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Helen Featherstone
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MATHEMATIZING, IDENTIFYING, AND AUTONOMOUS LEARNING: FOURTH GRADE STUDENTS ENGAGE MATHEMATICS

By

Marcy Britta Wood

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ABSTRACT

MATHEMATIZING, IDENTIFYING, AND AUTONOMOUS LEARNING: FOURTH GRADE STUDENTS ENGAGE MATHEMATICS

By

Marcy Britta Wood

Over the past decade, mathematics education researchers have used the lens of participation to explore how student activity relates to mathematical learning. While this lens is useful in drawing connections between mathematical activity that occurs during lessons and subsequent mathematical activity, it does not provide a specific conception of central concerns of mathematics education: student learning and thinking. Narrowing participation to communication and equating thinking with communication addresses this problem. Using this frame, discourse becomes the object of study and learning is defined as a change in discourse. However, what remains unexplained are questions about why students in similar circumstances engage in different activity or why the same student engages in different activity on different occasions. Researchers have turned to identity to tie student activity to who students think they are. By examining the interplay of discourse about who students think they are (identifying) and discourse about mathematical objects (mathematizing), it is possible to explore, elaborate, and clarify connections across mathematical activity, identifying activity, and learning.

One tool that supports the analysis of mathematizing, identifying, and learning is a theory of autonomous learning. Autonomy is a central goal of reform mathematics education, yet it has not been well defined or well studied in classroom settings. Defining autonomous learning as a constellation of mathematizing and identifying activities reflecting curiosity creates a tool for evaluating students' learning activities and suggesting how mathematics educators might support more mathematically desirable learning outcomes.

This study used the lenses of mathematizing, identifying, and autonomous learning to probe the discourse of three fourth grade students engaged in a mathematics lesson. The analysis of discourse evaluated changes in each student's mathematical discourse across the lesson, examined mathematizing and identifying activities might be grouped and described as a kind of learning, and made connections between those changes in discourse and kinds of learning. Finally, the kinds of learning and changes in mathematical discourse were examined using the theoretical lens of autonomy to determine what advantages the activities of autonomous learning might have for the development of a mathematically desirable discourse.

The findings show five kinds of learning: engaged, directed, covert, watchful, and guided. Each was characterized by a distinctive group of intertwined identifying and mathematizing activities that could be connected to changes in discourse. The kinds of learning also varied in their resemblance to the activities of autonomous learning, with the kind of learning which was the most autonomous exhibiting the development of a more mathematically desirable discourse.

The findings carry implications for how mathematics educators might construct learning situations to encourage identifying and mathematizing to best support student learning. They also suggest that the lenses of mathematizing, identifying, and autonomous learning are useful tools for examining complexities of students' actions and interactions as they learn mathematics.

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CHAPTER 1

INTRODUCTION

> *I stop by Dquan's desk. He is a fourth grade student working on a math problem that asks him to decide whether a rectangle has been divided into thirds. Part of the solution to this problem includes careful attention to a triangular shape arising from the partitions inside the rectangle. Dquan has written his solution to this problem. However, when I ask him what he has done, he is unable to explain his work. He is even unable to read his written explanation. I puzzle for a moment about how he could have so much done and yet have so little to say about it. Then my eye strays to Dquan's neighbor Corey whose work looks almost identical to Dquan's, except for one peculiar change of words: In the sentence where Corey has written "triangle", Dquan has written "Hiangle". I look more carefully at Corey's paper and realize that from the side, the "tr" in Corey's "triangle" does look remarkably like Dquan's "H". Clearly Dquan has been copying what he sees on his neighbor's paper. (02/14/06)*

Dquan's copying was not inherently problematic: He was a student in a reform-oriented mathematics classroom in which the teacher encouraged students to use each other as resources. The teacher felt that students learned better when they had opportunities to work together, support each other, and talk through their mathematical ideas. Unfortunately, Dquan did not seem to be using the opportunity to interact with his neighbor in ways that supported the kind of mathematical learning his teacher imagined. Rather than learn mathematics from or with his neighbor, Dquan seemed to be avoiding mathematical learning and focusing instead upon getting something written on his paper.

This vignette is not atypical of elementary mathematics classrooms. It illustrates problems of mathematical learning that teachers and education researchers have thought about, worried about, and worked on for many years. Why couldn't Dquan explain the mathematics of the task he was working on? Why was he copying from Corey instead of discussing the mathematics of the problem with him? Was Dquan learning mathematics by copying the right answers? What would need to change so that Dquan would engage in learning mathematics in meaningful ways? These are the types of questions that have motivated my work as a researcher. As a classroom teacher and as a classroom researcher, I have been interested in finding ways to make sense of and support the learning of students who, like Dquan, are engaging in mathematics lessons in ways that result in less mathematically desirable learning than their peers. This dissertation is my articulation, exploration, and demonstration of research lenses that I have found particularly helpful in beginning to answer my questions about mathematical learning. The remainder of this chapter provides an overview of how I'm framing the problem of mathematical learning. I follow that problem statement with a summary of each of the subsequent chapters in this dissertation.

Problem Statement

Within the last decade, researchers have begun to answer questions like my questions about Dquan by considering how learning is an inherently social and cultural activity. Rather than frame the individual as context-independent, researchers have worked to develop theories that view learning as becoming a participant in a communal practice (Sfard, 1998; Wenger, 1998). When applied to Dquan's situation, these sociocultural or participationist theories would consider how Dquan's participation was supported or constrained by the social, cultural, and historical context of his classroom and his group. For example, researchers using a participation lens might ask how interactions with Corey have influenced Dquan's activity or how Dquan's experience of mathematics as a cultural activity impacted his engagement in learning.

The participationist lens is broad. It does not define what counts as participation or what student activities should be studied. In particular, this broad net provides no specific conception of student thinking or student learning. In order to operationalize these terms, I have used Anna Sfard's commognitive framework (2008) which –as the name suggests – defines cognition as communication. Thus, student thinking is communication, which need not be verbal and need not be with others. When thinking is communication, the study of thinking becomes the study of discourse, with discourse encompassing verbal and nonverbal communication. Learning is also defined in terms of discourse, as a change in discourse. Finally, Sfard has tailored this framework to the field of mathematics education by using the term mathematizing to indicate participation in discourse involving mathematical objects, whether that participation is mathematically appropriate or not.

The commognitive framework raises more specific questions about Dquan's activity in the opening vignette. What mathematizing did Dquan demonstrate? Did he use mathematical words or work with visual objects? What features of the task was he able to talk about? Has his discourse about fractions changed from earlier in the lesson or the unit? While these questions would help us understand more about Dquan's thinking and learning, they do not help us understand why he was copying from Corey. He could have asked Corey to explain the problem to him, he could have asked the teacher for help, or he could have listened in on the conversation of other students to determine their answer. The commognitive framework does not help us make sense of why Dquan engaged in the specific activity of copying.

To answer questions like these about student activity, researchers have turned to identity. By examining how student activity is related to who students think they are, researchers have begun to make connections between student mathematical activity, identity, and learning (Boaler & Greeno, 2000; Jilk, 2007; Martin, 2000; Nasir, 2002; Sfard & Prusak, 2005). Identity, like participation, has historically been defined in ways that do not specifically indicate what it is or what should be studied (Sfard & Prusak, 2005). In order to operationalize identity, I have used the definition proposed by Sfard and Prusak (2005). They define identity as a significant, endorsable, and reified narrative about a person. They elaborate on this definition:

> The reifying quality comes with the use of verbs such as *be, have* or *can* rather than *do,* and with the adverbs *always, never, usually,* and so forth, that stress repetitiveness of actions. A story about a person counts as *endorsable* if the identity-builder, when asked, would say that it faithfully reflects the state of affairs in the world. A narrative is regarded as *significant* if any change in it is likely to affect the storyteller's feelings about the identified person. (Sfard & Prusak, 2005, p. 16-17, italics in original)

The use of narrative as a definition for identity allows me to emphasize two ways in which I am elaborating others' use of identity. In order to illustrate these two features, let me offer a second story about Dquan. The day before I observed Dquan copying from Corey, Andy, Dquan, and I were sitting near each other at a table.

> *Andy had realized that he could create smaller fractions by doubling the denominator and halving the area of a known fraction. He was attempting to calculate the new dominator for a fraction that was half of 1/96. He spoke as he worked, "And so ninety-six times two equals. Hmm. Nine plus nine is eighteen." Dquan noticed Andy's work and spoke to me, "You don't plus those. Don't you just time those? Nine times nine. Cause he said nine plus nine." I explained that Andy was saying ninety-six times two which meant ninety-six two times. I continued, explaining that Andy was adding ninety-six plus ninety-six. I then asked Dquan if nine times two and nine plus nine would have the same answer. Dquan replied, "I guess" and then started working on a piece of paper, multiplying nine times two and adding nine plus nine. After he finished these calculations, he exclaimed, "Oh! They do equal the same thing." (02/13/06)*

Dquan's activity on this day was quite different from the copying described in the opening vignette. While there might be many reasons for this, use of an identity lens suggests that in this moment Dquan was identifying differently than he was in the opening vignette. Rather than rely on someone else's mathematical thinking, Dquan identified as someone who could ask questions and investigate ideas about mathematics.

Other studies linking mathematical activity and identity have not used identity in a way that would capture Dquan's shift in identities between these two vignettes. For example, Nasir (2002) emphasized how an individual's identity changed as they learned so that as the person became more proficient at an activity, they identified with others who were also more proficient. This conception of change in identity does not capture the ways in which a person's identification might differ as situations change across lessons or even within a lesson. Researchers have not analyzed student activity for changes in identity from moment to moment as context shifts. This example of Dquan illustrates why this more static conception of identity is problematic: Dquan could be identified very differently depending upon whether his identity was based upon the first vignette or the second. An identity based on the first vignette would fail to capture Dquan as someone who investigates mathematical conjectures and might only imagine Dquan as someone who copies. Instead of declaring one identity for Dquan as he learns mathematics, it is important to capture Dquan's identities in both moments and to study how the identities arise in the moment. This might allow for the construction of learning environments that encourage investigation rather than copying. Thus my analysis of identity emphasizes the ways in which identity seems stable and yet is fluid and susceptible to change from moment to moment.

My construction of identity also emphasizes how identity arises from interactions among students. Other studies of identity have not examined the role of student interactions. For example, Boaler and Greeno (2000) considered how students came to identify as math people or not math people based upon the match between their conceptions of themselves as learners and the activities required in their math classes. That analysis does not contribute to making sense of the difference in Dquan's activity between the two vignettes. I draw upon positioning theory (Harré & van Langenhove, 1999; van Langenhove & Harré, 1999) to elaborate how interactions among individuals convey identities. For example, when I explained to Dquan why Andy was adding nine plus nine, I asked Dquan if he thought adding two nines would have the same outcome as multiplying nine by two. I could have told Dquan that the outcome of multiplying by two and adding two nines would be the same or I could have told Dquan that he didn't need to know what Andy was doing. However, my reply to Dquan suggested a story in which he was identified as someone who would be interested in and able to determine the outcome of mathematical operations.

This study draws upon this elaborated notion of identity to examine how students' mathematizing and identifying are connected to their interactions and to their learning of mathematics. As the vignettes with Dquan illustrate, close examination of student activity has the potential to reveal surprising variation and complexity in student identity and mathematizing that could have important implications for learning.

This mathematizing and identifying lens has implications for reform mathematics education. The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), a guiding document for reform in K-12 mathematics education, suggests that student interaction is essential for learning mathematics with understanding. To facilitate interaction among students, teachers frequently place students in groups to complete instructional tasks. However, students don't always learn well in group settings. Researchers have examined the challenges of students learning from students through a variety of lenses such as status (Cohen, 1994), helping behaviors (Webb & Mastergeorge, 2003) and distribution of resources such as information (Buchs *et al.*, 2004). However, researchers have not yet explored the role of identity in facilitating or constraining learning in group settings. Because this project focuses on mathematizing and identifying as it occurs in small groups, this project may add information that could help scholars and teachers to make sense of group learning and to design more productive group work situations.

Finally, the mathematizing and identifying lens I propose also helps us to study and elaborate a central goal of reform mathematics education: autonomy. Mathematics educators have stated the importance of autonomy in learning mathematics (Ben-Zvi & Sfard, 2007; Kamii, 1994; National Council of Teachers of Mathematics, 2000; Piaget, 1948/1973; Warfield *et al.*, 2005; Yackel & Cobb, 1996). I became interested in autonomy as I looked for a way to capture the differences in student activity such as those demonstrated by Dquan above. In the second vignette, Dquan acted in ways that were consistent with descriptions of autonomy. He "[took] control of his learning", "explore[d] mathematical ideas", and "reflect[ed] on his thinking" (National Council of Teachers of Mathematics, 2000, p. 21). He wondered about Andy's mathematics and, when I answered his question, he did not merely accept my explanation, but investigated the mathematics for himself.

Elaboration of the activity of autonomous learning through the lens of mathematizing and identifying offered the possibility of capturing details of student activity that seemed mathematically desirable and analyzing and suggesting changes to activity that seemed less desirable. Thus I expanded my study of student learning to consider the ways in which student activity resembled the activity of autonomous learning and how that activity connected to changes in mathematical discourse.

Overview of the Dissertation

In the subsequent chapters of this dissertation, I elaborate my theoretical framework and present my analysis of data before describing the implications of this framework for research and teaching.

In Chapter 2, I add details to the theoretical framework sketched above. I begin by explaining how participationist theories contribute to examination of learning mathematics and then note challenges of using those theories. I discuss how Sfard's commognitive framework (2008) narrows participation to communication and focuses analysis of mathematical learning on discourse. I then describe why and how identity is important to making sense of student activity and I explain my construction of identity. Finally, I connect mathematizing and identifying to autonomous learning, constructing a definition for autonomous learning and describing the mathematizing and identifying activities that indicate autonomous learning.

In Chapter 3, I describe my research process. I discuss the theoretical basis for my methods and then describe the setting for this study including the school, classroom, teacher, and students. This description is followed by details of the data I collected, my analysis procedures, and limitations of my methods.

Chapter 4 orients the reader to the findings chapters. In this chapter, I provide a brief description of research on the teaching and learning of area, the mathematical focus of the lesson I examined. I then outline the lessons in the unit and indicate the position of the focal lesson in the unit. Finally, I describe the focal lesson in detail and lay out features of the mathematical discourse we might ideally hope that students would be able to use by the end of the lesson.

Chapters 5, 6, and 7 present my findings. Each chapter considers the learning of an individual student during the mathematics lesson. First, I analyze the change in each student's mathematical discourse across the lesson. I then characterize the kind of learning in which each student engages and explore how the mathematizing and identifying of the student compares to autonomous learning. The conclusion of each chapter describes connections between mathematizing, identifying, and changes in discourse. I also evaluate the ways in which the student demonstrates (or not) autonomous learning and discuss implications of their mathematizing, identifying, and change in discourse for our understanding of autonomy and learning mathematics.

Finally, in Chapter 8, I look across the three findings chapters to answer my research questions. I discuss my findings, elaborate implications for teachers and researchers, and discuss limitations and future research directions.

CHAPTER 2

THEORETICAL FRAMEWORK

In this chapter, I will build the theoretical background for my model by examining a series of theoretical lenses and empirical studies. I intend to show how my model integrates and extends the work of several other researchers. As I move from lens to lens, I also refer back to Dquan and the two vignettes from Chapter 1 as a way of showing how this progression of lenses allows an increasingly complex analysis of classroom events.

The Participationist Lens

Over the past decades, some educational researchers have adopted a perspective on learning that emphasizes the ways in which learning arises from and is connected to participation. Theories using this *participationist perspective* (Sfard, 2006) include *situative/ situated* (Boaler, 2002a; J. S. Brown *et al.*, 1989; Lave & Wenger, 1991), *social* (Schoenfeld, 1999; Wenger, 1998), and *sociocultural* (Cobb & Hodge, 2002). These theories define learning in terms of participation, as the increased ability to participate in the activities of a community (Greeno *et al.*, 1996).

The participationist perspective is frequently contrasted with cognitive or acquisition-oriented lenses which view learning as something individuals acquire (Sfard, 1998). For example, an acquisitionist might ask *whether* a student learned and *what* the student learned with responses to these questions framed in terms of immaterial, knowledge-oriented objects that the learner might obtain. In contrast, the participationist perspective assumes that people are always learning (Wenger, 1998) which eliminates questions about *whether* students learned. Instead, this perspective asks *what* the student learned and seeks to answer that question in terms of activity and interaction that arise from examining *how* the student participated. For example, Boaler (2002b) examined differences in learning between traditional and reform mathematics classrooms. Using an acquisitionist lens, she found that the students in the reform classrooms appeared to have learned more than the students in the traditional classrooms. However, as she examined student learning through a participationist lens (2002a), she concluded that both groups of students – those in traditional classrooms and those in reform classrooms – learned a great deal but that *what* they learned was different. The students in traditional classrooms learned "to watch and faithfully reproduce procedures and … to follow different textbook cues that allowed them to be successful as they worked through their books" (2002a, p. 43). In contrast, the students from the reform classrooms learned to use mathematics in a variety of different, open-ended contexts. The participationist lens also enabled Boaler to see how this difference in participation resulted in differences in test performance between the two groups of students. The test that Boaler used to compare the students required the ability to use mathematical knowledge in diverse ways and as part of authentic activity. The students from the reform classrooms had learned mathematics in ways that resembled the kinds of engagement necessary to successfully answer the test questions. In contrast, the repetitive learning of the traditional students did not give them experiences that were useful on the test. As a consequence, the students from the reform classrooms outperformed the students from the traditional classrooms on the test. Boaler concluded from this analysis that the ways in which students participate in learning mathematics has consequences for what they learn and the circumstances in which they can utilize their learning.

While participationist theories have been helpful in making sense of how student learning varies as students participate in different activities and different contexts, the broad brush of participation does not provide a means to specifically explore an important concern of mathematics educators: student thinking. In order to make statements about the mathematics Dquan understood, the vignette must be examined with a lens that explicitly defines thinking in such a way that makes it accessible to researchers. Furthermore, in keeping with participationist tenets, thinking, which is usually conceptualized as originating from within the individual, must be defined in a way that acknowledges how individual activity arises from and connects back to the activities of the community. One theory that appears to be particularly promising for defining thinking in ways that meet these challenges is Anna Sfard's commognitive framework (2001, 2008).

Narrowing Participation to Communication

Sfard's framework defines thinking as communicating with oneself. Equating thinking with communicating acknowledges the recursive relationship between the individual and the community: Thinking is the individualization of the communication of the community. This individualized communication might then be introduced to the community where it could alter the communication of the community. For example, as Dquan's class studied fractions, one student, Andy, realized that new fractions could be created by doubling the denominator and halving the size of an already existing fraction. Andy's thinking was connected to the mathematical practices established in this classroom: Students had been constructing familiar fractions by determining their area. Andy individualized this community practice by exploring how new fractions could be constructed from these familiar fractions. Andy shared his thinking with the class and although his explanation was initially resisted, some of the students eventually showed interest in exploring his ideas, thus establishing the possibility that Andy's thinking might affect the activity of his classroom community.

Defining thinking as communicating makes discourse the object of study. Discourse is broadly defined as communication, so gestures and other nonverbal or nonlinguistic means of communicating are considered in a commognitive analysis. By focusing on communication, the commognitive framework solves one dilemma of participationist frameworks: setting boundaries on what to observe and how to focus an analysis. Rather than make everything a student does a potential object for analysis, the commognitive framework concentrates on communication involving mathematical objects. These communications, whether mathematically appropriate or not, are called mathematizing.

Using Sfard's framework to analyze the vignette with Dquan would lead to careful examination of Dquan's mathematizing: what he communicated about mathematics through writing, speaking, or gesturing. For example, Dquan's written word "Hiangle" would be investigated for its relationship to mathematical discourses including discourses specific to this classroom. A researcher might want to explore the possibility that "Hiangle" was Dquan's individualization of the classroom discourse around triangles. Dquan's discourse as he explained (or, in this case, was unable to explain) his use of the word would provide clues to Dquan's thinking.

If the researcher had access to other moments in which Dquan communicated about fractions, she might be able to make claims about a central, if not the central, phenomenon of interest to mathematics educators: learning. Defining thinking as communicating suggests that learning, as an outcome, can be defined as a change in discourse. As students engage in the process of learning, they are working (even if tacitly) to change their communication with themselves and with others. The effectiveness of the process of learning can be evaluated by comparing the resulting change in discourse with the mathematically desirable discourse. In the specific case of mathematics, a desirable outcome of learning means that the learner is more able to engage in an appropriate mathematical conversation with him/herself and others (Ben-Zvi & Sfard, 2007). Dquan could be said to have learned if his communication around the mathematics task changed so that he used a more mathematical discourse to explain whether a given rectangle was partitioned into thirds.

The commognitive framework lists four features of discourse that can be used in determining what changed in Dquan's discourse (or that of another learner) and whether that change is more mathematically desirable. These features are mathematical word use, visual mediators, discursive routines, and endorsed narratives. I will briefly describe how Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) define each feature.

Word use considers how speakers use mathematical words. Ben-Yehuda et al. (2005) examined each mathematical word uttered by their research participants. They specifically considered whether mathematical words referred to written or spoken symbols or to independently existing entities. For example, using area, a student could use *unit* to refer to a repeated shape partitioning a figure drawn on paper into equal pieces or to a concept not tied to a physical object (i.e. a unit of measurement, which is not necessarily pointing to something concrete). When the discourse referred to independently existing entities, the discourse is objectified, meaning that the learner refers to the mathematical entity as an object separate from any concrete representation. Objectified discourses are also impersonal. Rather than talk about what they did (i.e. I counted an area of eight), students using objectified discourses use the mathematical words as the subjects of utterances (i.e. The area of the figure is eight). Ben-Yehuda et al. (2005) note that discourse of formal mathematics and that used by mathematicians is impersonal and objectified and that students who used objectified discourses have been found to have more effective mathematical performances. Thus, it seems important to evaluate a discourse for objectivity as that could mean that the students using the discourse have made advances toward a more mathematical and academically productive way of talking about mathematics.

A second feature of discourse, *visual mediators,* are symbols, icons, or concrete things that are the objects of the discourse (Ben-Yehuda et al., 2005). They are what the discourse refers to, the objects (concrete or imagined) about which individuals are communicating. For example, as students compare the area of two figures, they will refer to the figures in their discourse. In order for the sentence, "They both have two," to be meaningful to everyone involved in the conversation, all interlocutors must understand that "they" refers to the two figures (and that two refers to the area in units). The figures thus mediate communication in this conversation. Mediators can also be iconic (such as graphs or other pictures) or symbolic (such as ½ or any other mathematical symbol, whether present in written form or spoken).

Discursive routines are "a set of meta-rules that specify both *when* and *how* repetitive discursive action is employed" (Ben-Yehuda et al., 2005, p. 203, italics in original). As interlocutors develop a discourse, patterns emerge in what to say and when. A common example of a discursive routine occurs when two friends meet each other. Seeing a friend for the first time during the day (*when*) initiates a routine of greetings and counter greetings (*how*):

Friend 1: "Hello. How are you?"

Friend 2: "Fine. How are you?"

Friend 1: "Good."

This routine is complete after this exchange and the friends are free to engage in a different discourse, which will have its own set of meta-rules and routines. Mathematical discourses also have routines. For example, when asked for the area of a figure partitioned into square units, a student might count the number of square units and then announce the number. Having stated this piece of information the student would then stop. The student has initiated a routine in response to a question embedded in a context. The student proceeded through the routine to an answer and then terminated the routine. Thus discursive routines have rules that govern when to initiate them, what to do and say, and when to stop.

The fourth feature of mathematical discourses is the *endorsed narrative*. The use of narrative in the commognitive framework differs somewhat from the use of narrative in literacy. Narratives in literature are usually accounts of events. In contrast, narratives in the commognitive framework are

> a series of utterances, spoken or written, that is framed as a description of objects, of relations between objects, or processes with or by objects, and is subject to endorsement or rejection, that is, to being labeled as "true" or "false" (Sfard, 2008, p. 300)

Endorsed narratives are statements that participants in the discourse believe to be true. Math facts and definitions are two examples. Other endorsed narratives may be unique to the particular discourse. For example, students and their teacher might develop a classroom discourse around a particular task. Features of that task might be incorporated in endorsed narratives. For example, the statement, "Two triangles make a square" is a narrative because it describes a process with mathematical objects. It is not endorsable across any two triangles. However, in the context of task in which all triangles are congruent, isosceles, right triangles, this statement is true and thus, when individuals act as though they accept this narrative as true, it is an endorsed narrative.

These four discursive features are helpful in examining changes in student discourse and how student discourse differs from mathematical discourse. Using these tools to make these comparisons allows claims about outcomes of student learning. If these tools were applied to Dquan's discourse, we could make claims about what he learned and in what ways his learning outcome was mathematically desirable. However, these tools and this framework as elaborated thus far does not provide a means for explaining why Dquan's activity was so different from his neighbor's. Looking closely at Dquan's mathematical discourse cannot answer questions about why Dquan chose to copy from Corey instead of engaging in the same work or why Dquan did not ask Corey to explain the problem. The commognitive framework also does not address how Dquan's activity might result from interactions with Corey or others. To answer questions about how students act and why they might act in different ways, researchers have turned to the notion of identity.

Mathematical Activity and Identity

By examining how student activity is related to who students they think they are, researchers have begun to make connections between student mathematical activity, identity, and learning (Boaler & Greeno, 2000; Jilk, 2007; Martin, 2000; Nasir, 2002; Sfard & Prusak, 2005). For example, Boaler and Greeno (2000) demonstrated the interplay between mathematical activity, learning, and identity in their study of Advanced Placement (AP) Calculus students. They interviewed several students who were learning in classrooms characterized by Boaler and Greeno as didactic. The rhythm of these rooms was predictable from day to day: The students reviewed homework, watched the teacher demonstrate the next procedure, and then practiced the procedure using problems from the textbook. Students were discouraged from discussing the mathematics or the problems and were instead directed to focus on their own work. Boaler and Greeno interviewed several successful students from these classes. They found that some of the students liked the classes and were identified as "math people" while several of the students expressed a dislike for the ways of learning required by the class. This second group of students noted that their math class asked them to act in ways that were not congruent with how they saw themselves. They felt they were creative, verbal, and independent and believed that if they continued in mathematics they would need to give up these ways of being. According to Boaler and Greeno, the identities of this second group of students were not consistent with the mathematical activity required in their mathematics class. As such, they did not identify as math people. Both groups of students were successful in their calculus classes. However, their individual experiences of the classes varied, as did their identification as math people. Boaler and Greeno's study suggests that Dquan's identification of himself as a mathematical learner might be related to the match between his identity as a learner and the mathematical activities required in his class.

The work of Boaler and Greeno suggests that students may participate in similar mathematical activities and come to different conclusions about their identities. The work of Sfard and Prusak (2005) adds more complexity to the relationship between identity and mathematical activity: Rather than focus on how activity influences identity, their work suggests that individuals' identities influence their mathematical activity. Sfard and Prusak looked at differences in activity and identity between two groups of Israeli high school students, the OldTimers and the NewComers. The two groups had very different ideas about both who they were to become (what Sfard and Prusak refer to as designated identities) and about how to engage in mathematical learning. The NewComers felt that learning mathematics was essential part of becoming completely human. In order to be a whole person, they needed to understand mathematics. In contrast, the OldTimers felt that who they were to become was fluid and could change. Mathematics was important because it could open doors, but it was not essential to who they were. When each group of students engaged in learning mathematics, the ways in which they participated were substantially different. When the NewComers studied, they worked until they felt they understood the mathematical content. When the OldTimers studied, they focused on producing written evidence that they had done the assigned work, but they did not test themselves or double check their understanding. Sfard and Prusak suggest that differences in these students' ideas about their designated identities could account for these differences in their mathematical activity.

Sfard and Prusak's proposal that identity influences mathematical activity is supported by the findings of other researchers exploring connections between racial/ethnic and gender identity and activity. Some of these other researchers hail from the field of psychology, which means that their research methods are different from those of educational researchers. However, their work is helpful in thinking about connections between identity and activity. One team of researchers, McGlone and Aronson (2006), started with the assumption that individuals have multiple identities. For example, a person may identify as a woman, as an Asian American, and as a student at an elite private school. (Multiple identities are important to my model. I will have more to say about them later.) McGlone and Aronson proposed that reminding students of particular identities could impact students' performance on tests. They worked with male and female undergraduates and gave them one of three tasks that asked them to reflect upon their experiences relative to their gender, their elite private college, or the region of the country in which they currently live (the control group). This task was meant to evoke a particular identity (gender or elite college student). The researchers followed this identity task with a spatial reasoning test. When they compared the women's results from the different identity tasks, they found that the group of women who answered questions about their college scored the highest. The control group was the middle score and the gender experiences group had the lowest score.

McGlone and Aronson use "stereotype threat" (Steele & Aronson, 1995) to interpret these differences in performance. Stereotype threat suggests that individuals may underperform in intellectual tasks because they are worried about "confirming or being judged by a negative social stereotype" (Steele & Aronson, 1995p. 797). McGlone and Aronson suggest that asking some of the women to reflect upon their experiences as a woman made gender identity salient for them. When these women worked on the spatial reasoning test, they underperformed because of their apprehension about confirming the stereotype that women do poorly on spatial reasoning tests. In contrast, the women who were asked about their college experiences were not in a context where their gender was salient. Instead, they were reminded that they were high status students at a private institution. McGlone and Aronson argue that priming this student identity seems to have thwarted effects of stereotype threat and even enhanced student performance.

This work of McGlone and Aronson draws upon and adds to the work of other researchers who have found that racial/ethnic identity, like gender identity, can also positively or negatively affect performance. Shih, Pittinsky, and Ambady (1999) conducted a similar study with Asian-American women. They gave the women a task that asked about either their gender experiences, their family heritage including other languages spoken at home, or their preferences for cable television (the control group). The women were then given a math test. The analysis of the math test results showed that the lowest performing group was the group asked about their gender experiences. The highest performing group was the group asked about their family heritage. Like McGlone and Aronson, Shih, Pittinsky, and Ambady concluded that reminding women of their gender identity invoked a stereotype threat that undermined their performance. In contrast, reminding women of their Asian-American identity invoked a positive stereotype that enhanced performance.

The studies of McGlone and Aronson and Shih, Pittinsky, and Ambady provide evidence that complements Sfard and Prusak's suggestion that identity affects mathematical activity and can be directly manipulated. Their work also extends that of Sfard and Prusak by providing evidence that context can evoke particular identities. (This extension will be important later when I discuss the importance of interactions.) Taken together with Boaler and Greeno's study on calculus students, this research illustrates the bidirectional relationship between identity and mathematical activity and implies that questions about learning cannot just focus on mathematical activity but must also consider student's identity. As students learn and change their activity, their ability to do new things can lead to a new or changed identity. Also, a student's identity may lead them to seek changes in their mathematical activity and thus changes in their learning. These relationships between identity, mathematical activity and learning are supported by Nasir's study (2002) of mathematical learning among African-American students.

Nasir focused on students' out of school experiences with mathematics, observing students as they played basketball and dominoes. She found that as basketball players became more engaged in basketball, they became more interested and better skilled at calculating their performance statistics, which lead to more engagement in basketball and a shift in identity that reflected this increased engagement. As the students began to see themselves as serious basketball players, they set new goals for themselves and worked to change their participation to reach those goals. Thus, as the students learned more about playing basketball, including calculating their statistics, their identity as basketball players shifted and as they identified as serious basketball players, they learned more about basketball and statistics. Drawing from her observations of these students, Nasir argued that learning, identity, and mathematical activity are intertwined and simultaneously affect each other.

The research studies linking identity and mathematical activity that I reference above are primarily focused on elaborating identity. As such, they do not provide a specific definition of mathematical activity, participation, or learning. None of them draw upon the commognitive framework or limit their examination of mathematical activity to mathematical communication (although two of them (Nasir, 2002; Sfard & Prusak, 2005) include students' communication as they engage in mathematics in their analysis of mathematical activity). Their findings are still useful in constructing this theoretical framework. However, my use of a more narrow conception of mathematical activity means that my study may add more complexity and/or different features to understanding the relationship between mathematical activity, identity, and learning mathematics. In order to emphasize my focus on mathematical activity as communication about mathematical objects, I will begin to use the words mathematizing or mathematizing activity instead of mathematical activity.

