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ABSTRACT

Generalizations of the Reduced Distance in the Ricci Flow - Monotonicity and Applications

By

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The evolution of a Riemannian metric by the Ricci flow has been found to be very powerful in studying and classifying manifolds of certain curvature conditions. However, the flow typically develops singularities in finite time, which need to be understood. Quantities monotone in time are a common tool to study geometric flows near singularities.

We define a reduced distance function based at a point at the singular time of a Ricci flow on a complete n-dimensional manifold M. Our curvature bound assumption is weaker than the generic type I condition. We show that the corresponding reduced volume based at singular time is monotone along the flow. Since the quantity being constant implies that the flow is a gradient shrinking soliton, type I singularities can be modeled by those special solutions. We also show the monotonicity of the reduced volume arising from the reduced distance to a compact submanifold of M, and we similarly extend that notion to singular time.

To my parents,

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CHAPTER 1

Introduction

Given an $n-$dimensional Riemannian manifold M^n, Hamilton [16] introduced the evolution of an initial metric $g(0)$ on M^n through a family of metrics $g(t)$ according to

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)}.$$

Such a family $(M^n, g(t))$, $t \in [0, T)$ is called a Ricci flow. The analysis of this quasilinear second order (weakly) parabolic partial differential equation requires understanding the competition between the heat equation type (smoothing) behavior of the evolution and the development of singularities. In general, Ricci flows in dimensions $n \geq 3$ develop singularities and do not limit to a constant curvature metric (after renormalization). To classify manifolds in a particular case in dimension 4, Hamilton [19] introduced the concept of "Ricci flow with surgery" to continue the flow beyond singularities. Pursuing this idea in arbitrary dimensions without strong curvature assumptions requires a careful analysis of the singularities developing in Ricci flow. This is where one of Perelman's main contributions comes in. For any $p \in M^n$ and $0 < t_0$ less than the singular time T, the "reduced distance" l_{p,t_0} arises from a space-time version of Riemannian geometry adapted to the Ricci flow in all dimensions [26]. It is a locally Lipschitz function on $M^n \times [0, t_0]$ and satisfies important partial differential inequalities. Those imply that the "reduced volume based at (p, t_0)"

$$\tilde{V}_{p,t_0}(\bar{t}) := \int_{M^n} \left(4\pi(t_0 - \bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,t_0}(q,\bar{t})}\, dvol_{g(\bar{t})}(q)$$

is monotone nondecreasing in $\bar{t}$ along the Ricci flow on $[0, t_0]$. Moreover, $\tilde{V}_{p,t_0}(\bar{t})$ is constant in $\bar{t}$ if and only if $(M^n, g(t))$ is isometric to Euclidean space with the flat (non-evolving) metric.

The reduced volume is the key analytic tool to prove the "No Local Collapsing Theorem", which is essential to extract Cheeger-Gromov type limit flows from a sequence of rescaled Ricci flows around a singularity. For limit flows with nonnegative curvature operator arising from finite time singularities, Perelman uses the reduced distance again to obtain blow-down limits that are gradient shrinking solitons, i.e. only evolve by shrinking in scale and changing by diffeomorphisms generated by a gradient vector field. By classifying gradient shrinking solitons in dimension 3, this fully explains the structure of singularities and allows one to perform surgery to continue the flow [27].

Perelman's argument is 3-dimensional in several key steps. On the other hand, Hamilton conjectured that gradient shrinking solitons arise as blow-up solutions for type I singularities, i.e. as rescaling limits around singularities of Ricci flows satisfying the (generic) curvature bound

$$\sup_{M^n} |Rm_{g(t)}|_{g(t)} \leq \frac{C}{T-t}.$$

If such a rescaling limit is compact, the conjecture follows from [29]. The proof uses the monotonicity of Perelman's entropy functional: the scaling properties of the entropy imply that the functional is constant on the limit flow, which is precisely the case on gradient shrinking solitons.

Unlike the entropy functional, the reduced volume is defined for any complete Ricci flow (of any dimension). After presenting the necessary background in Chapter

2, we extend in Chapter 3 the reduced distance and volume to the singular time T to allow for gradient shrinking solitons other than flat Euclidean space to appear in the equality case of the monotonicity formula. To do this, we introduce a new (mild) curvature bound, which is more general than the type I assumption:

Definition 1.0.1 *Let $T < \infty$. A complete $n-$dimensional Ricci flow $(M^n, g(t))$ on $[0, T)$ is said to be of* ***type A*** *if there exist $C > 0$ and $r \in [1, \frac{3}{2})$ such that for all $t \in [0, T)$*

$$\sup_{M^n} |Rm_{g(t)}|_{g(t)} \leq \frac{C}{(T-t)^r}.$$

To our knowledge, it is not known whether there are closed maximal Ricci flows on $[0, T)$ that are not of type A. In [11], and in this dissertation in Section 3.3, we prove the following main

Theorem 1.0.2 *Let $(M^n, g(t))$ be a complete $n-$dimensional Ricci flow on $[0, T)$ of type A. Also let $p \in M^n$ and $t_i \nearrow T$. Then there exists a locally Lipschitz* ***reduced distance based at singular time*** *$l_{p,T} : M^n \times (0, T) \to \mathbb{R}$, which is a subsequential limit*

$$l_{p,t_i} \xrightarrow{C^0_{loc}(M^n \times (0,T))} l_{p,T}$$

and for all $(q, \bar{t}) \in M^n \times (0, T)$ satisfies the partial differential inequality

$$-\frac{\partial}{\partial \bar{t}} l_{p,T}(q, \bar{t}) - \Delta_{g(\bar{t})} l_{p,T}(q, \bar{t}) + |\nabla^{g(\bar{t})} l_{p,T}(q, \bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(T-\bar{t})} \geq 0$$

in the sense of distributions.

We then define for $\bar{t} \in (0, T)$ a **reduced volume based at singular time** (p, T) by

$$\tilde{V}_{p,T}(\bar{t}) := \int_{M^n} \left((4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,T}(q,\bar{t})} dvol_{g(\bar{t})}(q),$$

and show any reduced volume based at singular time is monotone nondecreasing in $\bar{t}$. This uses the differential inequality in the theorem. Moreover, $\tilde{V}_{p,T}(\bar{t}) \leq 1$ for all

$\bar{t} \in (0, T)$, and if $\tilde{V}_{p,T}(\bar{t})$ is constant in $\bar{t}$ then $(M^n, g(t))$ is a gradient shrinking soliton (see the precise statement in Section 3.4).

In Chapter 4 we illustrate how the usefulness of the reduced volume for analyzing the structure of singularities is increased by generalizing the reduced volume defined at smooth points in space-time to points at the singular time T. It also is a more direct analogue to Huisken's well-known monotonicity formula for the mean curvature flow [20]. We show how one can use the reduced volume monotonicity based at singular time to prove Hamilton's conjecture without the restriction of compact blow-ups. This has recently been done in [23], where similar estimates to define $l_{p,T}$ have been obtained under the assumptions that the Ricci flow is of type I (and $\kappa-$noncollapsed).

As a further generalization, in Chapter 5, we define and prove the monotonicity of the reduced volume arising from a reduced distance to a lower-dimensional submanifold S of M^n. It turns out that this quantity can also be extended to singular base time. Our discussion is motivated by examples of singular sets, which are of lower dimension than M^n.

The dissertation finishes with conclusions in Chapter 6. We mention an alternative definition of a reduced distance based at singular time, as well as future directions of further understanding and employing the generalizations of Perelman's reduced distance introduced in Chapters 3 and 5.

CHAPTER 2

Background on Ricci flow

In this chapter we will present the background material on Ricci flow needed in the subsequent chapters. After a discussion of important solutions to the Ricci flow, we will mainly focus on the reduced distance and its properties. The natural way the reduced distance arises is backward in time. While the forward time formulation chosen here is different from Perelman's original paper [26], it prepares the discussion in the later chapters. We finish the chapter with a discussion of the reduced volume and its monotonicity.

2.1 The equation and its properties

Let $(M^n, g(t))$, $t \in [0, T)$, $0 < T < \infty$ be a 1-parameter family of complete oriented connected n-dimensional Riemannian manifolds with bounded (sectional) curvature solving the equation

$$\frac{\partial g}{\partial t} = -2Ric_{g(t)}, \tag{2.1}$$

where $Ric_{g(t)}$ denotes the Ricci curvature tensor of the metric $g(t)$. We call a family with the assumptions as above a **Ricci flow on** $[0, T)$ throughout this thesis. In particular, we will always assume that the curvature for each $g(t)$ is bounded.

It follows from work by Hamilton [16], DeTurck [9], Shi [31][32] as well as Chen and Zhu [5] that for a given complete Riemannian manifold with bounded curvature (M^n, g_0), there exists a unique solution to this quasilinear second order weakly parabolic equation (2.1) with $g(0) = g_0$ on a time interval $[0, T)$ and the solution is maximal if and only if

$$\lim_{t \nearrow T} \sup_{M^n} |Rm_{g(t)}|_{g(t)} = \infty,$$

where $Rm_{g(t)}$ denotes the full curvature tensor of $g(t)$.

If we let $p \in M^n$, and $\{x^i\}$ normal coordinates near $p \in M^n$, then we can compute at p that

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} = 3\Delta g_{ij}, \tag{2.2}$$

where R_{ij} are the components of the Ricci tensor, and Δ is the coordinate Laplacian. While equation (2.2) only holds at p, and hence doesn't prove short-time existence, it indicates that the Ricci flow is the "heat equation for the Riemannian metric". This explains the many results of special curvature cone conditions that are preserved under the flow, as well as convergence to asymptotically round metrics under positivity assumptions, e.g. (2-)positive curvature operator [2]. In general, however, the non-linearity of the equation causes the formation of singularities in finite time, before the manifold tends to constant curvature. This happens even when the topology is simple.

2.2 Special solutions

In this section we will discuss special solutions to the Ricci flow (2.1). In particular, gradient shrinking solitons will play a key role in the further discussions, namely in the equality case of the monotone quantities.

The proper fixed points of the equation are **Ricci flat solutions:** Any g_0 with

$Ric_{g_0} \equiv 0$ gives rise to a solution $g(t) = g_0$. The metric does not evolve with time. A trivial example is the flat metric $g_{\mathbb{R}^n}$ on Euclidean space $\mathbb{R}^n$.

Because of the scaling and diffeomorphism invariance of the Ricci tensor, we next consider generalized fixed points of the equation.

2.2.1 Einstein solutions

Let (M^n, g_0), $n > 1$, be an Einstein manifold, i.e. g_0 satisfies

$$Ric_{g_0} = \frac{R_{g_0}}{n} g_0.$$

where R_{g_0} denotes the constant scalar curvature of g_0. Then

$$g(t) = (1 - \frac{2}{n} R_{g_0}\, t)\, g_0 \tag{2.3}$$

is a solution to the Ricci flow, where $g(t)$ is Einstein for each t. It only changes by scaling. Einstein manifolds are fixed points of the volume normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric_{g(t)} + \frac{2}{n} \frac{\int_M R_{g(t)} dvol_{g(t)}}{Vol(g(t))} g(t). \tag{2.4}$$

If $R_{g_0} > 0$, the solution (2.3) exists on the time interval $(-\infty, \frac{n}{2R_{g_0}})$. With $T := \frac{n}{2R_{g_0}}$, we then get

$$R_{g(t)} = \frac{n}{2(T-t)} \qquad \text{and} \qquad Ric_{g(t)} = \frac{1}{2(T-t)} g(t). \tag{2.5}$$

Example 2.2.1 *The* ***shrinking round*** *n*-***sphere*** $(S^n, g(t))$, $n > 1$, *on the maximal time interval* $(-\infty, T)$ *is given by*

$$g(t) = 2(n-1)(T-t) g_{S^n},$$

where g_{S^n} *denotes the standard round metric on* S^n *of sectional curvature 1.*

2.2.2 Gradient shrinking solitons

Motivated by Perelman's perspective in [26] and [27], we make the following

Definition 2.2.2 *A Riemannian manifold* (M^n, g) *is called a* ***gradient shrinking soliton*** *if there exists a* ***potential function*** $f : M \to \mathbb{R}$ *such that*

$$Ric_g + \nabla^g \nabla^g f - \frac{1}{2} g = 0, \tag{2.6}$$

where $\nabla^g \nabla^g f$ *denotes the covariant Hessian of* f. *We will typically write* (M^n, g, f).

Remark 2.2.3 *By making the definition as above, we assume that gradient shrinking solitons come at a certain scale: A more general definition requires* (M^n, g, f) *to satisfy*

$$Ric_g + \nabla^g \nabla^g f - \frac{\lambda}{2} g = 0,$$

where $\lambda > 0$. *We can always scale such an* f *and* g *to obtain (2.6).*

If $\nabla^g f$ determines a complete vector field, then we get a corresponding Ricci flow $g(t)$ on $(-\infty, T)$ with $g(T-1) = g$, called **gradient shrinking soliton in canonical form**, in the following way [7]: The 1-parameter family of diffeomorphisms ϕ_t of M^n generated by integrating the vector field $\frac{1}{T-t} \nabla^g f$ (with $\phi_{T-1} = id$) gives us

$$g(t) = (T-t) \phi_t^* g. \tag{2.7}$$

With the potential function $f(t) = \phi_t^* f$, the solution $g(t)$ satisfies the analogue of (2.6) on $(-\infty, T)$, i.e.

$$Ric_{g(t)} + \nabla^{g(t)} \nabla^{g(t)} f(t) - \frac{1}{2(T-t)} g(t) = 0. \tag{2.8}$$

Also, $f(t)$ satisfies

$$\frac{\partial f}{\partial t} = |\nabla^{g(t)} f(t)|^2_{g(t)}. \tag{2.9}$$

As we will always assume that (M^n, g) has bounded curvature, it follows from (2.6) that $\nabla^g f$ is at most linear, and in particular complete. Given a gradient shrinking soliton, we can therefore always obtain the corresponding gradient shrinking soliton in canonical form.

Remark 2.2.4 *We would like to make the following clarification, which we have not seen carefully discussed in the literature: If we have a Ricci flow $g(t)$ on $[0, T)$ which together with a 1-parameter family of functions $f(t)$ satisfies (2.8), this clearly defines a gradient shrinking soliton according to Definition 2.2.2 by considering the equation at $t = T - 1$. If $\nabla^{g(T-1)} f(T-1)$ is complete, we can conclude by uniqueness of the Ricci flow [5] that the corresponding Ricci flow in canonical form equals $g(t)$ on $[T-1, T)$. Moreover, if (2.9) holds for the original family of functions $f(t)$, then the given triple $(M^n, g(t), f(t))$ is the gradient shrinking soliton in canonical form constructed from $g(T-1)$ as explained above.*

Solitons are "self-similar" solutions, since they evolve only by scaling and diffeomorphism. Their existence is to be expected due to the diffeomorphism invariance of the Ricci flow equation (2.1). We can regard them as generalized fixed points, i.e. fixed points of the volume normalized Ricci (2.4) flow on the space of metrics modulo the diffeomorphism group.

Without going into a more detailed discussion, we mention that by changing the minus sign in (2.6) to "+" (or dropping the metric term), one can analogously define **gradient expanding** (or **steady**) **solitons**. Also, if the vector field generating the diffeomorphisms ϕ_t is not a gradient vector field, one gets a more general notion of solitons. It follows from [26] (in the compact case) and [23] (in the complete case) that any soliton is a gradient shrinking soliton for some (possibly different) vector field.

Example 2.2.5 *By (2.5) and Remark 2.2.4 we see that any Einstein solution with positive scalar curvature can be regarded as a gradient shrinking soliton in canonical form with* $f \equiv 0$ *(or* f *such that* ∇f *is a Killing vector field) and scaled so that* $\frac{R_0}{n} = \frac{1}{2}$.

Example 2.2.6 *Consider the non-evolving Ricci flow* $(\mathbb{R}^n, g(t) = g_{\mathbb{R}^n})$. *With* $f(x,t) = \frac{|x|^2}{4(T-t)}$, *we have*

$$\nabla\nabla f(t) = \frac{1}{2(T-t)} g_{\mathbb{R}^n},$$

which makes flat Euclidean space into a gradient shrinking soliton in canonical form, called ***Gaussian soliton****. It will arise in an important context in Section 2.4.*

Example 2.2.7 *Consider the* ***round shrinking cylinder***

$$\left(S^{n-k} \times \mathbb{R}^k, g(t) = 2(n-k-1)(T-t) g_{S^{n-k}} + g_{\mathbb{R}^k}\right),$$

where $n - k \geq 2$. *It follows easily from examples 2.2.1,2.2.5 and 2.2.6 that this is a solution to Ricci flow which is a gradient shrinking soliton in canonical form with potential*

$$f(\theta, x, t) = \frac{|x|^2_{g_{\mathbb{R}^k}}}{4(T-t)},$$

where $\theta \in S^{n-k}$ *and* $x \in \mathbb{R}^k$.

Example 2.2.8 *Example 2.2.7 is a special case of the following: We can construct gradient shrinking solitons as products of Einstein solutions* N^{n-k} *of positive scalar curvature with flat Euclidean space* $\mathbb{R}^k$. *It is an interesting question to ask when gradient shrinking solitons are of that form [28].*

We will now discuss an important equation satisfied by gradient shrinking solitons: Let $(M^n, g(t), f(t))$ be a gradient shrinking soliton in canonical form on $(-\infty, T)$. Taking the trace of equation (2.8) gives

$$R_{g(t)} + \Delta_{g(t)} f(t) - \frac{n}{2(T-t)} = 0. \tag{2.10}$$

We conclude from equations (2.9) and (2.10) that

$$-\frac{\partial f}{\partial t} - \Delta_{g(t)} f(t) + |\nabla^{g(t)} f(t)|^2_{g(t)} - R_{g(t)} + \frac{n}{2(T-t)} = 0. \tag{2.11}$$

In Perelman's point of view [26], if we let

$$u(x,t) := \big(4\pi(T-t)\big)^{-\frac{n}{2}} e^{-f(x,t)}, \tag{2.12}$$

where $x \in M^n$ and $t \in (-\infty, T)$, then a straight forward computation shows that (2.11) is equivalent to

$$\Box^*_{g(t)} u(x,t) = 0. \tag{2.13}$$

Here

$$\Box^*_{g(t)} = -\frac{\partial}{\partial t} - \Delta_{g(t)} + R_{g(t)}$$

denotes the formal adjoint of the heat operator $\Box_{g(t)} = \frac{\partial}{\partial t} - \Delta_{g(t)}$ under the Ricci flow. This observation will play a key role in the equality case of the reduced volume monotonicity.

