This is to certify that the
dissertation entitled

TOPOLOGICAL INVARIANTS OF CONTACT STRUCTURES
AND PLANAR OPEN BOOKS

presented by

MEHMET FIRAT ARIKAN

has been accepted towards fulfillment
of the requirements for the

Ph.D. degree in MATHEMATICS

Major Professor's Signature

14 / April / 08

Date

TOPOLOGICAL INVARIANTS OF CONTACT STRUCTURES AND PLANAR OPEN BOOKS

By

Mehmet Fırat Arıkan

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2008

ABSTRACT

TOPOLOGICAL INVARIANTS OF CONTACT STRUCTURES AND PLANAR OPEN BOOKS

By

Mehmet Fırat Arıkan

The algorithm given by Akbulut and Ozbagci constructs an explicit open book decomposition on a contact three-manifold described by a contact surgery on a link in the three-sphere. In this work, we first improve this algorithm by using Giroux's contact cell decomposition process. Our algorithm gives a better upper bound for the recently defined "support genus invariant" of contact structures. Secondly, we find the complete list of all contact structures (up to isotopy) on closed three-manifolds which can be supported by an open book having planar pages with three (but not less) boundary components. Among these contact structures we also distinguish tight ones from those which are overtwisted. Finally, we study contact structures supported by open books whose pages are four−punctured sphere. Among these contact structures we prove that a certain family is holomorphically fillable using lantern relation, and show the overtwistedness of certain families using the study of right-veering diffeomorphisms.

To my Love and Family

ACKNOWLEDGMENTS

First, I would like to express my sincere gratitude to my supervisor, Selman Akbulut, for his invaluable guidance, encouragement, and for the motivation he has given throughout the research. I also would like to thank Onur Ağırseven, Selahi Durusoy, Çağrı Karakurt, Burak Özbağcı, Arda Buğra Özer, and András Stipsicz for their suggestions and helpful conversations.

I want to thank my parents for their support.

I am also grateful to my friend Mehmet Öztan for his technical support.

Lastly, but not least, my deepest love and respect to my wife, Setenay Arıkan, who was the main source of my creativity and supported me all the way. Without her encouragement, this thesis would not be possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Introduction

In the past decade, the study of open book decompositions (open books) in contact geometry and related fields became more and more important in order to understand the connection between geometry and topology. In 1975, Thurston and Winkelnkemper [TW] constructed a special contact structure on a given closed 3-manifold equipped with an open book decomposition in the sense that certain compatibility conditions are satisfied. In 2002, Giroux [Gi] proved the existence of a $1-1$ correspondence between open books and compatible contact structures on a fixed closed 3-manifold modulo certain equivalence relations. Based on this correspondence, in 2002, Ozsvath and Szabo [OSz] defined Heegaard Floer contact invariant (also called $\mathrm{OS}-$contact invariant) using open books. $\mathrm{OS}-$invariant has allowed for a much better understanding of tight but not fillable contact structures. It is nonzero for a Stein fillable structure and it vanishes for an overtwisted one. In particular, if the invariant is nonzero, then the contact structure is tight. Later (in 2005) Honda, Kazez and Matic [HKM1] initiated the study of right-veering diffeomorphisms of surfaces with boundary. They proved that a contact structure on a closed 3-manifold is tight if and only if the monodromy map of any compatible open book is right-veering. Recently (in 2006), Etnyre and Ozbagci [EO] introduced two purely topological invariants of a given contact structure based on the above correspondence. Let (M, ξ) be a closed oriented 3-manifold with the contact structure ξ, and let (S, h) be an open book (decomposition) of M which is compatible with ξ. In this case, we also say that (S, h) supports ξ (for the definitions see the next chapter). Based on Giroux's correspondence theorem (Theorem 1.3.2), we can ask two natural questions:

(1) What is the possible minimal page genus $g(S) = \text{genus}(S)$?

(2) What is the possible minimal number of boundary components of a page S with $g(S)$ minimal?

In [EO], two topological invariants $sg(\xi)$ and $bn(\xi)$ were defined to be the answers. More precisely, we have:

$$sg(\xi) = \min\{\ g(S) \mid (S, h) \text{ an open book decomposition supporting } \xi\},$$

called *the support genus* of ξ, and

$$bn(\xi) = \min\{\ |\partial S| \mid (S, h) \text{ an open book supporting } \xi \text{ and } g(S) = sg(\xi)\},$$

called *the binding number* of ξ. There are some partial results about these invariants. For instance,

Theorem 0.0.1 ([Et1]) *If (M, ξ) is overtwisted, then $sg(\xi) = 0$.* □

Also the contact structures with support genus zero and the binding number less than or equal to two are classified in [EO] (see Theorem 3.0.1). Unlike the overtwisted case, there is not much known yet for $sg(\xi)$ when ξ is tight. On the other hand, if we, furthermore, require that ξ is Stein fillable, then an algorithm to find an open book supporting ξ was given in [AO]. Although their construction is explicit, the pages of the resulting open books arise as Seifert surfaces of torus knots or links, and so this algorithm is far from even approximating the number $sg(\xi)$. In [St], the same algorithm was generalized to the case where ξ need not to be Stein fillable (or even tight), but the pages are still of large genera.

Based on the result of Ding and Geiges [DG] (2004), and by the result of Gompf [Gm] (1998) on Stein surfaces, we can study the contact structures and their possible fillings by using Legendrian framed link diagrams describing them. It is not clear, in general, yet that what property of such a diagram determines the tightness or the type of the filling. However, once we obtain a compatible open book, we can encode the surgery data by means of its monodromy, and then the monodromy can determine the tightness and the type of the filling for some (and maybe for all) such diagrams. To study overtwistedness, right-veering diffeomorphisms are very handy. On the other hand, OS$-$contact invariant and its recent interpretation, $EH-$invariant (defined in [HKM2]), can be used to determine the tightness for some cases.

As an another approach, we can fix an abstract surface and study all possible monodromies to understand which ones give tight structures and which ones not. Planar surfaces (the surfaces of genus zero) are the best choices to start with for such an approach. By Theorem 0.0.1, if $sg(\xi) \neq 0$, then ξ is tight. Therefore, understanding planar open books almost solves the tightness classification of contact structures. We remark that a monodromy which is not a right-veering can not support a tight structure. In fact, the following conjecture is the main motivation of the author throughout this work.

Conjecture 0.0.2 *If the monodromy of an open book with minimal page genus and binding number is right-veering, then the compatible contact structure is tight.*

This dissertation thesis is organized as follows: In the preliminaries chapter (Chapter 1), we will give the basic definitions and facts which will be used in the rest of the work. In Chapter 2, we will present an algorithm which finds a reasonable upper bound for $sg(\xi)$ using the given contact surgery diagram of ξ. We will also give examples to understand how efficient the new algorithm is. In Chapter 3, the complete list of contact structures with $sg(\xi) = 0$ and $bn(\xi) = 3$ will be given by analyzing the mapping class group of the surface so called "pair of pants". In Chapter 4, we will explore the planar contact structures with $bn(\xi) \leq 4$: After studying the mapping class group of "four$-$punctured sphere", we will prove that a certain family is holomorphically fillable, and also show the overtwistedness for some other family.

Chapter 1

Preliminaries

This chapter provides the definitions, notation and facts that will be used in the next chapters. Some well-known material is included in order to make the presentation self-contained.

1.1 Contact Structures and open book decompositions

A 1-form $\alpha \in \Omega^1(M)$ on a 3-dimensional oriented manifold M is called a *contact form* if it satisfies $\alpha \wedge d\alpha \neq 0$. An *oriented contact structure* on M is then a hyperplane field ξ which can be globally written as the kernel of a contact 1-form α. We will always assume that ξ is a *positive* contact structure, that is, $\alpha \wedge d\alpha > 0$. Note that this is equivalent to asking that $d\alpha$ be positive definite on the plane field ξ, ie., $d\alpha|_\xi > 0$. Two contact structures ξ_0, ξ_1 on a 3-manifold are said to be *isotopic* if there exists a 1-parameter family ξ_t $(0 \leq t \leq 1)$ of contact structures joining them. We say that two contact 3-manifolds (M_1, ξ_1) and (M_2, ξ_2) are *contactomorphic* if there exists a diffeomorphism $f : M_1 \longrightarrow M_2$ such that $f_*(\xi_1) = \xi_2$. Note that isotopic contact structures give contactomorphic contact manifolds by Gray's Theorem, and any contact 3-manifold is locally contactomorphic to $(\mathbb{R}^3, \xi_0)$ where *standard contact structure* ξ_0 on $\mathbb{R}^3$ with coordinates (x, y, z) is given as the kernel of $\alpha_0 = dz + xdy$. The standard contact structure ξ_{st} on the 3-sphere $S^3 = \{(r_1, r_2, \theta_1, \theta_2) : r_1^2 + r_2^2 = 1\} \subset \mathbb{C}^2$ is given as the kernel of $\alpha_{st} = r_1^2 d\theta_1 + r_2^2 d\theta_2$. One basic fact is that $(\mathbb{R}^3, \xi_0)$ is contactomorphic to $(S^3 \setminus \{pt\}, \xi_{st})$. For more details on contact geometry, we refer the reader to [Ge], [Et3], and [MS].

An *open book decomposition* of a closed 3-manifold M is a pair (L, f) where L is an oriented link in M, called the *binding*, and $f : M \setminus L \to S^1$ is a fibration such that $f^{-1}(t)$ is the interior of a compact oriented surface $S_t \subset M$ and $\partial S_t = L$ for all $t \in S^1$. In such a case, L is also called a *fibered link* in M. The surface $S = S_t$, for any t, is called the *page* of the open book. The *monodromy* of an open book (L, f) is given by the return map of a flow transverse to the pages (all diffeomorphic to S) and meridional near the binding, which is an element $h \in Aut(S, \partial S)$, the group of (isotopy classes of) diffeomorphisms of S which restrict to the identity on ∂S . The group $Aut(S, \partial S)$ is also said to be the *mapping class group* of S, and denoted by $\Gamma(S)$.

An open book can also be described as follows. First consider the mapping torus

$$S(h) = [0, 1] \times S/(1, x) \sim (0, h(x))$$

where S is a compact oriented surface with $n = |\partial S|$ boundary components and h is an element of $Aut(S, \partial S)$ as above. Since h is the identity map on ∂S, the boundary $\partial S(h)$ of the mapping torus $S(h)$ can be canonically identified with n copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $[0,1]/(0 \sim 1)$ and the second one comes from a component of ∂S. Now we glue in n copies of $D^2 \times S^1$ to cap off $S(h)$ so that ∂D^2 is identified with $S^1 = [0,1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of ∂S. Thus we get a closed 3-manifold

$$M = M_{(S,h)} := S(h) \cup_n D^2 \times S^1$$

equipped with an open book decomposition (S, h) whose binding is the union of the core circles in the $D^2 \times S^1$'s that we glue to $S(h)$ to obtain M. To summarize, an element $h \in Aut(S, \partial S)$ determines a 3-manifold $M = M_{(S,h)}$ together with an "abstract" open book decomposition (S, h) on it. For further details on these subjects, see [Gd], and [Et2].

Throughout the work, D_γ will denote the right Dehn twist along the simple closed curve γ, and most of the time we will write γ instead D_γ for simplicity. We want to state the following classical fact which will be used in Section 4.1. We also give a proof since the author couldn't find the given version of the theorem in the literature.

Theorem 1.1.1 *Let S be any surface with nonempty boundary, and let $\sigma, h \in Aut(S, \partial S)$. Then there exists a contactomorphism*

$$(M_{(S,h)}, \xi_{(S,h)}) \cong (M'_{(S,\sigma h \sigma^{-1})}, \xi'_{(S,\sigma h \sigma^{-1})}).$$

Proof: The proof based on the idea of breaking up the monodromy $\sigma h \sigma^{-1}$ into pieces as depicted in Figure 1.1. First take each glued solid torus (around each binding component) out from both $(M_{(S,h)}, \xi_{(S,h)})$ and $(M'_{(S,\sigma h\sigma^{-1})}, \xi'_{(S,\sigma h\sigma^{-1})})$ to get the mapping tori (S,h) and $S(\sigma h\sigma^{-1})$. By breaking the monodromy $\sigma h\sigma^{-1}$, the mapping torus $S(\sigma h\sigma^{-1}) = [0,1] \times S/(1,x) \sim (0, \sigma h\sigma^{-1}(x))$ can be constructed also as follows: We write

$$S(\sigma h\sigma^{-1}) = (\coprod_{i=1}^{4} S_i)/\sim,$$

where $S_i = S \times [\frac{i-1}{4}, \frac{i}{4}]$ and $\sim$ is the equivalence relation that glues $S \times \{\frac{1}{4}\}$ in S_1 to $S \times \{\frac{1}{4}\}$ in S_2 by σ, glues $S \times \{\frac{1}{2}\}$ in S_2 to $S \times \{\frac{1}{2}\}$ in S_3 by h, glues $S \times \{\frac{3}{4}\}$ in S_3 to $S \times \{\frac{3}{4}\}$ in S_4 by σ^{-1}, glues $S \times \{1\}$ in S_4 to $S \times \{0\}$ in S_1 by the identity map id. (See the picture on the left in Figure 1.1.)

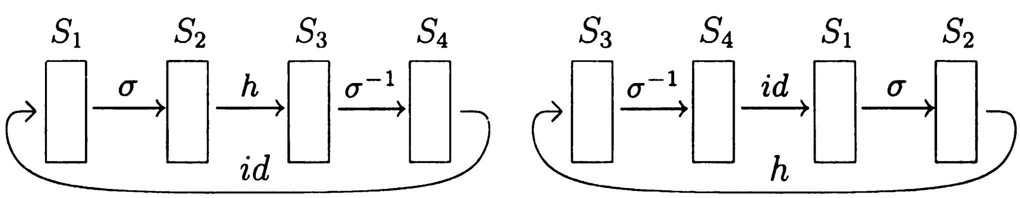

Figure 1.1. Mapping torus $S(\sigma h\sigma^{-1})$, before and after the cyclic permutation.

Since $S(\sigma h\sigma^{-1})$ is a fiber bundle over the circle S^1, we are free to change its monodromy by any cyclic permutation. Therefore, the monodromy element $\sigma^{-1} \cdot id \cdot \sigma h = h$ also gives the same fiber bundle $S(\sigma h\sigma^{-1})$ (the picture on the right in Figure 1.1 shows the new configuration of $S(\sigma h \sigma^{-1})$ after the cyclic permutation). Therefore, $S(h) = S(\sigma h \sigma^{-1})$. By gluing all solid tori back, we conclude that $(M_{(S,h)}, \xi_{(S,h)})$ is contactomorphic to $(M'_{(S,\sigma h \sigma^{-1})}, \xi'_{(S,\sigma h \sigma^{-1})})$. □

1.2 Legendrian knots and contact surgery

A *Legendrian knot* K in a contact 3-manifold (M, ξ) is a knot that is everywhere tangent to ξ. Any Legendrian knot comes with a canonical *contact framing* (or *Thurston-Bennequin framing*), which is defined by a vector field along K that is transverse to ξ. We call (M, ξ) (or just ξ) *overtwisted* if it contains an embedded disc $D \approx D^2 \subset M$ with boundary $\partial D \approx S^1$ a Legendrian knot whose contact framing equals the framing it receives from the disc D. If no such disc exists, the contact structure ξ is called *tight.*

For any $p, q \in \mathbb{Z}$, a contact (r)-surgery $(r = p/q)$ along a Legendrian knot K in a contact manifold (M, ξ) was first described in [DG]. It was proved in [Ho] that if $r = 1/k$ with $k \in \mathbb{Z}$, then the resulting contact structure is unique up to isotopy. In particular, a contact (± 1)-surgery along a Legendrian knot K on a contact manifold (M, ξ) determines a unique surgered contact manifold which will be denoted by $(M, \xi)_{(K, \pm 1)}$. The most general result along these lines is:

Theorem 1.2.1 ([DG]) *Every closed contact* 3*-manifold* (M, ξ) *can be* *obtained via contact* (± 1)*-surgery on a Legendrian link in* (S^3, ξ_{st}). $\square$

Any closed contact 3-manifold (M, ξ) can be described by a *contact surgery diagram* drawn in $(\mathbb{R}^3, \xi_0) \subset (S^3, \xi_{st})$. By Theorem 1.2.1, there is a contact surgery diagram for (M, ξ) such that the contact surgery coefficient of any Legendrian knot in the diagram is ± 1. For any oriented Legendrian knot K in $(\mathbb{R}^3, \xi_0)$, we compute the *Thurston-Bennequin number* $tb(K)$, and the *rotation number* $rot(K)$ as

$$tb(K) = bb(K) - (\#\text{ of left cusps of K}),$$
$$rot(K) = \frac{1}{2}[(\#\text{ of downward cusps}) - (\#\text{ of upward cusps})]$$

where $bb(K)$ is the blackboard framing of K.

If a contact surgery diagram for (M, ξ) is given, we can also get the smooth surgery diagram for the underlying 3-manifold M. Indeed, for a Legendrian knot K in a contact surgery diagram, we have:

$$\text{Smooth surgery coeff. of } K = \text{Contact surgery coeff. of } K + tb(K)$$

For more details see [OSt] and [Gm].

1.3 Compatibility and stabilization

A contact structure ξ on a 3-manifold M is said to be *supported by an open book* (L, f) if ξ is isotopic to a contact structure given by a 1-form α such that

1. $d\alpha$ is a positive area form on each page $S \approx f^{-1}(\mathrm{pt})$ of the open book and

2. $\alpha > 0$ on L (Recall that L and the pages are oriented.)

When this holds, we also say that the open book (L, f) is *compatible with the contact structure* ξ on M.

Definition 1.3.1 *A positive (resp., negative) stabilization* $S_K^+(S, h)$ *(resp.,* $S_K^-(S, h)$*) of an abstract open book* (S, h) *is the open book*

1. *with page* $S' = S \cup$ *1-handle and*

2. *monodromy* $h' = h \circ D_K$ *(resp.,* $h' = h \circ D_K^{-1}$*) where* D_K *is a right-handed Dehn twist along a curve* K *in* S' *that intersects the co-core of the 1-handle exactly once.*

Based on the result of Thurston and Winkelnkemper [TW] which introduced open books into the contact geometry, Giroux proved the following theorem strengthening the link between open books and contact structures.

Theorem 1.3.2 ([Gi]) *Let* M *be a closed oriented* 3*-manifold. Then there is a one-to-one correspondence between oriented contact structures on* M *up to isotopy and open book decompositions of* M *up to positive stabilizations: Two contact structures supported by the same open book are isotopic, and two open books supporting the same contact structure have a common positive stabilization.* $\square$

For a given fixed open book (S, h) of a 3-manifold M, there exists a unique compatible contact structure up to isotopy on $M = M_{(S,h)}$ by Theorem 1.3.2. We will denote this contact structure by $\xi_{(S,h)}$. Therefore, an open book (S, h) determines a unique contact manifold $(M_{(S,h)}, \xi_{(S,h)})$ up to contactomorphism. We will shorten the notation as (M_h, ξ_h) if the surface S is clear from the content.

Taking a positive stabilization of (S, h) is actually taking a special Murasugi sum of (S, h) with the positive Hopf band (H^+, D_γ) where $\gamma \subset H^+$ is the core circle. Taking a Murasugi sum of two open books corresponds to taking the connect sum of 3-manifolds associated to the open books. The proofs of the following facts can be found in [Gd], [Et2].

Theorem 1.3.3 *We have*

$$(M_{S^+_K(S,h)}, \xi_{S^+_K(S,h)}) \cong (M_{(S,h)}, \xi_{(S,h)}) \#(S^3, \xi_{st}) \cong (M_{(S,h)}, \xi_{(S,h)}). \quad \square$$

1.4 Symplectic and Stein Fillings

A contact manifold (M, ξ) is called *symplectically fillable* if there is a compact symplectic 4–manifold (X, ω) (that is, ω is a non-degenerate and closed 2−form on X) such that $\partial X = M$ and $\omega|_\xi \neq 0$. A Stein manifold is a triple (X, J, ψ) where J is a complex structure on X, $\psi : X \to \mathbb{R}$, and the 2−form $\omega_\psi = -d(d\psi \circ J)$ is non-degenerate. We say (M, ξ) is called *Stein (holomorphically) fillable* if there is a Stein manifold (X, J, ψ) such that ψ is bounded from below, M is a non-critical level of ψ and $-(d\psi \circ J)$ is a contact form for ξ. We note that any Stein filling (X, J, ψ) of (M, ξ) can be also considered as the symplectic filling (X, ω_ψ). See [Et2] or [OSt] for the related facts. We will use the following theorem later.

Theorem 1.4.1 ([EG]) *Any symplectically fillable contact structure is tight.* ($\Rightarrow$ *Any holomorphically fillable contact structure is tight.*) $\square$

Following fact was first implied in [LP], and then in [AO]. The given version below is due to Giroux and Matveyev. For a proof, see [OSt].

Theorem 1.4.2 *A contact structure ξ on M is holomorphically fillable if and only if ξ is supported by some open book whose monodromy admits a factorization into positive Dehn twists only.* $\square$

1.5 Monodromy and surgery diagrams

Given a contact surgery diagram for a closed contact 3-manifold (M, ξ), we want to construct an open book compatible with ξ. One implication of Theorem 1.2.1 is that one can obtain such a compatible open book by starting with a compatible open book of (S^3, ξ_{st}), and then interpreting the effects of surgeries (yielding (M, ξ)) in terms of open books. However, we first have to realize each surgery curve (in the given surgery diagram of (M, ξ)) as a Legendrian curve sitting on a page of some open book supporting (S^3, ξ_{st}). We refer the reader to Section 5 in [Et2] for a proof of the following theorem.

Theorem 1.5.1 *Let (S, h) be an open book supporting the contact manifold (M, ξ). If K is a Legendrian knot on the page S of the open book, then*

$$(M, \xi)_{(K,\,\pm 1)} = (M_{(S,\, h\circ D_K^{\mp})}, \xi_{(S,\, h\circ D_K^{\mp})}). \quad \square$$

1.6 Contact cell decompositions

The exploration of contact cell decompositions in the study of open books was originally initiated by Gabai [Ga], and then developed by Giroux [Gi]. We want to give several definitions and facts carefully.

Let (M, ξ) be any contact 3-manifold, and $K \subset M$ be a Legendrian knot. The *twisting number* $tw(K, Fr)$ of K with respect to a given framing Fr is defined to be the number of counterclockwise 2π twists of ξ along K, relative to Fr. In particular, if K sits on a surface $S \subset M$, and Fr_S is the surface framing of K given by S, then we write $tw(K, S)$ for $tw(K, Fr_S)$. If $K = \partial S$, then we have $tw(K, S) = tb(K)$ (by the definition of tb).

Definition 1.6.1 *A contact cell decomposition of a contact $3-$manifold (M, ξ) is a finite CW-decomposition of M such that*

(1) the 1-skeleton is a Legendrian graph,

(2) each 2-cell D satisfies $tw(\partial D, D) = -1$, and

(3) ξ is tight when restricted to each 3-cell.

Definition 1.6.2 *Given any Legendrian graph G in (M, ξ), the ribbon of G is a compact surface $R = R_G$ satisfying*

1. *R retracts onto G,*

2. *$T_pR = \xi_p$ for all $p \in G$, and*

3. *$T_pR \neq \xi_p$ for all $p \in R \setminus G$.*

For a proof of the following theorem we refer the reader to [Gd] and [Et2].

Theorem 1.6.3 *Given a closed contact 3−manifold (M, ξ), the ribbon of the 1-skeleton of any contact cell decomposition is a page of an open book supporting ξ.* $\square$

1.7 Right-veering diffeomorphisms

We recall the right-veering diffeomorphisms originally introduced in [HKM1]. If S is a compact oriented surface with $\partial S \neq \emptyset$, the submonoid $Veer(S, \partial S)$ of right-veering elements in $Aut(S, \partial S)$ is defined as follows: Let α and β be isotopy classes (relative to the endpoints) of properly embedded oriented arcs $[0,1] \to S$ with a common initial point $\alpha(0) = \beta(0) = x \in \partial S$. Let $\pi : \tilde{S} \to S$ be the universal cover of S (the interior of $\tilde{S}$ will always be $\mathbb{R}^2$ since S has at least one boundary component), and let $\tilde{x} \in \partial\tilde{S}$ be a lift of $x \in \partial S$. Take lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}$. $\tilde{\alpha}$ divides $\tilde{S}$ into two regions – the region "to the left" and the region "to the right". We say that β is *to the right* of α, denoted $\alpha \geq \beta$, if either $\alpha = \beta$ (and hence $\tilde{\alpha}(1) = \tilde{\beta}(1)$), or $\tilde{\beta}(1)$ is in the region to the right (Figure 1.2).

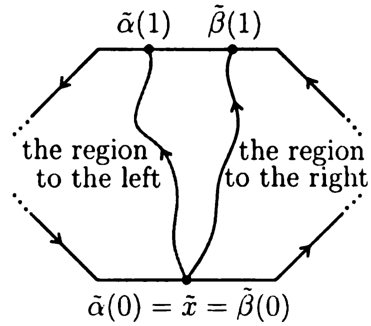

Figure 1.2. Lifts of α and β in the universal cover $\tilde{S}$.

As an alternative to passing to the universal cover, we first isotop α and β, while fixing their endpoints, so that they intersect transversely (this include the endpoints) and with the fewest possible number of intersections. Then β is to the right of α if the tangent vectors $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation on S at x.

Definition 1.7.1 *Let $h : S \to S$ be a diffeomorphism that restricts to the identity map on ∂S. Let α be a properly embedded oriented arc starting at a basepoint $x \in \partial S$. Then h is* right-veering *(that is, $h \in Veer(S, \partial S)$) if for every choice of basepoint $x \in \partial S$ and every choice of α based at x, $h(\alpha)$ is to the right of α (at x). If C is a boundary component of S, we say is h is* right-veering with respect to C *if $h(\alpha)$ is to the right of α for all α starting at a point on C.*

It turns out that $Veer(S, \partial S)$ is a submonoid and we have the inclusions:

$$Dehn^+(S, \partial S) \subset Veer(S, \partial S) \subset Aut(S, \partial S).$$

We will use the following two results of [HKM1].