A Definition for Identity

The relationship between mathematizing activity, identity, and learning can be further clarified through a definition of identity. Because the definition of identity organizes the connections between the elements of my model, I will devote some space to developing and elaborating my definition. Sfard and Prusak (2005) note that many researchers (in particular Gee (2001) and Holland, Lachiotte, Skinner, and Cain (1998)) have suggested "who a person is" as a definition for identity. While this definition feels like a reasonable match with an intuitive sense of what identity might be, it does not require or necessarily evoke a connection with participation. In addition, "who a person is" suggests that identity is a stable essence of a person while I have already proposed that identity is dynamic and changes with mathematizing (and that mathematizing changes with identity). Finally, my definition of identity must also account for stereotypes and the ways in which one person has multiple identities.

In order to meet the requirements listed above, I am drawing upon the definition of identity proposed by Sfard and Prusak (2005). They suggest that identity is a collection of reifying, significant, and endorsable narratives about a person. They elaborate on this definition:

> The reifying quality comes with the use of verbs such as *be, have* or *can* rather than *do,* and with the adverbs *always, never, usually,* and so forth, that stress repetitiveness of actions. A story about a person counts as *endorsable* if the identity-builder, when asked, would say that it faithfully reflects the state of affairs in the world. A narrative is regarded as *significant* if any change in it is likely to affect the storyteller's feelings about the identified person. (Sfard & Prusak, 2005, p. 16-17, italics in original)

For example, statements like "I am a woman" or "I am a math person" are identities for me: I believe them to be both true and important to how I feel about myself. Sfard and Prusak's definition captures the link between mathematizing and identity through the concept of *reified narratives*. Because this concept is vital to the articulation of mathematizing and identity, I will spend some time elaborating the terms narrative, reification, and reified narrative.

As I noted in my discussion of the four discursive features above, narrative has a variety of definitions arising from its use across multiple research disciplines and traditions. My use of narrative to define identity draws upon the definition of researchers whose research interest and genre of data are similar to my own. Ochs and Capps (2001) studied narratives arising in everyday conversations. They were interested in how impromptu, co-constructed narratives help people understand themselves and others with whom they interact. Their work parallels my focus on everyday conversations between students as they negotiate their participation and identities in math class.

Ochs and Capps define narrative as an account of life events. They note that narratives may have description, chronology, evaluation, and explanation, but that that these elements do not have to be explicitly evident in order for a person's speech to qualify as a narrative. Ochs and Capps' flexible definition of narrative means that minimal statements are also narratives as long as they are an account of an event. Other researchers have also argued this point. Dino Felluga (2003) along with Michael Riffaterre (1990) argues that short sentences like "The road is clear" are narratives because they suggest some sequence of events. The reader is drawn to wonder why the road is clear, whether we should cross the road, or whether the clear road is a moment of calm. This notion of a minimal narrative means that statements like "I am a math person" are also narratives because they conjure images of what such a person has done or might do. This connection between these minimal narratives and potential actions will become important in my discussion of identities below.

Ochs and Capps' definition of narrative points to an important feature of narrative: Narrative is not a straightforward list of everything that occurred but is instead the narrator's abridged and edited *account*. This reworking of events is a necessary part of how humans make sense of what happened. Events are not meaningful in themselves. Instead, people work to understand events by focusing on particular parts of the event and weaving these parts with different framings until the narrative offers an explanation of the event that matches other narratives the author already has (Ochs & Capps, 2001). For example, in the opening vignette, I told a particular narrative about Dquan. However, I could have told a different story. I could have said that Dquan was worried that he would look stupid if he asked Corey to explain what he was doing, so he was copying as a way of saving face. Another alternative was to say that Dquan didn't want to extend the mental effort necessary to figure out the math and so he was copying. Either of these scenarios is a possibility. However, I settled on my narrative because it resonates with other narratives I have including narratives about myself, Dquan, his neighbor, this math classroom, fourth grade students, and math learners. In addition, my narrative reflects my ideas about narratives written for academic audiences, which means that I included as little explicit interpretation as possible. However, if I told a story about this event to Dquan's teacher, my narrative might incorporate more interpretation and labeling of Dquan. I might say that Dquan doesn't think he's smart or I might draw upon other narratives that the teacher has told me about Dquan. Thus, I abridge and edit my account of events in order to make sense of the event in light of narratives I have about both the people directly involved in the event and the people involved in listening to my account of the event.

Sfard and Prusak's concept of reified narrative unites narrative with another mechanism people use to making sense of the world: reification. According to Wenger (1998), reification is "the process of giving form to our experience by producing objects that congeal this experience into 'thingness' (p. 58). My mathematizing-identifying-learning model is an example of reification. As I construct this model, I am making an object that freezes the transient actions of mathematizing and identifying and allows

those actions to be manipulated, examined, and connected in ways that are not possible when they are conceptualized as fleeting experiences.

As people tell stories, they tend to summarize the story's action into statements that describe or label people. These statements reify activity into human conditions. For example, at the end of Dquan's vignette, I could have added, "Dquan was reluctant to engage math." This sentence condenses Dquan's copying and his inability to explain the mathematics of the problem into a tidy description and interpretation of my perception of Dquan's problem. Once I have generated this reified narrative, I no longer need to tell the entire story about Dquan's actions. Instead, I have a short sentence that captures the activity and turns it into a thing (Dquan's reluctance) that I can use to make sense of my interactions with Dquan.

This short sentence along with other significant and endorsable reified narratives about Dquan is an identity of Dquan. It is important to note that as narratives, identities have an author, an identified person, and an audience (Sfard & Prusak, 2005). For any identified person, there can be multiple identities as different authors tell different identities to different audiences. For example, depending upon my audience, I might identify Dquan as a reluctant participant, as a low student, or as an African-American boy. Dquan might also alter his identity depending upon his audience, narrating himself as a talented football player, a caring friend, or an incompetent math student.

These multiple identities point to one of the complexities of identity: It is situated and dynamic yet it appears to be a stable essence of a person. I can identify Dquan in multiple ways at different times, each time making an endorsable and significant identification matched to the situation. Dquan is an African-American boy when I'm talking about demographics in a case study. He is a reluctant student when I'm discussing his activities during math class with his teacher. He is a low student when his teacher suggests that he receive extra tutoring support. Each of these different and changing identities seems to be a stable and essential part of Dquan. This sensation of a stable essence arises from identity's connection to reification. Wenger (1998) notes that the process of reification projects understanding into the world in such a way that it appears disconnected from its human origin. The reified object seems to be a real, independent thing rather than a subjective human creation. Also, as the object loses its history of creation, it becomes timeless. Thus as narratives become reified into identities, they lose their human author and gain a sense of permanence. No longer are they one person's perspective on a transient occurrence. Instead they are narratives that that stretch through time and become something the person has always done and will always do.

However, because identities maintain their tie to actions, they are still situated. Thus, a particular context can evoke a particular identity and not others. For example, Dquan's identity as a football player does not enter my introductory vignette because *my interpretation* of the context and situation in the vignette did not evoke the narrative of Dquan as a football player. If I wrote about Dquan at recess, I might see his identity as a football player as more relevant to my narrative. (This is not to say that Dquan's identity as a football player is irrelevant when he is doing math. He might see this identity as central to how he talks and interacts in math class. After careful study, I might also be able to construct a narrative in which the football player identity is significant in the context of the math class. However, I want emphasize that identities are dynamic and situated and that there might be times when Dquan is seen as a football player and other times when he is seen as a reluctant student.)

Thus far, I have elaborated upon the ways that my definition of identity accounts for multiple identities and can be dynamic yet stable. I noted above that my definition of identity must also account for stereotypes. It must be able to describe how invocations of stereotypes work through identity to affect mathematizing activity as demonstrated in the work of McGlone and Aronson (2006) and Shih, Pittinsky, and Ambady (1999). A stereotype is a widely-circulating reified narrative that oversimplifies or exaggerates the actions of a population of people. For some people, when they interact with members of this population, these reified narratives are significant and endorsable and thus constitute identities. This construction of stereotype relies upon two important features of identity that I have yet to elaborate: identities as the tip of the iceberg and identities as the union of the individual and the collective. I will develop both features and then return to stereotype.

Wenger (1998) described reified objects as "the tip of the iceberg" (p. 61). What is visible (or in the case of speech, audible) is connected to a more detailed and hidden story of actions. Identities as reified narratives can be thought of as the tip of a narrative iceberg: An identification of a person comes with expectations for how the identified person has acted and will act. For example, if I say that Dquan is a low student, to the extent that others agree with me, they will use that identity to construct Dquan's past and future narratives as a student. Another important feature of this iceberg metaphor is that because the actions are hidden, the nature of the underwater iceberg is may be different for different people. If I identify Dquan as a boy, I might be reifying a narrative about

Dquan's preferences for recess activities. However, others might use that identity to imply that Dquan's participation in class resembles that of research about boys.

These different variations on the "underwater" narrative are an important outcome of identity's union of the individual and the collective. Because the individual and the collective commingle in narrative, I will begin by describing narrative's role in this union before moving to identity. As I noted in my discussion of narrative above, an account of events is constrained by its resonance with other narratives. While multiple stories might be told about an event, the narrator who is interested in making sense of events and communicating with others will construct an account of events that matches other narratives that person has about themselves and others. These other narratives come from a variety of sources. They may arise in a local community such as a classroom or school or the narrator may draw upon national or international narratives. As the narrator uses these other narratives, they incorporate collective perspectives into their individual account of events. Also, as individual narratives are told and retold, they become part of the narrative possibilities other authors can use in crafting their own narratives. Thus the individual creation becomes a collective resource. This "communalization of the individual" (Sfard, 2006, p. 23) is important because it provides a mechanism for variation in narratives. It means that as the individual creates a narrative, s/he is not necessarily constrained by the most popular national narrative. Instead, there are a variety of narratives circulating, allowing for many different individual narratives.

Identities as endorsable reified narratives capture this same union of the individual and collective. When I state, "I am a woman" I draw upon circulating identities of women to make sense of and communicate my understanding of my actions.

As narratives about individual women circulate and are reified (which especially happens when the story is about an extraordinarily positive or negative event), they become identities that are collectively available. For example, as stories are told and reified about U.S. Speaker of the House Nancy Pelosi or NASA astronaut and alleged kidnapper Lisa Marie Nowak, those individual identities (being a Pelosi or being a Nowak) could become widely available and used to identify other women.

While this merging of the individual and collective provides occasions for individuals to alter the collective understanding, it also provides a mechanism for the collective understanding to limit the possibilities for the individual. Stereotypes are an example of this. I defined stereotypes as widely circulating, oversimplified, reified narratives about a population of people. As identities, they are imposed upon people based upon the perception that they are members of a population rather than based upon individual actions. As reified narratives, they are the tip of the narrative iceberg and thus imply a particular set of past and future actions. However, because identities, as the tip of the iceberg, can have different narratives attached to them, a stereotyped person may agree with the reified narrative (such as African American, woman, or firefighter), but have a different set of actions tied to that narrative. For example, as I discussed earlier, one stereotype about women is that they are not as capable as men at tasks involving mathematics (Shih et al., 1999). Someone could identify me as woman and I would agree with that identity, but not with the accompanying stereotype about mathematical ability. Instead, for me, being a woman is tied to narratives arising from a wide variety of experiences I have had in which being a woman was salient. However, as research in stereotype threat has shown (McGlone & Aronson, 2006; Shih et al., 1999; Steele &

Aronson, 1995), people are not immune to the narratives of stereotypes: If a person is concerned that they might confirm or be evaluated by the stereotype, they may actually enact the stereotype narrative. Thus I may underperform on a mathematical task if I feel that the task might confirm the stereotype about women even though my identity as a woman and my past actions include narratives about strong mathematical abilities.

I engaged in this discussion of stereotype because I used research on stereotypes to claim that identity affects mathematizing and I wanted to demonstrate how my narrative definition of identity accommodates the connections between stereotype, identity, and mathematizing activity. I want to conclude this discussion of my identity definition with a summary of the ways in which identity and mathematical activity are intertwined. Both begin with a person's actions. Activities involving communications about mathematical objects are mathematizing activities. An account of these actions, can become an identity. In addition, identities, because they imply future actions, constrain and enable options for mathematizing activity. Because learning arises from mathematizing, the connection between identity and mathematizing is critical to learning. As identity encourages participation in activities that lead to changes in mathematical discourse, identity is implicated in learning.

Using mathematizing and identity together raises new questions about Dquan's activity in the opening vignette. How has Dquan identified himself and how have others identified him so that copying is the appropriate form of activity for his identities? What identities are constructed by Dquan and others based upon narratives about his copying? What identities do Dquan and Corey have for Corey and how do those identities lead to Dquan's decision to copy from Corey? Thus far, these questions assume my perspective on events. However, Dquan and Corey might have very different narratives to describe what happened. What narratives about Dquan's participation might be told by each of them? What different identities would arise from those narratives?

Underlying most of these questions is the notion that identifying is not solely an individual activity. Dquan might identify himself in a variety of ways, but he also has identities that are written by Corey, his other classmates, his teacher, his family, and many others including myself. My framework as articulated so far assumes that an identity composed by someone else might affect an individual's activity. For example, Dquan's copying may arise from the ways in which his peers or his teacher have identified him even though Dquan's peers or teacher may not have specifically articulated their identifications of him. However, my framework does not so far offer an explanation of how one person's activity might be affected by other people's identifications of him or her nor does it articulate how individuals can communicate identifications of each other through interactions that don't necessarily include reified narratives. To do this work of tying together mathematical activity and identification, I draw on positioning theory.

Moving from Identity to Identifying

Positioning theory is an appropriate tool for extending my mathematizing-identity framework because it uses story in ways that are parallel to my use of narrative in the construction of identity. Positioning theory assumes that an individual's actions can be interpreted in multiple ways. In order to make sense of what is happening, people use stories and they position themselves and any others involved in the activity in the story (van Langenhove & Harré, 1999). For example, if one student says, "Let me see your work" to another student, the second student might understand those words in different ways depending upon how s/he positions her/himself and the other student in a story arising from those words. The second student might draw upon a story in which s/he and the speaker are friendly peers in which case the second student could understand "Let me see your work" as a plea for assistance. Or, the second student might use a story of intimidation and position the speaker as a bully in which case the second student could understand the words as a command that must be obeyed. The story is important to understanding because it interprets actions and explains who people are.

This use of story and position is similar to my use of narrative and identity except that positioning theory assumes that all people involved in an interaction are identified through the narrative/story enacted in the interaction, even if they are not explicitly identified using a reified narrative. Instead of relying only on reified narratives, identification through positioning is based upon the cluster of utterances and actions that are appropriate for that person in context of the story (Harré & van Langenhove, 1999). This connection between actions and identity draws upon the "tip of the iceberg" metaphor discussed earlier. However, in this instance, the actions that are the underwater part of the iceberg are used to determine what identity might be at the tip. For example, if a student wants to be identified as a bully, they might march up to another student and gruffly declare, "Let me see your work." The bully is enacting a story of what bullies do, part of which involves positioning others as helpless victims. Neither the bully nor the student victim has been identified using a reified narrative, but if the bully has been successful in their posturing, gesturing, and speech, they have communicated both their identity as a bully and their identification of the other student as a helpless victim. In addition, the bully may also identify anyone else in the room as either allies of the bully or other potential victims by either laughing and winking at the bystanders or by scowling menacingly at them.

In positioning theory, even narratives about a single person identify both the author and the audience of the narrative in addition to the subject of the narrative. In my opening vignette about Dquan, I was identifying myself as someone who is interested in elementary students who aren't engaging in mathematical learning and I was also identifying my readers as concerned about struggling students and what I might have to say about them.

While positioning theory provides a mechanism for identifying everyone involved in a conversation, it also provides a means for individuals to negotiate their identities. In the example of the bully, the person identified as the helpless victim could dispute this identity by refusing to act in the ways that helpless victims might act. Instead of meekly handing over their work, the "helpless victim" could stand up and firmly say, "No." This move shifts everyone's positions. The "bully" has been identified by the student as another student and not someone with particular power and the "helpless victim" has become an assertive person. This positioning and repositioning could continue as the "bully" and the "helpless victim" negotiate this interaction.

If the "helpless victim" had accepted this identification instead of disputing it, they would have indicated their acceptance through their actions and speech including their interactions with the bully. In other words, the helpless victim's activity would demonstrate their identification as the helpless victim while it would also assert the bully's identification as a bully. Thus, positioning theory unites interaction, activity, and identity by describing how individual's communication when viewed by others indicates how the individual is identifying themselves and others. This identification then opens pathways for other individuals to act and interact in ways that either affirm this identity or declare another identity.

Sfard's recent work on identity (2007) elaborates ways of identifying that are consistent with the indirect story-telling assumptions of positioning theory. Sfard has described three ways in which people identify: direct, indirect-verbal, and enacted. *Direct* identifying occurs as a person tells a reifying story about the identified person. I primarily referred to this way of identifying as I elaborated my definition of identity in the sections above. *Indirect-verbal* identifying is when a story is told about a person that does not include reifying statements. Finally, a person may identify him/herself or another through other activities that do not include story telling. Sfard calls this type of identifying *enacted.* As students interact in classrooms, they rarely directly identify each other. Instead, most identifications are enacted or in-direct verbal. I have used positioning theory to provide a framework for elaborating these last two ways of identifying.

Positioning theory also supports the specific vocabulary I am using to capture my thinking about identities. Rather than describe individuals as *having* identities, I draw upon Sfard's recent work (2007) and purposefully use the words *identifying*, *enacting identities*, *identification,* and *engaging in identifying activity*. These ways of talking are consistent with the notion of positioning, which emphasizes the ways in which identities are constructed by individuals in response to situations.

Positioning also captures the ways in which identities may appear to be stable but are also situated and dynamic. An individual, through activity, both offers and authors positions for him/herself and others. The positions are offered because they are consistent with reified narratives the individual has about him/herself and others, but those narratives have been authored by the individual so while the reification suggests permanence, the authorship indicates the human activity which could author and offer another identity in another situation.

In summary, I define identifying as participation in discourse that communicates a significant, endorsable, and reifying narrative about an individual. This narrative need not be directly communicated: It may be suggested by what an individual says or does. It may be accepted or rejected by other individuals involved in the communication process. Finally, it suggests a storyline that guides the activities of the identified individual.

Theoretical Links Between Mathematizing and Identifying

My framework of identifying has many parallels with my description of mathematizing. Because identifying arises from communication of stories, it, like mathematizing, arises in discourse. Identifying and mathematizing are both focused on the intersection of the individual and the community. Both are also central to my definition of learning as a change in discourse. Let me elaborate this point. As a student engages in learning, he or she both communicates about mathematics and identifies him/herself as a kind of learner of mathematics. His/her communication about mathematics simultaneously tells an identity story and provides the opportunity of changing the learner's mathematical discourse. Identifying that does not address a mathematical object (and so is not mathematizing) tells how the student and others in his/her audience might engage in mathematizing. For example, the student might proclaim to another student, "You're so fast at math." This statement is about a student and not about a mathematical object, but it identifies both the speaker and the addressee relative to their mathematizing. It implies that the addressee is a fast mathematizer while the speaker is slower. This difference in speed might also be tied to perceptions of a difference in competence so the speaker might also be identifying the addressee as a more competent mathematizer. This statement about speed could affect the subsequent mathematizing of the speaker and addressee: The addressee might accept the identification as fast and then turn to see what the speaker is doing so that the speaker is feeling slower. The addressee might offer some mathematical advice that benefits the speaker. The addressee might also agree that he/she is fast and then race ahead through more problems, leaving the speaker to feel slower, less competent, and less enthusiastic about engaging in mathematizing. This example shows how identifying, even when it is not directly about a mathematical object, can have implications for students' engagement with mathematical objects. These connections between identifying and mathematizing suggest that learning arises from the interplay of these two activities (Sfard, 2007).

Theoretically, the claim that learning is the interplay of mathematizing and identifying seems to be quite productive in making sense of connections between students' thinking about themselves as learners of mathematics and their thinking about mathematics. However, this theoretical work has not yet been well explored in classroom data. It is one of the goals of this dissertation to elaborate how this interplay among identifying, mathematizing, and learning. For example, in the second vignette, I identified Dquan as interested in and capable of exploring a mathematical idea about the relationship of multiplying and adding. Dquan, by investigating this mathematical idea, both tacitly agreed with my identifying statement and engaged in mathematical thinking. His subsequent words, "Oh! They do equal the same thing," seemed to suggest that

Dquan had changed his thinking about multiplying and adding. He seemed to realize that adding two nines was the same as multiplying nine times two. While more investigation of Dquan's previous and subsequent discourse would further substantiate this claim, Dquan's statement seems to indicate different thinking from when he asked me why Andy was adding nine plus nine. This analysis of this vignette with Dquan shows how Dquan has engaged in the process of learning by both identifying and mathematizing. Furthermore, it suggests that his participation in learning was effective as it changed his discourse so that it was more mathematically appropriate.

In the three cases I present as Chapters 5, 6, and 7 of this dissertation, I investigate this claim further. Specifically, in each chapter, I will elaborate a different kind of learning and show how that kind of learning consists of a particular constellation of mathematizing and identifying activities. I will also relate the kind of learning to the effectiveness of the learning.

My analysis of learning, mathematizing and identifying includes one final piece that might best be described by returning to the second vignette with Dquan. In this vignette, Dquan engaged in mathematics in ways that reflected descriptions of ideal mathematical learning. He worked to make sense of the Andy's mathematics. He started a conversation about something that seemed contradictory to him and he investigated my response, rather than accepting my words based on my authority as an adult. Dquan, through his questions and exploration, displayed curiosity about Andy's mathematical ideas. Dquan's activity in this second vignette is important because it resembles the descriptions of autonomous learning that mathematics educators have proclaimed as the goal of school mathematics (Ben-Zvi & Sfard, 2007; Kamii, 1994; National Council of

Teachers of Mathematics, 2000; Piaget, 1948/1973; Warfield *et al.*, 2005; Yackel & Cobb, 1996). In the next section of this chapter, I elaborate my definition for autonomous learning.

Autonomous Learning

In 1948, Jean Piaget proposed that autonomy should be the goal of education and he elaborated this idea in the context of learning mathematics (Kamii, 1994; Piaget, 1948/1973). Constance Kamii reintroduced his ideas to mathematics education in the 1980's and early 1990's through a series of books focusing on children learning mathematics (i.e., Kamii, 1985, 1989; i.e., Kamii, 1994). More recent work of mathematics educators cites the work of Kamii and Piaget, concurs that autonomy is a major goal of mathematics education, and begins to describe autonomous learning (Ben-Zvi & Sfard, 2007; Warfield et al., 2005; Yackel & Cobb, 1996). However, this recent work has not elaborated the activities of autonomous learning or explored the concept in classroom settings. For example, Yackel and Cobb (1996) and Warfield et al. (2005) offer brief definitions of autonomy and Yackel and Cobb describe how autonomy is related to judging mathematical contributions, but they do not explicitly explore the connection between student autonomy and learning mathematics. Given the stated importance of autonomy to mathematics education and the possibility that autonomy could be a useful lens for differentiating and explaining student learning such as Dquan's activity in the two vignettes from Chapter 1, I propose that it is critical to further investigate autonomous learning.

Researchers seem generally to concur with Kamii's explanation of Piaget's definition of autonomy as the ability to think for oneself and to decide between truth and untruth (Kamii, 1994). My definition of autonomous learning acknowledges the importance of the individual's thinking. However, I define autonomous learning in terms of the learner's focus on and desire to understand experiences including both physical experiences and interactions with others. I draw upon the theory of learning elaborated above to define autonomous learning in terms of the interplay of mathematizing and identifying. Specifically, I explore autonomous learning as a constellation of mathematizing and identifying activities reflecting curiosity about how things are – both what others think/say and what seems to be true.

My use of curiosity is purposeful because it captures the ways in which autonomous learners compare their thinking with their observations and experiences. Curiosity requires that the learner verify whether his or her thinking adequately describes, accounts for, or matches his or her experiences. The Oxford English Dictionary defines curiosity as "the desire or inclination to know or learn about anything, [especially] what is novel or strange; a feeling of interest leading one to inquire about anything" ("Curiosity", 2008). The emphasis on the novel or strange highlights the autonomous learner's passion to make sense of ideas that he/she does not yet understand.

John Dewey argued that curiosity was necessary for reflective thought as it provided the awareness and the store of facts and experiences from which thoughts would arise (Dewey, 1910). Dewey characterized intellectual curiosity as "interest in *problems* provoked by the observation of things" (p. 33, italics in original, 1910). For Dewey, observation was not enough: The curious learner formulated questions about those observations and was actively exploring and seeking material for thought. My definition of autonomous learning draws upon these notions of the learner as active, problem seeking, and interested in the novel or strange. Curiosity means that the learner is not satisfied with memorization or answers for the sake of other's approval: The learner's autonomy is in part due to the learner's own passion for understanding.

The autonomous learner's curiosity leads him/her to explore problems arising from observation of discourse. He/she investigates mathematical discourses and verifies that the learner's communication about those discourses is compatible with the communication of others who are proficient in the discourse. This verification ensures that the learner's interpretation of the discourse is consistent with other's use of the discourse and reflects the learner's attempts to make sense of the other's discourse. It ensures that the learner is decentering (Piaget, 1932/1960) or is doing more than reflecting their own thinking: They are making sense of other's thinking, whether the other is the teacher, the textbook, or other students. Autonomous learning also involves the quest for mathematical truth. The learner uses logic to examine a discourse for contradictions both within the discourse and between different discourses. The learner then works to resolve any contradictions by altering their thinking or by proposing changes or additions to the discourses.

This notion of autonomous learning is consistent with the picture of autonomous learning portrayed by Ben-Zvi and Sfard (2007). They emphasize that autonomous learners explore the discourse of experts working to make the *discourse-for-others* into a *discourse-for-oneself*. Ben-Zvi and Sfard define discourse-for-oneself as "a discourse to which … the learner would turn spontaneously whenever it may help her in solving her own problems" (p. 77). For example, the learner's peers might try to explain to the learner that two triangles can be put together to make a square that can be used as a unit for determining area. In order to make this statement into a discourse-for-oneself, the learner needs to investigate this discourse. Perhaps he/she needs to assemble triangles into squares to see how they relate to area. Perhaps he/she needs to attempt to determine the area of a figure that is partitioned into triangles and squares. As the learner examines this discourse and incorporates the discourse in his/her thinking, the discourse becomes a discourse for the learner.

The concept of discourse-for-oneself is similar to Wenger's (1998) concept of *ownership of meaning*. Wenger defines ownership of meaning as "the degree to which we can make use of, affect, control, modify, or in general, assert as ours the meanings that we negotiate" (p. 200). As the learner transforms a discourse into a discourse-for-him/herself, he/she becomes able to use that discourse to solve problems and accomplish other tasks. This use of the discourse for the learner's own ends demonstrates Wenger's ownership of meaning. The discourse is not merely words mechanically recited to get the approval of another person, such as a teacher. Instead, when a learner owns the meaning of a discourse, the discourse becomes a tool for the learner. Ownership of meaning acknowledges the social nature of discourse: Ownership means the learner can use the discourse to accomplish social ends. If a learner's use of a discourse is not understood or is rejected by others, then the learner does not own the meaning in a way that is valued within the community. Wenger's concept of ownership of meaning clarifies that Ben-Zvi and Sfard's notion of discourse-for-oneself does not mean the discourse is only used to communicate with oneself or solve the individual's problems in isolation. Instead discourse-for-oneself emphasizes that the learner can communicate with herself as well as with others using the discourse. Discourse-for-oneself and ownership of meaning are consistent with my definition of autonomous learning. The process of autonomous learning results in a discourse-for-oneself and ownership of meaning as the learner mathematizes and identifies in ways that reflect curiosity about others' discourses and mathematical truth.

As the learner engages in autonomous learning, he/she performs several identifying and mathematizing activities that support and reflect his/her curiosity about mathematical discourses. I will elaborate these discursive activities and explain how they are connected to autonomous learning.

An individual engaged in autonomous learning identifies the audience of his/her discourse in particular ways. The learner interacts with his/her audience in ways that suggest they should provide support, evaluation, or verification of the learner's investigation. For example, the autonomous learner might position his/her audience as experts in the discourse who can provide feedback on the fidelity of the learner's interpretation of the discourse. As another example, the learner might position the audience as co-learners who are interested in what he/she has to say and who can ask questions to clarify the learner's meaning. In the eyes of the autonomous learner, the audience is not a gatekeeper whose approval must be won through the meaningless performance of a ritual. In addition, the learner him/herself is an important member of his or her audience. As the autonomous learner communicates with him/herself, he/she can make sense of the discourse for him/herself. This understanding for oneself is an important goal for the autonomous learner.

The next three discursive activities of autonomous learning are connected to the activity of mathematizing. First, a person engaging in autonomous learning faithfully adopts the mathematical discourse of those whom she considers as expert interlocutors. This does not mean that the learner is not critical of the discourse. However, before the learner can critique or examine another's discourse he/she must be sure that his/her interpretation and use of the discourse is consistent with other's use. As I described earlier in this chapter, the commognitive framework describes four features that make discourses distinct: word use, visual mediators, discursive routines, and endorsed narratives. Faithful adoption of the discourse means that the learner's use of the discourse is consistent with others' use of the discourse across these four features.

In addition to adopting the discourse, individuals engaged in autonomous learning also produce new discursive features that build from the adopted features of the discourse. For example, a discourse might focus on assembling triangles into squares to be counted. The learner might realize that two right triangles assembled into a square are each half of the square and might start to talk about each triangle as half. Use of the word "half" both builds from the original discourse and the way triangles were used in this discourse and is also a new word use for this discourse. Production of new discursive features arises from and contributes to investigation of the original discourse. The autonomous learner's curiosity about how people use a discourse and how the discourse relates to truth leads to explorations about what could be communicated using a discourse. Exploration of what is possible with a discourse could lead to new discursive features. Thus one more question to ask while investigating autonomous learning is whether the learner mathematizes in ways that build from and are novel to the discourse as used by others.

The final mathematizing move related to autonomous learning is substantiation of discursive narratives. Sfard (2008) defines a narrative as "a series of utterances, spoken or written, that is framed as a description of objects, of relations between objects, or processes with or by objects, and is subject to … being labeled as 'true' or 'false'" (p. 300). Examples of narratives include statements such as "two triangles make a square" or "this triangle is half a square." Narratives can also include statements about processes like "you count the squares to figure out the area." For example, in the second vignette above, Dquan substantiated the narrative, "nine times two is the same as nine plus nine." This statement is a narrative because it describes two processes and their relationship and, as Dquan demonstrated, can be labeled as true or false. Sfard notes that discourses have meta-rules that direct their substantiation. In mathematics for example, discursive narratives can be substantiated by a mathematical proof.

An examination of autonomous learning considers two aspects of substantiation: whether the learner relies upon her own judgment and how well the learner employs the meta-rules of the discourse. In school settings, students may not use their own discursive resources to substantiate a narrative. Instead, they may rely upon other individuals such as the teacher, another student, or the answer key in the back of the textbook. According to Piaget (1932/1960), this reliance on others for right answers is an indication of heteronomy, regulation or governance by others, and thus is not a trait of the autonomous, or self-regulating, learner (who might co-learn with others, but does not rely on other's authority to substantiate narratives). My construction of autonomy and substantiation does not define heteronomy and autonomy as discrete categories in which the learner's reliance upon another person means they are not autonomous. Instead, I construct heteronomy and autonomy as ends of a spectrum. The learner could rely entirely upon another to verify a statement in which case they are not investigating a discourse or exhibiting curiosity. In this case, the learner is heteronomous and not autonomous. However, the learner could ask another for an explanation of a statement. If the learner explores the substantiation supplied by another or uses this substantiation to work through their own substantiation, then the learner is more autonomous. Autonomy does not mean that the learner does not interact at all with other individuals, but it does require that the individual engage in their own examination of the discourse, even if that examination is initiated or supported by another.