We will also need the following fact: If $(M^n, g(t), f(t))$ is a gradient shrinking soliton in canonical form, then it follows from the contracted second Bianchi identity and (2.8) that there exists a constants $C(t)$, such that

$$R_{g(t)} + |\nabla^{g(t)} f(t)|^2_{g(t)} - \frac{f(t)}{T-t} = C(t). \tag{2.14}$$

2.3 Perelman's reduced distance

In this section we will briefly discuss Perelman's reduced distance for the Ricci flow. Contrary to [26], we will use forward time notation rather than backward time, since it will come more natural when we consider a sequence of different base-times in Chapter 3. The monotonicity of Perelman's reduced volume will then be described in Section 2.4. While the results are due to Perelman, there are by now several references detailing his work, e.g. [30], [21], [36], [6], [22].

Definition 2.3.1 *Let* $(M^n, g(t))$ *be a Ricci flow on* $[0,T)$. *For any curve* $\gamma : [\bar{t}, t_0] \to M$, *where* $0 < \bar{t} < t_0 < T$, *we define the* ***$\mathcal{L}-$length*** *of* γ *by*

$$\mathcal{L}(\gamma) := \int_{\bar{t}}^{t_0} \sqrt{t_0 - t}\left(|\dot{\gamma}(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))\right)dt.$$

Figure 2.1 illustrates this definition.

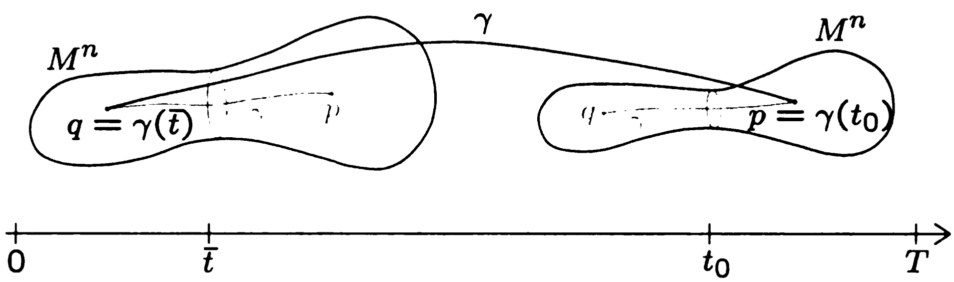

Figure 2.1: $\mathcal{L}-$length of a curve γ in M^n between $(q, \bar{t})$ and (p, t_0).

At a curve γ, the first variation of $\mathcal{L}$ with fixed endpoints is given by

$$\delta_Y \mathcal{L}(\gamma) = \langle 2\sqrt{t_0 - t}\,\dot{\gamma}(t), Y(t)\rangle \,\Big|_{\bar{t}}^{t_0} \tag{2.15}$$

$$- \int_{\bar{t}}^{t_0} 2\sqrt{t_0 - t}\langle \nabla^{g(t)}_{\dot{\gamma}(t)}\dot{\gamma}(t) - \frac{1}{2(t_0 - t)}\dot{\gamma}(t) - 2Ric_{g(t)}(\dot{\gamma}(t), \cdot)^{\#} - \frac{1}{2}\nabla^{g(t)}R_{g(t)}, Y(t)\rangle dt.$$

$Y(t)$ is the variational vector field along $\gamma(t)$ with $Y(\bar{t}) = Y(t_0) = 0$ and $\#$ denotes the metric dual, which we will omit. The Euler-Lagrange equation is called **$\mathcal{L}$-geodesic equation** and given by

$$\nabla^{g(t)}_{\dot{\gamma}(t)}\dot{\gamma}(t) - \frac{1}{2(t_0 - t)}\dot{\gamma}(t) - 2Ric_{g(t)}(\dot{\gamma}(t), \cdot) - \frac{1}{2}\nabla^{g(t)}R_{g(t)} = 0. \tag{2.16}$$

Smooth solutions to (2.16) are called **$\mathcal{L}$-geodesics.** Multiplying by $t_0 - t$, we can rewrite (2.16) to get rid of the unbounded coefficient of the second term:

$$\nabla^{g(t)}_{\sqrt{t_0 - t}\dot{\gamma}(t)}(\sqrt{t_0 - t}\dot{\gamma}(t)) - 2\sqrt{t_0 - t}Ric_{g(t)}(\sqrt{t_0 - t}\dot{\gamma}(t), \cdot) - \frac{1}{2}(t_0 - t)\nabla^{g(t)}R_{g(t)} = 0. \tag{2.17}$$

Using the direct method in the calculus of variations after a change of variables in $\mathcal{L}$, as well as the ODE (2.17) together with the assumed curvature bound and Shi's derivative estimates [32] (see also Appendix A), one obtains the following

Proposition 2.3.2 *For any $(q,\bar{t})$ and (p,t_0) with $p,q \in M^n$ and $0<\bar{t}<t_0<T$ there exists an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic $\gamma : [\bar{t},t_0] \to M$ with $\gamma(\bar{t}) = q$ and $\gamma(t_0) = p$. Moreover, the **initial vector** $\lim_{t\nearrow t_0} \sqrt{t_0-t}\,\dot{\gamma}(t) \in T_pM$ exists.*

Definition 2.3.3 *For $(q,\bar{t})$ and (p,t_0) as in the Proposition 2.3.2, we define*

*(i) **the $\mathcal{L}-$distance from** $(q,\bar{t})$ **to** (p,t_0)*

$$L_{p,t_0}(q,\bar{t}) := \inf\{\mathcal{L}(\gamma) \,|\, \gamma : [\bar{t},t_0] \to M, \gamma(\bar{t}) = q, \gamma(t_0) = p\},$$

*(ii) **the reduced distance based at** (p,t_0)*

$$l_{p,t_0}(q,\bar{t}) := \frac{L_{p,t_0}(q,\bar{t})}{2\sqrt{t_0-\bar{t}}},$$

(iii) and

$$v_{p,t_0}(q,\bar{t}) := (4\pi(t_0-\bar{t}))^{-\frac{n}{2}} e^{-l_{p,t_0}(q,\bar{t})}.$$

We will fix $(p,t_0) \in M^n \times (0,T)$ and regard L_{p,t_0}, l_{p,t_0} and v_{p,t_0} as functions on space-time $M^n \times (0,t_0)$.

By studying the second variation of $\mathcal{L}$ and obtaining a Laplacian comparison theorem for the reduced distance, Perelman derives the following three differential (in)equalities:

Theorem 2.3.4 *Fix $(p,t_0) \in M^n \times (0,T)$. Then for all $(q,\bar{t}) \in M^n \times (0,t_0)$ the following hold:*

$$-\frac{\partial}{\partial \bar{t}} l_{p,t_0}(q,\bar{t}) - \Delta_{g(\bar{t})} l_{p,t_0}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{p,t_0}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(t_0-\bar{t})} \geq 0, \tag{2.18}$$

or equivalently , $\qquad \Box^*_{g(\bar{t})} v_{p,t_0}(q,\bar{t}) \leq 0.$ (2.19)

$$-|\nabla^{g(\bar{t})} l_{p,t_0}(q,\bar{t})|^2_{g(\bar{t})} + R_{g(\bar{t})}(q) + \frac{l_{p,t_0}(q,\bar{t}) - n}{t_0 - \bar{t}} + 2\Delta_{g(\bar{t})} l_{p,t_0}(q,\bar{t}) \leq 0. \quad (2.20)$$

$$-2\frac{\partial}{\partial \bar{t}} l_{p,t_0}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{p,t_0}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{l_{p,t_0}(q,\bar{t})}{t_0 - \bar{t}} = 0. \quad (2.21)$$

Remark 2.3.5 *It can be shown (see e.g. [36]) that* $L_{p,t_0}(q,\bar{t})$ *is locally Lipschitz on* $M^n \times (0,t_0)$. *Hence, the same is true for* $l_{p,t_0}(q,\bar{t})$ *and* $v_{p,t_0}(q,\bar{t})$, *and the functions are differentiable a.e. in* q *and* $\bar{t}$. *The (in)equalities in Theorem 2.3.4 therefore hold in the barrier sense, or in the weak sense. In particular, they hold in the sense of distributions. For (2.18) this means that for any nonnegative* $\phi \in C^\infty_{cpt}(M^n \times (0,t_0))$

$$\int_0^T \int_M \Big((-\tfrac{\partial}{\partial t} l_{p,t_0} + |\nabla^{g(t)} l_{p,t_0}|^2_{g(t)} - R_{g(t)} + \frac{n}{2(t_0 - t)}) \phi + \nabla^{g(t)} l_{p,t_0} \cdot \nabla^{g(t)} \phi \Big) dvol_{g(t)} dt \geq 0. \quad (2.22)$$

In fact, $l_{p,t_0}(q,\bar{t})$ *is smooth away from a closed subset* $C \subset M^n \times (0,t_0)$, *and* $C \cap M^n \times \{t\}$ *has zero Lebesgue measure in* $M^n \times \{t\}$ *for every* $t \in (0,t_0)$.

2.4 Reduced volume and monotonicity

We can now define Perelman's reduced volume.

Definition 2.4.1 *Let* $(M^n, g(t))$ *be a Ricci flow on* $[0,T)$, *and* $(p,t_0) \in M^n \times (0,T)$. *Then for each* $\bar{t} \in (0,t_0)$ *we define the* ***reduced volume based at*** (p,t_0) *by*

$$\tilde{V}_{p,t_0}(\bar{t}) := \int_M v_{p,t_0}(q,\bar{t})\, dvol_{g(\bar{t})}(q) = \int_M \left(4\pi(t_0 - \bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,t_0}(q,\bar{t})}\, dvol_{g(\bar{t})}(q).$$

Example 2.4.2 *For the Gaussian soliton (Example 2.2.6) one computes*

$$l_{p,t_0}(q,\bar{t}) = \frac{|q-p|^2}{4(t_0 - \bar{t})},$$

and hence

$$\tilde{V}_{p,t_0}(\bar{t}) = \int_{\mathbb{R}^n} \left(4\pi(t_0 - \bar{t})\right)^{-\frac{n}{2}} e^{-\frac{|q-p|^2}{4(t_0 - \bar{t})}}\, dq = 1.$$

The monotonicity of the reduced volume along the Ricci flow (see also Figure 2.2) is now essentially a consequence of inequality (2.19) in Theorem 2.3.4:

Corollary 2.4.3 (Monotonicity of the reduced volume)

Under the same assumptions as in Definition 2.4.1, we have

(i) $\frac{d}{dt}\tilde{V}_{p,t_0}(\bar{t}) \geq 0$,

(ii) $\lim_{\bar{t}\nearrow t_0} \tilde{V}_{p,t_0}(\bar{t}) = 1$,

(iii) $\tilde{V}_{p,t_0}(\bar{t}_1) = \tilde{V}_{p,t_0}(\bar{t}_2)$ *for* $0 < \bar{t}_1 < \bar{t}_2 < t_0$ *if and only if* $(M^n, g(t))$ *is isometric to the Gaussian soliton* $(\mathbb{R}^n, g_{\mathbb{R}^n})$ *and* $\tilde{V}_{p,t_0}(\bar{t}) \equiv 1$.

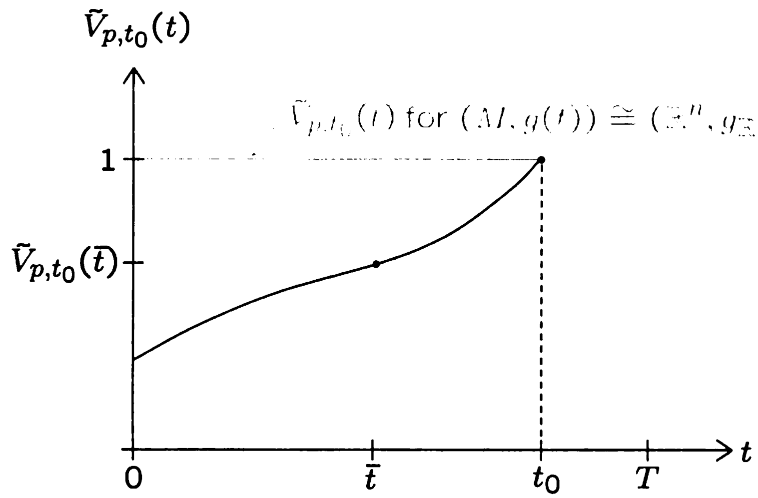

Figure 2.2: Reduced volume monotonicity based at (p, t_0) for some Ricci flow $(M^n, g(t))$ on $[0, T)$.

The reduced distance and reduced volume monotonicity are one of the most important analytic ingredients for Perelman's study of singularities in the Ricci flow [26]. First, the reduced volume is used to prove the "No Local Collapsing Theorem", which is necessary to guarantee Cheeger-Gromov-Hamilton convergence of rescaled flows to a singularity model. In a next step, the reduced distance is again the essential tool to construct asymptotic solitons for ancient solutions. Those help in the classification of singularities. In the next chapter, we will extend the reduced distance and volume to base points at singular time.

CHAPTER 3

Reduced distance and monotonicity based at singular time

3.1 Motivation

In the previous chapter, we defined the reduced volume $\tilde{V}_{p,t_0}$ for smooth points in space-time (p, t_0). We now discuss our main motivation to allow the base time t_0 in the reduced volume to be the singular time T of a Ricci flow.

Results of the type of Corollary 2.4.3 are known for other monotone quantities in geometric evolution equations, e.g. for Perelman's $\lambda-$ and $\mu-$functionals for the Ricci flow [26], or Huisken's monotonicity formula for the mean curvature flow [20]. In all of those, solitons arise in the equality case. Note also that the equality of Harnack expressions in the Ricci flow similarly identifies gradient expanding and steady solitons, see e.g. [17], [4], [24].

In the proof of Corollary 2.4.3 (iii), it is the fact that at time t_0 the curvature is bounded, that limits the equality case of the monotonicity to the Gaussian soliton.

If we are able to base the reduced distance and volume at singular time (p, T) for a maximal Ricci flow on $[0, T)$, depending on the base point $p \in M^n$, we expect to get gradient shrinking solitons other than the Gaussian soliton, whenever this generalized reduced volume is constant. We will prove this in Corollary 3.4.3. Before that, in Section 3.3, we will define a reduced distance based at singular time and study its properties. Our approach is as follows:

Let $(M^n, g(t))$ be a maximal Ricci flow on $[0, T)$. Let $t_i \nearrow T$ and $p \in M^n$. Then for all $(q, \bar{t}) \in M^n \times (0, T)$ the reduced distance $l_{p,t_i}(q, \bar{t})$ is defined for large enough i and the differential inequality (2.18) holds for each such i. This raises two questions:

(i) Does there exist a good limit $l_{p,T} := \lim_{t_i \nearrow T} l_{p,t_i}$?

(ii) Does a differential inequality analogous to (2.18) hold for $l_{p,T}$?

In Figure 3.1, we sketch the idea to define a reduced distance based at a point p at singular time T.

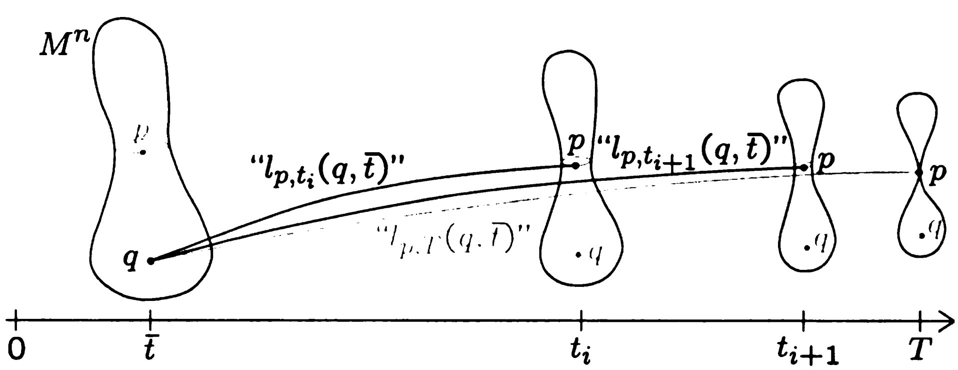

Figure 3.1: Convergence of the reduced distances $l_{p,t_i}(q, \bar{t})$ from q and p based at times t_i approaching the singular time T.

While the discussion is motivated by a base point p in the singular set (see Definition 4.2.2) in order to apply the quantity $l_{p,T}$ to the analysis of singularities as we discuss in Chapter 4, this will not be a requirement in our main result in this section. We will address the issue of the choice of base points in Sections 3.5.2 and 4.2. In general, to answer the two questions above positively, we need to mildly strengthen the bounded curvature assumption to a new curvature condition.

3.2 Type A Ricci flow solutions

Definition 3.2.1 *A Ricci flow $(M^n, g(t))$ on $[0,T)$ is said to be of* ***type A*** *if there exist $C > 0$ and $r \in [1, \frac{3}{2})$ such that for all $t \in [0,T)$*

$$\sup_{M^n} |Rm_{g(t)}|_{g(t)} \leq \frac{C}{(T-t)^r}.$$

Note that for $r = 1$ this includes the well-known **type I** condition. From the maximum principle for $|Rm_{g(t)}|_{g(t)}$ it follows that for a closed maximal Ricci flow on the interval $[0,T)$

$$\max_{M^n} |Rm_{g(t)}|_{g(t)} \geq \frac{1}{8(T-t)},$$

which implies that curvature blow-up with $r < 1$ is impossible. On the other hand, the type I condition is assumed to be generic. To our knowledge it is not known whether there are closed maximal Ricci flows which are not of type A. The only known example of a type II (i.e. not of type I) closed Ricci flow is the degenerate neckpinch [15], but its curvature blow-up rate is not known. In [8], an example of a type II Ricci flow on $\mathbb{R}^2$ is shown to blow up proportional to $\frac{1}{(T-t)^2}$.