Theorem 1.7.2 ([HKM1]) *A contact structure* (M, ξ) *is tight if and only if all of its compatible open book decompositions* (S, h) *have right-veering* $h \in Veer(S, \partial S) \subset Aut(S, \partial S)$. $\square$

1.8 Homotopy invariants of contact structures

The set of oriented 2−plane fields on a given 3-manifold M is identified with the space $Vect(M)$ of nonzero vector fields on M. $v_1, v_2 \in Vect(M)$ are called *homologous* (denoted by $v_1 \sim v_2$) if v_1 is homotopic to v_2 in $M \backslash B$ for some 3−ball B in M. The space $Spin^c(M)$ of all spin^c structures on M is the defined to be the quotient space $Vect(M)/\sim$. Therefore, any contact structure ξ on M defines a spin^c structure $\mathbf{t}_\xi \in Spin^c(M)$ which depends only on the homotopy class of ξ. As the first invariant of ξ, we will use the first Chern class $c_1(\xi) \in H^2(M; \mathbb{Z})$ (considering ξ as a complex line bundle on M). For a spin^c structure $\mathbf{t}_\xi$, whose first Chern class $c_1(\mathbf{t}_\xi)(:= c_1(\xi))$ is torsion, the obstruction to homotopy of two 2-plane fields (contact structures) both inducing $\mathbf{t}_\xi$ can be captured by a single number. This obstruction is the 3*-dimensional invariant* $d_3(\xi)$ of ξ). To compute $d_3(\xi)$, first recall an *almost complex manifold* is a pair (X, J) where $J : TX \to TX$ is such that $J^2 = -id_{TX}$. Now suppose that a compact almost complex 4-manifold (X, J) is given such that $\partial X = M$, and ξ is the complex tangencies in TM, i.e., $\xi = TM \cap J(TM)$. Let $\sigma(X), \chi(X)$ denote the signature and Euler characteristic of X, respectively. Then we have

Theorem 1.8.1 ([Gm]) *If $c_1(\xi)$ is a torsion class, then the rational number*

$$d_3(\xi) = \frac{1}{4}(c_1^2(X,J) - 3\sigma(X) - 2\chi(X))$$

is an invariant of the homotopy type of the 2-plane field ξ. Moreover, two 2-plane fields ξ and η with $\mathbf{t}_\xi = \mathbf{t}_\eta$ and $c_1(\xi) = c_1(\eta)$ a torsion class are homotopic if and only if $d_3(\xi) = d_3(\eta)$. $\square$

As a result of this fact, if (M, ξ) is given by a contact (± 1)-surgery on a link sitting in (S^3, ξ_{st}), then we can compute $d_3(\xi)$ using

Corollary 1.8.2 ([DGS]) *Suppose that (M, ξ), with $c_1(\xi)$ torsion, is given by a contact (± 1)-surgery on a Legendrian link $\mathbb{L} \subset (S^3, \xi_{st})$ with $tb(K) \neq 0$ for each $K \subset \mathbb{L}$ on which we perform contact $(+1)$-surgery. Let X be a 4-manifold such that $\partial X = M$. Then*

$$d_3(\xi) = \frac{1}{4}(c^2 - 3\sigma(X) - 2\chi(X)) + s,$$

where s denotes the number of components in $\mathbb{L}$ on which we perform $(+1)$-surgery, and $c \in H^2(X;\mathbb{Z})$ is the cohomology class determined by $c(\Sigma_K) = rot(K)$ for each $K \subset \mathbb{L}$, and $[\Sigma_K]$ is the homology class in $H_2(X)$ obtained by gluing the Seifert surface of K with the core disc of the 2-handle corresponding K. $\square$

We use the above formula as follows: Suppose $\mathbb{L}$ has k components. Write $\mathbb{L} = \sqcup_i{}^k K_i$. By converting all contact surgery coefficients to the topological ones, and smoothing each cusp in the diagram, we get a framed link (call it $\mathbb{L}$ again) describing a simply connected 4-manifold X such that $\partial X = M$. Using this description, we compute

$$\chi(X) = 1 + k, \text{ and } \sigma(X) = \sigma(\mathcal{A}_{\mathbb{L}})$$

where $\mathcal{A}_{\mathbb{L}}$ is the linking matrix of $\mathbb{L}$. Using the duality, the number c^2 is computed as

$$c^2 = (PD(c))^2 = [b_1\ b_2 \cdots b_k]\mathcal{A}_{\mathbb{L}}[b_1\ b_2 \cdots b_k]^T$$

where $PD(c) \in H_2(X, \partial X; \mathbb{Z})$ is the Poincaré dual of c, the row matrix $[b_1\ b_2 \cdots b_k]$ is the unique solution to the linear system

$$\mathcal{A}_{\mathbb{L}}[b_1\ b_2 \cdots b_k]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_k)]^T.$$

Here the superscript "T" denotes the transpose operation in the space of matrices. See [DGS], [Gm] for more details.

Chapter 2

An upper bound for the support genus invariant

In this chapter, we will present an explicit construction of a supporting open book (with considerably less genus) for a given contact surgery diagram of any contact structure ξ. Of course, because of Theorem 0.0.1, our algorithm makes more sense for the tight structures than the overtwisted ones. Moreover, it depends on a choice of the contact surgery diagram describing ξ. Nevertheless, it gives better and more reasonable upper bound for $sg(\xi)$ (when ξ is tight) as we will see from our examples in Section 2.2.

Let L be any Legendrian link given in $(\mathbb{R}^3, \xi_0 = ker(\alpha_0 = dz + xdy)) \subset (S^3, \xi_{st})$. L can be represented by a special diagram $\mathcal{D}$ called *a square bridge diagram* of L (see [Ly]). We will consider $\mathcal{D}$ as an abstract diagram such that

1. $\mathcal{D}$ consists of horizontal line segments $h_1, ..., h_p$, and vertical line segments $v_1, ..., v_q$ for some integers $p \geq 2$, $q \geq 2$,
2. there is no collinearity in $\{h_1, \dots, h_p\}$, and in $\{v_1, \dots, v_q\}$.

3. each h_i (resp., each v_j) intersects two vertical (resp., horizontal) line segments of $\mathcal{D}$ at its two endpoints (called *corners* of $\mathcal{D}$), and

4. any interior intersection (called *junction* of $\mathcal{D}$) is understood to be a virtual crossing of $\mathcal{D}$ where the horizontal line segment is passing over the vertical one.

We depict Legendrian right trefoil and the corresponding $\mathcal{D}$ in Figure 2.1.

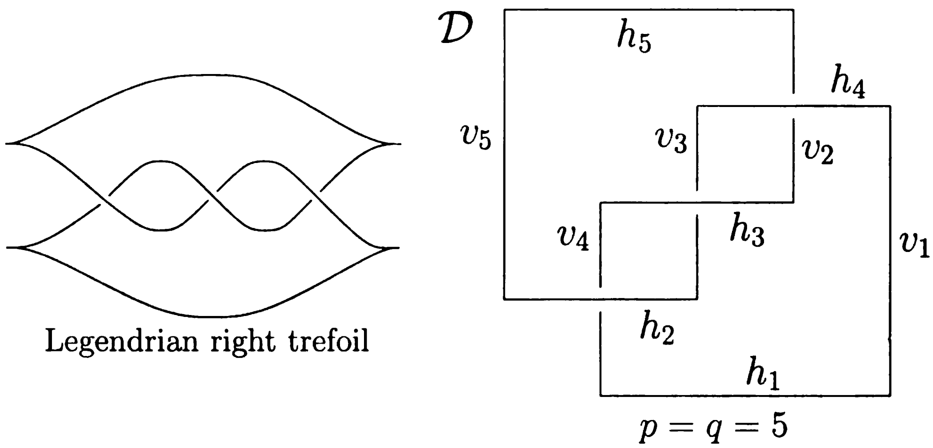

Figure 2.1. The square bridge diagram $\mathcal{D}$ for the Legendrian right trefoil.

Clearly, for any front projection of a Legendrian link, we can associate a square bridge diagram $\mathcal{D}$. Using such a diagram $\mathcal{D}$, the following two facts were first proved in [AO], and later made more explicit in [Pl]. Below versions are from the latter:

Lemma 2.0.3 *Given a Legendrian link L in $(\mathbb{R}^3, \xi_0)$, there exists a torus link $T_{p,q}$ (with p and q as above) transverse to ξ_0 such that its Seifert* *surface $F_{p,q}$ contains L, $d\alpha_0$ is an area form on $F_{p,q}$, and L does not separate $F_{p,q}$.* □

Proposition 2.0.4 *Given L and $F_{p,q}$ as above, there exist an open book decomposition of S^3 with page $F_{p,q}$ such that:*

1. *the induced contact structure ξ is isotopic to ξ_0;*
2. *the link L is contained in one of the page $F_{p,q}$, and does not separate it;*
3. *L is Legendrian with respect to ξ;*
4. *there exist an isotopy which fixes L and takes ξ to ξ_0, so the Legendrian type of the link is the same with respect to ξ and ξ_0;*
5. *the framing of L given by the page $F_{p,q}$ of the open book is the same as the contact framing.* □

Being a Seifert surface of a torus link, $F_{p,q}$ is of large genera. In Section 2.1, we will construct another open book $\mathcal{OB}$ supporting (S^3, ξ_{st}) such that its page F arises as a subsurface of $F_{p,q}$ (with considerably less genera), and given Legendrian link L sits on F as how it sits on the page $F_{p,q}$ of the construction used in [AO] and [Pl]. The page F of the open book $\mathcal{OB}$ will arise as the ribbon of the 1-skeleton of an appropriate contact cell decomposition for (S^3, ξ_{st}). As in [Pl], our construction will keep the given link L Legendrian with respect to the standard contact structure ξ_{st}. The following theorem summarize our algorithm:

Theorem 2.0.5 *Given L and $F_{p,q}$ as above, there exists a contact cell decomposition Δ of (S^3, ξ_{st}) such that*

1. *L is contained in the Legendrian 1-skeleton G of Δ.*
2. *The ribbon F of the 1-skeleton G is a subsurface of $F_{p,q}$ (p and q as above).*
3. *The framing of L coming from F is equal to its contact framing $tb(L)$.*
4. *If $p > 3$ and $q > 3$, then the genus $g(F)$ of F is strictly less than the genus $g(F_{p,q})$ of $F_{p,q}$.*

As an immediate consequence (see Corollary 2.1.1), we get an explicit description of an open book supporting (S^3, ξ) whose page F contains L with the correct framing. Therefore, if $(M^{\pm}, \xi^{\pm})$ is given by contact (± 1)-surgery on L (such a surgery diagram exists for any closed contact 3-manifold by Theorem 1.2.1), we get an open book supporting $\xi^{\pm}$ with page F by Theorem 1.5.1. Hence, $g(F)$ improves the upper bound for $sg(\xi)$ as $g(F) < g(F_{p,q})$ (for $p > 3$, $q > 3$). It will be clear from our examples in Section 2.2 that this is indeed a good improvement.

The following lemma will be used in the next section.

Lemma 2.0.6 *Let Δ be a contact cell decomposition of a closed contact 3-manifold (M, ξ) with the $1-$skeleton G. Let U be a 3-cell in Δ. Consider two Legendrian arcs $I \subset \partial U$ and $J \subset U$ such that*

1. $I \subset G$,

2. $J \cap \partial U = \partial J = \partial I$,

3. $C = I \cup_\partial J$ *is a Legendrian unknot with* $tb(C) = -1$.

Set $G' = G \cup J$. *Then there exists another contact cell decomposition* Δ' *of* (M, ξ) *such that* G' *is the 1-skeleton of* Δ'

Proof: The interior of the 3−cell U is contactomorphic to $(\mathbb{R}^3, \xi_0)$. Therefore, there exists an embedded disk D in U such that $\partial D = C$ and $int(D) \subset int(U)$ as depicted in Figure 2.2(a). We have $tw(\partial D, D) = -1$ since $tb(C) = -1$. As we are working in $(\mathbb{R}^3, \xi_0)$, there exist two C^∞-small perturbations of D fixing $\partial D = C$ such that perturbed disks intersect each other only along their common boundary C. In other words, we can find two isotopies $H_1, H_2 : [0,1] \times D \longrightarrow U$ such that for each $i = 1, 2$ we have

1. $H_i(t, .)$ fixes $\partial D = C$ pointwise for all $t \in [0, 1]$,
2. $H_i(0, D) = Id_D$ where Id_D is the identity map on D,
3. $H_i(1, D) = D_i$ where each D_i is an embedded disk in U with $int(D_i) \subset int(U)$,
4. $D \cap D_1 \cap D_2 = C$ (see Figure 2.2(b)).

Note that $tw(\partial D_i, D_i) = tw(C, D_i) = -1$ for $i = 1, 2$. This holds because each D_i is a small perturbation of D, so the number of counterclockwise twists of ξ (along K) relative to Fr_{D_i} is equal to the one relative to Fr_D.

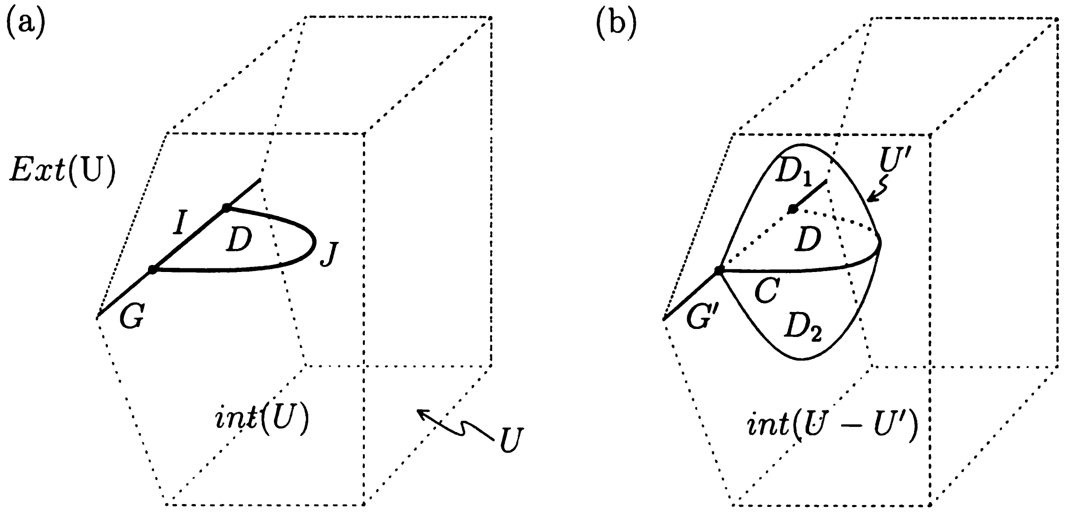

Figure 2.2. Constructing a new contact cell decomposition.

Next, we introduce $G' = G \cup J$ as the 1-skeleton of the new contact cell decomposition Δ'. In $M - int(U)$, we define the 2- and 3- skeletons of Δ' to be those of Δ . However, we change the cell structure of $int(U)$ as follows: We add 2-cells D_1, D_2 to the 2-skeleton of Δ' (note that they both satisfy the twisting condition in Definition 1.6.1). Consider the 2-sphere $S = D_1 \dot{\cup} D_2$ where the union is taken along the common boundary C. Let U' be the 3-ball with $\partial U' = S$. Note that $\xi|_{U'}$ is tight as $U' \subset U$ and $\xi|U$ is tight. We add U' and $U - U'$ to the 3-skeleton of Δ' (note that $U - U'$ can be considered as a 3-cell because observe that $int(U - U')$ is homeomorphic to the interior of a 3-ball as in Figure 2.2(b)). Hence, we established another contact cell decomposition of (M, ξ) whose 1-skeleton is $G' = G \cup J$. (Equivalently, by Theorem 1.3.3, we are taking the connect sum of (M, ξ) with (S^3, ξ_{st}) along U'.) $\square$

2.1 The algorithm (The proof of Theorem 2.0.5)

Proof: By translating L in $(\mathbb{R}^3, \xi_0)$ if necessary (without changing its contact type), we can assume that the front projection of L onto the yz-plane lying in the second quadrant $\{\ (y,z) \mid y < 0,\ z > 0\}$. After an appropriate Legendrian isotopy, we can assume that L consists of the line segments contained in the lines

$$k_i = \{x = 1, z = -y + a_i\},\ i = 1, \ldots, p,$$
$$l_j = \{x = -1, z = y + b_j\},\ j = 1, \ldots, q$$

for some $a_1 < a_2 < \cdots < a_p$, $0 < b_1 < b_2 < \cdots < b_q$, and also the line segments (parallel to the x-axis) joining certain k_i's to certain l_j's. In this representation, L seems to have corners. However, any corner of L can be made smooth by a Legendrian isotopy changing only a very small neighborhood of that corner.

Let $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the projection onto the yz-plane. Then we obtain the square bridge diagram $D = \pi(L)$ of L such that D consists of the line segments

$$\begin{aligned} h_i \subset \pi(k_i) &= \{x = 0, z = -y + a_i\}, \quad i = 1, \ldots, p, \\ v_j \subset \pi(l_j) &= \{x = 0, z = y + b_j\}, \quad j = 1, \ldots, q. \end{aligned}$$

Notice that D bounds a polygonal region P in the second quadrant of the yz-plane, and divides it into finitely many subregions $P_1, \ldots, P_m$ (see Figure 2.3-(a)).

Throughout the proof, we will assume that the link L is not split (that is, the region P has only one connected component). Such a restriction on L will not affect the generality of our construction (see Remark 2.1.2).

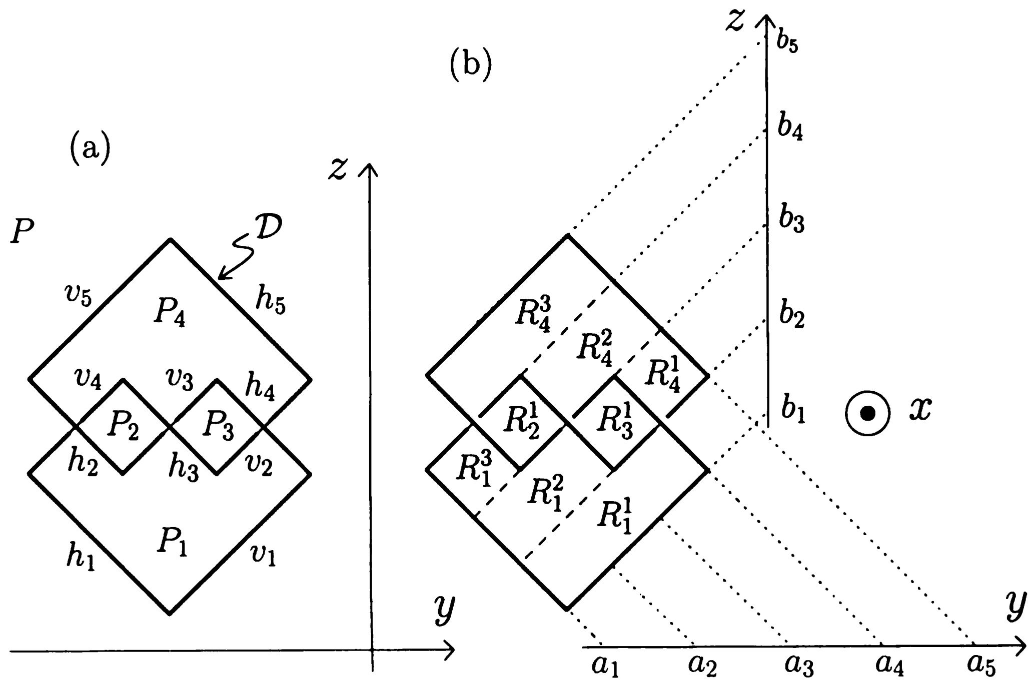

Figure 2.3. The region P for the right trefoil knot and its division.

Now we decompose P into finite number of ordered rectangular subregions as follows: The collection $\{\pi(l_j) \mid j = 1, \ldots, q\}$ cuts each P_k into finitely many rectangular regions $R_k^1, \ldots, R_k^{m_k}$. Consider the set $\mathfrak{P}$ of all such rectangles in P. That is, we define

$$\mathfrak{P} \doteq \{\, R_k^l \mid k = 1, \ldots, m, \quad l = 1, \ldots, m_k \}.$$

Clearly $\mathfrak{P}$ decomposes P into rectangular regions (see Figure 2.3-(b)). The boundary of an arbitrary element R_k^l in $\mathfrak{P}$ consists of four edges: Two of them are the subsets of the lines $\pi(l_{j(k,l)})$, $\pi(l_{j(k,l)+1})$, and the other two are the subsets of the line segments $h_{i1(k,l)}$, $h_{i2(k,l)}$ where $1 \leq i_1(k,l) < i_2(k,l) \leq p$ and $1 \leq j(k,l) < j(k,l) + 1 \leq q$ (see Figure 2.4).

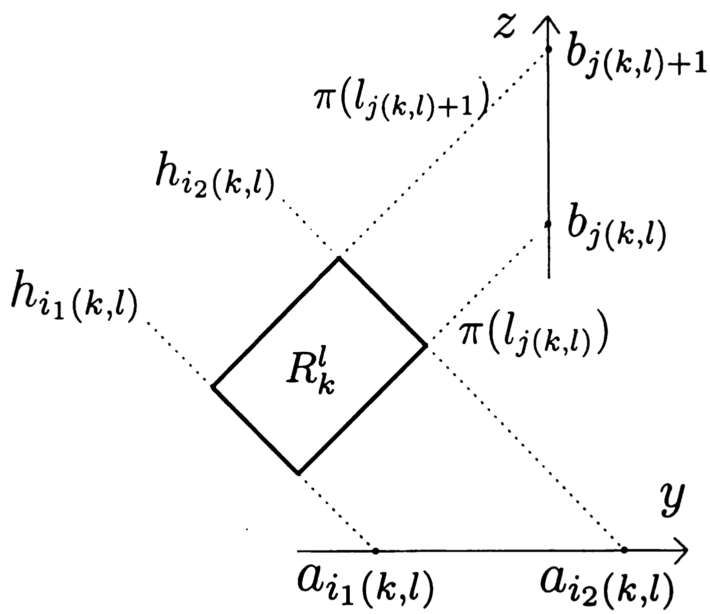

Figure 2.4. An arbitrary element R_k^l in $\mathfrak{P}$.

Since the region P has one connected component, the following holds for the set $\mathfrak{P}$:

($\star$) Any element of $\mathfrak{P}$ has at least one common vertex with another element of $\mathfrak{P}$.

By $(\star)$, we can rename the elements of $\mathfrak{P}$ by putting some order on them so that any element of $\mathfrak{P}$ has at least one vertex in common with the union of all rectangles coming before itself with respect to the chosen order. More precisely, we can write

$$\mathfrak{P} = \{ R_k \mid k = 1, \dots, N\}$$

(N is the total number of rectangles in $\mathfrak{P}$) such that each R_k has at least one vertex in common with the union $R_1 \cup \cdots \cup R_{k-1}$.

Equivalently, we can construct the polygonal region P by introducing the building rectangles (R_k's) one by one in the order given by the index set $\{1, 2, \dots, N\}$. In particular, this eliminates one of the indexes, i.e., we can use R_k's instead of R^l_k's. In Figure 2.5, how we build P is depicted for the right trefoil knot (compare it with the previous picture given for P in Figure 2.3-(b)).

Using the representation $P = R_1 \cup R_2 \cup \cdots \cup R_N$, we will construct the contact cell decomposition (CCD) Δ. Consider the following infinite strips which are parallel to the x-axis (they can be considered as the unions of "small" contact planes along k_i's and l_j's):

$$\begin{aligned} S_i^+ &= \{1-\epsilon \leq x \leq 1+\epsilon, \ z = y + a_i\}, \ i = 1, \dots, p, \\ S_j^- &= \{-1-\epsilon \leq x \leq -1+\epsilon, \ z = -y + b_j\}, \ j = 1, \dots, q. \end{aligned}$$

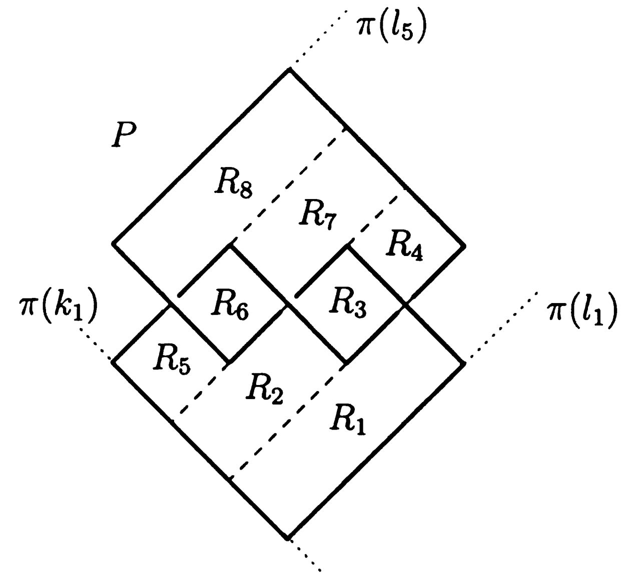

Figure 2.5. The region P for the right trefoil knot.

Note that $\pi(S_i^+) = \pi(k_i)$ and $\pi(S_j^-) = \pi(l_j)$. Let $R_k \subset P$ be given. Then we can write

$$\partial R_k = C_k^1 \cup C_k^2 \cup C_k^3 \cup C_k^4$$

where $C_k^1 \subset \pi(k_{i1})$, $C_k^2 \subset \pi(l_j)$, $C_k^3 \subset \pi(k_{i2})$, $C_k^4 \subset \pi(l_{j+1})$ for some $1 \leq i_1 < i_2 \leq p$ and $1 \leq j \leq q$. Lift $C_k^1, C_k^2, C_k^3, C_k^4$ (along the x-axis) so that the resulting lifts (which will be denoted by the same letters) are disjoint Legendrian arcs contained in $k_{i1}, l_j, k_{i2}, l_{j+1}$ and sitting on the corresponding strips $S_{i1}^+, S_j^-, S_{i2}^+, S_{j+1}^-$. For $l = 1, 2, 3, 4$, consider Legendrian linear arcs I_k^l (parallel to the x-axis) running between the endpoints of C_k^l's as in Figure 2.6-(a)&(b). Along each I_k^l the contact planes make a 90° left-twist. Let B_k^l be the narrow band obtained by following the contact planes along I_k^l. Then define F_k to be the surface constructed by taking the union of the compact subsets of the above strips (containing corresponding C_k^l's) with the bands B_k^l's (see Figure 2.6-(b)). C_k^l's and I_k^l's together build a Legendrian unknot γ_k in $(\mathbb{R}^3, \xi_0)$, i.e., we set

$$\gamma_k = C_k^1 \cup I_k^1 \cup C_k^2 \cup I_k^2 \cup C_k^3 \cup I_k^3 \cup C_k^4 \cup I_k^4.$$

Note that $\pi(\gamma_k) = \partial R_k$, γ_k sits on the surface F_k, and F_k deformation retracts onto γ_k. Indeed, by taking all strips and bands in the construction small enough, we may assume that contact planes are tangent to the surface F_k only along the core circle γ_k. Thus, F_k is the ribbon of γ_k. Observe that, topologically, F_k is a positive (left-handed) Hopf band.

Let $f_k : R_k \longrightarrow \mathbb{R}^3$ be a function modelled by $(a, b) \mapsto c = a^2 - b^2$ (for an appropriate choice of coordinates). The image $f_k(R_k)$ is, topologically, a disk, and a compact subset of a saddle surface. Deform $f_k(R_k)$ to another "saddle" disk D_k such that $\partial D_k = \gamma_k$ (see Figure 2.6-(c)). We observe here that $tw(\gamma_k, D_k) = -1$ because along γ_k, contact planes rotate 90° in the counter-clockwise direction exactly four times which makes one full left-twist (enough to count the twists of the ribbon F_k since F_k rotates with the contact planes along γ_k !).

