kind of a model. I'm going to do this [Draws a long, narrow rectangle.] I'm going to use a long, skinny model. Do you guys think we can handle that today?

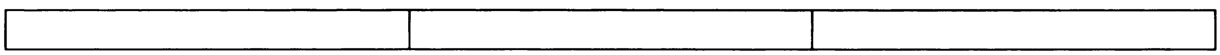
(2) *Ss: Yes.*

(3) *T: Okay.*

(4) *S1: No.*

(5) *T: No? Where's that good attitude? [Laughter] Okay. This is the whole goal, right? So it says that wow, after four days, we reached two thirds of that, who can come up and show what that would look like on a long, skinny model? Steven, you want to do that for us? What does two thirds look like on this kind of a model?*

(6) *S2: Something like that [Divides linear area model into thirds]*



(7) *T: Can you show us where two thirds would be, though? Fill it right up to two thirds. Does everybody agree?*

[Shades two of the thirds on the linear area model using red marker]

(8) *Ss: Yeah, yup.*

(9) *T: So, so far, so good. Okay. Thanks, Steven. They wonder what fraction of the goal they raised each day on average [points to problem displayed by overhead projector], so to figure this out they said, let's figure out what one fourth of that two thirds is. So this [points to vertical line at right end of shading] is all they've reached, is this much of their goal. What if I want to figure out what one fourth of that red part [the shaded part] is? Steven, will you pick somebody to help us do that?*

(10) *S2: Michael.*

(11) *T: Here, Michael [hands Michael a marker]. How can we figure out what one fourth of that is up there [points to model]?*

[Student 2 divides the shaded two thirds into two parts]

(12) T: Remember, you want one fourth of the red part [shaded part].

(13) S3: Oh, one fourth of it?

[Student 3 erases lines he has added]

(14) T: Yup.

(15) S3: The red part?

(16) T: Yup.

(17) S3: I don't know. I can't

(18) T: You can ask for help if you want to ask for help.

(19) S3: I don't want to mess up on it, though.

(20) T: Do you want to ask somebody to help you out, Bud?

(21) S3: Yeah.

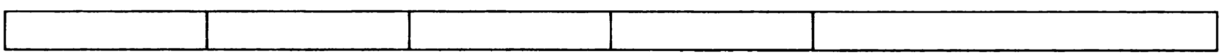
(22) T: Okay. There's lots of people with their hands up and they'd be glad to help you.

(23) S3: Graham.

(24) T: It's a little trickier looking at a model like this instead of that big box, isn't it?

(25) S4: Okay. So now one fourth of two thirds

[Student 4 redraws the lines cutting each third in half]



(26) T [to Student 3]: I think that's just what you were doing, right?

(27) S4: Now, here would be one fourth [colors in one-fourth of the two-thirds with black marker].

(28) *T: I think that's exactly what Michael was doing, wasn't it?*

(29) *Ss: Yeah.*

(30) *T: So we just want a fourth of just that red part. So we're kind of ignoring the part that's not there. Thanks for helping, Graham. So if this isn't there [covers up the unshaded part with arm], here's four parts [points to shaded parts] and he got one of them, right? So you were absolutely doing the right thing there. So now the question is, though, what fraction of the whole goal did we get each day? So, Chloe, what fraction of that whole bar is this double piece that we have there [points to double-shaded piece]?*

(31) *S5: One fourth?*

(32) *T: One fourth [writes "$\frac{1}{4}$" under drawing]? Would you guys agree?*

(33) *Ss: No.*

(34) *T: Okay. Talk to Chloe if you disagree.*

(35) *S6: I think it should be one fifth because, uh, you have to add an extra, no, wait, I think it should be*

(36) *S?: I think it should be two twelfths because*

(37) *S?: Yeah.*

(38) *T: Is this picture showing two twelfths?*

(39) *S?: I think, no*

(40) *T: What is this picture [points to linear area model] showing?*

(41) *S7: Four times three*

(42) *T: Well, but I want, I want to talk about this drawing though.*

(43) *S?: It shows one sixth.*

(44) *S?: Yeah, one sixth.*

(45) T: *It shows one sixth or one fourth [writes "$\frac{1}{6}$" under drawing]?*

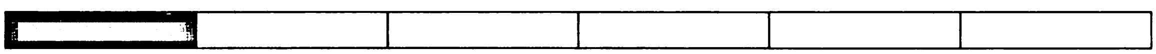
[Many students speaking at once]

(46) T: *Okay. Hold on. One at a time. One at a time. What is this drawing showing? Is this shaded part one fourth of the whole bar, or one sixth of the whole bar?*

(47) C: *One sixth.*

(48) T: *But prove that, though. Okay. Go ahead. Can you prove that this is one sixth?*

(49) S8: *[Student walks to board, gets marker from teacher] Because, if you, just like on the one big bar that we had, if you just end up splitting that one [divides the unshaded section in half], it ends up giving you six pieces, and so, like, if we use those big bars, it would just show you like you had one sixth.*



(50) S?: *I told you it was two twelfths. That's equivalent. [Laughter]*

(51) S?: *That's what it is.*

(52) T: *Chloe, what do you think about that? Do you think that it's one fourth or do you think it's showing one sixth of the whole bar?*

(53) S5: *One sixth.*

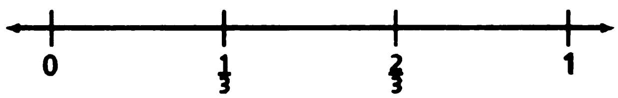
In this construction of the linear area model, the students use techniques similar to those suggested in the written curriculum (i.e., vertical lines and shading). In fact, such techniques are applied in all uses of the linear area model in the enacted curriculum.

Twenty-nine percent of the iconic visual mediators included in the written curriculum are number lines whereas number lines are absent in the enacted curriculum. The written curriculum acknowledges that number lines can be partitioned as described for the linear area model above; however, they also point out an "alternative approach" for partitioning number lines for fraction multiplication. The Teacher's Guide (p. 66) poses the problem and demonstrates the "alternative approach:

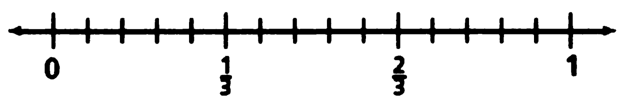
Example 145 (WC).

- *Let's look at $\frac{1}{5} \times \frac{2}{3}$. How would you start if you wanted to use a number line model?*

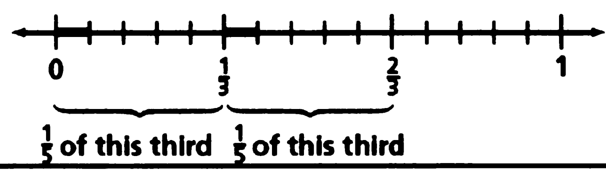
(Draw a number line, and label 0 and 1. Next, partition the number line into thirds, and mark $\frac{1}{3}$ and $\frac{2}{3}$.)



- *How could you repartition to find $\frac{1}{5}$ of $\frac{2}{3}$?* (You could break each third into five equal parts.)



- *What is $\frac{1}{5}$ of two thirds?* (Each $\frac{1}{5}$ of one third is $\frac{1}{15}$, so twice that would be $\frac{2}{15}$.)



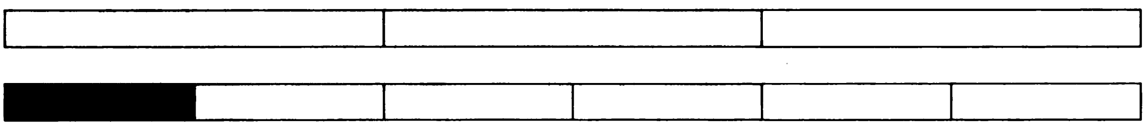
This approach parallels the most common use of the rectangular area model because the model is partitioned into A x B pieces where A and B are the denominators of the fractions. Even though the enacted curriculum does not use numbers lines per se, this approach is used with the linear area models in several situations. In fact, the conversation included earlier in which "$\frac{1}{4}$ of $\frac{2}{3}$" is modeled on a linear area model (see Example 144) continues with a student suggesting this "alternative approach." Recall that the final agreed upon solution is "$\frac{1}{6}$." Example 146 presents the linear area model as it appeared at the conclusion of the discussion in Example 144 along with the continuing conversation:

Example 146 (EC).

(54) T: Okay, so we'll rethink this guy then, right? Okay, now, Elliot, why two twelfths? You said two twelfths.

(55) S1: I was imagining the whole bar in thirds, as it was first

[The teacher draws the original model of two-thirds on the board above the other model]

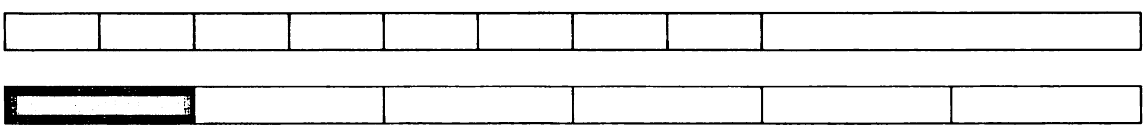


(56) T: Okay.

(57) S1: Then you cut it into fourths and then cut each of those thirds into four pieces.

(58) T: You want to cut each of these into four pieces?

[The teacher divides each of the shaded thirds into four pieces in the top model]



(59) S1: Yes.

(60) T: Oh. So, if I split this [points at leftmost third] into four pieces and this [points at the other shaded third] into four pieces, how is that going to help me get one fourth of the two thirds? How is that going to help me?

(61) *S1: Well, I was actually, I was just thinking of the one third. I wasn't just thinking of one fourth of both of them, but*

(62) *T: Oh, okay. So you were just going to get a fourth of this one [points at leftmost third]?*

(63) *S1: Yeah.*

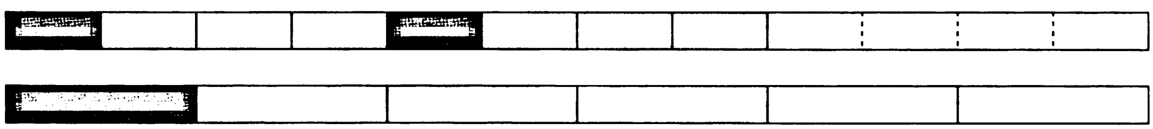
(64) *T: And how is that going to help us get a fourth of two thirds?*

(65) *S1: 'Cause it's just, if you double that then it would be your answer for that.*

(66) *T: Oh.*

(67) *S?: 'Cause that's a third and get a fourth from another third.*

(68) *T: Oh. So I've got a fourth from this third [shades in one-fourth of the leftmost shaded third] and a fourth from this third [shades in one-fourth of the other shaded third], so together, and I'm going to do what Karen just did down here [divides the unshaded third into fourths using dashed vertical lines], I would have two of the twelve pieces?*



(69) *Ss: Yeah.*

(70) *T: Are those different, two twelfths and one sixth?*

(71) *Ss: No.*

(72) *S?: They're not different.*

(73) *T: They're not different?*

(74) *S?: They just look different but they're the same - but they're equivalent.*

(75) *T: So if I took this piece [points to shaded fourth in the center third] and I moved it over to here [points next to the shaded fourth in the leftmost third], would it be the same size as this one sixth [points to shaded one-sixth in the bottom model]?*

(76) *Ss: Yes.*

(77) T: Okay. So either way would be okay?

(78) C: Um hmm.

This discussion in the enacted curriculum compares the two approaches suggested in the written curriculum and the class concludes that the answers are equivalent. It could be argued that although the enacted curriculum never uses a number line model, the methods of partitioning number lines advocated for in the written curriculum are present in conjunction with linear area models in the enacted curriculum. In addition, the simple presence of number line models is not enough to accomplish the goals of objectification. If learning mathematics is a change in participation in mathematical discourse, it is the ways in which the number line is talked about that is the key to this development. In order to reify fractions, it is necessary to encapsulate "1" and "6" into the number "$\frac{1}{6}$" and although number lines can facilitate this process, it is also possible to use number lines in ways in which the talk is of parts and pieces instead of numbers.

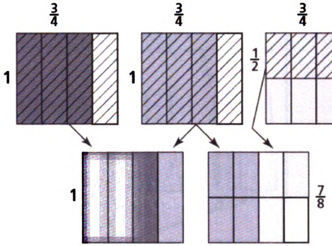
Examples 133-146 certainly highlight the question of the affordances and limitations of certain types of iconic mediators and prompt the question, "Which iconic mediators best accomplish the goals set forth in the written curriculum for this Investigation?" It is not the purpose of this project to definitively answer this question, rather to discuss the visual mediators present in the written and enacted curricula; however, its importance seems indisputable.

Symbolic Mediators

Seventy-six percent and 58% of the visual mediators in the written and enacted curricula respectively are symbolic mediators (i.e., number sentences). In many cases, the symbolic mediators are constructed from the iconic mediators. In these cases, students are encouraged first to model the situation and then to write the symbolic version from that model. An example of such use from the written curriculum is provided here:

Example 147 (WC).

For $2\frac{1}{2} \times \frac{3}{4}$, the area model could be as shown below.



$$2\frac{1}{2} \times \frac{3}{4} = \frac{5}{2} \times \frac{3}{4} = 1\frac{7}{8}$$

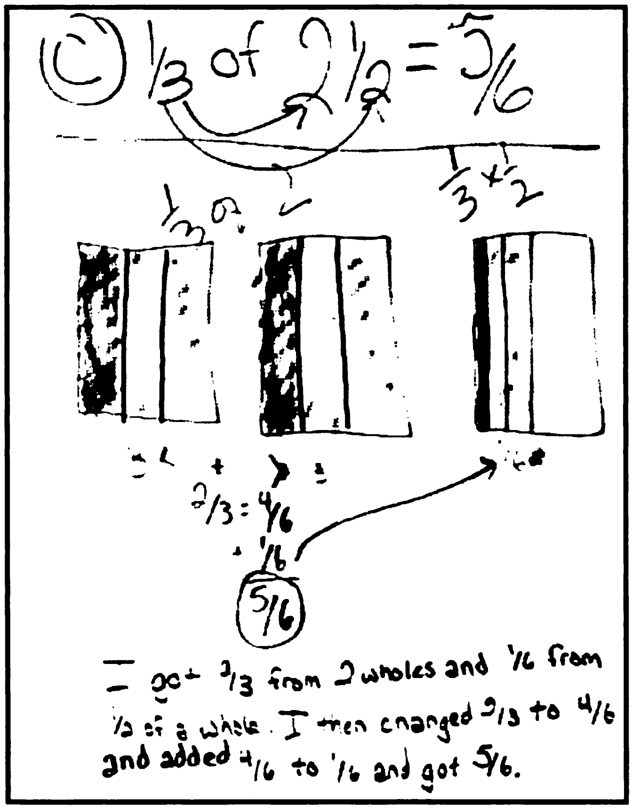
In Example 147, the symbolic mediator is completed only after the situation has been modeled with an iconic mediator. The written curriculum also suggests the use of symbolic mediators to help students see patterns. For example:

Example 148 (WC).
"It is helpful to write the number sentence by the models. By doing this, students typically notice that you can multiply the numerators and denominators to find the product." (TG, p. 61)

In Example 148, the symbolic mediator is used to illustrate the traditional algorithm for fraction multiplication. That is, seeing several symbolic mediators, students may identify the pattern of multiplying numerators and denominators.

Similar uses of symbolic visual mediators are seen in the enacted curriculum. An example of deriving the symbolic mediator from the iconic mediator is provided here:

Example 149 (EC - Student Work).



First, the rectangular area model is constructed showing "$\frac{1}{3}$ of 2" and "$\frac{1}{3}$ x $\frac{1}{2}$," then the model is used to construct the final number sentence "$\frac{1}{3}$ x $2\frac{1}{2} = \frac{5}{6}$." It is interesting here to note the use of "of" and "x" in this diagram. The relationship between "of" and "x" is discussed in other sections (e.g., pp. 50-54 and 118-119); however, the transition between a quasi-symbolic mediator (e.g., "$\frac{3}{4}$ of $\frac{1}{2} = \frac{3}{8}$") and a standard symbolic mediator ("$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$") is common in the enacted curriculum. In fact, the teacher often writes "X" on top of "of" on student work to emphasize the relationship (see Example 17 on p. 53). Symbolic mediators are also used to identify the traditional algorithm for fraction multiplication in the enacted curriculum. In fact, it is exactly after the teacher has changed several quasi-symbolic mediators to standard symbolic mediators on Day 2 that a student notices the pattern and suggests the traditional algorithm.

Example 150 (EC).

(1) *T: Instead of saying nine tenths of a sixth, I can also say nine tenths times a sixth [writes "X" on top of "of" in "$\frac{9}{10}$ of $\frac{1}{6} = \frac{9}{60}$." Go ahead, Trevor.*

(2) *S: I think it's times the numerator and numerator and then times the denominator and denominator.*

Examples 147-150 indicate similar uses of symbolic mediators in the written and enacted curriculum even though they are relatively more common in the written curriculum.