Example 3.2.2 *Let $(M^n, g(t), f(t))$ be a complete gradient shrinking soliton in canonical form with bounded curvature. Then it follows directly from equation (2.7) that the flow is of type A, in fact of type I.*

3.3 Reduced distance based at singular time

In this section, we state and prove the main

Theorem 3.3.1 *Let $(M^n, g(t))$ be a Ricci flow on $[0, T)$ of type A. Also let $p \in M^n$ and $t_i \nearrow T$. Then there exists a locally Lipschitz function*

$$l_{p,T} : M^n \times (0, T) \to \mathbb{R},$$

which is a subsequential limit

$$l_{p,t_i} \xrightarrow{C^0_{loc}(M^n \times (0,T))} l_{p,T}$$

and which satisfies the differential inequality analogous to (2.18), i.e. for all $(q, \bar{t}) \in M^n \times (0, T)$

$$-\frac{\partial}{\partial \bar{t}} l_{p,T}(q, \bar{t}) - \Delta_{g(\bar{t})} l_{p,T}(q, \bar{t}) + |\nabla^{g(\bar{t})} l_{p,T}(q, \bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(T - \bar{t})} \geq 0 \quad (3.1)$$

holds in the sense of distributions. Equivalently,

$$\Box^*_{g(\bar{t})} v_{p,T}(q, \bar{t}) \leq 0, \quad (3.2)$$

where

$$v_{p,T}(q, \bar{t}) := \left(4\pi(T - \bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,T}(q, \bar{t})}.$$

Remark 3.3.2 *This theorem has been independently obtained in [23] under the type I assumption (and $\kappa-$ noncollapsedness) using very similar techniques.*

Theorem 3.3.1 allows us to make the following

Definition 3.3.3 *Under the assumptions of Theorem 3.3.1 we define*

$$l_{p,T} : M^n \times (0, T) \to \mathbb{R}$$

to be ***a reduced distance based at singular time*** *(p, T)* ***in the Ricci flow*** *$(M^n, g(t))$.*

We now give the proof of Theorem 3.3.1.

Proof. To simplify notation let $l_i := l_{p,t_i}(q,\bar{t})$ and $L_i := L_{p,t_i}(q,\bar{t})$. The proof will be in 3 steps.

1. First, we will derive a basic uniform bound on l_i on compact subsets $K = K_1 \times [a,b] \subset M^n \times (0,T)$. By definition of l_i it suffices to show such a bound for L_i on K. Let $\eta : [0,1] \to M$ be a $g(0)-$geodesic with $\eta(0) = q$ and $\eta(1) = p$. Fix $k \in (b,T)$. As depicted in Figure 3.2, consider the curve

$$\gamma(t) := \begin{cases} \eta(\frac{t-\bar{t}}{k-\bar{t}}) & t \in [\bar{t},k] \\ p & t \in (k,t_i]. \end{cases}$$

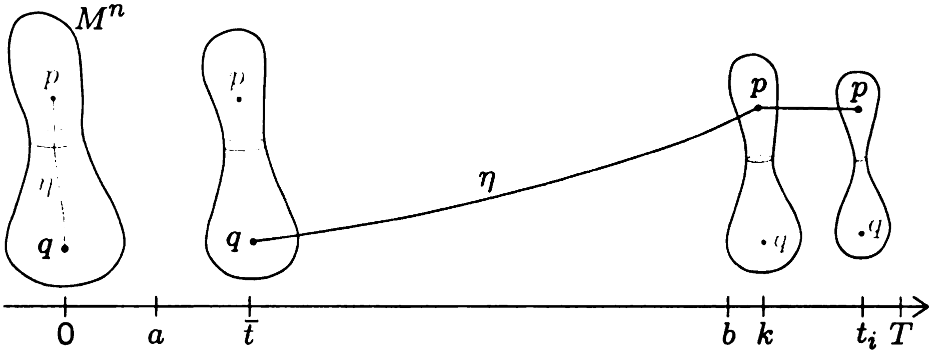

Figure 3.2: Construction of $\gamma(t)$ for a uniform bound on $L_{p,t_i}(q,\bar{t})$.

Since $|\eta'(s)|^2_{g(0)} = c$ for a constant c, the uniform equivalence of the metrics along the Ricci flow on $[0,k]$ yields a constant D such that $|\eta'(s)|^2_{g(t)} \leq D$. The type A assumption implies that there exist constants $C > 0$ and $r \in [1, \frac{3}{2})$ such that

$|R| \le \frac{C}{(T-t)^r}$. If $L_i(q,\bar{t}) \ge 0$, we get the following estimate:

$$|L_i(q,\bar{t})| \le \left|\int_{\bar{t}}^{t_i} \sqrt{t_i - t}\big(|\dot{\gamma}(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))\big)dt\right| \qquad (3.3)$$
$$\le \int_{\bar{t}}^{k} \frac{\sqrt{t_i - t}}{(k-\bar{t})^2}\left|\eta'\left(\frac{t-\bar{t}}{k-\bar{t}}\right)\right|^2_{g(t)} dt + C\int_{\bar{t}}^{T} \frac{\sqrt{t_i - t}}{(T-t)^r} dt$$
$$\le \frac{D\sqrt{T}}{k-b} + \frac{2C}{3-2r} T^{\frac{3}{2}-r} =: E.$$

If $L_i(q,\bar{t}) < 0$, we let $\gamma(t)$ be an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic such that $L_i(q,\bar{t}) = \mathcal{L}(\gamma)$. Then we get the same bound:

$$L_i(q,\bar{t}) = \int_{\bar{t}}^{t_i} \sqrt{t_i - t}\big(|\dot{\gamma}(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))\big)dt$$
$$\ge -\int_{\bar{t}}^{t_i} \sqrt{t_i - t}\,|R_{g(t)}(\gamma(t))|\,dt$$
$$\ge -E.$$

Thus, we have uniform bounds in i on any given compact subset K.

2. Next we derive uniform derivative bounds for L_i on compact subsets K of space-time as above. We first prove

Lemma 3.3.4 *Under the assumptions of Theorem 3.3.1 let $\gamma_i(t)$ be an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesics from $(q,\bar{t})$ to (p,t_i), where $(q,\bar{t}) \in K$. Then there exists a constant G independent of i such that for all $t \in [\bar{t},k]$*

$$|\sqrt{t_i - t}\,\dot{\gamma}_i(t)|^2_{g(t)} \le G.$$

Proof. Denote by $V_i(t) = \sqrt{t_i - t}\,\dot{\gamma}_i(t)$. Using the $\mathcal{L}-$geodesic equation (2.17) we compute

$$\begin{aligned}
\frac{d}{dt}|V_i(t)|^2_{g(t)} &= -2Ric(V_i(t),V_i(t)) + 2\langle \nabla_{\dot{\gamma}_i(t)} V_i(t), V_i(t)\rangle_t \\
&= -2Ric(V_i(t),V_i(t)) + \frac{2}{\sqrt{t_i - t}}\langle \nabla_{V_i(t)} V_i(t), V_i(t)\rangle_t \\
&= -2Ric(V_i(t),V_i(t)) \\
&\quad + \frac{2}{\sqrt{t_i - t}}\langle 2\sqrt{t_i - t}Ric_{g(t)}(V_i(t),\cdot)^{\#} + \frac{1}{2}(t_i - t)\nabla^{g(t)} R_{g(t)}, V_i(t)\rangle_t \\
&= 2Ric(V_i(t),V_i(t)) + \sqrt{t_i - t}\,\langle \nabla^{g(t)} R_{g(t)}, V_i(t)\rangle_t \\
&\leq 2\frac{C_1}{(T-t)^r}|V_i(t)|^2_{g(t)} + \frac{C_2}{(T-t)^{r-\frac{1}{2}}}|V_i(t)|_{g(t)}, \qquad (3.4)
\end{aligned}$$

where in the last inequality, since $t \geq \bar{t} > 0$, we used Shi's derivative estimates [32] (see also Appendix A) combined with the type A assumption to bound $\nabla^{g(t)} R_{g(t)}$. Note that C_1, C_2 are constants depending on the type A constant C, n and $\bar{t}$, but are independent of i.

Before integrating this ordinary differential inequality to conclude the proof of the lemma, we need to get uniform bounds on each $V_i(t)$ for some t in a compact set of time. By definition of $\mathcal{L}$, and using again the type A assumption we estimate

$$\begin{aligned}
\int_{\bar{t}}^{t_i} \frac{1}{\sqrt{t_i - t}}|V_i(t)|^2_{g(t)}dt &= \mathcal{L}(\gamma_i) - \int_{\bar{t}}^{t_i} \sqrt{t_i - t}\, R_{g(t)}(\gamma_i(t))dt \\
&\leq \mathcal{L}(\gamma_i) + \frac{2C}{3-2r}T^{\frac{3}{2}-r}.
\end{aligned}$$

Now the integral mean value theorem yields the existence of $\hat{t}_i \in [\bar{t}, k]$, such that

$$\begin{aligned}
\frac{1}{\sqrt{t_i - \hat{t}_i}}|V_i(\hat{t}_i)|^2_{g(\hat{t}_i)} &= \frac{1}{k-\bar{t}}\int_{\bar{t}}^{k} \frac{1}{\sqrt{t_i - t}}|V_i(t)|^2_{g(t)}dt \\
&\leq \frac{1}{k-\bar{t}}\int_{\bar{t}}^{t_i} \frac{1}{\sqrt{t_i - t}}|V_i(t)|^2_{g(t)}dt \\
&\leq \frac{1}{k-\bar{t}}\left(\mathcal{L}(\gamma_i) + \frac{2C}{3-2r}T^{\frac{3}{2}-r}\right),
\end{aligned}$$

or equivalently

$$\begin{aligned}|V_i(\hat{t}_i)|^2_{g(\hat{t}_i)} &= \frac{\sqrt{t_i-\hat{t}_i}}{k-\bar{t}}(\mathcal{L}(\gamma_i)+\frac{2C}{3-2r}T^{\frac{3}{2}-r})\\ &\leq \frac{\sqrt{T}}{k-b}(E+\frac{2C}{3-2r}T^{\frac{3}{2}-r}) =: F,\end{aligned}$$

since by choice of γ_i the bound (3.3) holds for $\mathcal{L}(\gamma_i)$.

W.l.o.g. we can assume that $|V_i(t)|_{g(t)} \geq 1$ and estimate (3.4) for $t \in [a,k]$ to get

$$\frac{d}{dt}|V_i(t)|^2_{g(t)} \leq \left(2\frac{C_1}{(T-k)^r}+\frac{C_2}{(T-k)^{r-\frac{1}{2}}}\right)|V_i(t)|^2_{g(t)} = C_3|V_i(t)|^2_{g(t)}.$$

Integrating this, we conclude that for all $t \in [\bar{t},k]$

$$|V_i(t)|^2_{g(t)} \leq Fe^{C_3(t-\hat{t}_i)} \leq Fe^{C_3T} =: G. \tag{3.5}$$

This proves the Lemma. □

To get the gradient bounds for L_i recall that from (2.15)

$$\nabla^{g(t)}L_i(q,\bar{t}) = -2\sqrt{t_i-\bar{t}}\,\dot{\gamma}_i(\bar{t}).$$

With Lemma 3.3.4 we obtain for $(q,\bar{t}) \in K$

$$|\nabla^{g(t)}L_i(q,\bar{t})|_{g(\bar{t})} = 2|\sqrt{t_i-\bar{t}}\,\dot{\gamma}_i(\bar{t})|_{g(\bar{t})} \leq 2\sqrt{G}. \tag{3.6}$$

For the time derivative bound for L_i, we compute

$$\begin{aligned}\frac{\partial}{\partial\bar{t}}L_i(q,\bar{t}) &= \frac{d}{d\bar{t}}L_i(q,\bar{t}) - \langle\nabla^{g(\bar{t})}L_i(q,\bar{t}),\dot{\gamma}_i(\bar{t})\rangle_{\bar{t}}\\ &= -\sqrt{t_i-\bar{t}}\,(|\dot{\gamma}_i(\bar{t})|^2_{g(\bar{t})}+R_{g(\bar{t})}(\gamma(\bar{t}))) + 2\sqrt{t_i-\bar{t}}\,|\dot{\gamma}(\bar{t})|^2_{g(\bar{t})}\\ &= \frac{1}{\sqrt{t_i-\bar{t}}}|\sqrt{t_i-\bar{t}}\,\dot{\gamma}_i(\bar{t})|^2_{g(\bar{t})} - \sqrt{t_i-\bar{t}}R_{g(\bar{t})}(\gamma(\bar{t})).\end{aligned}$$

Using the type A assumption and Lemma 3.3.4, we get the time derivative bound for $(q,\bar{t}) \in K$

$$|\frac{\partial}{\partial\bar{t}}L_i(q,\bar{t})| \leq \frac{G}{\sqrt{k-b}}+\frac{C}{(T-b)^{r-\frac{1}{2}}}. \tag{3.7}$$

Recall that by Remark 2.3.5 each l_i is locally Lipschitz on $M^n \times (0, t_i)$. The above bounds show that l_i are in fact uniformly locally bounded and Lipschitz on $M^n \times (0, T)$. Hence, there exists a locally Lipschitz function

$$l_{p,T} : M^n \times (0, T) \to \mathbb{R}$$

such that a subsequence, denoted l_i again, converges to a $l_{p,T}$ in $C^0_{loc}(M^n \times (0, T))$. This proves the first part of the theorem.

3. To prove that the differential inequality (3.1) holds in the sense of distributions, we first note that $l_i \in W^{1,2}_{loc}(M^n \times (0, T))$ and the bounds derived above imply

$$|l_i|_{W^{1,2}_{loc}(M^n \times (0,T))} < C.$$

W.l.o.g. we can assume that

$$l_i \rightharpoonup l_{p,T} \tag{3.8}$$

weakly in $W^{1,2}_{loc}(M^n \times (0, T))$. This implies for all $(V, \psi) \in W^{1,2}_{loc}(M^n \times (0, T), \mathbb{R}^{n+1})$

$$\int_0^T \int_M \nabla^{g(t)} l_i \cdot V + \frac{\partial}{\partial t} l_i \, \psi \, dvol_{g(t)} dt \to \int_0^T \int_M \nabla^{g(t)} l_{p,T} \cdot V + \frac{\partial}{\partial t} l_{p,T} \, \psi \, dvol_{g(t)} dt$$

In particular, if for nonnegative $\phi \in C^\infty_{cpt}(M^n \times (0, T))$, we let $\psi = -\phi$ and $V = \nabla^{g(t)}\phi$, we get the distributional convergence

$$\begin{aligned} &\int_0^T \int_M \nabla^{g(t)} l_i \cdot \nabla^{g(t)} \phi - \frac{\partial}{\partial t} l_i \, \phi \, dvol_{g(t)} dt \\ &\to \int_0^T \int_M \nabla^{g(t)} l_{p,T} \cdot \nabla^{g(t)} \phi - \frac{\partial}{\partial t} l_{p,T} \, \phi \, dvol_{g(t)} dt. \end{aligned}$$

Comparing with the distributional formulation (2.22), we see that we now only need to show that for all nonnegative $\phi \in C^\infty_{cpt}(M^n \times (0, T))$

$$\int_0^T \int_M |\nabla^{g(t)} l_i|^2_{g(t)} \, \phi \, dvol_{g(t)} dt \to \int_0^T \int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)} \, \phi \, dvol_{g(t)} dt.$$

It suffices to show for each $t \in (0,T)$ and nonnegative $\phi \in C^{\infty}_{cpt}(M^n)$

$$\int_M |\nabla^{g(t)} l_i|^2_{g(t)}\, \phi\, dvol_{g(t)} \to \int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)}\, \phi\, dvol_{g(t)}.$$

(3.8) implies the weak L^2 convergence of $\sqrt{\phi}\,\nabla^{g(t)} l_i \rightharpoonup \sqrt{\phi}\,\nabla^{g(t)} l_{p,T}$, and hence

$$\int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)}\, \phi\, dvol_{g(t)} \le \liminf_{i\to\infty} \int_M |\nabla^{g(t)} l_i|^2_{g(t)}\, \phi\, dvol_{g(t)}.$$

We now show the other direction

$$\limsup_{i\to\infty} \int_M |\nabla^{g(t)} l_i|^2_{g(t)}\, \phi\, dvol_{g(t)} \le \int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)}\, \phi\, dvol_{g(t)}$$

using an argument similar to Lemma 9.21 in [22]. We rewrite

$$\begin{aligned} &\limsup_{i\to\infty} \int_M \left(|\nabla^{g(t)} l_i|^2_{g(t)} - |\nabla^{g(t)} l_{p,T}|^2_{g(t)}\right) \phi\, dvol_{g(t)} \\ = \; &\limsup_{i\to\infty} \Big(\int_M \langle \nabla^{g(t)}(l_i - l_{p,T}), \phi \nabla^{g(t)} l_{p,T} \rangle_t\, dvol_{g(t)} \\ &+ \int_M \langle \nabla^{g(t)}(l_i - l_{p,T}), \phi \nabla^{g(t)} l_i \rangle_t\, dvol_{g(t)} \Big). \end{aligned} \tag{3.9}$$

We can approximate $\phi \nabla^{g(t)} l_{p,T}$ by $V_j \in C^{\infty}_{cpt}(M, \mathbb{R}^n)$ and conclude by weak L^2 convergence of $\nabla^{g(t)} l_i$ that the first integral goes to zero as $i \to \infty$. For the second integral, we first use the $C^0_{loc}-$ convergence of $l_i \to l_{p,T}$ to get a sequence $\epsilon_i \searrow 0$, such that on $\mathrm{supp}(\phi)$ we have

$$l_{p,T} - l_i + \epsilon_i > 0.$$

Then the second integral above equals

$$\limsup_{i\to\infty} \int_M \langle \nabla^{g(t)}(l_i - l_{p,T} - \epsilon_i), \phi \nabla^{g(t)} l_i \rangle_t\, dvol_{g(t)}.$$

We multiply Perelman's differential inequality (2.20) for l_i by ϕ and write it in the distributional sense for a nonnegative $\psi \in C^{\infty}_{cpt}(M^n)$:

$$-\int_M \langle \nabla^{g(t)}(\psi\phi), \nabla^{g(t)} l_i \rangle_t\, dvol_{g(t)} \le \int_M \frac{\psi\phi}{2}\left(|\nabla^{g(t)} l_i|^2_{g(t)} - R_{g(t)} - \frac{l_i - n}{t_i - t}\right) dvol_{g(t)}.$$

By approximation in $W^{1,2}$, we can take $\psi = l_{p,T} - l_i + \epsilon_i \geq 0$ to be only locally Lipschitz, i.e. conclude

$$\begin{aligned}
&\int_M \langle \nabla^{g(t)}((l_i - l_{p,T} - \epsilon_i)\phi), \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \\
&\leq \int_M \frac{(l_{p,T} - l_i + \epsilon_i)\phi}{2} (|\nabla^{g(t)} l_i|^2_{g(t)} - R_{g(t)} - \frac{l_i - n}{t_i - t}) dvol_{g(t)}.
\end{aligned}$$