We repeat the above process for each rectangle R_k in P and get the set

$$\mathfrak{D} = \{\ D_k \mid D_k \approx f_k(R_k),\ k = 1, \ldots, N\}$$

consisting of the saddle disks. Note that by the construction of $\mathfrak{D}$, we have the property:

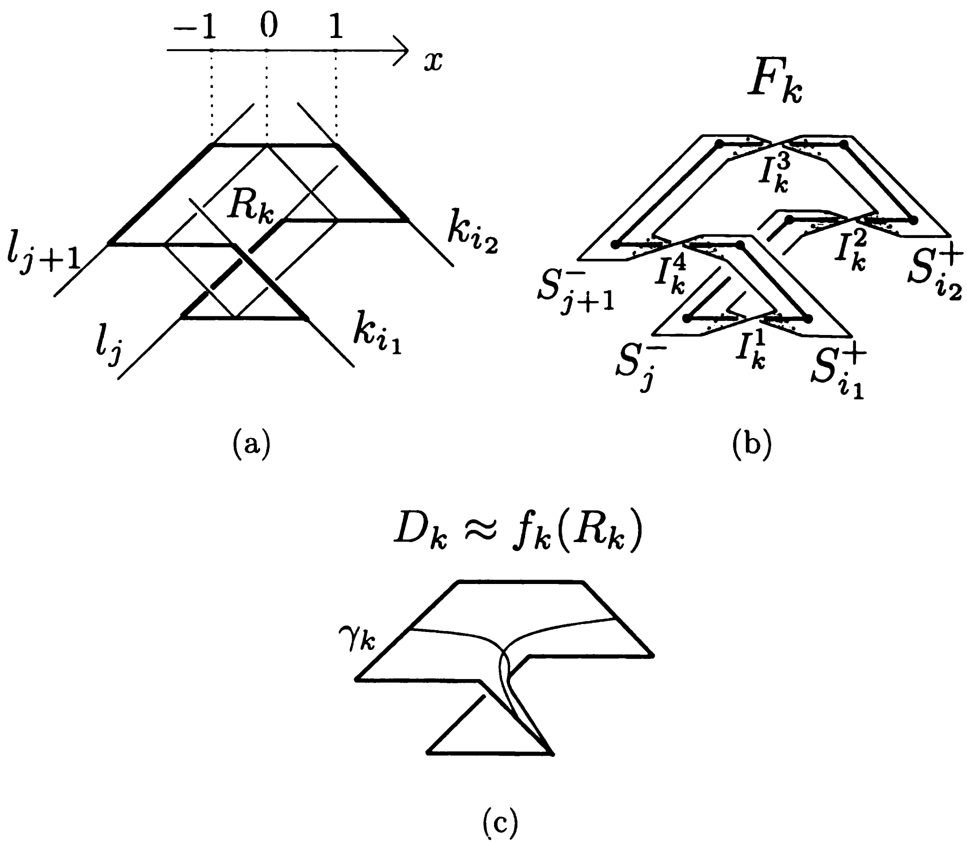

Figure 2.6. (a) The Legendrian unknot γ_k, (b) the ribbon F_k, (c) the disk D_k (shaded bands in (b) are the bands B_k^l's).

($*$) *If any two elements of* $\mathfrak{D}$ *intersect each other, then they must intersect along a contractible subset (a contractible union of linear arcs) of their boundaries.*

For instance, if the corresponding two rectangles (for two intersecting disks in $\mathfrak{D}$) have only one common vertex, then those disks intersect each other along the (contractible) line segment parallel to the x-axis which is projected (by the map π) onto that vertex.

For each k, let D'_k be a disk constructed by perturbing D_k slightly by an isotopy fixing only the boundary of D_k. Therefore, we have

$$(**)\ \partial D_k = \gamma_k = \partial D'_k\ , \qquad int(D_k) \cap int(D'_k) = \emptyset\ ,\ and$$
$$tw(\gamma_k, D'_k) = -1 = tw(\gamma_k, D_k).$$

In the following, we will define a sequence $\{\ \Delta_k \mid k = 1, \ldots, N\ \}$ of CCD's for (S^3, ξ_{st}). Δ_k^1, Δ_k^2, and Δ_k^3 will denote the 1-skeleton, 2-skeleton, and 3-skeleton of Δ_k, respectively. First, take $\Delta_1^1 = \gamma_1$, and $\Delta_1^2 = D_1 \cup_{\gamma 1} D'_1$. By $(**)$, Δ_1 satisfies the conditions (1) and (2) of Definition 1.6.1. By the construction, any pair of disks D_k, D'_k (together) bounds a Darboux ball (tight 3-cell) U_k in the tight manifold $(\mathbb{R}^3, \xi_0)$. Therefore, if we take $\Delta_1^3 = U_1 \cup_\partial (S^3 - U_1)$, we also achieve the condition (3) in Definition 1.6.1 (the boundary union " $\cup_\partial$" is taken along $\partial U_1 = S^2 = \partial(S^3 - U_1))$. Thus, Δ_1 is a CCD for (S^3, ξ_{st}).

Inductively, we define Δ_k from Δ_{k-1} by setting

$$\begin{aligned}
\Delta_k^1 &= \Delta_{k-1}^1 \cup \gamma_k = \gamma_1 \cup \cdots \cup \gamma_{k-1} \cup \gamma_k, \\
\Delta_k^2 &= \Delta_{k-1}^2 \cup D_k \cup_{\gamma k} D'_k = D_1 \cup_{\gamma 1} D'_1 \cup \cdots \cup D_{k-1} \cup_{\gamma(k-1)} D'_{k-1} \cup D_k \cup_{\gamma k} D'_k, \\
\Delta_k^3 &= U_1 \cup \cdots \cup U_{k-1} \cup U_k \cup_\partial (S^3 - U_1 \cup \cdots \cup U_{k-1} \cup U_k)
\end{aligned}$$

Actually, at each step of the induction, we are applying Lemma 2.0.6 to Δ_{k-1} to get Δ_k. We should make several remarks: First, by the construction of γ_k's, the set

$$(\gamma_1 \cup \cdots \cup \gamma_{k-1}) \cap \gamma_k$$

is a contractible union of finitely many arcs. Therefore, the union $\Delta^1_{k-1} \cup \gamma_k$ should be understood to be a set-theoretical union (not a topological gluing!) which means that we are attaching only the (connected) part $(\gamma_k \setminus \Delta^1_{k-1})$ of γ_k to construct the new 1-skeleton Δ^1_k. In terms of the language of Lemma 2.0.6, we are setting $I = \Delta^1_{k-1} \setminus \gamma_k$ and $J = \gamma_k \setminus \Delta^1_{k-1}$. Secondly, we have to show that $\Delta^2_k = \Delta^2_{k-1} \cup D_k \cup_{\gamma_k} D'_k$ can be realized as the 2-skeleton of a CCD: Inductively, we can achieve the twisting condition on 2-cells by using $(**)$. The fact that any two intersecting 2-cells in Δ^2_k intersect each other along some subset of the 1-skeleton Δ^1_k is guaranteed by the property $(*)$ if they have different index numbers, and guaranteed by $(**)$ if they are of the same index. Thirdly, we have to guarantee that 3-cells meet correctly: It is clear that $U_1, \dots, U_k$ meet with each other along subsets of the 1-skeleton $\Delta^1_k (\subset \Delta^2_k)$. Observe that $\partial(U_1 \cup \cdots \cup U_k) = S^2$ for any $k = 1, \dots, N$ by $(*)$ and $(**)$. Therefore, we can always consider the complementary Darboux ball $S^3 - U_1 \cup \cdots \cup U_{k-1} \cup U_k$, and glue it to $U_1 \cup \cdots \cup U_k$ along their common boundary 2-sphere. Hence, we have seen that Δ_k is a CCD for (S^3, ξ_{st}) with Legendrian 1-skeleton $\Delta^1_k = \gamma_1 \cup \cdots \cup \gamma_k$.

To understand the ribbon, say Σ_k, of Δ^1_k, observe that when we glue the part $\gamma_k \setminus \Delta^1_{k-1}$ of γ_k to Δ^1_{k-1}, actually we are attaching a 1-handle (whose core interval is $(\gamma_k \setminus \Delta^1_{k-1}) \setminus \Sigma_{k-1}$) to the old ribbon Σ_{k-1} (indeed, this corresponds to a positive stabilization). We choose the 1-handle in such a way that it also rotates with the contact planes. This is equivalent to extending Σ_{k-1} to a new surface by attaching the missing part (the part which retracts onto $(\gamma_k \setminus \Delta^1_{k-1}) \setminus \Sigma_{k-1}$) of F_k given in Figure 2.6-(c). The new surface is the ribbon Σ_k of the new 1-skeleton Δ_k^1.

By taking $k = N$, we get a CCD Δ_N of (S^3, ξ_{st}). By the construction, γ_k's are only piecewise smooth. We need a smooth embedding of L into the 1-skeleton Δ_N^1 (the union of all γ_k's). Away from some small neighborhood of the common corners of Δ_N^1 and L (recall that L had corners before the Legendrian isotopies), L is smoothly embedded in Δ_N^1. Around any common corner, we slightly perturb Δ_N^1 using the isotopy used for smoothing that corner of L. This guaranties the smooth Legendrian embedding of L into the Legendrian graph $\Delta_N^1 = \cup_{k=1}^N \gamma_k$. Similarly, any other corner in Δ_N^1 (which is not in L) can be made smooth using an appropriate Legendrian isotopy.

As L is contained in the 1-skeleton Δ_N^1, L sits (as a smooth Legendrian link) on the ribbon Σ_N. Note that during the process we do not change the contact type of L, so the contact (Thurston-Bennequin) framing of L is still the same as what it was at the beginning. On the other hand, consider tubular neighborhood $N(L)$ of L in Σ_N. Being a subsurface of the ribbon Σ_N, $N(L)$ is the ribbon of L. By definition, the contact framing of any component of L is the one coming from the ribbon of that component. Therefore, the contact framing and the $N(L)$-framing of L are the same. Since $N(L) \subset \Sigma_N$, the framing which L gets from the ribbon Σ_N is the same as the contact framing of L. Finally, we observe that Σ_N is a subsurface of the Seifert surface $F_{p,q}$ of the torus link (or knot) $T_{p,q}$. To see this, note that P is contained in the rectangular region, say $P_{p,q}$, enclosed by the lines $\pi(k_1), \pi(k_p), \pi(l_1), \pi(l_q)$. Divide $P_{p,q}$ into the rectangular subregions using the lines $\pi(k_i), \pi(l_j)$, $i = 1, \dots, p$, $j = 1, \dots, q$. Note that there are exactly pq rectangles in the division. If we repeat the above process using this division of $P_{p,q}$, we get another CCD for (S^3, ξ_{st}) with the ribbon $F_{p,q}$. Clearly, $F_{p,q}$ contains our ribbon Σ_N as a subsurface (indeed, there are extra bands and parts of strips in $F_{p,q}$ which are not in Σ_N).

Thus, (1), (2) and (3) of the theorem are proved once we set $\Delta = \Delta_N$, (and so $G = \Delta_N^1$, $F = \Sigma_N$). To prove (4), recall that we are assuming $p > 3$, $q > 3$. Then consider

$$\kappa \doteq \text{total number of intersection points of all } \pi(l_j)\text{'s with all } h_i\text{'s.}$$

That is, we define $\kappa \doteq |\{\pi(l_j) \mid j = 1, \dots, q\} \cap \{h_i \mid i = 1, \dots, p\}\,|$. Notice that κ is the number of bands used in the construction of the ribbon F, and also that if $\mathcal{D}$ (so P) is not a single rectangle (equivalently $p > 2$, $q > 2$), then $\kappa < pq$. Since there are $p + q$ disks in F, we compute the Euler characteristic and genus of F as

$$\chi(F) = p + q - \kappa = 2 - 2g(F) - |\partial F| \Rightarrow g(F) = \frac{2 - p - q}{2} + \frac{\kappa}{2} - \frac{|\partial F|}{2}.$$

Similarly, there are $p + q$ disks and pq bands in $F_{p,q}$, so we get

$$\chi(F_{p,q}) = p + q - pq = 2 - 2g(F_{p,q}) - |\partial F_{p,q}| \Rightarrow g(F_{p,q}) = \frac{2 - p - q}{2} + \frac{pq}{2} - \frac{|\partial F_{p,q}|}{2}.$$

Observe that $|\partial F_{p,q}|$ divides the greatest common divisor $gcd(p, q)$ of p and q, so

$$|\partial F_{p,q}| \leq gcd(p, q) \leq p \Longrightarrow g(F_{p,q}) \geq \frac{2 - p - q}{2} + \frac{pq}{2} - \frac{p}{2}.$$

Therefore, to conclude $g(F) < g(F_{p,q})$, it suffices to show that

$$pq - \kappa > p - |\partial F|.$$

To show the latter, we will show $pq - \kappa - p \geq 0$ (this will be enough since $|\partial F| \neq 0$). Observe that $pq - \kappa$ is the number of bands (along x-axis) in $F_{p,q}$ which we omit to get the ribbon F. Therefore, we need to see that at least p bands are omitted in the construction of F: The set of all bands (along x-axis) in $F_{p,q}$ corresponds to the set

$$\{\pi(l_j) \mid j = 1, \ldots, q\} \cap \{\pi(k_i) \mid i = 1, \ldots, p\}.$$

Notice that while constructing F we omit at least 2 bands corresponding to the intersections of the lines $\pi(k_1), \pi(k_p)$ with the family $\{\pi(l_j) \mid j = 1, \ldots, q\}$ (in some cases, one of these bands might correspond to the intersection of the lines $\pi(k_2)$ or $\pi(k_{p-1})$ with $\pi(l_1)$ or $\pi(l_q)$, but the following argument still works because in such a case we can omit at least 2 bands corresponding to two points on $\pi(k_2)$ or $\pi(k_{p-1})$). For the remaining $p - 2$ line segments $h_2, \ldots, h_{p-1}$, there are two cases: Either each h_i, for $i = 2, \ldots, p-1$ has at least one endpoint contained on a line other than $\pi(l_1)$ or $\pi(l_q)$, or there exists a unique $h_i, 1 < i < p$, such that its endpoints are on $\pi(l_1)$ and $\pi(l_q)$ (such an h_i must be unique since no two v_j's are collinear !). If the first holds, then that endpoint corresponds to the intersection of h_i with $\pi(l_j)$ for some $j \neq 1, q$. Then the band corresponding to either $\pi(k_i) \cap \pi(l_{j-1})$ or $\pi(k_i) \cap \pi(l_{j+1})$ is omitted in the construction of F (recall how we divide P into rectangular regions). If the second holds, then there is at least one line segment $h_{i'}$, which belongs to the same component of L containing h_i, such that we omit at least 2 points on $\pi(k_{i'})$ (this is true again since no two v_j's are collinear). Hence, in any case, we omit at least p bands from $F_{p,q}$ to get F. This completes the proof of Theorem 2.0.5. □

Corollary 2.1.1 *Given L and $F_{p,q}$ as in Theorem 2.0.5, there exists an open book decomposition $\mathcal{OB}$ of (S^3, ξ_{st}) such that*

1. *L lies (as a Legendrian link) on a page F of $\mathcal{OB}$,*
2. *The page F is a subsurface of $F_{p,q}$*
3. *The page framing of L coming from F is equal to its contact framing,*
4. *If $p > 3$ and $q > 3$, then $g(F)$ is strictly less than $g(F_{p,q})$, and*
5. *The monodromy h of $\mathcal{OB}$ is given by $h = t_{\gamma 1} \circ \cdots \circ t_{\gamma N}$ where γ_k is the Legendrian unknot constructed in the proof of Theorem 2.0.5, and $t_{\gamma k}$ denotes the positive (right-handed) Dehn twist along γ_k.*

Proof: The proofs of (1), (2), (3), and (4) immediately follow from Theorem 2.0.5 and Lemma 1.6.3. To prove (5), observe that by adding the missing part of each γ_k to the previous 1-skeleton, and by extending the previous ribbon by attaching the ribbon of the missing part of γ_k (which is topologically a 1-handle), we actually positively stabilize the old ribbon with the positive Hopf band $(H^+, t_{\gamma k})$. Therefore, (5) follows. □

With a little more care, sometimes we can decrease the number of 2-cells in the final 2-skeleton. Also the algorithm can be modified for split links:

Remark 2.1.2 *Under the notation used in the proof of Theorem 2.0.5, we have the following:*

1. *Suppose that the link L is split (so P has at least two connected components). Then we can modify the above algorithm so that Theorem 2.0.5 still holds.*

2. *Let T_j denote the row (or set) of rectangles (or elements) in P (or in $\mathfrak{P}$) with bottom edges lying on the fixed line $\pi(l_j)$. Consider two consecutive rows T_j, T_{j+1} lying between the lines $\pi(l_j), \pi(l_{j+1})$, and $\pi(l_{j+2})$. Let $R \in T_j$ and $R' \subset T_{j+1}$ be two rectangles in P with boundaries given as*

$$\partial R = C_1 \cup C_2 \cup C_3 \cup C_4, \quad \partial R' = C_1' \cup C_2' \cup C_3' \cup C_4'$$

   Suppose that R and R' have one common boundary component lying on $\pi(l_{j+1})$, and two of the other components lie on the same lines $\pi(k_{i1}), \pi(k_{i2})$ as in Figure 2.7. Let $\gamma, \gamma' \subset \Delta_N^1$ and $D, D' \subset \Delta_N$ be the corresponding Legendrian unknots and 2-cells of the CCD Δ_N coming from R, R'. That is,

$$\partial D = \gamma, \ \partial D' = \gamma', \text{ and } \pi(D) = R, \ \pi(D') = R'$$

Suppose also that $L \cap \gamma \cap \gamma' = \emptyset$. Then in the construction of Δ_N, we can replace $R, R' \subset P$ with a single rectangle $R'' = R \cup R'$. Equivalently, we can take out $\gamma \cap \gamma'$ from Δ_N^1, and replace D, D' by a single saddle disk D'' with $\partial D'' = (\gamma \cup \gamma') \setminus (\gamma \cap \gamma')$.

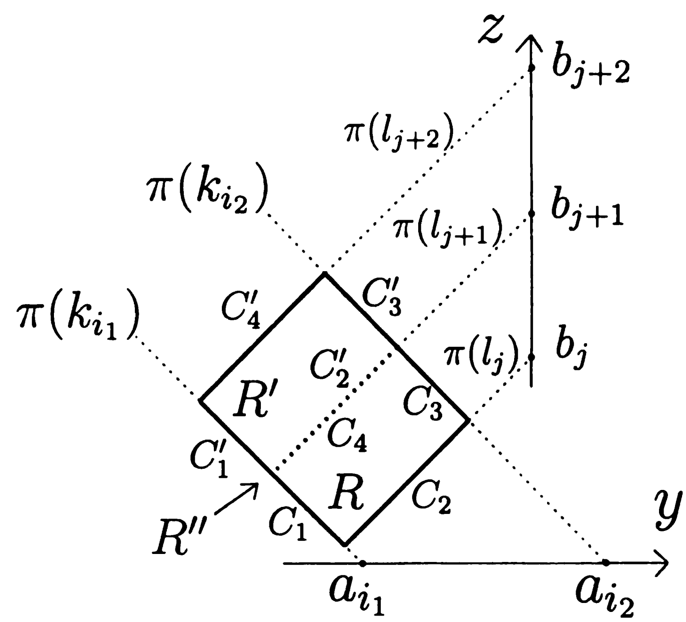

Figure 2.7. Replacing R, R' with their union R''.

Proof: To prove each statement, we need to show that CCD structure and all the conclusions in Theorem 2.0.5 are preserved after changing Δ_N the way described in the statement.

To prove (1), let $P^{(1)}, \ldots, P^{(m)}$ be the separate components of P. After putting the corresponding separate components of L into appropriate positions (without changing their contact type) in $(\mathbb{R}^3, \xi_0)$, we may assume that the projection

$$P = P^{(1)} \cup \cdots \cup P^{(m)}$$

of L onto the second quadrant of the yz-plane is given similar as the one which we illustrated in Figure 2.8.

In such a projection, we require two important properties:

1. $P^{(1)}, \ldots, P^{(m)}$ are located from left to right in the given order in the region bounded by the lines $\pi(k_1)$, $\pi(l_1)$, and $\pi(l_q)$.

2. Each of $P^{(1)}, \ldots, P^{(m)}$ has at least one edge on the line $\pi(l_1)$.

If the components $P^{(1)} \ldots P^{(m)}$ remain separate, then our construction in Theorem 2.0.5 cannot work (the complement of the union of 3-cells corresponding to the rectangles in P would not be a Darboux ball; it would be a genus m handle body). So we have to make sure that any component $P^{(l)}$ is connected to the some other via some bridge consisting of rectangles. We choose only one rectangle for each bridge as follows: Let A_l be the rectangle in T_1 (the row between $\pi(l_1)$ and $\pi(l_2)$) connecting $P^{(l)}$ to $P^{(l+1)}$ for $l = 1, \ldots, m-1$ (see Figure 2.8). Now, by adding 2- and 3-cells (corresponding to $A_1, \ldots, A_{m-1}$), we can extend the CCD Δ_N to get another CCD for (S^3, ξ_{st}). Therefore, we have modified our construction when L is split.

To prove (2), if we replace D'' in the way described above, then by the construction of Δ_N^3, we also replace two 3-cells with a single 3-cell whose boundary is the union of D'' and its isotopic copy. This alteration of Δ_N^3 does not change the fact that the boundary of the union of all

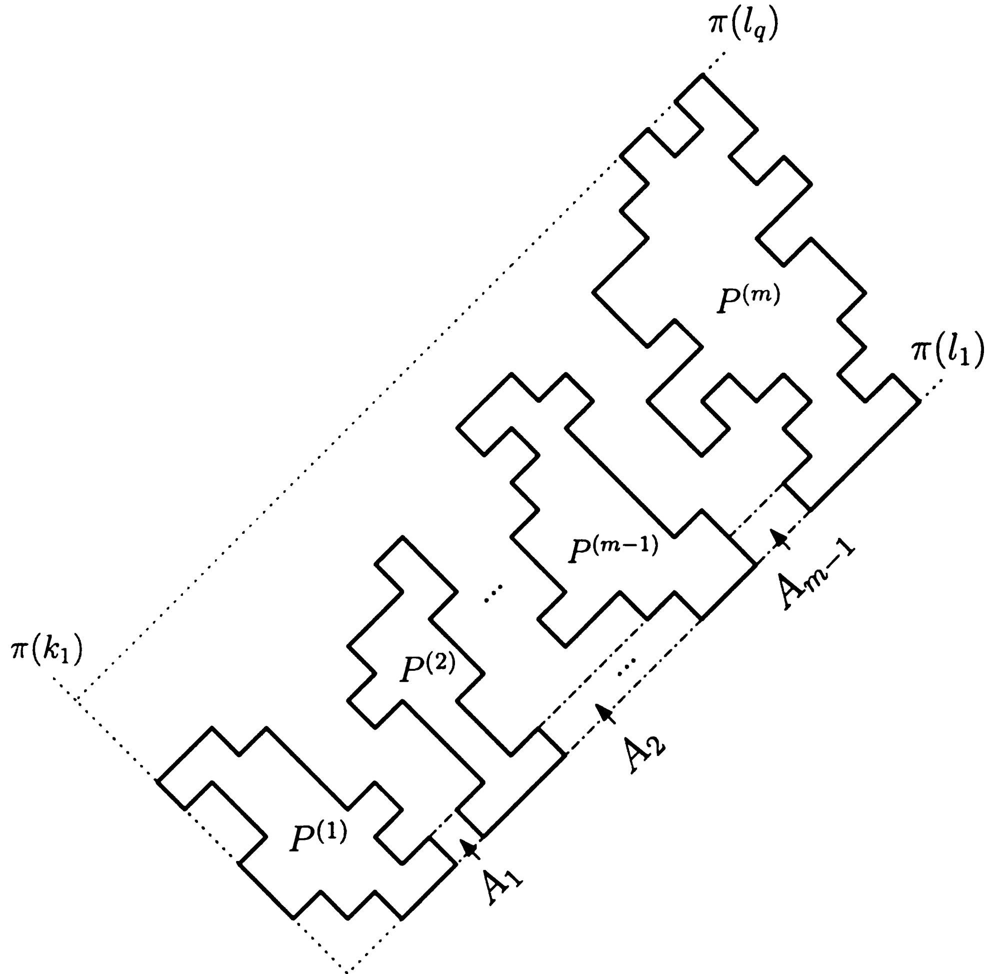

Figure 2.8. Modifying the algorithm for the case when L is split.

3-cells coming from all pairs of saddle disks is still homeomorphic to a 2-sphere S^2, Therefore, we can still complete this union to S^3 by gluing a complementary Darboux ball. Thus, we still have a CCD. Note that $\gamma \cap \gamma'$ is taken away from the 1-skeleton. However, since $L \cap \gamma \cap \gamma' = \emptyset$, the new 1-skeleton still contains L. Observe also that this process does not change the ribbon $N(L)$ of L. Hence, the same conclusions in Theorem 2.0.5 are satisfied by the new CCD. $\square$

2.2 Examples

Example I. As the first example, let us finish the one which we have already started in the previous section. Consider the Legendrian right trefoil knot L (Figure 2.1) and the corresponding region P given in Figure 2.5. Then we construct the 1-skeleton, the saddle disks, and the ribbon of the CCD Δ as in Figure 2.10.

In Figure 2.10-(a), we show how to construct the 1-skeleton $G = \Delta^1$ of Δ starting from a single Legendrian arc (labelled by the number " 0 "). We add Legendrian arcs labelled by the pairs of numbers "1, 1", ..., "8, 8" to the picture one by one (in this order). Each pair determines the endpoints of the corresponding arc. These arcs represent the cores of the 1-handles building the page F (the ribbon of G) of the corresponding open book $\mathcal{OB}$. Note that by attaching each 1-handle, we (positively) stabilize the previous ribbon by the positive Hopf band $(H_k^+, t_{\gamma k})$ where γ_k is the boundary of the saddle disk D_k as before. Therefore, the monodromy h of $\mathcal{OB}$ supporting (S^3, ξ_{st}) is given by

$$h = t_{\gamma 1} \circ \cdots \circ t_{\gamma 8}$$

where $t_{\gamma k} \in Aut(F, \partial F)$ denotes the positive (right-handed) Dehn twist along γ_k. To compute the genus g_F of F, observe that F is constructed by attaching eight 1-handles (bands) to a disk, and $|\partial F| = 3$ where $|\partial F|$ is the number of boundary components of F. Therefore,

$$\chi(F) = 1 - 8 = 2 - 2g_F - |\partial F| \Longrightarrow g_F = 3.$$

Now suppose that $(M_1^{\pm}, \xi_1^{\pm})$ is obtained by performing contact (± 1)-surgery on L. Clearly, the trefoil knot L sits as a Legendrian curve on F by our construction, so by Theorem 1.5.1, we get the open book (F, h_1) supporting ξ with monodromy

$$h_1 = t_{\gamma 1} \circ \cdots \circ t_{\gamma 8} \circ t_L^{\mp 1} \in Aut(F, \partial F).$$

Hence, we get an upper bound for the support genus invariant of ξ_1, namely,

$$sg(\xi_1) \leq 3 = g_F.$$

We note that the upper bound, which we can get for this particular case, from [AO] and [St] is 6 where the page of the open book is the Seifert surface $F_{5,5}$ of the $(5,5)$-torus link (see Figure 2.9).