Summary

The question addressed in this summary is, "What does an investigation of the visual mediators in the written and enacted curricula allow us to see?" That is, "What do we know now about the relationship between the written and enacted curricula that we did not know before?"[49] Recall that visual mediators are artifacts created specially for

[49] The statements in this summary and throughout this analysis are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days in the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or

the sake of mathematical communication. Table 16 summarizes the findings from the investigation of visual mediators for the purposes of examination for overall conclusions.

Table 16.

Summary of "Visual Mediators" Analysis

	Written Curriculum	Enacted Curriculum
Analysis of References to Visual Mediators (i.e., word analysis)	26% of references to visual mediators are "general" references	14% of references to visual mediators are "general" references
	94% of references to visual mediators refer to iconic mediators (remaining 6% refer to symbolic mediators)	81% of references to visual mediators refer to iconic mediators (18% refer to symbolic mediators and 1% to concrete mediators)
	-----	The one reference to concrete mediators in the enacted curriculum is "blocks"
	The most common general word referencing iconic mediator is "model" (64%)	The most common general word referencing iconic mediator is "drawing" (45%)
	Neither curricula use "representation" as a label for an iconic mediator	
	The most common function of references to iconic mediators are "constructing" and "mediational flexibility" (frequencies range from 29% to 35%)	

the enacted curriculum as a whole. Rather, my statements highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.

Table 16 (cont'd)

	Written Curriculum	Enacted Curriculum
Analysis of References to Visual Mediators (i.e., word analysis)	Rectangular area models are the most common iconic mediators in both curricula	
	28% of iconic mediators are number line models	0% of iconic mediators are number line models
	The most common function of references to symbolic mediators is "writing"	
	Virtually the same set of verbs associated with visual mediators is present in both curricula	
	Both curricula include references that relate one iconic mediator to another, one symbolic mediator to another, and an iconic mediator to a symbolic mediator	
Analysis of Visual Mediators (i.e., visual mediator analysis)	The primary use of visual mediators in both curricula is the construction of endorsed narratives	
	76% of the visual mediators are symbolic mediators (remaining 24% are iconic mediators)	58% of the visual mediators are symbolic mediators (41% refer to iconic mediators and 1% to concrete mediators)
	-----	The one use of concrete mediators in the enacted curriculum involve the use of blocks
	Both curricula include rectangular area models and linear area models	

Table 16 (cont'd)

<table>
<tr><th></th><th>Written Curriculum</th><th>Enacted Curriculum</th></tr>
<tr><td rowspan="6">Analysis of Visual Mediators (i.e., visual mediator analysis)</td><td colspan="2">Rectangular area models are the most common type of iconic mediator in both curricula (53% and 60% respectively)</td></tr>
<tr><td>Includes contextual pictures and number lines</td><td>-----</td></tr>
<tr><td colspan="2">In both curricula, instructions regarding the use of rectangular area models are quite detailed early in the week and much briefer later in the week</td></tr>
<tr><td colspan="2">Both curricula use "coloring" and vertical and horizontal lines (for "cutting" and "dividing" with rectangular area models, dividing the rectangle into A x B pieces where A and B are the denominators of the two fractions</td></tr>
<tr><td>-----</td><td>Students use alternative cutting methods in addition to dividing the rectangle into A x B pieces where A and B are the denominators of the two fractions</td></tr>
<tr><td colspan="2">Both curricula use "coloring" and vertical lines for partitioning the linear area models, dividing the model into A x B pieces where A and B are the denominators of the two fractions; an alternative partitioning approach is used as well</td></tr>
</table>

Table 16 (cont'd)

	Written Curriculum	Enacted Curriculum
Analysis of Visual Mediators (i.e., visual mediator analysis)	Use of the number line includes partitioning in ways similar to the linear area model; however, coloring is not used	-----
	In both curricula, symbolic mediators are constructed using information gained from constructing iconic mediators	
	Both curricula include both “quasi-symbolic” mediators (e.g., “$\frac{3}{4}$ of $\frac{1}{2} = \frac{3}{8}$”) and “standard symbolic” mediators (e.g., “$\frac{3}{4} \text{ x } \frac{1}{2} = \frac{3}{8}$”)	

Note. Dashes (i.e., “-----“) indicate that the category is not applicable to the designated curriculum.

Table 16 and the more detailed analysis included within the chapter highlight many discursive similarities and differences between the written and enacted curricula with regard to visual mediators. The problem with such a table is that some of the richness and interpretation present within the text of the chapter is lost. However, the summary provided within the table allows a look at the data as a whole.

It seems that there are far more similarities between the written and enacted curricula in this analysis (presented in Table 16) than are present in the analysis of mathematical words (presented in Table 10 on p. 124). From the information presented in Table 16, I conclude that the use of visual mediators in the written and enacted curricula is similar. Of particular importance is that both curricula address the goals of mediational flexibility and fluency with symbolic mediators in similar ways. That is, students are introduced to a variety of visual mediators, expected to recognize the relationships between them, and transition from one to another.

One notable exception to this overall similarity is the lack of number lines in the enacted curriculum.[50] This is particularly important for the purposes of objectification because the number line affords opportunities to reify fractions (i.e., discuss them as numbers). Area models (whether rectangular or linear) do not necessarily afford these same opportunities for reification because the discourse if more likely to include words such as "part" or "piece" and phrases such as "X of Y" which do less to promote fractions as numbers. This greater opportunity for reification in the written curriculum supports similar earlier conclusions in Chapters 4 (Goals of the Written and Enacted Curricula) and 5 (Mathematical Words in the Written and Enacted Curricula).

[50] This difference is highlighted in Table 16.

CHAPTER 7: ENDORSED NARRATIVES IN THE WRITTEN AND ENACTED CURRICULA

Endorsed narratives are the third mathematical feature under consideration in this investigation. A narrative is defined as "any text, spoken or written, that is framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to endorsement or rejection, that is, being labeled true or false" (Sfard, 2008, p. 176). A "narrative" that has been established as true within some mathematical discourse community has been "endorsed." Endorsed narratives in formal mathematical discourse include, but are not limited to, axioms, definitions, and theorems. Narratives can be classified as object-level (e.g., "$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$") or meta-level (e.g., "multiply the numerator of the first fraction by the numerator of the second fraction to get the numerator of their product").

There is a wide variety of ways in which narratives can be endorsed depending on the mathematical context. On one end of the continuum, professional mathematicians adhere to very specific rules regarding the endorsement of narratives. That is, mathematicians produce fully endorsed narratives using formal deduction. This takes the form of a proof containing a series of endorsed narratives sequentially linked through strict deductive processes.[51] Induction and abduction are often used by mathematicians as heuristics for investigating mathematics, but neither by itself can serve to endorse a narrative in scholarly mathematics. In contrast, empirical evidence is often used to endorse narratives in colloquial mathematics. For example, children might use counting marbles as a procedure to "prove" that they have more marbles than their playmates. One

[51] This is the tradition in mathematics; however, there are current debates regarding what counts as proof in mathematics. For example, can computer-generated data serve as proof?

of the goals of school mathematics is to move students from the everyday use of empirical or quasi-empirical evidence to more formal mathematical methods of narrative endorsement (Sfard, 2008).

Much of the discourse in mathematics involves (1) recalling, (2) constructing, and (3) substantiating narratives. One can imagine a mathematics classroom in which students are asked to *recall* basic mathematical facts such as "3 x 6 = 18," *construct* the number sentence "45 + 86 = 131," and *substantiate* the narrative, "The probability of choosing a jack from a standard deck of cards is $\frac{1}{13}$." As illustrated here, *recalling* involves summoning a previously endorsed narrative, *constructing* results in a newly endorsed narrative, and *substantiating* addresses the ways in which decisions are made about whether to endorse a narrative (i.e., decide that it is true) (Sfard, 2008, p. 291). Clearly the nature of these processes varies both quantitatively and qualitatively depending on many factors including the age and grade level of the students. For example, many first grade students may *construct* the narrative "2 + 7 = 9," whereas a fourth grade student may *recall* the same narrative. Similarly, fourth grade students and eleventh grade students may have different methods by which to *substantiate* the narrative. Narratives and their endorsement play a key role in mathematics. That is, using previously substantiated narratives to construct new narratives is the work of professional mathematicians.

Given this, it seems appropriate to examine recalling, constructing, and substantiating narratives in the comparison of the mathematical features of the written and enacted curricula. This section first introduces the types of narratives present in the written and enacted curricula and then describes the ways in which recalling, constructing, and substantiating play out in relation to these narratives.

Types of Narratives

Only those narratives directly related to fraction multiplication are considered in this analysis. That is, the narrative is either an example of fraction multiplication (object-level narrative) or addresses the derivation or magnitude of the product of fraction multiplication (meta-level narrative). As with the mathematical features discussed earlier, it is necessary to explicate the analytic decisions made for the process of identifying the endorsed narratives in the written and enacted curricula. In the case of the written curriculum, many narratives are *opened* in the Student Guide (e.g., "$\frac{2}{3} \text{x} \frac{1}{5}$") and *closed* in the Teacher Guide (e.g., "$= \frac{2}{15}$").[52] In these cases, the *opening* and *closure* of the narratives were conjoined (e.g., "$\frac{2}{3} \text{x} \frac{1}{5} = \frac{2}{15}$") and included in the analysis. This decision is consistent with the conceptualization of the written curriculum as the union of the Teacher's Guide and the Student Guide.

Given the multi-modal nature (i.e., it contains both spoken and written forms of communication) of the enacted curriculum and the general complexity of spoken language, several analytic decisions applicable to the enacted curriculum are described here.[53] First, many narratives are included multiple times in close proximity to one another within the enacted curriculum. This repetition takes several forms, including

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[52] Throughout this analysis, "opening" will refer to the "factor x factor" component of the narrative (e.g., "$\frac{2}{3} \text{x} \frac{1}{5}$") and "closure" will refer to the "= product" component of the narrative (e.g., "$= \frac{2}{15}$").

[53] The written form of the enacted curriculum includes the teacher and students writing on paper, posters, transparencies, and the whiteboard.

saying the narrative as well as writing the narrative, saying the narrative more than once in adjacent utterances, and writing the narrative in two forms, hereafter referred to as "symbolic form" (e.g., "$\frac{3}{4} x \frac{1}{2} = \frac{3}{8}$") and "quasi-symbolic form" (e.g., "$\frac{3}{4}$ of $\frac{1}{2} = \frac{3}{8}$"). In the first two cases (i.e., saying and writing, saying more than once in adjacent utterances), the narrative was counted only once in this analysis. In the case of saying and writing, the written form was included because this best served the comparison with the written curriculum. It can be argued that these repetitions are important and indicate emphasis on particular narratives; however, the major purpose of this analysis is to compare the written and enacted curricula and such redundancy is not a feature of the written curriculum. Therefore, the exclusion of these redundancies is appropriate for this comparative study. In the final case (i.e., writing the narrative in symbolic form and quasi-symbolic form), both forms were counted because both forms appear in the written and enacted curricula.

Several times in the enacted curriculum, narratives are proposed in which the opening ("$\frac{2}{3}$ of $\frac{1}{2}$") is the same, but the closure of the narrative is different (e.g., "$\frac{2}{3}$ of $\frac{1}{2} = \frac{2}{6}$," "$\frac{2}{3}$ of $\frac{1}{2} = \frac{4}{6}$"). Note that in these examples, one of the narratives is closed with "$= \frac{2}{6}$" and the other is closed with "$= \frac{4}{6}$." In some of these cases (as in this one), the narratives are non-equivalent (i.e., one of them is mathematically incorrect). In other cases, the narratives are equivalent (e.g., "$\frac{2}{3}$ of $\frac{1}{2} = \frac{2}{6}$," "$\frac{2}{3}$ of $\frac{1}{2} = \frac{4}{12}$"). In these cases, whether equivalent or not, the proposed narratives are included in the analysis. These instances of "conflicting narratives" will be elaborated later in this section.

134 and 108 narratives in the written and enacted curricula respectively are directly related to fraction multiplication. Here, I examine these narratives in several ways to provide a more elaborated picture of the narratives. Figure 23 summarizes the relative frequencies of meta-level and object-level narratives in the written and enacted curricula.

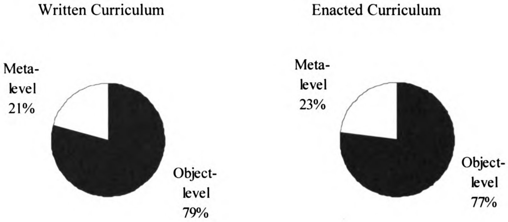


Figure 23. Types of narratives in the written and enacted curricula.

The relative frequencies of object-level and meta-level narratives related to fraction multiplication in the written and enacted curricula are very similar. In both cases, object-level narratives make up more than 75% of the narratives. This may indicate that students are expected, in both curricula, to multiply fractions more than to talk about fraction multiplication. Next, I describe the nature of each of these types of narratives in greater detail.

Object-level Narratives

All object-level narratives in both the written and enacted curricula involve the multiplication of two numbers, at least one of which is a fraction or includes a fraction (i.e., a fraction or a mixed number). Table 17 summarizes the relative frequencies of the types of numbers involved in these narratives in both curricula.

Table 17.

Relative Frequencies of the Types of Numbers involved in Object-level Narratives in the Written and Enacted Curricula

Type	Sample Narrative	Written Curriculum	Enacted Curriculum
Fraction x Fraction	"$\frac{1}{3} \text{x} \frac{2}{3} = \frac{2}{9}$"	50%	69%
Fraction x Whole Number	"$\frac{1}{2} \text{x } 2 = 1$"	14%	6%
Fraction x Mixed Number	"$\frac{1}{3} \text{x } 2\frac{1}{2} = \frac{5}{6}$"	15%	8%
Mixed Number x Whole Number	"$1\frac{2}{3} \text{x } 12 = 20$"	3%	1%
Mixed Number x Mixed Number	"$2\frac{1}{2} \text{x } 1\frac{1}{6} = 2\frac{11}{12}$"	18%	16%

Note. Order of the numbers was not considered (i.e., "$2\frac{1}{3} x \frac{2}{3} = 1\frac{5}{9}$" and "$\frac{2}{3} x\ 2\frac{1}{3} = 1\frac{5}{9}$" were treated as the same type).

"Fraction x Fraction" object-level narratives are proportionally more common in the enacted curriculum than in the written curriculum (69% compared to 50%). Three other types ("Fraction x Whole Number," "Fraction x Mixed Number," and "Mixed Number x Mixed Number") are represented in approximately 15% of the object-level narratives in the written curriculum. Only one other type ("Mixed Number x Mixed Number") receives this much attention in the enacted curriculum. Figures 24 and 25 indicate the way in which these object-level narratives are distributed across the five days in the written and enacted curricula. Figure 24 presents the distribution of the "Fraction x Fraction" type and Figure 25 presents the distribution of the "Other" types (all types except "Fraction x Fraction").

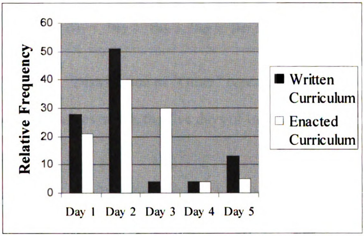


Figure 24. Distribution of "Fraction x Fraction" type object-level narratives across the five days of Investigation 3 in the written and enacted curricula.

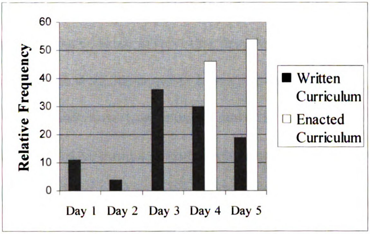


Figure 25. Distribution of "Other" type (i.e., all types except Fraction x Fraction) object-level narratives across the five days of Investigation 3 in the written and enacted curricula.

Figures 24 and 25 indicate that in both the written and enacted curricula, "Fraction x Fraction" narratives are more common early in the Investigation and the other types become more prevalent later in the Investigation. Most notable, however, is the distinct separation of "Fraction x Fraction" from the other types in the enacted curriculum. In fact, the other types are completely absent in the enacted curriculum during the first three days and the "Fraction x Fraction" type is virtually absent during the last two days of the Investigation. When taken together, Table 17 and Figures 24 and 25 indicate that both the types of object-level narratives and their distribution across the Investigation differ between the written and enacted curricula.

In order to further compare the types of object-level narratives in the written and enacted curricula, I classified the narratives into "simple" and "complex" categories. The simple object-level narratives include three forms: (1) "A x B = C," (2) "A of B = C," and (3) "A x B ≈ C," where A and B are fractions, mixed numbers, or whole numbers, and where no more than one of A or B is a whole number. The complex object-level narratives include any forms other than these three basic forms. In both the written and enacted curricula, the complex forms present in the curricula illustrate the Distributive Property and converting a mixed number into an equivalent improper fraction in order to facilitate multiplication. Figure 26 summarizes the relative frequencies of simple and complex object-level narratives in the written and enacted curricula.