Since the right-hand integrand is bounded on $\text{supp}(\phi)$ and

$$l_{p,T} - l_i + \epsilon_i \to 0$$

uniformly, we obtain

$$\limsup_{i\to\infty} \int_M \langle \nabla^{g(t)}((l_i - l_{p,T} - \epsilon_i)\phi), \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \leq 0.$$

Now, inserting the characteristic function $\chi_{supp(\phi)}$, we can rewrite

$$\begin{aligned}
&\int_M \langle \nabla^{g(t)}((l_i - l_{p,T} - \epsilon_i)\phi), \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \\
= &\int_M \langle \nabla^{g(t)}(l_i - l_{p,T} - \epsilon_i)\phi, \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \\
&+ \int_M \langle (l_i - l_{p,T} - \epsilon_i)\nabla^{g(t)}\phi, \chi_{supp(\phi)} \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \\
= &\int_M \langle \nabla^{g(t)}(l_i - l_{p,T}), \phi \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \\
&+ \int_M \frac{1}{2}(l_i - l_{p,T} - \epsilon_i)\big(|\nabla^{g(t)}\phi|^2_{g(t)} + |\chi_{supp(\phi)} \nabla^{g(t)} l_i|^2_{g(t)}\big) \, dvol_{g(t)}.
\end{aligned}$$

As before, the last integral tends to zero because of the uniform convergence of

$$l_{p,T} - l_i + \epsilon_i \to 0.$$

This implies that the second term in (3.9) satisfies

$$\limsup_{i\to\infty} \int_M \langle \nabla^{g(t)}(l_i - l_{p,T}), \phi \nabla^{g(t)} l_i \rangle_t \, dvol_{g(t)} \leq 0$$

and finishes the proof. □

The proof of Theorem 3.3.1 implies that the analogues of the differential (in)equalities (2.20) and (2.21) hold for a reduced distance based at singular time as well. We summarize this in the the following

Corollary 3.3.5 *For $l_{p,T}$ as in Theorem 3.3.1 and $(q,\bar{t}) \in M^n \times (0,T)$, the following three (in)equalities hold:*

$$-\frac{\partial}{\partial \bar{t}} l_{p,T}(q,\bar{t}) - \Delta_{g(\bar{t})} l_{p,T}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{p,T}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(T-\bar{t})} \geq 0, \quad (3.10)$$

$$\text{or equivalently ,} \qquad \Box^*_{g(\bar{t})} v_{p,T}(q,\bar{t}) \leq 0. \quad (3.11)$$

$$-|\nabla^{g(\bar{t})} l_{p,T}(q,\bar{t})|^2_{g(\bar{t})} + R_{g(\bar{t})}(q) + \frac{l_{p,T}(q,\bar{t}) - n}{T-\bar{t}} + 2\Delta_{g(\bar{t})} l_{p,T}(q,\bar{t}) \leq 0. \quad (3.12)$$

$$-2\frac{\partial}{\partial \bar{t}} l_{p,T}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{p,T}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{l_{p,T}(q,\bar{t})}{T-\bar{t}} = 0. \quad (3.13)$$

3.4 Reduced volume monotonicity based at singular time

Definition 3.4.1 *Let $(M^n, g(t))$ be a Ricci flow on $[0,T)$ of type A. Let $p \in M^n$, $t_i \nearrow T$, and $l_{p,T}$ and $v_{p,T}$ as in Theorem 3.3.1. Then we define **a reduced volume based at singular time** (p,T) by*

$$\tilde{V}_{p,T}(\bar{t}) := \int_M v_{p,T}(q,\bar{t}) dvol_{g(\bar{t})}(q) = \int_M \left((4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,T}(q,\bar{t})} dvol_{g(\bar{t})}(q).$$

Remark 3.4.2 *The finiteness of $\tilde{V}_{p,T}(\bar{t})$ for any Ricci flow $(M^n, g(t))$ of type A and any fixed $\bar{t} \in (0,T)$ follows from Fatou's lemma and the finiteness of Perelman's reduced volume as follows:*

$$\begin{aligned}
\tilde{V}_{p,T}(\bar{t}) &= \int_M \left((4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-\lim_{t_i \nearrow T} l_{p,t_i}(q,\bar{t})} dvol_{g(\bar{t})}(q) \\
&= \int_M \lim_{t_i \nearrow T} \left(\left((4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,t_i}(q,\bar{t})} \right) dvol_{g(\bar{t})}(q) \\
&\leq \liminf_{t_i \nearrow T} \underbrace{\tilde{V}_{p,t_i}(\bar{t})}_{\leq 1} \\
&\leq 1.
\end{aligned}$$

Now we state the analogue of Perelman's monotonicity (Corollary 2.4.3) for the reduced volume based at singular time. Compare also Figure 3.3.

Corollary 3.4.3 (Monotonicity based at singular time)

Under the assumptions as in Definition 3.4.1 we have

(i) $\frac{d}{dt}\tilde{V}_{p,T}(\bar{t}) \geq 0$,

(ii) $\lim_{\bar{t}\nearrow T}\tilde{V}_{p,T}(\bar{t}) \leq 1$,

(iii) If $\tilde{V}_{p,T}(\bar{t}_1) = \tilde{V}_{p,T}(\bar{t}_2)$ *for* $0 < \bar{t}_1 < \bar{t}_2 < T$, *then* $(M^n, g(t), l_{p,T}(\cdot, t))$ *is a gradient shrinking soliton in canonical form.*

Remark 3.4.4 *Similar results have also been obtained in [23] under the type I assumption (and* $\kappa-$ *noncollapsedness).*

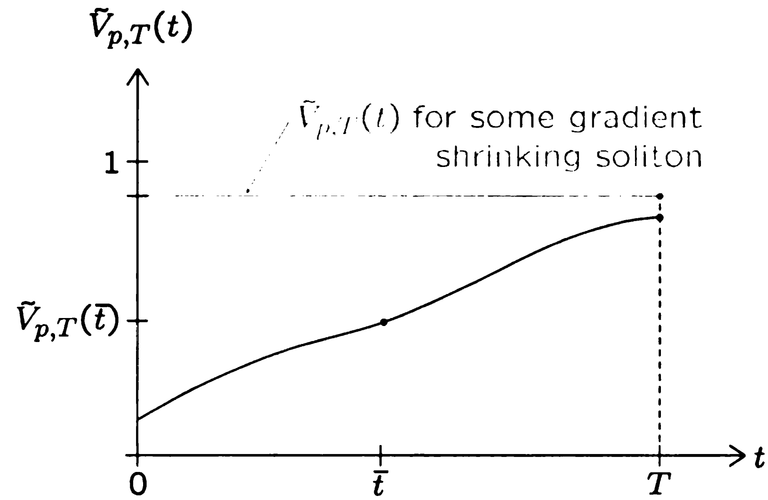

Figure 3.3: Reduced volume monotonicity based at singular time (p, T) for some Ricci flow $(M^n, g(t))$ on $[0, T)$.

Proof. (i) If M^n is compact and $l_{p,T}$ is smooth the proof follows directly from (3.2) in Theorem 3.3.1:

$$\begin{aligned}\frac{d}{d\bar{t}}\tilde{V}_{p,T}(\bar{t}) &= \int_M (\frac{\partial}{\partial \bar{t}} v_{p,T} - R) dvol_{g(\bar{t})} \\ &= \int_M (\frac{\partial}{\partial \bar{t}} v_{p,T} + \Delta v_{p,T} - R) dvol_{g(\bar{t})} \\ &= \int_M -\Box^* v_{p,T} dvol_{g(\bar{t})} \\ &\geq 0.\end{aligned} \tag{3.14}$$

In the general case, we need to justify the differentiation under the integral and the adding in of the Laplacian term. For both arguments, we need to bound

$$\int_M e^{-l_{p,T}} |l_{p,T}| dvol_{g(\bar{t})} < \infty \tag{3.15}$$

for fixed time $\bar{t} \in (0, T)$. From Remark 3.4.2, we know that

$$\int_M e^{-l_{p,T}} dvol_{g(\bar{t})} < \infty, \quad \text{and hence} \quad \int_M e^{-|l_{p,T}|} dvol_{g(\bar{t})} < \infty.$$

Since the proof of the finiteness of Perelman's reduced volume V_{p,t_i} relies on comparison of the reduced distance $l_{p,ti}$ with the square of the distance function, we also get

$$\int_M e^{-\frac{1}{2} l_{p,T}} dvol_{g(\bar{t})} < \infty, \quad \text{and hence} \quad \int_M e^{-\frac{1}{2}|l_{p,T}|} dvol_{g(\bar{t})} < \infty.$$

Now let $N := \{q \in M \,|\, l_{p,T}(q, \bar{t}) \geq 0\}$. Then

$$\int_M e^{-l_{p,T}} |l_{p,T}| dvol_{g(\bar{t})} = \int_{M-N} e^{-l_{p,T}} |l_{p,T}| dvol_{g(\bar{t})} + \int_N e^{-|l_{p,T}|} |l_{p,T}| dvol_{g(\bar{t})},$$

where the second term is finite since $\frac{1}{2}|l_{p,T}| \leq e^{\frac{1}{2}|l_{p,T}|}$. The first integral is bounded because the type A curvature bound yields an upper bound on $|l_{p,T}|$ on N by dropping the energy part in the $\mathcal{L}-$functional. This proves (3.15).

The proof of Lemma 3.3.4 implies that in fact there exist constants C_1 and C_2 (depending on $\bar{t}$) such that

$$|\nabla^{g(\bar{t})} l_{p,T}|^2_{g(\bar{t})} \leq C_1 |l_{p,T}| + C_2.$$

(Note that $L_{p,T}$ gets bounded on compact sets by the constant E in that proof.) This implies

$$\int_M e^{-l_{p,T}} |\nabla^{g(\bar{t})} l_{p,T}|^2_{g(\bar{t})} dvol_{g(\bar{t})} < \infty. \tag{3.16}$$

We lastly show that also

$$\int_M e^{-l_{p,T}} |\frac{\partial}{\partial \bar{t}} l_{p,T}| dvol_{g(\bar{t})} < \infty. \tag{3.17}$$

The differential equality (3.13)

$$-2\frac{\partial}{\partial \bar{t}} l_{p,T} + |\nabla^{g(\bar{t})} l_{p,T}|^2_{g(\bar{t})} - R_{g(\bar{t})} + \frac{l_{p,T}}{T - \bar{t}} = 0$$

can be used to bound $|\frac{\partial}{\partial \bar{t}} l_{p,T}|$. Then (3.17) follows from (3.15) and (3.16) and the bounded curvature assumption.

Now we can justify what corresponds to the adding in of the Laplacian term in (3.14) in the distributional setting. As in (2.22), Theorem 3.3.1 implies that for fixed $\bar{t} \in (0, T)$ and for any nonnegative $\phi \in C^\infty_{cpt}(M^n)$

$$\int_M \Big(\big(-\frac{\partial}{\partial \bar{t}} l_{p,T} + |\nabla^{g(\bar{t})} l_{p,T}|^2_{g(\bar{t})} - R_{g(\bar{t})} + \frac{n}{2(T - \bar{t})} \big) \phi + \nabla^{g(\bar{t})} l_{p,T} \cdot \nabla^{g(\bar{t})} \phi \Big) dvol_{g(\bar{t})} \geq 0.$$

Note that

$$\nabla^{g(\bar{t})} (e^{-l_{p,T}}) = -e^{-l_{p,T}} \nabla^{g(\bar{t})} l_{p,T},$$

so using the bounds for the integrals proved above and approximating $\phi = e^{-l_{p,T}}$ by $\phi_j \in C^\infty_{cpt}(M^n)$, we conclude

$$\int_M \big(-\frac{\partial}{\partial \bar{t}} l_{p,T} - R_{g(\bar{t})} + \frac{n}{2(T - \bar{t})} \big) e^{-l_{p,T}} dvol_{g(\bar{t})} \geq 0.$$

Up to the factor of $\big(4\pi(T - \bar{t})\big)^{-\frac{n}{2}}$ this is the integral we obtain when differentiating the integrand in the reduced volume based at (p, T). To conclude (i), we only have to justify the differentiation under the integral sign. We will do so using the standard dominated convergence argument. We bound, at $\bar{t} \in (0, T)$, the reduced volume integrand difference quotient $D(q,\bar{t},h)$, rewritten in integral form, by an integrable function. Note that

$$D(q,\bar{t},h)=\frac{1}{h}\int_0^h\Big(-\frac{\partial}{\partial \bar{t}}l_{p,T}(q,\bar{t}+s)-R_{g(\bar{t}+s)}(q)$$

$$+\frac{n}{2(T-\bar{t}-s)}\Big)\big(4\pi(T-\bar{t}-s)\big)^{-\frac{n}{2}}e^{-l_{p,T}(q,\bar{t}+s)}\frac{dvol_{g(\bar{t}+s)}(q)}{dvol_{g(\bar{t})}(q)}ds$$

(compare e.g. [6] for the argument in Perelman's case). Now from Remark 3.4.2 and the bound (3.17) as well as the exponential growth bound of the volume form under Ricci flow, which makes the quotient

$$\frac{dvol_{g(\bar{t}+s)}(q)}{dvol_{g(\bar{t})}(q)}$$

bounded near $\bar{t}$ (where we have bounded curvature), we conclude that $D(q,\bar{t},h)$ is dominated by an integrable function for h near 0.

(ii) This follows from (i) and Remark 3.4.2.

(iii) From the initial computation in (i), which we have justified for noncompact M^n and in the distributional setting, we get

$$0=\tilde{V}_{p,T}(\bar{t}_2)-\tilde{V}_{p,T}(\bar{t}_1)=\int_{\bar{t}_1}^{\bar{t}_2}\frac{d}{d\bar{t}}\tilde{V}_{p,T}(\bar{t})d\bar{t}=\int_{\bar{t}_1}^{\bar{t}_2}\int_M -\Box^* v_{p,T}dvol_{g(\bar{t})}d\bar{t}.$$

From (3.2) we conclude that in the sense of distributions

$$\Box^* v_{p,T}\equiv 0, \tag{3.18}$$

and hence by parabolic regularity $l_{p,T}$ is smooth. Recall equation (3.13)

$$-2\frac{\partial}{\partial \bar{t}}l_{p,T}+|\nabla^{g(\bar{t})}l_{p,T}|^2_{g(\bar{t})}-R_{g(\bar{t})}+\frac{l_{p,T}}{T-\bar{t}}\equiv 0.$$

Subtracting $2\cdot(3.18)$ from this, we get

$$w_{p,T}:=\Big((T-\bar{t})(2\Delta_{g(\bar{t})}l_{p,T}-|\nabla^{g(\bar{t})}l_{p,T}|^2_{g(\bar{t})}+R_{g(\bar{t})})+l_{p,T}-n\Big)v_{p,T}\equiv 0.$$

Hence,

$$0 \equiv \Box^*_{g(\bar{t})} u_{p,T} = -2(T-\bar{t}) \left| Ric_{g(\bar{t})} + \nabla^{g(\bar{t})}\nabla^{g(\bar{t})} l_{p,T} - \frac{1}{2(T-\bar{t})} g(\bar{t}) \right|^2_{g(\bar{t})} v_{p,T},$$

where the last equality follows from a formal computation as in Proposition 9.1 [26]. Because of $v_{p,T} > 0$, we conclude

$$Ric_{g(\bar{t})} + \nabla^{g(\bar{t})}\nabla^{g(\bar{t})} l_{p,T} - \frac{1}{2(T-\bar{t})} g(\bar{t}) = 0. \tag{3.19}$$

Plugging the trace of (3.19) into (3.18), we obtain

$$\frac{\partial l_{p,T}}{\partial \bar{t}} = |\nabla^{g(\bar{t})} l_{p,T}(\bar{t})|^2_{g(\bar{t})}. \tag{3.20}$$

By Remark 2.2.4, this implies that $(M^n, g(t), l_{p,T}(\,\cdot\,, t))$ is a gradient shrinking soliton in canonical form. □

Remark 3.4.5 *We will discuss some issues arising with the inverse of the statement in Corollary 3.4.3 (iii) in Section 3.6.*

3.5 Examples

3.5.1 Einstein solutions

For Einstein solutions $(M^n, g(t))$ on $(0,T)$ with $R_{g(0)} > 0$ as in Example 2.2.1, it follows from an explicit computation (see [6]) that for $(p, t_0) \in M^n \times (0,T)$

$$l_{p,t_0}(q,\bar{t}) = \frac{n}{2}\left(1 - \frac{\arctan\sqrt{\frac{t_0-\bar{t}}{T-t_0}}}{\sqrt{\frac{t_0-\bar{t}}{T-t_0}}}\right) + \frac{d^2_{\bar{t}}(p,q)}{4(T-\bar{t})\sqrt{\frac{t_0-\bar{t}}{T-t_0}}\arctan\sqrt{\frac{t_0-\bar{t}}{T-t_0}}}. \tag{3.21}$$

We can easily conclude the following: if $t_i \nearrow T$, then

$$l_{p,t_i} \to \frac{n}{2}$$

uniformly on $M^n \times [0,T)$. In particular, any reduced distance based at any (p,T) is given by

$$l_{p,T}(q,\bar{t}) = \frac{n}{2}. \tag{3.22}$$

Then the differential inequalities in Corollary 3.3.5 become equalities for the constant function $l_{p,T}$, because of equation (2.5). In particular, $\tilde{V}_{p,T}(\bar{t}) \equiv const.$ Let us compute its value using (2.3):

$$\begin{aligned} \tilde{V}_{p,T}(\bar{t}) &= \left(4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-\frac{n}{2}} Vol(g(\bar{t})) \\ &= (4\pi e T)^{-\frac{n}{2}} Vol(g_0) \\ &= \left(\frac{R_{g_0}}{2\pi e n}\right)^{\frac{n}{2}} Vol(g_0) \end{aligned} \tag{3.23}$$

For the standard shrinking round n-sphere (S^n, g_{S^n}), we therefore get

$$\tilde{V}_{p,T}(\bar{t}) = \left(\frac{n(n-1)}{2\pi e n}\right)^{\frac{n}{2}} Vol(g_{S^n}) = \left(\frac{n-1}{2\pi e}\right)^{\frac{n}{2}} Vol(g_{S^n}).$$

In the following proposition, we show that the condition $l_{p,T} \equiv \frac{n}{2}$ characterizes Einstein solutions uniquely.