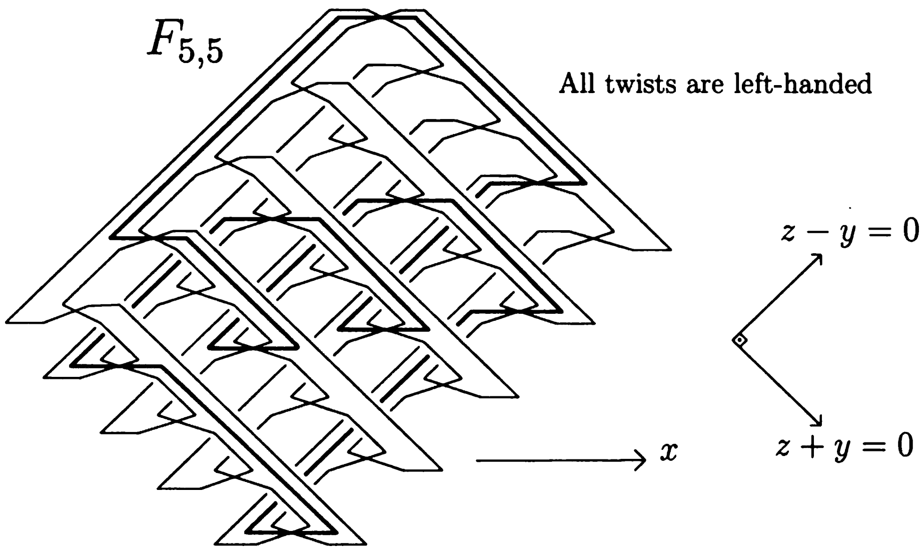

Figure 2.9. Legendrian right trefoil knot sitting on $F_{5,5}$.

Example II. Consider the Legendrian figure-eight knot L, and its square bridge position given in Figure 2.11-(a) and (b). We get the corresponding region P in Figure 2.11-(c). Using Remark 2.1.2 we replace R_5 and R_8 with a single saddle disk. So this changes the set $\mathfrak{P}$. Reindexing the rectangles in $\mathfrak{P}$, we get the decomposition in Figure 2.12 which will be used to construct the CCD Δ.

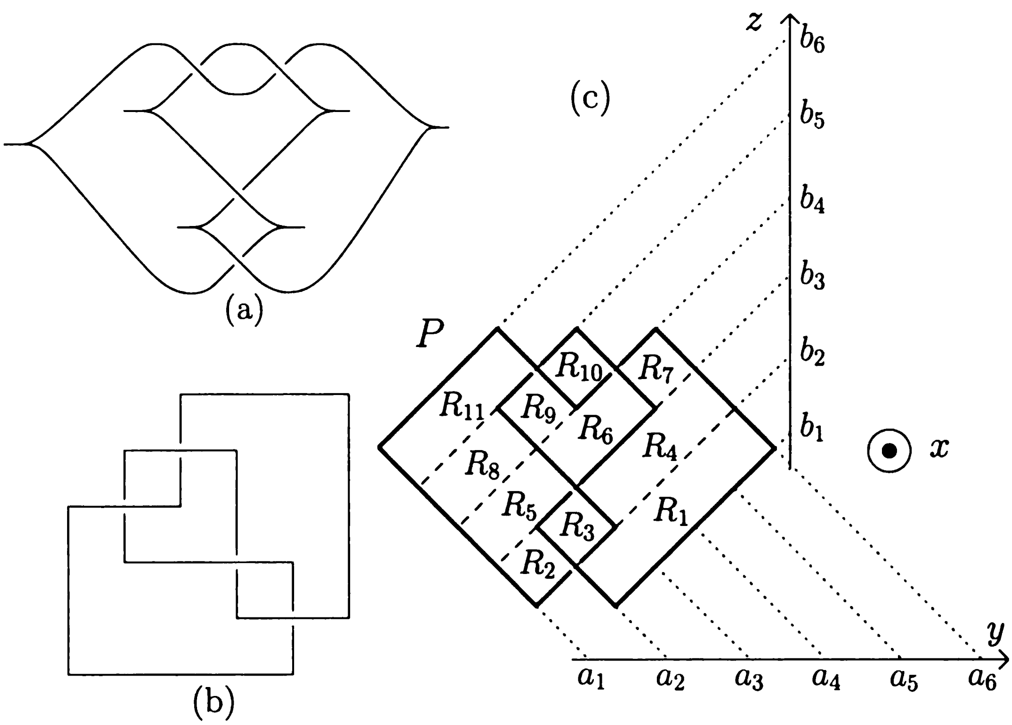

Figure 2.11. (a),(b) Legendrian figure-eight knot, (c) the region P.

In Figure 2.14-(a), similar to Example I, we construct the 1-skeleton $G = \Delta^1$ of Δ again by attaching Legendrian arcs (labelled by the pairs of numbers "1, 1", ..., "10, 10") to the initial arc (labelled by the number "0") in the

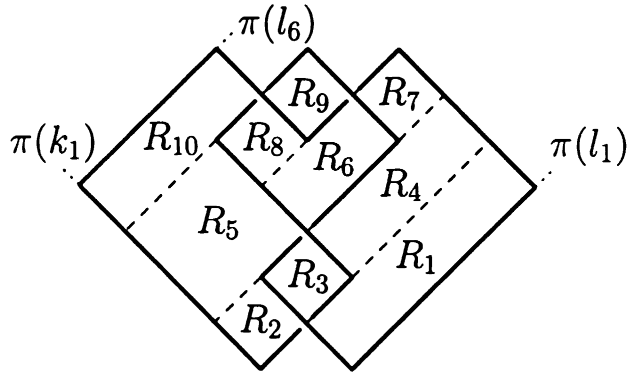

Figure 2.12. Modifying the region P.

given order. Again each pair determines the endpoints of the corresponding arc, and the cores of the 1-handles building the page F (of the corresponding open book $\mathcal{OB}$). Once again attaching each 1-handle is equivalent to (positively) stabilizing the previous ribbon by the positive Hopf band $(H_k^+, t_{\gamma k})$ for $k = 1, \dots, 10$. Therefore, the monodromy h of $\mathcal{OB}$ supporting (S^3, ξ_{st}) is given by

$$h = t_{\gamma 1} \circ \cdots \circ t_{\gamma 10}$$

To compute the genus g_F of F, observe that F is constructed by attaching ten 1-handles (bands) to a disk, and $|\partial F| = 5$. Therefore,

$$\chi(F) = 1 - 10 = 2 - 2g_F - |\partial F| \Longrightarrow g_F = 3.$$

Let $(M_2^{\pm}, \xi_2^{\pm})$ be a contact manifold obtained by performing contact (± 1)-surgery on the figure-8 knot L. Since L sits as a Legendrian curve on F by our construction, Theorem 1.5.1 gives an open book (F, h_2) supporting

ξ_2 with monodromy

$$h_2 = t_{\gamma 1} \circ \cdots \circ t_{\gamma 10} \circ t_L^{\mp 1} \in Aut(F, \partial F).$$

Therefore, we get the upper bound $sg(\xi_2) \leq 3 = g_F$. Once again we note that the smallest possible upper bound, which we can get for this particular case, using the method of [AO] and [St] is 10 where the page of the open book is the Seifert surface $F_{6,6}$ of the $(6,6)$-torus link (see Figure 2.13).

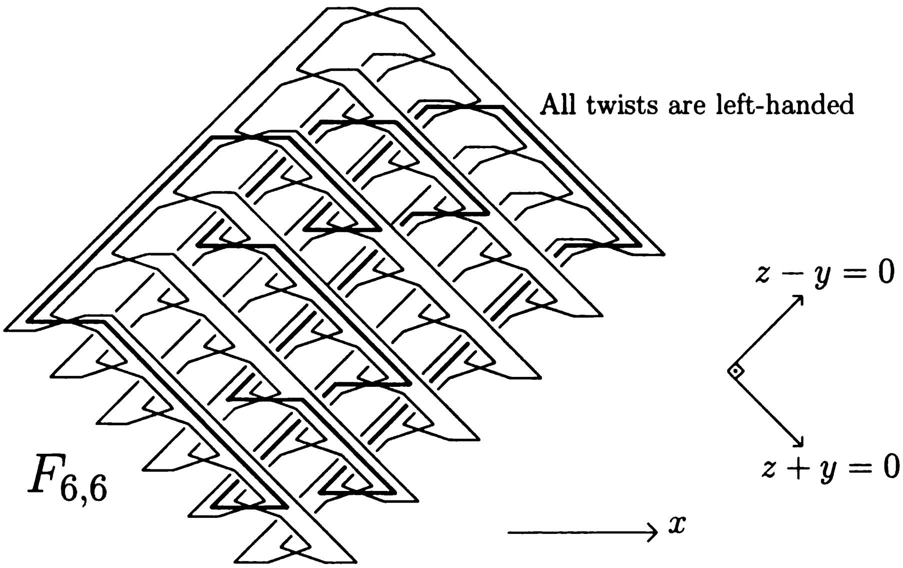

Figure 2.13. The figure-eight knot on $F_{6,6}$.

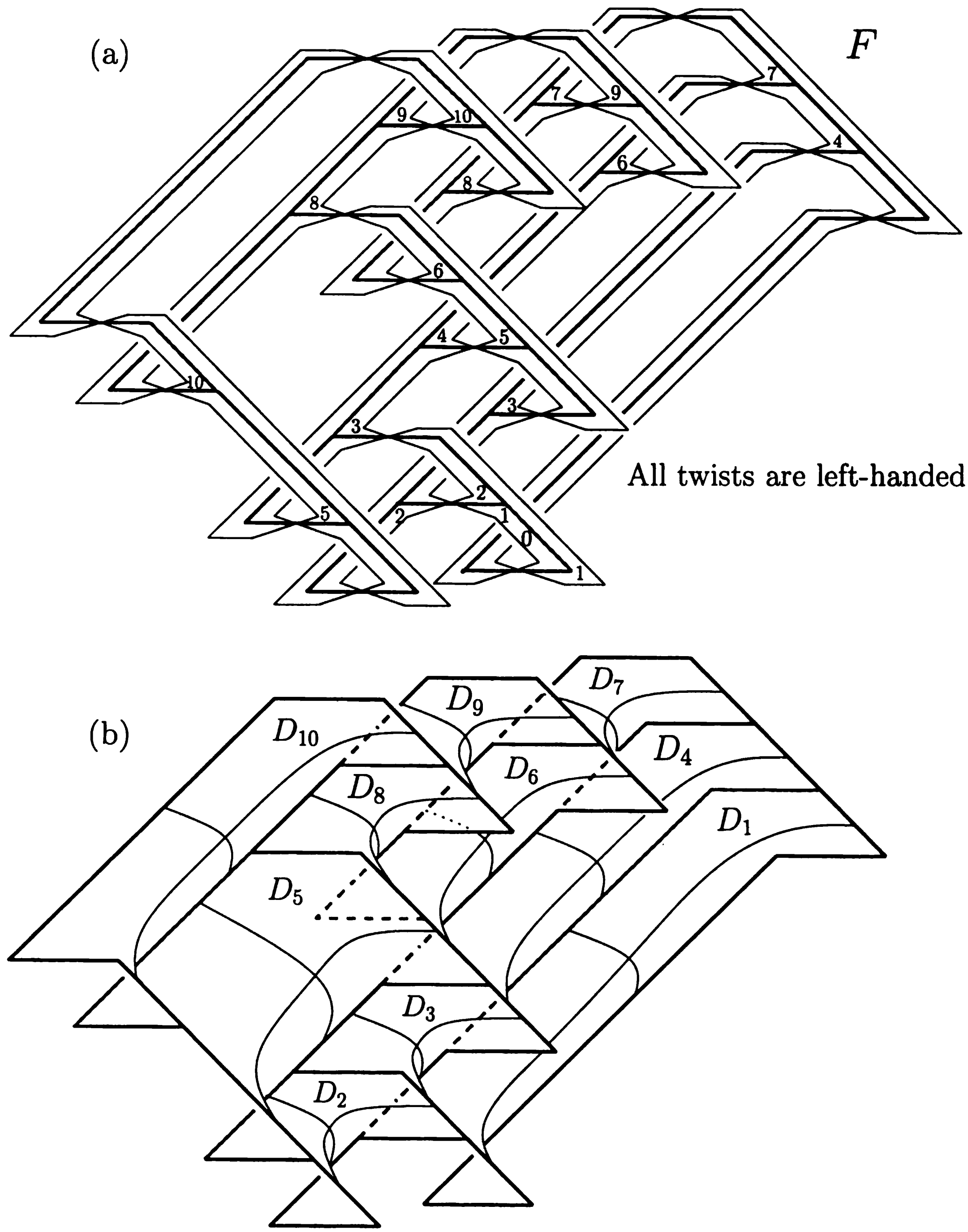

Figure 2.14. (a) The page F for the figure-eight knot, (b) construction of Δ and saddle disks.

Chapter 3

Planar contact structures with binding number three

In this chapter, we will obtain a complete list of contact manifolds corresponding to a fixed support genus and a fixed binding number. To get such complete list, we consider all possible monodromy maps. Throughout the chapter $L(m, n)$ stands for the lens space obtained by $-m/n$ rational surgery on an unknot. The first step in this direction is the following result given in [EO].

Theorem 3.0.1 ([EO]) *Suppose ξ is a contact structure on a 3-manifold M that is supported by a planar open book (i.e., $sg(\xi) = 0$). Then*

1. *If $bn(\xi) = 1$, then ξ is the standard tight contact structure on S^3.*
2. *If $bn(\xi) = 2$ and ξ is tight, then ξ is the unique tight contact structure on the lens space $L(m, m-1) = L(m, -1)$ for some $m \in \mathbb{Z}_+ \cup \{0\}$.*
3. *If $bn(\xi) = 2$ and ξ is overtwisted then ξ is the overtwisted contact structure on $L(m, 1)$, for some $m \in \mathbb{Z}_+$, with $e(\xi) = 0$ and $d_3(\xi) = -\frac{1}{4}m + \frac{3}{4}$ where $e(\xi)$ and $d_3(\xi)$ denotes the Euler class and*

d_3− invariant of ξ, respectively. When m is even then the refinement of $e(\xi)$ is given by $\Gamma(\xi)(\mathfrak{s}) = \dfrac{m}{2}$ where $\mathfrak{s}$ is the unique spin structure on $L(m,1)$ that extends over a two handle attached to a μ with framing zero. Here we are thinking of $L(m,1)$ as $-m$ surgery on an unknot and μ is the meridian to the unknot. □

We remark that Theorem 3.0.1 gives the complete list of all contact 3-manifolds which can be supported by planar open books whose pages have at most 2 boundary components. Next step in this direction should be to find all contact 3-manifolds (M, ξ) such that $sg(\xi) = 0$ and $bn(\xi) = 3$. We will not only get all such contact structures, but also distinguish tight ones by looking at the monodromy maps of their corresponding open books (See Theorem 3.0.2 and Theorem 3.0.4).

Let Σ be a planar compact oriented surface with $|\partial\Sigma| = 3$. Consider the boundary parallel curves a, b, c in Σ as in the Figure 3.1. Throughout the chapter, Σ will always stand for this surface whose abstract picture is given below. Consider $Aut(\Sigma, \partial\Sigma)$, the group of (isotopy classes of) diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$ (such diffeomorphisms are automatically orientation-preserving).

It is known (see [Bi]) that

$$Aut(\Sigma, \partial\Sigma) = \mathbb{Z}\langle D_a\rangle \oplus \mathbb{Z}\langle D_b\rangle \oplus \mathbb{Z}\langle D_c\rangle \cong \mathbb{Z}^3$$

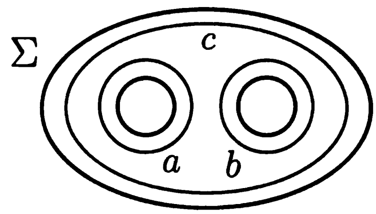

Figure 3.1. The surface Σ and the curves giving the generators of $Aut(\Sigma, \partial\Sigma)$.

where D_a, D_b, D_c denote positive Dehn twists along the curves a, b, c given as in Figure 3.1. In the rest of the paper, we will not make any distinction between isotopy classes of arcs/curves/maps and the individual arcs/curves/maps.

We start with studying the group $Aut(\Sigma, \partial\Sigma)$ in details. Since generators commute with each other, we have that

$$Aut(\Sigma, \partial\Sigma) = \{{D_a}^p {D_b}^q {D_c}^r \,|\, p, q, r \in \mathbb{Z}\}.$$

For any given $p, q, r \in \mathbb{Z}$, let $Y(p, q, r)$ denote the smooth 3-manifold given by the smooth surgery diagram in Figure 3.2 (diagram on the left). It is an easy exercise to check that $Y(p, q, r)$ is indeed diffeomorphic to Seifert fibered manifold given in Figure 3.2 (diagram on the right).

Now we state the following theorem characterizing all closed contact 3-manifolds whose contact structures supported by open books $(\Sigma, \phi = {D_a}^p {D_b}^q {D_c}^r)$.

Theorem 3.0.2 *Let (M, ξ) be a contact manifold supported by the open book (Σ, ϕ) where $\phi = {D_a}^p {D_b}^q {D_c}^r \in Aut(\Sigma, \partial\Sigma)$ for fixed integers p, q, r.*

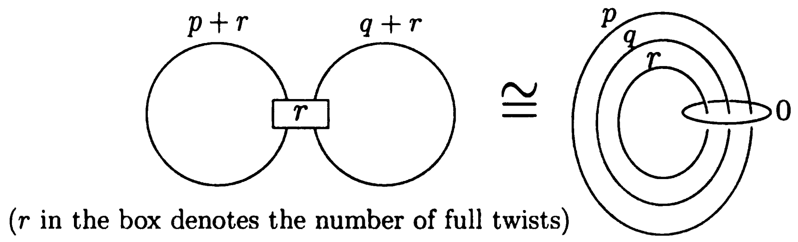

Figure 3.2. Seifert fibered manifold $Y(p,q,r)$.

Then (M,ξ) is contactomorphic to $(Y(p,q,r),\xi_{p,q,r})$ where $\xi_{p,q,r}$ is the contact structure on $Y(p,q,r)$ given by the contact surgery diagram in Figure 3.3. Moreover,

(1) ξ is tight (in fact holomorphically fillable) if $p \geq 0, q \geq 0, r \geq 0$, and

(2) ξ is overtwisted otherwise.

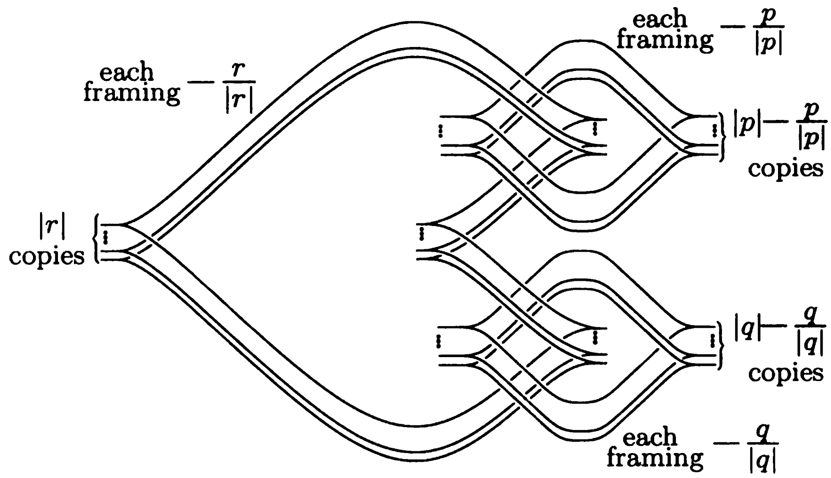

Figure 3.3. Contact manifold $(Y(p,q,r),\xi_{p,q,r})$.

Remark 3.0.3 *In Figure 3.3, if* $r = 0$, *then we completely delete the family corresponding to* r *from the diagram, so we are left with two families of Legendrian curves which do not link to each other, and so the contact surgery diagram gives a contact structure on the connected sum of two lens spaces. However, if* $p = 0$ *(or* $q = 0$*), then we replace the Legendrian family corresponding to* p *(or* q*) by a single Legendrian unknot with tb number equal to* -1, *and we do* $(+1)$*-contact surgery on the new unknot. Note also that Figure 3.3 is symmetric with respect to* p *and* q. *This reduces the number of cases in the proof of Theorem 3.0.4.*

Of course not all $\xi_{p,q,r}$ have binding number three. We will prove

Theorem 3.0.4 *Let* (M, ξ) *be a closed contact* 3*-manifold with* $sg(\xi) = 0$ *and* $bn(\xi) = 3$. *Then* (M, ξ) *is contactomorphic to some* $(Y(p,q,r), \xi_{p,q,r})$ *satisfying the following conditions:*

1. *If* $r = 0$, *then* $p \neq 1$ *and* $q \neq 1$.

2. *If* $r = 1$, *then* $p \notin \{-1, 0\}$ *and* $q \notin \{-1, 0\}$.

3. *If* $r = -1$, *then* $p \neq 1$ *and* $q \neq 1$.

4. *If* $|r| \geq 2$, *then* $pq \neq -1$ *and* $(p,q) \notin \{(1,0),(0,1)\}$.

Suppose that (M, ξ) is a closed contact 3-manifold with $sg(\xi) = 0$, $bn(\xi) = 3$, and let $c_1 = c_1(\xi) \in H^2(M;\mathbb{Z})$ denote the first Chern class, and $d_3 = d_3(\xi)$ denote the 3-dimensional invariant (which lies in $\mathbb{Q}$ whenever c_1 is a torsion class in $H^2(M;\mathbb{Z})$). Using c_1 and d_3, we can distinguish these structures in most of the cases. In fact, we have either M is a lens space, or a connected sum of lens spaces, or a Seifert fibered manifold with three singular fibers. If one of the first two holds, then using the tables given in Section 3.1, 3.2 and 3.3, one can get the complete list of all possible (M, ξ) without any repetition. That is, the contact structures in the list are all distinct pairwise and unique up to isotopy. On the other hand, if the third holds, we can also study them whenever c_1 is a torsion class. More discussion will be given later in Section 3.3.

We first prove that the submonoids $Dehn^+(\Sigma, \partial\Sigma)$ and $Veer(\Sigma, \partial\Sigma)$ are actually the same in our particular case.

Lemma 3.0.5 *$Dehn^+(\Sigma, \partial\Sigma) = Veer(\Sigma, \partial\Sigma)$ for the surface Σ given in Figure 3.1.*

Proof: The inclusion $Dehn^+(S, \partial S) \subset Veer(S, \partial S)$ is true for a general compact oriented surface S with boundary (see Lemma 2.5. in [?] for the proof). Now, suppose that $\phi \in Veer(\Sigma, \partial\Sigma) \subset Aut(\Sigma, \partial\Sigma)$. Then we can write ϕ in the form

$$\phi = D_a{}^p D_b{}^q D_c{}^r \text{ for some } p, q, r \in \mathbb{Z}.$$

We will show that $p \geq 0, q \geq 0, r \geq 0$. Consider the properly embedded arc $\alpha \subset \Sigma$ one of whose end points is $x \in \partial\Sigma$ as shown in the Figure 3.4. Note that, for any $p, q, r \in \mathbb{Z}$, $D_c{}^r$ fixes α, and also any image $D_a{}^p D_b{}^q(\alpha)$ of α because c does not intersect any of these arcs. Assume at least one of p, q, or r is strictly negative. First assume that $p < 0$. Then consider two possible different images $\phi(\alpha) = D_a{}^p D_b{}^q(\alpha)$ of α corresponding to whether

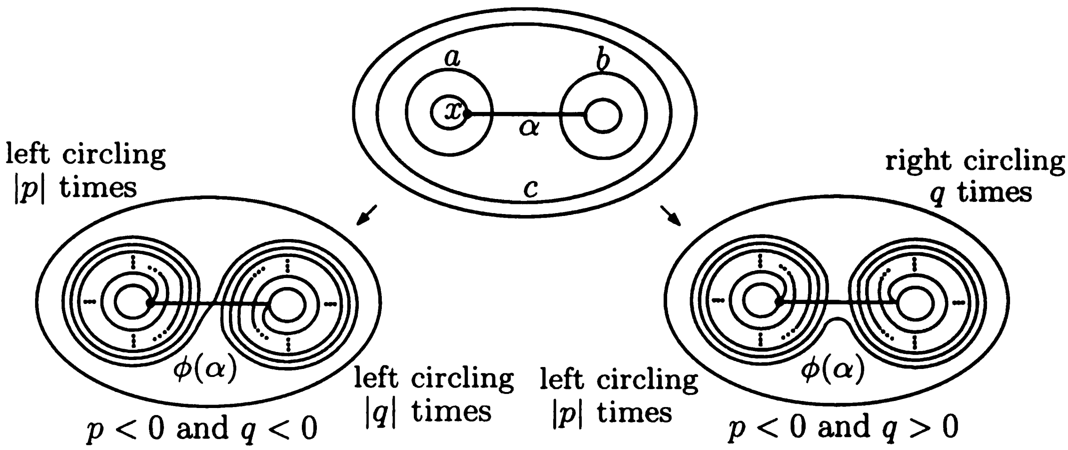

Figure 3.4. The arc α and its image $\phi(\alpha) = D_a{}^p D_b{}^q(\alpha)$.

$q < 0$ or $q > 0$ (See Figure 3.4). Since we are not allowed to rotate any boundary component, clearly $\phi(\alpha)$ is to the left of α at the boundary point x. Equivalently, $\phi(\alpha)$ is not to the right of α at x which implies that h is not right-veering with respect to the boundary component parallel to a. Therefore, $\phi \notin Veer(\Sigma, \partial\Sigma)$ which is a contradiction. Now by symmetry, we are also done for the case $q < 0$. Finally, exactly the same argument (with a different choice of arc one of whose end points is on the boundary component parallel to the curve c) will work for the case when $r < 0$. □

Lemma 3.0.6 *Let (M, ξ) be a contact manifold. Assume that ξ is supported by (Σ, ϕ) where $\phi \in Aut(\Sigma, \partial\Sigma)$. Then ξ is tight if and only if ξ is holomorpfically fillable.*

Proof: Assume that ξ is tight. Since $\phi \in Aut(\Sigma, \partial\Sigma)$, there exists integers p, q, r such that $\phi = D_a{}^p D_b{}^q D_c{}^r$. As ξ is tight, the monodromy of any open book supporting ξ is right-veering by Theorem 1.7.2. In particular, we have $\phi \in Veer(\Sigma, \partial\Sigma)$ since (Σ, ϕ) supports ξ. Therefore, $\phi \in Dehn^+(\Sigma, \partial\Sigma)$ by Lemma 3.0.5, and so $p \geq 0, q \geq 0, r \geq 0$. Thus, ξ is holomorphically fillable by Theorem 1.4.2. Converse statement is a consequence of Theorem 1.4.1. □

Now, the following corollary of Lemma 3.0.6 is immediate:

Corollary 3.0.7 *Let (M, ξ) be a contact manifold. Assume that ξ is supported by (Σ, ϕ) where $\phi \in Aut(\Sigma, \partial\Sigma)$. Then*

$$\xi \textit{ is tight} \iff \phi = {D_a}^p {D_b}^q {D_c}^r \textit{ with } p \geq 0, q \geq 0, r \geq 0. \quad \square$$

3.1 The proof of Theorem 3.0.2

Proof: Let (M, ξ) be a contact manifold supported by the open book $(\Sigma, \phi_{p,q,r})$ where $\phi_{p,q,r} = {D_a}^p {D_b}^q {D_c}^r \in Aut(\Sigma, \partial\Sigma)$ for $p, q, r \in \mathbb{Z}$. As explained in [EO], $(M, \xi) = (M_{(\Sigma, \phi_{p,q,r})}, \xi_{(\Sigma, \phi_{p,q,r})})$ is given by the contact surgery diagram in Figure 3.5. Then we apply the algorithm given in [DG] and [DGS] to convert each rational coefficient into ± 1's, and obtain the diagram given in Figure 3.3.