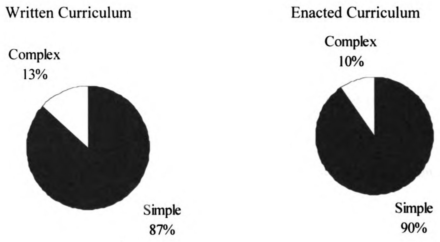


Figure 26. Types of object-level narratives in the written and enacted curricula.

The relative frequencies of simple and complex object-level narratives in the written and enacted curricula are very similar; nearly 90% of the object-level narratives in both curricula are simple.

Table 18 summarizes the relative frequencies of the three simple forms of object-level narratives present in the written and enacted curricula.

Table 18.

Relative Frequencies of three Forms of Simple Object-level Narratives

Form	Written Curriculum	Enacted Curriculum
"A x B = C"	47%	45%
"A of B = C"	16%	41%
"A x B ≈ C"	34%	13%

Table 18 indicates that the form "A x B = C" is represented in similar proportions in both the written and enacted curricula, in both cases accounting for nearly half of the simple object-level narratives. However, "A x B ≈ C" is more than twice as common in the written curriculum than in the enacted curriculum and conversely "A of B = C" is more than twice as common in the enacted curriculum than in the written curriculum. Both of these findings support conclusions make in earlier chapters. The greater relative frequency of simple object-level estimation narratives supports earlier findings that suggest that a greater emphasis on estimation is present in the written curriculum than in the enacted curriculum. The greater relative frequency of simple object-level narratives containing "of" in the enacted curriculum supports earlier findings that concluded that "x of y" is more common in the enacted curriculum. This discrepancy is important because estimation narratives have the potential to reify fractions while "X of Y" do not provide this opportunity. Figures 27, 28, and 29 summarize the distribution of the simple object-level narratives, A x B = C, A of B = C, and A x B ≈C, respectively across the five days included in this analysis.

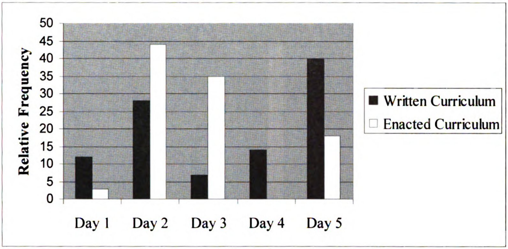


Figure 27. Distribution of "A x B = C" across the five days of Investigation 3 in the written and enacted curricula.

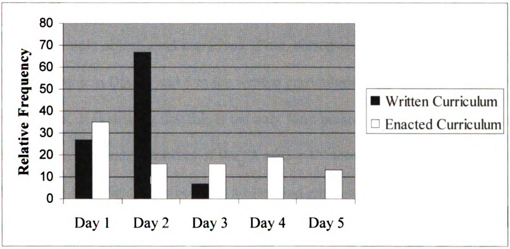


Figure 28. Distribution of "A of B = C" across the five days of Investigation 3 in the written and enacted curricula.

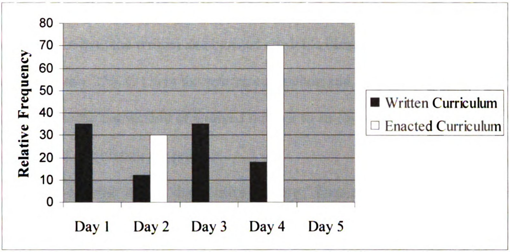


Figure 29. Distribution of "A x B ≈ C" across the five days of Investigation 3 in the written and enacted curricula.

Figures 27, 28, and 29 indicate interesting features of the distribution of each of the basic forms of simple object-level narratives. First, narratives of the "A x B = C" form are most common in Days 2 and 5 in the written curriculum and Days 2 and 3 in the enacted curriculum. It is striking that this is the only form present on Day 5 of the written curriculum. Second, narratives of the "A of B = C" form are more common in the early days of the Investigation in both the written and enacted curriculum; however, this tendency is much more pronounced in the written curriculum than in the enacted curriculum where such forms are present on each day. Finally, narratives of the form "A x B ≈ C" are present on four days in the written curriculum, but only on two days in the enacted curriculum. Together, Table 18 and Figures 27, 28, and 29 indicate that the written and enacted curricula vary in both the proportion of the three forms of simple object-level narratives and their distribution across the five days in this analysis.

As mentioned previously, 13% and 10% of the object-level narratives in the written and enacted curricula respectively are "complex." At least 75% (79% in the written curriculum and 75% in the enacted curriculum) of these complex object-level narratives in both the written and enacted curricula illustrate the distributive property and approximately 25% (21% in the written curriculum and 25% in the enacted curriculum) demonstrate the conversion of a mixed number to an equivalent improper fraction for the purposes of calculating the product.

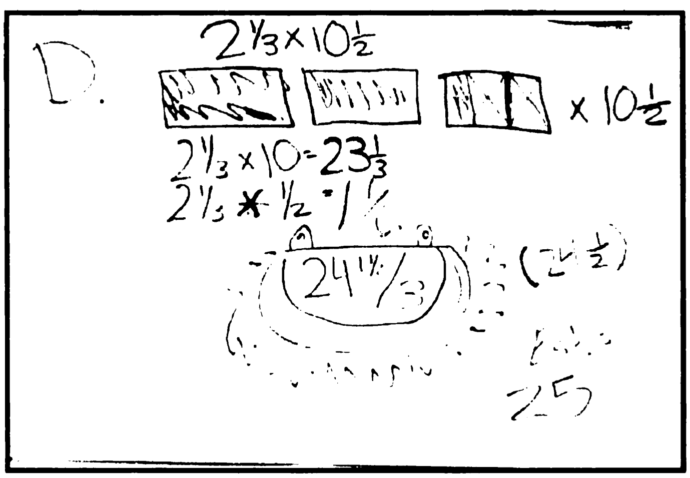
An example of the "distributive property" form of object-level narratives from the written curriculum is provided here:

Example 151 (WC).

"$2\frac{1}{3} x\ 10\frac{1}{2} = (2\frac{1}{3} x\ 10) + (2\frac{1}{3} x\ \frac{1}{2}) = 24\frac{1}{2}$" *(TG, p. 74)*

An example from the same category, the same problem as well, in the enacted curriculum is demonstrated by the following poster presented on Day 5:

Example 152 (EC - Student Work).



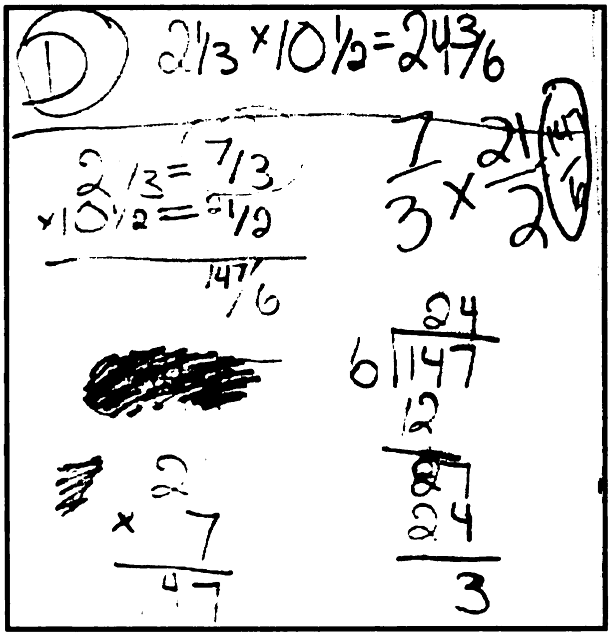
Examples 151 and 152 illustrate the same complex object-level narrative in the written and enacted curricula. Examples from the written and enacted curriculum of the second form of "complex" object-level narratives, converting a mixed number to an improper fraction for the purposes of finding the product, are provided here. First, from the written curriculum:

Example 153 (WC).

"$2\frac{1}{2} x \frac{3}{4} = \frac{5}{2} x \frac{3}{4} = 1\frac{7}{8}$" *(TG, p. 71)*

The following poster and associated explanation provide an example from Day 5 of this category in the enacted curriculum:

Example 154 (EC - Student Work).



S: "Okay. I did two thirds times ten halves [points to "$2\frac{1}{3}$" and "$10\frac{1}{2}$"] and then I changed two thirds [points to "$2\frac{1}{3}$"] to seven thirds and ten and a half to twenty one seconds, and then three times two is six and then seven times twenty one is a hundred forty seven and I showed that here [points to 21 x 7 = 147]. So I did six, which is my denominator, divided by one hundred forty seven [points to 147 + 6 = 24R3], which is my numerator, and then I got twenty four and three sixths [points to "$24\frac{3}{6}$"]."

In this explanation, the student describes the conversion of the mixed numbers to improper fractions followed by the use of the standard algorithm for fraction multiplication, albeit while displaying some confusion regarding reading mixed numbers (e.g., saying "two thirds" while pointing to "$2\frac{1}{3}$"). This language confusion, however, does not seem to impact the student's ability to complete the process of fraction multiplication correctly. The relative frequencies (see Figure 26 and discussion on p. 195) and Examples 151-154 indicate that the written and enacted curricula are similar, both quantitatively and qualitatively, with respect to "complex" object-level narratives.

Meta-level Narratives

As mentioned previously, meta-level narratives represent 21% and 23% of the narratives in the written and enacted curricula respectively. These narratives were classified into two types in the written curriculum, those that describe an algorithm for fraction multiplication (hereafter "algorithm" meta-level narratives) and those that describe the relationship between the magnitude of the factors and the product (hereafter "factor-product relationship" meta-level narratives). A third type of meta-level narrative present in the enacted curriculum describes the commutative property as it relates to fraction multiplication (hereafter "commutative" meta-level narrative). Table 19 summarizes the relative frequency of these types of meta-level narratives in the written and enacted curricula.

Table 19.

Relative Frequencies of Types of Meta-level Narratives in the Written and Enacted Curricula

Type of Meta-level Narrative	Written Curriculum	Enacted Curriculum
"Algorithm"	50%	84%
"Factor-Product Relationship"	50%	8%
"Commutative"	-----	8%

Note. Dashes (i.e., "-----") indicate that the category was not present in the designated curriculum.

Table 19 indicates that the enacted curriculum contains substantially more "algorithm" meta-level narratives than the written curriculum, and conversely the written curriculum contains substantially more "factor-product relationship" meta-level narratives than the enacted curriculum.

"Algorithm" meta-level narratives were categorized into those that describe the standard algorithm for multiplying fractions (i.e., $\frac{a}{b} x \frac{c}{d} = \frac{ac}{bd}$) and those that describe other algorithms, including procedures used for mixed numbers and whole numbers. In the written curriculum, 57% are of the first type and 43% are of the second type. The corresponding frequencies in the enacted curriculum are 48% and 52%. An example of each type of "algorithm" meta-level narrative from both the written and enacted curriculum is provided here ("standard" followed by "other"):

Example 155 (WC).
"If you multiply the numerators and multiply the denominators, you will get the product of the fractions." (TG, p. 70)

Example 156 (EC).
S: "What I realized is that when you plus it you are, uh, finding, like, common denominators, and then when you're, uh, timesing it, you can just, uh, you're timesing one of, the numerator by the, the numerator by the numerator, and then you're timesing the denominator by the denominator." (Day 2)

Example 157 (WC).
"When the denominator of the first number is equal to the numerator of the second number, just use the first number for the numerator and the second number for the denominator." (TG, p. 67)

Example 158 (EC).
T: "Oh my gosh. Now wouldn't that be nice if we could multiply any fraction by a whole number just by doing that [points to where she has just put 1 under 20 in "$\frac{1}{2} x \frac{20}{1} = \frac{20}{2} = 10$" on the whiteboard] and we could still use our algorithm?" (Day 5)

Example 157 from the written curriculum describes an algorithm for a special case of fraction multiplication (i.e., when the denominator of the first number is equal to the numerator of the second number), whereas Example 158 illustrates an algorithm for multiplying a fraction by a whole number. Note that Examples 155-157 suggest the reification of fractions because they involve a procedure for multiplying fractions, which implies that fractions are numbers. Example 158 goes a step further and explicitly refers to fractions as "numbers."

"Factor-Product Relationship" meta-level narratives make up 50% of the meta-level narratives in the written curriculum, but only 8% in the enacted curriculum. An example from the written curriculum is provided here:

Example 159 (WC).
"When you multiply a fraction by a fraction, the product is less than each of the two fractions you multiplied." (SG, p. 35)[54]

[54] This "Factor-Product Relationship" meta-level narrative was stated as a conjecture by Libby, a fictitious student in the written curriculum.

Similarly, the following excerpt from the enacted curriculum addresses the relationship between the magnitude of the factors and the product in fraction multiplication:

Example 160 (EC).

T: Well, that's why I said it's sometimes hard to think about multiplying with fractions. Because we're not getting like seven groups of five where we're getting this bigger number. We're taking a part of something that's already a part. So your, what's going to happen to your answer if you're taking a part of something that's already only a part? Is your answer going to get bigger, or is your answer going to get smaller?

S: Smaller. (Day 2)

The large discrepancy between the relative frequencies of "factor-product relationship" meta-level narratives in the written and enacted curricula (50% compared to 8%) supports earlier findings that the relationship between the magnitude of the factors and product of fraction multiplication receives more emphasis in the written curriculum than in the enacted curriculum. As with estimation object-level narratives, this is important because discussions of this relationship promote reification of fractions.

Finally, 8% of the meta-level narratives in the enacted curriculum address the Commutative Property. An example is illustrated in the following excerpt from Day 3:

Example 161 (EC).

(1) *S1: So it could be either way.*

(2) *T: So it could be either way [points to "$\frac{1}{4} x \frac{1}{2} = \frac{1}{8}$" and "$\frac{1}{2} x \frac{1}{4} = \frac{1}{8}$" written on the transparency]?*

(3) *S2: One fourth of one half because that's really what you took.*

(4) *T: Okay, you're right. That's really what he took. But what point is Collin bringing out that's an important one about multiplication?*

(5) *S3: It doesn't matter which, which way -*

(6) *T: Do you remember what you call that property? Remember, we talked about with addition?*

(7) *S1:* ***Commutative.***

(8) *T: You're good. Excellent. Excellent.*

Overall, Table 19 and Examples 155-161 indicate that although the relative frequency of meta-level narratives in the written and enacted curricula is quite similar (approximately 20-25%) and qualitatively many of the sample meta-level narratives are similar, the proportion of the meta-level narratives representing the three types varies greatly.

Recalling narratives

Students are expected, both in the written and enacted curricula, to *recall* previously endorsed narratives. For example, Problem 3.1 in the written curriculum states "In this investigation, you will relate what you already know about multiplication to situations using fractions" (SG, p. 32). This sentence is likely asking students to recall previously endorsed narratives related to whole number multiplication because this has been the students' primary experience (if not their entire experience) with multiplication. Similarly, in the enacted curricula (specifically early on Day 1) the class recalls narratives regarding whole number multiplication:

Example 162 (EC).

(1) *T: So what if I had something like this (writes "12 x 13" on the board), twelve times thirteen? How would you think of this problem right here? Grace, how would you think of this problem?*

(2) *S1: As twelve groups of thirteen.*

(3) *T: Twelve groups of thirteen. How many of you think about it like that? (Some students raise their hands). Okay? Can you tell me the answer to this problem right here (points to "8 x 3" written on the board), eight groups of three or eight plus eight plus eight? What's the answer, anybody?*

(4) Ss: Twenty-four.

In this example, the class is asked to recall the endorsed narrative "8 x 3 = 24." In addition, students are asked to recall that 24 is 8 groups of 3 or 8 + 8 + 8. Recalling endorsed narratives, however, does not explicitly take place during this Investigation, either in the written or enacted curricula, very often. It can be argued that every time students calculate the product of fractions using the standard algorithm they recall their basic multiplication facts; however, because these facts are not the focus of this investigation, recalling will take a backseat to constructing and substantiating narratives for the remainder of this chapter.

Constructing Narratives

Recall that *constructing* results in a new narrative that is subject to rejection or endorsement. Before examining the relationship between the written and enacted curricula in terms of the construction of narratives, it is important to note that the way in which a narrative is constructed proved to be more difficult to glean from the written curriculum than from the enacted curriculum where the construction is often captured in the classroom discourse. Statements such as Example 155 (see p. 202) are the exception to this. In this case, the algorithm is a prescription for constructing narratives. Some general statements made in the written curriculum provide subtle clues regarding the ways in which the authors are envisioning narrative construction in the classroom. For example:

Example 163 (WC).
"Students develop strategies for multiplying mixed numbers built on their previous work." (TG, p. 71)

Example 164 (WC).
"Use what you know about equivalence and multiplying fractions to first estimate, and then determine, the following products." (SG, p. 38)

Example 165 (WC).
"Allow students to find ways to make sense of the problem using the model." (TG, p. 60)

Example 163 indicates that students should use strategies developed "in their previous work" to construct narratives for multiplying mixed numbers. Example 164 again suggests using previous knowledge, here about equivalence and estimation, to construct narratives. Finally, Example 165 advocates for the use of "models" to construct narratives. These types of statements along with specific directions given in the Teacher's Guide and Student Guide provide evidence regarding the methods of narrative construction in the written curriculum. Because the method of construction in both curricula is largely dependent upon the type of narrative, this section will be organized around the classification of object-level and meta-level narratives.