Just as autonomous learners may access varying degrees of support from others in substantiating a discourse, they also exhibit varying degrees of proficiency at using the meta-rules of a discourse to substantiate its narratives. Meta-rules define the pattern of activity of a discourse (Sfard, 2008). They are the rules that guide how someone determines the truth of a statement. For example, the rules for determining whether people evolved from animals are very different in science and in the evangelical Protestant religions. Meta-rules reflect social conventions and are not necessarily imposed by external reality. A person who was familiar with a few basic narratives in mathematics could not reason from those narratives to an understanding of how mathematical truth is established. Instead, meta-rules are learned through interaction with individuals who are proficient at the discourse. As learners become more proficient in the meta-rules, they are better able to substantiate narratives in ways that are consistent with the discourse. As a consequence, they will better able to evaluate the narratives they produce in terms of the discourse and will become more autonomous in their ability to communicate using the discourse.

In school settings, the meta-rules for substantiation of a mathematical discourse may be different from mathematicians' meta-rules for substantiation. For example, in a mathematics lesson, verification of the area of a figure might require cutting apart the figure and arranging the pieces into squares to be counted. In contrast, a mathematician would accept multiple means of verifying the area of a figure and might not consider physically cutting apart the figure to be an acceptable means. This difference in substantiation means that student discourse should be evaluated to determine not only whom the learner involves in their substantiation and how that person is involved but also what rules the learner is using to guide substantiation.

In the paragraphs above, I've described four discursive activities (one related to identifying and three related to mathematizing) that characterize autonomous learning. These discursive activities provide a lens for examining student participation. For example, in the first vignette, Dquan cannot explain his work, which suggests that he was not an audience for his work. He copied what he wrote, so he was not producing new discourse. He did adopt some discourse, but only words. He could not recite narratives, enact routines, or point to visual mediators. Finally, he did not substantiate what he wrote, relying upon Corey to produce an answer that reflected the truth. In summary, Dquan did not enact autonomous learning in the first vignette. In contrast, in the second vignette, Dquan identified both himself and me as audience members. He asked me for information and then worked through two mathematical operations to determine the truth of my response. As he worked through the operations, he both substantiated my statement and produced his own discursive features (the numerical statements) that built on my statement. Dquan exhibited several mathematizing and identifying activities that were consistent with autonomous learning. In addition, the outcome of his investigation seemed to be a change in his discourse, indicating that learning happened. Dquan's activity and learning were very different in the two vignettes.

The theory of autonomous learning I articulated above seems effective in highlighting differences in mathematizing and identifying that could lead to more mathematically desirable learning outcomes. While a description of differences in mathematizing and identifying could help researchers and teachers promote certain kinds of mathematizing and identifying, a framework that theorizes the differences provides more analytic power for invoking change in teaching and learning. The final project of this dissertation is to use the lens of learning as mathematizing and identifying to craft a definition for and description of autonomous learning that is useful in analyzing student activity to sort out activity that promotes more effective learning. In each findings chapter, after I have described the kind of learning engaged in by each student, I analyze the learning for the four discursive activities described above and then draw conclusions about both autonomous learning and the effectiveness of the learning for each student.

In summary, this dissertation seeks to understand the consequences of students' classroom activities for the development of their mathematics. To do this, I use a discursive lens and define the process of learning as the interplay of communicating about mathematical objects (mathematizing) and communicating about individuals (identifying). The outcome of learning is defined as a change in discourse about mathematical objects and about the learner. Using these definitions, I probe the discourse of two groups of students engaged in a mathematics lesson. I focus on the learning (process and outcome) of three students and describe the kind of learning each student engaged in, relating the interplay of mathematizing and identifying and connecting each individual's learning process with the outcome of that process. Finally, I compare each student's mathematizing and identifying activity with a theory of autonomous learning in order to both elaborate the theory and explore how a lens of autonomous learning might help highlight limitations and affordances of each student's learning process.

Research Questions

The theoretical framework elaborated has guided my investigation of the learning of mathematics in an elementary classroom. My main research questions are:

- What effect do the activities of mathematizing and identifying have on one another and on the development of mathematics discourse?
- What is the advantage of autonomous learning for the development of mathematical discourse.

As a reminder of how I am using the terms in my questions, I offer the following short definitions: *Mathematizing* is participation in discourse (verbal and nonverbal) involving mathematical objects. *Identifying* is participation in discourse that communicates a significant, endorsable, and reifying narrative about an individual. Finally, *autonomous learning* is the constellation of mathematizing and identifying activities reflecting curiosity about how things are – both what others think/say and what seems to be true.

CHAPTER 3

METHODS

Overview

This chapter elaborates upon both the tools I employed and my sequence of work as I explored my research questions. I begin by elaborating my study design. I then describe details of my research site and my participants, my data collection procedures, and my data analysis.

Study Design

This study seeks to deepen our understanding of the sense making of students: As students talk about mathematics or make statements about themselves or others, they ascribe meaning to and inscribe meaning in each of these actions and interactions. These meanings then serve as the basis for choices about how to act, react, and ultimately learn about mathematical ideas. In order to make claims about what meanings students are making during mathematics lessons, my research is based upon interpretive methodologies that assume that people decide how to act and react based not upon other people's actions but instead upon the meanings or interpretations they make of other's actions (Erickson, 1986). For example, a student might decide to raise her hand to join the many other hands that are already raised. Her decision to raise her hand does not come from the *activity* of other students raising their hands but instead from her *interpretation of what it means* that other students have raised their hands. This assumption of interpretive methods matches my construction of narrative from my framework above: People create accounts of events that provide meaning for the event. They then react based upon that narrative. Thus an interpretive method not only supports the questions and assumptions in my study, it also parallels important features of my theoretical framework.

There are several different interpretive methods, each of which focuses on a different locus of meaning making. I have used case study because it provided a means for exploring an abstract social phenomenon by examining the phenomenon in a specific circumstance (Dyson & Genishi, 2005). In particular, focusing my study on four students in one classroom over a limited period of time allowed me to concentrate on the complex relations between the activities involved in learning mathematics (mathematizing and identifying) and the activity and outcome of learning. In addition, case study is founded upon the careful examination of the meaning-making of individuals. While it is not possible for an outside researcher to be entirely accurate in their interpretations of individual's meaning-making, the close examination of individuals provides the case study researcher with more opportunities to explore individual thinking than other research methods (Hodkinson & Hodkinson, 2001). Finally, case study supports the development of theoretical ideas (Hodkinson & Hodkinson, 2001). Case study supported my effort to elaborate a theory of autonomous learning by allowing me to work back and forth between detailed data and theory.

Case study describes my research approach and my assumptions about the sources of meaning. However, what I use to make my case about the activity of learning is the discourse, both verbal and nonverbal, among students in the classroom and between students and the teacher. My focus on discourse stems from my use of Anna Sfard's commognition framework (2008). Sfard defines thinking as communication with oneself and discourse as a specific type of communication regulated by rules for acting and reacting. Within this framework, the study of thinking is only possible through the study of discourse. Given this framework, my study of the process and outcome of mathematical learning must focus on discourse as both the means of engaging in the activity of learning mathematics and as the object and outcome of mathematics lessons. My study examines discourse during a mathematics lesson in order to construct the case of learning mathematics.

Context

Choosing a site

To learn more about interactions among students, I needed a classroom in which the students were allowed to interact during mathematics lessons. During a previous research project (Wood, 2007), I had collected data in Mrs. Smyer's classroom and I felt that her emphasis on small group work during mathematics lessons would provide an environment rich with student conversations. I had worked with Mrs. Smyer in a variety of capacities, both on research projects and as a cooperating teacher for my undergraduate students during their field experience over the previous four years. When I approached her about this project, she agreed to allow me to collect data in her room.

Just before I was to begin data collection, Mrs. Smyer was asked to move to a different classroom: She was needed to provide stability for a classroom of fourth grade students who had been taught by several different substitute teachers. Mrs. Smyer's previous class (a group of third grade students) would be taught by Ms. Cramer, who had just finished a year of student teaching in this third grade classroom.

Mrs. Smyer, and Ms. Cramer, and the principal were aware that the shift in Mrs. Smyer's classroom assignment had implications for this study and they invited me to visit both classrooms to determine which setting would be most appropriate. After visiting both classes, I decided to follow Mrs. Smyer into the fourth grade classroom. I considered several factors in making this decision. First, while Ms. Cramer was very competent beginning teacher, she was a novice teacher and I felt that if I stayed in the third grade classroom, my study would have to address her status as a new teacher. I felt this could be a distraction from my focus on student interactions. In addition, I felt that the trusting relationship I had established with Mrs. Smyer would provide me with more opportunities for and more flexibility in making observations of students.

Third, after visiting the fourth grade class, I felt Mrs. Smyer was going to be able to focus on teaching mathematics. This was a concern because her new fourth grade classroom had a reputation as a "bad" class. The class had their own recess and lunch times and while there were different explanations for this arrangement, Mrs. Smyer and other teachers felt that the isolation was a result of the misbehavior of these fourth grade students. I wondered whether Mrs. Smyer would need to focus more on managing student behavior and less on teaching mathematics or whether she would shift her teaching away from small groups and toward more individual work in order to manage student behavior. My first visit to this classroom alleviated my concerns: The students were in small groups and working on math and it was not evident to me why the class had a "bad" reputation. The students' behavior during this first visit proved to be fairly typical. While there were two occasions during my data collection in which Mrs. Smyer stopped the mathematics lesson to address the "disappointing" behavior of the students as a class, my concerns about the time spent on managing student behavior proved to be unnecessary.

In the end, collecting sufficient data was not a challenge: During the six weeks I collected data, I was able to observe 34 mathematics lessons and record over 70 hours of student interaction on videotape. Later in this chapter I will describe the details of the data I collected.

The Classroom Setting

I originally selected Mrs. Smyer's classroom because she encouraged students to interact as they worked on math tasks. However, the kind of small group interaction Mrs. Smyer encouraged and I sought for my data collection was not the norm for Mrs. Smyer's new fourth grade students. I will describe the changes Mrs. Smyer implemented as a means of conveying what the students had experienced and what changed for the students after Mrs. Smyer arrived. My data on the student's previous experience comes from an interview with their original fourth grade teacher, Mrs. Reeves. Because of the timing of the change in teachers, I was not able to observe the classroom before Mrs. Smyer arrived.

Mrs. Smyer changed the physical arrangement of the desks in the room. The desks had been arranged in rows of single desks. Mrs. Smyer grouped the desks into clusters of four desks. This physical rearrangement was tied to a change in how lessons were organized. Mrs. Reeves' lessons started with a whole group warm-up followed by individual work on problems. Mrs. Smyer started her lessons as a whole group but this was typically followed by work with a partner or a small group. Mrs. Smyer encouraged students to discuss their work with each other and support each other in making sense of the task and the mathematics.

The kind of mathematical work also changed after Mrs. Smyer arrived in the classroom. Mrs. Reeves focused on developing computational skills such as proficiency with the long division algorithm. She would model the algorithm, emphasizing the steps, where to place numbers, and what operation (i.e. multiply, subtract, and bring down) to perform. In contrast, Mrs. Smyer focused less on computational skills and more on concepts. While she occasionally gave students worksheets with similar problems, she also frequently gave students a task with one or two problems. Mrs. Smyer did not model algorithms, preferring instead to have students explain their own thinking about a problem.

Mrs. Smyer organized her math lessons into five different types of activities: math, problem solving, centers, bell work, and Big 4. A "math" activity was one in which the students worked in pairs on a specific task. The nature of the tasks varied across the six weeks including sorting and classifying cut-out shapes, rolling dice to determine probability, and completing worksheet pages on area and perimeter. "Problem-solving" activities usually involved the students working in groups of three to solve a single problem or logic task. During "centers" time, groups of three or four, the students rotated through five different centers, spending approximately 15 minutes at each one. Many centers featured games focused on mathematical concepts. Occasionally a center would involve a solving a mathematics problem with the teacher. Some centers were arts activities or focused on finishing work from earlier in the day. The last two math activities usually offered practice on computational skills. Bell work consisted of a worksheet students completed individually as they arrived in the morning. Big 4 usually occurred after lunch recess. Mrs. Smyer would have four math problems that reviewed concepts and skills on the overhead. Students copied and completed them independently.

The students were grouped differently for each of these activities. For bell work and Big 4, they sat at their regular classroom desks. They sat at different desks and worked with different students for centers, problem-solving, and "math".

In a typical week, students had centers once, problem solving twice, and "math" three times. They had bell work and Big 4 each day (although the bell work might not focus on math each day). This schedule meant that on some days, students might be involved in math activities at three or four different times. For example, students might complete bell work in the morning, followed by a math activity before lunch, and Big 4 and problem solving after lunch.

Over Mrs. Smyer's 15 years of teaching, she accumulated texts, lessons, games, and manipulatives from workshops, conferences, and research projects. She drew from across this wide variety of resources to construct the tasks and problems in her mathematics units. During the units I observed, she used the district-adopted curriculum, Harcourt (Maletsky *et al.*, 2004); *Mouse and Elephant* from Middle Grades Mathematics Project (Shroyer & Fitzgerald, 1991); and *Dot Paper Geometry* (Lund, 1980). She also used materials from a research project in which she was involved and she designed her own problems and tasks based upon her assessment of student needs.

School, Students, and Teacher

The fourth grade classroom in this study was part of a K-5th grade public school located in a residential area of a midsize midwestern city. The student population of the class was diverse in terms of both race/ethnicity and socioeconomic status. The classroom had 20 students. Eleven students identified as African American. Seven students identified as European American. One student identified as Asian American and one student identified as Hispanic. None of the students were labeled as English language learners. Twelve students received free or reduced rate lunches.

Most of Mrs. Smyer's 15 years of teaching experience was teaching fourth or fifth grade or a split 4/5 class. She did have some experience at the middle school level, teaching 7th and 8th grade math for a year and a half. Mrs. Smyer was actively involved in improving her teaching, participating in research projects in math and literacy education at the nearby university, attending workshops, and working towards a master's degree in education. Mrs. Smyer's principal described her as a highly effective teacher and her school district concurred, recognizing her as the outstanding elementary teacher of the year.

Focal Participants

In order to make sense of students' interpretations of activity and interactions, I focused my data collection on one group of students who sat together during the instructional time the teacher called "math". This group did not exist before I started data collection: The teacher allowed me to select four students for a focal group. She then arranged the other students into other groups. I chose the four focal students based upon interactions I observed during my first few visits to the classroom. I wanted some students who, like Dquan from the opening vignette, relied upon other students to support their mathematical work. I also wanted students who, like Dquan's neighbor Corey, served as teachers or supports for other students. By focusing my study on students who engaged in different mathematizing and identifying during the mathematics lesson, I hoped to make sense of the relationship between mathematizing, identifying, and learning.

I selected my focal students based primarily upon interactions that occurred during nine hours of observation over the first six days. My rationale for selection of three of the students is best demonstrated by one interaction that occurred during my fifth visit to the classroom. I will present the interaction in detail and then explain how this interaction guided my selection of students. I have included a transcript of this interaction below. While I will describe my transcription conventions later in this chapter, I will explain my use of parentheses now so that the transcript makes sense. In situations where I was not sure about what was said, I placed parentheses around the words that seemed to constitute the spoken discourse. In situations where I could not make out the spoken discourse I placed a question mark in the parentheses.

This transcript focuses on three students – Jakeel, Minerva, and Nerissa – who were sitting at the same cluster of desks, working on separate geometry tasks. Jakeel was trying to determine the name for a geometric solid he had pulled from an envelope. He asked Minerva for help.

Line Number	Speaker	Spoken Discourse	Gestures and Other Details
1	Jakeel	Is this a sphere?	*He holds up the cylinder and looks at Minerva. Minerva shakes her head.*
2	Nerissa	(It's a) sphere	
3	Jakeel	How do you spell it?	*He looks at Nerissa*
4	Nerissa	S-P-E-R-E (?) sphere	
5	Jakeel	Spell it again	
6	Nerissa	S	

7	Jakeel	S	*Jakeel repeats Nerissa*
8	Nerissa	P	
9	Jakeel	P	
10	Nerissa	E	
11	Jakeel	E	
12	Nerissa	R	
13	Minerva	That's not a sphere	*She is looking at Nerissa.*
14	Nerissa	Yes it is	
15	Minerva	It's a cylinder	
16	Nerissa	(?)	
17	Minerva	I wouldn't listen to her	*She looks at Jakeel*
18	Jakeel	I don't know who I can believe. You're my best friend	*He points at Minerva*
		and she's smart	*He points at Nerissa*

In this interaction, Jakeel identified himself as a learner by asking for help (Line 1). Both Minerva and Nerissa acted as teachers although they differed in both their answer to Jakeel's question and their enactment of teaching. Minerva's support for Jakeel consisted of shaking her head (Line 1) and telling him not to listen to Nerissa (Line 17). In contrast, Nerissa verbalized her answer. She further supported Jakeel by offering her spelling of *sphere.* However, Nerissa's answer was wrong (the shape was a cylinder) and her spelling was incorrect. Thus while Nerissa was a more willing teacher, what she taught was problematic. In Line 18, Jakeel articulated his identities for Minerva and Nerissa and how those identities complicated his choice of whose answer to use.

This interaction illustrates the reasons I elected to focus on Jakeel, Minerva, and Nerissa. Jakeel was willing to ask others for support in engaging mathematical tasks. Nerissa was willing to provide this support and Jakeel saw her as smart even though, in this instance, her support was not mathematically correct (Jakeel did not yet know whether she was right). Minerva's mathematizing was more mathematically correct than Nerissa and while she was willing to provide some support for Jakeel, her support was minimal and was very different from Nerissa's. Continued examination of the interactions among these three students seemed to offer possibilities for making sense of the relationship between mathematizing, identifying, and learning.

Once I had settled on these three students, I decided that I needed a fourth student whose mathematizing was more like that of Dquan and Jakeel than that of Nerissa or Minerva. My research interests have been driven by students' struggles to learn mathematics and it seemed that I should complete this group with a student who relied upon others rather than a student who was more self-sufficient. After reflecting over my observations, I selected Jessica. I had observed Jessica working with other students and noted her attempts to copy from her neighbor. I had also noted that a student complained that she was a difficult partner because she played too much. I felt that Jessica exhibited the kind of mathematizing that I wanted to further examine and so I included her as my fourth student.

Consent from families to include children as focal students in the study also influenced my choices. Although I obtained consent from all families and students except for one, some families indicated that they did not want their children to be focal students. The families of my four focal students and the students consented to be part of the study.

While Jakeel, Jessica, Minerva, and Nerissa were my focal students, three other students – Rebecca, Daren, and Bonita – play important roles in this study because of interactions that occurred between the focal students and these students as they worked in problem solving groups. As I will describe later in this chapter, I focused my analysis on a problem-solving activity. Because the students sit in different groups during math time and problem-solving time, my focal students were not grouped together for problem-solving time. Instead, Jakeel was with Rebecca and Daren. Minerva and Jessica were with Bonita. On this particular day, Nerissa was absent. Thus while my focal students (with the exception of Nerissa) are central in my analysis, these other students play important roles.

Researcher Role

Because my study focuses on student interaction, I worked to minimize my interactions with the students. I was not unfriendly, but I also did not initiate interactions or provide assistance resolving mathematical questions or other issues that arose. I assumed the role of "an unhelpful but attentive adult friend of children" (Dyson & Genishi, 2005, p. 52).

I met with the four focal students during the first day they were grouped together. I explained that I would be taping them each day and that the tapes were for my project and not for the teacher or their parents. I told them that while I would be watching and taping them, I couldn't answer any questions they had. Jakeel suggested that I was like a ghost and I agreed that was a useful way to think about my presence. I would be there, but I couldn't help them.

The students were usually appeared to ignore the camera and microphone, but they did occasional comment on my presence and on the video camera, sometimes speaking directly into the microphone or signaling to the camera.

Data Collection

I collected four types of data: student work, video-recordings of interviews with students and the teacher, audio-recordings and notes on informal conversations with the teacher, and several types of data documenting student interactions in the classroom.

Classroom Observations and Student Work

I will address two features of data collection around classroom observations: the time period (number of days) over which I collected data and the focus of data collection on each day. I originally intended to bound my data collection by collecting data across two instructional units, capturing one unit in its entirety and the beginning or end of a second unit depending upon the timing of my entry into the field. I thought it was important to collect data across two curricular units because students may participate in different ways when working with different mathematical content.

However, my timing shifted when Mrs. Smyer switched classrooms. This switch was made eight weeks before the end of the school year and I lost data collection time because I needed to gain consent and select focal students in this new classroom. Thus my data collection was bounded by Mrs. Smyer's time in this new classroom: I began data collection her second week in the classroom and continued through the end of school. During this time, I was able to collect data across three units: the end of a unit on geometric shapes, an entire unit on area, and a final unit on probability.

I taped and collected student work for problem-solving, math, and center time. However, I usually did not tape bell work or Big 4 because student work was primarily independent during these times and I was interested in student interactions. I focused on my focal group when they were together for math time. During problem-solving and centers, I used multiple cameras to capture each focal student in his/her separate group. In addition to videotaping, I also recorded “scratch notes” (Dyson & Genishi, 2005, p. 67) during my observations. These notes allowed me to supplement the videotape by recording details not captured on the tape such as what was happening at other groups, who was absent, and what other events happened in school that day.

At the conclusion of my data collection, I had over 70 hours of classroom video tape from 28 activities (centers, problem-solving, or math activities) collected during 20 days of observation.

Interviews

At the conclusion of my study, I interviewed each of my focal students as well as other students who had been involved in interesting interactions with the focal students. Because I did not use this interview data as part of my analysis I will summarize the interview questions rather than providing details. The interviews were semi-structured and asked about classroom events. Each interview was approximately an hour long. I asked the students to tell me about their experiences in school and describe their favorite part and least favorite part. I asked if they were good at math and who in their classroom was most like them in math. I also asked them to describe what math was like in their classroom. I had the students complete two tasks that resembled work they had during earlier mathematics lessons. Finally, I showed the students video clips from my data and asked them if they remembered the day and then asked them to narrate the moment.

I also interviewed Mrs. Smyer and Mrs. Reeves (the fourth graders’ first teacher) at the end of the study. Both interviews were semi-structured and lasted less than an hour. I asked Mrs. Reeves to describe her experiences with the class including how she taught math, what content she covered, and the history of substitute teachers in the room. I also asked her about her background as a teacher and any background information she had on each of the focal students. I asked Mrs. Smyer some of the same questions I asked Mrs. Reeves. She described the situation of the fourth grade class before she became their teacher, her background as a teacher, and background information on each of the focal students.

Informal Conversations with the Teacher

The teacher and I had several informal conversations about students and math lessons. These conversations were important to this study because they included the teacher's narrative of events that occurred when I was not present in the classroom. They also included the teacher's interpretation of events, which was helpful in making sense of students' activities. Because these conversations were spontaneous, I did not purposefully tape them. On some occasions when the conversation seemed particularly interesting and the situation was appropriate, I asked the teacher for permission to audiotape. For the other conversations, I made notes either as we spoke or shortly after our conversation.

Data Analysis

First Pass through the Data

Most of my data analysis focused on transcripts of my focal students' actions and interactions. However, these transcribed events are part of a larger context. In order to record this larger context and situate my transcriptions, I constructed written fieldnotes for each classroom observation. These fieldnotes drew upon my scratch notes, the student work, the videotape, and the informal conversations with the teacher. The fieldnotes were primarily descriptive, but they also included my reflections and questions about the events during the observation. These fieldnotes also included a first rough transcription of conversations involving any of the students in my focal group. For example, notes of whole class discussions reflected what my focal students were doing during the discussions, but do not include transcriptions of what was said unless my focal students were participating in the whole group conversation or in a side conversation.

I constructed separate fieldnotes for each lesson. I defined a lesson based upon the teacher's differentiation of math activities. Thus if the students participated in problem-solving and centers on the same day, I wrote separate fieldnotes for each. My fieldnotes totaled 369 single-spaced pages.

After I writing fieldnotes for each lesson, I wrote a reflective memo in which I recorded my thoughts and ideas on events in the fieldnotes. As I explored different ideas in these memos, I constructed labels for each idea. I then used these labels to track emerging themes across numerous memos and lessons.

Second Analytical Pass

This first pass allowed me to select moments that were particularly interesting for a second analytical pass through some of the data. This second pass considered the learning of one student during one lesson and was comprised of three parts: an exploration of the student's opportunity to learn as defined by a difference between the student's discourse at the beginning of the lesson and the presence of other, more mathematical discourses during the lesson; a description of the process of learning which told the sequence of events involving the focal student during the lesson; and, finally, an analysis of learning which identified factors that seem to support or limit the student's learning. As I conducted this second pass, I reviewed the video of each lesson I analyzed and constructed a more detailed transcript. I will describe the transcription process and my transcription conventions in the section that follows this description of my analysis.

I selected lessons for this second pass based upon my evaluation of the opportunity to learn for focal students in the lesson. I considered the student's initial discourse, the discourse of other students in the group, and the focal student's final discourse, looking for lessons in which the focal student's discourse changed during the lesson. The first lesson I chose to analyze was a problem-solving lesson on area in which Jakeel was the focal student. It was clear from the fieldnotes I generated during my first pass through the data that Jakeel's discourse had changed. In addition, I was also interested in how he involved himself in his learning. My first impression was that he was not copying from the other students but was asking them questions and working to make sense of their explanations. I felt that his change in discourse and his engagement in learning warranted a closer examination of this lesson.

As I indicated in the section on focal students, this lesson was a problem-solving lesson, which meant that my focal students were not grouped together. Jakeel was in one group. Minerva and Jessica were in another group and Nerissa was absent. After completing the analysis of Jakeel's learning, I looked at the interactions and learning in Minerva and Jessica's group for the same lesson. It became clear that Jessica's discourse changed during the lesson, but her learning activities were very different from Jakeel's activities. I decided that close examination of her activity in the context of the same lesson would provide an interesting and informative contrast to Jakeel's activities. I also analyzed a third lesson, again focusing on Jakeel's learning, but I found in that analysis that the data were insufficient to construct a satisfactory argument about Jakeel's learning and so I did not pursue this lesson beyond this second analysis.

Third Analytical Pass

As I examined my analyses of Jakeel and Jessica's learning, it became apparent that the factors I explored in this data could be sorted into the mathematizing in the group and the identifying in the group. This insight helped to structure a third pass, which also focused on the learning of one student during one lesson and had three parts: an exploration of the focal student's learning outcome as defined by changes from the student's initial to his/her final discourse; a description of the identifying and mathematizing that occurred in the group and was related to the student's learning process; and a comparison of the student's learning process and the activities of autonomous learning. In the next few paragraphs, I will describe the analysis of identifying, mathematizing, and autonomous learning.

Identifying and mathematizing were analyzed using different protocols. However, because I was interested in learning activities, my analyses of identifying and mathematizing were isolated to portions of the transcript in which students were engaged with mathematical objects. For example, I analyzed portions of the transcript in which students were talking about the task or working on cutting and gluing the figures, but I did not analyze students' conversations about church, soccer, or lyrics to rap music. To analyze identifying activities, I partitioned each turn into message units.

Bloome, Carter, Christian, Otto, and Shuart-Faris (2005) define a message unit as the smallest discursive unit that communicates meaning. They emphasized the perspective of the listener in making decisions about where to bound message units.

However, because I am interested in both the intention of the speaker and the listener's interpretations, I considered the speaker's and the listener's perspectives when making decisions about where message units started and ended.

Message units are different from turns. Turns are bounded by changes in speakers. A single turn might contain just one message unit or it might contain many. For example, in the transcript I used above to introduce my focal participants, each of the first 17 numbered lines is a turn. Each is also a message unit because it communicates meaning. Line 18 contains 3 units: "I don't know who I can believe", "you're my best friend", and "and she's smart". By partitioning the turn into 3 units, I can consider what Jakeel might be trying to accomplish with each unit and with the units together. I can also consider how Minerva and Nerissa might interpret each unit.

In my analysis, I examined message units and any accompanying gestures or nonverbal communication. For each unit, I considered the range of possible meanings for the speaker and the listeners, seeking to answer the question about what story the speaker and listener(s) might construct from the unit and how each person might be identified or might identify him/herself within that story. When I found units that identified the focal student as a learner in a consistent way, I pulled those units together, labeled them as a kind of learning, and wrote a description of that identification in my analysis.

I want to emphasize that my analysis of identifying only included units and turns that contained talk that was entirely mathematical. I was interested in not only moments where students explicitly identified themselves and each other (like Line 18 above, when Jakeel called Minerva his best friend and Nerissa smart) but also in more subtle identifying activity such as Line 1 when Minerva responded to Jakeel's question by shaking her head and Line 2 when Nerissa provided a verbal answer. These different responses suggest that Minerva and Nerissa were communicating to Jakeel different notions of their enactment of and willingness to be his teachers. Thus my analysis of identifying activity included units in which students' talk was entire mathematical like counting or talking about two halves making a whole.

I also examined these same lines through a mathematizing lens. I wanted to consider what mathematical discourse was available, how that discourse resembled the desired mathematical discourse, and how changes in student discourse resembled both the available and the desired mathematical discourse. In order to compare these different discourses, I used the four discursive features Sfard (2008) elaborated in her commognition framework (See Chapter 2). In my analysis I note differences and similarities in these four features between different units of mathematical talk and between mathematical talk and the desired mathematical discourse. These differences and similarities allow me to make claims about student learning.

The third part of the third analytical pass explored autonomous learning. This analysis developed as I attempted to characterize the differences between students' learning processes. Jakeel seemed to enact learning activity that was more like the activity encouraged by reform mathematics pedagogies. I developed a lens for describing autonomous learning as I worked to differentiate Jakeel's activity from that of other students. I then used that lens to analyze the activity of each student. The features of autonomous learning that I used in my analysis are described in Chapter 2. They are: identification of the audience and adoption, production, and substantiation of mathematical discourse. The development of the autonomous learning lens was a recursive process involving using data to build theory and theory to analyze data.

I conducted this third analytical pass on three lessons. I returned to the lesson I used in my second analysis and reexamined Jakeel's learning and Jessica's learning. As I explored Jessica's learning, I realized that Minerva's learning activities (Minerva was in the same group as Jessica) were also quite different from Jakeel's and Jessica's. Minerva's learning activities seemed important to explore not only because they were different, but also because they added complexity to the analysis and the theory of autonomous learning. The third analytical pass resulted in three detailed and lengthy memos, one about each student's learning, that I rearranged and shorted to construct the three findings chapters in this document.

Transcriptions.

As I worked on my second analytical pass through my data, I constructed detailed transcriptions of each group during the entire lesson. During the second and third analytical pass, I reviewed the portions of video I was analyzing and added more detail to the transcriptions. I separated the transcript into numbered turns. I used turns to separate and number the data because I wanted a way to track the sequence of events and where each event occurred in the lesson, including the portions of the lesson in which students were not involved with mathematical objects.

As the transcript included earlier in this chapter demonstrates, I used a table format to organize line (turn) number, the speaker, the spoken discourse, and gestures or other details. The fourth column primarily contains nonverbal gestures but I also used this space to describe verbalizations that were not words and to note indications mood or other details.

My transcription conventions draw upon Jefferson's transcription notation as described by Atkinson and Heritage (1984). and are indicated in Table 3.1.

Table 3.1 Transcription conventions

Symbol	Interpretation
()	Uncertainty about what was said
(?)	More uncertainty about what was said or the speech was unintelligible
(pause)	Pause in speech, approximate length of pause is included
=	Interrupted speech
[	Overlapping speech

Limitations

Any research method has limitations and affordances. The goal of the researcher is to design a study that uses the affordances of the methods to illuminate a research problem while minimizing the effects of the limitations of the methods. As I described earlier, I have used cases to organize this study because they allow close examination of a complex phenomenon. However this close examination means that the claims I make are not generalizable to larger populations based upon this study alone. I cannot claim that the learning activities I describe in this study will occur in other similar circumstances. I also cannot make causal claims based upon careful control of variables. My findings describe the activities I see in the groups I studied and suggest the possibility of influences and effects, but I cannot claim that these same factors are at play across other situations.