Proposition 3.5.1 *Let $(M^n, g(t))$ be a maximal type A Ricci flow on $[0,T)$, $p \in M^n$ and $l_{p,T}$ a reduced distance based at singular time (p,T). Then we have the following:*

(i) If $l_{p,T} \equiv c$ on $M^n \times (0,T)$, then $R_{g(t)} = \frac{c}{T-t}$, $c \leq \frac{n}{2}$, and M^n is compact.

(ii) $l_{p,T} \equiv \frac{n}{2}$ if and only if $(M^n, g(t))$ is a (compact) Einstein solution with $R_{g_0} > 0$.

Proof. (i) It follows directly from (3.13) that $R_{g(t)} = \frac{c}{T-t}$, and combined with (3.1) that $c \leq \frac{n}{2}$. The compactness of M^n follows from the finiteness of $\tilde{V}_{p,T}$ by Remark 3.4.2.

(ii) We have seen for Einstein solutions $l_{p,T} \equiv \frac{n}{2}$. On the other hand, if $l_{p,T} \equiv \frac{n}{2}$, then it follows from (i) that (3.1) becomes an equality, and from the proof of Corollary 3.4.3 that the corresponding reduced volume $\tilde{V}_{p,T}$ is constant. We conclude $(M^n, g(t))$ is a gradient shrinking soliton with potential function $\frac{n}{2}$, i.e.

$$Ric_{g(t)} = \frac{1}{2(T-t)} g(t). \quad \square$$

3.5.2 Gradient shrinking solitons

Recall from equation (2.11) that the potential $f(t)$ of a gradient shrinking soliton in canonical form $(M^n, g(t), f(t))$ satisfies the differential equality

$$-\frac{\partial f}{\partial t} - \Delta_{g(t)} f(t) + |\nabla^{g(t)} f(t)|^2_{g(t)} - R_{g(t)} + \frac{n}{2(T-t)} = 0.$$

Since by Theorem 3.3.1 the reduced distance based at singular time $l_{p,T}$ for some $p \in M^n$ satisfies the same relation as an inequality, it is natural to attempt to relate $f(t)$ and $l_{p,T}(\cdot, t)$.

Before we proceed, we discuss a normalization of the soliton potential, which preserves its canonical form. Let $(M^n, g(t), f(t))$ be a complete gradient shrinking soliton in canonical form on $(-\infty, T)$ with bounded curvature. Recall equation (2.14):

$$R_{g(t)} + |\nabla^{g(t)} f(t)|^2_{g(t)} - \frac{f(t)}{T-t} = C(t)$$

We can use this equation to normalize the soliton potential $f(t)$, a thought which was triggered in discussions with Xiaodong Wang. To keep the soliton in canonical form, however, we only **normalize** such that

$$R_{g(T-1)} + |\nabla^{g(T-1)} f(T-1)|^2_{g(T-1)} - f(T-1) = 0, \tag{3.24}$$

and replace the original soliton potential $f(t)$ by the gradient shrinking soliton in canonical form obtained from $f(T-1)$, as discussed in Remark 2.2.4. From here on, we will assume a given gradient shrinking soliton is normalized by (3.24).

Remark 3.5.2 *We note the gradient shrinking solitons in canonical form* $(M^n, g(t), l_{p,T}(\cdot, t))$ *arising in the equality case of the reduced volume based at singular time* (p, T) *are in fact normalized for all* $t \in (0, T)$, *i.e. satisfy*

$$R_{g(t)} + |\nabla^{g(t)} l_{p,T}(\cdot, t)|^2_{g(t)} - \frac{l_{p,T}(\cdot, t)}{T - t} = 0.$$

This follows directly by plugging (3.20) into (3.13).

We will next show that for compact gradient shrinking solitons $f = l_{p,T}$ holds.

Theorem 3.5.3 *Let* $(M^n, g(t), f(t))$ *be a compact normalized gradient shrinking soliton in canonical form on* $(-\infty, T)$, *and let* $p \in M$. *If* $l_{p,T}$ *is a reduced distance based at singular time* (p, T), *then it is independent of* p *and*

$$l_{p,T}(\cdot, t) = f(t).$$

Remark 3.5.4 *While the theorem is mentioned in the literature [10], [3], we are not aware of any published proof. There are however results [6] using an alternative definition of a reduced distance based at singular time* l, *which we will briefly discuss in Section 6.1. The results can be improved to obtain* $f = l$, *rather than* $f = l + c$, *when using the above normalization. Our proof utilizes techniques from Section 4.1.*

Proof. We know from (2.7) that

$$g(t) = (T - t)\phi_t^* g(T - 1). \tag{3.25}$$

In particular $g(t)$ is of type I. For $p \in M^n$, let $l_{p,T}$ and $\tilde{V}_{p,T}$ a reduced distance and corresponding reduced volume based at singular time, respectively. We refer to Section 4.1 for more details of the following argument: For $t_j \nearrow T$, consider the rescaled metrics pointed at p and compute using (3.25):

$$\begin{aligned} g_j(t) &= \frac{1}{T - t_j} g(T + (T - t_j)t) \\ &= \frac{1}{T - t_j} \big(T - (T + (T - t_j)t)\big)\, \phi^*_{T+(T-t_j)t}\, g(T - 1) \\ &= -t\, \phi^*_{T+(T-t_j)t}\, g(T - 1). \end{aligned}$$

With the definition of Cheeger-Gromov convergence, this implies that the rescaling limit $(M^n_\infty, g_\infty(t), p_\infty)$ is compact. In particular, we can assume $M^n_\infty = M^n$ and $p_\infty = p$.

Next, note that $g_j(-1) = \phi^*_{t_j} g(T-1)$, i.e.

$$g_j(-1) \cong g(T-1)$$

are isometric for all j. We can therefore identify the reduced distances defined on the rescaled metrics by

$$l^j_{p,0}(q, \bar{t}) := l_{p,T}(q, T + (T - t_j)\bar{t})$$

using the diffeomorphism. After shifting time from T to 0, we have

$$l^j_{p,0} \cong l_{p,T}.$$

Next, we conclude by Cheeger-Gromov convergence

$$g_\infty(-1) \cong g(T-1),$$

which gives us the identification

$$l^\infty_{p,0} \cong l_{p,T},$$

where $l^\infty_{p,0}$ is the limit as extracted in Section 4.1 from the reduced distances based at singular time $l^j_{p,0}$ of the rescaled metrics. By Theorem 4.1.3, $(M^n_\infty, g_\infty(t), l^\infty_{p,0}(\cdot, t)) \cong (M^n, g(t), l_{p,T}(\cdot, t))$ is a normalized gradient shrinking soliton in canonical form.

To see $l_{p,T}(\cdot, t) = f(t)$, note that both potentials satisfy equation (2.8) for the same Ricci flow. Hence, hence

$$\nabla^{g(t)}\nabla^{g(t)}(l_{p,T}(\cdot, t) - f(t)) = 0.$$

As observed by Xiaodong Wang, this implies $l_{p,T}(\cdot, t) = f(t) + c$, unless the manifold splits off a line in the direction of $\nabla^{g(t)}(l_{p,T}(\cdot, t) - f(t))$, which we can exclude since we assumed compactness of M^n. Since both soliton potentials are normalized, we conclude

$$l_{p,T}(\,\cdot\,,t) = f(t). \quad \square$$

Given a normalized complete gradient shrinking soliton in canonical form $(M^n, g(t), f(t))$, in general for a base point $p \in M^n$ and a reduced distance $l_{p,T}$ based at singular time, we cannot expect to conclude $f(t) = l_{p,T}(\,\cdot\,,t)$. The following example (see also Figure 3.4) illustrates, that the choice of base point p is one issue.

Example 3.5.5 *Let $(N^{n-k}, g_N(t))$, $0 < k < n-1$ be an Einstein solution on $(-\infty, T)$ with $R_{g_0} > 0$. As in Example 2.2.8, we consider the normalized gradient shrinking soliton in canonical form given by*

$$\Big(N^{n-k} \times \mathbb{R}^k,\ g(t) = g_N(t) + g_{\mathbb{R}^k},\ f(\theta, x, t) = \frac{|x|^2_{g_{\mathbb{R}^k}}}{4(T-t)} + \frac{n-k}{2}\Big),$$

where $\theta \in N^{n-k}$ and $x \in \mathbb{R}^k$. Note that the constant $\frac{n-k}{2}$ normalizes the soliton by equation (3.24).

We can then conclude by the obvious additivity of the $\mathcal{L}-$distance for product manifolds, as well as Example 2.4.2 and (3.22), that for $p = (\theta_0, x_0)$ and $q = (\theta, x) \in N^{n-k} \times \mathbb{R}^k$

$$l_{p,T}(q,t) = \frac{n-k}{2} + \frac{|x - x_0|^2_{\mathbb{R}^k}}{4(T-t)}.$$

Hence, the equality $f(t) = l_{p,T}(\,\cdot\,,t)$ holds if and only if we choose the base point $p \in N^{n-k} \times \{0\}$.

Using Example 2.4.2 and (3.23), the corresponding reduced volume based at *singular time can be directly computed:*

$$
\begin{aligned}
\tilde{V}_{p,T}(t) &= \int_{N^{n-k}} \int_{\mathbb{R}^k} (4\pi(T-t))^{-\frac{n}{2}} e^{-l_{p,T}(\theta,x,t)}\, dvol_{g_N(t)}(\theta)\, dx^k \\
&= \int_{N^{n-k}} (4\pi(T-t))^{-\frac{n-k}{2}} e^{-\frac{n-k}{2}}\, dvol_{g_N(t)} \\
&\quad \cdot \int_{\mathbb{R}^k} (4\pi(T-t))^{-\frac{k}{2}} e^{-\frac{|x-x_0|^2_{\mathbb{R}^k}}{4(T-t)}}\, dx^k \qquad (3.26) \\
&= \left(\frac{R_{g_0}}{2\pi e n}\right)^{\frac{n-k}{2}} Vol(g_0) \qquad (3.27)
\end{aligned}
$$

In particular, it is constant in t*, independent of the base point* p *and the sequence* $\{t_i\}$ *used to define* $l_{p,T}$*, as well as equal to the reduced volume based at singular time for the Einstein part* $(N^{n-k}, g_N(t))$ *of the soliton.*

We also make the observation that while $f \neq l_{p,T}$*, we do have*

$$
\tilde{V}_{p,T}(t) = \int_{N^{n-k}} \int_{\mathbb{R}^k} (4\pi(T-t))^{-\frac{n}{2}} e^{-f(\theta,x,t)}\, dvol_{g_N(t)}(\theta)\, dx^k, \qquad (3.28)
$$

so in examples of this kind, we can define the reduced volume in terms of the soliton potential.

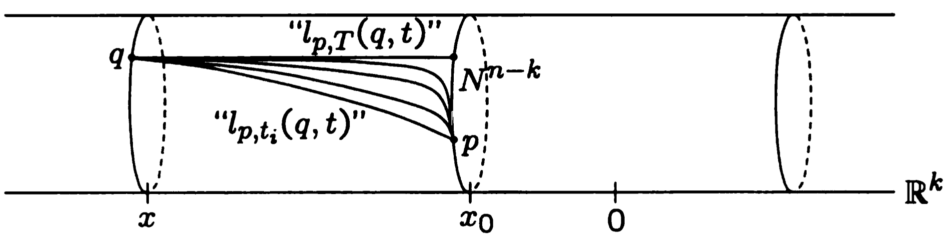

Figure 3.4: Einstein solution on $N^{n-k} \times \mathbb{R}^k$ as an example of a normalized gradient shrinking soliton in canonical form with $f(t) \neq l_{p,T}(\cdot, t) + c$.

While the choice of a suitable base point p in the very special Example 3.5.5 is obvious, that choice of base point $p \in M^n$ such that $f = l_{p,T}$ might not even exist for general complete gradient shrinking solitons, but depend on the structure of the soliton, and in particular the critical points of f.

For complete gradient shrinking solitons with quadratic curvature decay and non-collapsedness assumption, which converge to an incomplete metric cone which is smooth away from the parabolic vertex (consisting of a collapsed compact set), Cao, Hamilton and Ilmanen state in [3] that the reduced distance and volume based at singular time are well-defined. Also, this reduced volume is constant. They use this result to define a notion of density for such shrinking solitons, which we generalize to type A solutions in Section 4.2.

3.6 The inverse of Corollary 3.4.3 (iii)

Recall that Corollary 3.4.3 (iii) stated that if a reduced volume based at singular time is constant, the corresponding Ricci flow is a gradient shrinking soliton in canonical form (which is in fact normalized).

The inverse statement holds, whenever for some $p \in M^n$, we have $f = l_{p,T}$. More generally, the following is true:

Proposition 3.6.1 *If $(M^n, g(t), f(t))$ is a gradient shrinking soliton in canonical form on $(-\infty, T)$, and $f(t) = l_{p,T}(\cdot, t) + c$ for some $p \in M^n$, $c \in \mathbb{R}$, then $\tilde{V}_{p,T}(t)$ is constant in t.*

Proof. From Example 3.2.2, we get that the type A assumption is satisfied. Since $l_{p,T} = f$ is smooth, the inequality (3.1) becomes the equality (2.11), i.e.

$$\Box^*_{g(t)} v_{p,T} = 0.$$

From the arguments in the proof of Corollary 3.4.3 (i), we conclude

$$\frac{d}{dt}\tilde{V}_{p,T}(t) = 0. \quad \Box$$

Remark 3.6.2 *As discussed in Section 3.5.2, compact gradient shrinking solitons in canonical form will satisfy the assumptions of the proposition for any $p \in M^n$, even without being normalized. Similarly, this is the case for solitons of the form in Example 3.5.5 with the correct choice of base point, as well as the ones in [3].*

CHAPTER 4

Applications of monotonicity based at singular time

One of the main applications of monotone quantities in geometric evolution equations is the analysis of singularities. We will first show in Section 4.1 how the notion of a reduced distance based at singular time can be used to model type I singularities by gradient shrinking solitons. Results of that kind are very important, as they reduce the study of singularities in the Ricci flow to the classification of gradient shrinking solitons.

In Section 4.2, based on the reduced volume and its extension to singular time, we will define a notion of density for any point in space-time in the Ricci flow. This is motivated by the Gaussian density in mean curvature flow based on Huisken's monotonicity formula [20] [34], which can abstractly be regarded as an analogue to the monotonicity of a reduced volume based at singular time.

4.1 Rescaling limits of type I Ricci flows are gradient shrinking solitons

We will describe in this section one of the main motivations for extending the notion of the reduced distance to singular time, as done in Chapter 3.

Let $(M^n, g(t)), t \in [0,T), T < \infty$ be a complete maximal type I Ricci flow, and let $p \in M^n$. We rescale the flow parabolically: Let $t_j \nearrow T$ and consider the sequence of pointed 1-parameter families of Riemannian manifolds $(M^n, g_j(t), p)$, where

$$g_j(t) := \frac{1}{T - t_j} g(T + (T - t_j)t). \tag{4.1}$$

Then $g_j(t)$ is a Ricci flow on $[-\frac{T}{T-t_j}, 0)$ for each j. Because of the type I curvature bound, we have at any $x \in M^n$ that

$$\begin{aligned} |Rm_{g_j(t)}|_{g_j(t)}(x) &= (T - t_j)|Rm_{g(T+(T-t_j)t)}|_{g(T+(T-t_j)t)}(x) \\ &\leq (T - t_j)\frac{C}{T - (T + (T - t_j)t)} \\ &= \frac{C}{t}. \end{aligned} \tag{4.2}$$

This gives a uniform curvature bound on compact subsets of $(-\infty, 0)$. Together with Perelman's no local collapsing theorem (which here also holds for complete M^n because of the uniform (in t) lower bound on the reduced volume as described below), we can use the Cheeger-Gromov-Hamilton Compactness Theorem B.0.2 to extract a complete pointed subsequential limit Ricci flow $(M^n_\infty, g_\infty(t), p_\infty)$ on $(-\infty, 0)$, which is still type I. An example of the convergence of rescales around a neckpinch to a shrinking cylinder is shown in Figure 4.1.

As ancient solutions are expected to have a very special structure, Hamilton [18] conjectured (under a different rescaling procedure) that $(M^n_\infty, g_\infty(t))$ is a gradient shrinking soliton.

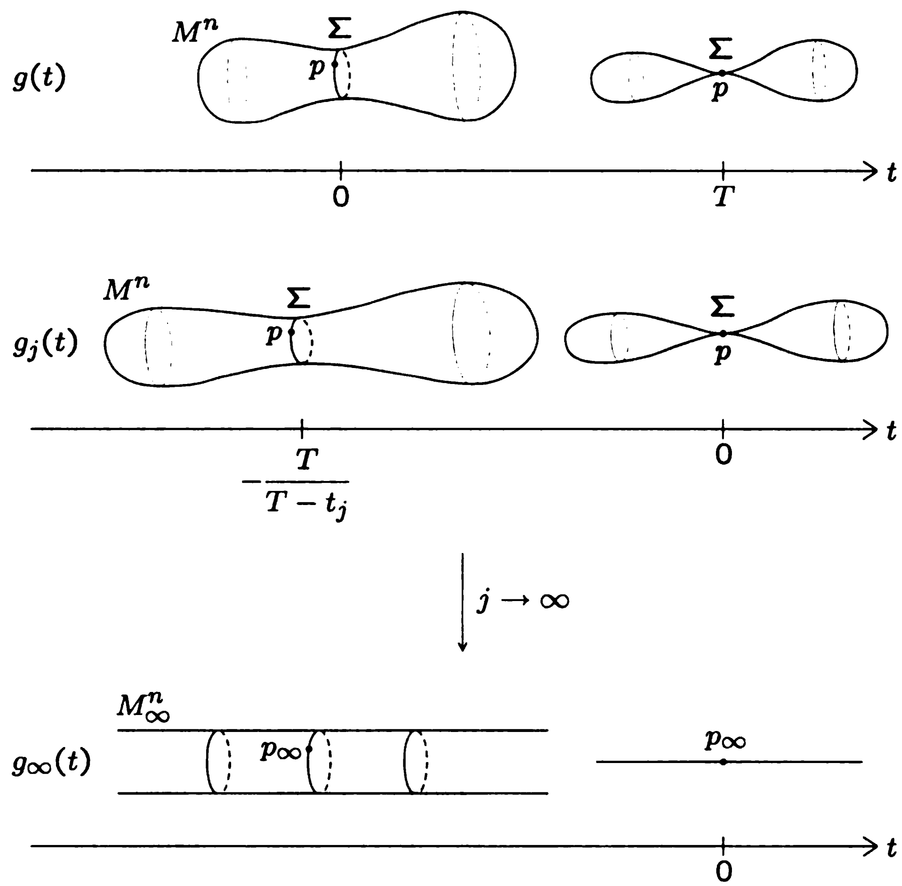

Figure 4.1: Parabolic rescalings $(M^n, g_j(t))$ of a neckpinch $(M^n, g(t))$ and pointed Cheeger-Gromov-Hamilton convergence near a singular point p to a shrinking cylinder $(M^n, g_\infty(t))$.