Construction of Object-Level Narratives

The construction of object-level narratives varies depending whether the narratives come from contextual or non-contextual problems.[55] For example, consider the following contextual question (i.e., Question B-3 from Problem 3.2) in the written curriculum:

Example 166 (WC)
Blaine drives the machine that paints stripes along the highway. He plans to paint a stripe that is $\frac{9}{10}$ of a mile long. He is $\frac{2}{3}$ of the way done when he runs out of paint. How long is the stripe he painted? (SG, p. 35)

[55] Recall that "contextual" here is used for Questions that are associated with a "real life" situation.

In this Question, the *opening* of the narrative (i.e., “$\frac{2}{3} \text{ x } \frac{9}{10}$”) is constructed from the context of the problem and the *closure* of the narrative (e.g., “$= \frac{18}{30}$”) may be constructed using a variety of strategies. In contrast, consider this non-contextual Question (Question A-1.d. from Problem 3.2) in the written curriculum:

Example 167 (WC):
“Use your algorithm to multiply $\frac{5}{6} x \frac{3}{4}$” (SG, p. 39)

The Question provides the *opening* of the narrative and requires only the construction of the *closure* of the narrative (e.g., “$= \frac{5}{8}$”). In this case, the question specifies the use of an algorithm for this construction. Given the importance of whether the narrative comes from a contextual Question or a non-contextual Question, this section begins with a comparison of the relative frequencies of narratives associated with contextual and non-contextual questions in the written and enacted curricula (see Figure 30).

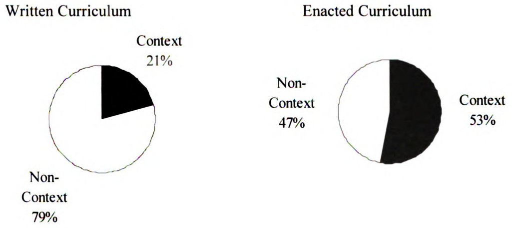


Figure 30. Relative frequencies of narratives associated with contextual and non-contextual Questions in the written and enacted curricula.

Figure 30 indicates that less than one-fourth of the narratives in the written curriculum are associated with contextual Questions compared to approximately half of the narratives in the enacted curriculum. Figures 31 and 32 illustrate the distribution of the narratives associated with contextual and non-contextual Questions respectively across the five days.

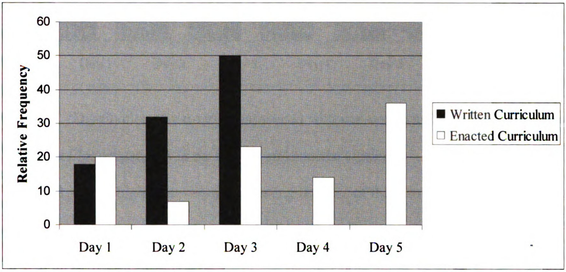


Figure 31. Distribution of narratives associated with contextual Questions across the five days of Investigation 3 in the written and enacted curricula.

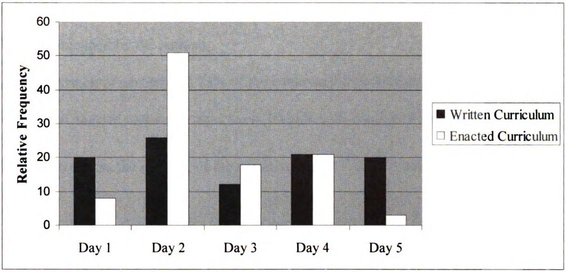


Figure 32. Distribution of narratives associated with non-contextual Questions across the five days of Investigation 3 in the written and enacted curricula.

Figures 31 and 32 highlight several interesting features of the distribution. For example, the relative frequency of narratives associated with contextual Questions in the written curriculum increases steadily over Days 1-3, reaches a maximum of 50% on Day 3, and then disappears completely. That is, there are no contextual narratives presented in the written curriculum on Days 4 and 5. Narratives associated with contextual Questions are present on all days in the enacted curriculum and are actually most prevalent on Day 5.

The narratives associated with non-contextual Questions are represented quite consistently in the written curriculum, approximately 20% each day with a slight dip on Day 3. It is also notable that more than 50% of the non-contextual narratives in the enacted curriculum occur on Day 2. Some of these differences may be attributable to the fact that Problems 3.4 and 3.5 were not enacted in the classroom during these five days. The possibility remains; however, that the students may have had more experience with contextual narratives and less experience with non-contextual narratives proportionally than was intended by the authors.

Sixty-eight percent and 73% of the contextual object-level narratives in the written and enacted curricula respectively include both context and iconic mediators in their construction, the *opening* of the narrative (e.g., "$\frac{1}{2}$ of $\frac{3}{8}$") constructed from the context of the Question and the *closure* of the narrative (e.g., "$= \frac{3}{16}$") constructed using an iconic mediator representing the Question.[56] Two examples are provided here in which the written curriculum (speaking to the student in Example 168 and the teacher in Example 169) encourages the use of iconic mediators for constructing narratives.

> ***Example 168 (WC).***
> *"Remember, to make sense of a situation you can* ***draw a model*** *or change a fraction to an equivalent form." (SG, p. 32)*
>
> ***Example 169 (WC).***
> *"Help them connect what is happening* ***in the drawing*** *to what is happening in the problem." (TG, p. 61)*

The use of iconic mediators for completing the construction of object-level narratives in contextual questions is also encouraged in the enacted curriculum. This is evidenced by questions asked by the teacher, such as: "Are you guys following **his drawing**?" on Day 1 and "But what is **your drawing** showing?" on Day 2. The following excerpt from Day 1 contains a discussion in the enacted curriculum in which a narrative is constructed from a contextual question during a whole class discussion. The poster presented by a group of students is also included here.

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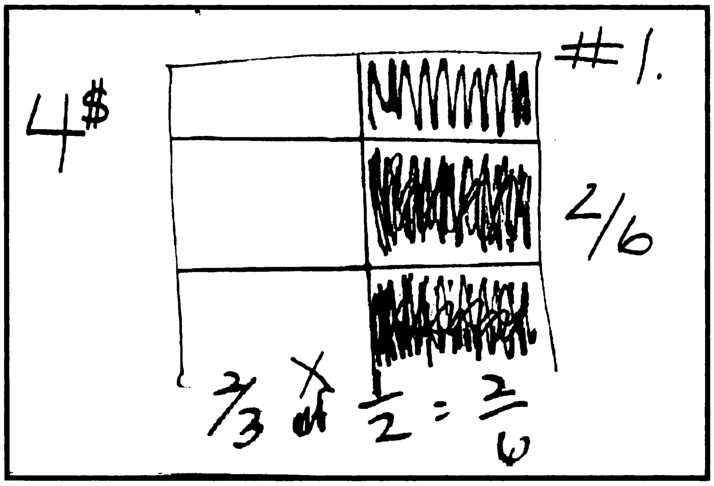
[56] I will not provide details here regarding the ways in which the models were used; model use is detailed in the "Visual Mediators" section of this chapter.

Example 170 (EC - Student Work).



(1) T: *Okay. Can we go over there to number 1? This is off that extra sheet of problems. "Jennifer wants to buy some brownies. When she gets there the pan is half full. She buys two thirds of that pan." So talk about what you guys did.*

(2) S1: *Um, well, there, we knew that she, there was, it was half full, so we split it into half [waves hand down middle of picture], and then we, she bought two thirds of it, so we shaded, we made thirds first [points to each line dividing the rectangle into thirds], and then we shaded in two [points to shaded two thirds], and then, um, she, it said what fraction of the pan she bought, and she would buy two sixths of the pan because if you split all these into thirds [points to picture] you'll get two sixths.*

(3) T: *So can somebody say in words what she bought? She bought what? What fraction of what fraction? Anybody. What'd she buy?*

(4) Ss: *Two sixths.*

(5) T: *But that's the answer. But what did she buy?*

(6) S?: *Two*

(7) T: *Two*

(8) S?: *Two thirds*

(9) T: *Two thirds of what?*

(10) S?: *Of a half.*

(11) *T: Can you write that down right by your problem. She bought two thirds of one half. Can you write that next to it [S1 writes "2" on whiteboard next to poster]? You might have to write it underneath [S1 erases "2" on whiteboard]. I don't know if you can squeeze that in there. Underneath your drawing. Two thirds of what?*

(12) *S?: One half*

(13) *T: So two thirds of one half [S1 writes "$\frac{2}{3}$ of $\frac{1}{2}$" underneath picture], and, and what was two thirds of a half? How much of that pan, you guys?*

(14) *S?: Two thirds*

(15) *T: What answer did they get?*

(16) *S?: Two sixths*

(17) *T: Could you write "equals one, equals two sixths" [S1 writes "$=\frac{2}{6}$"] so we can look at that for a minute?*

The length of Example 170 illustrates the complexity of constructing narratives in a classroom. Here the class constructed "$\frac{2}{3}$ of $\frac{1}{2} = \frac{2}{6}$" using a contextual Question and an iconic mediator.

The other common method of construction of object-level narratives occurs in the enacted curriculum. These cases, comprising the construction of 23% of the objective-level narratives, are those in which the contextual object-level narrative is constructed using a previously-constructed narrative. The student work included in Example 170 represents this type. The narrative on the original poster was "$\frac{2}{3}$ of $\frac{1}{2} = \frac{2}{6}$." On Day 2, an "X" was placed over the "of," constructing a new narrative "$\frac{2}{3} \times \frac{1}{2} = \frac{2}{6}$" (see Figure 20 on p. 146). Construction of a new object-level narrative from a previously-constructed narrative does not occur in the written curriculum; however, the use of previously-constructed narratives to construct new meta-level narratives is common in both the written and enacted curricula. This type of narrative construction is very important as this is the way that narratives are constructed in formal mathematics. That is, previously proven statements are used in the construction of new statements. More examples of this type will be discussed in the next section, *Construction of Meta-Level Narratives*.

As discussed earlier, in non-contextual object-level narratives, the *opening* of the narrative is provided in the written curriculum (and therefore also accessed in the enacted curriculum if the question is completed by the class). The *closure* of the construction of these narratives occurs in two primary ways in both the written and enacted curricula. The first method uses an iconic mediator in the same way described previously in the construction of contextual object-level narratives (see description on p. 207). The only difference is whether or not the *opening* is given (non-contextual) or needs to be derived from a contextual situation. This method of construction is used in 21% and 59% of the non-contextual object-level narratives in the written and enacted curricula respectively.

The second method is the use of estimation strategies to complete the construction of object-level narratives when the *opening* of the narrative is given. This estimation method is used in 33% and 23% of the non-contextual object-level narratives in the written and enacted curricula respectively. An example of this method occurs in both the written and enacted curricula in association with the "Getting Ready" in Problem 3.3 (SG, p. 36):

Example 171 (WC).

> **Getting Ready for Problem 3.3**
>
> Estimate each product to the nearest whole number (1, 2, 3, . . .).
>
> $\frac{1}{2} \times 2\frac{9}{10}$ $\quad 1\frac{1}{2} \times 2\frac{9}{10}$ $\quad 2\frac{1}{2} \times \frac{4}{7}$ $\quad 3\frac{1}{4} \times 2\frac{11}{12}$
>
> Will the actual product be greater than or less than your whole number estimate?

The following excerpt from the written curriculum provides suggestions for the use of estimation strategies for narrative construction with reference to the first and second of the four fraction multiplication questions here:

> ***Example 172 (WC).***
> *"There are different ways to approach estimation. For example, with the first problem in the Getting Ready,* $\frac{1}{2}$ *x* $2\frac{9}{10}$*, students might use a benchmark strategy. Since* $2\frac{9}{10}$ *is almost 3, a reasonable estimate would be* $\frac{1}{2}$ *of 3 or* $1\frac{1}{2}$*. In the second problem,* $1\frac{1}{2}$ *x* $2\frac{9}{10}$*, students might use a combination of benchmarking and distribution." (TG, p. 71)*

The first question (i.e., "$\frac{1}{2}$ x $2\frac{9}{10}$") referenced here is enacted in a similar way in the classroom on Day 4. That is, the same method of narrative construction (i.e., estimation) is used. An excerpt from this discussion is included here:

Example 173 (EC).

(1) *T: We're just going to estimate how much this might be before we try to figure out some sort of a strategy. Could you help me with an estimate? If I said I was going to get one half of two and nine tenths, Graham, how could we think about estimating that answer?*

(2) *S1: Well, nine tenths, that's close to a whole, so, one. Yeah, so*

(3) *T: So what, Graham?*

(4) *S1: Nine tenths is close to a whole.*

(5) *T: Okay.*

(6) *S1: So you could round that up into a whole and then two would be a three.*

(7) *T: So you would call this [points to "$2\frac{9}{10}$"] about three?*

(8) *S1: Yeah.*

(9) *T: And then can you think about what it would be to be half of three?*

(10) *S1: Um*

(11) *T: What's half of three?*

(12) *S1: One and a half.*

Examples 172-173 included here illustrate several interesting points regarding the construction of object-level narratives in the written and enacted curricula. The use of iconic mediators in the construction of contextual object-level narratives is common in both the written and enacted curricula (68% and 70% respectively). The frequency of use of iconic mediators in the construction of non-contextual object-level narratives, however, is quite different, 21% of the constructions in the written curriculum and 59% of the constructions in the enacted curriculum. In addition to using iconic mediators, 33% of the contextual object-level narratives in the written curriculum and 23% in the enacted curricula are constructed using estimation strategies. Furthermore, 23% of the contextual object-level narratives in the enacted curriculum are constructed using previously-constructed narratives. This greater emphasis on estimation in the written curriculum supports earlier findings. Finally, if all iconic mediator use for the construction of object-level narratives (both contextual and non-contextual) is combined, the relative frequencies for the written and enacted curriculum are strikingly different, 31% and 66% respectively. This seems to indicate a greater variety of construction techniques in the written curriculum.

Construction of Meta-Level Narratives

Recall that there are two types of meta-level narratives in the written curriculum ("algorithm" and "factor-product relationship"). The enacted curriculum includes these two types as well as a third type ("commutative"). The vast majority of "algorithm" meta-level narratives are constructed using previously-constructed narratives, 79% and 90% in the written and enacted curricula respectively. That is, it is expected that students will reason inductively and generalize from a series of fraction multiplication object-level narratives to a general pattern or algorithm. Evidence of this method of construction in the written curriculum is provided here in the form of a suggestion from the Teacher's Guide:

> ***Example 174 (WC).***
> *"It is helpful to write the number sentence by the models. By doing this, students typically notice that you can multiply the numerators and denominators to find the product." (TG, p. 61)*

On Day 2 in the enacted curriculum, the teacher returns to a series of six posters and changes the form of the narratives from "of" narratives to "X" narratives (see Example 17 on p. 53). The associated exchange is included here:

> ***Example 175 (EC).***
>
> *T: Okay. We're going to come back and look at all these, 'cause you know what I'm going to do? I'm going to change that word "of" to a mathematic symbol. [Changes "of" to "X" on six posters] What you guys have done here, I love the way that you're all looking at me, I can tell you're listening. Thank you very much. When I take a part of another part, I can write that as a*

multiplication problem. Instead of saying nine tenths of a sixth, I can also say nine tenths times a sixth.

S1: I think it's times the numerator and numerator and then times the denominator and denominator.

In fact, this exchange demonstrates the desired response from the written curriculum, that the student, when shown a series of these previously-constructed object-level narratives may construct a meta-level "algorithm" narrative.

In the written curriculum, 71% of the "factor-product relationship" meta-level narratives are constructed by "exploration." Remember that these narratives address the relationship between the magnitude of the factors and product of a fraction multiplication object-level narrative. The written curriculum suggests that students explore object-level narratives in order to construct meta-level narratives about these relationships. For example, the following excerpt appears in the written curriculum:

Example 176 (WC).
"You might want to ***explore*** *what happens when one factor is less than 1 and the other is greater than 1." (TG, p. 70)*

This "exploration" method of construction is not very different than that of using previously-constructed narratives. The subtle distinction is that the previously-constructed narratives to be considered are not specified in the curriculum. That is, it is left to the students and teacher to decide which object-level narratives to construct and "explore."

All of the "factor-product relationship" and "commutative" meta-level narratives in the enacted curriculum are constructed using previously-constructed narratives (only four such narratives are constructed in the enacted curriculum). On Day 1, the teacher points to a series of posters containing object-level narratives and states:

Example 177 (EC).