Also, my close examination of two groups in one lesson means that I do not consider the effects of outside influences on the activity in this group. While I incorporate information gleaned from across the classroom data I collected, I do not speculate about how the students' home circumstances, previous life experiences, or even prior experiences during the day might impact their activity during the lesson. Because I am an outsider attempting to make sense of other's meanings, it is possible that there are activities that I incorrectly interpreted because I lacked background information. I attempted to counter this possibility by proposing and evaluating multiple interpretations, discussing my data with other researchers, and looking for patterns in activities in order to provide multiple points of data for an interpretation. These efforts reduce, but do not eliminate, the likelihood that some of my interpretations are incorrect.

An additional problem with my research methods is justification of the selection of data. I purposefully selected lessons that offered varying perspectives on learning including autonomous learning in order to elaborate the detailed relationships between mathematizing, identifying, and learning. However, if I had focused on different lessons or collected data in the third grade classroom instead of the fourth grade classroom, the findings from this study would probably elaborate different aspects of learning mathematics. While this study could and should be expanded by examining other cases of learning and by testing the theories in other populations and settings, the methods of my study were justified in spite of their limitations: It was necessary to conduct a detailed examination of a limited number of contexts in order to develop the theories and findings that might be examined in future studies. This examination allowed the development and elaboration of theory and ideas that can then be further explored through other studies and other methods.

CHAPTER 4

INTRODUCTION TO THE FINDINGS

This chapter serves as an introduction to my findings. Each findings chapter (Chapters 5, 6, and 7), I examine the discourse of one child as he/she works with a group to solve a mathematical task involving area. In this chapter, I provide background for the analyses in the subsequent chapters. This chapter begins with a review of research on area that is relevant to focal lesson of this study. I then provide an overview of the area unit taught by the teacher. Finally, I describe in detail the task assigned in the lesson.

Relevant Research on Area

I limit my discussion of research on area to four points that are important to this study. Three points focus on the construction, iteration, and counting of units of area measurement. The fourth point relates to conservation of area as a shape is rearranged. Because the math lesson described in this study uses figures that have been partitioned to square grids (For example, see Figures 4.1 and 4.2), much of my presentation of research focuses on how elementary age students use grids to determine area.

In order to measure the area of a figure, students must recognize that area is an attribute describing the quantity of two-dimensional space inside the figure and they must designate and iterate a unit so that it covers the space. Designating and iterating a unit can pose challenges for elementary age children (Lehrer, 2003). According to Battista (2003), children may have difficulties structuring spaces, which means that they struggle to construct a regular grid that partitions a space into square units. Instead of drawing regularly spaced lines to construct rows and columns, some students draw each square independently or are inconsistent in their placement of lines. The resulting spaces may be inconsistent in size and shape and, when counted, may not result in an accurate measurement for area.

When students are presented with figures that have already been partitioned into square grids, they may still have problems measuring area. If the figure is nonrectangular or does not otherwise match the squares on the grid (for example, see J in Figure 4.2), then some of the grid spaces will be partial squares. In order to accurately determine the area of such a figure, students need to count the partial units in terms of the whole units. Some students may not recognize the need for identical units and may mix units as they count (Lehrer, 2003; Lehrer *et al.*, 1998). For example, students may count all units, partial squares and full squares, as one unit each. If a shape (like J in Figure 4.2) contains spaces that are squares and triangles, students may count both as equal units (Piaget *et al.*, 1960).

Even when a figure is entirely filled with square units, quantification of area may be problematic. While the students may accurately count the square spaces, they may be focused on enumerating squares as discrete objects, rather than on understanding and quantifying the spaces as units of area (Clements, 2003; Lehrer, 2003). If students think about squares as discrete objects, they may not understand area as a measure of space. For example, they may instead think about area as how many squares a shape can hold. When students fail to see area as a measure of space, they may fail to consider some of the important aspects of measurement such as the need to account for all of the space in the figure. For example, when determining the area of a nonrectangular figure, they may only count the grid spaces that are complete squares. Rather than think about area as a measure of space, they may understand area as a count of how many squares happen to be in the figure.

Finally, elementary students may struggle to make sense of how area can remain constant as a shape is cut into pieces and rearranged. For example, Piaget, Inhelder, and Szeminska (1960) presented children with rectangles partitioned into squares. They rearranged the squares and asked the children if the rectangle was the same size. Some of the children replied that the number of squares was the same but that the area was not. The children were able to keep track of the number of squares, but they were unable to conserve the area of the figure as its pieces were rearranged.

Students' struggles with using identical units, counting partial units, relating area to space, and conserving area are visible throughout my data. Some of these struggles served as the basis for the teacher's decisions about structuring the focal lesson. Others become apparent as the students worked on the focal lesson. It is important to my study to recognize these struggles not because I attempt to add to the research corpus on children's understanding of area, but instead because awareness of the challenges of learning area helps us to evaluate students' mathematical discourses to determine their progress toward understanding area.

The remaining sections of this chapter present information on the focal lesson including an overview of the unit on area and details of the lesson.

Overview of the Unit on Area

The unit on area consisted of eight lessons. In addition, there was one day of centers in which two of the centers involved review of area concepts. The unit was preceded by a unit on geometric shapes and was followed by lessons on probability.

The area unit began with a whole group introduction/review of the concepts of area and perimeter. Students used square tiles to determine the area and perimeter of a variety of shapes. The teacher designed the next two lessons to emphasize the relationship between area and perimeter. Students were to construct all possible rectangles given a certain number of squares and determine the perimeter of each rectangle. The next lesson focused on determining area. The teacher gave the students several shapes (which were called rugs) and asked which covered more floor. The students needed to draw grids within the shapes. Most of the shapes were rectangular although one was triangular. Some students struggled to make regular grids and the teacher felt that problems with drawing grids were preventing students from exploring questions about identifying and counting units. The teacher designed the next lesson – my focal lesson – with this concern in mind. She gave the students pairs of pre-grided figures and asked them to compare the area of the figures. Because this is my focal lesson, it is described in more detail in the section below.

The math activity for the following day involved two centers, both focused on area. One was a review of the lesson prior to my focal lesson. In another center, students were to make a figure using square units and then determine the area of the figure. The following two lessons involved determining the area (and sometimes perimeter) of figures on geoboards. The first lesson involved rectangular shapes. The second day involved triangular shapes. (This second geoboard lesson is mentioned in Chapter 5.) The last lesson in the area unit was a review of area and geometric shapes. This sequence of lessons is listed in Table 4.1. The table has information on whether the lessons were organized as Problem-Solving, "Math", or Centers. The table also includes math activities that occurred during the same time as the lesson but whose content did not include area.

In constructing these lessons, the teacher drew upon four resources: the fourth grade Harcourt math textbook (Maletsky *et al.*, 2004), *Mouse and Elephant* Middle Grades Mathematics Project curriculum (Shroyer & Fitzgerald, 1991), *Dot Paper Geometry* (Lund, 1980), and materials from a university research project.

Table 4.1 Sequence of lessons involving mathematics during area unit

Day	**Math Lessons**
Week 1	
Tuesday	Math: Introduction to area and perimeter
	Problem-Solving: Recognizing Rhombi
Wednesday	Power outage during school, math canceled
Thursday	Math: Making different rectangles with an area of 12, measuring perimeter
Friday	Problem-Solving: Making different rectangles with an area of 24, measuring perimeter
	Centers (none involving area)
Week 2	
Monday	Substitute teacher: Independent work, not on area
Tuesday	Substitute teacher: Independent work, not on area
Wednesday	Math: Comparing areas of different shapes
Thursday	Problem-Solving: Comparing areas of different shapes (focal lesson)
Friday	Centers: One center is review of the problem from Wednesday, in another center students use squares to make a shape and then determine the area
Week 3	
Monday	Math: Area and perimeter of rectangles using geoboards
Tuesday	Math: Area of triangles using geoboards
	Problem-Solving: Logic problem with coins
Wednesday	Review worksheets on geometry including geometric shapes and area

Description of the Focal Lesson

The focal lesson occurred on the Thursday of the second week of unit. In the previous lesson, students were to answer the following questions which were printed on a worksheet: *Which of these rugs covers more of the floor? Which covers the least? Do any cover the same amount?* The "rugs" were four shapes printed beneath the questions: three rectangles and one triangle. Many students struggled to find the area of the triangular rug, prompting the teacher to construct the task described below – the task for the focal lesson.

The teacher organized this task as a Problem-Solving lesson. During Problem-Solving, the students worked in groups of three. Instead of giving instructions about the lesson to the whole class, the teacher gave each group a one page "Task Card" which listed materials needed, the task, group expectations, and the final product. Students were to read the task card to determine their work for the lesson.

The theme of measuring rugs continued into this lesson, as indicated by the information on the task card:

Materials
shape papers, scissors, glue, individual worksheet

Task
As a group, find which of the rugs covers more area or if they cover the same amount. First compare rugs H and I. Prove how they are the same or different. Once you are done with this, do the same task with rugs J and K. When you are finished with that you may come up and grab the last worksheet and read what you are supposed to do.

Group Expectations
1. Everyone must show their work on an individual worksheet.
2. No one can touch their worksheet.
3. You may help by teaching but not by doing.
4. Everyone in the group has to agree on an answer before going and getting the next paper.

Final Product
1. Use your paper to individually record your solution to the problem.

Challenge
Make up your own problem as a group.

To complete the task, students received scissors, glue, individual worksheets, and copies of shape papers. The individual worksheets had a blank for the student's name and the following words: "Directions: Write how you solved the problem and show the answer that you chose below." The shape papers were half sheets of paper with two figures on them. Students initially received papers with Figures H and I on them (see Figure 4.1). Papers containing Figures J and K (see Figure 4.2) were kept at the front of the room. Students were allowed to retrieve those papers and begin work on them as a group after the teacher approved each individual student's work with Figures H and I.

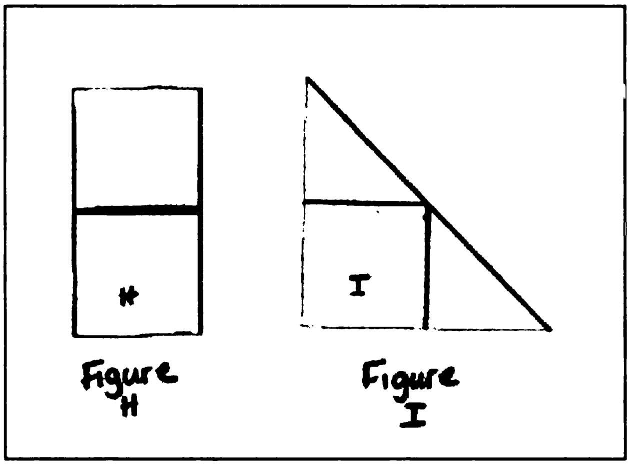

Figure 4.1 Scanned image of Figures H and I. Figure labels were handwritten by the teacher.

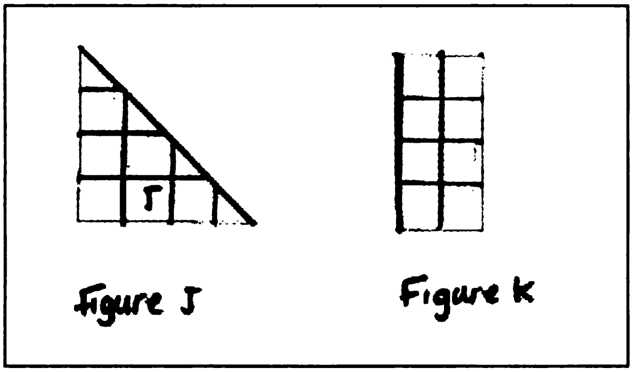

Figure 4.2 Scanned image of Figures J and K. Figure labels were handwritten by the teacher.

Desired Mathematical Discourse

To complete this task, students needed to determine which rug or figure covered more area. To do this for Figures H and I, they could cut out each rug, cut Figure I into one square and two triangles, and then rearrange the pieces of Figure I so that they cover Figure H. The student could then state that because the shapes are congruent or fit on top of each other without any excess, the figures have the same area. However, the teacher designed this lesson so that students would wrestle with an issue that arose during the previous lesson involving determining the area of a non-rectangular figure. During the previous lesson, the students drew a grid on the non-rectangular figure. Many of the grids drawn by the students were irregular and all of the grids contained spaces that were not square. Most of the students then proceeded to determine area by counting each space in the grid as one, without accounting for differences in the amount of area represented by each space. The teacher's intention for this task was to help students recognize that they could not count partial units as whole units. Specifically, she wanted students to assemble the triangular partial units into whole square units and then count each square.

Thus, to solve this problem using the mathematically desirable discourse, students first needed to identify squares as the unit of measurement. They then needed to account for the triangular partial units present in Figures I and J. The teacher expected the students to physically (by cutting) assemble the partial units into whole units and then count the units in each shape. Finally, students would state both the area of each shape in square units and which figure covers more area. Successful mathematizing does not require that students be explicit about each step in order. For example, students might not state that they are using squares as the measurement unit until the conclusion of their work when they state the area of each shape. As we shall see in the three subsequent findings chapters, students experienced varying degrees of success in using the desired mathematical discourse.

CHAPTER 5

ENGAGED LEARNING AND DIRECTED LEARNING

Overview

This case presents the learning of one student, Jakeel, whose identifying and mathematizing changed dramatically during the course of the lesson. He shifted from *engaged learning* which supported his work to craft a discourse-for-himself to *directed learning* in which his mathematical discourse was entirely a discourse-for-others. I first present the outcome of Jakeel's learning by contrasting his initial discourse with his final discourse. I then describe how engaged learning was enacted by Jakeel and supported by his peers and how this learning shifted to directed learning and back to engaged learning. I conclude by discussing the implications of these different processes of learning for Jakeel's participation in autonomous learning.

Summary of the Lesson

My analysis of Jakeel's learning uses several excerpts taken from the transcript of Jakeel's group. The excerpts are not presented in chronological order: Instead, I used excerpts from across the lesson as necessary to support my argument. In an effort to orient my readers to the position of the excerpts in the flow of the lesson, I have summarized the main events in the lesson and constructed a table that describes the content of the lesson and the excerpts and shows where the excerpts occur in the lesson (See Table 5.1).

The teacher began the lesson by explaining the lesson and her expectations for behavior to the whole group. She then dismissed students to work in small groups. Rebecca, Daren, and Jakeel gathered at a table in the back of the room and started to read the directions for the task. The teacher came to the group and asked questions about the task. Once she was satisfied that they understood what to do, she allowed them to get their materials. Jakeel picked up the worksheets with Figures H and I on them and the Daren and Rebecca began to work. Jakeel asked Daren and Rebecca what they were doing and Daren explained how he was thinking about the area of Figures H and I. Rebecca finished writing her solution to the problem and tried to get Jakeel to copy her answer. Jakeel refused and constructed his own solution. When he was done, Rebecca told him that wasn't what he was supposed to do and told him he would have to redo his work (See Figure 5.1). He began again on a separate piece of paper and, with Rebecca's assistance, constructed another response to the task involving Figures H and I (See Figure 5.2).

The teacher came to the group and approved their answers for Figures H and I and allowed the students to get the worksheet with Figures J and K. When the teacher returned to the group later, Jakeel told the teacher he was done with Figures J and K (See Figure 5.2). However, he was unable to demonstrate how count the spaces in Figure J so the area was the same as Figure K. Rebecca explained her thinking about how to count the spaces and after several counting attempts and explanations from Rebecca and Daren, Jakeel was able to successfully count the spaces in Figure J for the teacher.

Table 5.1 matches this summary with the excerpts I use in my analysis. I begin my analysis by considering how Jakeel's discourse changed over the course of this lesson.

Table 5.1 Transcript excerpts and main events in lesson

Line number	Excerpt Number	Excerpt Summary	Main Events
0			The teacher explains the lesson .
			The students move to their small groups and start work. Jakeel, Rebecca, and Daren read the task card and get their materials.
100			
			The students begin work on the first problem with Figures H and I.
	Excerpt 6	Jakeel involves himself in the conversation at his table	
	Excerpt 1	Jakeel's initial discourse	
	Excerpt 7	Jakeel refuses Rebecca's directions	
200	Excerpt 9	Jakeel uses *rectangle*	
	Excerpt 10	Rebecca tells Jakeel he has to rewrite	
			Rebecca tells Jakeel his answer for Figures H and I is wrong and tells him what to write.
300			
	Excerpt 11	Jakeel asks what he was supposed to write	The students work on Figures J and K.
400			
500			
	Excerpt 2	Jakeel's final discourse: the same	Jakeel counts the spaces in Figure J, eventually counting them correctly
	Excerpt 8	Jakeel counts and Rebecca explains	
600			

Table 5.1 con't

	Excerpt 3	Jakeel's final discourse: pairing triangles	
	Excerpt 4	Jakeel counts eight	
			The students clean up.

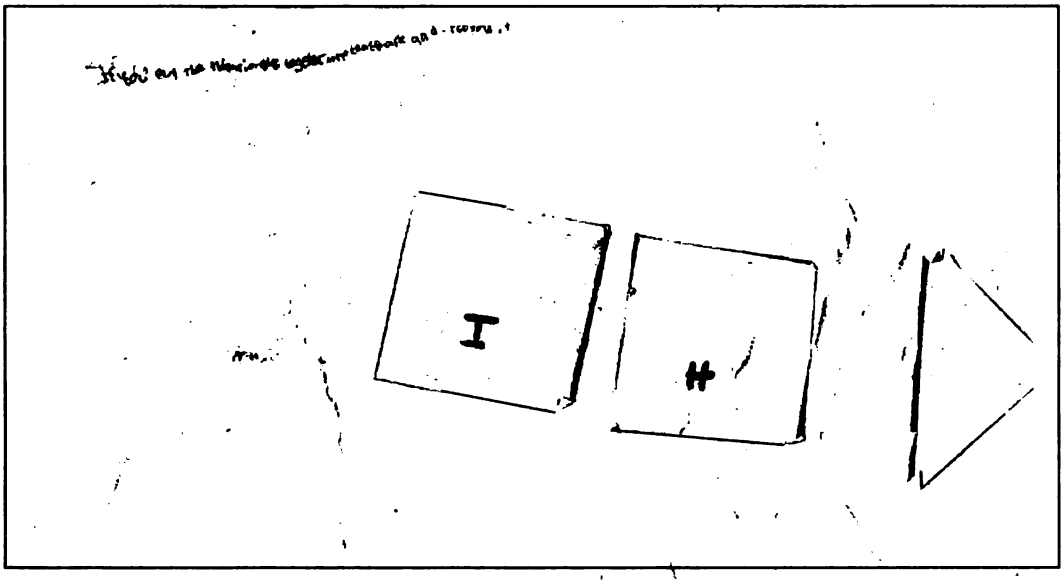

Figure 5.1 Jakeel's initial work on Figures H and I
This is student work. The text in the top left reads, "If you put the two triangles togeter [sic] it make [sic] a square and it covers it".

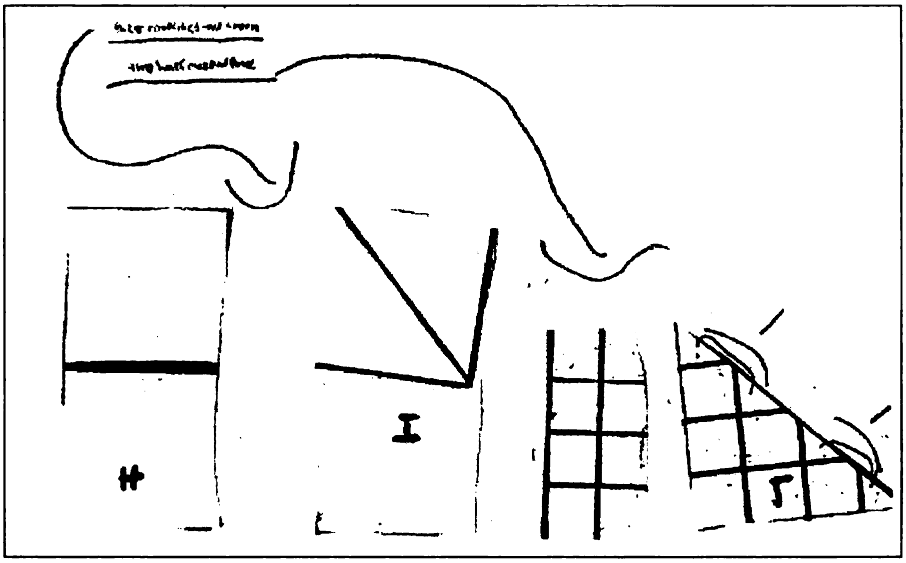

Figure 5.2 Jakeel's final work on Figures H, I, J, and K
This is student work. The text in the top left reads, "they cover the same amount" and, on the second line, "two half [sic] make [sic] a square".

Outcome of Learning

Initial Mathematical Discourse

The first time Jakeel spoke about the mathematics of this task occurred 13 minutes after the beginning of small group work time. All three students – Rebecca, Daren, and Jakeel – were working on comparing the Figures H and I. The teacher arrived at the group and began asking questions about their work. Rebecca responded that H and I cover the same amount and the teacher turned to Jakeel to ask for an explanation.

Excerpt 1

176	Teacher	Jakeel. What do you think? How can you prove this? What could you do? What materials do you have that could prove this?	*She picks up a paper with Figures H and I.*
177	Jakeel	H and I. They both have squares.	*He points at the figures on the paper the teacher is holding.*
178	Teacher	Okay but are they the same?	*The teacher puts the paper down.*
179	Jakeel	Yes	
180	Teacher	Do they cover the same?	*Daren and Rebecca raise their hands.*
181	Rebecca	I know	
182	Teacher	I mean Jakeel. How	
183	Jakeel	Cause it's a square	*Jakeel points at the full square in Figure I.*
184	Rebecca	Cut this out	*She takes a paper with H and I on it and picks up scissors.*
185	Teacher	But this isn't a square	*She points at the two triangles in Figure I.*
186	Jakeel	No, so that's why it won't cover the same.	

With his "Yes" on Line 179, Jakeel initially responded to the teacher that Figures H and I had the same area. While this answer was mathematically correct, Jakeel was unable to offer a mathematically appropriate explanation for why the two figures covered the same amount. His explanation, "Cause it's a square" (Line 183) suggested that he was thinking about squares. The response provided a label for a geometric space, but it did not explain how squares were important to area or to comparing areas.

Jakeel also did not attempt to explain how the two triangles in Figure I were part of the area. He does not mention or indicate the triangles, raising the question of whether he thinks the triangles are important for determining area. In addition, when pressed by the teacher (Line 185), he changed his answer, responding that the figures "won't cover the same" (line 186). While this response was less mathematically accurate than the "Yes" (Line 179) Jakeel first provided, it did begin to elaborate more upon notions of area. Jakeel's use of the word "cover" implied that he wasn't only thinking of the squares as labels for geometric spaces (Line 183), but that he recognized that an important attribute of squares was that they offered coverage. The notion of coverage is consistent with the notion of area as a measure of two-dimensional space. In addition, his use of "the same" (along with his use of "both" in Line 177) indicated a comparison of the two figures. A comparison is a component of the desired discourse. However, Jakeel's comparison in this initial discourse seems to focus on descriptive geometry (the squares) and not on quantifying or evaluating area.

In summary, Jakeel's initial discourse offered some mathematically appropriate notions of area, but it was not the desired discourse. He wavered in his response about whether the figures "covered the same." He did not specifically label squares as the unit of measurement for determining area. He also did not quantify the squares or offer a way to account for the two triangular spaces in Figure I. He did indirectly indicate that area is related to coverage.

Final Mathematical Discourse

By the end of the lesson, Jakeel's discourse had changed. I will draw upon four moments to illustrate this change. Three moments come near each other at the end of this lesson. The fourth moment is from a subsequent lesson that occurred five days later. The first example of Jakeel's discourse occurred 40 minutes into the small group work time. Rebecca, Daren, and Jakeel had finished their work comparing Figures H and I and were close to completing their work on Figures J and K. The teacher came to the group and looked at Jakeel's work.

Excerpt 2

534	Teacher	Okay. So, what's the area of J? (2 second pause)	*The teacher points to Jakeel's paper.*
535	Daren	It's um same as=	
536	Teacher	What's the area of J?	*The teacher looks at Jakeel.*
537	Jakeel	The same as um	*Jakeel points at Figure K on his paper.*
538	Teacher	What is it? Tell me what it is.	
539	Jakeel	Eight	

This excerpt suggests that Jakeel knew that Figures J and K had the same area (Line 537) and that area was eight (Line 539). It is possible that Jakeel's statement in Line 537 was an echo of Daren's statement (Line 535). However, in the following moment, it became apparent that Jakeel expected both figures to have an area of eight.

One final point about this excerpt is that Jakeel was not explicit about what eight was a count of. In other words, he does not elaborate on his eight with any noun that would indicate what he was counting. This lack of the object counted was a consistent trend in Jakeel's final discourse.

Just after Excerpt 2 above, the teacher asked Jakeel to demonstrate how the area of the Figure J was eight. Jakeel counted the spaces in the figure but counted to ten. Rebecca asked if she could help him and the teacher left as Rebecca began her explanation. I have not included this transcript in this section because Jakeel's discourse was very similar to his discourse in Excerpt 3 below except that Excerpt 3 illustrates some additional features of his final discourse. I felt that it would be redundant to include both.

Excerpt 3 occurred almost 10 minutes after Excerpt 2. The teacher returned to the group and asked Jakeel again about his work.

Excerpt 3

621	Teacher	Okay, show me on here which two I put together	She points at Figure J.
622	Jakeel	Those two	Jakeel points at two triangles in Figure J.
623	Teacher	Can you draw a line or something	Jakeel draws a line connecting two triangles.
624	Jakeel	Yeah	
625	Teacher	To show me? Okay and what's the other two?	Jakeel draws another line, connecting the two remaining triangles.

Jakeel's actions and words indicate that he was putting together pairs of triangles. This action was important because it indicated that Jakeel was working with the triangles. This is a shift from his original discourse in which he only discussed squares. However, his verbal mathematical discourse in this excerpt is limited. His only mathematical word is "two". Similar to Excerpt 2, he does not label what object the two enumerates. Thus although I have referred to the objects as triangles, Jakeel has provided no information about how he is thinking about what he is counting.

This excerpt continued with more questions from the teacher.

Excerpt 3, continued

625	Teacher	Now show me how it's the same area as that	The teacher points to Figure K.
626	Jakeel	Okay look. one, two, three, four, five, six, seven, eight.	He points to each square in Figure K.
627	Teacher	Now count that one	She points to Figure J.
628	Jakeel	one, two, three, four, five, six, seven, eight.	He starts at the top of Figure J and counts the spaces in each row, counting each space, including each triangle as one. He pauses when he reaches eight. He doesn't count the last square or triangle. Rebecca leans in toward Jakeel. Daren watches him.
		(Hold on?) Oh yeah. one, two, three, four, five, six, seven, eight.	Jakeel counts Figure J in a more random pattern, but again counts the triangles as one. Again he doesn't count each space.

When Jakeel counted Figure K, he successfully enumerated the area as eight (Line 626). Because he did not describe what he was counting, we cannot be sure what he was enumerating, but he was nonetheless confident that the area of K was eight. When he counted the area of Figure J (Line 628), he also stopped counting at eight. However, stopping at eight was not a signal that he had counted the entire area. Instead, Jakeel stopped at eight even though he still had two uncounted spaces. His hesitation and re-counting indicated that he expected to end on the number eight. Jakeel's actions here were consistent with his words from Excerpt 2 when he stated that the figures had the same area (Line 537) and that the area was eight (line 539).

In this last transcript, it was apparent that while Jakeel was physically putting the triangles into squares, he had not yet integrated the pairing of the triangles into his routine for determining area. Rather than count each triangle as half or count two triangles together as one square, Jakeel counted each space as one, as a result, he was unable to accurately determine the area of Figure J. As above, his discourse contains only counting numbers. Thus my reference to triangles and squares may not represent how Jakeel identified the spaces.

The teacher left the group shortly after the moment above and Rebecca and Daren worked with Jakeel on counting the spaces in Figure J. During the next four minutes, Rebecca and Daren explained how to count the area and Jakeel practiced. The teacher returned two times and each time Jakeel counted each space as he did in the transcript above. On Jakeel's fourth attempt in front of the teacher, he counted the area:

Excerpt 4

668	Jakeel	One	This word is elongated as Jakeel simultaneously puts his index and middle fingers on two triangles.
		Two, three, four, five, six, seven	He counts each square, pointing with his index finger to each square.
		Eight	He puts index and middle fingers on the last two triangles.

By pointing to two triangles with two fingers, Jakeel was able to count the triangles as one and as a result, he was able to accurately count the area of Figure J. This counting was a change from Jakeel's discourse at the beginning of the lesson. While Jakeel spoke about squares at the beginning of the lesson, he did not talk about or in anyway indicate the two triangles. He also did not indicate that he understood area as something to be counted. Thus his final discourse was a change from his initial discourse and more like the desired course because he enumerated the spaces in the figures and he included in that enumeration the triangles. In addition, his counting of the triangles demonstrated that he understood that two triangles were the same as one square. One last way in which Jakeel's final discourse was like the desired discourse was that Jakeel was able to use counting to establish that the two figures were the same.

There is one more important point about Jakeel's final discourse. It is unclear from his discourse what he understood about area. His only mathematical words are the counting words one through eight. While he was responding to the teacher's request to show the area, he does not mention the word area. In addition, he does not label what he is counting. It is possible that Jakeel was counting squares, but it was also possible that he was counting the number of times he pointed (with one finger or two). If Jakeel was thinking of his counting as the number of squares, he still might not be thinking of the squares as units of area measurement: Instead, he could be thinking about them as objects to be counted. Thus while Jakeel demonstrated many of the features of the desired discourse, it is unclear what he understood about area and units of area measurement.

One final piece of evidence of Jakeel's change in discourse was his activity on a subsequent lesson. The following week, the students participated in another lesson involving finding the area of triangular figures. (See Table 4.1 for a sequence of lessons. The case study lesson occurred on Thursday of Week 2. This subsequent lesson occurred on Tuesday of Week 3.) The students had worksheets with triangles drawn on a grid (See Figure 5.3 for an example). They used rubber bands to recreate the triangles on geoboards. They then determined the area of the triangles. One of the triangular figures on the worksheet (the middle problem from Figure 5.3) was identical to Figure I from this lesson.

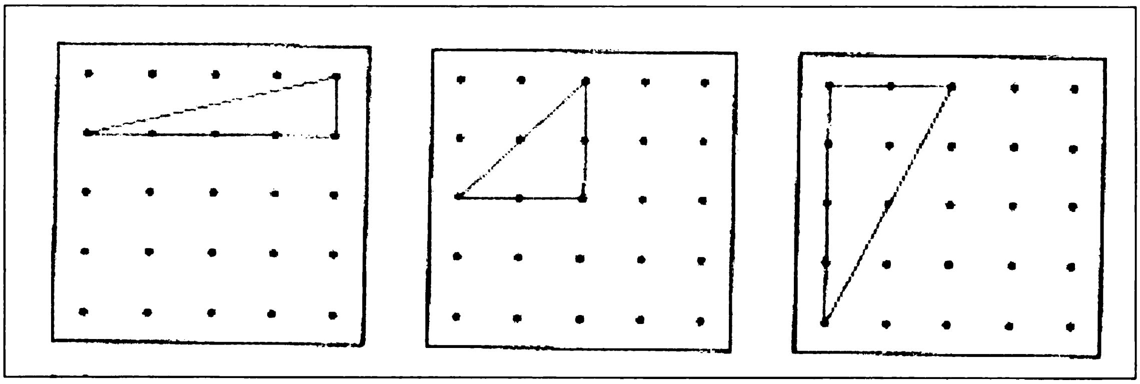

Figure 5.3 Portion of worksheet on area of triangular figures.

Jakeel was working with Minerva and Jessica. He created the triangle from the second problem on his geoboard and divided it into a square and two smaller triangles with rubber bands (See Figure 5.4). Jessica was working on the same problem and was unsure how to determine the area of the figure. The teacher asked Jakeel to show Jessica the squares in the figure.