To determine the topological (or smooth) type of (M, ξ), we start with the diagram in Figure 3.3. Then by converting the contact surgery coefficients into the smooth surgery coefficients, we get the corresponding smooth surgery diagram in Figure 3.6 where each curve is an unknot.

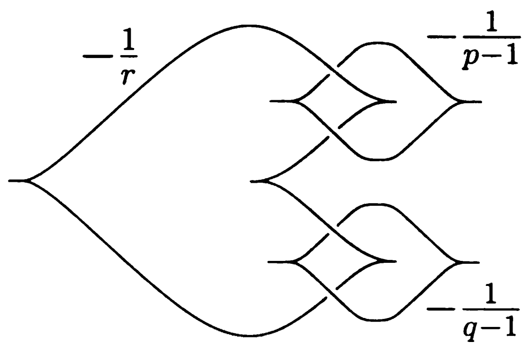

Figure 3.5. Contact surgery diagram corresponding to $(\Sigma, \phi_{p,q,r})$.

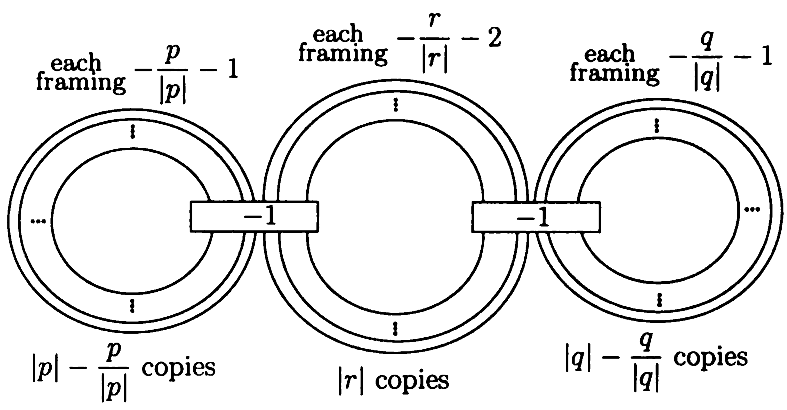

Figure 3.6. Smooth surgery diagram corresponding to Figure 3.3.

Now we modify this diagram using a sequence of blow-ups and blow-downs. These operations do not change smooth type of M. We first blow up the diagram twice so that we unlink two -1 twists. Then we blow down each unknot in the most left and the most right families. Finally we blow down each unknot of the family in the middle. We illustrate these operations in Figure 3.7. To keep track the surgery framings, we note that each blow-up increases the framing of any unknot by 1 if the unknot passes through

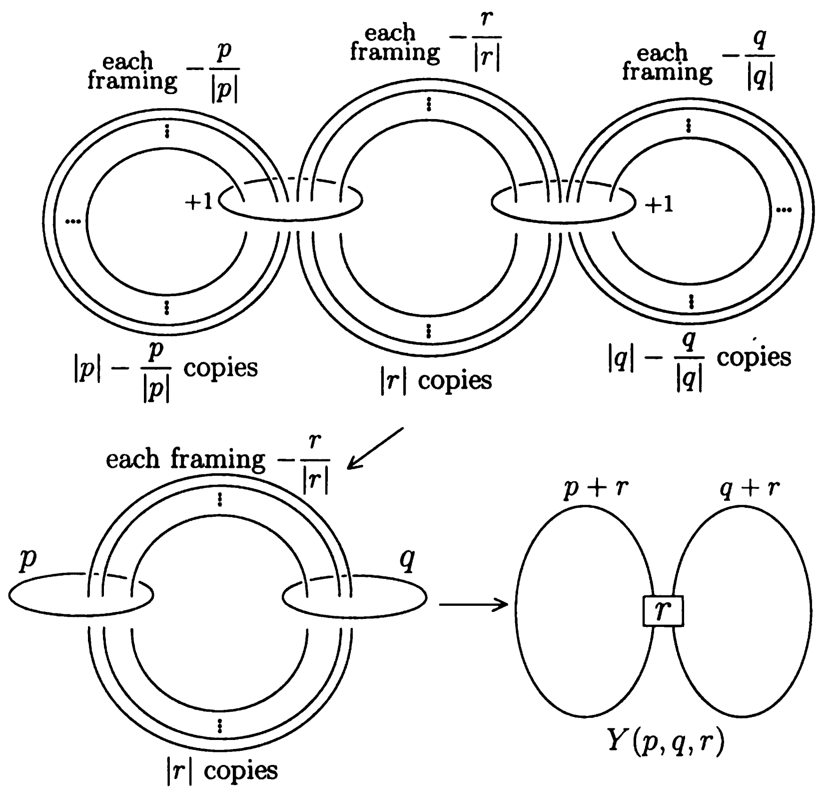

Figure 3.7. Sequence of blow-ups and blow-downs.

the corresponding twist box in Figure 3.6. So we get the first diagram in Figure 3.7. Blowing each member down on the left (resp. right) decreases the framing of the left (resp. right) $+1$-unknot by $-\frac{p}{|p|}$ (resp. $-\frac{q}{|q|}$). Since there are $|p|-\frac{p}{|p|}$ blow-downs on the left and $|q|-\frac{q}{|q|}$ blow-downs on the right, we get the second diagram in Figure 3.7. Finally, if we blow down each $(-\frac{r}{|r|})$-unknot in the middle family, we get the last diagram. Note that each blow-down decreases the framing by $-\frac{r}{|r|}$, and introduces a $\frac{r}{|r|}$ full twist. Hence, we showed that (M, ξ) is contactomorphic to $(Y(p,q,r), \xi_{p,q,r})$. The statements (1) and (2) are the consequences of

Corollary 3.0.7. □

We now examine the special case where $Y(p,q,r)$ is homeomorphic to 3-sphere S^3. The following lemma lists all planar contact structures on S^3 with binding number less than or equal to three.

Lemma 3.1.1 *Suppose that $(Y(p,q,r), \xi_{p,q,r})$ is contactomorphic to (S^3, ξ) for some contact structure ξ on S^3. Then Table 3.1 lists all possible values of (p,q,r), the corresponding ξ (in terms of the d_3-invariant), and its binding number.*

r	p	q	$d_3(\xi)$	$bn(\xi)$
-3	-2	1	$-1/2$	3
-2	-3	1	$-1/2$	3
-1	1	any q	$1/2$	2
-1	3	2	$3/2$	3
0	1	1	$-1/2$ (tight)	1
0	-1	-1	$3/2$	3
0	1	-1	$1/2$	2
1	0	1	$-1/2$ (tight)	1
1	-1	any q	$1/2$	2
1	-2	-3	$-1/2$	3
2	3	-1	$3/2$	3
3	2	-1	$3/2$	3
$\lvert r\rvert \geq 2$	1	-1	$1/2$	2

Table 3.1. All planar contact structures on S^3 with binding number ≤ 3.

Proof: The proof is the direct consequence of the discussion given in the proof of Lemma 5.5 in [EO]. We remark that the interchanging p and

q does not affect the contact structure in Figure 3.3, so we do not list the possibilities for (p, q, r) that differ by switching p and q. Note that in Table 3.1 there are only two contact structures (up to isotopy) on S^3 with binding number 3, namely, the ones with d_3-invariants $-1/2$ and $3/2$. $\square$

3.2 The proof of Theorem 3.0.4

Proof: We will use the results of Theorem 3.0.1, Theorem 3.0.2, and Lemma 3.1.1. Consider the 3-sphere S^3 in Theorem 3.0.1 as the lens space $L(1, \pm 1)$. By Theorem 3.0.1, for any contact manifold (Y, η) with $sg(\eta) = 0$ and $bn(\eta) \leq 2$, we have either

1. $(Y, \eta) \cong (S^3, \xi_{st})$ if $bn(\eta) = 1$,
2. $(Y, \eta) \cong (L(m, -1), \eta_m)$ for some $m \geq 2$ if $bn(\eta) = 2$, and η is tight,
3. $(Y, \eta) \cong (L(m, 1), \eta_m)$ for some $m \geq 0$ if $bn(\eta) = 2$, and η is overtwisted (for $m \neq 0$).

where η_m is the contact structure on the lens space $L(m, -1)$ (or $L(m, 1)$) given by the contact surgery diagram consisting of a single family of Legendrian unknots (with Thurston-Benequen number -1) such that each member links all the other members of the family once, and each contact surgery coefficient is -1 (if η_m is tight) or $+1$ (if η_m is overtwisted). These are illustrated by the diagrams $(*)$ and $(\star)$ in Figure 3.8, respectively. Notice the exceptional cases: $m = 1$ in $(*)$, and $m = 0$ in $(\star)$.

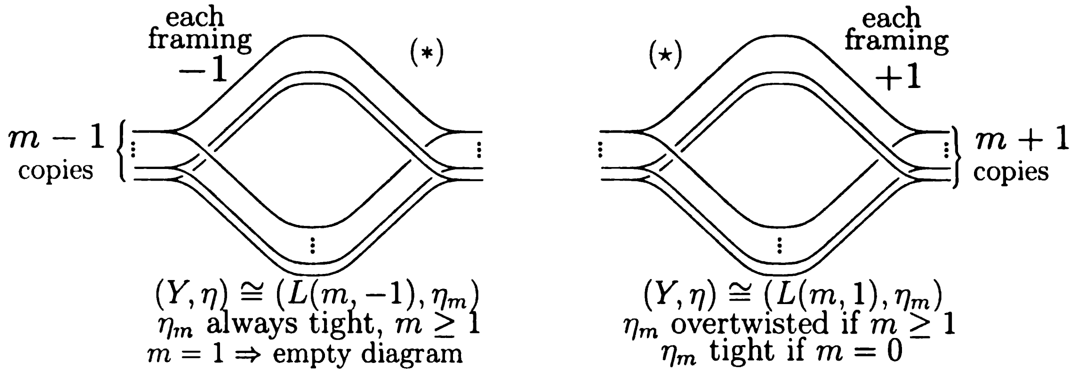

Figure 3.8. Contact surgery diagrams for (Y,η).

Now, if (M,ξ) is a contact manifold with $sg(\xi)=0$ and $bn(\xi)=3$, then by the definitions of these invariants there exists an open book (Σ,ϕ) supporting ξ. Therefore, by Theorem 3.0.2, (M,ξ) is contactomorphic to $(Y(p,q,r),\xi_{p,q,r})$ for some $p,q,r\in\mathbb{Z}$, and the contact surgery diagram of ξ is given in Figure 3.3. However, p,q,r can not be arbitrary integers because there are several cases where the diagram in Figure 3.3 reduces to either $(*)$ or $(\star)$ in Figure 3.8 for some m. So for those values of p,q,r, (M,ξ) can not be contactomorphic to $(Y(p,q,r),\xi_{p,q,r})\cong(Y,\eta)$ because $bn(\xi)=3\neq 2\geq bn(\eta)$. Therefore, we have to determine those cases.

If $|p|\geq 2$ and $|q|\geq 2$, then the only triples (p,q,r) giving $L(m,\pm1)'s$ are $(-2,q,1)$ and $(2,q,-1)$. Furthermore, if we assume also that $|r|>1$, then the Seifert fibered manifolds $Y(p,q,r)$ are not homeomorphic to even a lens space $L(m,n)$ for any m,n (for instance, see Chapter 5 in [Or]). As a result, we immediately obtain $bn(\xi_{p,q,r})=3$ for $|p|\geq 2$ and $|q|\geq 2$ and $|r|\geq 2$. Therefore, to finish the proof of the theorem, it is enough to analyze the cases where $|p| < 2$ or $|q| < 2$, and the cases $(-2, q, 1)$ and $(2, q, -1)$ for any q. As we remarked before, we do not need to list the possibilities for (p, q, r) that differ by switching p and q. We first consider $r = 0,\ \pm 1,\ \pm 2$, and then the cases $r > 2$ and $r < -2$. In Table 3.2 - 3.8, we list all possible (M, ξ) for each of these cases.

Remark 3.2.1 *To determine the binding number $bn(\xi)$ in any row of any table below, we simply first check the topological type of the manifold under consideration. If $M \approx S^3$, we determine the corresponding binding number using Table 3.1. If the topological type is not $L(m, 1)$ or $L(m, -1)$, then we immediately get that $bn(\xi) = 3$. If $M \approx L(m, 1)$ with $m > 1$, then we first compute $c_1(\xi)$. If $c_1(\xi) \neq 0$, then $bn(\xi) = 3$ as $c_1(\eta_m) = 0$ for any η_m given above. If $c_1(\xi)$=0, we compute the $d_3(\xi)$ using the 4-manifold defined by the surgery diagram in Figure 3.6. (Indeed, we can use the formula for d_3 given in Corollary 1.8.2 as long as $c_1(\xi)$ is torsion. In particular, whenever $H^2(M)$ is finite, then d_3 is computable). Then if $d_3(\xi) = d_3(\eta_m) = (-m + 3)/4$, then ξ is isotopic to η_m which implies that $bn(\xi) = 2$ by Theorem 3.0.1. Otherwise $bn(\xi) = 3$. In the case that $M \approx L(m, -1)$ with $m > 1$, we first ask if ξ is tight. If it is tight (which is the case if and only if $p \geq 0, q \geq 0, r \geq 0$), then $bn(\xi) = 2$ (again by Theorem 3.0.1) since the tight structure on $L(m, -1)$ is unique (upto isotopy). If it is overtwisted (which is the case if and only if at least one of p, q, r is negative), then $bn(\xi) = 3$ because ξ is not covered in Theorem 3.0.1. As a final remark, sometimes the contact structure ξ can be viewed as a positive stabilization* *of some η_m. For these cases we immediately obtain that $bn(\xi) = 2$ because positive stabilizations do not change the isotopy classes of contact structures.*

To compute the d_3-invariant of $\xi_{p,q,r}$ (for $c_1(\xi_{p,q,r})$ torsion), we will use the $(n+1) \times (n+1)$ matrices A_n $(n \geq 1)$, B_n $(n \geq 1)$, and C_n $(n \geq 4)$ given below. It is a standard exercise to check that

1. $\sigma(A_n) = n - 1$ if $n \geq 1$, and $\sigma(C_n) = n - 1$ if $n \geq 4$.

2. $\sigma(B_n) = n - 3$ if $n \geq 3$, and $\sigma(B_n) = 0$ if $n = 1, 2$.

3. The system $A_n[\mathbf{b}]_{n+1}^T = [\mathbf{0}]_{n+1}^T$ has trivial solution $[\mathbf{b}]_{n+1}^T = [\mathbf{0}]_{n+1}^T$ where

   $[\mathbf{b}]_{n+1} = [b_1\ b_2 \cdots b_{n+1}]$, $[\mathbf{0}]_{n+1} = [0\ 0 \cdots 0]$ are $(n+1) \times 1$ row matrices.

$$A_n = \begin{bmatrix} 0 & -1 & -1 & \dots & -1 \\ -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \quad B_n = \begin{bmatrix} 0 & -1 & -1 & \dots & -1 \\ -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & -1 \\ -1 & -1 & \cdots & -1 & -2 \end{bmatrix} \quad C_n = \begin{bmatrix} 0 & -1 & -1 & \dots & -1 \\ -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & -1 \\ -1 & -1 & \cdots & -1 & 2 \end{bmatrix}$$

In some cases, A_n appears (as a block matrix) in the linking matrix $\mathcal{L}_{p,q,r}$ of the framed link $\mathbb{L}_{p,q,r}$ given in Figure 3.6. On the other hand, B_n and C_n are very handy when we diagonalize $\mathcal{L}_{p,q,r}$ to find its signature. As we discussed before, the link $\mathbb{L}_{p,q,r}$ defines a 4-manifold $X_{p,q,r}$ with $\partial X = M$. So we have

$$
\begin{aligned}
\sigma(X_{p,q,r}) &= \sigma(\mathcal{L}_{p,q,r}), \\
\chi(X_{p,q,r}) &= 1 + (\#\text{ of components of } \mathbb{L}_{p,q,r}), \\
c^2 &= [\mathbf{b}]_k \mathcal{L}_{p,q,r} [\mathbf{b}]_k^T
\end{aligned}
$$

where $[\mathbf{b}]_k^T$ is the solution to the linear system

$$\mathcal{L}_{p,q,r}[\mathbf{b}]_k^T = [rot(K_1)\ rot(K_2) \cdots rot(K_k)]^T$$

with $K_1, K_2, \cdots K_k$ being the components of $\mathbb{L}_{p,q,r}$.

To compute the first Chern class $c_1(\xi_{p,q,r}) \in H^2(M)$, note that in Figure 3.3, the rotation number of any member in the family corresponding to r is ± 1 (depending on how we orient them). We will always orient them so that their rotation numbers are all $+1$. On the other hand, the rotation number is 0 for any member in the family corresponding to p and q. Therefore, $c_1(\xi_{p,q,r}) = PD^{-1}(\mu_1 + \mu_2 + \cdots + \mu_{|r|})$ where $\mu_i \in H_1(M)$ is the class of the meridian of the Legendrian knot K_i in the family corresponding to r. Then we compute $H_1(M)$ (which is isomorphic to $H^2(M)$ by Poincaré duality) as

$$H_1(M) = \langle\ \mu_1, \mu_2, \cdots, \mu_k |\ \mathcal{L}_{p,q,r}[\mu]_k^T = [\mathbf{0}]_k^T\ \rangle$$

where $[\mu]_k = [\mu_1\ \mu_2 \cdots \mu_k]$ is the $k \times 1$ row matrix. The final step is to understand $PD(c_1(\xi_{p,q,r})) = \mu_1 + \mu_2 + \cdots + \mu_{|r|}$ in this presentation of $H_1(M)$.

In **Table 3.2**, we need to compute the binding number $bn(\xi)$ for the rows 5, 12. For the other rows, see Remark 3.2.1.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
0	-1	-1	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
0	-1	0	$S^3 \# S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z}$	1
0	-1	1	S^3	2	$(\star)$ $m = 1$	$0 \in \{0\}$	$1/2$
0	-1	$q \geq 2$	$S^3 \# L(q, -1)$	3	Figure 3	[0]	$(q+1)/4$
0	-1	$q \leq -2$	$S^3 \# L(\lvert q\rvert, 1)$	3	Figure 3	[0]	$(-\lvert q\rvert + 7)/4$
0	0	0	$\#_2 S^1 \times S^2$	3	Figure 3	$[0] \in \mathbb{Z} \oplus \mathbb{Z}$	$1/2$
0	0	1	$S^1 \times S^2 \# S^3$	2	$(\star)$ $m = 0$	$0 \in \mathbb{Z}$	0
0	0	$q \geq 2$	$S^1 \times S^2 \# L(q, -1)$	3	Figure 3	[0]	$(q-1)/4$
0	0	$q \leq -2$	$S^1 \times S^2 \# L(\lvert q\rvert, 1)$	3	Figure 3	[0]	$(-\lvert q\rvert + 5)/4$
0	1	1	S^3	1	$(*)$ $m = 1$	$0 \in \{0\}$	$-1/2$
0	1	$q \geq 2$	$S^3 \# L(q, -1)$	2	$(*)$ $m = q$	[0]	$(q-3)/4$
0	1	$q \leq -2$	$S^3 \# L(\lvert q\rvert, 1)$	2	$(\star)$ $m = \lvert q\rvert$	[0]	$(-\lvert q\rvert + 3)/4$

Table 3.2. The case $r = 0$ ($|p| < 2$ or $|q| < 2$).

• If $p = -1, q \leq -2, r = 0$, we need to compute $d_3(\xi_{-1,q,0})$ as $c_1(\xi_{-1,q,0}) = 0$: We have

$$\mathcal{L}_{-1,q,0} = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_{|q|+1} \end{pmatrix}.$$

The contact structure $\xi_{-1,q,0}$, and $\mathbb{L}_{-1,q,0}$ describing $X_{-1,q,0}$ are given in Figure 3.9. We compute that $s = |q| + 3$, $c^2 = 0$, $\chi(X_{-1,q,0}) = |q| + 4$, and

$$\sigma(X_{-1,q,0}) = \sigma(A_1) + \sigma(A_{|q|}) = 0 + |q| - 1 = |q| - 1,$$

and so we obtain $d_3(\xi_{-1,q,0}) = (-|q| + 7)/4$ by Corollary 1.8.2. Therefore, $\xi_{-1,q,0}$ is not isotopic to $\eta_{|q|}$ as $d_3(\eta_{|q|}) = (-|q|+3)/4$. Hence, $bn(\xi_{-1,q,0}) = 3$ for any $q \leq -2$ by Theorem 3.0.1.

• If $p = 1, q \leq -2, r = 0$, we have $(\Sigma, \phi_{1,q,0}) = S_a^+(H^+, D_b^q)$ (recall the identification of Σ and the curves a, b, c in Figure 3.1). Therefore, $\xi_{1,q,0} \cong \eta_{|q|}$

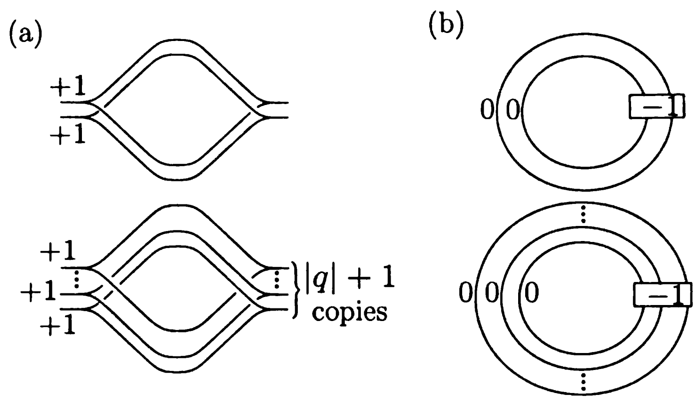

Figure 3.9. (a) $\xi_{-1,q,0}$ on $S^3 \# L(|q|,1) \approx L(|q|,1)$, (b) the link $\mathbb{L}_{-1,q,0}$.

since (H^+, D_b^q) supports the overtwisted structure $\eta_{|q|}$ on $L(|q|,1)$. Hence, $bn(\xi_{1,q,0}) = 2$ for $q \leq -2$.

In **Table 3.3**, we need to compute the binding number $bn(\xi)$ for the rows 1 and 9. For the other rows, see Remark 3.2.1.

• If $p = -2, q \leq -4, r = 1$, let K_i's be the components (with the given orientations) of $\mathbb{L}_{-2,q,1}$ as in Figure 3.10. Then we obtain the linking matrix

$$\mathcal{L}_{-2,q,1} = \begin{pmatrix} -3 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 0 & -1 & 0 & \cdots & 0 \\ -1 & -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & \cdot & \cdot & & & \\ -1 & 0 & 0 & 0 & & & \end{pmatrix}.$$

r	p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
1	-2	$q \leq -4$	$L(\lvert q+2\rvert, 1)$	3	$[\lvert q\rvert - 4]$	$(-q^2 - 7q - 14)/(-4q - 8)$
1	-2	-3	S^3	3	$0 \in \{0\}$	$-1/2$
1	-2	-2	$S^1 \times S^2$	3	$-2 \in \mathbb{Z}$	$\notin Q$
1	-2	$q \geq 2$	$L(q+2, -1)$	3	$[q]$	$(q^2 + q + 2)/(4q + 8)$
1	-1	any q	S^3	2	$0 \in \{0\}$	$1/2$
1	0	0	$S^1 \times S^2$	2	$0 \in \mathbb{Z}$	0
1	0	1	S^3	1	$0 \in \{0\}$	$-1/2$
1	0	$q \geq 2$	$L(q, -1)$	2	$[0]$	$(q-3)/4$
1	0	$q \leq -1$	$L(\lvert q\rvert, 1)$	2	$[0]$	$(-\lvert q\rvert + 3)/4$
1	1	-2	$L(3, -1)$	3	$[1]$	$1/3$
1	1	1	$L(3, 1)$	3	$[1]$	$-1/3$
1	1	$q \leq -3$	$L(2q+1, -q-1)$	3	$[\lvert q\rvert - 1]$	$(-q^2 - 4q - 2)/(-4q - 2)$
1	1	$q \geq 2$	$L(2q+1, -q-1)$	3	$[q+1]$	$(q^2 - 2q - 1)/(4q + 2)$

Table 3.3. The case $r = 1$, $|p| < 2$ or $|q| < 2$ (and the case $(p, q, r) = (-2, q, 1)$).

It is not hard to see that

$$H_1(M) = \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+5} \,|\, \mathcal{L}_{-2,q,1}[\mu]^T_{|q|+5} = [\mathbf{0}]^T_{|q|+5} \,\rangle$$
$$= \langle\, \mu_2 \,|\, (|q|-2)\mu_2 = 0 \,\rangle \cong \mathbb{Z}_{|q|-2},$$

and $\mu_1 = (|q| - 4)\mu_2$. Therefore,

$$c_1(\xi_{-2,q,1}) = PD^{-1}(\mu_1) = PD^{-1}(|q| - 4)\mu_2 = [|q| - 4] \in \mathbb{Z}_{|q|-2}.$$

Thus, if $q < -4$, then $\xi_{-2,q,1}$ is not isotopic to $\eta_{|q+2|}$ as $c_1(\eta_{|q+2|}) = 0$ implying that $bn(\xi_{-2,q,1}) = 3$ by Theorem 3.0.1. If $q = -4$, we compute that $d_3(\xi_{-2,-4,1}) = -1/4 \neq 1/4 = d_3(\eta_2)$, so $bn(\xi_{-2,-4,1}) = 3$.

- If $p = 0, q \leq -1, r = 1$, we have $(\Sigma, \phi_{0,q,1}) = S^+_c(H^+, D^q_b)$ (again recall the identification of Σ and the curves a, b, c in Figure 3.1). There-

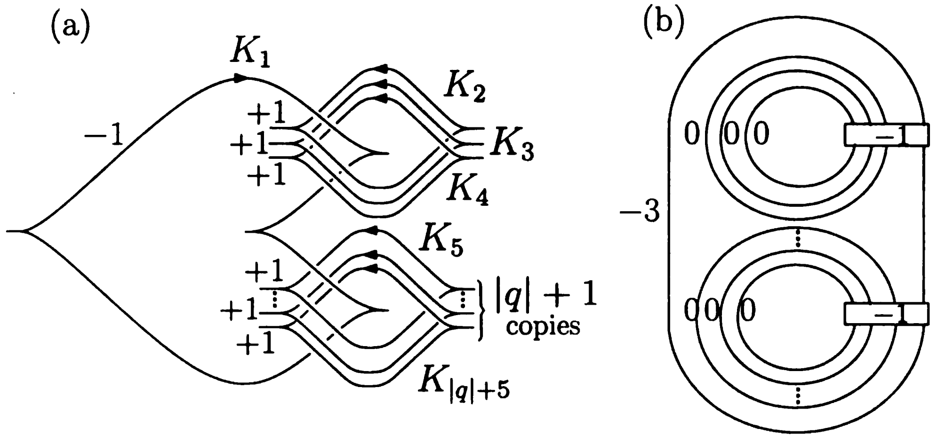

Figure 3.10. (a) $\xi_{-2,q,1}$ on $L(|q+2|,1)$, (b) the link $\mathbb{L}_{-2,q,1}$.

fore, $\xi_{0,q,1} \cong \eta_{|q|}$ since (H^+, D_b^q) supports the overtwisted structure $\eta_{|q|}$ on $L(|q|,1)$. Hence, $bn(\xi_{0,q,1}) = 2$ for $q < 0$.