T: "You know what. I'm noticing that every time. I'm noticing, that, ooh, I got a little two sixths, or a third. I got three eighths, I got two twenty oneths. My answer got so little."

She is likely using these products of fraction multiplication to begin to encourage students to think about the magnitude of the product when multiplying two fractions less than one.

Overall, both qualitatively and quantitatively, the construction of meta-level narratives in the written and enacted curricula is quite similar. Nearly all of the meta-level narratives are constructed by noticing patterns in previously-constructed narratives. This result is in keeping with earlier findings that indicated similarities between the written and enacted curricula with regard to meta-level narratives.

Substantiating Narratives

As stated previously, one of the primary purposes of school mathematics is to change the ways in which children *substantiate* (and therefore endorse) narratives, moving them from the use of empirical or quasi-empirical evidence to more formal mathematical justification. Given that proof is the gold standard of substantiation in scholarly mathematics, I began my investigation by searching for "prove" and its derivatives (e.g., proving) in the written and enacted curricula. It occurs four times in the written curriculum and six times in the enacted curriculum. All four uses in the written curriculum refer to the use of iconic mediators for proving. Two examples are given here:

Example 178 (WC).

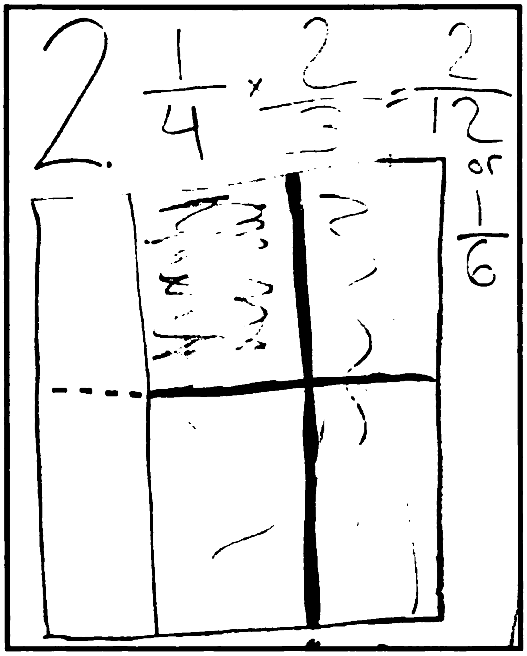
"Draw a ***picture to prove*** *that your calculation makes sense." (SG, p. 38)*

Example 179 (WC).

"Who can use their ***model to prove*** *that the answer* $\frac{8}{12}$ *is sensible?" (TG, p. 76)*

Four of the six uses of "prove" and its derivatives in the enacted curriculum are similar in nature to the Examples 178 and 179, one such use is illustrated in the following excerpt (including the corresponding student poster from Day 3):

Example 180 (EC - Student Work).

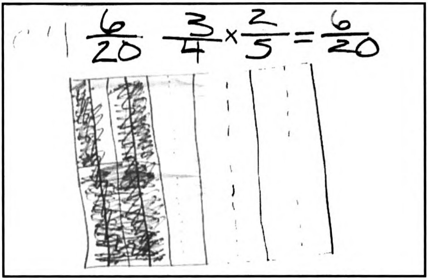


T: Go ahead. Can you ***prove*** *that this is one sixth?*

S: Because, if you, just like on the one big bar that we had, if you just end up splitting that one, it ends up giving you six pieces, and so, like, if we use those big bars, it would just show you like you had one sixth.

One of the other two instances of "prove" occurs on Day 2 in the enacted curriculum. In this case, using the standard algorithm for fraction multiplication for proving is suggested. The student in this excerpt has just finished constructing the narrative "$\frac{3}{4} x \frac{2}{5} = \frac{6}{20}$" as shown in the poster included here:

Example 181 (EC - Student Work).



(1) *S1: And there's four pieces in each of the fifths and we colored in three pieces in each of those fifths. That's what we did.*

(2) *T: Okay.*

(3) *S1: Another way to* ***prove*** *it would be three times two equals six and then five*

(4) *T: Right. But do we know that that for sure works?*

(5) *Ss: No.*

(6) *T: I guess that's what we're trying to figure out with our drawings.*

In this excerpt, the student suggests another method of proving (i.e., using the algorithm), but the teacher points out that the class has not yet fully endorsed the algorithm narrative, suggesting that it is unavailable for substantiation at this point. The final use of "prove" is a statement that allows student choice of substantiation method. It occurs on Day 4 in the enacted curriculum:

Example 182 (EC).

T: *"Okay, alrighty. I know some of you are really thinking hard about this. Now I'm going to push you to think even harder, okay? I'm going to give you a couple of situations, and what you're going to do at your table is you're going to talk about is the answer to the problem going to be more or less than one whole pan of brownies. That's all you have to do is talk about it. However you want to* ***prove*** *it is up to your table."*

These instances of "prove" provide a glimpse into the methods of substantiation in the written and enacted curricula. An analysis of these methods will be the focus of the remainder of this section.

I found the line between narrative construction and substantiation to be blurry in the analysis of the written and enacted curriculum. The methods of narrative construction are in most cases also used for narrative substantiation. For example, the previous section indicated that iconic mediators are often used to construct narratives (in particular the "$= \frac{5}{9}$" portion of the narrative) once "$\frac{2}{3} x \frac{5}{6}$" is constructed from a contextual problem. It is also true that iconic mediators are used to substantiate narratives. That is, if a student questions a narrative (e.g., "$\frac{3}{4} \text{x} \frac{1}{3} = \frac{3}{12}$"), the same iconic mediator used to construct the narrative may also be used to substantiate the narrative. Therefore, for the purposes of these analyses, I am conceptualizing these constructs temporally. That is, constructing is conceived as resulting in a narrative and any discussion that occurs about the narrative following its construction is conceived as substantiation. Therefore, this analysis will be framed using the narrative categories described in *Types of Narratives* and consider any discussions that occur after the narratives are presented or stated. In the written curriculum, substantiation is also discussed in terms of how students are expected to support their narratives once they have been constructed.

As might be expected given the findings summarized in *Constructing Narratives* and the blurry line described in the previous paragraph, the primary methods for substantiating narratives are quite predictable. More than 80% of the object-level and meta-level narratives (88% and 81% in the written and enacted curricula respectively) are substantiated using either iconic mediators or estimation strategies. In the written curriculum, the 88% breaks down into 47% and 41% for iconic mediators and estimation strategies respectively. As one might expect given the previous discussions regarding the lesser emphasis on estimation in the enacted curriculum, the breakdown is quite different, 68% and 13% for iconic mediators and estimation strategies respectively.

Although individual narratives are substantiated using one method (often the same method used to construct it), another interesting phenomenon is present in both the written and enacted curricula. In the written curriculum, students are encouraged to substantiate 25% of the narratives with a method beyond the particular method of substantiation associated with the narrative. The method involves using the production of the same or similar narratives using several strategies as further substantiation of the narratives. For example, in Problem 3.3, students are asked to estimate an answer, create a "model," and write a number sentence for a given contextual problem. Problem 3.4 asks students to use these same three strategies in a non-contextual problem. The agreement of these three methods (i.e., estimating, "modeling," and writing a number sentence) is used to further substantiate the narratives. That is, if you do the same Question in three different ways and get the same or similar (in the case of estimation) answers, there is a good chance that you are correct and the narrative can be endorsed. There are statements in the written curriculum that provide additional support for this phenomenon:

> ***Example 183 (WC).***
> *"Help them use **estimation** to decide if their **models** and computations are reasonable." (TG, p. 66)*

> ***Example 184 (WC).***
> *"You can also **estimate** to see if **your answer makes sense**." (SG, p. 32)*

"Agreement" between related narratives as substantiation is also present in the enacted curriculum. For example, a discussion (Example 185) ensues after the incomplete narrative "$2\frac{1}{2} \times \frac{4}{7}$" is displayed using the overhead projector on Day 4. First, a student uses the Commutative Property to rearrange the problem and construct the remainder of the narrative.

Example 185 (EC).

(1) *T: What if I have something like this? What if I have two and a half groups that are four sevenths [shows "$2\frac{1}{2} x \frac{4}{7}$"], or maybe I want to think about it like four sevenths of a group that two and a half. How are we going to think about, how are you going to think about estimating something like, just estimating?*

(2) *S?: I think it*

(3) *T: Connor, can you see okay? How, how can we think about estimating that one? Katie, you have any ideas about how to estimate that one?*

(4) *S1: Well, um, I think it might be, um, well, 'cause if it was four and four sevenths, um, times two and one half it would be like four sevenths of two and a half.*

(5) *T: Okay, so you're thinking about it like this [writes "$\frac{4}{7} x\ 2\frac{1}{2}$"]?*

(6) *S1: Yeah. So*

(7) T: Okay.

(8) S1: And four sevenths is about half so I think it'd be like two.

(9) T: So you're thinking of half of two and a half [crosses out "$\frac{4}{7}$" and writes "$\frac{1}{2}$"].

(10) S1: Yeah.

(11) T: So what would a half of two and a half be?

(12) S1: Oh, wait. It would be, um, one and one fourth.

The resulting narrative from this conversation is "$\frac{4}{7} \times 2\frac{1}{2} \approx 1\frac{1}{4}$." The conversation then turns to a debate regarding the Commutative Property and whether or not they are allowed to rearrange the factors when constructing narratives involving mixed numbers. The class decides to construct the original given narrative to see if the answers are the same, thus demonstrating the strategy of "agreement." The conversation continues:

Example 186 (EC).

(1) T: Hmmm. Let's think about this for a second. This is two and a half groups of, about how much is this [points to "$\frac{4}{7}$"]?

(2) Ss: Half.

(3) T: Okay [crosses out "$\frac{4}{7}$" and writes "$\frac{1}{2}$"]. So let's think about this for a second. If I have two and a half halves, how much do I have? Okay. How about if I get rid of this [covers the "$\frac{1}{2}$" in the "$2\frac{1}{2}$"]? Can you figure out two halves?

(4) S?: You have about

(5) Ss: A whole. A whole.

(6) T: *What's two halves?*

(7) Ss: *A whole.*

(8) T: *One whole [holds up index finger]. Could I get a half of a half [gestures between the "$\frac{1}{2}$" in $2\frac{1}{2}$ and the "$\frac{1}{2}$" on its own]?*

(9) S?: *It'd be the same.*

(10) S?: *It would be zero fourths.*

(11) T: *Careful. What's a half of a half?*

(12) S?: *One fourth.*

This conversation eventually constructs the narrative "$2\frac{1}{2} \text{x} \frac{4}{7} \approx 1\frac{1}{4}$". After a bit more discussion, a student suggests another yet another way to think about it:

Example 187 (EC).

(1) S1: *To make it less confusing, maybe if you changed the two wholes into halves, like, for, five halves*

(2) T: *Oh*

(3) S1: *So maybe*

(4) T: *Would that help us?*

(5) S?: *No.*

(6) T: *What if we called this [points to "$2\frac{1}{2} x \frac{4}{7}$"] five halves of four sevenths [writes "$\frac{5}{2} x \frac{4}{7}$"]? But could you multiply that using our algorithm?*

(7) Ss: *Yeah.*

(8) *T: Let's do it just to see what we get. Could we do five halves of four sevenths [writes "$\frac{5}{2} x \frac{4}{7}$"]? What would we end up with?*

(9) *S?: Fourteen twentieths*

(10) *S?: Twenty fourteenths*

(11) *T: Actually, twenty fourteenths [writes "$= \frac{20}{14}$"], right? And what is that the same as?*

(12) *S: One whole and six fourteenths [writes "$1 \frac{6}{14}$"].*

Example 187 constructs the narrative "$\frac{5}{2} x \frac{4}{7} = \frac{20}{14} = 1 \frac{6}{14}$." Examples 185-187 individually suggest various methods of narrative substantiation; however, taken as a whole they suggest the method of "agreement." That is, the class seems to believe that if they can make these narratives the same or similar (in the case of estimation), the agreement serves to substantiate (at least to some extent) the narrative. The excerpt that follows illustrates the conversation in which the class discusses whether these narratives ("$\frac{4}{7} \text{x } 2\frac{1}{2} \approx 1\frac{1}{4}$," "$2\frac{1}{2} x \frac{4}{7} \approx 1\frac{1}{4}$," and "$\frac{5}{2} x \frac{4}{7} = \frac{20}{14} = 1\frac{6}{14}$") are close enough to serve as substantiation for one another:

Example 188 (EC).

(1) *T: Is that about - now these are estimates. Is that close to this estimate [points to $1\frac{1}{4}$ in each estimate]?*

(2) *Ss: Yes.*

(3) *T: Or no?*

(4) *S1: Not really.*

(5) *T: No?*

(6) *Ss: Yeah, yes.*

(7) *T: Isn't that about -*

(8) *S2: 'Cause you've got to divide -*

(9) *T: Almost one and a half, and that's one and a fourth. Nope. We're not even close? Oh, okay. What do you think?*

(10) *S3: I think we are pretty close, because, first of all, it's not yet a half.*

(11) *T: Okay.*

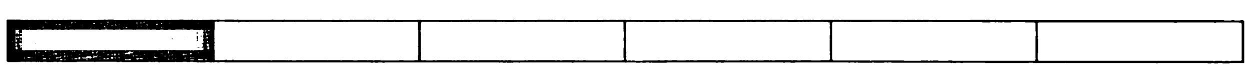
(12) *S3: So, um, I think we'll only be like a few fourteenths off and that's not very big of a chunk because -*

(13) *T: Okay.*

(14) *S4: Fourteenths are very, are kind of small, small pieces.*

A closely related type of "agreement" substantiation occurs when students produce equivalent narratives using slightly different iconic mediators. For example, the incomplete narrative "$\frac{1}{4}$ of $\frac{2}{3}$" was written on the whiteboard on Day 3. First, a student shaded two-thirds of the bar (indicated here by light grey shading):

Then the student divides the two-thirds into fourths and shades one of them (dark grey shading). The student also cuts the unshaded third in half:



These activities construct the narrative "$\frac{1}{4}$ of $\frac{2}{3} = \frac{1}{6}$." Another student suggests the following to complete the same narrative "$\frac{1}{4}$ of $\frac{2}{3}$." He begins in the same way as the first diagram presented above (i.e., shading two-thirds of the bar). He then divides each of the thirds into fourths and shades one fourth of each third (indicated in dark grey):

This method establishes the narrative "$\frac{1}{4}$ of $\frac{2}{3} = \frac{2}{12}$." The following discussion established their equivalence in the class:

Example 189 (EC).

(1) *T: Are those different, two twelfths and one sixth?*

(2) *S1: No. They're not different.*

(3) *T: They're not different?*

(4) *S1: They just look different but they're the same - but they're equivalent.*

(5) *T: So if I took this piece [points to one of the small dark gray pieces in the third diagram] and I moved it over to here [points to the other small dark gray piece], would it be the same size as this one sixth [points to the dark gray piece in the second diagram]?*

(6) *Ss: Yes.*

(7) *T: Okay. So either way would be okay?*

(8) *Ss: Um hmm.*

In the same way mentioned previously, these equivalent narratives substantiate one another by the very fact of their equivalence. That is, two different methods with the same or equivalent answers further substantiate a narrative.

A final version of the "agreement" substantiation of narratives in the enacted curriculum involves the "agreement" of the class. That is, it seems that consensus or convincing your classmates serves as a method of narrative substantiation. The examples provided here illustrate the teacher questioning students regarding their agreement:

Example 190 (EC).
T: "How many of you agree with Trevor that I ate two sixths? How many of you disagree with him?" (Day 1)

Example 191 (EC).
T: "Does everybody agree with her about that?" (Day 1)

Example 192 (EC).
T: "Do you guys agree with that?" (Day 2)

In addition, statements like the following further indicate the use of consensus as a method of substantiation:

Example 193 (EC).
T: "Talk to them [your classmates]. They're the ones you have to convince." (Day 1)

Example 194 (EC).
T: "You need to convince your classmates it's less than a whole." (Day 2)

Although this method of substantiation does not occur in quite the same way in the written curriculum, there are three instances (SG, pp. 35, 37, and 38) in which fictitious students are created that either disagree with one another or are presenting a strategy to the "live" students in the classroom for their appraisal. One such example from the written curriculum is included here:

Example 195 (WC).

Yuri and Paula are trying to find the following product.

$$2\frac{2}{3} \times \frac{1}{4}$$

Yuri says that if he rewrites $2\frac{2}{3}$, he can use what he knows about multiplying fractions. He writes:

$$\frac{8}{3} \times \frac{1}{4}$$

Paula asks, "Can you do that? Are those two problems the same?"

What do you think about Yuri's idea? Are the two multiplication problems equivalent?

The creation of these fictitious students and asking for agreement or disagreement from the "live" students in the written curriculum may have been intended to serve many purposes (e.g., It may be easier for "live" students to disagree with "virtual" students than their own peers). However, the creation of these fictitious students and the questions associated with them seem to indicate the possibility that consensus can serve as a form of substantiation in the written curriculum as well.