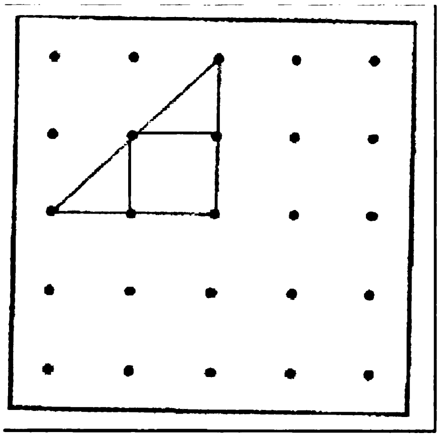

Figure 5.4 Illustration of Jakeel's geoboard showing the triangular figure partitioned into a square and two rectangles.

Excerpt 5

450A[1]	Teacher	Now show her Ja Jakeel. Show her how it makes a square.	Jakeel turns the geoboard so he can see it. He puts his index and middle fingers on the two smaller triangles. He then turns it back to Jessica.
451A	Jakeel	See these two make a square	
452A	Minerva	Duh duh duh	These are rude noises that seem to convey that Jessica should already understand. Minerva glances at the teacher after she makes the noises. Jakeel removes his fingers from the board and places his middle finger in the full square.
453A	Jakeel	And so it's two squares	

In this excerpt, Jakeel was able to use words to describe what had only been a pointing action in the previous lesson. In the previous lesson, Jakeel had pointed to the

[1] I have added the letter A to each of these line numbers as a reminder that this excerpt and these line numbers are not from the same lesson as the other excerpts in this chapter.

two small triangles in the figure and counted them as one, without stating what they were one of. In this lesson, in addition to pointing in the same way, he was able to state that the two triangles made a square (Line 451). In addition, rather than only count the area, Jakeel specifically stated that there were two squares (Line 453). He still was not explicitly connecting the squares with area, so it is uncertain whether he was only thinking about counting squares or whether he understood that counting squares was a measurement of area. However, this excerpt shows similarities to Jakeel's activity in the previous lesson suggesting that he was able to draw upon that activity from the previous lesson and continue to change his mathematical discourse.

In summary, over the course of the lesson comparing the various figures, Jakeel learned to count partial units in triangular figures as one and he was able to draw upon and elaborate this participation in a subsequent lesson. While Jakeel's final discourse was not as mathematically rich as what was indicated in the description of the desired discourse, he did learn during this lesson in ways that were helpful in his later mathematical work.

The next section will look at the process of learning and describe the kind of learning activity enacted by Jakeel and supported by his group during the lesson on Figures H, I, J and K.

Learning Activity

I have partitioned Jakeel's learning activities in this lesson into three phases. Initially, he enacted what I describe as engaged learning. He then went through a period of directed learning before returning again to engaged learning. Because I describe the process of learning as the interplay of identifying and mathematizing, I will describe each kind of learning in terms of the identifying and mathematizing activity of Jakeel. I will also describe how Rebecca and Daren mathematized and how they identified Jakeel. I use the five characteristics of autonomous learning as categories to help organize my analysis. Thus for each kind of learning, I illustrate the key activities that distinguish the learning, explain how Jakeel identifies the audience for his discourse, elaborate the available and adopted mathematical discourse, discuss Jakeel's production of variations on discursive features, and finally describe Jakeel's substantiation of the discourse.

Engaged Learning

According to the Oxford English Dictionary, one definition for engage is to involve ("Engage", 2008). This sense of involvement captures Jakeel's activity as he enacts engaged learning. Jakeel was involved with learning, not only in terms of actively mathematizing and making his own sense of mathematical ideas but also in terms of contributing to and exploring the ideas of his peers. My label of engaged learning also draws upon Etienne Wenger's (1998) concept of engagement. In particular, I want to emphasize Wenger's notion of engagement as mutual. For Wenger, this means that all individuals are moving towards fully participating in a community by actively making meaning and contributing to practices. As I will illustrate below, for Jakeel, learning was a mutual activity and not an independent or ritualistic activity: He interacted with others, contributed ideas, worked to make his own sense of discourse, and relied upon others to support his quest for understanding.

An Example of Engaged Learning

My presentation of Jakeel's engaged learning begins with a detailed exploration of one excerpt in which Jakeel and his peers illustrate several aspects of engaged learning. This excerpt occurred toward the beginning of the small group work, before any of the exchanges included above. Rebecca and Daren started writing their answers to the task involving Figures H and I while Jakeel looked around for his pencil. When Jakeel realized that Rebecca and Daren were working, he was puzzled.

Excerpt 6

147	Jakeel	How we gonna get the answer if we didn't even read the paper?	*He looks at Daren.*

With this question, Jakeel identified himself as part of a group working together to find the answer. His "we" made him an active partner in this process. He also articulated the first step toward getting the answer, reading the paper. Jakeel could have asked what Rebecca and Daren were doing or he could have watched them to see what they did, but instead, he asked a question that positioned him as a contributing member of the group.

Daren responded to Jakeel.

148	Daren	We know. Look.	*He picks up a paper with Figures H and I and points at the figures.*
		This is half. If we put this together that's two squares.	*He points to the triangles in Figure I.*
149	Jakeel	That's a obtuse.[2]	*Jakeel points at Figure I.*

Daren's response acknowledged Jakeel's question and positioned Jakeel as interested in making sense of the task. Daren could have ignored or dismissed Jakeel's question (as we see other students doing in the next two chapters). He also could have told Jakeel to copy his answer. Instead, Daren explained his thinking including demonstrating how his words point to the picture of the figures. Daren, like Jakeel, also

[2] During the previous unit on geometry, Jakeel erroneously began to call right triangles "obtuse". He was consistent in this mislabeling, suggesting that his use of obtuse in this moment, while mathematically incorrect, correctly designated Figure I as being in the same category as other triangles Jakeel called obtuse.

used "we" (Line 148) which might suggest to Jakeel that he was part of the group and that he could also work with the figures in the same way as Daren. Daren's actions in this excerpt supported Jakeel in being engaged in learning by positioning him as actively interested in making sense of the task.

In Line 149, Jakeel demonstrated that his engagement in the learning process by involving himself in the conversation. Rather than accept Daren's explanation as the only important mathematical information, Jakeel suggested he also knew information and had something to contribute. It is unclear how Jakeel was positioning Daren with this utterance. He could have been trying to teach Daren or compensate him for sharing his ideas by sharing one of his own. It is also possible that Jakeel was attempting to indicate to Daren that he knew some information related to the task so that Daren would not think he had to explain everything about the task. One message Jakeel's utterance seemed to convey was that he was attending to and interested in mathematical features of the task.

Again, Daren responded to Jakeel.

150	Daren	We don't care about that! (?) Look, look.	*Jakeel has picked up his own copy of Figures H and I.*
		You see how that	*Daren points at the paper Jakeel is holding.*
151	Jakeel	This is I	*He points at his paper.*

Daren's response to Jakeel (Line 150) evaluated Jakeel's contribution and labeled it as irrelevant. While this labeling could be seen as suggesting that Jakeel was incapable, Daren continued his explanation. Thus Daren's utterance supported Jakeel's sense making by redirecting him to relevant mathematics and continuing to position him as someone who can understand Daren's thinking.

Jakeel indicated his continued interest in the task by picking up his paper (Line 150) and contributing again to the conversation (Line 151). With his statement, "This is I" Jakeel indicated that he was attending to the figures on the paper and that he was an active partner in this conversation. He also demonstrated that he could make contributions that were more relevant to the task. Daren responded to Jakeel by continuing his explanation.

152	Daren	You see the one square and that's got two. You put these together. That's two squares. So they cover the same area.	*Daren is pointing at Jakeel's paper. It is not clear what he is pointing at. After this statement, Daren turns to his paper.*
153	Jakeel	Can you say that again?	*Daren puts his hands on his face. Jakeel looks at his paper.*

With this question, Jakeel acknowledged that he did not understand what Daren had said and he indicated he wanted to hear the explanation again. It is possible that Jakeel wanted Daren to repeat his words so that he could imitate them or write them down rather than so that he could hear them again in an attempt to make sense of them. However, this possibility seems unlikely given Jakeel's previous contributions to the conversation and given the other possibilities for what Jakeel might have said. For example, Jakeel might have asked Daren to give him the answer or he might have shrugged his shoulders and tried to copy what Daren had written. Instead, this question seems to reflect Jakeel's desire to make sense of Daren's explanation.

Jakeel's engagement in learning was also illustrated by his refusal to participate or be positioned as someone who only wants the answer. For example, Rebecca suggested that Jakeel could copy her answer.

Excerpt 7

193	Rebecca	So let's cut these out. Wait. We'll write first and then we'll cut. Okay	*Rebecca picks up her pencil. Daren and Jakeel also stop cutting and start writing.*
		And then H. H has two squares. Hold on let me finish writing this and then I'll show you what I'm writing.	*Jakeel is writing and erasing*
		Like I	*Rebecca puts her paper in front of Jakeel*
		See Jakeel. Jakeel, look.	*She points at her paper.*
194	Jakeel	I don't want to write that	*Jakeel pushes the paper away*
		I know what=	
195	Rebecca	You just won't write	

Rebecca placed her paper in front of Jakeel and directed him to look at her answer (Line 193). However, Jakeel was emphatic in his refusal, stating that he didn't want to write what she had written (Line 194). In addition, the beginning of his next sentence suggested that he knew what to write. Indeed, he was writing at the time Rebecca placed her paper in front of him and he continued to write at the end of this interaction.

This excerpt was one of three interactions between Rebecca and Jakeel in which Rebecca tried and failed to get Jakeel to follow her directions. In one instance, she was very insistent that he and Daren copy her work, placing her paper between Jakeel and Daren and commanding, "Write that. Write that. People write this." (Line 160). She continued to tell Daren and Jakeel to "write this" (Lines 167 and 169). Jakeel at first deflected Rebecca's command by teasing her about how she said his name. The teacher then arrived and redirected the conversation.

In the second interaction, Rebecca told Jakeel to make a square with the small triangles he had cut from Figure I and tried to move his pieces for him. He told her he knew what to do and proved it by putting the two triangles together. Both of these examples, along with Excerpt 7, demonstrate Jakeel's refusal to be positioned by Rebecca as a passive learner who did as he was told. Instead, he wanted to be actively involved and responsible for making decisions and making sense of interactions. His refusal to follow her directions, coupled with his eagerness to contribute to conversations as described earlier show how Jakeel was participating in engaged learning.

In the next few sections, I will discuss Jakeel's audience and the mathematizing in the group. My elaboration of these other features will continue to add nuance and texture to my description of Jakeel's engaged learning.

Identifying the Audience

In Chapter 2, I discussed the ways in which the autonomous learner identifies his or her audience as important to the mathematizing of the learner. This relationship between audience and mathematizing holds for other kinds of learning as well. In this section, I describe Jakeel's audience for his learning activity, emphasizing the members of Jakeel's learning audience and how he identified them.

An important member of Jakeel's learning audience was himself. While much of his discourse was directed to others, there were moments when he was clearly talking to himself. This was most apparent as he tried to make sense of how to count the area of Figure J. This excerpt occurred almost 40 minutes into the small group time. The teacher has just asked Jakeel what the area was of Figure J. This excerpt continues Excerpt 2 from above. I have included a large portion of transcript here so that I can use it to make several points later.

Excerpt 8

538	Teacher	What is it? Tell me what it is.	
539	Jakeel	Eight	
540	Teacher	How is that eight? I can't tell that's eight.	
541	Jakeel	Because one, two, three, four, five, six, seven, eight, nine, t-.	*Jakeel points to the spaces in Figure J as he counts. He points once at each space with the pinky of his right hand.*
		Hold on.	*Rebecca has her hand up. Jakeel points at each space in K with his pinky. This motion suggests that he is silently counting*
542	Rebecca	Can I tell him?	*Rebecca talks to the teacher. The teacher then leaves.*
543	Jakeel	It's	
544	Rebecca	Okay, Jakeel. You see those little triangles.	*She points to Figure J.*
545	Jakeel	Yes	
546	Rebecca	Those are half a squares. Look and if you put these together, those equal a square	*Rebecca points to Figure J on Jakeel's paper.*
		and so it's eight, it's eight and eight.	*Rebecca points to Figures J and K.*
547	Jakeel	Oh. one=	*Jakeel is pointing to spaces in Figure J. Rebecca interrupts him.*

548	Rebecca	So write that if you put the two half squares together it makes a square. If you put two half squares together, it makes a square. Write that on your paper. Write that. Then you're done Jakeel.	*Jakeel points at the spaces in Figure J. His motions suggest that he is silently counting. It looks like he points at squares with one finger and at triangles with two fingers.*
549	Jakeel	Oh, yeah!	

There are two moments in this excerpt when Jakeel's audience for his discourse was primarily himself. On Lines 541 and 548, Jakeel silently pointed to the spaces in Figures K (Line 541) and J (Line 548). Although Jakeel didn't audibly utter any words, his actions were consistent with counting to himself. His "Oh, yeah!" on Line 549 seemed to indicate that he was successful in his counting. This utterance could have been in response to Rebecca's words (Line 548). However, Jakeel was looking at his paper and his gestures while Rebecca explained which suggested that his exclamation was a proclamation of his success counting rather than a declaration of his understanding of Rebecca. Jakeel's exclamation provides further evidence that he was an audience for himself.

There were other moments in which Jakeel included himself as an audience for his discourse. On three other occasions, he demonstrated his counting of the spaces in Figure J for the teacher. On two of those occasions, Jakeel stopped his counting before he had counted all of the spaces. He was not interrupted or distracted by anyone else. Instead, he stopped each time he reached eight. His voluntary halt to his counting indicates that he was monitoring his activity and not relying on others to evaluate whether he had reached the right number. Thus as Jakeel enacted engaged learning, he was an important audience for his discourse.

This excerpt illustrates that the teacher was also a member of Jakeel's audience. He initially counted the spaces in Figure J in response to the teacher's question about the area (Line 540), so this first counting was specifically enacted for the teacher. However, it was not enacted so that the teacher could tell Jakeel whether he was right or wrong. Jakeel knew that he should end his counting on eight. Instead of positioning the classroom teacher as the person who should tell him if he was right or wrong or the person who could help him figure out what he should do differently, Jakeel positioned her as someone he performed for. He needed or wanted to show her that he knew how to count the spaces in Figure J and he persisted in his counting and in demonstrating his counting to her until he finally was correct in his counting (See Excerpts 3 and 4 for examples of Jakeel's counting for the teacher).

Jakeel's positioning of the teacher contrasts with his positioning of Rebecca and Daren. These two peers were evaluators and explainers for Jakeel. They would listen to his discourse and then provide him with feedback including their thinking about the mathematics. Excerpt 8 provides an example of this. Jakeel counted the spaces in Figure J for the teacher, but Rebecca observed his counting and was eager to explain to Jakeel how to think about the spaces in order to end on a count of eight. Jakeel was willing to allow her to explain: He responded to her question about seeing the triangles (Lines 544 and 545) and, after her explanation, was successful in altering his counting. This interaction is in contrast to the next interaction in which Rebecca repeated her thinking, but couched it as a command. In Line 548, Rebecca was no longer the helpful explainer but was instead directing Jakeel about what to write. As I will demonstrate later when I describe directed learning, Jakeel was not always successful in rejecting Rebecca's commands. However, at this moment, when Jakeel was enacting engaged learning, he ignored Rebecca's command and focused on figuring out the counting, refusing to allow her to occupy a position as director.

Daren and Rebecca served as evaluators and explainers on several occasions. Excerpt 6 above provides examples of Daren evaluating and explaining. On Line 150, Daren rejected Jakeel's contribution that the figure was obtuse and then explained his thinking. Also, after each time that Jakeel incorrectly counted Figure J, either Rebecca or Daren explained their thinking about counting the triangular spaces. Jakeel paid attention to these explanations, asked questions when he didn't understand (For example, see Excerpt 6, Line 153), and, when he did make sense of their explanations, used them in formulating his next utterance.

In summary, the audience for Jakeel's discourse had three different types of members. First, Jakeel was a member. He listened to himself and compared his discourse to his expected outcome. The teacher was a second member. She watched Jakeel's performances but was not responsible for explicitly evaluating him. Instead, his peers served as this third type of audience member. Rebecca and Daren were evaluators and explainers for Jakeel, supporting him in altering his discourse. A common theme across these audience members was an emphasis on supporting Jakeel in developing his own discourse.

Mathematizing

The previous section of this chapter focused characterized engaged learning and described Jakeel's audience for engaged learning. This next section focuses more specifically on the mathematical discourse, examining Jakeel's adoption of the discourse, production of new discourse, and substantiation of the discourse.

Adoption of discourse. Adoption of discourse means using the same narratives, routines, words, and visual mediators as others using the discourse. There is evidence that Jakeel adopted a portion of the mathematical discourse available to him as he enacted engaged learning. I will first describe the available mathematical discourse and then evaluate Jakeel's adoption of that discourse.

During the small group time in this lesson, two important sources of mathematical discourse for Jakeel were Rebecca and Daren. Rebecca's mathematical discourse was the most mathematically sophisticated of the three students. This sophistication was apparent as Rebecca read her written answer to the task involving Figures H and I to Daren and Jakeel.

160	Rebecca	Whatever. H and I cover the same amount. Figure I has one square and two half squares that equals two squares.	*Rebecca reads from her paper.*

While this statement did not specifically state that Rebecca was comparing the area of the two rugs or the relationship between area and squares, the statement used mathematical words to communicate how the partial units should be counted toward the total number of squares. Rebecca specifically labeled the partial units as half squares. By using the word half, Rebecca acknowledged the relationship between the partial shapes and the whole. In contrast, when the partial shapes are labeled as triangles, there is no indication of how they might be related to the squares or how they might be counted in the area. Additionally, Rebecca's use of "half" as part of a number sentence ("one square and two half squares that equals two squares") offers further evidence that her use of half was mathematically purposeful. She did not mean that the half units were just some part of a square. Instead, the two halves could be quantified, combined, and then added to one to equal two squares.

Rebecca's statement in Line 160 hints at a narrative that she used repeatedly in her interactions with Jakeel. On five occasions, when Jakeel was enacting engaged learning, Rebecca articulated a variation of "If you put two half squares together, it makes a square" (Line 548). This statement was frequently accompanied by pointing at the small triangles in either Figure I or J. Thus Jakeel had multiple opportunities to hear these words and see how Rebecca linked them to an image.

Like Rebecca, Daren also used "half" to describe the partial units in Figure I. For example, he when he explained his thinking about the task to Jakeel in Excerpt 6, Line 148, he stated, "This is half. If we put this together that's two squares." This statement suggests that Daren like Rebecca, was thinking that two halves together would count as a square. Daren's discourse was also like Rebecca's in his gestures. In each of his interactions with Jakeel, he physically and verbally pointed toward the figures on the page. For example, in Line 148 above, Daren pointed at Figure I as he spoke, "This is half. If we put this together that's two squares." Daren explicitly connected his words with an image.

Daren was the most explicit about connecting area with counting squares. Later in the Excerpt 6, Daren explained to Jakeel, "You see the one square and that's got two. You put these together. That's two squares. So they cover the same area." (Line 152). By enumerating the squares and then stating, "they cover the same area" Daren began to connect counting squares with area although he did not explicitly state the connection.

Daren's mention of area was only the second time (and only one of seven times) that area was mentioned during this entire small group time. The teacher used the word *area* five times, each time as either part of a question or request directed to Jakeel (i.e. Line 534: What's the area of J? and Line 431: Show me how it's the same area as that.). While the teacher required Jakeel to react to her use of *area,* she did not explicitly connect it with any unit of measurement, nor did she require that Jakeel or the other students specifically label what they were counting as area.

In summary, the mathematical discourse available to Jakeel had instances in which the objects counted were labeled as squares and in which tenuous connections were made between area and the unit of measurement. What was repeatedly emphasized in this group, both through words (especially the use of half) and through pointing, was that two of the triangular spaces should be counted together as one space.

Jakeel only adopted a limited portion of the mathematical discourse available in his group. As I noted in the section on his final mathematizing, there was no evidence that he made explicit connections between counting spaces and area. He also did not label what he was counting or use the word half, except on one occasion that I will discuss in the section on directed learning below. Instead, the portion of the mathematical discourse that Jakeel visibly adopted was counting the two triangular spaces as one. In particular, what he adopted from Rebecca and Daren was simultaneously pointing at the two triangular spaces while uttering a single counting word. While I will have more to say later about the specifics of this pointing and how Jakeel intertwined adoption and production, I want to emphasize the evidence suggesting that Jakeel's pointing and uttering were an adoption of Rebecca and Daren's discourse. As I described in the section on his final mathematizing, Jakeel was not able to appropriately count the area of the triangular spaces until his fourth attempt. After each of his attempts, either Rebecca or Daren re-explained with words and gestures that the two half squares should be counted as one square (For example, see Excerpt 8, Line 546). Jakeel's attention to and repeated need for Rebecca and Daren's discourse provide evidence that Jakeel's pointing and counting was at least in part an adoption of Rebecca and Daren's discourse.

Before I discuss Jakeel's production of discourse, I want to make two comments about his adoption of discourse. First, while there was no evidence that he adopted other portions of the discourse available to him, it is possible that if Jakeel had been asked different questions or had more time he might have demonstrated adoption of more of Rebecca and Daren's discourse. Second, the portion of their discourse that he visibly adopted reflected the teacher's primary learning objective for this lesson. She wanted students to learn to appropriately account for partial units when determining area. While it is uncertain that Jakeel's thinking included partial units, he was able to discriminate between different size spaces and count them in ways that appropriately considered their difference in size. Although it seems that Jakeel's adoption of Rebecca and Daren's discourse was limited in terms of number of words, narratives, or routines, what he adopted was significant in terms of the teacher's mathematical objective for the lesson.

Production of discourse. Production of discourse refers to the learner's construction and use of narratives, routines, mediators, or words that were not already part of the discourse presented to the learner. As I hinted above, when Jakeel enacted engaged learning, he intertwined adoption and production of discourse. The best example of this was his counting of the spaces in Figure J. He eventually adopted the routine of counting the two triangle spaces as one. However, he produced the hand gestures that enabled him to correctly count. Rebecca and Daren pointed to the two triangular spaces by using an index finger from each hand, drawing a line to connect them, or pointing back and forth between the spaces with one figure. When Jakeel accurately counted the spaces, he used his index and middle fingers to simultaneously point to the triangular spaces, counting one number when he pointed with the two fingers. When he counted the squares, he pointed with his index finger and counted one number (See Excerpt 4, Line 668). These gestures were his invention and were a change from his initial counting in which he pointed at each space, triangles and squares, with one finger (his pinky) or with a pencil and counted each space as one. Jakeel's gesture demonstrates the integration of production (of the gesture) with adoption of routine of counting two squares as one.

Another aspect of Jakeel's production of discourse was his word use. Jakeel rarely copied or imitated another person's words. Instead, he translated Rebecca and Daren's words into his own words. One example of this is the word *half.* Although Rebecca and Daren frequently used "half" as a label for the partial units in Figure I and Figure J, Jakeel only once used that word. This one use was during the portion of the lesson when he was allowing Rebecca to direct his learning. He did not speak the word *half*, but rather wrote it on his paper as part of a sentence Rebecca dictated to him. Instead of the word half, the only time Jakeel explicitly labeled the partial units, he called them "triangles." He wrote this on his paper shortly after the conversation captured in Excerpt 7. He was working on the task with Figures H and I. Rebecca encouraged him to copy her writing, but he refused, writing his own words, "If you put the two triangles together it makes a square and it covers it." While Jakeel only used triangles only once, it was the first time anyone in the group had labeled the partial units using their geometric name. It provides evidence that Jakeel, as he enacted engaged learning, was determined to do his own learning and to make his own sense rather than blindly adopt the discourse of others.

Here is a second example of Jakeel forging his own discourse. He had finished writing on his paper and was cutting out Figure I. The teacher came to the group and began asking questions.

Excerpt 9

205	Teacher	So what are you doing, Jakeel?	
206	Jakeel	I'm gonna cut these out.	*He is holding the paper with Figures H and I. He points at it with his scissors.*
207	Teacher	Why are you gonna cut them out?	
208	Jakeel	Because I have to make these two a rectangle just like these three.	*He points at his paper with his scissors. His motions are hidden from the camera by his paper.*
209	Teacher	Okay. Why are you making it a rectangle?	
210	Jakeel	Because if you don't it won't cover the whole thing.	

In Line 208, Jakeel stated that he had to make a rectangle. While his motions were blocked from the camera by his paper, it seems that the "two" he wanted to make into a rectangle were the two triangles from Figure I. Instead of using *square* as the other students had, Jakeel called the square a rectangle. In this line, Jakeel indicated that if he assembled the triangles into a rectangle (a square), they would be just like the other three squares already on the paper (one square in Figure I and two in Figure H). Jakeel's use of *rectangle*, like his use of triangle above was the only time (except for the teacher's echo of the word in Line 146) the word was used throughout the lesson. This again reflected

Jakeel's propensity to use his own words. Also, with his mention of "these three," Jakeel was also the only student in this group to count all of the full squares together. While counting the three squares together did not relate to calculating or comparing the area of the figures, it does reflect Jakeel's sense of the situation and his stance that he should communicate his ideas.

Jakeel's reluctance to use the words of others while simultaneously using their same image suggests that Jakeel focused more upon the visual communication from Rebecca and Daren than upon their verbal communication. If Jakeel attended primarily to Rebecca and Daren's gestures (rather than their verbalizations) and linked those gestures to his own words, then his verbal communication is more likely to consist of words and narratives he produced. This is in contrast to Jakeel's attention to words when he enacts directed learning as I will later describe.

Substantiation of narratives. A final feature of mathematical discourse that I will explore is the substantiation of narratives. Substantiation is how a person decides whether a statement reflects the state of affairs (Sfard, 2008). When Jakeel enacts engaged learning he relies upon others to support his substantiation of statements. For example, Rebecca supported Jakeel's substantiation of the counting of Figure J. In Excerpt 8 above, Jakeel counted ten spaces in Figure I. In Lines 544 and 546, Rebecca explained how to count so that the outcome was eight: "Okay, Jakeel. You see those little triangles…. Those are half a squares. Look and if you put these together, those equal a square and so it's eight, it's eight and eight." Rebecca motioned to the figures on the paper as she spoke.

There are two examples of substantiation linked to this excerpt. First, Rebecca's explanation was her substantiation of the narrative that Figure J was eight. She explained how she thought about the triangles as half squares which meant that two triangles could be put together to make a square. Rebecca did not include in her substantiation counting the spaces. Instead, she seemed to imply that the important evidence that Figure J had eight was that two triangles could be counted together as a square. With this explanation, Rebecca implied that substantiation was an appropriate activity and that Jakeel could make sense of this substantiation. Furthermore, her substantiation relied upon the visual image. She used "see" and "look" and pronouns like "it" which were only sensible in reference to the visual. Thus another possible message in Rebecca's substantiation was that it was important to link visual and the verbal communication.

Jakeel could have attempted to memorize Rebecca's substantiation or to interpret her words into his own words to offer as an explanation to the teacher. However, as Jakeel enacted engaged learning, he constructed his own substantiation of the narrative that the count for Figure J was eight. He altered his gestures so that he differentiating the triangular and square spaces and then he was able to arrive at eight as he counted (See Excerpt 4). Jakeel's substantiation incorporated Rebecca's explanation of how to count the two triangles, but was his own routine for counting. Thus Jakeel relied upon Rebecca to support his thinking but he did not limit his thinking to Rebecca's ideas. Likewise, Rebecca's explanation positioned Jakeel as a learner who could use her ideas as a tool for altering his own thinking.

Jakeel's enactment of engaged learning meant that he was an active learner, asking questions, listening to feedback, and positioning his peers as teachers. He also displayed his thinking, contributing his ideas to the conversation and demonstrating his responses to questions. He insisted upon doing his own work and not copying from others and he attended to his own discourse, monitoring his counting to see if it matched his expectations. While he faithfully adopted some pieces of his peer's discourse, especially the visual discourse, he tended to insert his own words rather than use the same verbal communication as Rebecca and Daren. Finally, although he used others to help substantiate the discourse, but he did not use their authority as the substantiation. Instead, he built from their explanations to construct his own substantiation. His peers supported Jakeel's engaged learning by positioning him as someone who could make sense of their explanations. Their explanations modeled discourse and connected the words to visual objects, but did not explicitly state the answer, which created an opportunity for Jakeel to use their thinking to construct his answer. As Jakeel enacted engaged learning, he was involved in actively mathematizing and identifying as a learner who wanted to make his own sense of mathematical ideas.

For the majority of this lesson (in terms of time and number of turns), Jakeel enacted engaged learning. However, during the middle of the lesson, his activity shifted to directed learning. This shift involved changes in his identifying and mathematizing activity as well as the identifying and mathematizing activity of one of his peers, Rebecca. I will first describe the circumstances leading to the shift in participation and then characterize directed learning as enacted by Jakeel and Rebecca.

Departure from Engaged Learning

During the early part of the lesson, Rebecca often (6 times) told Jakeel what to do and write. Each time, Jakeel either ignored her words or he insisted that he would do his own work. For an example, see Excerpt 7, Lines 193 and 194. In spite of Jakeel's resolve to do his own work, Rebecca was eventually able to position him as needing to copy from her: After Jakeel had finished his work on Figure H and I (See Figure 4.1), Rebecca looked at his work.

Excerpt 10

254	Rebecca	You wasn't supposed to do that Jakeel. Jakeel.	*She takes his paper*
		Jakeel what you was supposed to dooo is so this. What we did.	*She puts her paper so Jakeel can see it and she points to it.*
255	Jakeel	Oh. Oh.	*He looks at Rebecca's paper.*
256	Rebecca	So you have to take it and rewrite it	*She tries to pull up Jakeel's glued pieces.*
		and I'll cut these out for you.	
		Rewrite what you wrote here on there.	*She points at an extra paper.*
257	Jakeel	I don't want to rewrite.	
258	Rebecca	Well you have to because you messed up.	*Jakeel picks up his pencil and starts writing. Rebecca starts cutting.*

Rebecca's feedback on Line 254 did not offer Jakeel an explanation or an indication of how he might think about the content of his work. Instead, as a rationale for her negative evaluation of his work, Rebecca told Jakeel that he wasn't doing what he was supposed to be doing. This evaluation did not invite Jakeel to rethink his work or suggest that he could make sense of what Rebecca had done. Instead the authority for deciding what to do was embedded in "supposed" without any indication of how Jakeel might make sense of what he was supposed to do.

Jakeel offered some resistance to Rebecca's directive, stating, "I don't want to rewrite" (Line 257). However, Rebecca's rejoinder (Line 258) left him with little room for negotiation. Jakeel accepted this positioning, picked up his pencil and started to write (Line 258). At this moment, Jakeel's participation shifted from engaged learning to directed learning. For most of the next 17 minutes, Rebecca directed Jakeel's work on the task. He did what she said and he asked her to tell him what to do. In the next sections, I elaborate the features of directed learning as enacted by Jakeel and supported by Rebecca.

Directed Learning

In contrast to engaged learning, when Jakeel engaged in directed learning, he was more passive and reactive, allowing Rebecca to tell him what to do. The excerpt below occurred after the excerpt above in which Rebecca told Jakeel he had to redo his work. Rebecca told Jakeel what to write, but he was distracted by a conversation with Daren. He asked Rebecca to tell him again what to write.

Excerpt 11

319	Jakeel	What am I supposed to write again?	*He talks to Rebecca.*
320	Rebecca	You're playing	
321	Jakeel	What am I supposed to write? (Well you play too much)	
322	Rebecca	They H and I cover the same amount of floor.	
323	Daren	Go get the other J and K.	*Daren talks to Jakeel.*
324	Rebecca	I mean just cover the same amount	*Jakeel writes. The teacher comes by.*
325	Daren	We need J and K	*Daren talks to the teacher.*
326	Teacher	So go up to get J and K.	*Daren gets up and leaves.*

327	Rebecca	H and I cover the same amount of floor. No just they cover the same amount.	*She watches Jakeel write.*
		Same. Amount. Do you try to write small? All right. That's all you gotta write.	*Rebecca reads as Jakeel writes. Daren returns with the papers. Rebecca reaches for the papers and hands them out.*

In Line 319, Jakeel asked Rebecca what he should write and then wrote what she dictated on Line 322 and 327 ("They cover the same amount"). He did not attempt to use different words, interject his own ideas, or connect Rebecca's words to a visual image. Instead, his activity with mathematical objects was limited to following Rebecca's directions.