In general, M^n_∞ and M^n are different smooth manifolds. The definition of Cheeger-Gromov convergence implies that if M^n_∞ is compact, then in fact $M^n_\infty = M^n$. It follows from [29], that Hamilton's conjecture holds in this case:

Theorem 4.1.1 *The rescaling limit $(M^n_\infty, g_\infty(t))$ obtained as described above is a gradient shrinking soliton, if M^n_∞ is compact.*

The proof uses the monotonicity of Perelman's entropy functional, which is only defined on compact manifolds.

We will now describe our idea on how to use the monotonicity of a reduced volume based at singular time to obtain a proof of the conjecture without the very restrictive assumption of M^n_∞ being compact: Let $l_{p,T}$ be any reduced distance based at singular time (p, T) for the Ricci flow $(M^n, g(t))$ on $[0, T)$ as defined in Chapter 3. For each $(q, \bar{t}) \in M^n \times (-\infty, 0)$, consider for large enough j

$$l^j_{p,0}(q, \bar{t}) := l_{p,T}(q, T + (T - t_j)\bar{t}), \tag{4.3}$$

which is a reduced distance based at singular time $(p, 0)$ for the Ricci flow $(M^n, g_j(t))$ on $[-\frac{T}{T-t_j}, 0)$ because of the scaling properties of the reduced distance.

$$\tilde{V}^j_{p,0}(\bar{t}) = \tilde{V}_{p,T}(T + (T - t_j)\bar{t}). \tag{4.4}$$

We can conclude from (4.4) that

$$\tilde{V}^j_{p,0} \xrightarrow{j \to \infty} \lim_{t \nearrow T} \tilde{V}_{p,T}(t)$$

uniformly on compact subsets of $(-\infty, 0)$. Hence by Corollary 3.4.3, $\tilde{V}^j_{p,0}$ converge to a constant in $(0, 1]$. Figure 4.2 illustrates this convergence.

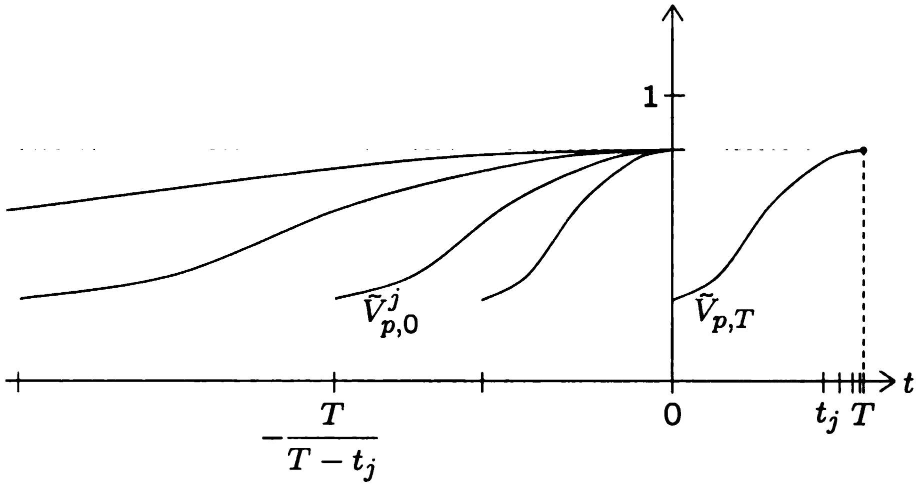

Figure 4.2: Convergence of the rescaled reduced volumes $\tilde{V}^j_{p,0}$ based at singular time to the constant $\lim_{t\nearrow T} \tilde{V}_{p,T}(t)$.

We would like to conclude that there exists a locally Lipschitz function $l^{\infty}_{p_{\infty},0}$ on the limit manifold $M^n_{\infty} \times (-\infty, 0)$, such that

$$l^j_{p,0} \xrightarrow{C^0_{loc}} l^{\infty}_{p_{\infty},0},$$

and which satisfies the differential (in)equalities of Corollary 3.3.5. Since its corresponding formal reduced volume $V^{\infty}_{p_{\infty},0}$ is constant, those are equalities for $l^{\infty}_{p_{\infty},0}$, and we can conclude that $(M^n_{\infty}, g_{\infty}(t), l^{\infty}_{p_{\infty},0}(\,\cdot\,,t))$ is a normalized gradient shrinking soliton in canonical form. To do this, we need uniform bounds on the sequence of functions $l^j_{p,0}$, as in the proof of Theorem 3.3.1. However, those bounds must survive under Cheeger-Gromov convergence.

In [23], this problem has been worked on independently, and estimates have been derived to define a "singular reduced length", which is a reduced distance based at singular time as we constructed in Section 3.3. Those estimates are derived by comparison with a minimizing geodesic, and hence, are in terms of the Riemannian distance away from the singular time. As such, they are preserved under Cheeger-Gromov convergence. In our notation, they read as follows:

Theorem 4.1.2 *Let $(M^n, g(t))$ be a complete type I Ricci flow on $[0,T)$, and let $(p, t_0) \in M^n \times [0,T)$. Then there exist $C > 0$ (only dependent on n and the type I constant) such that*

(i) $\frac{1}{C}(1 + \frac{d_{\bar{t}}(p,q)}{\sqrt{t_0 - \bar{t}}})^2 - C \leq l_{p,t_0}(q,\bar{t}) \leq C(1 + \frac{d_{\bar{t}}(p,q)}{\sqrt{t_0 - \bar{t}}})^2,$

(ii) $|\nabla^{g(\bar{t})} l_{p,t_0}(q,\bar{t})|_{g(\bar{t})}(q) \leq \frac{C}{\sqrt{t_0 - \bar{t}}}(1 + \frac{d_{\bar{t}}(p,q)}{\sqrt{t_0 - \bar{t}}}),$

(iii) $|\frac{\partial}{\partial \bar{t}} l_{p,t_0}(q,\bar{t})|_{g(\bar{t})}(q) \leq \frac{C}{t_0 - \bar{t}}(1 + \frac{d_{\bar{t}}(p,q)}{\sqrt{t_0 - \bar{t}}})^2.$

In particular, those estimates are satisfied by any reduced distance based at singular time $l_{p,T}$ by replacing t_0 by T in Theorem 4.1.2. For the metrics $g_j(t)$ defined in (4.1), we have the same type I bound (see (4.2)). We can therefore conclude from Theorem 4.1.2 that

$$l^j_{p,0}(q,\bar{t}) \leq C(1 + \frac{d_{\bar{t}}(p,q)}{\sqrt{-\bar{t}}})^2.$$

In the same way, the other bounds hold and are preserved under Cheeger-Gromov convergence. This allows us to extract $l^{\infty}_{p_\infty,0}$ as we described above, which satisfies Corollary 3.3.5 as equalities and is normalized for the same reasons as explained in Remark 3.5.2.

We summarize the above discussions by stating the theorem, which is stated slightly differently in [23]:

Theorem 4.1.3 *Let $(M^n, g(t), p)$, $t \in [0,T)$, $T < \infty$, $p \in M^n$ be a complete maximal pointed type I Ricci flow. For $t_j \nearrow T$, consider $g_j(t) := \frac{1}{T - t_j} g(T + (T - t_j)t)$*

on $M^n \times [-\frac{T}{T-t_j}, 0)$*. Then there exists a Cheeger-Gromov-Hamilton limit Ricci flow* $(M^n_\infty, g_\infty(t), p_\infty)$ *on* $(-\infty, 0)$ *and a potential function* $l^\infty_{p_\infty,0}$ *on* $M^n \times (0,T)$*, such that* $(M^n_\infty, g_\infty(t), l^\infty_{p_\infty,0}(\,\cdot\,,t))$ *is a normalized gradient shrinking soliton in canonical form with bounded curvature.*

We saw in the proof of Theorem 3.5.3, that if $(M^n, g(t), f(t))$ is a compact normalized gradient shrinking soliton in canonical form, then the limit soliton constructed above equals the original soliton, and moreover, the soliton potentials are equal.

4.2 Definition of a density

Consider a type A Ricci flow $(M^n, g(t))$ on $[0,T)$, and let $p \in M^n$. For a sequence $t_i \nearrow T$ defining a reduced distance based at singular time $l_{p,T}$, we have from Definition 3.4.1 that

$$\tilde{V}_{p,T}(\bar{t}) = \int_M \lim_{t_i \nearrow T} \left(((4\pi(T-\bar{t}))^{-\frac{n}{2}} e^{-l_{p,t_i}(q,\bar{t})} \right) dvol_{g(\bar{t})}(q).$$

Using the uniform lower estimate on l_{p,t_i} by the square of the distance function in Theorem 4.1.2, we actually obtain from the dominated convergence theorem that

$$\tilde{V}_{p,T}(\bar{t}) = \lim_{t_i \nearrow T} \tilde{V}_{p,t_i}(\bar{t}),$$

which is illustrated in Figure 4.3.

We will now consider points in the closure of space-time, i.e. in $M^n \times [0,T]$, to include the singular time.

Definition 4.2.1 *Let* $(M^n, g(t))$ *be a type A Ricci flow on* $[0,T)$*. For any* $(p, t_0) \in M^n \times [0,T]$ *and a reduced distance* l_{p,t_0}*, we define* ***a density of*** (p,t_0) ***in the Ricci flow*** $(M^n, g(t))$ *by*

$$\theta_{p,t_0} = \lim_{\bar{t} \nearrow t_0} \tilde{V}_{p,t_0}(\bar{t}) \in (0,1].$$

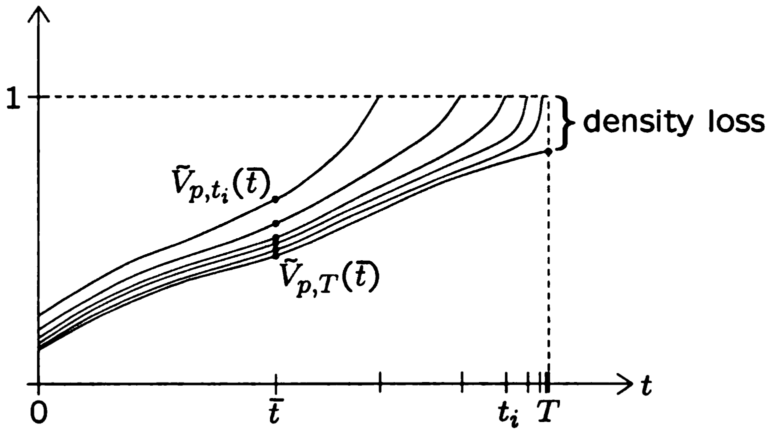

Figure 4.3: Convergence of the reduced volumes $\tilde{V}_{p,t_i}(\bar{t})$ to a reduced volume based at singular time $\tilde{V}_{p,T}(\bar{t})$ and density loss.

If $t_0 < T$, then θ_{p,t_0} is well-defined since the reduced distance l_{p,t_0} is unique, and we have $\theta_{p,t_0} = 1$ by Corollary 2.4.3. For $t_0 = T$ the density might depend on the choice of the reduced distance functions, which in general is only defined as a subsequential limit. If $(M^n, g(t), f(t))$ is a compact gradient shrinking soliton, then by Theorem 3.5.3 the density $\theta_{p,T}$ is well-defined, in fact even independent of the base point p. More general, for products of Einstein solutions with Euclidean space as in Example 3.5.5, we also obtain that the density is well-defined by equation (3.28). This provides a collection of densities $\in (0, 1]$ occuring in Ricci flows, and the values can be computed by (3.26). In [3], Cao, Hamilton and Ilmanen define an analogous density for gradient shrinking solitons satisfying their assumptions.

In general, for complete type I Ricci flows $(M^n, g(t))$ on $[0, T)$, it follows from Theorem 4.1.3 that for $p \in M^n$ and choice of $l_{p,T}$ there exists a gradient shrinking soliton in canonical form $(M^n_\infty, g_\infty(t), l^\infty(t))$, which models the singularity at p and whose potential function satisfies

$$\theta_{p,T} = \int_{M^n} (4\pi(T-t))^{-\frac{n}{2}} e^{-l^\infty} dvol_{g(t)}$$

for any t. This is depicted in Figure 4.4. While the integral is constant in t, it is only a formal reduced volume, i.e. l^∞ might not be a reduced distance. This could even be the case when $(M^n, g(t))$ is a gradient shrinking soliton in canonical form if the limit soliton is not isometric to it.

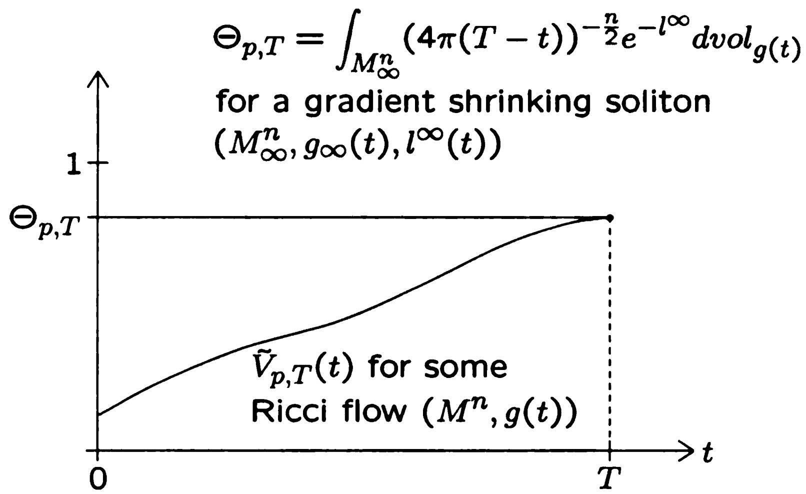

Figure 4.4: Relation between density $\theta_{p,T}$ for a type I Ricci flow $(M^n, g(t))$ and limit gradient shrinking soliton $(M^n_\infty, g(t), l^\infty(t))$.

To distinguish points $p \in M^n$ where the curvature blows up at the singular time T, from those where the curvature stays bounded, we make the following

Definition 4.2.2 *The* ***singular set*** $\Sigma \subset M^n$ *for a maximal Ricci flow* $(M^n, g(t))$ *on* $[0, T)$ *is given by*

$$\Sigma := \{q \in M \mid |Rm_{g(t)}(q)|_{g(t)} \xrightarrow{t \to T} \infty\}.$$

Remark 4.2.3 *The points in the singular set* Σ *are equivalently those, where the metric* $g(t)$ *becomes degenerate as* $t \to T$.

We can prove

Proposition 4.2.4 *Let* $(M^n, g(t))$ *be a maximal type I Ricci flow on* $[0, T)$ *with singular set* Σ. *If* $p \in M^n \backslash \Sigma$, *then* $\theta_{p,T} = 1$.

Proof. Give $p \in M^n \backslash \Sigma$, we rescale $g(t)$ for a sequence $t_j \nearrow T$ by

$$g_j(t) := \frac{1}{T - t_j} g(T + (T - t_j)t).$$

The bounded curvature of $g(t)$ near p for all t implies the Cheeger-Gromov-Hamilton limit flow is

$$(M^n_\infty, g_\infty(t)) \cong (\mathbb{R}^n, g_{\mathbb{R}^n}),$$

which by Theorem 4.1.3 is a gradient shrinking soliton in canonical form on flat $\mathbb{R}^n$. The soliton equation (2.8) immediately implies this is the Gaussian soliton, and hence, $\theta_{p,T} = 1$ by the discussions above. □

We generally expect that the structure of a singularity at $p \in \Sigma$ determines a "density loss", so that $\theta_{p,T} < 1$ as shown in Figure 4.3,

CHAPTER 5

Reduced distance and monotonicity based at a submanifold

5.1 Motivation

Our main reason to study a reduced distance based at a point (p, T) at singular time in Chapter 3 was the application to singularity analysis in Chapter 4. In particular, the case where $p \in \Sigma$, the singular set of the Ricci flow $(M^n, g(t))$, is of interest. In this chapter, we are motivated by the case where the singular set Σ is a lower dimensional submanifold of M^n. In general, not much is known about the size of the singular set for a Ricci flow. The considerations here were triggered by the example of a neckpinch as shown in Figure 5.1. We will extend the notions of the reduced distance and volume from being based at a point $p \in M^n$ to being based at any submanifold $S \subset M^n$.

First, we briefly discuss the neckpinch in Figure 5.1. In dimension $n = 2$ the picture is actually false, since the curvature in the neck is negative. For $n \geq 3$ the

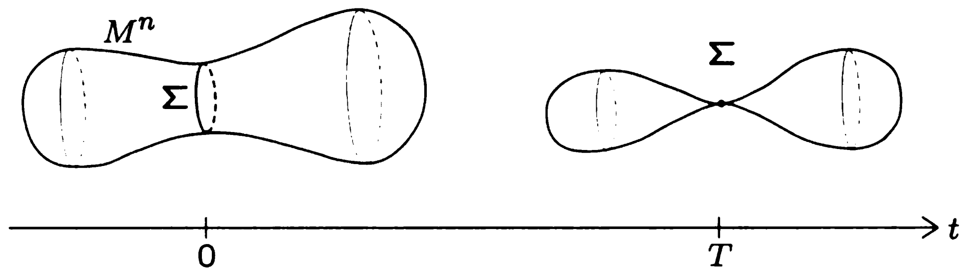

Figure 5.1: Formation of a neckpinch.

existence of symmetric neckpinches has been rigorously shown in [1]. The authors identify the n-sphere S^n minus two points with $(-1,1) \times S^{n-1}$ and consider the evolution of a metric of the form

$$g = \phi(x)^2 dx^2 + \psi(x)^2 g_{S^{n-1}}, \tag{5.1}$$

where as before $g_{S^{n-1}}$ denotes the standard round metric on S^{n-1} (see Figure 5.2.)