Summary

The question addressed in this summary is, "What does an investigation of the endorsed narratives in the written and enacted curricula allow us to see?" That is, "What do we know now about the relationship between the written and enacted curricula that we did not know before?"[57] Recall that a narrative is "any text, spoken or written, that is framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to endorsement or rejection, that is, being labeled true or false" (Sfard, 2008, p. 176). Table 20 summarizes the findings from the investigation of endorsed narratives for the purposes of examination for overall conclusions.

[57] The statements in this summary and throughout this analysis are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days in the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or the enacted curriculum as a whole. Rather, my statements highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.

Table 20.

Summary of "Endorsed Narratives" Analysis

	Written Curriculum	Enacted Curriculum
Types of Narratives	Includes 138 narratives	Includes 108 narratives
	Approximately three-fourths of the narratives in both curricula are object-level narratives (the remainder are meta-level narratives	
	50% of the object-level narratives are "Fraction x Fraction" (other combinations that represent at least 10% include "Fraction x Whole Number," "Fraction x Mixed Number," and "Mixed Number x Mixed Number")	69% of the object-level narratives are "Fraction x Fraction" (other combinations that represent at least 10% include "Mixed Number x Mixed Number")
	"Fraction x Fraction" narratives occur early in the week in both curricula	
	The other types of object-level narratives (not "Fraction x Fraction") are present on all five days	The other types of object-level narratives (not "Fraction x Fraction") are present on Days 4 and 5
	Nearly 90% of all object-level narratives are "simple" (the remainder are "complex")	

Table 20 (cont'd)

	Written Curriculum	Enacted Curriculum
Types of Narratives	Nearly half of all simple object-level narratives are of the form "A x B = C," where A and B are fractions, whole number, or mixed numbers (A and B are not both whole numbers)	
	The remainder of the simple object-level narratives are primarily of the form "A x B ≈ C"	The remainder of the simple object-level narratives are primarily of the form "A of B = C" (i.e., quasi-symbolic form)
	Simple object-level narratives of the form "A x B = C" are most common on Days 2 and 5	Simple object-level narratives of the form "A x B = C" are most common on Days 2 and 3
	Simple object-level narratives of the form "A of B = C" are most common on Days 1 and 2	Simple object-level narratives of the form "A of B = C" occur across all five days
	Simple object-level narratives of the form "A x B ≈ C" occurs on Days 1-4	Simple object-level narratives of the form "A x B ≈ C" occurs on Days 2 and 4

Table 20 (cont'd)

	Written Curriculum	Enacted Curriculum
Types of Narratives	Approximately 75% of the complex object-level narratives in both curricula address the Distributive Property (the remainder address the conversion of a mixed number to an improper fraction for the purposes of fraction multiplication)	
	50% of the meta-level narratives represent the "algorithm" type and 50% represent the "factor-product relationship" type	84% of the meta-level narratives represent the "algorithm" type (8% represent the "factor-product relationship" type and 8% represent the "commutative" type)
	Approximately half of the "algorithm" meta-level narratives address the traditional algorithm for fraction multiplication and the other half address other algorithms	
Constructing Narratives	Object-level narratives associated with contextual Questions represent 21% of the narratives	Object-level narratives associated with non-contextual Questions represent 53% of the narratives

Table 20 (cont'd)

	Written Curriculum	Enacted Curriculum
Constructing Narratives	Object-level narratives associated with contextual Questions occur on Days 1-3	Object-level narratives associated with contextual Questions occur across all five days
	The construction of object-level narratives varies depending on whether it is associated with a contextual or non-contextual Question	
	Approximately 70% of the contextual object-level narratives are constructed using both context and iconic mediators	
	-----	23% of the contextual object-level narratives are constructed from previously-constructed narratives
	21% of the non-contextual object-level narratives are completed using iconic mediators	59% of the non-contextual object-level narratives are completed using iconic mediators
	33% of the non-contextual object-level narratives are completed using estimation strategies	23% of the non-contextual object-level narratives are completed using estimation strategies

Table 20 (cont'd)

	Written Curriculum	Enacted Curriculum
Constructing Narratives	31% of all object-level narratives are constructed using iconic mediators	66% of all object-level narratives are constructed using iconic mediators
	Nearly all "algorithm" meta-level narratives are constructed using previously-constructed narratives (79% and 90% respectively)	
	71% of the "factor-product relationship" meta-level narratives are constructed using "exploration"	The "factor-product relationship" meta-level narratives are constructed using previously-constructed narratives
	-----	The "commutative" meta-level narratives are constructed using previously-constructed narratives
Substantiating Narratives	Nearly all uses of "prove" refer to proving using iconic mediators	
	47% and 41% of narratives are substantiated using iconic mediators and estimation strategies respectively	68% and 13% of narratives are substantiated using iconic mediators and estimation strategies respectively

Table 20 (cont'd)

	Written Curriculum	Enacted Curriculum
Substantiating Narratives	Both curricula use "agreement" between strategies for substantiating narratives	
	Both curricula use "consensus" for substantiating narratives	

Note. Dashes (i.e., "-----") indicate that the category is not applicable to the designated curriculum.

Table 20 and the more detailed analysis included within the chapter highlight many discursive similarities and differences between the written and enacted curricula with regard to endorsed narratives. The problem with such a table is that some of the richness and interpretation present within the text of the chapter is lost. However, the summary provided within the table allows a look at the data as a whole.

The comparison between the written and enacted curricula in Table 20 seems to indicate several substantial differences. First, several points seem to indicate that the enacted curriculum contains more experiences with object-level narratives addressing fractions and fewer experiences with object-level narratives including combinations of fractions, mixed numbers, and whole numbers. Similar results emerged from the mathematical words analysis (see Chapter 5). This difference in experiences with a broader range of numbers in the written curriculum may further limit students' opportunities to reify fractions because operations with other types of numbers (e.g., mixed numbers, whole numbers) reinforces the use of fractions as numbers.

Table 20 also includes evidence of more experiences with narratives addressing estimation (as a type of simple object-level narrative and for constructing and substantiating narratives) and the relationship between the magnitude of the factors and the product in fraction multiplication in the written curriculum than in the enacted curriculum. This supports similar results from Chapter 4 (Goals of the Written and Enacted Curricula), Chapter 5 (Mathematical Words in the Written and Enacted Curricula), and Chapter 6 (Visual Mediators in the Written and Enacted Curricula). In addition, the table indicates that the enacted curriculum provides more experiences with quasi-symbolic narratives (supports conclusions in Chapter 4 and 5) and narratives associated with contextual Questions and iconic mediators.[58] That is, experiences with estimation and relating the magnitude of the factors and product seem more likely to lead to fraction objectification than quasi-symbolic narratives (e.g., "$\frac{1}{3}$ of $\frac{2}{3} = \frac{2}{9}$") and the use of iconic mediators because the former use fractions and numbers and the latter use fractions as descriptors or parts. This analysis lends additional support to the conclusion that objectification is supported more substantially in the written curriculum than in the enacted curriculum.

[58] These differences are highlighted in Table 20.

CHAPTER 8: MATHEMATICAL ROUTINES IN THE WRITTEN AND ENACTED CURRICULA

Routines are the fourth and final mathematical feature to be examined in the written and enacted curricula. Routines are "well-defined repetitive patterns in interlocutors' actions, characteristic of a given discourse" (Sfard, 2007, p. 574). In this case, the discourse is mathematical in nature and the context is a sixth grade classroom. It is expected that the mathematical routines found in classrooms will differ to a greater or lesser extent from routines used by professional mathematicians depending on the grade level of the students. The preceding three chapters include analyses of routines in conjunction with the other mathematical features under investigation. The ubiquitous nature of routines makes this inevitable. In particular, thus far I have examined the routines associated with word use (e.g., emphasizing "of" as a proxy for fraction multiplication), the routine ways in which visual mediators are used (e.g., partitioning area and number line models), and the routines of narrative construction and substantiation (e.g., constructing meta-level narratives from previously endorsed object-level narratives). Therefore, these routines will not be further explicated in this chapter except as they naturally arise in the discussion.

Questioning is a widely used mathematical routine in both the written and enacted curricula (and arguably in scholarly mathematics) which has only been mentioned in passing thus far. Danielson (2005) noted the pervasiveness of questioning in both the written and enacted curricula in his study of the introduction to linearity in *Connected Mathematics*. In fact, the *Connected Mathematics* Teacher's Guides contain a section entitled "Suggested Questions" for each Problem in the Investigation. Given the fact that "learning mathematics" is conceptualized here as changing discourse practices, the questions included in the written and enacted curricula are worthy of examination. That is, "What discursive practices are elicited through questions in the written and enacted curriculum?" The discussion here is limited to questions that address the processes and products of fraction multiplication.

I compared the ranking of common question words (Who, What, When, Where, Why, and How) in the written and enacted curricula. Such a comparison is somewhat problematic because the written curriculum contains 837 distinct words and the enacted curriculum contains 1293 distinct words. To account for this difference, each word ranking in the written curriculum was multiplied by $\frac{1293}{837}$ to "scale it up" for comparison purposes. Table 21 provides the ranking for the common question words in both curricula. For example, "What" is the 31^{st} most common word in the written curriculum (after being scaled up) and the 13^{th} most common word in the enacted curriculum (of 1293 distinct words).

Table 21.

Rankings of Common Question Words in the Written and Enacted Curricula

Question Word	Written Curriculum	Enacted Curriculum
What	31	13
How	46	23
When	59	123
Why	120	97
Where	141	138
Who	334	299

Several notable features are illustrated in Table 21. First, each of the curricula has three question words in the top 100 words in the text. "What" and "How" are included in this top tier in both curricula. However, the third question word in the written curriculum is "When" whereas the third in the enacted curriculum is "Why." Aside from this exception, the order of the remaining question word rankings is the same between the two texts. In addition, with the exception of "When," all six words rank higher in the enacted curriculum than in the written curriculum. This may be a result of the differential nature of the written and enacted curricula or it may indicate a greater emphasis on questioning in the enacted curriculum than in the written curriculum. It should be noted that the context in which these words are used was not considered in this analysis (i.e., whether they were used in questions or embedded in statements), only the presence of the words themselves.

The written text of Investigation 3 includes a total of 110 suggested questions (an average of 22 questions per day). These questions are located in the Student Guide and Teacher's Guide, which include 22 and 88 questions respectively. In the Teacher's Guide, questions are included both in the "Suggested Questions" section mentioned earlier, as well as throughout the text. Usually the questions have a question format; however, several questions are mentioned as suggestions in statement form. Both types are included in this analysis. An example of the latter form follows:

Example 196 (WC).
"Ask students to explain how each of these is different from the fraction multiplication problems they have solved already." (TG, p. 71)

Given the complexity of spoken language in the enacted curriculum, several analytic decisions were made regarding which questions to include in the analysis. First, only questions related to mathematics were included. For example, "Did you watch *American Idol* last night?" was not included. Second, several questions in the enacted curriculum are rhetorical in nature (e.g., "Can I ask you something about multiplication?"). Questions of this type were excluded from the analysis. Third, it is common in the enacted curriculum to make a statement into a question by adding "okay" or "right" on the end. For example, "There are three pieces, right?" is for all intents and purposes a statement with a request for confirmation. Therefore, questions of this type were also excluded from the analysis. Finally, as mentioned in previous sections, it is very common to repeat utterances (e.g., questions) in spoken language. Therefore, any question repeated in the same turn or in any two consecutive turns by the same person was only included once in the analysis. In Example 197 from Day 4, the teacher asks the same question more than once within a turn.

Example 197 (EC).
T: "Okay. I tell you what. Let's look at one more and let's see if we can just figure out what it's saying. Not necessarily how to solve it, but ***what is it saying if I have three and a fourth times two and eleven twelfths? What does that mean? What does that mean?"***

Here, "what is it saying if I have three and a fourth times two and eleven twelfths?" "What does that mean?" and "What does that mean?" are all asking students for the meaning of mixed number multiplication. Therefore, only the first question from this utterance is included in the analysis. Similarly, the following excerpt from Day 2 provides an example in which the same question is asked in consecutive turns.

Example 198 (EC).

S1: First I would divide it into fifths and then, if I have this and then we would split these two pieces into thirds, and then two thirds of that would be, like, um, these parts.

*T: **Where's your one third?***

S1: I messed up.

*T: **Where's your one third?** Can you show us, okay.*

The teacher asks the question "Where's your one third" in consecutive turns here. Therefore, only the first of these questions was included in the analysis. Using these guidelines in the analysis of the enacted curriculum, 579 questions were identified over the course of the five days (an average of 116 questions per day).

To examine the relationship between the questions in the written and enacted curricula, the remainder of this chapter will present a series of analyses of the 110 questions in the written curriculum and the 579 questions in the enacted curriculum. These analyses include: (1) Leading Words of Questions, (2) Elicited Answers to Questions, (3) Mathematical Processes addressed by Questions, and (4) Miscellaneous Questions.

Leading Words of Questions

For this analysis, the first word of each question in the written and enacted curricula was noted. In several cases in both curricula the question was reframed in order to put the "question word" first. For example, in "For Question A, how did you first mark your brownie pan?" (TG, p. 61), "For Question A" was ignored and the first word was recorded as "How." Similarly, "And then" was ignored and "Did" was recorded as the first word in the following question from Day 5: "And then did she just use our algorithm, multiply the numerators and multiply the denominators?" Figure 33 summarizes the relative frequencies of the first words of questions that are present in at least 10% of either the written or enacted curriculum or both.

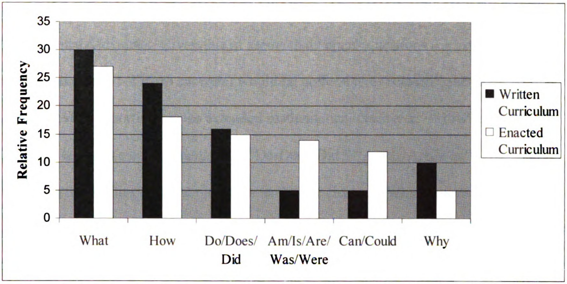


Figure 33. Relative frequencies of leading words of questions in the written and enacted curricula.

Interestingly, "What" and "How" are the most common in both curricula (each representing between 17% and 30%). This supports the earlier findings from the ranking in each curriculum. In both cases, however, the relative frequency in the written curriculum is greater than in the enacted curriculum. This does not support the previous finding in which both "What" and "How" are ranked higher in the enacted curriculum and may suggest that the word uses in the enacted curriculum are not from direct questions. This rationale also applies to "Why" because the Figure 33 indicates that it is twice as common in the written curriculum as in the enacted curriculum. Remember that its ranking is slightly higher in the enacted curriculum. The relative frequency of "Do/Does/Did" is very similar between the two curricula (approximately 15%). Finally, both "Am/Is/Are/Was/Were" and "Can/Could" are more common in the enacted curriculum.

Here, "learning mathematics" is defined as changes in participation in mathematical discourse. Given this, it seems that "open" questions (i.e., questions that require more than 1-2 word answers) would be preferable to "closed" questions, as they would provide opportunities for extended mathematical discourse. "How" and "Why" questions tend to be open, whereas the "Do/Does/Did," "Am/Is/Are/Was/Were," and "Can/Could" families of question words tend to be closed. "What" questions are not obviously open or closed and they appear in fairly equal frequencies in both curricula, therefore they were not included in this calculation. That is, 34% and 23% of the questions in the written and enacted curricula respectively are open questions. The next section describes a closely related analysis.

Elicited Answers to Questions

The questions in the written and enacted curricula ask students either to say something or to do something. Again, if learning mathematics is conceptualized as a change in ways of participating in mathematical discourse, then what students are expected to say and/or do is of critical importance. In particular, if students are expected primarily to "do," then it is questionable how much mathematics learning can take place. Figure 34 indicates the proportion of questions in the curricula that expect students to "say" or "do."

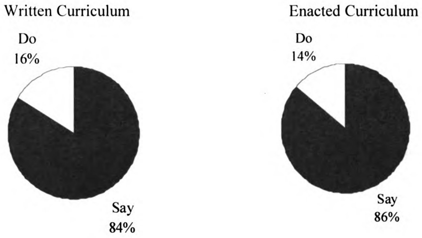


Figure 34. Relative frequencies of questions asking student to "say" something versus those asking students to "do" something.

Figure 34 indicates that questions expecting students to "say" and "do" are present in similar frequencies in the written and enacted curricula. In both cases, questions asking students to "say" are much more common than questions asking students to "do."

The questions were classified into *what* the students are expected to say or do. In making these decisions, I utilized the context of the question, including the answer given in the enacted curriculum or the "possible response" in the written curriculum. When expected to "say" something, several types of responses are elicited. These include explanations, 1-2 word answers, yes or no, a number, a number sentence, or a question. Table 22 provides examples from the written and enacted curricula of questions eliciting each of these responses.

Table 22.