Jakeel's activity when he enacted directed learning was markedly different from when he was enacting engaged learning. As an engaged learner, he refused to follow Rebecca's directions, telling her that he knew what to do and that he wouldn't do what she asked (Excerpt 7, Line 194). He asked questions about explanations and contributed to the conversation (Excerpt 6). In contrast, as he and Rebecca enacted directed learning, they both focused on Jakeel's physical activity, whether it was writing, cutting, or not playing around, rather than on his engagement with the mathematical discourse.

In the time that Jakeel engaged directed learning, there were five instances in which Rebecca told Jakeel what to do or write and he agreed. Each instance was similar to Excerpt 11 with Rebecca giving directions for Jakeel to follow. Because of the similarities across these instances, I primarily draw upon Excerpt 11 to describe the features of directed learning and how it is different from engaged learning.

Identifying the Audience

Jakeel's audience for directed learning was more limited than his audience for engaged learning. In engaged learning, he addressed his peers, his classroom teacher, and himself. In contrast, when Jakeel enacted directed learning, his audience for his content discourse was either the classroom teacher or his peer, Rebecca. There was no indication that he was attempting to communicate with himself. Instead, his communication around the mathematical content was to write on his paper what Rebecca dictated (See Excerpt 7, Line 327). This writing was done at Rebecca's request, thus Rebecca was a member of Jakeel's audience. Since the writing was done on a paper to turn into the teacher, the classroom teacher was also Jakeel's audience. Jakeel, however, did not intend this writing to be a communication with himself: He did not refer back to it. He did not question it, examine it, or repeat it to himself. It might be argued that he understood what he wrote and so he did not need to examine it. However, he asked Rebecca what he was "supposed to write" (Line 321), implying that Jakeel did not know what to write. Furthermore, the word "supposed" carries implications of expectation or requirement, suggesting that Jakeel was more concerned with how an authority might evaluate his work than understanding it himself.

Rebecca's role as a member of Jakeel's audience was to monitor his physical activity, telling him what to do and ensuring compliance. By doing this, Rebecca served as a director for Jakeel's activity. One example of Rebecca acting as a director was her feedback to Jakeel. In Excerpt 11, Rebecca accused Jakeel of playing (Line 320). She had told him to write, but Jakeel had talked with Daren rather than follow her direction. Jakeel confirmed Rebecca's role as director by asking what he was supposed to write. (Although he indicated that he wasn't entirely pleased with her identification of him as playing by retorting that she plays, too.) This interaction focused Jakeel back on the physical activity he was supposed to be doing. Rebecca's feedback was not an explanation or a correction of mathematical discourse. Instead, it was feedback on Jakeel's progress toward the final product of the task. By contrast, when Jakeel enacted engaged learning, the feedback he received was an explanation of the thinking of his peers, directed toward problems with Jakeel's mathematical discourse. Thus as Jakeel enacted directed learning, he provided support for Rebecca's identification of herself as a director of Jakeel's physical activity.

Mathematizing. As Jakeel enacted directed learning, his use of mathematical discourse was limited to writing the words Rebecca dictated. Excerpt 11 is typical of Jakeel's discourse during directed learning. He spoke twice and each time he asked Rebecca what he should write (Lines 319 and 321). His mathematical discourse was limited to writing "They cover the same" (Lines 324 through 327). At other times when he enacted directed learning, Jakeel either wrote what Rebecca dictated or cut figures as she directed. As a result of Jakeel's limited mathematical discourse, Jakeel's adoption, production, and substantiation of discourse were also limited. I will examine each of these aspects of mathematizing.

Evidence of Jakeel's adoption of discourse would be Jakeel's use of the same narratives, routines, words, and visual mediators as another. However, because Jakeel had limited mathematical discourse, there is no evidence that he adopted mathematical discourse as he enacted directed learning. His writing of Rebecca's words might be construed as adoption. However, as I noted earlier, Jakeel wrote words as Rebecca dictated. He seemed to be primarily recording Rebecca's words rather than using them. Also, in both instances in which Jakeel wrote Rebecca's words, his next activity after finishing his writing was playing with his glue. Jakeel's quick shift from writing to nonmathematical activity suggests that Jakeel was focused on getting his writing done rather than on exploring or investigating mathematical discourse, providing further evidence that Jakeel's writing does not reflect adoption of Rebecca's discourse.

Production of discourse is predicated on adoption of discourse. In order to generate new discourse, the learner must first use accurately use the mathematical discourse and then elaborate on that discourse to generate new narratives, routines, visual images, or word use. As Jakeel enacted directed discourse, he did not demonstrate adoption of discourse and thus it was not possible for him to produce discourse.

Finally, as Jakeel enacted directed learning, he did not directly substantiate mathematical statements. He did not attempt to verify the truth of what he wrote or what others said. Instead, he relied upon Rebecca to provide him with the right answer. For example, in Excerpt 11, he asked Rebecca what he was "supposed" to write (Lines 319 and 321). His use of *supposed* indicated that Jakeel was focused on what was the expected or required activity. If Jakeel had been attempting to substantiate mathematical discourse, he might have asked how Rebecca knew that H and I covered the same amount of floor or he might have counted or otherwise compared H and I to verify Rebecca's statement. Instead, he wrote what she dictated and then began to play with his glue. In each moment in which Jakeel enacted directed learning, he did what Rebecca directed but did not engage in any activity that verified any statements she made.

Jakeel's mathematizing was an important characteristic of directed learning. He and Rebecca focused on his physical activity rather than on his thinking about mathematics. As a consequence of this focus, as Jakeel enacted directed learning, he did not adopt or produce discourse and his substantiation of discourse relied upon Rebecca's authority.

Transition Back to Engaged Learning

Jakeel's enactment of directed learning did not last until the end of the lesson. Seventeen minutes after Rebecca was first successful in directing him, he switched back to engaged learning. This return to engaged learning was evident in Excerpt 8. Prior to this moment, Rebecca first directed Jakeel to cut out Figures J and K and then told him to stop while she determined, out loud, the areas of J and K. She announced that they were each eight and were the same (Line 358, not in this document). She then gave Jakeel permission to cut out the figures: "Okay you can cut them up now. Don't cut out the little squares" (Line 378). Jakeel followed these directions, cut out the figures and glued them on his paper. At this point, his obedience to Rebecca's orders reflected his enactment of directed learning.

However, after Jakeel had finished cutting and gluing, he transitioned back to engaged learning as the teacher asked him questions about his work. Excerpts 2 and 8 show this transition. The teacher asked Jakeel about the area of Figure J (Excerpt 2, Line 534). He pointed at Figure K and replied that it was the same (Excerpt 2, Line 537) and that it was eight (Excerpt 2, Line 539). The teacher asked Jakeel to show her how Figure J was eight (Excerpt 8, Line 540). Jakeel counted the spaces in the figure, stopping as he started to count ten. He said, "Hold on" and then counted the spaces in Figure K (Excerpt 8, Line 541).

At this moment, Jakeel was not following orders or directions. He was evaluating his own discourse and puzzling over his counting. He stated that Figures J and K should have the same count – a count of 8 – yet his counting was yielding a larger number. This discrepancy over what he counted for J, what he thought the count should be, and what he counted for K (Excerpt 8, Line 541) seemed to raise questions for Jakeel. His pursuit of those questions demonstrated his shift back to engaged learning. He continued to demonstrate engaged learning through the end of the lesson, working on counting Figure J, receiving input from Rebecca and Daren and then recounting until he was able to count to eight in front of the teacher (Excerpt 4).

This transition from directed back to engaged learning is an important piece of the next section of this chapter, an exploration of the autonomous learning features of Jakeel's activity.

Autonomous Learning

Autonomous learning is the constellation of identifying and mathematizing activities that reflect curiosity about what others think/say and what seems to be true. As elaborated in my theoretical framework, curiosity is demonstrated by an interest in what is novel, strange, or problematic. Evidence of curiosity can be found in the way the learners identify their audience and in their adoption, production, and substantiation of mathematical discourse. In this section, I will refer back to my descriptions of engaged learning and directed learning to make claims about what features of Jakeel's learning activity are consistent with autonomous learning.

Jakeel's enactment of engaged learning contained several features were similar to the activities of autonomous learning. His audience supported his exploration of the problem of how to count eight when counting Figure J. He included himself in his audience, demonstrating that his discourse was a discourse for himself. His peers provided explanations and feedback and Jakeel monitored his own discourse. Jakeel adopted and produced discourse, drawing upon Rebecca and Daren's illustration of how to count the triangular spaces in Figure I to construct his own gestures for successfully counting and thus resolve the problem of how to reach 8 and not 10 when counting. Finally, Jakeel drew upon Rebecca and Daren's substantiation of the discourse to create his own substantiation. He was not willing to accept Rebecca's statement about how to count the triangular spaces in Figure J was as adequate proof that Figure J was 8. Instead, he wanted to establish for himself that he could count the spaces and arrive at 8.

While Jakeel's engaged learning activity paralleled the activity of autonomous learning in multiple ways, one important way in which he did not enact autonomous learning was his lack of attention to the mathematical words used by Rebecca and Daren. For example, Rebecca and Daren used *half* several times in their interactions with Jakeel. This word labeled the triangular spaces in Figures I and J in a way that captured the mathematical relationship between the triangular spaces and the square spaces. Jakeel did not use the word half when he enacted engaged learning. He did not frame Rebecca and Daren's use of *half* as novel or strange. He did not adopt the word or produce new discourse that integrated the word. His substantiation did not involve the word or reflect his attempts to understand how *half* was an accurate description of the triangular spaces. His discourse does not offer moments in which he explores or shows curiosity about Rebecca and Daren's use of *half*. While we cannot be certain that Jakeel's discourse would have changed if he had been curious about *half*, his lack of curiosity meant that there was not the possibility that he would explore the word and it's implications for his own discourse.

Jakeel's curiosity about Rebecca and Daren's discourse seemed to focus primarily upon their visual communication. He explored how to point and count the pointing gestures, but he did not attempt to use mathematical words to describe his pointing or how it resulted in a different count for Figure J, suggesting that he did not explore Rebecca's verbal discourse in counting the triangular spaces. As I describe above, Jakeel connected the visual with his own words, but his learning outcome might have been different if he had been more curious about their words.

As Jakeel enacted engaged learning, his activity paralleled autonomous learning in many ways. In contrast, his enactment of directed learning did not demonstrate any of the activities of autonomous learning. His audience directed his activity. He did not adopt discourse as a tool for himself. He did not produce discourse and his substantiation relied upon Rebecca's authority

Discussion

Research that has examined identity in mathematics lessons has focused on a more macro view of identity and on identity as reported by individuals. These studies, such as Boaler and Greeno's examination of identity in high school math (2000) and Jilk's exploration of the intersection of salient identities and mathematics pedagogy (2007), described student's self-reported identities relative to learning math. This study demonstrates what a microanalysis of discourse can reveal about identity and learning mathematics. The analysis of engaged learning and directed learning presented in this chapter demonstrates four claims about identity and learning mathematics.

Intertwining of Identity and Mathematizing

First, this analysis suggests that identity and mathematizing are inextricably intertwined. As students mathematize, they construct identities for themselves and others. These identities then offer possibilities for the mathematizing of both the speaker and any others who are identified. For example, in Excerpt 8 Line 546, Rebecca explained to Jakeel how he could think about the triangular spaces in Figure J in order to get the count of 8. This mathematizing identified Jakeel as interested in and capable of making sense of how to count Figure J. It also provided him with the information he needed to alter his counting gestures in order to count 8. As Jakeel responded to Rebecca, he identified her as a teacher whose mathematizing could help him solve his problem of how to count. In this instance, the mathematizing and identifying co-occurred and worked to construct compatible learning and teaching narratives for Jakeel and Rebecca.

Mathematizing can also identify individuals in ways that constrain mathematical discourse. In Excerpt 11, Rebecca told Jakeel what to write on his paper. Rebecca's mathematizing was a statement of the answer for Jakeel to write. It identified Jakeel as someone who was to write words and not someone who was to explore the mathematical discourse. It identified Rebecca as someone who stated answers, but not someone who explained or engaged other people's mathematical discourse. Rebecca's mathematizing and Jakeel's willingness to write her words limited the mathematical discourse for both of them. This limitation included not only the quantity of discourse, but also the range of discursive features and the way in which they used the discourse. As Rebecca dictated words, she did not try a variety of explanations to help Jakeel understand. She also did not use gestures to indicate how her words referred to visual objects. She focused on supplying Jakeel with the words to write. Jakeel was not a learner. He was a recorder. As a recorder, he did not need to engage the discourse so that he could later use it to solve his own problems. He needed to produce the appropriate outcome on his paper in this moment. In this example, the ways in which Rebecca and Jakeel mathematized identified each of them in ways that promoted a particular, limited way of using mathematical discourse.

Finally, individuals can engage in identifying activity that does not refer to mathematical objects, but constructs narratives that constrain or enable mathematizing. For example, in Excerpt 11, Line 320, Rebecca told Jakeel that he was playing. Rebecca's statement was not about mathematics although it addressed her perception of Jakeel's lack of mathematical activity. By commenting on Jakeel's playing, she implied not only that he was not working but also that he needed someone to monitor his activity and direct him to appropriate activity. By focusing on Jakeel's writing, Rebecca limited his mathematizing to the physical activity of writing words. Jakeel's next statement, in which he asks what he was supposed to write (Line 321), indicated that he accepted her limitations on his mathematizing. Rebecca could have asked Jakeel what he thought the answer was or how he was thinking about the problem or she could have explained how she was thinking about the problem. While these might not be typical fourth grade reactions to this situation, I suggest them in order to emphasize how Rebecca's accusation that Jakeel was playing framed Jakeel in a certain way that suggested he should mathematize in a certain way: By commenting on Jakeel's playing, Rebecca identified herself as a director and Jakeel as someone who should be directed and their mathematizing became about the physical activity.

Kinds of Learning

Many conceptions of identities relative to learning math describe the learner as either a math person or not a math person (see for example, Boaler and Greeno, 2000). Because students (and others) use these identities to explain their engagement (or reluctance to engage) mathematics, they are important identities to study. However, the identities of math person or not a math person do not capture the different ways in which people mathematize and identify as they enact these distinctions. Just as the use of a higher power microscope lens changes the image so that new features are visible, the microanalysis of student activity performed in this study can capture important details not visible at lower resolution. Therefore, a second important claim made by this analysis is that student's identifications relative to learning math can be described in more detail and in ways that capture important differences in activities in which students might engage. This study of the learning activities of Jakeel found two qualitatively different ways in which Jakeel mathematized and identified: engaged learning and directed learning. The analysis of these two kinds of learning shows how interactions with peers are important in proposing and enacting these ways of learning. For example, Jakeel could not have enacted directed learning without Rebecca's directions. This increased detail in thinking about student mathematical learning also provides a means to examine what specific mathematizing and identifying activities might support or constrain effective learning of mathematics. Finally, this more detailed description of learning and identities offers an alternative to the binary distinction between math person/not a math person that may force students into categories that don't capture how they think about themselves as learners of mathematics. Given what we have seen here, it would be hard to fit Jakeel into one of these two.

Fluid Identities

This analysis also captures the ways in which student's identities may shift during the course of a lesson. Jakeel was an engaged learner for most of this lesson. However, toward the middle of the lesson, he enacted an identity of directed learner and then returned to engaged learner. This shift in identities indicates a change in mathematizing and talk about oneself and others. While it is possible that this change is new discourse and reflects learning, it is also possible the change reflects enactment of a different, but not new narrative about the person and their mathematical activity. For example, in a previous lesson, Jakeel had enacted directed learning. Thus his enactment of the learning in this instance did not reflect a change to a new discourse but rather a shift to an identifying discourse that he had previously used. Thus this analysis shows that students may enact multiple identities as they learn mathematics.

Development of Mathematical Discourse

A final claim resulting from my analysis is that certain kinds of learning activity results in the development of a more mathematical discourse. Over the course of this lesson, Jakeel's discourse changed such that by the end of the less he could differentiate the triangular spaces from the square spaces and count two triangular spaces as one square space. This change of discourse seemed to occur as Jakeel enacted the mathematizing and identifying of engaged learning. Jakeel's enactment of engaged learning involved identifying and working on a problem whose solution required Jakeel to change his mathematical discourse. He became interested in the problem of how to count the spaces in Figure J. He listened to Rebecca's explanation and then modified his gestures. He counted and recounted, adjusting his pointing, until his count was only eight. Both Rebecca and Jakeel identified Jakeel as an involved and interested problem solver who could solve his problem by changing his discourse. By the conclusion of the lesson, after a time in which Jakeel enacted engaged learning and was supported by Rebecca, he changed his discourse.

In contrast, when Jakeel enacted directed learning, his mathematical discourse was limited and showed no evidence of developing. As a directed learner, the problem Jakeel worked on was doing what Rebecca asked him to do. He and Rebecca both emphasized Jakeel's physical activity. As a result, Jakeel was not identified and did not identify himself in a way that required him to work towards changing his mathematical discourse. Thus, when Jakeel enacted directed learning, his mathematizing and identifying and that of his peers focused him on activity that was not about changing or developing his mathematical discourse.

I have defined learning as a change in discourse and noted that effective learning means that the discourse has changed to be more like the desired discourse. Jakeel's discourse changed as he enacted engaged learning so that his discourse was more like the desired discourse: He accurately differentiated the triangular spaces from the square spaces in his counting. This was in contrast to the lack of a change in discourse as Jakeel enacted directed learning. This analysis suggests that Jakeel's engaged learning activity resulted in more effective learning than his directed learning activity. This conclusion demonstrates an important benefit of this study. The close analysis of discourse allows for the coordination of mathematizing, identifying, and learning so that outcomes of learning can be linked to mathematizing and identifying moves. This linkage provides a powerful tool for studying mathematical learning in classroom situations.

An important component in guiding my decisions about how to examine mathematizing, identifying, and learning was my theoretical frame for autonomous learning. By elaborating autonomous learning, I was able to develop a lens to examine what mathematizing and identifying moves might be most desirable. I was then able to use each case to clarify my autonomous learning framework. In the next section of this discussion I will describe how the framework of autonomous learning supported the examination of Jakeel's learning and how Jakeel's learning added detail and complexity to the autonomous learning framework.

Autonomous Learning

My framework for examining autonomous learning emphasizes the ways in which individuals adopt, produce, and substantiate discourse with the construction of discourse-for-oneself as the end result. This focus on discourse helped me to describe and differentiate engaged learning and directed learning: Jakeel's engagement with discourse was very different as he enacted engaged learning and directed learning. In particular, Jakeel's work to adopt, produce, and substantiate discourse as an engaged learner demonstrated that he was developing the mathematical discourse as a discourse-for-himself. In contrast, when he acted as a directed learner, his mathematical discourse was a discourse-for-others. Jakeel demonstrated that his discourse during engaged learning was a discourse for himself by including himself in his audience, by adopting Rebecca's visual discourse and then using it as a tool to produce his own routine for counting Figure J, and by working to substantiate the count of Figure J using his own discourse. These features of Jakeel's activity show the ways in which Jakeel was enacting autonomous learning as he enacted engaged learning. In contrast, as Jakeel enacted directed learning, he did not adopt or produce discourse and he relied upon Rebecca for substantiation. Also, he was not an audience for his discourse. Thus his enactment of directed learning did not include any features of autonomous learning.

Using the lens of autonomous learning to examine Jakeel's enactment of engaged learning and directed learning helps highlight which of Jakeel's activities may have been most helpful in supporting the development of his mathematical discourse. The change in Jakeel's mathematical discourse occurred during the time in which his activities most closely resembled the mathematizing and identifying of autonomous learning. For example, Jakeel's was curious about counting Figure J. He worked to monitor and change his discourse for himself. He attended to, incorporated, and modified the explanations of others. The overlap between the activities of autonomous learning and Jakeel's development of a more mathematical discourse suggest that the enactment of autonomous learning may support students in learning mathematics.

Ironically, Jakeel's written discourse as he enacted directed learning was more like the desired mathematical discourse than his discourse as he enacted engaged learning. For example, while directed by Rebecca, he wrote, "[T]wo halves make a square". This statement because it included half and square as labels for the spaces in Figure J, was more mathematical than Jakeel's pointing and counting of the spaces in Figure J. However, the discourse Jakeel wrote was not a discourse-for-himself and there was no evidence that he could use this discourse to solve his own problems. Thus even though the discourse was more desirable, Jakeel was not engaged in a kind of learning that made the more desirable discourse a useful tool. This suggests that it is insufficient to consider what students say or write as an indication of their ownership of those words. Instead, the context in which students engage in discourse and their ability to wield the discourse as a tool is an important factor in evaluating student learning.

The lens of autonomous learning also helps us to critique Jakeel's engaged learning. While Jakeel's discourse became more like the desired mathematical discourse as he enacted engaged learning, he did not adopt the available verbal discourse in the same way that he adopted the visual discourse. For example, if Jakeel had been more curious about Rebecca's use of *half* or the ways in which Daren verbally linked squares and area, he might have adopted those portions of Rebecca and Daren's discourse in ways that could have helped him add those features to his discourse-for-himself. If Jakeel had adopted more of Rebecca and Daren's verbal discourse, his mathematical discourse might have become even more like the desired discourse. Using the framework of autonomous learning to consider how Jakeel's activity was or was not autonomous offers a tool for considering ways in which Jakeel's mathematical discourse might be further developed.

While the lens of autonomous learning provided insights into Jakeel's learning, the case of Jakeel's learning adds complexity to the framework of autonomous learning. As Ben-Zvi and Sfard (2007) note, autonomy is typically construed as relying upon oneself. For example, the definition of autonomy used by Warfield, Wood, and Lehman (2005) is "students are capable of thinking about mathematical ideas without having the ideas 'explained' to them and of solving mathematical problems without being shown a method by another person" (p. 440). According to this definition, Jakeel's activity could not be described as autonomous because he relied upon Rebecca and Daren to provide him with feedback and explanations. However, Piaget's description of autonomy, as describe in Chapter 2, suggests that cooperative learning is an essential piece of autonomy because it helps the learner decenter from their perspective and realize that there are other ways of seeing, doing, and understanding (Kamii, 1994). This description of learning together focuses on autonomy as an intellectual activity rather than as a physical activity. Rather than examine who the learner learns with, Piaget's autonomy asks how the learner is engaging with ideas. According to this conception of autonomy, Jakeel's engaged learning could be classified as autonomous. He examines the ideas of Rebecca and Jacob and works to incorporate their feedback and explanations into his discourse.

However, Piaget emphasizes that learners should not be coerced by those with whom they learn: Learners should be equals, learning together (Kamii, 1994). This condition does not describe the learning situation with Jakeel. Rebecca and Daren were not learning with Jakeel, they were teaching Jakeel. These students were not on the same footing regarding discourse around area. Their unequal footing was the reason that Jakeel could learn from them. Ben-Zvi and Sfard (2007) note that autonomy is a possible outcome for learners who are taught by other students. They elaborate a learner-teacher agreement in which the teacher is able to use a discourse that the learner does not yet know. The learner must learn from the teacher. This situation can result in autonomous learning if the learner works to make the new discourse a discourse for him/herself through critical examination of the discourse. Jakeel and Rebecca and Daren offer an example of Ben-Zvi and Sfard's learning-teaching agreement and thus offer evidence that autonomous learning, contrary to the conceptions of it as requiring the learner to be working alone, can, and in some situations must, arise when one student learns from another.

CHAPTER 6

COVERT LEARNING

Overview

This case offers a contrast to the case in the proceeding chapter. Jakeel was explicit about his learning. As he enacted engaged learning, he identified as someone who was interested in understanding and coming to know. Even as he enacted directed learning, he identified as not knowing what to do. In contrast, the focal student in this case, Minerva, enacted covert learning in which she identified herself as already knowing. The words *covert learning* emphasize her need to learn even as she presents as not needing to learn. Minerva's enactment of covert learning raises important questions about the roles of independence and knowing in autonomous learning.

As in the last case, I begin by summarizing the lesson and then describing the outcome of Minerva's learning. I then elaborate covert learning and describe the identifying and mathematizing of covert learning before discussing the implications of covert learning for autonomy and for the development of mathematical discourse.

Summary of the Lesson

As in the last chapter, this analysis of Minerva's learning uses excerpts from across the transcript of the lesson in her group. Table 6.1 summarizes the lesson and indicates the position of the excerpts relative to the turns in the lesson. In the next few paragraphs, I provide a summary of the main events in Minerva's group.

The teacher dismissed the students to work in small groups. Minerva, Bonita, and Jessica gathered at a cluster of four desks near the entrance to the classroom. After reading the task card and gathering the materials for the task, they began work cutting out the figures. The teacher came to the group and asked questions about the area of the two figures. During this conversation, Bonita explained that the area was two because two triangles made a square. Minerva disagreed with this. Bonita and Jessica explained how two triangles could make a square. Minerva constructed her first solution which consisted of Figures H and I glued down intact and a sentence about area. Minerva's work on this first solution (and her next two solutions) is in Figure 6.1. The teacher came to the group and told Minerva that her paper didn't show the area. Minerva constructed a second solution that had Figure H and the two triangles from Figure I arranged in a square. This solution was missing the square from Figure I. In addition, Minerva had numbered Figure H as one and the two triangles from Figure I as one, counting two altogether. The teacher told Minerva that her solution didn't prove anything and Minerva started over on her third solution. This solution had all of the pieces from Figure I with the two triangles arranged as a square. It also had Figure H intact. Minerva had numbered each square in Figure H as one and indicated that there were two. As she numbered the pieces of Figure I, she first counted each triangle as one, but she then erased these numbers and wrote "1" across both triangles and indicated that the triangles and the square from Figure I were two altogether.

The teacher told Minerva that this solution did not show the whole rug, so Minerva started over for the fourth time. In this solution, she glued the pieces from Figure I next to the pieces from Figure H constructing a large square. (See Figure 6.2) She indicated that the area of the both figures together was four. She and the teacher discussed this solution and Minerva was able to get the teacher's permission to go to the next task (involving Figures J and K). The teacher then realized that Jessica did not yet have the answer to the task with Figures H and I. She instructed Bonita and Minerva to help Jessica, which they reluctantly did. They all then started work on Figures J and K. Minerva's solved this task by cutting the small triangles and arranging them into squares, placing Figures J and K next to each other, and counting each of the squares to determine the area. This work is also displayed on Figure 6.2.

Table 6.1 Transcript excerpts and main events in lesson

Line number	Excerpt Number	Excerpt Summary	Main Events
0			The teacher explains the lesson .
			The students move to their small groups and start work.
100			
	Excerpt 1	Bonita states two triangles make a square. Minerva disagrees.	
200	Excerpt 4	Bonita and Jessica explain about the triangles. Minerva says "Nah huhn".	
			Minerva works on her first solution.
	Excerpt 6	Solution 1. Minerva is done. The teacher disagrees.	
			Minerva works on her second solution.
300			
	Excerpt 7	Solution 2. The teacher evaluates Minerva's second solution.	
			Minerva works on her third solution.
	Excerpt 5	Bonita states, "They're the same."	
	Excerpt 2	Solution 3. Minerva states she's gotta explain it.	
400			

Table 6.1 con't

	Excerpt 8	The teacher asks how to show the whole rug. Minerva says, "Put it together."	
500	Excerpt 3	Solution 4. Minerva talks about connecting it.	Minerva works on her fourth solution.
			The teacher approves Bonita and Minerva's solution to Figures H and I.
	Excerpt 9	Minerva teaches Jessica.	
600			Minerva, Bonita, and Jessica work on Figures J and K.

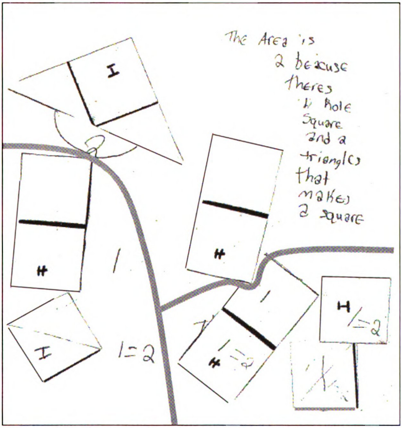

Figure 6.1 Minerva's first three solutions to the task involving Figures H and I. This is student work. The first solution is in the top center. The writing reads, "The Area is 2 beacuse [sic] theres [sic] 1 hole [sic] square and 2 triangles that makes a square". Minerva's second solution is on the bottom left. Her third solution is on the bottom right.

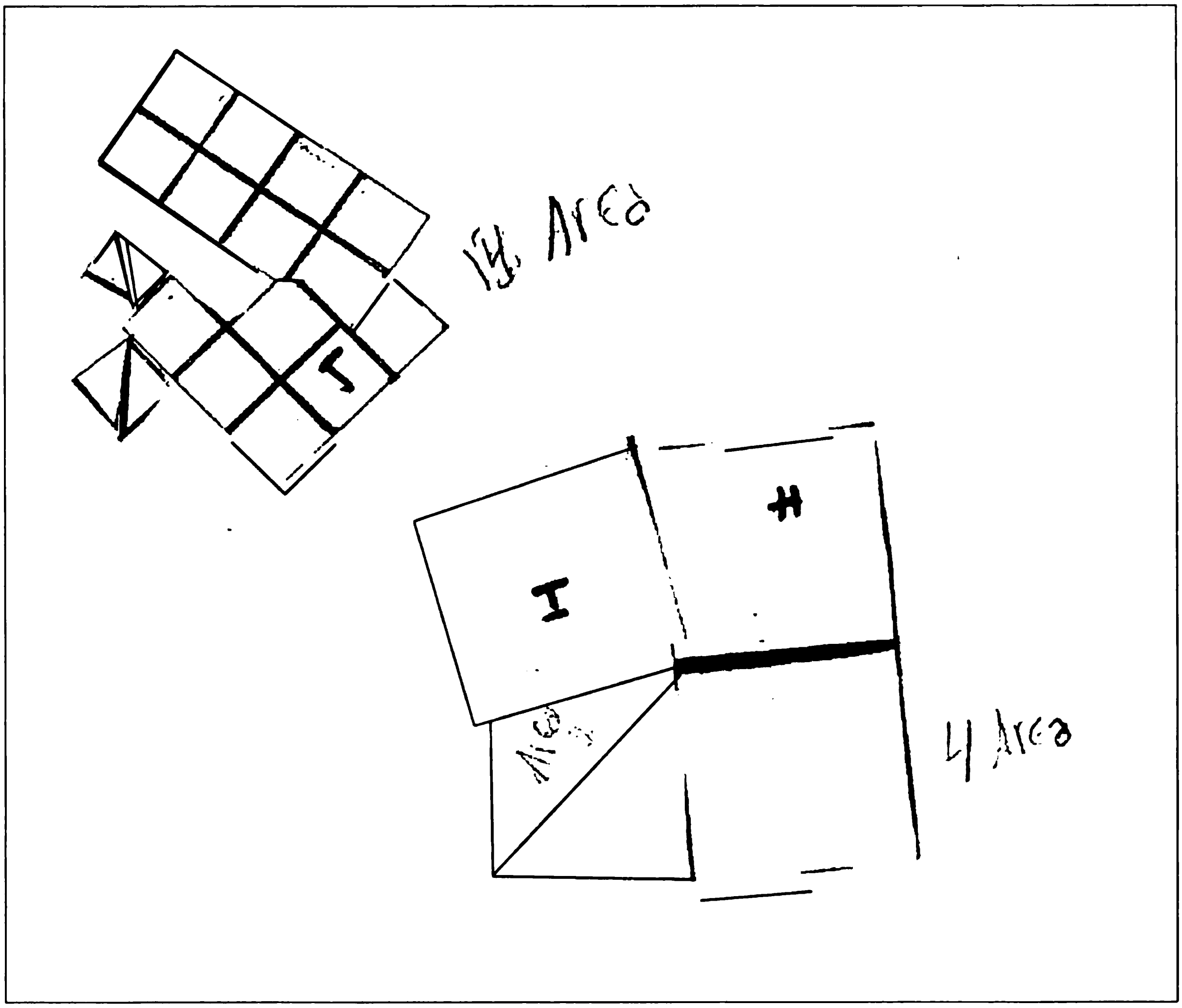

Figure 6.2 Minerva's final work
This is student work. Minerva's fourth solution is on the bottom right. Her solution to the task with Figures J and K is on the top left. The writing toward the top of the page reads, "14 Area". The bottom right reads, "4 Area". The writing in the triangle (bottom middle) is "Area 2".