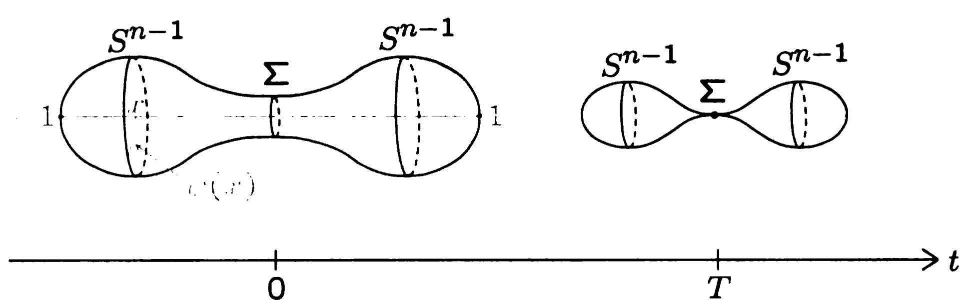

Figure 5.2: A symmetric neckpinch.

For a neckpinch the singular set Σ is of lower dimension than the manifold M^n. Motivated by this, we will study $\mathcal{L}-$geodesics, which are minimizing the $\mathcal{L}-$length to any given $k-$dimensional submanifold S of M^n, where $k < n$ and $(M^n, g(t))$ is a Ricci flow on $[0, T)$. We show that the monotonicity of the corresponding reduced volume still holds, and will also extend it to singular time.

Remark 5.1.1 *Besides the symmetric neckpinch, there are other examples where similar pinching behavior occurs on compact lower dimensional submanifolds of non-compact manifolds, see e.g. [33] and [13].*

5.2 Reduced distance based at a submanifold

In this section, we study the reduced distance based at a lower dimensional submanifold S of M^n.

Definition 5.2.1 *For a Ricci flow $(M^n, g(t))$ on $[0, T)$ and a compact (not necessarily connected) embedded $k-$dimensional submanifold $S \subset M^n$, $0 \leq k < n$, we define for given $t_0 \in (0, T)$ and any $(q, \bar{t}) \in M^n \times (0, t_0)$*

(i) ***the $\mathcal{L}$-distance from*** *$(q, \bar{t})$* ***to*** *(S, t_0) by*

$$L_{S,t_0}(q, \bar{t}) := \inf_{p \in S}\{L_{p,t_0}(q, \bar{t})\},$$

(ii) ***the reduced distance based at*** *(S, t_0) by*

$$l_{S,t_0}(q, \bar{t}) := \frac{L_{S,t_0}(q, \bar{t})}{2\sqrt{t_0 - \bar{t}}},$$

(iii) and

$$v_{S,t_0}(q, \bar{t}) := (4\pi(t_0 - \bar{t}))^{-\frac{n}{2}} e^{-l_{S,t_0}(q,\bar{t})}.$$

Remark 5.2.2

(i) The compactness of S implies that the infimum in (i) is in fact assumed for some $p \in S$, i.e. $L_{S,t_0}(q, \bar{t})$ is given by the $\mathcal{L}-$length of an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic from $(q, \bar{t})$ to (p, t_0) for some $p \in S$ as shown in Figure 5.3.

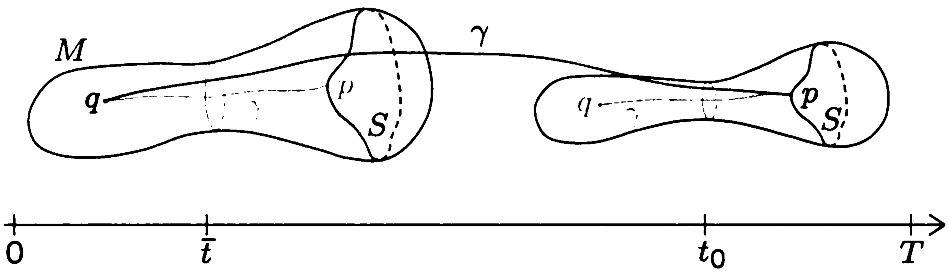

Figure 5.3: The $\mathcal{L}$-distance from $(q, \bar{t})$ to (S, t_0) assumed by an $\mathcal{L}$-minimizing $\mathcal{L}-$geodesic γ for some Ricci flow $(M^n, g(t))$ with submanifold $S \subset M^n$.

(ii) It follows trivially from Definition 2.3.3 that

$$l_{S,t_0}(q, \bar{t}) = \inf_{p \in S} \{l_{p,t_0}(q, \bar{t})\}$$

and

$$v_{S,t_0}(q, \bar{t}) = \sup_{p \in S} \{v_{p,t_0}(q, \bar{t})\}.$$

To justify the picture in Figure 5.3, we first state and prove the following

Lemma 5.2.3 *We make the assumptions as in Definition 5.2.1. If $\gamma : [\bar{t}, t_0] \to M^n$ with $\gamma(\bar{t}) = q$ and $\gamma(t_0) = p \in S$ is an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic such that $\mathcal{L}(\gamma) = L_{S,t_0}(q, \bar{t})$, then γ is orthogonal to S at $t = t_0$.*

Proof. By the assumption $\mathcal{L}(\gamma) = L_{S,t_0}(q, \bar{t})$, we consider the following variational problem: Let

$$\phi : (-\epsilon, \epsilon) \times [\bar{t}, t_0] \to M^n$$

be a (C^1) variation with

$$\begin{aligned} \phi(s, t_0) &\in S, \\ \phi(s, \bar{t}) &= q, \\ \text{and } \phi(0, t) &= \gamma(t). \end{aligned}$$

Then the first variation formula (2.15) for $\mathcal{L}$ at the $\mathcal{L}-$geodesic γ implies

$$0 = \delta_Y \mathcal{L}(\gamma) = \frac{d}{ds}\Big|_{s=0} \mathcal{L}(\phi(s,t)) = \langle 2\sqrt{t_0 - t}\,\dot{\gamma}(t), Y(t)\rangle_{g(t)} \,|_{\bar{t}}^{t_0}, \qquad (5.2)$$

where

$$\begin{aligned} Y(t) &= \frac{\partial \phi}{\partial s}|_{s=0}(s,t), \\ Y(\bar{t}) &= 0, \\ \text{and } Y(t_0) &\in T_{\gamma(t_0)}S. \end{aligned}$$

Recall that by Proposition 2.3.2, the initial vector

$$w := \lim_{t \to t_0} \sqrt{t_0 - t}\,\dot{\gamma}(t) \in T_{\gamma(t_0)}M$$

exists. Hence, we conclude from (5.2) that

$$0 = \lim_{t \to t_0} \langle \sqrt{t_0 - t}\,\dot{\gamma}(t), Y(t)\rangle_{g(t)} = \langle w, Y(t_0)\rangle_{g(t_0)}.$$

This holds for all $Y(t_0) \in T_{\gamma(t_0)}S$ (arising from any variation ϕ as above) and implies that $w \in (T_{\gamma(t_0)}S)^{\perp}$, the normal space to S at $\gamma(t_0)$ in $(M^n, g(t_0))$. □

Remark 5.2.4 *The case $k = 0$ is included in the above proof; S is then a finite collection of points in M^n.*

The main result of this section is that the three differential (in)equalities for l_{p,t_0} in Theorem 2.3.4 also hold for l_{S,t_0} :

Theorem 5.2.5 *Under the assumptions of Definition 5.2.1, let $t_0 \in (0,T)$. Then for all $(q,\bar{t}) \in M^n \times (0,t_0)$ the following three (in)equalities hold:*

$$-\frac{\partial}{\partial \bar{t}} l_{S,t_0}(q,\bar{t}) - \Delta_{g(\bar{t})} l_{S,t_0}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{S,t_0}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(t_0 - \bar{t})} \geq 0, \qquad (5.3)$$

or equivalently , $$\Box^*_{g(\bar t)} v_{S,t_0}(q,\bar t) \le 0. \tag{5.4}$$

$$-|\nabla^{g(\bar t)} l_{S,t_0}(q,\bar t)|^2_{g(\bar t)} + R_{g(\bar t)}(q) + \frac{l_{S,t_0}(q,\bar t) - n}{t_0 - \bar t} + 2\Delta_{g(\bar t)} l_{S,t_0}(q,\bar t) \le 0. \tag{5.5}$$

$$-2\frac{\partial}{\partial \bar t} l_{S,t_0}(q,\bar t) + |\nabla^{g(\bar t)} l_{S,t_0}(q,\bar t)|^2_{g(\bar t)} - R_{g(\bar t)}(q) + \frac{l_{S,t_0}(q,\bar t)}{t_0 - \bar t} = 0. \tag{5.6}$$

Remark 5.2.6 *Similarly to the reduced distance l_{p,t_0}, the reduced distance based at a submanifold l_{S,t_0} is locally Lipschitz on $M^n \times (0,t_0)$, and in fact smooth away from a closed set of Lebesgue measure zero for all t. Hence, the (in)equalities hold in the weak sense, or in particular in the sense of distributions. Compare Remark 2.3.5.*

The proof of Theorem 5.2.5 will follow from the next two propositions.

Proposition 5.2.7 *Under the assumptions of Definition 5.2.1, let $t_0 \in (0,T)$. Then for all $(q,\bar t) \in M^n \times (0,t_0)$ we have*

(i) $|\nabla l_{S,t_0}(q,\bar t)|^2 = -R_{g(\bar t)}(q) + \frac{1}{t_0-\bar t} l_{S,t_0}(q,\bar t) - (t_0-\bar t)^{-\frac{3}{2}} K(\gamma,\bar t)$,

(ii) $\frac{\partial}{\partial \bar t} l_{S,t_0}(q,\bar t) = -R_{g(\bar t)}(q) + \frac{1}{t_0-\bar t} l_{S,t_0}(q,\bar t) - \frac{1}{2}(t_0-\bar t)^{-\frac{3}{2}} K(\gamma,\bar t)$,

where

$$K(\gamma,\bar t) := \int_{\bar t}^{t_0} (t_0 - t)^{\frac{3}{2}} H(\dot\gamma(t),t)\,dt,$$

$$H(X,t) := \frac{\partial}{\partial t} R_{g(t)} + \frac{1}{t} R_{g(t)} - 2\langle \nabla R, X\rangle + 2Ric(X,X),$$

and $\gamma(t)$ is as in Lemma 5.2.3.

Remark 5.2.8 *The equalities in the proposition were observed for l_{p,t_0} by Perelman. $H(X,t)$ is Hamilton's Harnack expression, which is nonnegative for all vector fields X along any complete Ricci flow $(M^n, g(t))$ with bounded nonnegative curvature operator. We will not need any properties of this expression, as will become clear in the proof of Theorem 5.2.5.*

Proof. It follows from Lemma 5.2.3 and the first variation formula (2.15) that

$$\nabla L_{S,t_0}(q,\bar{t}) = -2\sqrt{t_0-\bar{t}}\,\dot\gamma(\bar{t}).$$

Hence,

$$|\nabla L_{S,t_0}(q,\bar{t})|^2 = 4(t_0-\bar{t})\,|\dot\gamma(t)|^2 = -4(t_0-\bar{t})R_{g(\bar{t})} + 4(t_0-\bar{t})(R_{g(\bar{t})} + |\dot\gamma(t)|^2). \quad (5.7)$$

We also get

$$\begin{aligned} \frac{\partial}{\partial \bar{t}} L_{S,t_0}(q,\bar{t}) &= \frac{d}{d\bar{t}} L_{S,t_0}(q,\bar{t}) - \langle \nabla L_{S,t_0}(q,\bar{t}), \dot\gamma(\bar{t})\rangle \\ &= -\sqrt{t_0-\bar{t}}\,(|\dot\gamma(\bar{t})|^2 + R_{g(\bar{t})}(q)) + 2\sqrt{t_0-\bar{t}}\,|\dot\gamma(\bar{t})|^2 \\ &= -2\sqrt{t_0-\bar{t}}\,R_{g(\bar{t})}(q) + \sqrt{t_0-\bar{t}}\,(R_{g(\bar{t})}(q) + |\dot\gamma(\bar{t})|^2). \quad (5.8) \end{aligned}$$

We can use Perelman's [26] computation along the $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic $\gamma(t)$ as in Lemma 5.2.3, which in our context reads

$$(t_0-\bar{t})^{\frac{3}{2}}\,(R_{g(\bar{t})}(q) + |\dot\gamma(\bar{t})|^2) = -K(\gamma,\bar{t}) + \frac{1}{2}L_{S,t_0}(q,\bar{t}). \quad (5.9)$$

Plugging (5.9) into (5.7) and (5.8), we obtain

$$|\nabla L_{S,t_0}(q,\bar{t})|^2 = -4(t_0-\bar{t})R_{g(\bar{t})} + \frac{4}{\sqrt{t_0-\bar{t}}}(\frac{1}{2}L_{S,t_0}(q,\bar{t}) - K(\gamma,\bar{t}))$$

$$\text{and} \quad \frac{\partial}{\partial \bar{t}} L_{S,t_0}(q,\bar{t}) = -2\sqrt{t_0-\bar{t}}\,R_{g(\bar{t})}(q) + \frac{1}{t_0-\bar{t}}(\frac{1}{2}L_{S,t_0}(q,\bar{t}) - K(\gamma,\bar{t})),$$

and therefore

$$|\nabla l_{S,t_0}(q,\bar{t})|^2 = -R_{g(\bar{t})}(q) + \frac{1}{t_0-\bar{t}}\,l_{S,t_0}(q,\bar{t}) - (t_0-\bar{t})^{-\frac{3}{2}}K(\gamma,\bar{t})$$

$$\text{and} \quad \frac{\partial}{\partial \bar{t}} l_{S,t_0}(q,\bar{t}) = -R_{g(\bar{t})}(q) + \frac{1}{t_0-\bar{t}}\,l_{S,t_0}(q,\bar{t}) - \frac{1}{2}(t_0-\bar{t})^{-\frac{3}{2}}K(\gamma,\bar{t}). \quad \square$$

To obtain the monotonicity result of the next section, we will now show that the crucial differential inequality (5.4) holds for the reduced distance to a submanifold.

Proposition 5.2.9 *Under the assumptions of Definition 5.2.1, we have*

$$\Box^*_{g(\bar{t})} v_{S,t_0}(q,\bar{t}) \leq 0$$

in the weak sense.

Proof. We argue by contradiction: Assume there exists a (small) parabolic cylinder $P = U \times [t_2, t_1) \subset M^n \times (0, t_0)$, U open, such that for all $0 \leq \phi \in C^2_{cpt}(M^n \times (0, t_0))$ with support in P

$$\iint_P v_{S,t_0} \Box\phi \, dvol_{g(t)} dt > 0.$$

Inverting time ($\tau := t_0 - t$) implies that $-v_{S,t_0}$ is strictly subparabolic in the weak sense of Friedman [14], and we will apply his (strong) maximum principle several times to derive a contradiction. By Remark 5.2.2 (ii)

$$v_{S,t_0}(q,\bar{t}) = \sup_{p \in S} \{v_{p,t_0}(q,\bar{t})\}.$$

Let $p \in S$, $v_p := v_{p,t_0}(q,\bar{t})$ and $v_S := v_{S,t_0}(q,\bar{t})$. Then $v_p \leq v_{S,t_0}$, and we know from (2.19) that

$$\Box^* v_p \leq 0$$

in the weak sense. Now let $\Gamma := \bar{P} \backslash P$ and w_p be a solution to

$$\begin{cases} \Box^* w_p &= 0 \quad \text{in P} \\ w_p|_\Gamma &= v_p|_\Gamma. \end{cases}$$

Hence,

$$\Box^*(v_p - w_p) \leq 0,$$

and the maximum principle implies

$$v_p \leq w_p \text{ in } \bar{P}. \tag{5.10}$$

Similarly, let w_S be a solution to

$$\begin{cases} \Box^* w_S &= 0 \quad \text{in P} \\ w_S|_\Gamma &= v_S|_\Gamma. \end{cases}$$

Since

$$\Box^*(w_p - w_S) = 0$$

and $w_p|_\Gamma = v_p|_\Gamma \leq v_S|_\Gamma = w_S|_\Gamma$, the maximum principle implies that

$$w_p \leq w_S \text{ in } \bar{P}. \tag{5.11}$$

As $p \in S$ was arbitrary, we conclude from (5.10) and (5.11) that

$$v_S \leq w_S \text{ in } \bar{P}.$$

Using $v_S|_\Gamma = w_S|_\Gamma$ and the maximum principle again, this contradicts the assumption that in the weak sense in P

$$\Box^*(v_S - w_S) > 0,$$

which finishes the proof. □

Remark 5.2.10 *In the above proof, we only needed the weak maximum principle, which follows trivially from Friedman's strong maximum principle. Moreover, by Remarks 2.3.5 and 5.2.6, we could have assumed v_p and v_S are smooth and P, and purely worked in the smooth category.*

Proof of Theorem 5.2.5. The first differential inequality is equivalent to Proposition 5.2.9 by definition of v_{S,t_0}.

We obtain equation (5.6) by subtracting (i) $-2\cdot$ (ii) in Proposition 5.2.7, since then the term involving $K(\gamma, \bar{t})$ cancels.

Finally, the inequality (5.5) follows as $-2\cdot(5.3)+(5.6)$. □

Remark 5.2.11 *By combining the ideas of Perelman's original proof for the first two equalities with the comparison principle proof for the differential equality, we avoid a lengthy discussion of a Laplacian comparison theorem for L_{S,t_0} in the setting of the submanifold S.*

5.3 Reduced volume monotonicity based at a submanifold

The purpose of this section is to show the monotonicity of the reduced volume corresponding to the reduced distance based at a submanifold S. Therefore, we now make a further generalization of Perelman's reduced volume.

Definition 5.3.1 *Under the assumptions of Definition 5.2.1, we define for each* $\bar{t} \in (0, t_0)$ *the* ***reduced volume based at*** (S, t_0) *by*

$$\tilde{V}_{S,t_0}(\bar{t}) := \int_M v_{S,t_0}(q, \bar{t})\, dvol_{g(\bar{t})}(q).$$

By definition, we have for any $p \in S$ and $\bar{t} \in (0, t_0)$ that

$$\tilde{V}_{S,t_0}(\bar{t}) \geq \tilde{V}_{p,t_0}(\bar{t}).$$

To guarantee the existence of the reduced volume based at (S, t_0), we prove the following

Lemma 5.3.2 $\tilde{V}_{S,t_0}(\bar{t})$ *as in Definition 5.3.1 is finite for any* $\bar{t} \in (0, t_0)$.