In **Table 3.4**, we need to determine the binding number $bn(\xi)$ for the rows 4, 7, 9, and 11. For the other rows, see Remark 3.2.1.

• If $p = 2, q \leq -2, r = -1$, then using the corresponding matrix $\mathcal{L}_{2,q,-1}$, we have

$$\begin{aligned} H_1(M) &= \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+3} |\; \mathcal{L}_{2,q,-1}[\mu]^T_{|q|+3} = [\mathbf{0}]^T_{|q|+3} \,\rangle \\ &= \langle\, \mu_2 |\; (|q|+2)\mu_2 = 0 \,\rangle \cong \mathbb{Z}_{|q|+2}, \end{aligned}$$

and $\mu_1 = |q|\mu_2$. Therefore,

$$c_1(\xi_{2,q,-1}) = PD^{-1}(\mu_1) = PD^{-1}(|q|\mu_2) = [|q|] \in \mathbb{Z}_{|q|+2}.$$

Thus, if $q \leq -2$, then $\xi_{2,q,-1}$ is not isotopic to $\eta_{|q-2|}$ as $c_1(\eta_{|q-2|}) = 0$ implying that $bn(\xi_{2,q,-1}) = 3$ by Theorem 3.0.1.

r	p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-1	2	$q \geq 4$	$L(q-2,-1)$	3	$[q-4]$	$(-q^2+3q-6)/(-4q+8)$
-1	2	3	S^3	3	$0 \in \{0\}$	$3/2$
-1	2	2	$S^1 \times S^2$	3	$-2 \in \mathbb{Z}$	$\notin Q$
-1	2	$q \leq -2$	$L(\lvert q-2\rvert,1)$	3	$[\lvert q\rvert]$	$(-q^2-3q+6)/(-4q+8)$
-1	1	any q	S^3	2	$0 \in \{0\}$	$1/2$
-1	0	0	$S^1 \times S^2$	3	$0 \in \mathbb{Z}$	1
-1	0	-1	S^3	3	$0 \in \{0\}$	$3/2$
-1	0	$q > 1$	$L(q,-1)$	3	$[0]$	$(q+1)/4$
-1	0	$q < -1$	$L(\lvert q\rvert,1)$	3	$[0]$	$(-\lvert q\rvert+7)/4$
-1	-1	-1	$L(3,-1)$	3	$[1]$	$4/3$
-1	-1	2	$L(3,1)$	3	$[1]$	$2/3$
-1	-1	$q \leq -2$	$L(-2q+1,-q+1)$	3	$[\lvert q\rvert+1]$	$(-q^2-6q+3)/(-4q+2)$
-1	-1	$q \geq 3$	$L(-2q+1,-q+1)$	3	$[q-1]$	$(-q^2)/(-4q+2)$

Table 3.4. The case $r=-1$, $|p|<2$ or $|q|<2$ (and the case $(p,q,r)=(2,q,-1)$).

• If $p=0, q \leq -1, r=-1$ (the rows 7 or 9), then $c_1(\xi_{0,q,-1})=0$ and so we need to compute $d_3(\xi_{0,q,-1})$. Let K_i's be the components of $\mathbb{L}_{0,q,-1}$ as in Figure 3.11. Then

$$\mathcal{L}_{0,q,-1} = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & & & \\ -1 & 0 & & & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & & & \\ 0 & 0 & & & \end{pmatrix}$$

By diagonalizing the first two rows of $\mathcal{L}_{0,q,-1}$, we obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,q,-1}) = \sigma(A_{|q|}) = |q| - 1$. The contact surgery diagram for $\xi_{0,q,-1}$ and the corresponding 4-manifold $X_{0,q,-1}$ (with $\partial X_{0,q,-1} = M$) are given in Figure 3.11.

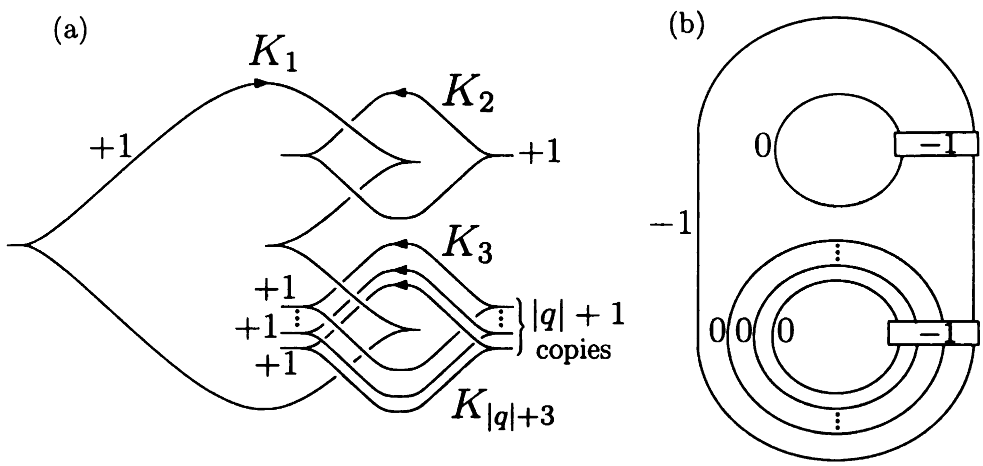

Figure 3.11. (a) $\xi_{0,q,-1}$ on $L(|q|,1)$, (b) the link $\mathbb{L}_{0,q,-1}$.

Then the system $\mathcal{L}_{0,q,-1}[\mathbf{b}]^T = [rot(K_1) \ \ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1 \ 0 \ 0 \cdots 0]^T$ has the solution $[\mathbf{b}] = [0 \ -1 \ 0 \cdots 0]$, and so $c^2 = 0$. Moreover, $\chi(X_{0,q,-1}) = |q| + 4$ and $s = |q| + 3$. Therefore, we obtain $d_3(\xi_{0,q,-1}) = (-|q| + 7)/4$ implying that $\xi_{0,q,-1}$ is not isotopic to $\eta_{|q|}$ as $d_3(\eta_{|q|}) = (-|q| + 3)/4$. Hence, $bn(\xi_{0,q,-1}) = 3$ by Theorem 3.0.1.

- If $p = -1, q = 2, r = -1$, we have $c_1(\xi_{-1,2,-1}) = [1]$ implying that $bn(\xi_{-1,2,-1}) = 3$. To see this, note that $c_1(\xi -1, 2, -1) = PD^{-1}(\mu_1)$ where μ_1 is the meridian of the surgery curve corresponding K_1. Then using

$$\mathcal{L}_{-1,2,-1} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

we get $H_1(M) = \langle\ \mu_1, \mu_2, \mu_3, \mu_4 |\ \mathcal{L}_{-1,2,-1}[\mu]_4^T = [\mathbf{0}]_4^T\ \rangle = \langle\ \mu_2 |\ 3\mu_2 = 0\ \rangle \cong \mathbb{Z}_3$, and $\mu_1 = -2\mu_2$. Therefore, we compute

$$c_1(\xi_{-1,2,-1}) = PD^{-1}(\mu_1) = PD^{-1}(-2\mu_2) = [-2] \in \mathbb{Z}_3 \equiv [1] \in \mathbb{Z}_3.$$

r	p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
2	-1	$q \leq -3$	$L(\|q-2\|, 1)$	3	[2]	$(-q^2-3q+6)/(-4q+8)$
2	-1	$q = -1$	$L(3, 1)$	3	[2]	$-5/6$
2	-1	$q = -2$	$L(4, 1)$	3	[2]	$-1/4$
2	-1	1	S^3	2	$0 \in \{0\}$	$1/2$
2	-1	2	$S^1 \times S^2$	3	$2 \in \mathbb{Z}$	$\notin Q$
2	-1	3	S^3	3	$0 \in \{0\}$	$3/2$
2	-1	$q \geq 4$	$L(q-2, -1)$	3	$[q-4]$	$(q^2-3q+6)/(4q-8)$
2	0	0	$S^1 \times S^2 \# L(2, -1)$	3	[0]	$1/4$
2	0	1	$S^3 \# L(2, -1)$	2	[0]	$-1/4$
2	0	$q > 1$	$L(q, -1) \# L(2, -1)$	3	[0]	$(q-2)/4$
2	0	$q < 0$	$L(\|q\|, 1) \# L(2, -1)$	3	[0]	$(-\|q\|+4)/4$
2	1	-2	$L(4, -1)$	3	[2]	$1/2$
2	1	1	$L(-5, 2)$	3	[2]	$-1/10$
2	1	$q \leq -3$	$L(-3q-2, q+1)$	3	[2]	$(3q^2+15q+10)/(12q+8)$
2	1	$q \geq 2$	$L(-3q-2, q+1)$	3	[2]	$(3q^2-3q-2)/(12q+8)$

Table 3.5. The case $r = 2$ ($|p| < 2$ or $|q| < 2$).

In **Table 3.5**, we need to compute the binding number $bn(\xi)$ for the rows 1, 2, and 3. For the other rows, again see Remark 3.2.1.

For the first three rows in Table 3.5, the contact structure $\xi_{-1,q,2}$ on $L(|q-2|,1)$ and the link $\mathbb{L}_{-1,q,2}$ $(q \leq -1)$ are given in Figure 3.12. We write the linking matrix $\mathcal{L}_{-1,q,2}$ as the matrix on the left below. It is not hard to see that $c_1(\xi_{-1,q,2}) = [2] \in \mathbb{Z}_{|q-2|}$, and so $bn(\xi_{-1,q,2}) = 3$. As an illustration we will compute $d_3(\xi_{-1,q,2})$ (even though it is not necessary for the proof). The matrix on the right below is obtained by diagonalizing the first two rows of $\mathcal{L}_{-1,q,2}$. So we compute $\sigma(\mathcal{L}_{-1,q,2}) = 2 + \sigma(A_1) + \sigma(B_{|q|})$ which is $|q|-1$ if $q \leq -3$, and is equal to 2 if $q = -1,-2$ (recall $\sigma(B_n)$ is $n-3$ if $n \geq 3$, and 0 if $n = 1,2$). By a standard calculation, the system

$$\mathcal{L}_{-1,q,2} = \left(\begin{array}{cc:cc:cccc}
-3 & -2 & -1 & -1 & -1 & -1 & \cdots & -1 \\
-2 & -3 & -1 & -1 & -1 & -1 & \cdots & -1 \\ \hdashline
-1 & -1 & \multicolumn{2}{c:}{\multirow{2}{*}{A_1}} & 0 & 0 & \cdots & 0 \\
-1 & -1 & & & 0 & 0 & \cdots & 0 \\ \hdashline
-1 & -1 & 0 & 0 & & & & \\
-1 & -1 & 0 & 0 & & & & \\
\vdots & \vdots & \vdots & \vdots & & \multicolumn{2}{c}{A_{|q|}} & \\
-1 & -1 & 0 & 0 & & & &
\end{array}\right) \longrightarrow \left(\begin{array}{cc:cc:cccc}
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1/2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hdashline
0 & 0 & \multicolumn{2}{c:}{\multirow{2}{*}{A_1}} & 0 & 0 & \cdots & 0 \\
0 & 0 & & & 0 & 0 & \cdots & 0 \\ \hdashline
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & & \\
\vdots & \vdots & \vdots & \vdots & & \multicolumn{2}{c}{B_{|q|}} & \\
0 & 0 & 0 & 0 & & & &
\end{array}\right)$$

$$\mathcal{L}_{-1,q,2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+5})]^T = [1\ 1\ 0 \cdots 0]^T$$

has the solution

$$[\mathbf{b}] = [\frac{|q|}{|q|+2}\ \frac{|q|}{|q|+2}\ \frac{-2|q|}{|q|+2}\ \frac{-2|q|}{|q|+2}\ \frac{-2}{|q|+2} \cdots \frac{-2}{|q|+2}]$$

for $q \leq -1$, and so we compute

$$c^2 = [\mathbf{b}]\mathcal{L}_{-1,q,2}[\mathbf{b}]^T = 2|q|/(|q|+2).$$

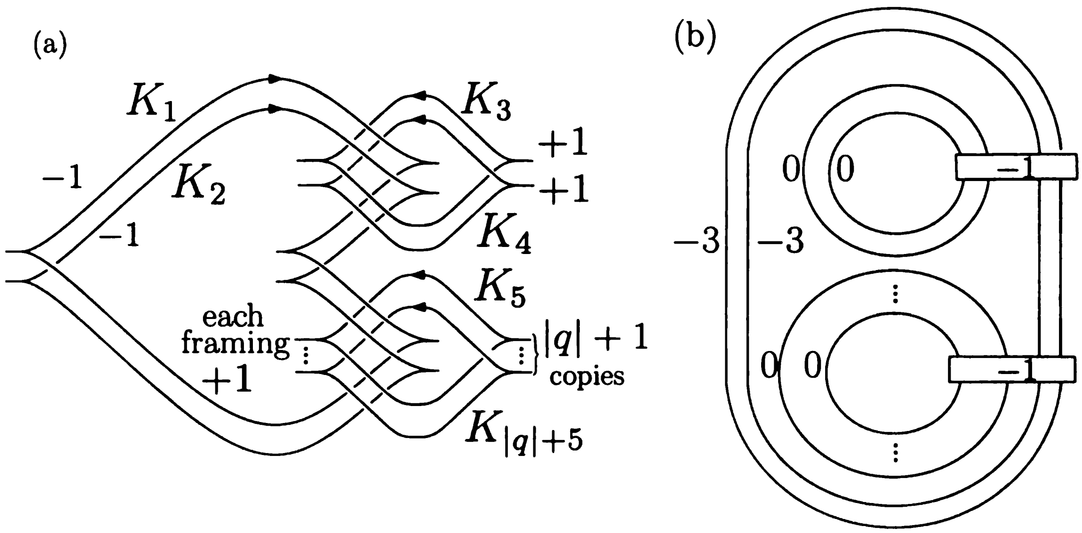

Figure 3.12. (a) $\xi_{-1,q,2}$ on $L(|q-2|,1)$ for $q \leq -1$, (b) the link $\mathbb{L}_{-1,q,2}$.

- If $p=-1, q=-1, r=2$, then $c^2 = 2/3$, $\sigma(X_{-1,-1,2}) = 2$, $\chi(X_{-1,-1,2}) = 7$, and $s = 4$. So we get $d_3(\xi_{-1,-1,2}) = -5/6$.
- If $p=-1, q=-2, r=2$, then $c^2 = 1$, $\sigma(X_{-1,-2,2}) = 2$, $\chi(X_{-1,-2,2}) = 8$, and $s = 5$. Therefore, we get $d_3(\xi_{-1,-2,2}) = -1/4$.
- If $p=-1, q \leq -3, r=2$, then $c^2 = 2|q|/(|q|+2)$, $\sigma(X_{-1,q,2}) = |q|-1$, $\chi(X_{-1,q,2}) = |q|+6$, and $s = |q|+3$. So we obtain

$$d_3(\xi_{-1,q,2}) = \frac{-q^2-3q+6}{-4q+8}.$$

r	p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-2	1	$q \geq 3$	$L(q+2,-1)$	3	$[2]$	$(q^2+q+2)/(4q+8)$
-2	1	2	$L(4,-1)$	3	$[0]$	$1/2$
-2	1	1	$L(3,-1)$	3	$[2]$	$1/3$
-2	1	-1	S^3	2	$0 \in \{0\}$	$1/2$
-2	1	-2	$S^1 \times S^2$	3	$2 \in \mathbb{Z}$	$\notin Q$
-2	1	-3	S^3	3	$0 \in \{0\}$	$-1/2$
-2	1	$q \leq -4$	$L(\lvert q+2\rvert,1)$	3	$[\lvert q\rvert - 4]$	$(-q^2-7q-14)/(-4q-8)$
-2	0	0	$S^1 \times S^2 \# L(2,1)$	3	$[0]$	$3/4$
-2	0	1	$S^3 \# L(2,1)$	2	$[0]$	$1/4$
-2	0	-1	$S^3 \# L(2,1)$	3	$[0]$	$5/4$
-2	0	$q \geq 2$	$L(q,-1)\#L(2,1)$	3	$[0]$	$q/4$
-2	0	$q \leq -2$	$L(\lvert q\rvert,1)\#L(2,1)$	3	$[0]$	$(-\lvert q\rvert+6)/4$
-2	-1	2	$L(4,1)$	3	$[0]$	$1/2$
-2	-1	-1	$L(-5,-2)$	3	$[2]$	$11/10$
-2	-1	$q \leq -2$	$L(3q-2,q-1)$	3	$[2]$	$(-3q^2-15q+10)/(-12q+8)$
-2	-1	$\lvert q\rvert \geq 3$	$L(3q-2,q-1)$	3	$[2]$	$(-3q^2+3q-2)/(-12q+8)$

Table 3.6. The case $r = -2$ ($|p| < 2$ or $|q| < 2$).

In **Table 3.6**, we need to compute the binding number $bn(\xi)$ for the rows 7, 9, 10, and 13. For the other rows, see Remark 3.2.1.

- If $p = 1, q \leq -4, r = -2$, the contact structure $\xi_{1,q,-2}$ on $L(|q+2|,1)$ and the link $\mathbb{L}_{1,q,-2}$ are given in Figure 3.13.

We will first compute that $c_1(\xi_{1,q,-2}) = [|q|-4] \in \mathbb{Z}_{|q|-2}$ (so $bn(\xi_{1,q,-2}) = 3$), and then (even though it is not necessary for the proof) we will evaluate $d_3(\xi_{1,q,-2})$ as an another sample computation. Using $\mathcal{L}_{1,q,-2}$ (on the left

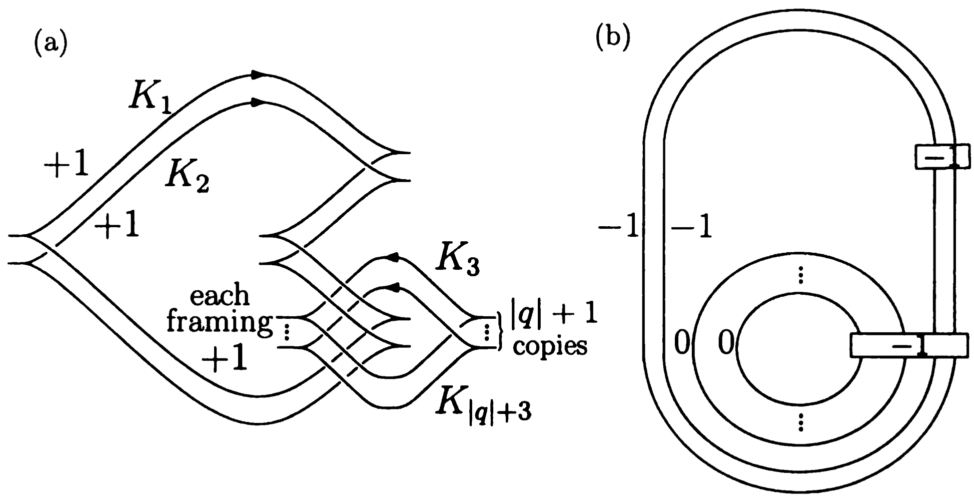

Figure 3.13. (a) $\xi_{1,q,-2}$ on $L(|q+2|,1)$ for $q<-3$, (b) the link $\mathbb{L}_{1,q,-2}$.

below), we have

$$\begin{aligned} H_1(M) &= \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+3} |\ \mathcal{L}_{1,q,-2}[\mu]^T_{|q|+3} = [\mathbf{0}]^T_{|q|+3} \,\rangle \\ &= \langle\, \mu_1, \mu_3 |\ -3\mu_1 - (|q|+1)\mu_3 = 0, -2\mu_1 - |q|\mu_3 = 0 \,\rangle \\ &= \langle\, \mu_3 |\ (|q|-2)\mu_3 = 0 \,\rangle \cong \mathbb{Z}_{|q|-2}, \end{aligned}$$

and also we have $\mu_1 = \mu_2 = -\mu_3$. Therefore, we obtain

$$c_1(\xi_{2,q,-1}) = PD^{-1}(\mu_1+\mu_2) = PD^{-1}(-2\mu_3) = -2 \equiv [|q|-4] \in \mathbb{Z}_{|q|-2}.$$

The matrix on the right below is obtained by diagonalizing the first two rows of $\mathcal{L}_{1,q,-2}$. So we compute $\sigma(\mathcal{L}_{1,q,-2}) = 0 + \sigma(C_{|q|}) = |q|-1$ (recall $\sigma(C_n) = n-1$ if $n \geq 2$).

By a standard calculation, the system

$$\mathcal{L}_{1,q,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1\ 1\ 0 \cdots 0]^T$$

$$\mathcal{L}_{1,q,-2} = \left[\begin{array}{cc:ccc} -1 & -2 & -1 & \dots & -1 \\ -2 & -1 & -1 & \cdots & -1 \\ \hdashline -1 & -1 & & & \\ \vdots & \vdots & & A_{|q|} & \\ -1 & -1 & & & \end{array}\right] \longrightarrow \left[\begin{array}{cc:ccc} 2 & 0 & 0 & \dots & 0 \\ 0 & -1/2 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & & & \\ \vdots & \vdots & & C_{|q|} & \\ 0 & 0 & & & \end{array}\right]$$

has the solution $[\mathbf{b}] = [\frac{-|q|}{|q|-2}\ \frac{-|q|}{|q|-2}\ \frac{2}{|q|-2}\ \cdots\ \frac{2}{|q|-2}]$, and so we obtain

$$c^2 = [\mathbf{b}]\mathcal{L}_{1,q,-2}[\mathbf{b}]^T = -2|q|/(|q|-2).$$

Moreover, $\chi(X_{1,q,-2}) = |q|+4$, and $s = |q|+3$. So we compute

$$d_3(\xi_{1,q,-2}) = \frac{-q^2-7q-14}{-4q-8}.$$

- If $p=0, q=1, r=-2$, then $\xi_{0,1,-2}$ and $\mathbb{L}_{0,1,-2}$ are given in Figure 3.14.

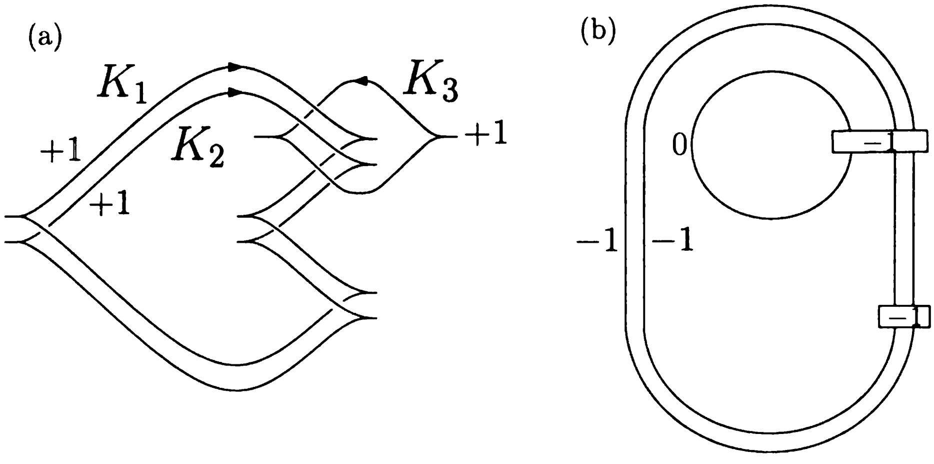

Figure 3.14. (a) $\xi_{0,1,-2}$ on $S^3\#L(2,1) \approx L(2,1)$, (b) the link $\mathbb{L}_{0,1,-2}$.

One can get $c_1(\xi_{0,1,-2}) = 0$, so we need $d_3(\xi_{0,1,-2})$. The corresponding linking matrix is

$$\mathcal{L}_{0,1,-2} = \begin{pmatrix} -1 & -2 & -1 \\ -2 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{0,1,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,1,-2}) = 1$. We find that the system $\mathcal{L}_{0,1,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)]^T = [1\ 1\ 0]^T$ has the solution $[\mathbf{b}] = [0\ 0\ -1]$, and so $c^2 = 0$. Also we have $\chi(X_{0,1,-2}) = 4$ and $s = 3$. So we get $d_3(\xi_{0,1,-2}) = 1/4 = d_3(\eta_2)$ which implies that $\xi_{0,1,-2}$ is isotopic to η_2. Thus, $bn(\xi_{0,1,-2}) = 2$ by Theorem 3.0.1.

• If $p = 0, q = -1, r = -2$, then the contact structure $\xi_{0,1,-2}$ on $L(2,1)$ and the link $\mathbb{L}_{0,-1,-2}$ describing $X_{0,-1,-2}$ are given in Figure 3.15. It is easy to check $c_1(\xi_{0,-1,-2}) = 0$, so we compute $d_3(\xi_{0,-1,-2})$:

The corresponding linking matrix is

$$\mathcal{L}_{0,-1,-2} = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

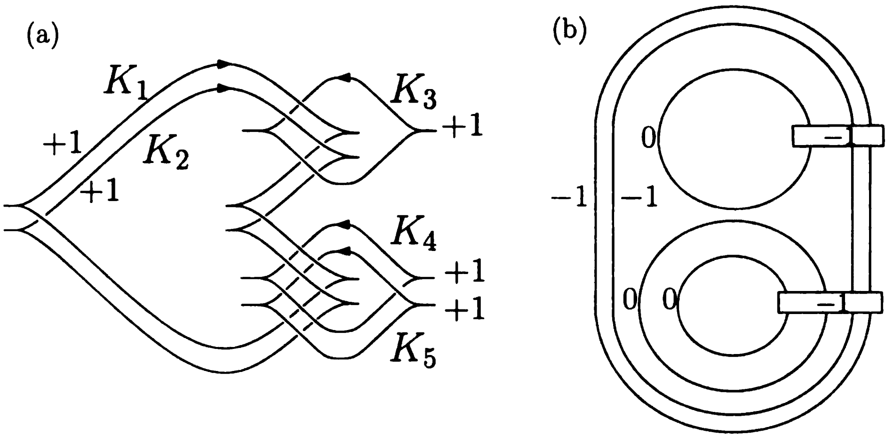

Figure 3.15. (a) $\xi_{0,-1,-2}$ on $S^3\#L(2,1) \approx L(2,1)$, (b) the link $\mathbb{L}_{0,-1,-2}$.

We diagonalize $\mathcal{L}_{0,-1,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,-1,-2}) = 1$. The system

$$\mathcal{L}_{0,-1,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)\ rot(K_4)\ rot(K_5)]^T = [1\ \ 1\ \ 0\ \ 0\ \ 0]^T$$

has the solution $[\mathbf{b}] = [0\ \ 0\ -1\ \ 0\ \ 0]$ which yields $c^2 = 0$. Also we have $\chi(X_{0,-1,-2}) = 4$ and $s = 3$. So we get $d_3(\xi_{0,-1,-2}) = 5/4 \neq 1/4 = d_3(\eta_2)$. Therefore, $\xi_{0,-1,-2}$ is not isotopic to η_2, and so $bn(\xi_{0,-1,-2}) = 3$ by Theorem 3.0.1.

- If $p = -1, q = 2, r = -2$, then the contact structure $\xi_{-1,2,-2}$ on $L(4,1)$ and the link $\mathbb{L}_{-1,2,-2}$ describing $X_{-1,2,-2}$ are given in Figure 3.16. We compute that $c_1(\xi_{-1,2,-2}) = 0$, so we need to find $d_3(\xi_{-1,2,-2})$.