Outcome of Learning

During this lesson, Minerva's solution to the task changed several times, becoming more and then less like the desired discourse. Across these different solutions, one feature remained constant and was a change from her initial discourse: her statement and/or demonstration (through rearranging the pieces of the figures) that two triangles make a square. In the following sections, I present Minerva's initial discourse, the moment in which her discourse is most like the desired discourse, and then her final discourse. Across these three samples of discourse, I will emphasize Minerva's use of the narrative of two triangles making a square.

Initial Discourse

At the beginning of the small group work, there was little evidence of Minerva's discourse around area. This first example occurred toward the beginning of the small group work. Minerva has told the teacher that they need help because they don't know what to do.

Excerpt 1

191	Teacher	So what does the question say that we have to try to do?	
192	Bonita	Compare them	
193	Teacher	Let's read that task again.	
194	Bonita	First compare rugs H and I.	*Bonita reads from the directions on the task card.*
195	Teacher	How can I compare them do you think?	
196	Minerva	They're different shapes.	
197	Teacher	Yeah so how could I compare them	
198	Minerva	The area	

Minerva's response of "area" as a way to compare Figure H and I was mathematically appropriate. However, the response was not detailed or elaborate. As a result, it is difficult to determine what Minerva was thinking about area. The conversation with the teacher continued.

199	Teacher	Okay you're going to compare their area. How could I compare it?
200	Bonita	Both the areas, they both have two.

201	Teacher	How do you know?	
202	Bonita	Because two triangles make a square	*Bonita points at the two triangles in Figure I.*
203	Minerva	No it doesn't	
204	Bonita	Uhhunh.	*This is an affirmative utterance.*
205	Jessica	Yes it does!	*The teacher leaves the group*
206	Minerva	Nah hunh	*This is a negative utterance.*

Minerva's statement on Line 203 was a direct contradiction of Bonita's statement that "two triangles make a square" (Line 202). While Minerva did not elaborate upon the reasons for her disagreement, it was possible that Minerva either disagreed that two triangles made a square or disagreed that the information was helpful in determining the area of the figures. In either event, at this moment it seemed that Minerva was not thinking about how triangles and squares would be useful in determining the area of the figures.

Approaching the Desired Discourse

Excerpt 2 occurred approximately 16 minutes after Excerpt 1. During that time, Minerva produced two solutions to the task involving Figures H and I. Her first solution was primarily written. It included the statement "The area is 2 beacuse [sic] theres [sic] 1 hole square and 2 triangles that makes a square" along with Figures H and I glued as whole figures with no alterations. In this solution, Minerva explicitly used the narrative of two triangles making a square in her writing, but she did not demonstrate this narrative with the figures. The teacher asked Minerva to "show her" how the area was two.

Minerva complied, generating her second solution. In this solution, she cut the two small triangles from Figure I and arranged them as a square. She glued them next to another copy of Figure H that she glued intact. She explained to the teacher, "One, two equals two. One and one equals two" (Line 288). As she said this she pointed first to Figure H and then to the two triangles joined as a square. She seemed to be suggesting that the answer of two came from counting Figure H as one and the two triangles of Figure I as another one, resulting in a total count of two. In this solution, Minerva illustrated by cutting and arranging the triangles from Figure I how the two triangles could be a square, but her numerical solution of two included all of Figure H and just part of Figure I, raising questions about what how she thought about area and what her intention was in counting. When the teacher examined and critiqued this work, stating to Minerva, " There's no proof of to me of how they're [Figures H and I] the same or different. How am I going to prove they are the same or different? Minerva. How am I going to prove it?" (Line 337).

After this input from the teacher, Minerva worked on a third solution. She glued Figure H on her paper and then spoke to Bonita as she wrote. (See the bottom right of Figure 6.1 for Minerva's final product.)

Excerpt 2

378	Minerva	One two equals two. I gotta explain it for her.	*Minerva talks as she writes on her paper. She writes on Figure H, "1" in the top square, and 1=2" in the bottom square. "Her" refers to the teacher.*

While Minerva did not describe her counting as determining the area of Figure H, her activity suggested that she was counting the separate pieces of Figure H and determining a number for Figure H. After writing on Figure H, Minerva cut out and cut apart the pieces of Figure I. She glued the triangles on her paper and then the square. These shapes were glued so that they touched, but they were not aligned into a rectangle. (See Figure 6.1.) Minerva wrote “1” on one triangle from Figure I and “1=2” on the other triangle. Minerva did not describe or explain this work to anyone, so any interpretation relies solely upon her actions. This initial writing seemed to be a count of each triangle from Figure I.

Minerva next moved her pencil to the square from Figure I. She held the pencil as if to write, but instead she hesitated, and then erased what she had written in the triangles. She wrote again on the triangles, drawing a “1” across both triangles and writing “1=2” in the square from Figure I. This erasing and rewriting suggests that when Minerva realized she should count the triangles together as one and that the square and the joined triangles counted as two. Like her second solution, this third solution provided an illustration of how two triangles could make a square. Unlike her second solution, Minerva’s counting in this solution demonstrated that she was counting these two triangles together in the same way that she counted the single square. Thus in this third solution, Minerva used both a picture and words to indicate that two triangles made a square.

Minerva’s writing “1=2” on the square from Figure I was similar to the writing she just completed on Figure H (Excerpt 2), suggesting that she was counting the “squares” in Figure I in the same way as the squares in Figure H. At this moment, Minerva seemed to be indicating that Figures H and I were the same, that they had the same count. While this comparison was part of the desired mathematical discourse, Minerva was not explicit about what object she was counting, how this counting related to area, or whether the figures had the same area. Thus while this discourse had many of the desired mathematical features, it still lacked some components of the desired discourse.

Final Discourse

Minerva's final solution to the task involving Figures H and I was constructed almost 38 minutes into the small group time. The teacher critiqued her third solution, telling Minerva that it didn't "show what the area is of the whole rug. This is only part of the rug" (Line 477). The teacher pointed to the bottom of Minerva's paper (Figure 6.1) as she said this. The teacher's statement is hard to interpret because Minerva's work contains all of the pieces from both figures. It seems that the teacher was suggesting Minerva did not have all of Figure I. The teacher may also have been trying to articulate that Minerva should have the pieces of Figure I arranged together as a rectangle (like Figure H) instead of adjacent to each other as Minerva had placed them. Minerva suggested that she could show the whole rug by "put[ting] it together" (Line 478).

This next excerpt shows how Minerva understood her words "Put it together" (Line 478). After the teacher left the group, Minerva constructed her final solution by cutting out Figures H and I again. She put glue on the back of the square from Figure I and placed the square so that it overlapped Figure H, sticking Figure H to the paper (See Figure6.2). The teacher arrived at the group.

Excerpt 3

492	Teacher	So this is H, right?	*The teacher points at Minerva's paper.*
493	Minerva	No, that's H. This is I.	*Minerva points at the shapes on her paper.*
494	Teacher	So I'm wondering why you have I.	

495	Minerva	I'm connecting it	
496	Teacher	Why are you connecting it?	
497	Minerva	Because you told us to.	
498	Teacher	Did I say connect it?	
498	Minerva	Yes.	
499	Teacher	Did I say connect it?	
500	Minerva	Yes.	
501	Teacher	I said I want you to prove how this is. I didn't say connect it. If I said connect it, I was wrong.	*Minerva groans and pulls Figure H from her paper. The square from Figure I comes up with it.*

Minerva interpreted the teacher's request to show "the whole rug" to mean that she should connect or glue Figure H and Figure I together. By telling Minerva that she was wrong if she said to connect it, the teacher indicated that connecting was not what she intended (Line 501). However, Minerva continued to arrange the two figures together and then to determine the area as the count of both figures.

Minerva's understanding that the two figures should be put together is apparent in her final work this task and on the next task involving Figures J and K. The bottom right portion of Figure 6.2 shows how she arranged the pieces of Figures H and I. After the conversation in Excerpt 3 above, Minerva arranged the pieces of figure I and wrote "Area 2" on one of the triangles. Minerva placed Figure H as shown in Figure 6.2. Minerva then wrote "4 Area" to the right of the connected figures. Minerva's positioning of the two figures and writing "4 Area" indicated that she was thinking about the area as both figures together.

Her solution to the next problem also reflects her thinking about area as both figures. The work at the top left of Figure 6.2 shows how Minerva arranged the figures from the next problem. She cut the small triangles from Figure J and assembled them into two squares. She then placed the pieces of both figures together and counted the spaces in each, writing "14 Area." The area of both figures together should be 16. It is not apparent from the video how Minerva counts to arrive at 14, but it is apparent that she counted both figures in determining her number.) In both of these solutions, Minerva demonstrated that two triangles make a square through her arrangement of two triangles into squares and her counting of the triangles together as one.

A last example of Minerva's understanding of area as joining the two figures together is Minerva's directions to Jessica. Jessica had Figure H glued on her paper and was trying to determine what to do with the pieces of Figure I. Jessica had the square from Figure I in her hand. Minerva told her, "Connect it to the two squares" (Line 585). Minerva meant for Jessica to put the square from I next to the two squares in Figure H already glued on Jessica's paper. While Minerva does not specifically tell Jessica to count the squares in both figures together, her command to Jessica, coupled with her use of the "4 Area" and "14 Area" in her work shows that Minerva was thinking that area needed to be both figures together.

At the start of the lesson, Minerva did not seem to understand that putting two triangles together into a square could be an important piece of solving the task of comparing the areas of the figures. However, she quickly incorporated that narrative into her talk about and her work on area: Each of her solutions demonstrated that information either in writing or in her arrangement of the pieces of the figures. Her use of "two triangles make a square" demonstrated her learning. In addition, it demonstrated learning that was an important part of the desired discourse: This narrative and Minerva's use of it gave her a means of translating all of the pieces of Figure I into the same unit, which could then be counted. However, Minerva's final discourse also included a routine that was not part of the desired discourse. She seemed to learn that the arrangement of the pieces of both figures was important: It was important to connect the figures and count them together to determine area. From Minerva's actions, it is difficult to determine how joining the two figures was related to her understanding of area. It is possible that she interpreted area as a mathematical operation like addition in which separate pieces (in this case from the two figures) were combined and counted. Whether this was her understanding, it is clear that her final discourse, while more mathematical in some ways, was mathematically troubling in others.

I would like to suggest that this final discourse, both the desirable and undesirable components, was an outcome of Minerva's mathematizing and identifying activities during the lesson. In the next section, I describe Minerva's learning activities and begin to explain how they might account for the change in her discourse.

Learning Activity: Covert Learning

The word covert suggests something that is concealed, hidden, or disguised. Covert learning is meant to imply that learning is happening without the learner openly identifying as a learner or explicitly identifying someone as a teacher. Someone who is engaging in covert learning is attempting to conceal their need to learn and their activity of learning. The engaged learning Jakeel enacted is in contrast to covert learning. In engaged learning, the learner identifies and is identified by others as someone who is working to make sense of discourse and activity. In directed learning, the learner identifies/is identified as someone who needs to be told what to do. In contrast, covert learning means that the learner recognizes (even if only tacitly and not explicitly) the situation as one in which he/she can/needs to learn but that the learner does not want to explicitly identify as a learner. He/she does not want others to position him/her as a learner. Thus the learner makes moves that simultaneously communicate that he/she is not a learner while also watching and listening to others in order to develop his/her own ideas.

Two Illustrations of Covert Learning

Throughout this lesson, Minerva engaged in covert learning: she did not explicitly identify herself as a learner or another as a teacher and yet her subsequent actions indicated a change in her discourse. In some instances, the change in discourse might be attributable to an interaction with or between others. In other instances, Minerva seemed to be developing her own discourse. The most apparent example of covert learning as a result of an interaction with others occurred at the beginning of the small group work and was included above as Excerpt 1. The teacher asked Bonita how she knew that the areas of both figures are two (Line 201). Bonita responded, "Because two triangles make a square" (Line 202). Minerva, without hesitation, exclaimed, "No it doesn't" (Line 203). Minerva's disagreement was emphatic and explicitly directed at Bonita's statement. It is unclear what Minerva finds objectionable about Bonita's statement. However, it is clear from Minerva's tone that she does not agree with Bonita's statement.

As the interaction continued, Minerva's tone changed. In Line 206, Minerva continued to articulate disagreement with her peers. However, her utterance ("Nah huhn") was not as pointed or articulate as in Line 203. Instead, her "Nah huhn" had a more generalized, mocking tone. Minerva did not argue a specific point, but instead offered a more generic counter to Bonita and Jessica's insistence that Bonita's first statement was correct. Minerva continued to respond to Jessica and Bonita with "Nah huhn":

Excerpt 4

207	Jessica	Yes it does!	
208	Minerva	Nah hunh	
209	Jessica	Let me see, let me show you	
210	Bonita	You cut this right off	*Bonita is holding Figure I. She points to one triangle in Figure I*
		and put it there	*and motions next to the other triangle in the figure.*

In Lines 209 and 210, Jessica and Bonita both explicitly positioned Minerva as a learner and themselves as teachers. Jessica's "Let me show you" and Bonita's demonstration of how to rearrange the figure communicate their understanding that Minerva did not understand something they both understood.

211	Minerva	Nah hunh	
212	Jessica	Let me see. See. Look it. Let me show her that it can make a triangle.	*Jessica takes Bonita's Figure I and her Figure I.*
213	Minerva	Nah hunh	*Minerva watches Jessica.*
214	Jessica	See. Look it.	*She places them together to make a large square.*
215	Minerva	Nah hunh, nah hunh, nah hunh	
216	Jessica	Stop it.	
217	Minerva	Nah hunh	*Minerva puts glue on the back of Figure I.*

Again in Lines 212 and 214, Jessica positioned herself as a teacher and Minerva as the learner. During this interaction, Minerva watched Jessica and Bonita as they each explained how "two triangles make a square". While Minerva's "Nah huhn" communicated that she disagreed with their utterances, the timing of Minerva's "Nah huhn" indicates that she was attending to what Bonita and Minerva were saying. She was not interrupting their words, but responding (albeit negatively) to each of their statements. Her "Nah huhn" was indiscriminate: She responded in the same way to each statement of Jessica and Bonita's regardless of the information, tone, or type of statement: Jessica's demonstration of "two triangles make a square" (Line 214) received the same response from Minerva as her request that Minerva stop her utterances (Line 216). Minerva continued to respond to Bonita and Minerva with "Nah huhn" for a full minute after the interaction above, uttering this phrase 32 times in total. Jessica and Bonita both asked her to stop, but Minerva continued in spite of their repeated requests.

The repeated and indiscriminate qualities of Minerva's "Nah hunh" suggest that she was not communicating disagreement with the specific details of Bonita and Jessica's discourse. Instead, the "Nah huhn" seemed to be an attempt to portray her initial disagreement in Line 203 as a jest or a joke. It seemed to suggest mockery of the serious, explanatory tone set used by Bonita and Jessica. By disagreeing or denying everything Bonita and Jessica said, Minerva's initial negative exclamation ("No it doesn't" Line 203) became more trivial. In addition, Minerva's "Nah huhn" seems to send the message that she does not need or want Bonita and Jessica's explanations. In spite of Minerva's disagreement with Bonita's statement that "two triangles make a square", Minerva adopted this narrative in her subsequent discourse. She incorporated these words into her first solution to the problem (See Figure 6.1 above) and she repeated these words on two separate occasions (Excerpt 5, Line 359 below, and 457, not included in this document). Also, with the exception of her first solution, each of Minerva's physical solutions to the task included the arrangement of the two triangles into a square. Minerva's use of Bonita's narrative and arrangement of the triangles suggests that Minerva learned from Bonita even while she professed disagreement with Bonita and Jessica's statements.

Before continuing to elaborate covert learning, I want to address the possibility that Minerva's initial disagreement (Line 203) might have been the result of misunderstanding or misspoken words. Perhaps Minerva understood and agreed with Bonita but misspoke in the moment of Line 203. This scenario is unlikely for several reasons. First, Minerva was absent during the previous lesson when the class first worked on determining the area of triangular figures. Thus it is unlikely that Minerva had recently (or possibly even previously) attempted to resolve the question of how to count triangular spaces. Second, Minerva's discourse prior to Bonita's statement did not indicate that Minerva had a solution to the task or recognized that she needed to account for the triangular spaces in Figure I. She had asked the teacher for help (Line 184) and stated that she didn't know what to do (Line 186). In addition, Bonita statement about the two triangles (Line 202) was the first time triangles had been mentioned in this group. It seems unlikely that Minerva was already thinking about how to work with the triangles. Finally, if Minerva misunderstood Bonita or misspoke, it seems that a more effective recovery would have been to offer her own explanation of how two triangles make a square, as an illustration that she really did understand. Instead, Minerva's "Nah huhn" were timed so that they punctuated, but did not interrupt the explanations of Bonita and Jessica. In addition, Minerva watched and listened to Bonita and Jessica. If she had been less interested in what they were saying and doing, she could have been more dismissive and turned away from them. Instead, she attended to their words and actions and only began to work on the problem after their explanations.

It seems that Minerva wanted to communicate that she didn't need Bonita and Jessica's explanations while at the same time making sure that she had an opportunity to hear and watch those explanations. It is this combination of sending a message of denial about learning while also watching to see what others do and say that suggests Minerva was attempting to conceal her learning.

Another example of covert learning occurred after Minerva completed her second solution. The teacher was sitting with Minerva's group asking questions about the task. She asked them first about the area of Figure H. Everyone in the group agreed that the area was two. She then asked about Figure I. Before presenting this excerpt, I want to comment that I will have more to say later about Minerva's affect during this excerpt. Her depressed tone is related to her moves to conceal her learning, but it is more relevant to an earlier interaction than to the interaction below.

Excerpt 5

354	Teacher	Okay. What is the area of Figure I?	
355	Bonita	Two	
356	Teacher	How do you know?	*She looks at Bonita.*
357	Jessica	One	
358	Teacher	Tell me why it's one	*She touches Jessica's arm.*
359	Minerva	Two triangles make a square.	*Minerva uses very little inflection as she says this. She is supporting her head with her hand. She is looking at the desk.*
360	Bonita	Two triangles make a square.	
361	Teacher	So then what does that mean?	
362	Minerva	There's another square there.	*Minerva puts her head down on her*

			arm.
363	Bonita	They're the same.	

The contrast between Minerva's statement in Line 362 and Bonita's statement in Line 363 is important. In her statement, Minerva communicated that the narrative about triangles and squares was important because putting the triangles together made another square. Minerva had demonstrated her understanding of this in her second solution when she glued the two triangles together into a square. However, Minerva had not yet articulated or demonstrated that Figures H and I were the same. Neither of her solutions at this point compared the two figures. In contrast, Bonita's statement reflected a comparison of the two figures. Because Minerva had not yet articulated a comparison of the figures, she might learn from Bonita's narrative.

364	Teacher	So how can I show it on my paper that they're the same?

With this statement, the teacher took up Bonita's discourse and indicated that talking about the figures as the same was appropriate.

365	Bonita	Write it down?	
366	Teacher	What materials do you have that would show you, help you to prove it?	
367	Bonita	A pencil	
368	Teacher	How do you think, Jessica?	
369	Jessica	Um	
370	Teacher	How could I show those two are the same?	*The teacher pulls out Jessica's paper. Jessica looks at it.*
		They said they're the same. Maybe they're not the same. You said they're not	*The teacher talks to Jessica.*

Again the teacher reinforced the narrative of the figures as the same. She then suggested to Jessica that they might not be.

371	Bonita	[They are	*Minerva has picked up her scissors and is cutting.*
372	Minerva	[They are. God	

This statement was the first time Minerva stated that the figures were the same, indicating a change in Minerva's discourse during this interaction. This change was subtle, so it could be argued that Minerva's utterance in Line 372 did not indicate learning, but was instead merely the first time she spoke these words. However, the notion that the two figures were the same was a significant change from her earlier solution in which she counted Figure H as one and the triangles from Figure I as one and then indicated that together they were two. As discussed earlier, this solution did not include the square from Figure I and did not compare the two figures. Describing the figures as the same was a shift in discourse from this solution that counted the two figures together.

In addition, these words were more than a mimicking of Bonita. Minerva illustrated that the two figures were the same in her next solution. In this third solution, Minerva marked "1" in each of the squares of each figure and indicated that both figures were two (See Figure 6.1). Rather than count the two figures together, Minerva showed how they both had the same number, demonstrating that they were the same. This change from counting together to counting the same provides further evidence that Minerva changed her discourse or learned during this interaction.

In spite of this learning, Minerva did not position herself as a learner during this interaction, nor did anyone else position her as a learner. Instead, she portrayed herself as knowledgeable and confident, communicating the expert response (Two triangles make a square) to the question of why the area of Figure I was two (Line 359) and taking up the approved discourse that the figures were the same (Line 372). Just as in Excerpts 1 and 4 (with the "Nah huhn"), in this excerpt, Minerva simultaneously communicated that she did not need to learn while also learning, again demonstrating covert learning.

Identifying the Audience

Minerva's audience was a key feature of enacting covert learning. She was very careful about her performance, working to convince her audience that she was knowledgeable. Presenting as knowledgeable meant that she could successfully complete the assigned work, a decision that was in the hands of the teacher. Thus teacher was the primary member of Minerva's covert learning audience. Minerva's peers were important for the ways in which they were explicitly not members of her mathematizing audience. Minerva was also rarely an audience for herself. I begin my exploration of the audience for Minerva's covert learning by describing her framing of the teacher. I then describe her peers and herself as audience members.

Teacher as Gatekeeper to Being "Done"

Minerva positioned the teacher as the person to persuade that she had finished her work. This was evident early in the small group work on the task. Minerva finished her first solution and announced this to the teacher:

Excerpt 6

257	Minerva	Done	*Minerva smiles at the teacher and writes her name on the top of her paper.*

Minerva's announcement and her smile indicate her confidence that she had completed the task. Minerva did not attempt to explain her work or her thinking. Instead, she wanted the teacher to know that she was finished. The teacher examined her paper.

258	Teacher	How is this? I can't tell anything about the area by this, Minerva	*She points at Minerva's paper.*

259	Minerva	I said the area's two.	*She writes "2" under the copy of Figure I on her page.*

In spite of the Minerva's pronouncement that she was done, the teacher asserted that Minerva had not yet answered the question (Line 258). Minerva might have replied to the teacher by explaining her thinking about area to the teacher. Instead, she emphasized what she had done and then added to her work (Line 259). Minerva could have invited the teacher to inquire about how Minerva knew the area was two. Instead, Minerva continued to position the teacher as the person to persuade that she was done, rather than as someone interested in her thinking.

The teacher disagreed:

260	Teacher	But how can you show me by this, how can you actually show me?	
261	Minerva	Cut it out.	*The teacher leaves the group. Minerva picks up a paper with H and I and starts cutting.*

The teacher's words in Line 260 support Minerva's enactment of covert learning: The teacher persisted with her message that Minerva's solution was inadequate. However, the teacher did not position Minerva as someone who needed to learn. Instead she implied that Minerva had more to show her. Minerva interpreted the teacher's words as a request for physical activity rather than for explanation of her thinking. She stated that she would cut (Line 261) to show how the area was two. Minerva's response also indicated that she had stopped trying to convince the teacher that she was done (which was a change from Line 259). Instead, she accepted the teacher's message that she had more work to do (not more or different explaining to do) and laid out what work she should do.

While this conversation was short, it demonstrates Minerva's emphasis on getting done and her framing of the teacher as someone to be persuaded that she was done. This emphasis is consistent with covert learning. Minerva presented herself as knowledgeable. She did not indicate that she didn't understand or ask for the teacher's help in making sense of area. Instead, her final response suggested that she knew both the answer and what more she needed to do to satisfy the teacher. The teacher's words can also be interpreted as agreeing with Minerva's positioning: Minerva did not need to learn from a peer or offer a better explanation. She needed to do more to show her solution.

The next time Minerva completed her solution to the task, the teacher's evaluation was more critical. Minerva had finished her second solution (See Figure 6.1) when she received the following feedback from the teacher.

Excerpt 7

335	Teacher	So what are you trying to do with these two rugs?	*There is no answer.*
		What are you trying to do with these two rugs?	*There is no answer.*
		Okay I'm trying to compare the area and see if they are the same or if one is bigger than the other. And you need to use these shapes, but you need to prove it on your paper.	*Minerva picks up her pencil and writes "I" on the two triangles on the bottom left of her paper.*
336	Minerva	I did!	

With this response, Minerva expressed both confidence that her answer was complete and disagreement with the teacher's evaluation. As in the example above, both Minerva and the teacher identified the teacher as someone who evaluated whether Minerva had done what she needed to do.

The teacher continued to evaluate Minerva's work:

337	Teacher	This does not prove anything to me. This is H. This is H. This is I. I don't even know what this is. There's no proof of me on here which one is,	*She points at various parts of Minerva's paper.*
		Do you think this is the same as this because I don't. There's no proof of to me of how they're the same or different. How am I going to prove they are the same or different? Minerva. How am I going to prove it?	*She points at the two triangles assembled as a square and then at Figure H. Minerva writes "I" on one of the two triangles on the bottom left of her paper. Minerva drops her pencil and puts her hands on her face.*
338	Minerva	I don't know.	

By stating "Do you think this is the same as this because I don't" (Line 337), the teacher indicated that Minerva's discourse was not the teacher's desired discourse and positioned Minerva as someone who needed to change what she had done. Again, the teacher did not explicitly label Minerva as a learner: She did not tell her that she wasn't making sense or that she needed to ask her peers for help in understanding the task. Instead her questions "How am I going to prove it?" implied that Minerva could generate an acceptable solution. Thus while the teacher did declare that Minerva had not adequately completed the task, she did not explicitly position Minerva as needing to learn.

Segue to Peers are Not Teachers

Minerva's response, "I don't know" (Line 338), was part of a pattern of responding to the teacher's feedback on her work. Minerva received negative feedback from the teacher on four of her solutions to the problem. Her response to the teacher on three of those four occasions was to state that she didn't know what to do, just as she did in Line 338 above. (The exception was her response after her first solution as elaborated above with Excerpt 6.)

Minerva's tone as she spoke her "I don't know" was depressed and deflated. She was slumped over her desk with her hands on her face. This response might be seen as reasonable in light of what could be described as harsh criticism from the teacher. The teacher did not cushion her feedback to Minerva. Instead she was direct in her disagreement, stating that Minerva offered "no proof" of how H and I were the same or different (Line 337). It is possible that the depressed tone of Minerva's response was in reaction to the teacher's harsh tone.

However, Minerva's depressed response may also be tied to her covert learning moves. First, as part of Minerva's portrayal of herself as someone who did not need to learn, it was important to be done with the task. The teacher's feedback indicated that not only was Minerva not done with the task, but also that her solution was wrong. The teacher was unequivocal in pointing out that she did not agree with Minerva that the two triangles and Figure H were the same (Line 337). In contrast to the teacher's feedback on her first solution, Minerva needed to do more than elaborate her solution: She needed a different proof. Not only was Minerva not done with the task, but the moves she had made to covertly learn what to do to solve the task had been inadequate.

Another reason for Minerva's depressed response may have been that she could not see a way out of her situation that would allow her to continue to overtly deny her need to learn. The teacher implied that Minerva could generate a solution (Lines 335 and 337), but Minerva's "I don't know" indicated that Minerva did not have another solution in mind. She was unwilling to admit that she needed help or that she didn't understand which left her with very few options for generating a new strategy for solving the task.

This difficulty demonstrates Minerva's reluctance to position her peers as part of her audience. If she had been willing to acknowledge that she was a learner and that her peers might have something to teach or might be able to learn with her, she could have turned to them to talk about the mathematics of the task. However, Minerva's enactment of covert learning meant that such an overt display of not-knowing was not possible. Instead, Minerva was eventually able to construct another solution for the task by attending carefully to interactions between the teacher and Bonita (See the discussion above accompanying Excerpt 5). This sequence of moves surrounding this second solution was repeated for Minerva's next solution. She generated the solution, received feedback from the teacher, insisted she was correct, responded with despair, and then covertly learned from interactions between other students and the teacher.

Minerva's reluctance to position others as her teacher was also evident from her lack of questions during the lesson. Minerva asked only two questions related to the task throughout the whole lesson: Very early in the small group work, Bonita stated that they had "to know how what rugs can cover up the floor" (Line 90). Minerva responded with "Rugs?" indicating that she wanted clarification that they were talking about rugs. Then, a short time later, Minerva asked the teacher, "What do we do? We need help." Throughout the lesson, Minerva did not ask for any other clarification of the task or of mathematical content. She did not ask for explanations or definitions. Any of these types of questions would have identified her as a learner and the person to whom the question was directed as her teacher. Indeed, in this second question, the one time Minerva asked for help, she used "we" rather than "I", providing additional evidence that she did not want to position herself alone as a learner in need of help. Minerva's lack of questions was consistent with her enactment of covert learning and her identification of others as not her teacher.

Minerva's reluctance to engage in peer teaching also included her reluctance to serve as the teacher of others. For example, when Minerva started on her second solution, she began to cut out another copy of Figures H and I. Jessica noticed this and asked, "Gotta cut another one out? Why are you cutting another one out?" (Line 263). Rather than respond to Jessica's question with an explanation, Minerva dismissed her question, saying, "Because I'm doing something" (Line 264). This response ended their conversation. (This conversation is analyzed further in the next chapter.) It is possible that Minerva did not yet have the words to explain what she was doing or why to Jessica or it may be that Minerva was embarrassed or disappointed that she had not yet finished the problem and did not want to be slowed down by explaining her work to Jessica. It is also possible that Minerva resisted teaching because she worried that teaching would expose her lack of understanding and show others that she actually needed to learn.

Regardless of her motivation, Minerva was unwilling to explicitly serve as a peer teacher. Minerva's reluctance to teach was displayed again later in the lesson. After Minerva had constructed her final solution to the task with Figures H and I, the teacher demanded that Minerva (and Bonita) help Jessica. The teacher had to discipline Minerva to get her to begin to help Jessica. Minerva's teaching was minimal. Instead of encouraging Jessica to explain her thinking, asking questions that would help Jessica understand the task, or inviting Jessica to ask her questions about the task, Minerva directed Jessica's actions with short ambiguous commands. Jessica was able to complete the task to Minerva's satisfaction, but she needed the support of Bonita and the teacher to do this work. I describe this interaction in more detail in the next chapter, but for now, I want to make the point that as Minerva engaged in covert learning, she was reluctant to identify herself as a teacher and others as learners.

Not an Audience for Herself

One final consideration for Minerva's audience is whether she identifies herself as a member of her audience. I noted in the previous chapter that Jakeel's identification of himself as an audience for his discourse seemed to support his learning. Thus it is important to consider whether Minerva seemed to talk to herself as she mathematized.

On one occasion, Minerva's gestures indicated that she seemed to have herself as an audience. This occasion occurred as Minerva was working on her third solution, the solution in which she marks a "1" in each square of Figures H and I (See Figure 6.1). Minerva first glued down Figure H and then wrote "1" in the top square and "1=2" in the bottom square. Minerva then glued the two triangles from Figure I and finally the square from Figure I. Here is my description of Minerva's actions after she glued the pieces from Figure I:

> She picks up her pencil and writes a "1" on one triangle from Figure I and "1=2" on the other triangle. She moves her pencil to the square from Figure I and hesitates. She then erases what she wrote in the triangles and draws a "1" across both triangles and writes "1=2" in the square from Figure I. She puts her pencil down. (Line 379).