Proof. Fix $\bar{t} \in (0, t_0)$. For any $q \in M$, there exists a $p \in S$ such that

$$l_{S,t_0}(q, \bar{t}) = l_{p,t_0}(q, \bar{t}).$$

The known estimate for the reduced distance in terms of the Riemannian distance at time t_0 (see e.g. [36]) gives

$$l_{p,t_0}(q, \bar{t}) \geq C_1 d^2_{t_0}(p, q) - C_2,$$

where $C_1 > 0$ and $C_2 > 0$ are constants only depending on $\bar{t}$ and the curvature bound. Using the equivalence of the metrics along the Ricci flow on $[0, t_0]$, we obtain for a possibly different constant C_1 (with dependencies as above) that

$$l_{S,t_0}(q, \bar{t}) \geq C_1 d^2_{\bar{t}}(p, q) - C_2,$$

and thus

$$l_{S,t_0}(q,\bar{t}) \geq C_1 d^2_{\bar{t}}(S,q) - C_2. \tag{5.12}$$

The finiteness of $\tilde{V}_{S,t_0}(\bar{t})$ now follows from the convergence of the integral

$$\int_M e^{-d^2_{\bar{t}}(S,q)}\, dvol_{g(\bar{t})}(q) < \infty$$

for compact $S \subset M^n$. □

Corollary 5.3.3 (Monotonicity based at a submanifold)

Under the same assumptions as in Definition 5.3.1, we have

(i) $\frac{d}{dt}\tilde{V}_{S,t_0}(\bar{t}) \geq 0$,

(ii) *If* $\tilde{V}_{S,t_0}(\bar{t}_1) = \tilde{V}_{S,t_0}(\bar{t}_2)$ *for* $0 < \bar{t}_1 < \bar{t}_2 < t_0$, *then* $(M^n, g(t), l_{S,t_0}(\cdot, t))$ *is a flat gradient shrinking soliton in canonical form.*

Proof. This is very similar to the proof of Corollary 3.4.3 (i) and (iii), since the differential (in)equalities in Theorem 5.2.5 hold. The technical ingredient in the setting here is the comparison of $l_{S,t_0}(q,\bar{t})$ with $d^2_{\bar{t}}(S,q)$ in (5.12).

To see the flatness of the gradient shrinking soliton $(M^n, g(t), l_{S,t_0}(\cdot, t))$ obtained in (ii), we first use Remark 2.2.4: For $t \in [t_0 - 1, t_0]$, we have

$$g(t) = (t_0 - t)\phi_t^* g(t_0 - 1).$$

Together with the fact that the curvature of $(M^n, g(t))$ is bounded by C on $[t_0 - 1, t_0]$, we obtain

$$\begin{aligned}
\sup_M |Rm_{g(t_0-1)}|_{g(t_0-1)} &= \sup_M |Rm_{\phi_t^* g(t_0-1)}|_{\phi_t^* g(t_0-1)} \\
&= \sup_M |Rm_{\frac{1}{t_0-t}g(t)}|_{\frac{1}{t_0-t}g(t)} \\
&= (t_0 - t)\sup_M |Rm_{g(t)}|_{g(t)} \\
&\leq (t_0 - t)C.
\end{aligned}$$

Letting $t \to t_0$ finishes the proof. □

5.4 Reduced distance and volume based at a submanifold at singular time

The purpose of this section is to show that similar to Section 3.3, we can define a reduced distance and volume based at a submanifold at singular time under the type A curvature bound assumption.

Theorem 5.4.1 *Let $(M^n, g(t))$ be a Ricci flow on $[0,T)$ of type A, and $S \subset M^n$ be a compact (not necessarily connected) embedded $k-$dimensional submanifold, $0 \le k < n$. Let also $t_i \nearrow T$. Then there exists a locally Lipschitz function*

$$l_{S,T} : M^n \times (0,T) \to \mathbb{R},$$

which is a subsequential limit

$$l_{S,t_i} \xrightarrow{C^0_{loc}(M^n \times (0,T))} l_{S,T}$$

and satisfies for all $(q,\bar{t}) \in M^n \times (0,T)$ the following differential (in)equalities:

$$-\frac{\partial}{\partial \bar{t}} l_{S,T}(q,\bar{t}) - \Delta_{g(\bar{t})} l_{S,T}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{S,T}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(T-\bar{t})} \ge 0, \tag{5.13}$$

or equivalently for $v_{S,T}(q,\bar{t}) := \left(4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-l_{S,T}(q,\bar{t})}$

$$\Box^*_{g(\bar{t})} v_{S,T}(q,\bar{t}) \le 0. \tag{5.14}$$

$$-|\nabla^{g(\bar{t})} l_{S,T}(q,\bar{t})|^2_{g(\bar{t})} + R_{g(\bar{t})}(q) + \frac{l_{S,T}(q,\bar{t}) - n}{T-\bar{t}} + 2\Delta_{g(\bar{t})} l_{S,T}(q,\bar{t}) \le 0. \tag{5.15}$$

$$-2\frac{\partial}{\partial \bar{t}} l_{S,T}(q,\bar{t}) + |\nabla^{g(\bar{t})} l_{S,T}(q,\bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{l_{S,T}(q,\bar{t})}{T-\bar{t}} = 0. \tag{5.16}$$

Proof. This proof is now almost identical to the proof of Theorem 3.3.1. For the uniform bound on L_{S,t_i}, we can use the same estimate: For $p \in S$, $(q,\bar{t}) \in M^n \times (0,T)$ and $L_{S,t_i}(q,\bar{t}) \ge 0$, we have

$$|L_{S,t_i}(q,\bar{t})| \le |L_{p,t_i}(q,\bar{t})|.$$

If $L_{S,t_i}(q,\bar{t}) < 0$ the bound follows from the same argument as in the proof of Theorem 3.3.1, with $\gamma(t)$ being an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic from $(q,\bar{t})$ to (S,t_i).

Similarly, the derivative bounds follow by letting $\gamma_i(t)$ in Lemma 3.3.4 be an $\mathcal{L}-$minimizing $\mathcal{L}-$geodesic from $(q,\bar{t})$ to (S,t_i). With the results about l_{S,t_0} we have stated earlier, in particular Lemma 5.2.3 and Theorem 5.2.5, the rest of the proof follows. □

With Theorem 5.4.1, we can make the following

Definition 5.4.2 *Under the assumptions of Theorem 5.4.1 we define*

$$l_{S,T} : M^n \times (0,T) \to \mathbb{R}$$

to be ***a reduced distance based at singular time*** (S,T) ***in the Ricci flow*** $(M^n, g(t))$.

Next, we define the corresponding reduced volume.

Definition 5.4.3 *Under the assumptions as in Theorem 5.4.1 we define* ***a reduced volume based at singular time*** (S,T) *by*

$$\tilde{V}_{S,T}(\bar{t}) := \int_M v_{S,T}(q,\bar{t})dvol_{g(\bar{t})}(q).$$

Remark 5.4.4 *Contrary to Remark 3.4.2, the finiteness of $\tilde{V}_{S,T}(\bar{t})$ does not follow by the same argument. However, we can transfer the estimates from Theorem 4.1.2 to the submanifold setting, in the same way as we did in the proof of Lemma 5.3.2. We conclude that for every $\bar{t}$, $\tilde{V}_{S,t_i}(\bar{t})$ subconverge to $\tilde{V}_{S,T}(\bar{t})$, which is finite.*

Now we state the monotonicity for this reduced volume.

Corollary 5.4.5 (Monotonicity based at a submanifold at singular time) *Under the assumptions as in Definition 5.4.3, we have*

(i) $\frac{d}{dt}\tilde{V}_{S,T}(\bar{t}) \geq 0,$

(ii) If $\tilde{V}_{S,T}(\bar{t}_1) = \tilde{V}_{S,T}(\bar{t}_2)$ *for* $0 < \bar{t}_1 < \bar{t}_2 < T$, *then* $(M^n, g(t), l_{S,T}(\cdot t))$ *is a gradient shrinking soliton in canonical form.*

Proof. This is now standard. □

5.5 Examples

We consider examples, to indicate that the structure of the submanifold S in the Ricci flow $(M^n, g(t))$ determines the behavior of the monotone quantities defined in the previous sections.

Example 5.5.1 *Consider the Ricci flow on* $[0, T)$ *from Example 3.5.5 on* $M^n = N^{n-k} \times \mathbb{R}^k$, *where* N^{n-k} *is an Einstein solution. Let*

$$S = N^{n-k} \times \{x_0\} \subset M^n.$$

Then we use (3.21) to compute for $t_i \in [0, T)$, $p = (\theta_0, x_0)$, *as well as* $q = (\theta, x) \in N^{n-k} \times \mathbb{R}^k$

$$\begin{aligned} l_{S,t_i}(q, \bar{t}) &= \inf_{p \in S} l_{p,t_i}(q, \bar{t}) \\ &= \inf_{\theta_0 \in N^{n-k}} l_{\theta_0, t_i}(\theta, \bar{t}) + \frac{|x - x_0|^2_{\mathbb{R}^k}}{4(t_i - \bar{t})} \\ &= \frac{n-k}{2}\Bigg(1 - \frac{\arctan\sqrt{\frac{t_i - \bar{t}}{T - t_i}}}{\sqrt{\frac{t_i - \bar{t}}{T - t_i}}}\Bigg) + \frac{|x - x_0|^2_{\mathbb{R}^k}}{4(t_i - \bar{t})} \end{aligned}$$

In particular, we find that for any $p = (\theta_0, x_0) \in M$

$$l_{S,T}(q,\bar{t}) = \frac{n-k}{2} + \frac{|x-x_0|^2_{\mathbb{R}^k}}{4(T-\bar{t})} = l_{p,T}(q,\bar{t}),$$

and therefore, as computed before,

$$\tilde{V}_{S,T}(\bar{t}) = \left(\frac{R_{g_0}}{2\pi en}\right)^{\frac{n}{2}} Vol(g_0) = \tilde{V}_{p,T}(\bar{t}),$$

i.e. we can say that

$$\theta_{S,T} = \left(\frac{R_{g_0}}{2\pi en}\right)^{\frac{n}{2}} Vol(g_0) = \theta_{p,T}.$$

Alternatively, in Example 5.5.1, we can immediately see from Lemma 5.2.3 and the $\mathcal{L}-$geodesic equation that $\mathcal{L}-$minimizing $\mathcal{L}-$geodesics from $(q,\bar{t})$ to (S,t_0) are constant in N^{n-k}. This is similarly the case for a symmetric neckpinch as discussed in Section 5.1. In fact, the symmetries in equation (5.1) imply

$$l_{p,T} = l_{\Sigma,T}$$

for a symmetric neckpinch, where $p \in \Sigma$. For general Ricci flows, however, we do not obtain this.

CHAPTER 6

Conclusions

6.1 Reduced distance based at a point at singular time

In Chapter 3 of this dissertation, we have extended the reduced distance to base points at singular time, and have shown that the associated reduced volume is monotone. We can make the following alternative definition of a reduced distance based at singular time, which we denote by l.

Definition 6.1.1 *Let $(M^n, g(t))$ be Ricci flow on $[0,T)$ of type A. For $p \in M^n$, we define*

$$l(q,\bar{t}) := \frac{1}{2\sqrt{T-\bar{t}}} \inf_{\gamma} \int_{\bar{t}}^{T} \sqrt{T-t}\big(|\dot{\gamma}(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))\big)\,dt,$$

where the infimum is taken over all (piecewise C^1) curves

$$\gamma : [\bar{t}, T] \to M^n,\ \gamma(\bar{t}) = q,\ \gamma(T) = p.$$

The infimum is well-defined, since the functional is bounded from below because of the type A assumption. In general, however, $l(q,\bar{t})$ will not be assumed by a minimizing curve $\gamma : [\bar{t}, T] \to M^n$, $\gamma(\bar{t}) = q$, $\gamma(T) = p$.

Understanding the relation between $l(q, \bar{t})$ and $l_{p,T}(q, \bar{t})$ might help to answer both, the question of uniqueness of $l_{p,T}$, i.e. independence of the sequence $\{t_i\}$ used to define it, as well as the question of the dependence of $l_{p,T}$ on the base point $p \in M^n$. This is of particular importance in the case of gradient shrinking solitons, where the relation between the soliton potential f and a reduced distance $l_{p,T}$ based at singular time needs to be further understood (in the noncompact case). The failure of f to be equal to $l_{p,T}$ seems to contain geometric information about the structure of the soliton, in particular the relation between the critical points of f and the choice of the base point p. Here, both the comparison geometry and the partial differential equations points of view of the reduced distance should be helpful. Techniques involving the reduced distance based at singular time, similar to the ones in this dissertation, are also a main tool in [23] to classify 4-dimensional solitons of nonnegative curvature operator.

We discussed applications to the singularity analysis using $l_{p,T}$ in Chapter 4. The uniqueness of the reduced distance based at singular time would imply a well-defined notion of the density $\theta_{p,T}$ in Section 4.2. Regularity theorems similar to White's local regularity theorem for the mean curvature flow [35] should then hold, i.e. using localized quantities one should get a generalization to singular time of the local regularity theorem in [25].

One can obtain similar results to those in this dissertation for the forward reduced distance and volume as defined in [12]. In this case, gradient expanding solitons replace the role of gradient shrinking solitons. We will not include further details here.

6.2 Reduced distance based at a submanifold

We have generalized the reduced distance to be based at a compact $k-$dimensional submanifold S, $0 \leq k < n$, of a Ricci flow on M^n in Chapter 5, and extended the associated reduced volume monotonicity to singular time. We mention that one can also develop the theory of $\mathcal{L}_S-$geodesics similar to Perelman's theory of $\mathcal{L}-$geodesics. In particular, for each $t' \in [\bar{t}, t_0)$, the **time t' $\mathcal{L}_S$-exponential map** is

$$\begin{aligned} \mathcal{L}_S exp_{t'}: \quad (TS)^{\perp} &\rightarrow M^n, \\ (p, v) &\mapsto \gamma_{p,v}(t'), \end{aligned}$$

where $p \in S \subset M^n$, $v \in (T_pS)^{\perp} \cong \mathbb{R}^{n-k}$ and $\gamma_{p,v}$ is the $\mathcal{L}-$geodesic with $\gamma_{p,v}(t_0) = p$ and initial vector v at t_0. One can then study the $\mathcal{L}_S$-cut locus and $\mathcal{L}_S$-Jacobi fields to obtain the expected results.

Similarly to the reduced volume based at a point at singular time, we expect

$$\lim_{\bar{t} \nearrow T} \tilde{V}_{p,T}(\bar{t}),$$

if finite, to carry information about how the submanifold S evolves under the Ricci flow on M^n. We can then define a density of a submanifold in the flow. When $S = \Sigma$, the singular set, this limit is related to the structure of the singularity as is the case for the reduced volume based at a point in the singular set at singular time. If the singular set does not collapse completely (unlike a neckpinch), this quantity should provide insight which the reduced volume based at a point cannot provide. Finally, we expect that one can weaken the smoothness assumptions on the set S, in particular to include more general (compact) singular sets.

APPENDIX

Appendix A

Shi's derivative estimates

For the convenience of the reader, we give the following statement of Shi's local derivative estimates [32], as it can be found for example in [7]:

Theorem A.0.1 *For any* $\alpha, K, r > 0$ *and* $n, m \in \mathbb{N}$, *there exists a constant* $C = C(\alpha, K, r, n, m)$ *such that if* M^n *is a manifold,* $p \in M^n$, *and* $g(t)$ *is a Ricci flow on* $U \times [0,T]$, $0 < T < \frac{\alpha}{K}$, *where* $\bar{B}_{g(0)}(p,r) \subset U$ *open, and if*

$$|Rm_{g(t)}(x,t)|_{g(t)} \leq K \qquad \text{for all } x \in U \text{ and } t \in [0,T],$$

then

$$|\nabla^m Rm_{g(t)}(y,t)|_{g(t)} \leq \frac{C(\alpha,K,r,n,m)}{t^{\frac{m}{2}}}$$

for all $y \in B_{g(0)}(p, \frac{r}{2})$ *and* $t \in (0,T]$.

Appendix B

Cheeger-Gromov-Hamilton Compactness Theorem

For statement and proof of the following compactness theorem, see for example [7]. It is Hamilton's adaptation of the well-known Cheeger-Gromov Compactness Theorem from Riemannian geometry to the Ricci flow.

Theorem B.0.2 (Cheeger-Gromov-Hamilton Compactness) *Let* $\{(M_i^n, g_i(t), x_i)\}, t \in (\alpha, \omega)$, *be a sequence of complete pointed solutions to the Ricci flow and* $t_0 \in (\alpha, \omega)$. *Assume that*

(i) $|Rm(g_i(t))|_{g_i(t)} \leq C$ *on* $M_i^n \times (\alpha, \omega)$ *for some constant* C *independent of* i *and*

(ii) $inj_{g_i(t_0)}(x_i) \geq \delta > 0$ *for some constant* δ *independent of* i.

Then there exists a subsequence of $\{(M_i^n, g_i(t), x_i)\}$, *which converges to a complete pointed solution* $\{(M_\infty^n, g_\infty(t), x_\infty)\}, t \in (\alpha, \omega)$, *to the Ricci flow and*

$$|Rm(g_\infty(t))|_{g_\infty(t)} \leq C \text{ on } M_\infty^n \times (\alpha, \omega).$$

The convergence is in the $C^p(g_\infty(t_0))$*-topology for each* $p \geq 0$, *i.e. there exists a sequence of open sets* $U_i \subset M^n_\infty$ *with* $x_\infty \in U_i$, $U_i \subset U_{i+1}$ *and* $\cup_i U_i = M^n_\infty$, *and there exists a sequence of diffeomorphisms* $F_i : U_i \to M^n_i$ *with* $V_i := F_i(U_i)$ *and* $F_i(x_\infty) = x_i$ *such that for every compact* $K \subset M^n_\infty$ *and* $I \subset (\alpha, \omega)$

$$F_i^* g_i(t) \to g_\infty(t) \text{ in the } C^p(K \times I, g_\infty) \text{ topology .}$$

The injectivity radius bound needed to extract the limit flow generally follows from Perelman's "No Local Collapsing Theorem."

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