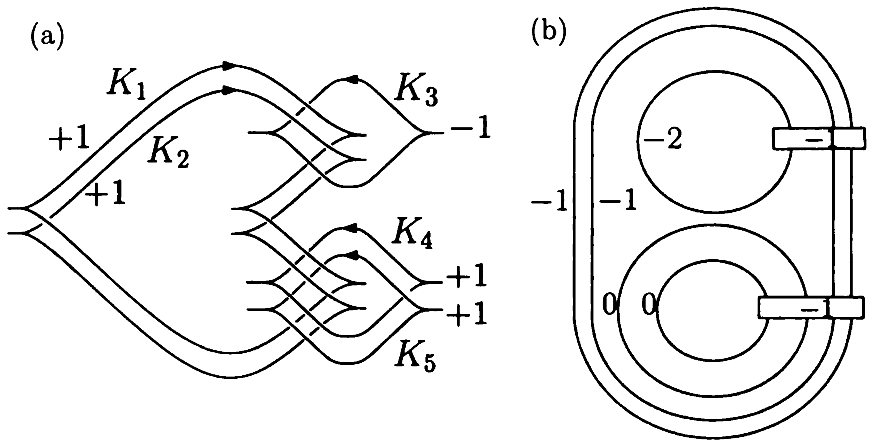

Figure 3.16. (a) $\xi_{-1,2,-2}$ on $L(4,1)$, (b) the link $\mathbb{L}_{-1,2,-2}$.

The corresponding linking matrix is

$$\mathcal{L}_{-1,2,-2} = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{-1,2,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{-1,2,-2}) = 1$. The system

$$\mathcal{L}_{-1,2,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)\ rot(K_4)\ rot(K_5)]^T = [1\ \ 1\ \ 0\ \ 0\ \ 0]^T$$

has the solution $[\mathbf{b}] = [1/2\ \ 1/2\ \ -1/2\ \ -1\ \ -1]$, so we compute $c^2 = 1$. Moreover, $\chi(X_{-1,2,-2}) = 6$ and $s = 4$. Then we get $d_3(\xi_{-1,2,-2}) = 1/2 \neq -1/4 = d_3(\eta_4)$. Therefore, $\xi_{-1,2,-2}$ is not isotopic to η_4, and so

$bn(\xi_{-1,2,-2}) = 3$ by Theorem 3.0.1.

p	q	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-1	0	$[0]$	$(r+1)/4$
-1	1	$0 \in \{0\}$	$1/2$
-1	2	$[2]$	$(r^2-3r+6)/(4r-8)$
-1	$q \leq -1$	$[r]$	$(q^2r+qr^2-q^2-r^2-q-r)/(4qr-4q-4r)$
-1	$q \geq 3$	$[r]$	$(q^2r+qr^2-q^2-r^2-6qr+5q+5r)/(4qr-4q-4r)$
0	0	$[0]$	$(r-1)/4$
0	1	$[0]$	$(-r+3)/4$
0	$q \leq -2$	$[0]$	$(q+r+2)/4$
0	$q \geq 2$	$[0]$	$(q+r-4)/4$
1	-2	$[r]$	$(r^2+r+2)/(4r+8)$
1	1	$[r]$	$(r^2-2r-1)/(4r+2)$
1	$q \leq -3$	$[r]$	$(q^2r+qr^2+q^2+r^2+4qr+3q+3r)/(4qr+4q+4r)$
1	$q \geq 2$	$[r]$	$(q^2r+qr^2+q^2+r^2-2qr-3q-3r)/(4qr+4q+4r)$

Table 3.7. The case $r > 2$ ($|p| < 2$ or $|q| < 2$).

In **Table 3.7**, we do not need any computation to find $bn(\xi)$: For any row, we can use Remark 3.2.1. For example, in the 1^{st} row, we have an overtwisted contact structure on the lens space $L(m, -1)$ for some $m \geq 1$. Therefore, the resulting contact manifold is not listed in Theorem 3.0.1, and hence we must have $bn(\xi) = 3$.

In **Table 3.8**, we need to compute the binding number $bn(\xi)$ for the rows 1, 3, 7, and 10. For the other rows, see Remark 3.2.1.

- If $p = 1, q = -2, r < -2$, $\xi_{1,-2,r}$ is an overtwisted contact structure on $L(|r+2|, 1)$. It is not hard to see that $c_1(\xi_{1,-2,r}) = [2] \in \mathbb{Z}_{|r|-2}$. Therefore,

p	q	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
1	-2	$[2]$	$(r^2+7r+14)/(4r+8)$
1	-1	$0 \in \{0\}$	$1/2$
1	0	$[0]$	$(-\lvert r\rvert+3)/4$
1	$q \leq -3$	$[\lvert r\rvert]$	$(q^2r+qr^2+q^2+r^2+10qr+9q+9r)/(4qr+4q+4r)$
1	$q \geq 1$	$[\lvert r\rvert]$	$(q^2r+qr^2+q^2+r^2+4qr+3q+3r)/(4qr+4q+4r)$
0	0	$[0]$	$(-\lvert r\rvert+5)/4$
0	-1	$[0]$	$(-\lvert r\rvert+7)/4$
0	$q \leq -2$	$[0]$	$(q+r+8)/4$
0	$q \geq 2$	$[0]$	$(q+r+2)/4$
-1	2	$[\lvert r\rvert]$	$(-r^2-3r+6)/(-4r+8)$
-1	-1	$[\lvert r\rvert]$	$(-r^2-6r+3)/(-4r+2)$
-1	$q \leq -2$	$[\lvert r\rvert]$	$(q^2r+qr^2-q^2-r^2+6qr-7q-7r)/(4qr-4q-4r)$
-1	$q \geq 3$	$[\lvert r\rvert]$	$(q^2r+qr^2-q^2-r^2-q-r)/(4qr-4q-4r)$

Table 3.8. The case $r < -2$ ($\lvert p\rvert < 2$ or $\lvert q\rvert < 2$).

we immediately get $bn(\xi_{1,q,-2}) = 3$ because $c_1(\eta_{\lvert r+2\rvert}) = 0$.

• If $p = 1, q = 0, r < -2$, the contact structure $\xi_{1,0,r}$ on $L(\lvert r\rvert, 1)$ and the link $\mathbb{L}_{1,0,r}$ are given in Figure 3.17. It is easy to see that $c_1(\xi_{1,0,r}) = 0 \in \mathbb{Z}_{\lvert r\rvert}$, so we need $d_3(\xi_{1,0,r})$: The corresponding linking matrix is on the left below. Diagonalize $\mathcal{L}_{1,0,r}$ to get the matrix on the right. Therefore,

$$\sigma(\mathcal{L}_{1,0,r}) = \lvert r\rvert - 1.$$

$$\mathcal{L}_{1,0,r} = \left[\begin{array}{ccccc:c} -1 & -2 & -2 & \dots & -2 & -1 \\ -2 & -1 & -2 & \dots & -2 & -1 \\ -2 & -2 & -1 & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & -1 & -2 & \\ -2 & -2 & \dots & -2 & -1 & -1 \\ \hdashline -1 & -1 & \dots & & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc:c} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 2 & 0 & \\ 0 & 0 & \dots & 0 & 1/2 & 0 \\ \hdashline 0 & 0 & \dots & & 0 & -3 \end{array}\right]$$

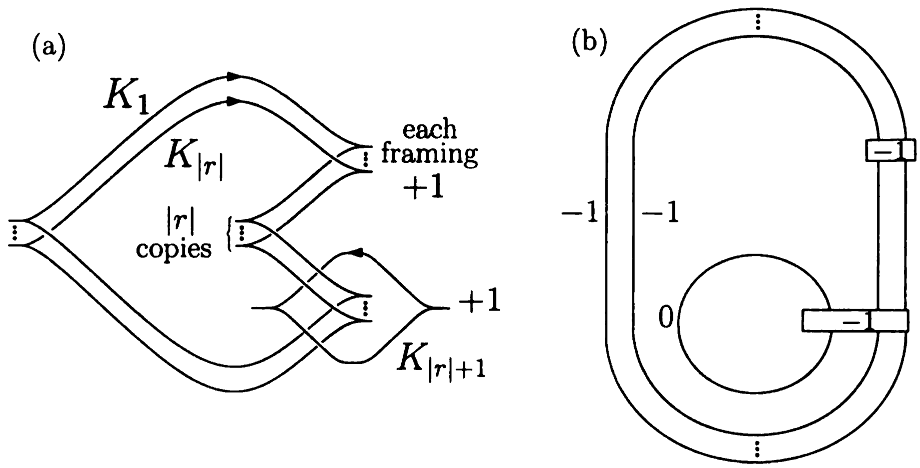

Figure 3.17. (a) $\xi_{1,0,r}$ on $L(|r|,1)$ for $r<-2$, (b) the link $\mathbb{L}_{1,0,r}$.

The system

$$\mathcal{L}_{1,0,r}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\cdots rot(K_{|q|+1})]^T = [1\cdots 1\ 0]^T$$

has the solution $[\mathbf{b}] = [0\cdots 0\ -1]$, and so we obtain $c^2 = 0$. Moreover, $\chi(X_{1,0,r}) = |r|+2$, and $s = |r|+1$. So we compute

$$d_3(\xi_{1,0,r}) = (-|r|+3)/4 = d_3(\eta_{|r|})$$

which implies that $\xi_{1,0,r}$ is isotopic to $\eta_{|r|}$ on $L(|r|,1)$. Thus, $bn(\xi_{1,0,r}) = 2$ by Theorem 3.0.1.

• If $p=0, q=-1, r<-2$, the contact structure $\xi_{0,-1,r}$ on $L(|r|,1)$ and the link $\mathbb{L}_{0,-1,r}$ are given in Figure 3.18. Again we have $c_1(\xi_{0,-1,r}) = [0] \in \mathbb{Z}_{|r|}$, so we need to find $d_3(\xi_{0,-1,r})$: We diagonalize $\mathcal{L}_{0,-1,r}$ and get the matrix on the right below. So, we conclude that $\sigma(\mathcal{L}_{0,-1,r}) = |r|-1$.

$$\mathcal{L}_{0,-1,r} = \left[\begin{array}{ccccc:ccc} -1 & -2 & -2 & \cdots & -2 & -1 & -1 & -1 \\ -2 & -1 & -2 & \cdots & -2 & -1 & -1 & -1 \\ -2 & -2 & -1 & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & -1 & -2 & & & \\ -2 & -2 & \cdots & -2 & -1 & -1 & -1 & -1 \\ \hdashline -1 & -1 & \cdots & & -1 & 0 & 0 & 0 \\ -1 & -1 & \cdots & & -1 & 0 & 0 & -1 \\ -1 & -1 & \cdots & & -1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc:ccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & 2 & 0 & & & \\ 0 & 0 & \cdots & 0 & 1/2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \cdots & & 0 & -3 & 0 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & 2 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & 0 & -1/2 \end{array}\right]$$

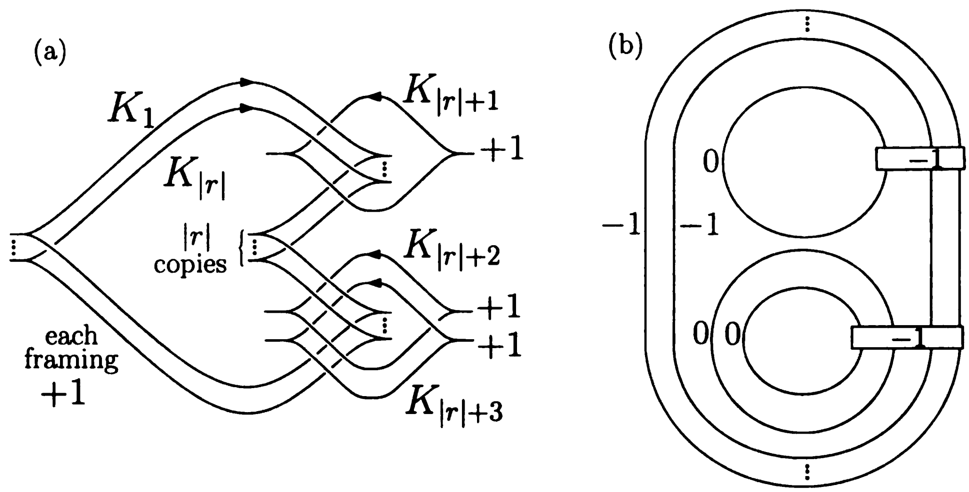

Figure 3.18. (a) $\xi_{0,-1,r}$ on $L(|r|,1)$ for $r < -2$, (b) the link $\mathbb{L}_{0,-1,r}$.

The system

$$\mathcal{L}_{0,-1,r}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1 \cdots 1\ 0\ 0\ 0]^T$$

has the solution $[\mathbf{b}] = [0 \cdots\ 0\ 1\ 0\ 0]$, so we get $c^2 = 0$. Also $\chi(X_{1,q,-2}) = |r| + 4$, and $s = |r| + 3$. So we compute $d_3(\xi_{0,-1,r}) = (-|r| + 7)/4 \neq (-|r| + 3)/4 = d_3(\eta_{|r|})$ implying that $\xi_{0,-1,r} \not\simeq \eta_{|r|}$ on $L(|r|,1)$. Hence,

$bn(\xi_{0,-1,r}) = 3$ by Theorem 3.0.1.

• If $p = -1, q = 2, r < -2$, we have $bn(\xi_{-1,2,r}) = 3$ because $c_1(\xi_{-1,2,r}) = [|r|] \in \mathbb{Z}_{|r|+2}$. We compute $c_1(\xi_{-1,2,r})$ as follows: We use the linking matrix $\mathcal{L}_{-1,2,r}$ to get the representation

$$\begin{aligned} H_1(M) &= \langle \mu_1, \mu_2, \cdots, \mu_{|r|+3} | \; \mathcal{L}_{-1,2,r}[\mu]^T_{|r|+3} = [\mathbf{0}]^T_{|r|+3} \rangle \\ &= \langle \mu_1 | \; (|r|+2)\mu_1 = 0 \rangle \cong \mathbb{Z}_{|r|+2}. \end{aligned}$$

Moreover, using the relations given by $\mathcal{L}_{-1,2,r}$ we have $\mu_1 = \mu_2 \cdots = \mu_{|r|}$ (μ_i's are the meridians as before). Therefore, we obtain

$$c_1(\xi_{-1,2,r}) = PD^{-1}(\mu_1 + \cdots + \mu_{|r|}) = PD^{-1}(|r|\mu_1) = [|r|] \in \mathbb{Z}_{|r|+2}.$$

To finish the proof, in each table above we find each particular case for (p, q, r) such that the corresponding contact structure $\xi_{p,q,r}$ has binding number 2. Note that the conditions on p, q, r given in the statement of the theorem excludes exactly these cases. This completes the proof of Theorem 3.0.4. □

3.3 Remarks on the remaining cases

Assume that $r = 0, \pm 1, |p| \geq 2, |q| \geq 2$. We list all possible contact structures in Table 3.9. These are the only remaining cases from which we still get lens spaces or their connected sums. Notice that we have already considered the cases $(-2, q, 1)$, and $(2, q, -1)$ in Tables 3.3 and 3.4, so we do not list them here.

r	p	q	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
0	$p \geq 2$	$q \geq 2$	$[0]$	$(p+q-4)/4$
0	$p \geq 2$	$q \leq -2$	$[0]$	$(p+q+2)/4$
0	$p \leq -2$	$q \leq -2$	$[0]$	$(p+q+8)/4$
1	$p \geq 2$	$q \geq 2$	$[-p]$	$(p^2q+pq^2+p^2+q^2-2pq-3p-3q)/(4pq+4p+4q)$
1	$p \geq 2$	$q < -2$	$[-p]$	$(-p^2q+pq^2+p^2+q^2+4pq+3p+3q)/(4pq+4p+4q)$
1	$p < -2$	$q < -2$	$[p]$	$(p^2q+pq^2+p^2+q^2+10pq+9p+9q)/(4pq+4p+4q)$
-1	$p > 2$	$q > 2$	$[-p]$	$(p^2q+pq^2-p^2-q^2-6pq+5p+5q)/(4pq-4p-4q)$
-1	$p > 2$	$q \leq -2$	$[-p]$	$(p^2q+pq^2-p^2-q^2-p-q)/(4pq-4p-4q)$
-1	$p \leq -2$	$q \leq -2$	$[p]$	$(p^2q+pq^2-p^2-q^2+6pq-7p-7q)/(4pq-4p-4q)$

Table 3.9. The case $r = 0, \pm 1$, $|p| \geq 2$, $|q| \geq 2$ ($bn(\xi) = 3$ in each row).

As we remarked at the beginning of the chapter (after Theorem 3.0.4) that one can obtain the complete list without any repetition: We first simply find all distinct homeomorphism types of the manifolds which we found in Table 3.2 through Table 3.9. Then on a fixed homeomorphism type we compare the pairs (c_1, d_3) coming from the tables to distinguish the contact structures.

Suppose now that M is a prime Seifert fibered manifold which is not a lens space. Then as we remarked before we have $|p| \geq 2, |q| \geq 2$, and $|r| \geq 2$. Then two such triples $(p,q,r), (p',q',r')$ give the same Seifert manifold Y if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'},$$

and (p', q', r') is a permutation of (p, q, r) (see [JN], for instance). Notice that we can drop the first condition in our case. Switching p and q does not change the contact manifold as we mentioned before. On the other hand, if we switch r and p (or r and q), we might have different contact structures on the same underlying topological manifold.

Another issue is that there are some cases where the first homology group $H_1(Y(p,q,r))$ is not finite. Indeed, consider the linking matrix $\mathcal{L}$ of the surgery diagram given on the right in Figure 3.2 as below.

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{pmatrix}$$

The determinant $\det(\mathcal{L}) = -r(p+q) - pq = 0$ implies that $r = -pq/(p+q)$. Thus, if $r \neq -pq/(p+q)$, then $H_1(Y(p,q,r))$ is finite, and so $d_3(\xi_{p,q,r})$ is still computable since $c_1(\xi_{p,q,r})$ is torsion. For instance, if $p \geq 2, q \geq 2, r \geq 2$ or $p \leq -2, q \leq -2, r \leq -2$, than $\det(\mathcal{L}) \neq 0$, and so we can distinguish the corresponding $\xi_{p,q,r}$ by computing the pair (c_1, d_3). Whereas if the sign of the one of p, q, r is different than the others', then we might have $\det(\mathcal{L}) = 0$. For instance, for the triples $(4,4,-2), (3,6,-2)$ and each nonzero integer multiples of them, $\det(A) = 0$. So more care is needed for these cases.

We would like to end the chapter by a sample computation. Assume that $\det(\mathcal{L}) \neq 0$, and that $r \leq 2, p \geq 2, q \leq 2$ (similar calculations apply for the other cases). We compute the first homology of $M \approx Y(p,q,r)$ as

$$\begin{aligned} H_1(M) &= \langle\, \mu_1, \mu_2, \cdots, \mu_{p+q+|r|} \,|\, \mathcal{L}_{p,q,r}[\mu]^T_{p+q+|r|} = [0]^T_{p+q+|r|} \,\rangle \\ &= \langle\, \mu_1, \mu_{|r|+1}, \mu_{p+|r|} \,|\, R_1, R_2, R_3 \,\rangle \end{aligned}$$

where the relations of the presentation are

$$\begin{array}{lrcrcrcl} R_1: & -(2|r|-1)\mu_1 & - & (p-1)\mu_{|r|+1} & - & (|q|+1)\mu_{p+|r|} & = & 0 \\ R_2: & & & p\,\mu_{|r|+1} & - & |q|\,\mu_{p+|r|} & = & 0 \\ R_3: & -|r|\,\mu_1 & - & p\,\mu_{|r|+1} & & & = & 0 \end{array}$$

While getting these relations, we also see that $\mu_1 = \mu_2 \cdots \mu_{|r|}$ (recall μ_i's are the meridians to the surgery curves in the family corresponding to r for $i = 1, \cdots, |r|$). Then using this presentation, and knowing that $c_1(\xi_{p,q,r}) = PD^{-1}(|r|\mu_1)$, we can evaluate (understand) $c_1(\xi_{p,q,r})$ in $H^2(M) \cong H_1(M)$. Now if $c_1(\xi_{p,q,r}) \in H^2(M)$ is a torsion class, then we can also compute $d_3(\xi_{p,q,r})$ as follows: By solving the corresponding linear system we get

$$c^2 = \frac{p|q||r|}{p|q| + p|r| - |q||r|}.$$

Moreover, we compute $\sigma(X_{p,q,r}) = \sigma(\mathcal{L}_{p,q,r}) = -p + |q| + |r|, \chi(X_{p,q,r}) = p + |q| + |r| + 1$, and $s = |q| + |r| + 1$. Hence, using Corollary 1.8.2, we obtain

$$d_3(\xi_{p,q,r}) = \frac{8pqr + p^2q + p^2r + 4pq^2 + 4qr^2 - pr^2 - q^2r - pq - pr - qr}{4pq + 4pr + 4qr}.$$

Chapter 4

Contact structures associated to four-punctured sphere

In this chapter, we consider the contact structures for $sg(\xi) = 0$ and $bn(\xi) \leq 4$. We first focus on the mapping class group of the four-punctured sphere in Section 4.1. In Section 4.2, we show certain families are holomorphically fillable. We show the overtwistedness of certain families in Section 4.3 where we also give alternative proofs of some results recently proved in [Y] (see Lemma 4.3.3, Remark 4.3.5, Lemma 4.3.6).

Let S be any surface with nonempty boundary as before. We will stick with the following convention: In $Aut(S, \partial S)$, we will multiply a new element from the right of the existing (pre-introduced) word although we compose the corresponding difeomorphisms of S from left. That is, if $\sigma, \gamma \in Aut(S, \partial S)$, then (denoting the corresponding diffeomorphisms with the same letters) we have

$$(\sigma \cdot \gamma)(x) = (\gamma \circ \sigma)(x) = \gamma(\sigma(x)) \text{ for } x \in S.$$

Let Σ be the four$-$punctured sphere obtained by deleting the interiors of four disks from the 2−sphere S^2 (see Figure 4.1). Let C_1, C_2, C_3, C_4 be the boundary components of Σ, and let a, b, c, d denote the simple closed curves parallel to the boundary components C_1, C_2, C_3, C_4, respectively. Also consider the simple closed curves e, f, g, h in Σ given as in Figure 4.1.

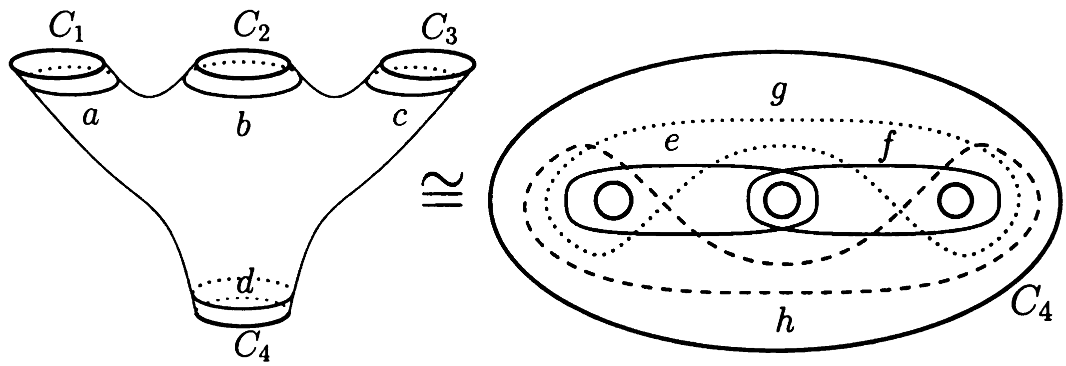

Figure 4.1. Four−punctured sphere Σ, and the simple closed curves.

Let $\phi \in Aut(\Sigma, \partial\Sigma)$ be any element. In Section 4.1, it will be clear that we can write

$$\phi = a^{r1} b^{r2} c^{r3} d^{r4} e^{m1} f^{n1} \cdots e^{ms} f^{ns}$$

for some integers m_i's and n_i's. Our main results are the following:

Theorem 4.0.1 *If* $min\{r_k\} \geq max\{-m, -n, 0\}$, *then* (M_ϕ, ξ_ϕ) *is holomorphically fillable, where* $\phi = a^{r1} b^{r2} c^{r3} d^{r4} e^{m1} f^{n1} \cdots e^{ms} f^{ns} \in Aut(\Sigma, \partial\Sigma)$ *and* $m = \Sigma_{i=1}^{s} m_i$ *and* $n = \Sigma_{i=1}^{s} n_i$.

Theorem 4.0.2 *The contact structure ξ_ϕ is overtwisted in the following cases:*

(1) $r_k < 0$ *for some* k,

(2) $r_k = 0$ *for some* k *and* $min\{m, n\} < 0$,

(3) $min\{r_k\} = 1$, $\{r_2 = 1$ *or* $r_4 = 1\}$, $min\{m, n\} < 0$ *and* $mn \geq 2$,

(4) $min\{r_k\} = 1$, $\{r_1 = 1$ *or* $r_3 = 1\}$, $min\{m, n\} < 0$ *and* $mn \geq 2$,

where $\phi = a^{r1}b^{r2}c^{r3}d^{r4}e^{m1}f^{n1}\cdots e^{ms}f^{ns} \in Aut(\Sigma, \partial\Sigma)$, $m = \Sigma_{i=1}^{s} m_i$ *and* $n = \Sigma_{i=1}^{s} n_i$.

4.1 Four – punctured sphere

For simplicity, we will denote the Dehn twist along any simple closed curve by the same letter we use for that curve.

Definition 4.1.1 *An element $\phi \in Aut(\Sigma, \partial\Sigma)$ is said to be in* ***reduced form*** *if there exists an unique integer $0 \leq s$ such that ϕ can be written as*

$$\phi = a^{r1}b^{r2}c^{r3}d^{r4}e^{m1}f^{n1}e^{m2}f^{n2}\cdots e^{ms}f^{ns}$$

where r_k, m_i, n_i are all integer for $1 \leq k \leq 4$, $1 \leq i \leq s$ with possibly m_1 or n_s zero.

Lemma 4.1.2 *Any element $\phi \in Aut(\Sigma, \partial\Sigma)$ can be written in reduced form.*

Proof: From braid group representation of full mapping class group, we know that the mapping class group $Aut(\Sigma, \partial\Sigma)$ can be generated by Dehn twists along the simple closed curves a, b, c, d, e, f, g, h given in Figure 4.1 (see [Bi] for details). Therefore, any element ϕ of $Aut(\Sigma, \partial\Sigma)$ can be written as a word consisting of only a, b, c, d, e, f, g, h and their inverses. Since $a, b, c, d,$ are in the center of $Aut(\Sigma, \partial\Sigma)$, we can bring them to any position we want. For the second part including e and f, we use the well-known *Lantern relation* (also known as *4-holed sphere relation*). In terms of our generators we will use two different Lantern relations. Namely, we have

$$gef = abcd \qquad \text{and} \qquad hfe = abcd.$$

These give $g = abcdf^{-1}e^{-1}$ and $h = abcde^{-1}f^{-1}$. Therefore, we can exchange any power of g and h in the word defining ϕ by some products of $a, b, c, d, e^{-1}, f^{-1}$ (and $a^{-1}, b^{-1}, c^{-1}, d^{-1}, e, f$ for negative powers of g and h). Combining (and canceling if there is any) the powers of e and f, and commuting the generators a, b, c, d, we get the reduced form of ϕ as claimed. $\square$

From now on, we will always consider the elements of $Aut(\Sigma, \partial\Sigma)$ in their reduced forms. In the following, we will say that two monodromy elements $h_1, h_2 \in Aut(S, \partial S)$ on the same surface S are *equivalent* if the contact manifolds $(M_{(S,h1)}, \xi_{(S,h1)}), (M_{(S,h2)}, \xi_{(S,h2)})$ are contactomorphic. We will denote this equivalence by $h_1 \sim h_2$.