Sample Questions eliciting Particular Response Types in the Written and Enacted Curricula

Response	Written Curriculum	Enacted Curriculum
Explanation	*"Why does it make sense that the product [of reciprocals] is always 1?" (TG, p. 81)*	*S: "Why did you write twenty twos, though?" (Day 5)*
1-2 Words	*"Would the estimate of $2\frac{1}{2}$ be an underestimate or an overestimate?" (TG, p. 76)*	*T: "Do you remember what you call that property?" (Day 3)*
Yes/No	*"Do you think Takoda's strategy works?" (SG, p. 38)*	*T: "So, is it fair to say that these two girls took each one of these thirds and split them into seven pieces?" (Day 1)*
Number	*"What fraction of a whole pan does Mr. Williams buy?" (SG, p. 33)*	*S: "So, how many thirds?" (Day 5)*
Number Sentence	-----	*T: "Did, what, what was our number sentence?" (Day 5)*
Question	-----	*T: "Any other questions for Trevor?" (Day 1)*

Note. Dashes (i.e., "-----") indicate that the category is not present in the designated curriculum.

Note that no questions in the written curriculum elicit a number sentence or a question. Figure 35 summarizes the relative frequency of each category of response in the written and enacted curricula.

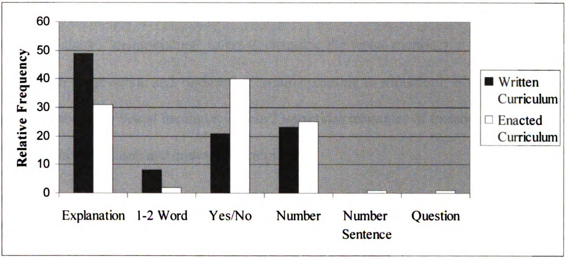


Figure 35. Relative frequencies of categories of elicited answers to questions in the written and enacted curricula.

Figure 35 indicates that explanations are the most commonly elicited type of response in the written curriculum (representing nearly 50% of all "saying" questions), whereas Yes/No responses are slightly more common than explanations in the enacted curriculum (40% compared to 31%). Questions eliciting a number occur in relatively similar proportions in the written and enacted curricula, 22% and 24% respectively. Figure 35 provides a slightly different picture than in the analysis of leading words (see Figure 33) because "explanation" questions are "open" and all other categories are "closed." In this analysis in which only "saying" questions are considered, the discrepancy between open and closed questions is greater. In the previous analysis, the relative frequencies of open questions in the written and enacted curricula were 34% and 23% respectively. In this analysis, the corresponding frequencies are 49% and 31%.

When a question expects the students to "do" something, the expected actions include calculating; constructing a symbolic mediator (e.g., a number sentence); constructing, manipulating, or indicating an iconic mediator (e.g., a diagram); and manipulating a concrete mediator (e.g., blocks). "Constructing" involves producing the visual mediator, "manipulating" refers to changing the visual mediator in some way once it has been constructed, and "indicating" entails pointing to something when asked a question about the visual mediator. Table 23 provides examples of questions from each category in the written and enacted curricula.

Table 23.

Sample Questions eliciting Particular Actions in the Written and Enacted Curricula

Action	Written Curriculum	Enacted Curriculum
Calculate	-----	*T: "But could you multiply that using our algorithm?" (Day 4)*
Construct Symbolic Mediator	*"Ask groups to write the number sentences for the problems on their transparency." (TG, p. 72)*	*T: "Can you write that [number sentence] next to it?" (Day 1)*
Construct Iconic Mediator	*"How would you represent $\frac{1}{4} x \frac{2}{3}$ on a number line?" (SG, p. 34)*	*T: "Who can come up and show what that would look like on a long, skinny model?" (Day 3)*

Table 23 (cont'd)

Action	Written Curriculum	Enacted Curriculum
Manipulate Iconic Mediator	*"What could you do in your drawing to make this clearer?" (TG, p. 60)*	*T: "Could you continue them as if this brownie were here, or this part of the goal was there?" (Day 4)*
Manipulate Concrete Mediator	-----	*T: "How much - could you get a half of them [blocks]? (Day 4)*
Indicate Iconic Mediator	*"Where do you see this [the numerators] on the brownie pan drawing?" (TG, p. 61)*	*T: "So where is one whole ounce?" (Day 5)*

Note. Dashes (i.e., "-----") indicate that the category is not present in the designated curriculum.

Table 23 indicates that the written curriculum does not contain any questions that ask students to calculate or to manipulate a concrete mediator. Figure 36 summarizes the relative frequencies of the types of "doing" responses to questions in the written and enacted curricula.

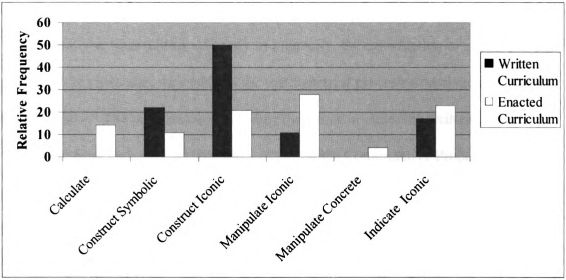


Figure 36. Relative frequencies of "doing" question responses in the written and enacted curricula.

Figure 36 illuminates several differences between the "doing" question responses in the written and enacted curricula. First, constructing iconic visual mediators is more than twice as common as any other expected "doing" question response in the written curriculum. In fact, half of all "doing" questions in the written curriculum expect this action. In contrast, several categories have relatively similar frequencies in the enacted curriculum. However, it should be noted that all three of the responses with the highest relative frequencies in the enacted curriculum involve iconic visual mediators (constructing, manipulating, and indicating). This is not to say, of course, that the Questions in the written curriculum does not expect students to calculate, only that the questions (i.e., sentences ending with "?") in the written curriculum do not ask students to calculate.[59]

[59] Recall that Questions (with a upper-case "Q") indicate specific exercises from the written curriculum.

Mathematical Processes Addressed by Questions

Most questions address, directly or indirectly, the construction or substantiation of narratives. In particular, they address the mathematical processes involved in both of these practices. The processes addressed by questions in either the written or enacted curricula or both include: (1) estimation, (2) using the Commutative Property, (3) decomposing numbers, (4) converting a mixed number to an improper fraction, (5) modeling (including concrete, iconic, and symbolic mediators), and (6) using an algorithm. There are also several questions that address mathematical processes more generally. Table 24 includes sample questions from the written and enacted curricula which exemplify questions addressing each process.

Table 24.

Sample Questions addressing Particular Mathematical Processes in the Written and Enacted Curricula

Process	Written Curriculum	Enacted Curriculum
Estimation	*"Who can explain how they* ***estimated*** $1\frac{1}{2} \times 2\frac{9}{10}$*?"(TG, p. 72)*	*T: "If I said I was going to get one half of two and nine tenths, Graham, how could we think about* ***estimating*** *that answer?" (Day 4)*
Commutative Property	-----	*T: "So can I switch my two factors around, my two numbers that I'm multiplying, around, or no?" (Day 4)*

Table 24 (cont'd)

Process	Written Curriculum	Enacted Curriculum
Decomposing Numbers	*"Would you want to use this strategy [distributive property] with the problem $3\frac{7}{8} \times 2\frac{5}{6}$?" (TG, p. 77)*	*T: "Could I do ten groups of two and a third and then a half of a group of two and a third?" (Day 5)*
Mixed Number → Improper Fraction	*"What do you think about Yuri's idea [to rewrite a mixed number as an improper fraction before multiplying]?" (SG, p. 37)*	*T: "Elliott, what do you think [about changing the mixed number to an improper fraction]?" (Day 5)*
Modeling	*"How does your **drawing** help someone see the part of the whole pan that is bought?" (TG, p. 60)*	*T: "So how far, what fraction of a whole mile is this one little piece right here [pointing to part of model]?" (Day 3)*
Algorithm	*"What observations can you make from Questions A and B that help you write an **algorithm** for multiplying fractions?" (SG, p. 35)*	*T: "Do you think that these are **the steps** that we should tape - take?" (Day 3)*

Table 24 (cont'd)

Process	Written Curriculum	Enacted Curriculum
General	*"**What method** did you use to solve the problem?" (TG, p. 81)*	*T: "Can you do this in a whole day, so **how could you figure out** what you do in a half of a day?" (Day 5)*

Note. Dashes (i.e., "-----") indicate that the category is not present in the designated curriculum.

Note that, as discussed earlier, the commutative property is not addressed by any questions in the written curriculum. Figure 37 summarizes the relative frequencies of the mathematical processes addressed in the questions. Some questions address more than one method; therefore, they were counted more than once (this type of question will be further discussed later in the section).

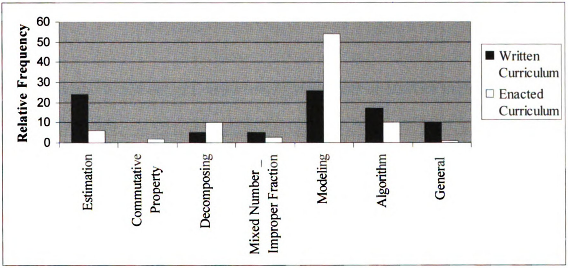


Figure 37. Relative frequency of mathematical processes addressed in questions in the written and enacted curricula.

Figure 37 indicates that more than 50% of the questions in the enacted curriculum address modeling (recall that this modeling may involve concrete, iconic, or symbolic visual mediators). In contrast, 25% of the questions in the written curriculum address estimation and another 25% address modeling. Much fewer questions in the enacted curriculum address estimation. This finding lends further support to the conjecture proposed and supported earlier that estimation receives more attention in the written curriculum than in the enacted curriculum. Questions addressing algorithms and general methods also represent at least 10% of the questions in the written curriculum. The same is true for decomposing numbers and algorithms in the enacted curriculum.

Of the questions that address more than one mathematical method, three categories emerged, including those that address the relationship between (1) iconic visual mediators, (2) symbolic visual mediators, (3) an iconic visual mediator and a symbolic visual mediator, and (4) other methods (e.g., estimation). Examples are provided in Table 25 representing each type in the written and enacted curricula.

Table 25.

Sample Questions addressing Comparison of Mathematical Methods in the Written and Enacted Curricula

Relationship	Written Curriculum	Enacted Curriculum
Iconic-Iconic	*"Who can share a number line model? [asked after a brownie pan model had been shared]?" (TG, p. 67)*	*"Is that the same thing that you guys saw Trevor do in the first drawing, or is this different?" (Day 1)*

Table 25 (cont'd)

Relationship	Written Curriculum	Enacted Curriculum
Symbolic-Symbolic	*"What does this [" $\frac{1}{2}$ x 3 = $1\frac{1}{2}$ "] tell you about the product for $\frac{1}{2}$ x $2\frac{9}{10}$? " (TG, p. 72)*	*"How is this problem [" $\frac{1}{2}$ of $\frac{2}{3}$ "] like that problem over there [" $\frac{1}{4}$ of $\frac{2}{3}$ "]?" (Day 2)*
Iconic-Symbolic	*"What part of your drawing in Question A shows the denominator?" (TG, p. 61)*	*"But does your work over here [number sentence] match this drawing?" (Day 2)*
Other	*"Does your exact answer seem reasonable given the estimate?" (TG, p. 72)*	*"Without multiplying it [only estimating it] can I know?" (Day 4)*

Twelve percent of the questions in the written curriculum address the relationships illustrated in Table 25 compared to 11% of the questions in the enacted curriculum. These questions seem important for the goal of mediational flexibility (i.e., the ability to transition between visual mediators).

Miscellaneous Questions

Questions that address the "answer" to a Question are very common in both the written and enacted curricula. In fact, 45% of the questions in the written curriculum address the answer to a Question compared to 23% of the questions in the enacted curriculum. These questions take several forms including asking for the answer itself, asking about the relative size of the answer, asking about the answer's relationship to another answer, etc. Examples given here provide a flavor of the variety of these questions in the written and enacted curricula:

Example 199 (WC).
"Who can use their model to prove that the answer $\frac{8}{12}$ is sensible?" (TG, p. 76)

Example 200 (EC).
T: "Do you think that it's one fourth or do you think it's showing one sixth of the whole bar?" (Day 3)

Example 201 (WC).
"What fraction of a whole pan does Aunt Serena buy?" (SG, p. 33)

Example 202 (EC).
T: "What fraction of the candy bar did I eat?" (Day 1)

Example 203 (WC).
"How did you come up with $2\frac{1}{2}$?" (TG, p. 71)

Example 204 (EC).
T: "How did you decide three eighths?" (Day 1)

Example 205 (WC).
"Does multiplication with fractions always lead to a product that is less than each factor? (TG, p. 67)

Example 206 (EC).
T: "Is your answer going to get bigger, or is your answer going to get smaller?" (Day 1)

Qualitatively the questions in the written and enacted curricula seem quite similar; however, quantitatively their relative frequencies are quite different (45% compared to 23%). This seems to indicate a greater focus in the enacted curriculum on process than on product.

Each of the following types of questions represents approximately 2-3% of all questions in both curricula with one exception which is noted below. The first type asks students about the meaning of fraction multiplication:

Example 207 (WC).
"What does it mean to find $\frac{1}{3}$ of $\frac{2}{3}$?" (TG, p. 60)

Example 208 (EC).
T: "What does that mean again, one third times one fourth?" (Day 2)

The second type asks students if they agree with an answer or strategy that has been suggested:

Example 209 (WC).
"Do you agree with this answer and the reasoning?" (TG, p. 72)

Example 210 (EC).
T: "Do you agree with that, Jacob, or no?" (Day 2)

The third type asks students if a suggested answer or strategy makes sense:

Example 211 (WC).
"Do you think Paula's strategy of rewriting the mixed numbers as fractions is sensible?" (TG, p. 80)

Example 212 (EC).
T: "Could you guys live with that, does that make sense?" (Day 3)

Finally, the fourth type asks students whether they are following or understand an answer or strategy (this type is present only in the enacted curriculum):

Example 213 (EC).
"Are you guys following his drawing?" (Day 1)

Example 214 (EC).
"Everybody understand what Katie did?" (Day 3)

Examples 207-212 indicate that these question types, quite similar in nature, are represented in both the written and enacted curricula (with the exception of the final type).

Summary

The question addressed in this summary is, "What does an investigation of the mathematical routines (in this case, the routine of asking questions) in the written and enacted curricula allow us to see?" That is, "What do we know now about the relationship between the written and enacted curricula that we did not know before?" Table 26 summarizes the findings from the investigation of mathematical routines for the purposes of examination for overall conclusions.[60]

Table 26.

Summary of "Mathematical Routines" (i.e., Questioning) Analysis

	Written Curriculum	Enacted Curriculum
Word Rankings	"What" and "How" are the first and second ranked question words in both curricula	
	"When" is the third ranked question word	"Why" is the third ranked question word

[60] The statements in this summary and throughout this analysis are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days in the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or the enacted curriculum as a whole. Rather, my statements highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.

Table 26 (cont'd)

	Written Curriculum	Enacted Curriculum
Leading Words of Questions	Average 22 questions/day	Average 116 questions/day
	"What" and "How" are the most common first word of a question in both curricula	
	10% of questions begin with "Why"	5% of questions begin with "Why"
	34% of the questions are open	23% of the questions are open
Elicited Answers to Questions	Approximately 85% of the questions ask students to "say" something rather than to "do" something	
	The most common elicited answer to "saying" questions is an "explanation" (49%)	The most common elicited answer to "saying" questions is "yes/no" (40%)
	Approximately 25% of the "saying" questions elicit a "number" in both curricula	
	49% of the "saying" questions are open	31% of the "saying" questions are open
	Nearly all elicited responses to "doing" questions involve visual mediators	

Table 26 (cont'd)

	Written Curriculum	Enacted Curriculum
Mathematical Processes addressed by Questions	26% of the questions address "modeling" and 24% address "estimation"	55% of the questions address "modeling" and 6% address "estimation"
	Approximately 10% of the questions address the relationship between two or more mathematical processes	
Miscellaneous Questions	45% of the questions address the answer to a Question	23% of the questions address the answer to a Question
	Both curricula include questions that ask for the meaning of fraction multiplication, requests for agreement/disagreement, and whether particular mathematics makes sense	

Table 26 and the more detailed analysis included within the chapter highlight many discursive similarities and differences between the questions included in the written and enacted curricula. The problem with such a table is that some of the richness and interpretation present within the text of this chapter is lost. However, the summary provided allows a look at the data as a whole.

Table 26 indicates a discrepancy in the written and enacted curricula between the quantity of questions asked in each. This table also highlights (in several places) that a greater proportion of the questions in the written curriculum require an explanation (i.e., are open questions). If learning mathematics involves changing participation in mathematical discourse, then the opportunities that "open" questions provide for students to participate actively in mathematical discourse (i.e., beyond short answers) make a difference in students' learning. That is, the 4-stage model of word use (See Figure 4 on p. 75) indicates the need to move beyond phrase-driven use and routine-driven use to reach object-driven use. Open questions provide opportunities for students to use words in a greater variety of ways, thus possibly facilitating objectification. In addition, questions that require explanation allow teachers greater access to the ways in which students are using words and therefore their stage of word use in particular situations. Finally, the table provides additional evidence that estimation is under-represented in the enacted curriculum compared to the written curriculum; this suggests that opportunities to objectify fractions are under-represented as well.[61]

[61] These differences are highlighted in Table 26.