Theorem 4.1.3 *Let $\phi = a^{r1}b^{r2}c^{r3}d^{r4}e^{m1}f^{n1}e^{m2}f^{n2}\cdots e^{ms}f^{ns} \in Aut(\Sigma, \partial\Sigma)$ be as before. Consider the element $\phi' = a^{r1}b^{r2}c^{r3}d^{r4}e^{m}f^{n}$ where $m = \Sigma_{i=1}^{s} m_i$ and $n = \Sigma_{i=1}^{s} n_i$. Then $\phi \sim \phi'$, i.e., there is a contactomorphism*

$$(M_{(\Sigma,\phi)}, \xi_{(\Sigma,\phi)}) \cong (M_{(\Sigma,\phi')}, \xi_{(\Sigma,\phi')}).$$

Proof: First consider only the last parts $\phi_0 = e^{m1}f^{n1}e^{m2}f^{n2}\cdots e^{ms}f^{ns}$ and $\phi_0' = e^{m}f^{n}$ of ϕ and ϕ'. We will show that $(M_{(\Sigma,\phi 0)}, \xi_{(\Sigma,\phi 0)})$ and $(M_{(\Sigma,\phi' 0)}, \xi_{(\Sigma,\phi' 0)})$ are contactomorphic (indeed they are isomorphic as open books). We will induct on s. All the equivalences in the induction follow from Theorem 1.1.1 and Theorem 1.5.1.

For $s = 2$, we have

$$e^{m1}f^{n1}e^{m2}f^{n2} \sim \overbrace{f^{-n1}\cdot e^{m1}f^{n1}}^{\sim e^{m1}} e^{m2}f^{n2}\cdot f^{n1} \sim e^{m1+m2}f^{n1+n2}$$

proving the first step of the induction. Now assume that the result is true for $s-1$, ie., $e^{m1}f^{n1}e^{m2}f^{n2}\cdots e^{m(s-1)}f^{n(s-1)} \sim e^{\overline{m}}f^{\overline{n}}$ where $\overline{m} = \Sigma_{i=1}^{s-1} m_i$ and $\overline{n} = \Sigma_{i=1}^{s-1} n_i$. Then

$$\begin{aligned} e^{m1}f^{n1}\cdots e^{m(s-1)}f^{n(s-1)}\cdot e^{ms}f^{ns} &\sim e^{\overline{m}}f^{\overline{n}}\cdot e^{ms}f^{ns} \\ &\sim \overbrace{f^{-\overline{n}}\cdot e^{\overline{m}}f^{\overline{n}}}^{\sim e^{\overline{m}}} e^{ms}f^{ns}\cdot f^{\overline{n}} \sim e^{\overline{m}+ms}f^{\overline{n}+ns} \end{aligned}$$

which proves the result for s. Therefore, we have showed that $\phi_0 \sim \phi_0'$. Now adding the same word $a^{r1}b^{r2}c^{r3}d^{r4}$ to both ϕ_0 and ϕ_0' give contactomorphic manifolds by Theorem 1.5.1, so $\phi \sim \phi'$ (Recall that $a^{r1}b^{r2}c^{r3}d^{r4}$ is a central element of $Aut(\Sigma, \partial\Sigma)$). $\Box$

4.2 Holomorphically fillable contact structures

In this short section, we will prove Theorem 4.0.1 using the lantern relation.

Proof: [Proof of Theorem 4.0.1] By Theorem 4.1.3, we will prove the statement for $\xi_{\phi'}$ where $\phi' = a^{r1}b^{r2}c^{r3}d^{r4}e^m f^n$. We assume that $m \leq n$, the other case is similar. If $m \geq 0$ then the result follows immediately from Theorem 1.4.2. Otherwise we first find a monodromy $\tilde{\phi}$ for which $(M_{(\Sigma,\phi')}, \xi_{(\Sigma,\phi')})$ is contactomorphic to $(M_{(\Sigma,\tilde{\phi})}, \xi_{(\Sigma,\tilde{\phi})})$ using Theorem 4.1.3 and then use the Lantern relations to write the monodromy $\tilde{\phi}$ as a product of positive Dehn twists as follows:

$$
\begin{aligned}
\tilde{\phi} &= a^{r1}b^{r2}c^{r3}d^{r4} \overbrace{(e^{-1}f^{-1})(e^{-1}f^{-1})\cdots(e^{-1}f^{-1})}^{-m \ times} f^{n-m} \\
&= a^{r1+m}b^{r2+m}c^{r3+m}d^{r4+m} \overbrace{(abcde^{-1}f^{-1})\cdots(abcde^{-1}f^{-1})}^{-m \ times} f^{n-m} \\
&= a^{r1+m}b^{r2+m}c^{r3+m}d^{r4+m} \overbrace{h \cdot h \cdots h}^{-m \ times} f^{n-m}.
\end{aligned}
$$

Theorem 1.4.2 implies that $\xi_{\tilde{\phi}}$ is holomorphically fillable. Hence, $\xi_{\phi'}$ and ξ_{ϕ} are also holomorphically fillable. $\Box$

4.3 Overtwisted contact structures

Among the contact structures ξ_ϕ with $\phi \in Aut(\Sigma, \partial\Sigma)$, we want to distinguish overtwisted ones. First, we prove three lemmas.

Lemma 4.3.1 *Let S be a planar hyperbolic surface with geodesic boundary $\partial S = \cup_{i=1}^{l} C_i$, $l \geq 4$. Suppose $h \in Aut(S, \partial S)$ and there is a properly embedded arc γ starting at $x \in C_i$, ending at C_j such that $h(\gamma)$ is to the left of γ at x and $i \neq j$. Then $(h \cdot D_\delta)(\gamma)$ is to the left of γ at $x \in C_i$ for any curve δ parallel to C_k with $k \neq i$.*

Proof: Isotoping if necessary, we may assume that γ and $h(\gamma)$ intersect minimally. We need to analyze two cases:

Case 1. Suppose $k \neq j$. Then we may assume $\gamma \cap \delta = \emptyset$, and so $h(\gamma) \cap \delta = \emptyset$. That is , D_δ fixes both γ and $h(\gamma)$. This implies that $D_\delta(h(\gamma)) = h(\gamma)$ is to the left of γ.

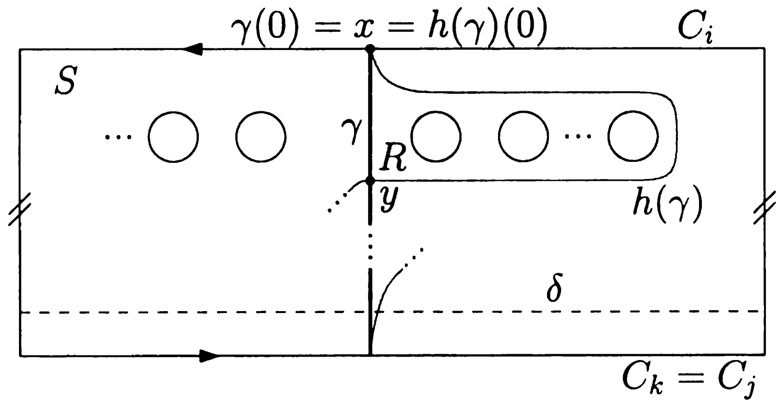

Figure 4.2. $h(\gamma)$ is to the left of γ (Left and right sides are identified).

Case 2. Suppose $k = j$. First note that $h \neq id_S$ since h is not a right veering. Therefore, there exists a region $R \subset S$ such that

1. R is an embedded disk punctured $r-$times for some $0 < r < m - 2$, and

2. $\partial R \subset \gamma \cup h(\gamma) \cup \partial S$.

Let $C_{i1}, \cdots, C_{ir}$ be the common components of ∂S and ∂R. We may assume that ∂R contains the common initial point x and the first intersection point y (of γ and $h(\gamma)$) coming right after x (See Figure 4.2).

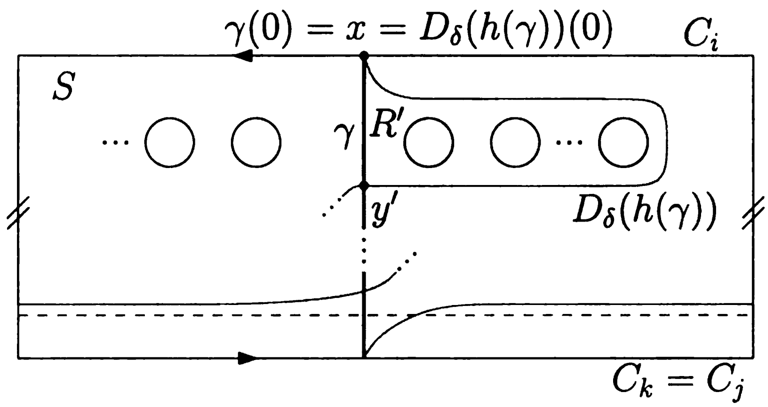

Figure 4.3. $D_\delta(h(\gamma))$ is to the left of γ (Left and right sides are identified).

Since the Dehn twist D_δ is isotopic to the identity outside of a small neighborhood of δ, the image $R' = D_\delta(R)$ is isotopic to R. In particular, $\partial R' \cap D_\delta(h(\gamma))$ is to the left of $\partial R' \cap \gamma$ (see Figure 4.3). Note that $D_\delta(h(\gamma))$ and γ are also intersecting minimally. Therefore, we conclude that $(h \cdot D_\delta)(\gamma) = D_\delta(h(\gamma))$ is to the left of γ. $\square$

The following corollary of Lemma 4.3.1 is immediate with the help of Theorem 1.7.2.

Corollary 4.3.2 *Let S be a planar hyperbolic surface with geodesic boundary $\partial S = \cup_{i=1}^{l} C_i$, $l \geq 4$. Suppose $h \in Aut(S, \partial S)$ is not right veering with respect to C_i for some i, and so the contact structure $\xi_{(S,h)}$ is overtwisted. Then the contact structure compatible with $(S, h \cdot {D_\delta}^k)$ is also overtwisted for any $k \in \mathbb{Z}_+$ and for any curve δ parallel to the boundary component which is different than C_i.* $\square$

Let us now interpret the notion of right-veering in terms of the circle at infinity as in [HKM1]. Let S be any hyperbolic surface with geodesic boundary ∂S. The universal cover $\pi : \tilde{S} \to S$ can be viewed as a subset of the Poincaré disk $D^2 = \mathbb{H}^2 \cup S^1_\infty$. Let C be a component of ∂S and L be a component of $\pi^{-1}(C)$. If $h \in Aut(S, \partial S)$, let $\tilde{h}$ be the lift of h that is the identity on L. The closure of $\tilde{S}$ in D^2 is a starlike disk. L is contained in $\partial \tilde{S}$. Denote its complement in $\partial \tilde{S}$ by L_∞. Orient L_∞ using the boundary orientation of $\tilde{S}$ and then linearly order the interval L_∞ via an orientation-preserving homeomorphism with $\mathbb{R}$. The lift $\tilde{h}$ induces a homeomorphism $h_\infty : L_\infty \to L_\infty$. Also, given two elements a, b in $Homeo^+(\mathbb{R})$, the group of orientation-preserving homeomorphisms of $\mathbb{R}$, we write $a \geq b$ if $a(z) \geq b(z)$ for all $z \in \mathbb{R}$ and $a > b$ if $a(z) > b(z)$ for all $z \in \mathbb{R}$. In this setting, an element h is rigth-veering with respect to C if $id \geq h_\infty$. Equivalently, if α

is any properly embedded curve starting at a point $\alpha(0) \in C$, and $\tilde{\alpha}$ is the lift of α starting at the lift $\tilde{\alpha}(0) \in L$ of x, then we have

$$h(\alpha) \text{ is to the right of } \alpha \iff \tilde{\alpha}(1) \geq h_\infty(\tilde{\alpha}(1))$$

Therefore, h is not right-veering with respect to C if there is an arc α starting at C such that we have $\tilde{\alpha}(1) < h_\infty(\tilde{\alpha}(1))$.

Lemma 4.3.3 *Let S be any hyperbolic surface with geodesic boundary ∂S. Suppose $h \in Aut(S, \partial S)$ and there is a properly embedded arc γ starting at $x \in C \subset \partial S$ such that $h(\gamma)$ is to the left of γ at x. Then $(h \cdot D_\sigma^{-1})(\gamma)$ is to the left of γ at $x \in C$ for any simple closed curve σ in S.*

Proof: Write σ for D_σ. Fix the identification of L_∞ with $\mathbb{R}$ as above. Consider the lift $\tilde{\gamma}$ and induced homeomorphisms $h_\infty, \sigma_\infty, \sigma_\infty^{-1} : L_\infty \to L_\infty$. Since $\sigma^{-1} \cdot \sigma = id_S$, we have

$$(\sigma^{-1} \cdot \sigma)_\infty = \sigma_\infty \circ \sigma_\infty^{-1} = (id_S)_\infty.$$

Therefore, σ_∞^{-1} must map any point in L_∞ to its left because σ is right-veering. In particular, $(h \cdot \sigma^{-1})_\infty(\tilde{\gamma}(1)) = \sigma_\infty^{-1}(h_\infty(\tilde{\gamma}(1)))$ is to the left of $h_\infty(\tilde{\gamma}(1))$ which is (by the assumption) to the left of $\tilde{\gamma}(1)$. That is, $(h \cdot \sigma^{-1})_\infty(\tilde{\gamma}(1)) > h_\infty(\tilde{\gamma}(1)) > \tilde{\gamma}(1)$. Hence, $(h \cdot \sigma^{-1})(\gamma)$ is to the left of γ.
$\square$

Corollary 4.3.4 *Let S be a planar hyperbolic surface with geodesic boundary $\partial S = \cup_{i=1}^{l} C_i$, $l \geq 4$. Suppose $h \in Aut(S, \partial S)$ is not right veering with* *respect to C_i for some i, and so the contact structure $\xi_{(S,h)}$ is overtwisted. Then the contact structure compatible with $(S, h \cdot {D_\sigma}^k)$ is also overtwisted for any $k \in \mathbb{Z}_-$ and for any simple closed curve σ in Σ.* $\square$

Remark 4.3.5 *The idea used in the proof of Lemma 4.3.3 gives a simple proof for Lemma 6. of [Y]. Moreover, the following lemma is given as Lemma 5 in [Y]. We want to give a different proof for it using the idea of the circle at infinity.*

Lemma 4.3.6 *Let S be a hyperbolic surface with geodesic boundary, and let $h \in Aut(S, \partial S)$ be a right-veering diffeomorphism. Then $h' = \sigma h \sigma^{-1}$ is right-veering for any $\sigma \in Aut(S, \partial S)$.*

Proof: Clearly, it is enough to consider the case when σ is a single Dehn twist. First, assume that σ is a positive Dehn twist. We need to show that h' is right-veering with respect to any boundary component of S. We will use the notations introduced in the previous paragraph. So fix the boundary component C, and an identification of L_∞ with $\mathbb{R}$ as above. Let α be any properly embedded curve in S starting at a point $\alpha(0) \in C$. Consider the lift $\tilde{\alpha}$ and induced homeomorphisms $h'_\infty, h_\infty, \sigma_\infty, \sigma_\infty^{-1} : L_\infty \to L_\infty$. From their definitions we have

$$h'_\infty(\tilde{\alpha}(1)) = \tilde{h}'(\tilde{\alpha}(1)) = \widetilde{\sigma h \sigma^{-1}}(\tilde{\alpha}(1)) = \tilde{\sigma}\tilde{h}\tilde{\sigma}^{-1}(\tilde{\alpha}(1)) = \sigma_\infty h_\infty \sigma_\infty^{-1}(\tilde{\alpha}(1))$$

Suppose that $\sigma_\infty^{-1}(\tilde{\alpha}(1)) = a \in L_\infty$ and $h_\infty(a) = b \in L_\infty$. Then since

$$\sigma_\infty(b) = ((\sigma^{-1})^{-1})_\infty(b) = (\sigma_\infty^{-1})^{-1}(b),$$

b must be mapped (by σ_∞) to a point in L_∞ which is to the right of $\tilde{\alpha}(1)$ as we illustrated in Figure 4.4. Equivalently, $\tilde{\alpha}(1) \geq \sigma_\infty h_\infty \sigma_\infty^{-1}(\tilde{\alpha}(1)) =$

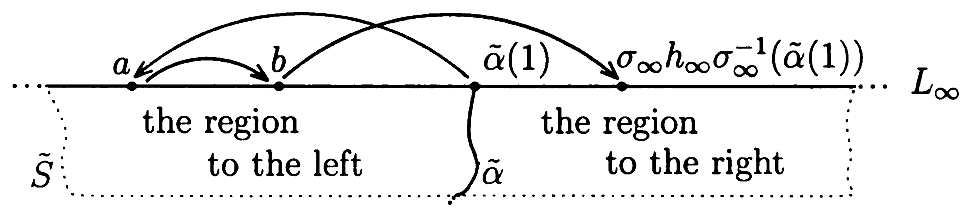

Figure 4.4. The point $\tilde{\alpha}(1) \in L_\infty \approx \mathbb{R}$, and how it is mapped to the right of itself.

$h'_\infty(\tilde{\alpha}(1))$ implying that h' is right-veering with respect to C. The proof of the case when σ is negative Dehn twist uses exactly the same argument, so we omit it. □

Now we can characterize the overtwisted structures stated in the introduction.

4.4 The proof of Theorem 4.0.2

Proof: By Theorem 4.1.3, we will prove the statements for $\xi_{\phi'}$ where

$$\phi' = a^{r1} b^{r2} c^{r3} d^{r4} e^m f^n.$$

To prove (1), consider the properly embedded curves $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ starting at the boundary components C_1, C_2, C_3, C_4, respectively, and their images under ϕ' as given in Figure 4.5. In all the pictures, we are assuming $m > 0$,

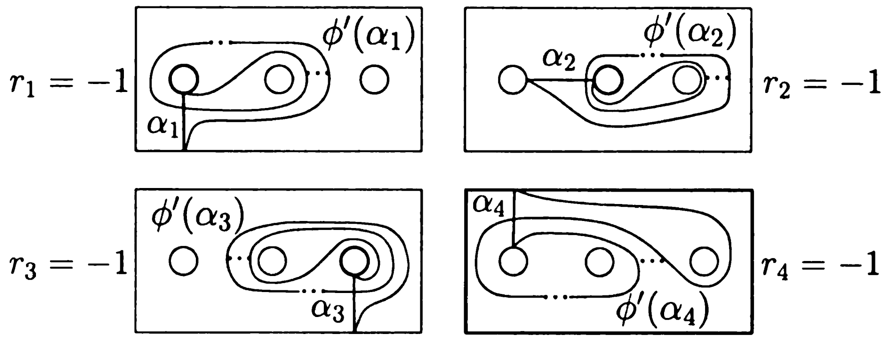

Figure 4.5. The curves α_k and their images under ϕ' in Σ.

$n > 0$, and $r_k = -1$ (otherwise the fact that ϕ' is not right-veering with respect to C_k is even more obvious). We can see from the pictures that if $r_k < 0$ for some k, then $\phi'(\alpha_k)$ is to the left of α_k , so ϕ' is not right-veering which implies by Theorem 1.7.2 that $\xi_{\phi'}$ is overtwisted. Note that in any picture in Figure 4.5, we are taking all the other r_k's to be zero. However, even if ϕ' has a factor of some positive power of Dehn twist along the boundary component other than C_k, $\phi'(\alpha_k)$ is still left to the α_k at their common starting point by Lemma 4.3.1, so $\xi_{\phi'}$ is overtwisted by Corollary 4.3.2.

To prove (2), consider the properly embedded curves $\beta_1, \beta_2, \beta_3, \beta_4$ starting at the boundary components C_1, C_2, C_3, C_4, respectively, and their images

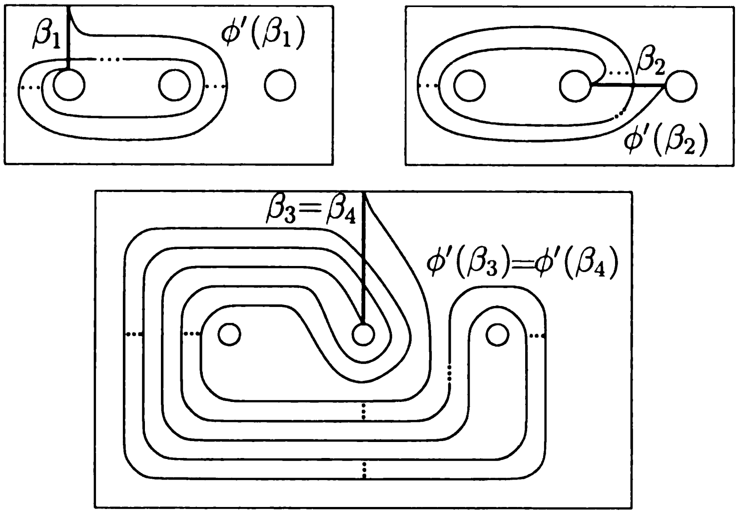

Figure 4.6. The curves β_k and their images under ϕ' in Σ.

under ϕ' as given in Figure 4.6. In all the pictures, we are assuming $m = -1$, $n > 0$, (again otherwise the fact that ϕ' is not right-veering with respect to C_k is even more obvious). We can see from the pictures that if $r_k = 0$ for some k, then $\phi'(\beta_k)$ is to the left of β_k , so ϕ' is not right-veering which implies again by Theorem 1.7.2 that $\xi_{\phi'}$ is overtwisted. Again, in all the pictures, we consider all the other r_k's to be zero, and if ϕ' has a factor of some positive power of Dehn twist along the boundary component other than C_k, $\phi'(\beta_k)$ is still left to the β_k at their common starting point by Lemma 4.3.1. Therefore, $\xi_{\phi'}$ is overtwisted by Corollary 4.3.2.

To prove (3), consider the curve γ running from C_2 to C_4 as in Figure 4.7. In the left picture each $r_k = 1, m = -2, n = -1$, and in the right one each $r_k = 1, m = -1, n = -2$. Clearly, the image $\phi'(\gamma)$ is to left of γ at both their common endpoints on C_2 and C_4, so $\xi_{\phi'}$ ($\phi' = abcde^{-2}f^{-1}$ or

$abcde^{-1}f^{-2}$) is overtwisted. In both cases, if we take r_1, r_3 and only one of r_2 and r_4 to be any positive integer, $\xi_{\phi'}$ is still overtwisted by Lemma 4.3.1 and Corollary 4.3.2. Moreover, for $m \leq -3, n \leq -3$ in both cases, $\xi_{\phi'}$ is overtwisted by Lemma 4.3.3 and Corollary 4.3.4.

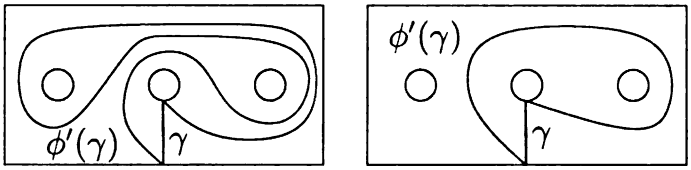

Figure 4.7. The curve γ and its images under two possible ϕ' in Σ.

The proof of (4) is similar to that of (3), so we will omit it. □

BIBLIOGRAPHY

BIBLIOGRAPHY

[AO] S. Akbulut, B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. **5** (2001), 319–334 (electronic).

[Bi] J. S. Birman, *Braids, links and mapping class groups*, Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

[DG] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 3, 583–598.

[DGS] F. Ding, H. Geiges and A. I. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math, **28** (2004) no. 1, 41–74.

[EG] Y. Eliashberg and M. Gromov *Convex symplectic manifolds*, Proc. of Symp. in Pure Math. **57** (1991), 135–162.

[Et1] J. B. Etnyre, *Planar open book decompositions and contact structures*, Int. Math. Res. Not. 2004, no.**79**, 4255–4267.

[Et2] J. B. Etnyre, *Lectures on open book decompositions and contact structures*, Floer homology, gauge theory, and low-dimensional topology, 103–141, Clay Math. Proc., 5, Amer. Math. Soc., Providence, RI, 2006.

[Et3] J. Etnyre, *Introductory Lectures on Contact Geometry*, Topology and geometry of manifolds (Athens, GA, 2001), 81–107, Proc. Sympos. Pure Math., 71, Amer. Math. Soc., Providence, RI, 2003.

[EO] J. Etnyre, and B. Ozbagci, *Invariants of Contact Structures from Open Books*, arXiv:math.GT/0605441, preprint 2006.

[Ga] D. Gabai, *Detecting fibred links in S^3*, Comment. Math. Helv., 61(4):519-555, 1986.

[Ge] H. Geiges, *Contact geometry*, Handbook of differential geometry. Vol. II, 315–382, Elsevier/North-Holland, Amsterdam, 2006.

[Gd] N Goodman, *Contact Structures and Open Books*, PhD thesis, University of Texas at Austin (2003)

[Gi] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the ICM, Beijing 2002, vol. 2, 405–414.

[Gm] R. E. Gompf, *Handlebody construction of Stein surfaces,* Ann. of Math. **148** (1998), 619–693.

[GS] R. E. Gompf, A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Math. **20**, Amer. Math. Soc., Providence, RI, 1999.

[HKM1] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. 169 (2007), no. 2, 427–449.

[HKM2] K. Honda, W. Kazez and G. Matić, *On the contact class in Heegaard Floer homology*, arXiv:math/0609734, preprint 2006.

[Ho] K. Honda, *On the classification of tight contact structures -I*, Geom. Topol. **4** (2000), 309–368 (electronic).

[JN] M. Jankins and W. D. Neumann, Lectures on Seifert manifolds, Brandeis Lecture Notes 2, Brandeis University (1981).

[MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 1995.

[LP] A. Loi, R. Piergallini, *Compact Stein surfaces with boundary as branched covers of B^4*, Invent. Math. **143** (2001), 325–348.

[Ly] H. Lyon, *Torus knots in the complements of links and surfaces*, Michigan Math. J. 27 (1980), 39-46.

[Or] P. Orlik, *Seifert Manifolds*, Lecture Notes in Mathematics (1972), Springer-Verlag, Berlin·Heidelberg·New York.

[OSt] B. Ozbagci, A. I. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Society Mathematical Studies, **13** (2004), Springer-Verlag, Berlin.

[OSz] P. Ozsváth, Z. Szabó, *Heegaard Floer homology and contact structures*, Duke Math. J. **129** (2005), no. 1, 39–61

[Pl] O. Plamenevskaya, *Contact structures with distinct Heegaard Floer invariants*, Math. Res. Lett., **11** (2004), 547-561.

[St] A. I. Stipsicz, *Surgery diagrams and open book decomposition of contact 3-manifolds*, Acta Math. Hungar, **108 (1-2)** (2005), 71-86.

[SW] S. Sarkar and J. Wang, *A combinatorial description of some Heegaard Floer homologies*, preprint (2006), math/0607777.

[TW] W. P. Thurston, H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.

[Y] E. Yilmaz, *A Note on Overtwisted Contact Structures*, arXiv:0712.4379, preprint 2008.