CHAPTER 9: DISCUSSION

In this chapter, I return to the question that motivated this study and those that guided my analysis. First, the question that served as the impetus for the study was: "Which mathematics curricula are the most effective for promoting student learning?" It can be argued that this study does little to answer the question because no recommendation for the "right" curricula will be given within its pages. The contribution of this study toward the answering of this question is more subtle. Efforts to answer this question in the mathematics education research community have come largely in the form of comparative curricular studies, including large quantitative studies, case studies, and curricular analyses, among others. In these studies, two or more curricula are compared using a variety of methods, and results often include the endorsement of a particular curriculum.

A complaint in the field regarding these studies has been that the credit or blame for the achievement (defined in myriad ways) of the students in the classrooms in which these curricula are being used is assigned to these curricula with little or no knowledge of the "fidelity of implementation" or "treatment integrity" of these curricula in the classrooms (e.g., National Research Council, 2004; Senk & Thompson, 2003).[62] In an attempt to address this concern, recent studies have conducted classroom observations and documented "implementation" in a variety of ways, including textbook-use diaries (in which students and teachers record the frequency with which they use their textbooks), table-of-contents implementation records (in which teachers note the chapters

[62] "Fidelity of implementation" and "treatment integrity" are in quotation marks here because they are not the language used in my study. Rather, I use "the relationship between the written and enacted curricula." They are, however, used in the literature within this conversation. Therefore, I use them here to position my study in the literature.

completed in the textbook), and various observation protocols (addressing instructional strategies, *standards*-based practices, etc.) (e.g., Post et al., 2008; Tarr et al., 2008).

In spite of these efforts, it still seemed that an in-depth focus on the way in which particular mathematics was being presented in the classroom compared to the way it was presented in the textbook was missing. I proposed this study to investigate the relationship between the written and enacted curricula in the context of a particular mathematical topic, fraction multiplication. I conducted this analysis using Sfard's Commognition framework (2008) in which the mathematical features of the curricula (i.e., mathematical words, visual mediators, endorsed narratives, and mathematical routines) are highlighted.[63] The results of this study confirm the importance of considering the mathematics when observing classrooms for "fidelity of implementation" or "integrity of treatment" of the curriculum. That is, methods that do not address the mathematics in the written and enacted curricula do not provide enough information to make claims regarding the effectiveness of a particular curriculum.

The question guiding my study was: What does an investigation of the key features of mathematical discourse, using the Commognition framework, in the written and enacted curricula reveal? This question served to maintain a mathematical focus in the study and as a reminder that the primary question was one of comparison between the written and enacted curricula. I have addressed these questions in each chapter summary with particular attention to the mathematical feature described there. In this discussion, I address the question more broadly.

[63] As earlier, I cite Sfard (2008) even though this framework has been elaborated in other publications, because this book represents her most recent and comprehensive rationale for and description of this theoretical framework.

I begin with a quick refresher course on Commognition to set up my later comments. First, recall that Commognition defines "learning mathematics" as changing participation in mathematical discourse. Second, the four key mathematical features highlighted in Commognition (along with their definitions) are:

- Mathematical words: Words that signify mathematical objects or processes
- Visual mediators: Symbolic artifacts, created specially for the sake of mathematical communication
- Endorsed narratives: Any text, spoken or written, which is framed as a description of objects, of relationships between processes with or by objects, and which is subject to endorsement or rejection, that is, to being labeled as true or false
- Mathematical routines: Repetitive patterns characteristic of mathematical discourse (Sfard, 2008)

It is important to note that these features interact with one other in a variety of ways. For example, endorsed narratives contain mathematical words, mathematical routines are apparent in the use of visual mediators, visual mediators are used in the construction of endorsed narratives, etc. The most interesting part of the investigation was the way in which the examination of each mathematical feature contributed to the richness of the comparison between the written and enacted curricula.

Finally, objectification is a key element in the Commognition framework. I will describe this part of the theory in some detail because it played a substantial role in my analysis and much of my later comments are framed using objectification. Sfard (2008) defines objectification as "a process in which a noun begins to be used as if it signifies an extradiscursive, self-sustained entity (object), independent of human agency" (p. 412). She describes the process as consisting of "two tightly related, but not inseparable sub-processes: reification and alienation" (p. 412). She defines reification and alienation as follows:

- Reification: Replacement of talk about processes with talk about objects
- Alienation: Using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings

The case of whole numbers illustrates the importance of objectification. A young child talks about a number as a process, that is "three" is not an object to them; rather it is the result of counting. This is demonstrated when you ask a child how many cookies she has and she counts them. Wherever she finishes when the last cookie has been counted (assuming she has mastered one-to-one correspondence) is her answer. A child will often then begin using "three" as an adjective. Her answer when you ask her how many cookies she has will be "three cookies." It is later (sometimes much later) before a number is objectified (i.e., takes on a life of its own).[64] When a child has objectified "three," she is able to operate on it without counting (i.e., "three" as a process) or associating it with cookies (i.e., "three" as an adjective). Instead, "three" is now an object for her and she is able to talk about it in new ways (e.g., "three" is greater than two). That is, it has a life of its own and does not depend on human agency for its object-ness (no counting, no cookies). In this case, the change in discourse that counts as learning is the transition from non-objectified ways of speaking to objectified ways of speaking.

Objectification is not straight-forward to detect; however, there are clues in the ways in which individuals speak that provide hints about how they are thinking. In the

[64]Commognition does not support a Platonic view of mathematical objects. That is, objectification is not meant to imply that mathematical objects actually exist and objectification represents their discovery. Rather, they are proposed as theoretical constructs to facilitate communication.

example in the previous paragraph, "three is greater than two," we see "three" used with "is" and "greater than." These are both clues that "three" has been objectified because "is" is used with objects (e.g., The cat is fat). That is, "three" is a noun. In addition, "greater than" is discourse used exclusively with numbers (i.e., the objectified use of "three"). In the "adjective" stage of the use of "three," we would be more likely to hear, "Three cookies are more than two cookies." Note here the use of "are" and "more than." In this case, "cookies" is the noun. Finally, "three is greater than two" is stated as a mathematical fact and is not dependent upon human agency to make it so. These types of hints were used throughout this analysis to detect the objectification of fractions and other mathematical words.

The objectification of fractions (the mathematical topic under investigation here) is perhaps even more complex. A fraction (e.g., "$\frac{5}{6}$") is an "encapsulation" of two whole numbers. Encapsulation is combining two objects (e.g., "5" and "6") so that they form a new object (i.e., a fraction). It is not a given that a student that has spent several years studying fractions has objectified them (i.e., he now sees "$\frac{5}{6}$" as a number in its own right). He may still see "five sixths" as five pieces (the pieces are sixths). If he has objectified "$\frac{5}{6}$," he is likely to say "five-sixths is …" If not, he is more likely to say "five sixths are …" I argue that this encapsulation (i.e., the objectification of fractions) is one of many to come in mathematical discourse. As the number system in which students are expected to operate grows, so does the expected level of encapsulation.

Down the road, we will expect these same students to treat "$\sqrt{-5}$" as a number in its own right and operate on it as such.

Given this background, I will summarize my analysis of the relationship between the written and enacted curricula through the lens of objectification. Recall that objectification has two requirements: reification and alienation. It is impossible to address these processes completely separately, but I will begin by saying a few general words about alienation in the written and enacted curricula.[65]

As stated previously, alienation is the depersonalization of mathematics in general, and in this case, fractions in particular. Personalization is a key component in both the written and enacted curricula. For example, the traditional algorithm for fraction multiplication is noticed by a student in the class, Trevor, as he examines several symbolic mediators (i.e., number sentences) constructed using an iconic mediator (i.e., diagram). Once he makes the suggestion, the algorithm is referred to as "Trevor's Way," Trevor's Idea," and "Trevor's Method" by both the teacher and students during the remainder of the week. That is, the mathematics in this case is personalized. In the written curriculum, fictitious students are used in several instances to suggest various strategies (e.g., Distributive Property). This is in contrast to a more traditional mathematics textbook that might state the algorithm or the Distributive Property as a rule in a box at the beginning of the lesson. Another example of personalization in the

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[65] The statements in this summary and throughout this analysis are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days in the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or the enacted curriculum as a whole. Rather, my statements highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.

curricula is the use of consensus as a strategy for narrative substantiation (i.e., proving). That is, agreement of the class often seems to be the standard for narrative endorsement (i.e., deciding whether a mathematical statement is true or false). Again, in the written curriculum, the questions posed to students regarding these fictitious students, is "Do you agree?" It is not my conclusion that alienation never occurs in the written or enacted curriculum. This is not the case; however, it is my claim that alienation of the mathematics is not the norm in either curriculum. It is not my intention to convince the reader whether or not personalization or depersonalization (i.e., alienation) is the "right" answer for teachers or curriculum developers, only to raise the question for discussion and to acknowledge that this choice likely affects how mathematics is learned.

The second component of objectification is reification. I argue here, and throughout this analysis, that more opportunities for fraction reification are present in the written curriculum than in the enacted curriculum. Several specific mathematical features of the written curriculum, that receive less emphasis in the enacted curriculum, have the potential to facilitate reification. First, four of the five major analyses in this study (see Chapters 4, 5, 7, and 8) indicate that estimation strategies for fraction multiplication feature more prominently in the written curriculum than in the enacted curriculum. Estimation strategies promote reification-type language (e.g. "$\frac{4}{5}$ is approximately 1") and therefore provide opportunities for students to use fractions in this way (i.e., as numbers). Second, two of the five major analyses in this study (see Chapters 4 and 7) suggest that exploring the relationship between the magnitude of the factors and the product of fraction multiplication receives more attention in the written curriculum than in the enacted curriculum. Like estimation, an exploration of this relationship likely encourages the reification of fractions because "factors" and "product" are words associated with numbers.

Third, three of the five major analyses in this study (Chapters 4, 5, and 7) revealed that the written curriculum provides more opportunities for fraction multiplication with combinations of fractions (proper and improper), whole number, and mixed numbers. That is, the majority of the fraction multiplication addressed in the enacted curriculum involves the multiplication of two proper fractions. The extension of fraction multiplication to include these combinations expands fraction use to move beyond "part of part" (e.g., "$\frac{1}{3} \times \frac{3}{4}$") which seems to be problematic for reification because it does not promote encapsulation. For example, "$\frac{1}{3} \times \frac{3}{4}$" is described, in both curricula as "$\frac{1}{3}$ of $\frac{3}{4}$." Iconic mediators are used to illustrate the meaning of this multiplication and the discussion turns out to be about "parts" and "pieces," rather than about numbers. Improper fractions are much more difficult to speak about in this way (i.e., "$\frac{1}{3} \times \frac{4}{3}$" does not easily translate into "$\frac{1}{3}$ of $\frac{4}{3}$"), therefore they may promote the reification of fractions. In addition, mixed numbers (e.g., "$2\frac{2}{3}$") are often known by students to be between two numbers (e.g., "$2\frac{2}{3}$ is between 2 and 3"), therefore mixed number multiplication may lead to discussions using "$2\frac{2}{3}$" as a number.

Finally, although the majority of the types of and uses of visual mediators are quite similar in the written and enacted curricula (see Chapters, 4, 6, 7, and 8), the one notable exception to this is the absence of number lines in the enacted curriculum. Of all of the visual mediators proposed in the written curriculum, the number line is the one with the most potential for reification of fractions because it is an iconic mediator created specifically for "numbers" and it does not appear in the enacted curriculum. Although the linear area model in the enacted curriculum is used in ways similar to a number line, the discussions still primarily include words such as "parts" and "pieces" and do not promote reifying fractions (i.e., fractions as numbers).

The use of the Commognition framework for these analyses revealed many discursive similarities and differences in the written and enacted curricula. An important feature of mathematical discourse, objectification, was highlighted in the five primary analyses (Chapters 4-8) and several mathematical differences emerged in the presentation of the mathematics between the two curricula. Recall that objectification is made up of two closely related processes, alienation and reification. I conclude that both the written and enacted curricula are largely personalized (i.e., not alienated); however, experiences that promote reification are more evident in the written curriculum than in the enacted curriculum.

CHAPTER 10: CONCLUSION

This study contributes to the field of mathematics education in several ways. It pilots an analytic method for investigating the relationship between the written and enacted curricula (i.e., another way to think about "fidelity of implementation" or "integrity of treatment") that focuses on the mathematics present in the curricula. This is important because we need methods that include mathematics when documenting "implementation" for use in comparative studies. This study is a step in that direction.

In addition, this study illustrates the usefulness of the Commognition framework for highlighting opportunities for objectification in the written and enacted curricula. That is, by conducting a detailed analysis of the mathematical features, differences between the mathematics in the written and enacted curricula emerged that may impact the ways in which students speak about, and therefore learn mathematics. This study also provides fodder for thought and discussion to mathematics educators interested in fractions in general or fraction operations more specifically. For example, should "fraction as number" take a backseat to "fraction as part-whole" while students are learning fraction multiplication or is it possible to address these two uses of fractions simultaneously? Finally, this study acknowledges students' agency in the classroom by conceptualizing the enacted curriculum as the union of the words and actions of the teacher and the words and actions of the students. This is in contrast to many studies of implementation that focus primarily on the words and actions of the teacher.

The primary limitations of this study have been stated throughout this dissertation, but will be reiterated here briefly. First, comparing two forms of curricula (i.e., written and enacted) is challenging. Hopefully, the analytic decisions made here regarding the challenges of comparing these two modes will be helpful to future researchers. Second, the primary perspective represented here is mine.[66] I acknowledge that another individual using a different (or even the same) framework may see things quite differently. Related to this, important mathematics may have been missed. That is, I cannot claim that any of these analyses are comprehensive in scope. Finally and most importantly, the results of this study cannot be generalized beyond the five days analyzed here. That includes generalization to the whole written curriculum or the whole enacted curriculum. These results are much less about this particular curriculum and its use in this particular classroom than about the questions the analysis allows us to ask and the discussions it allows us to have, all toward the improvement of mathematics teaching and learning. In addition, it is possible that reification of fractions is more present in the enacted curriculum on days before or after the five days included in this study. Even if this is true, questions addressing how and when opportunities for the reification of fractions are most appropriate are still worthy of discussion.

This study leaves us with a long list of questions and ideas for important further research. First, how would curriculum developers see these data? How would a teacher see these data? As mentioned previously, I have presented my analysis and interpretation of the data, but more perspectives would provide a more complete picture. For those interested in fraction multiplication, an investigation using this lens into other curricula (traditional or *standards*-based) or other classrooms (with teachers who have more/less experience with the curriculum) would be a next step. This work could also be extended into other mathematical topics. In addition, more discussions regarding the affordances

[66] I say "primary" here because my perspective has certainly been affected by my interactions with others and I have received input on my ideas from many individuals throughout this process.

and limitations of alienation are needed. What does personalization afford us? What does depersonalization (i.e., alienation) afford us? What are the tradeoffs?

Finally, we need to develop instruments for textbook analyses and classroom observations that are sensitive to the mathematics presented in the written and enacted curricula in order to better address the issue of "fidelity of implementation" or "treatment integrity." The commognition framework provides one possible set of mathematical features (i.e., mathematical words, visual mediators, endorsed narratives, and mathematical routines) to highlight in these instruments. In this particular case, I focused on objectification through the lens of these mathematical features, but in other cases, different aspects of mathematics might be brought to the fore.

These discussions regarding whether mathematics curriculum and classrooms promote objectification seem particularly important now given the move in many states toward an increase in high school mathematics requirements. With more students taking advanced mathematics, we need to focus more resources on increasing the chances for students' success in these challenging courses. Further investigation into both how objectification serves students in these advanced mathematical settings and how to facilitate objectification would be a useful research topic, because students who in the past needed familiarity with only whole numbers, integers, and rational numbers are now expected to operate with irrational and even complex numbers.

Additional questions for curriculum developers include: How can desired discourse practices be made more explicit in the curriculum? Some of this has already begun (e.g., "Suggested Questions" in *Connected Mathematics*), but what more can be done? Can a balance be struck between personalization and depersonalization of the mathematics? How can objectification be made more explicit in the curriculum? Is it beneficial to do this? Finally, how can objectification be presented in ways that are accessible and convincing to teachers? How can teachers best facilitate objectification at various levels and in various contexts?

In sum, this study provides opportunities for important conversations among mathematics education researchers, curriculum developers, and teachers, in which important and challenging mathematics is the focus. Such conversations, in this case addressing fraction multiplication through the lens of the Commognition framework, function to improve the work in all three domains, research, curriculum, and teaching, for these three communities ultimately serve the same master, the mathematics learner.

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