Minerva's initial writing on the triangle ("1" and "1=2") suggests that she was thinking about counting each triangle as one. However, when she went to write on the square, she changed her mind and marked both triangles together as 1. Writing "1=2" on the square suggests that she was counting the two triangles in the same way that she was counting the square, with the result that there were two. The hesitation and then change in

Minerva's writing suggests that she was an audience for her discourse. While she did not utter or write words, the change she made in mathematical symbols indicates that she was asking herself whether her communication made sense and then changing what she wrote when she found it lacking.

The solution Minerva constructed as she communicated with herself was the most mathematically correct solution she generated. She indicated that both figures could be counted in the same way to arrive at the same number, 2. While I cannot conclusively connect Minerva's thinking to herself with her increase in mathematical discourse, it is interesting that the two coincide.

Minerva does not provide other clear examples of communicating with herself. While this does not exclude the possibility that Minerva talked through her ideas in less visible ways, it does suggest that exploring her thoughts was not a frequent enough or central enough activity for her to engage in it in explicit ways. This is consistent with her enactment of covert learning: Minerva did not explicitly identify as a learner. She focused on getting done, convincing the teacher that she was done, and, when the teacher was unconvinced that she was done, using her peers as sources of ideas that she could use to generate new ways for getting done.

Mathematizing

In order to get done with the task, Minerva needed to demonstrate mathematizing that satisfied the teacher. To do this, she adopted discourse, produced discourse, and relied upon others to substantiate discourse.

Adoption of Discourse

Engagement as a covert learner required that Minerva adopt the discourse of others. She was not open about this adoption: Unlike Jakeel, she did not ask questions or appear to listen to explanations. However, the changes in her discourse provide evidence that she attended to and then used the discourse of others. When I first introduced Minerva's covert learning, I discussed in detail two examples of Minerva's adoption: her use of *two triangles make a square* (See the discussion of Excerpt 4) and *they're [both figures] the same* (See the discussion of Excerpt 5). In both of those examples, Bonita used discourse that Minerva later exhibited.

Another example of adoption of discourse is Minerva's use of "the area is 2". Bonita was the first person to announce the area of Figures H and I. In Excerpt 1, Line 200, Bonita stated, "Both the areas, they both have two." Minerva seemed to adopt this statement, using a variation of it in each of her solutions to the task involving Figures H and I. For example, in her first solution, she wrote, "[t]he area is 2". In her subsequent solutions, she emphasized counting to two. This statement seems to be something Minerva adopted rather than something she produced because the statement seemed to be a starting point for Minerva's solutions rather than the result. In her first solution, she started her written statement with "[t]he area is 2" and then proceeded to elaborate how the area could be two, but without explicitly connecting these words to the figures: "The area is 2 beacuse [sic] theres [sic] 1 hole square and 2 triangles that makes a square". Her next solution involved counting Figure H as one and the two triangles from Figure I as two and then declaring that the perimeter was two (Line 288, not included in this document). While I believe that Minerva meant area when she said perimeter, what she counted was not the area of either figure. Instead, she seemed to know that she needed to reach an answer of two and she was attempting to count different things to demonstrate that answer. Because Minerva could not initially demonstrate "the area is two," I believe that she adopted this statement from Bonita.

Production of Discourse

In addition to adopting discourse, Minerva also produced discourse. However, in many instances her production of discourse did not reflect faithful adoption of another's discourse. Instead of building from discourse she adopted from others, Minerva's production of discourse as she enacted covert learning was, at times, not consistent with others' discourses or the desired mathematical discourse.

For example, as I described in the section on Minerva's final discourse, she demonstrated a routine for determining area that involved cutting apart the pieces of the figures, joining triangles in pairs to make squares, connecting all of the pieces of both figures, and counting all of the squares of both figures to determine area (See Excerpt 3 and Figure 6.2). This routine had not been demonstrated by Bonita, Jessica, or the teacher. Instead, Minerva seemed to generate it in response to the teacher's evaluation of her third solution.

Minerva's third solution had the two triangles of Figure I joined as a square. Next to this square, Minerva glued the whole square from Figure I. She also glued Figure H intact nearby (See Figure 6.1 and the discussion accompanying Excerpt 2). The teacher looked at Minerva's work and told her, "But this doesn't show me anything" (Line 431). The teacher then asked Jessica several questions before turning back to Minerva's work. The teacher spoke to Minerva:

Excerpt 8

477	Teacher	What did you do on your paper that showed that two triangles equaled one?	*Minerva points to the bottom of her paper, perhaps to the two triangles assembled into a square.*
		So how could I show, what the area is of the whole rug. This is only part of the rug.	*The teacher points to the bottom of Minerva's paper.*
478	Minerva	Put it [together	

After this interaction, Minerva cut out Figures H and I, arranged the pieces so that they were connected, and counted all of the squares.

When the teacher asked her why she was doing this, she told the teacher that she was "connecting it" (Excerpt 3, Line 495) "[b]ecause you [the teacher] told us to" (Excerpt 3, Line 497). However, the teacher had not told Minerva to connect the pieces. Instead, the teacher had asked Minerva how she could show the area of "the whole rug" (Line 477). Minerva had translated the request to show the area of the whole rug into a request that she put all of the pieces of both figures together.

The teacher's confusion over why Minerva was connecting pieces of the two figures suggests that Minerva's production of discourse was not based upon a shared understanding of the teacher's comments about the whole rug. Minerva's reluctance to ask questions, talk about her work with others, or seek explanations prevented her from asking the teacher for clarification about what she wanted or what she meant by *the whole rug*. She also did not explain her thinking about her solution and so the teacher could not build from Minerva's thinking to convey her concerns about Minerva's solution. Instead, Minerva seemed to assume that she understood both the teacher's concern with her work and the meaning of *the whole rug* and she produced a routine that addressed that concern.

Minerva's production of discourse was similar to Jakeel's production in that she did some translation into her own words. For example, Minerva used "connecting" (Excerpt 3, Line 495) which was not a word used by anyone else in her group, including the teacher. Minerva, like Jakeel, also counted without verbally indicating what object she was enumerating (See for example Excerpt 2 and the accompanying description of Minerva's writing). Significantly, however, Minerva's production of discourse differs from Jakeel's in that Jakeel's discourse was clearly tied to a visible object while in most instances, Minerva's discourse lacked explicit links to images or gestures. For example, in Excerpt 8, Minerva states, "Put it together" (Line 478). This statement was not an echo of words uttered by anyone else during this lesson: They seemed to be Minerva's own production. However, Minerva's understanding of "it" is not clearly conveyed by any gesture and, as becomes evident in her final solution (discussed above), Minerva's understanding of *it* was quite different from the teacher's.

As another example, while directing Jessica on what to put on her paper, Minerva also used pronouns and did not clarify her words with gestures. The teacher had just asked Jessica how she knew the area of Figure I was two.

Excerpt 9.

555	Jessica	This is two because if you put two triangles together that's a square.	*As Jessica says triangles, she claps her hands together over her head.*
556	Teacher	But where on your paper does it show me that?	
557	Minerva	Gosh, Jessica, put it together.	

558	Teacher	Where on your paper does it show me that?	*With both hands, Jessica pats Figures H and I that she has glued on her paper.*
		But that doesn't show me that.	
559	Minerva	Put it on there	

Minerva's statements in Lines 557 and 559 were not connected any visible object. Minerva did not she motion with her hands to indicate what object Jessica might put together or put on there and Minerva's words were too vague to point to an object by themselves. "Put it together" in Line 557 might have been a reference to the two triangles mentioned by Jessica in Line 555. If this is what Minerva intended, then "it" should have been "them" in order to refer to the two triangles. Rather than support Jessica in understanding her intentions, Minerva's directives in Lines 557 and 559 required Jessica to make connections without providing enough information to be sure that Jessica would think about the same object as Minerva. This example was one of fifteen utterances by Minerva where her words either directed someone to do something or described what she had but without any accompanying gesture that would help more precisely communicate Minerva's message. In contrast, there were only three occasions in which Minerva used gestures to specifically indicate the object of her talk.

While Minerva produced her own discourse, this discourse was usually not explicitly connected to an image, leaving the object of her discourse unclear (for both the researcher and her listeners). This ambiguity could support Minerva's enactment of covert learning as it allowed her to produce discourse that could be interpreted in a variety of ways. Her listeners, including the teacher, could make assumptions that Minerva was talking about the same object they referred to or were thinking about. Thus

Minerva could appear to be knowledgeable without needing to be explicit about what she knew.

Substantiation of Narratives

The final aspect of Minerva's mathematizing I will probe is her substantiation of narratives. Substantiation is how the learner decides whether a narrative reflects the state of affairs (Sfard, 2008). For Minerva, whose focus was on appearing knowledgeable and finishing the task, it was important to use narratives that accomplished these two goals. Thus, for Minerva, narratives reflected the state of affairs if they seemed useful in completing the task. Thus, her substantiation depended upon her interactions with the teacher.

Her reliance upon the teacher for substantiation is evident in the interactions around Minerva's work. Minerva relied upon the teacher to determine whether her solution was adequate (See Excerpts 6 and 7). She would craft a solution, the teacher would critique it, and Minerva would start over with another solution. If Minerva's substantiation relied upon her own explanation or investigation of the narratives in her solution, she might have explained her solutions to the teacher, arguing for her solution and how it met the teacher's expectations, rather than only insisting that she was done. Furthermore, Minerva started over with each solution. Rather than build on what she had written or the figures she had glued, she cut out and glued additional copies of Figures H and I and wrote new text for each of her four solutions. It is possible that Minerva felt each solution was constructed from a logically flawed foundation and needed to be restarted in order to correct this flaw, but there was no moment in which she appeared to evaluate her solutions and she did not state what might be wrong. Finally, Minerva's first and third solutions had mathematically appropriate foundations. If she had been logically evaluating her solutions, looking to substantiate them based upon mathematical explanations, she might have realized that she had appropriate beginnings that needed to be elaborated rather than abandoned. Instead, she seemed to be relying upon the teacher to indicate whether her work, including any associated narratives, reflected the appropriate mathematics.

Relying upon the teacher to substantiate narratives and approve her work is consistent with Minerva's enactment of covert learning. The need to appear as knowing precludes activities that would demonstrate uncertainty or learning. Thus as Minerva enacts covert learning, she cannot engage in activities such as exploring or verifying a narrative. Instead, her decisions about whether narratives reflect the state of affairs relies upon her observation of how others, and in particular the teacher, respond to narratives.

Autonomy

At first blush, it seemed that Minerva enacted autonomous learning. She was focused on the mathematical task. She worked persistently and independently toward a solution and she was creative in generating multiple solutions. However, examination of her audience and her mathematizing suggest that very few of her activities could be described as autonomous learning.

The moment in which her mathematizing and identifying mostly closely resembled autonomous learning was as she worked on her third solution. I describe this in detail in the earlier section on Minerva as an audience for herself. As Minerva worked on this solution, there was a moment in which she hesitated and then changed her writing. In this moment, it seemed that she observed a problem with what she had written and then worked out how to change her discourse to address this problem. Her activities reflected autonomous learning in several ways. She identified herself as the audience for her discourse. It seemed that she attempted to substantiate her first narrative. When she found it lacking, she constructed a new solution that was logically consistent and drew upon narratives of triangles, squares, and area used by others. Unfortunately, the teacher did not recognize the ways in which the solution was mathematically appropriate and Minerva did not argue for or explain her work. Instead of continuing to act in ways that reflected autonomous learning, Minerva enacted covert learning and relied upon the teacher to approve her work.

As Minerva enacted covert learning, she did not demonstrate activities of autonomous learning. She did not position her audience as interested in explanations, investigations, or co-learning. She did not appear to be an audience for herself. As she enacted covert learning, she adopted some discourse and produced discourse, but she did not investigate either what she adopted or what she produced. As a result, she did not always use narratives and routines in ways that were consistent with other's use or that were mathematically appropriate. Her substantiation of discourse relied upon the teacher's approval rather than explanations. In summary, Minerva's enactment of covert learning, while it exhibited independence, was not an independence that enabled or reflected autonomous learning. Thus while Minerva could enact autonomous learning (as demonstrated by her work on the third solution), for much of this lesson, her learning activities were not autonomous.

Discussion

Intertwining of Mathematizing and Identifying

This case of covert learning adds evidence supporting my claim that mathematizing and identifying are intertwined. In the previous chapter, when I discussed mathematizing and identifying in reference engaged learning and directed learning, I emphasized the ways in which the mathematizing and identifying of one student, Jakeel, was related to other students' mathematizing with and identifying of him. In contrast, this case of covert learning demonstrates how one student (Minerva) identifies herself through her reactions to the mathematizing of other students.

Minerva identified herself as knowledgeable by resisting mathematizing that identified her as a learner or teacher and by covertly attending to mathematizing that might be useful later. Excerpt 4 illustrates both of these points. Bonita and Jessica attempted to explain to Minerva how two triangles make a square. Minerva, through her "Nah huhn," indicated that she was not willing to be identified as a learner and yet her subsequent work showed her use of the narrative *two triangles make a square.* The mathematizing of Bonita and Jessica positioned Minerva and provided discourse for her to adopt. She adopted this discourse, which allowed her to appear knowledgeable, while at the same time rejecting the ways in which the mathematizing positioned her as unknowledgeable.

This example also shows how Minerva's mathematizing communicated a certain identity. She used Bonita's narratives (See for example Excerpt 5, Lines 359 and 372) to convey that she understood the mathematics. Also, her ability to work independently on the problems demonstrated that she was mathematically competent. She did not need to ask questions or seek explanations. Instead, she was able to complete solutions to the task without seeming to need input from others. However, Minerva's struggles to construct a solution that satisfied the teacher demonstrated how this same identifying constrained her mathematizing. Minerva's appearance of competence, knowledge, and independence was not a mere façade. These identities were significant and endorsable for her. Evidence of this lies in her firm reaction each time the teacher evaluated her work. Minerva insisted that she was done. The teacher had to insist just as forcefully that she was not done before Minerva was willing to back down. (See for example Excerpts 6 and 7). Minerva's confidence in her mathematizing meant that she did not investigate her discourse or that of others. She did not ask questions or seek feedback on her work. As a consequence, Minerva's mathematizing was limited. Minerva's identity and mathematizing were intertwined so that each kind of activity had implications for the other.

Development of Mathematical Discourse

As I suggested above, Minerva's mathematizing and identifying activities were related to the development of her mathematical discourse. Final discourse shows that her discourse changed to include the narrative *two triangles make a square* and a routine for using it to assemble smaller triangles from the figures (I and J) into squares. Her discourse also changed to incorporate a less mathematically desirable routine, counting the squares of both figures together as the area. Each of these changes in her mathematical discourse can be traced back to her enactment of covert learning.

The first example in the section introducing covert learning described when Minerva first encountered the notion of how joining triangles might be important to solving this task. Bonita stated and then explained this narrative. (See Excerpts 1 and 4).

Minerva at first disagreed with Bonita (Line 203) and then resisted her explanation with a series of "Nah huhn"s. However, in her first solution, Minerva incorporated this narrative about triangles. Minerva's identifying as someone knowledgeable meant that she attended to and used this narrative when it seemed helpful, but it also meant that she did not explore more of Bonita's explanation. Thus Minerva's enactment of the identifying of covert learning meant that her discourse changed, but that the change was limited.

The less desirable routine that Minerva developed can also be tied to her activity of covert learning. She did not learn this routine by listening to another: She developed it in response to the teacher's request to show the area of the whole rug. This request came from the teacher's evaluation of Minerva's third solution. This solution was the most mathematically appropriate. It displayed both rugs and indicated how the area of each was two. However, the pieces of Figure I were not arranged together in a rectangle. (See Figure 6.1). The teacher incorrectly assumed that Minerva did not have all of Figure I in her solution. If Minerva had not been enacting covert learning, she might have asked the teacher what she meant or she might have attempted to argue that her solution did show the whole rug. However, Minerva's enactment of covert learning meant that she did not explain and she did not explicitly connect words with visual objects. Instead of probing the teacher's reaction to her work, Minerva assumed that she needed to construct a new solution in which the whole rug was Figures H and I together. It seems that Minerva's reluctance to investigate discourses as part of her enactment of covert learning meant that she abandoned her more mathematically correct solution and, without investigation or verification, developed a less mathematically desirable solution.

As Minerva engaged in covert learning, she developed both desirable and undesirable mathematical discourses. Her evaluation of discourses as a covert learner seemed to depend upon their role in getting the task completed to the teacher's satisfaction rather than upon their logical or mathematical consistency. Thus the development of her mathematical discourse was not guided by mathematical appropriateness but instead by her desire to identify as already knowing which was accompanied by limited mathematizing. This case of Minerva's covert learning demonstrates how identity and mathematizing intertwine with consequences for the development of the student's mathematical discourse.

Autonomous Learning

As I indicated earlier, Minerva rarely engaged in learning activities that could be described as autonomous learning. In addition, her final discourse was a shift away from a more mathematically desirable discourse and included a routine that was not mathematically desirable. Based upon these findings, it would be easy to portray Minerva as an unsuccessful student. However, her confidence, independence, and creativity in generating solutions are attributes of a successful student and might also be attributes one would ascribe to an autonomous learner. Exploration of this tension between the autonomous and non-autonomous aspects of Minerva's learning activities helps to clarify the activities of autonomous learning. I will focus on two particular tensions: independence versus decentering and knowing versus not knowing.

Independence Verses Decentering

Some researchers define autonomy in terms of independent action. For example, Warfield, Wood, and Lehman (2005) define autonomy to mean that "students are capable of thinking about mathematical ideas without having the ideas 'explained' to them and of solving mathematical problems without being shown a method by another person" (p. 440). This definition seems to be a very good match for Minerva's enactment of covert learning: She did not want to have ideas explained to her and she solved the mathematical task without relying on others. However, it would be troubling to label Minerva as autonomous given her solutions to the task and especially the ways in which her final mathematical discourse was not mathematically desirable.

Minverva's covert learning argues for a construction of autonomy and of autonomous learning that emphasizes a different aspect of independence. We want learners to be independent thinkers, but at the same time they need to learn canonical mathematical discourse. A construction of autonomy needs to negotiate this tension between individual learning and collective ideas. Piaget's theoretical construction of autonomy offers a means of negotiating this tension. For Piaget, a key feature of autonomy is decentering or learning to think about the perspectives of others (Kamii, 1994; Piaget, 1932/1960). Students become aware of and able to adopt other perspectives as they examine others' ideas and explore others' perspectives. Their thinking becomes less egocentric and subjective and more objective. They become capable of thinking in other ways and imagining other possibilities. Autonomy does not mean the learner is free to think anything. It means that the learner is capable of understanding and evaluating ideas including the ideas of others.

For Piaget, the only way to gain understanding of other's perspectives is to learn with and from others. Decentering is not possible in a coercive environment: When learners are obligated to act, they learn to evaluate ideas through the lens of reward and punishment rather than through rational examination of the logic of ideas. Instead, learners need cooperative learning environments in which "the obligation is to consider all viewpoints, to be coherent and rational, and to justify one's conclusions" (Kamii, 1994, p. 53).

When Minerva's learning activities are viewed through Piaget and Kamii's lens, it becomes clear that her activities were not autonomous. Minerva's focus on the teacher's approval meant that she constructed learning as coercive and not cooperative. She was focused on what she needed to do and evaluated ideas in terms of reaching the reward of being done rather than for their logical consistency. Although she adopted the ideas of others, she did not subject these ideas to rational examination. Thus while she was competent, independent, and creative as she enacted covert learning, these attributes came from her confidence in her ideas and not from her work to coordinate her ideas with the thinking of others. Minerva's covert learning helps to clarify that autonomous learning is only about independence in so far as it means that the learner has chosen for themselves to explore and logically evaluate the perspective of others.

This definition of autonomous learning also clarifies the meaning of "discourse-for-oneself". Sfard (2008) defines discourse-for-oneself as "a discourse in which one engages of one's own accord while trying to solve her own problems" (p. 297). Minerva might be said to have a discourse-for-herself: She produced her own narratives and routines and used them to solve her own problems. This discourse was meaningful to her. However, discourse-for-oneself does not mean one's own discourse. It means that the individual's discourse is decentered. It arises from critical examination of others' discourses (Ben-Zvi & Sfard, 2007). It is the result of adoption, production, and substantiation arising from investigation of the logical consistencies of other discourses. It means one can use the discourses of others as tool for oneself. As Minerva engaged covert learning, she did not logically examine other discourses. Instead, her discourse arose from her apparently unexamined interpretation of the situation. For example, she did not ask the teacher what she meant by "the whole rug." Minerva constructed her own understanding. Minerva's final discourse was also not a "discourse-for-others" (Sfard, 2008): The discourse was not sensible to others, including the teacher.

Minerva's enactment of covert learning clarifies that autonomy and autonomous learning does not mean that the individual learns in isolation or without assistance. Instead, autonomous learning is predicated upon the learner's interest in making sense of the discourse of others as well as of the situation or problem she investigates. As Minerva's covert learning demonstrates, an individual might develop a mathematical discourse without engaging in autonomous learning, but the development of a mathematically appropriate discourse-for-oneself requires exploration of other discourses.

Knowing and Not Knowing

Minerva's covert learning clarifies one more aspect of autonomous learning: the tension between knowing and not knowing. Minerva worked to appear knowledgeable: She appeared to avoid being seen as needing the help of her peers. Minerva's desire to be seen as knowing is consistent with the emphasis in schools. Students are seen as smart and are rewarded for having answers, not for needing or asking for help (Duckworth, 1996). This need for knowledge can become entangled in definitions of autonomy. For example, Warfield, Wood, and Lehman's (2005) definition of autonomy (See above) might be construed as already knowing or at least, not struggling to know: A student who can think about mathematical ideas without explanations from others and solve problems without being shown how by others must not engage in significant struggles to know. However, Duckworth (1996) notes that an important part of learning is what a person does when they don't know. Schools emphasize knowing and right answers, but students only know things because of what they have done when they didn't know. The definition of autonomous learning I have constructed also emphasizes not knowing. Curiosity is founded on recognition of what the learner does not know. The autonomous learner, because of his/her curiosity, embraces what is novel and strange, exploring problems of his/her own creation. This exploration leads to knowing, but not as end in itself. Instead knowing leads to more questions about what is not known.

Covert learning offers an important contrast to autonomous learning. In enacting covert learning, Minerva emphasized knowing and independence. While these might seem like positive, autonomous traits, this case shows how enacting knowing and independence and hiding learning resulted in a final discourse that was mathematically problematic. If Minerva had recognized and admitted to the teacher had her peers that she didn't know, had asked questions, and had explored her discourse and the discourse of others, she might have proceeded differently in her work on this area problem and learned a more mathematically appropriate discourse.

CHAPTER 7

WATCHFUL LEARNING AND GUIDED LEARNING

Overview

Minerva's group contained one other focal student: Jessica. This chapter explores the same lesson in the same group from the perspective of Jessica's learning. Like Jakeel, Jessica exhibited two different kinds of learning during the lesson. At the beginning of the lesson, she enacted *watchful learning*: During this part of the lesson she made decisions about what to do by observing Minerva and Bonita. Later in the lesson, the teacher facilitated a transition to *guided learning* in which Bonita, Minerva, and the teacher assisted Jessica in arranging the pieces of Figure I on her paper. Jessica's case shows the ways in which student interactions can constrain mathematizing and how students, through their mathematical activity, might identify in ways that are rejected by others.

Summary of the Lesson

This case examines the same lesson and same group as the previous case. I have summarized the activity and main events below. While much of this summary is the same as in the previous chapter, I have added details that are relevant to Jessica's work during the lesson.

After explaining her expectations for the lesson to the whole class, the teacher dismissed the students to work in small groups. Minerva, Bonita, and Jessica gathered at a cluster of four desks near the entrance to the classroom. After reading the task card and gathering the materials for the task, they began cutting out Figures H and I. The teacher came to the group and asked questions about the area of the two figures. During this conversation, Bonita explained that the area of both figures was two because two triangles made a square. Minerva disagreed with this. Bonita and Jessica explained to Minerva how two triangles could make a square.

After this explanation, Minerva attempted a variety of solutions to the task. Jessica watched her and mimicked her cutting and gluing activities. The teacher visited the group several times during the lesson and asked each student to explain her thinking. Minerva disengaged after the teacher criticized her work on her fourth solution and Jessica offered Minerva encouragement.

Bonita and Minerva finally received the teacher's permission to go to the next task (involving Figures J and K). However, the teacher then realized that Jessica did not yet have the answer to the task with Figures H and I. She instructed Bonita and Minerva to help Jessica, which they reluctantly did. After Jessica constructed the appropriate arrangement of the pieces of Figure I on her paper, they all started work on Figures J and K. During their work on Figures J and K, the teacher returned to the group and asked Jessica to talk about her solution for Figure I.

Table 7.1 Transcript excerpts and main events in lesson

Line number	Excerpt Number	Excerpt Summary	Main Events
0			The teacher explains the lesson .
			The students move to their small groups and start work.
	Excerpt 7	Jessica watches Minerva and Bonita. She talks to Bonita about cutting.	
100			

Table 7.1 con't

200	Excerpt 11	Bonita first uses *area is two* and *two triangles make a square.*	
			Minerva works on her first solution.
	Excerpt 8	Minerva cuts out the figures again. Jessica asks why	
			Minerva works on her second solution.
	Excerpt 9	Minerva gets the teacher's approval. Jessica copies her activity.	
300			
	Excerpts 1 & 2	Jessica first talks about the area of Figure I.	Minerva works on her third solution.
400	Excerpt 3	Jessica again talks about the area of Figure I.	
500	Excerpt 10	Jessica becomes a cheerleader for Minerva.	Minerva works on her fourth solution.
	Excerpts 12 & 13	The teacher approves Bonita's solution. Jessica celebrates.	
			The teacher approves Bonita and Minerva's solution to Figures H and I.
	Excerpts 4 &14	Jessica's final discourse on Figure I.	
	Excerpts 15-20	Bonita, Minerva, and the teacher guide Jessica.	
600			Minerva, Bonita, and Jessica work on Figures J and K.
	Excerpt 5	The teacher asks Jessica about Figure I again.	

Outcome of Learning

Initial Discourse

Jessica first talked about the mathematics of this problem 16 minutes after students began work on the task. The teacher came to the group and inquired about their work.

Excerpt 1

348	Teacher	How am I going to prove, Jessica, either one of these rugs is bigger than the other one or if they are the same? How am I going to prove it?	*She points to a paper with Figures H and I.*
349	Jessica	Find out the area	
350	Teacher	Okay, so what's the area of Figure H?	*Jessica points back and forth to the two squares in H.*
		What is it?	
351	Jessica	Two	

Jessica's motion (line 350) and her statement of "Two" indicated that she could determine the correct area of Figure H. It seemed by her motion that she was counting squares. However, she did not explicitly state that she was counting squares.

The teacher then asked her group about the area of Figure I.

Excerpt 2.

354	Teacher	Okay. What is the area of Figure I?	
355	Bonita	Two	
356	Teacher	How do you know?	*She looks at Bonita.*
357	Jessica	One	

At this moment, Jessica stated an incorrect area for Figure I. If she was counting squares, as she seemed to be doing when she determined the area of Figure H in Line 351, the response, "one" while still not the correct area, was a reasonable answer for the number of whole squares in the figure.

A short while later, the teacher again asked Jessica's group about the area of Figures H and I.

Excerpt 3

396	Teacher	How could I find out the area?
397	Bonita	We already found the area.

398	Jessica	If you count the squares.	
		This is two.	*She points at H with two fingers, putting one in each square.*
		That looks like three.	*She points at I, drawing a circle through the three pieces.*

In this excerpt, Jessica was more explicit about her routine for determining area. In line 398, she mentioned counting squares. She then demonstrated how counting squares would result in an area of two for Figure H and three for Figure I. Her determination of Figure H's area was consistent with her mathematizing in the previous excerpt (Lines 350 and 351). However, her statement of the area for Figure I changed from one in Line 357 to three in Line 398, indicating that Jessica was uncertain what to count as squares in Figure I. It appeared that in the first excerpt, she only counted whole squares (resulting in one),while in the second excerpt, she counted each of the spaces as a square (resulting in three).

Jessica's initial mathematizing had some of the features of the mathematically desirable discourse for this task: She explicitly identified squares as the unit of measurement and articulated a number for the area of each figure. However, in order to arrive at a more mathematical determination of area, Jessica needed a means of accounting for the two triangles in Figure I in a way that didn't either ignore them or count each one as a whole.

Final Discourse

By the end of the lesson, Jessica's mathematical talk included both a mathematically correct number for the area of Figure I as well as an explanation that began to account for the contribution of the two triangles to the area of the figure. However, careful examination of her words and gestures raises questions about Jessica's understanding of her own mathematical talk. In order to demonstrate Jessica's final discourse, I use excerpts of transcript from two events at the end of the lesson.

This first event occurred toward the end of the group's work on Figures H and I, 43 minutes into the small group work time. The teacher had just approved Bonita and Minerva's solutions. She turned to Jessica and asked her to explain her answer. Figure 7.1 is a drawing of Jessica's paper at this moment.

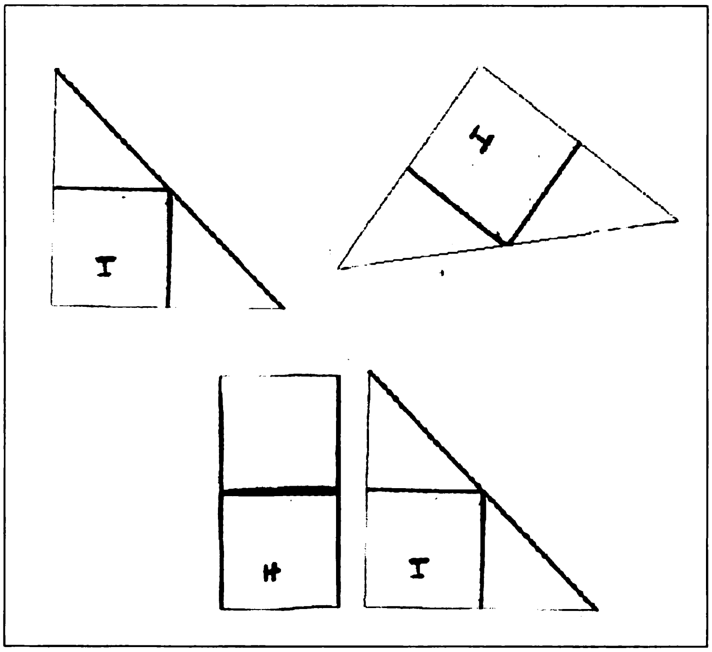

Figure 7.1 Drawing of Jessica's work at Excerpt 4.
This is student work.

Excerpt 4

550	Teacher	What's, what's not, what's the area of this Jessica?	*She points to Figure I glued on the bottom of Jessica's paper.*
551	Jessica	It's two.	
552	Teacher	How is it two?	
553	Jessica	Because (?)	*She points to all of the pieces of I at the top of her paper.*
554	Teacher	I don't know how that's two.	
555	Jessica	This is two because if you put two triangles together that's a square.	*As Jessica says triangles, she claps her hands together over her head.*

556	Teacher	But where on your paper does it show me that?	
557	Minerva	Gosh, Jessica, put it together.	
558	Teacher	Where on your paper does it show me that?	*With both hands, Jessica pats Figures H and I glued on the bottom of her paper.*
		But that doesn't show me that.	

Jessica demonstrated in Lines 551 and 555 that she could state the correct area of Figure I. In Line 555, she also stated a narrative ("if you put two triangles together that's a square") that began to explain why the area of Figure I was two. However, this explanation was incomplete because it did not show or describe how putting two triangles together resulted in an area of two for Figure I. In addition, as the teacher noted in Lines 556 and 558, Jessica could not point to any of her work to show how Figure I was two or how two triangles made a square. Instead of pointing to her paper or to Figure I, when Jessica spoke about triangles she clapped her hands over her head (Line 555). Thus while Jessica had words to talk about the area of Figure I, her use of these words did not seem to refer to any particular object.

Jessica had a series of other exchanges with the teacher and Minerva shortly after the interaction above. These exchanges provided further evidence that Jessica was not connecting her words about triangles and squares (Line 555) with the parts of Figure I. For example, the teacher and Minerva talked about how Jessica could "make it two" (Line 564, not included). They were referring to Figure I. However, Jessica picked up Figure H and began to cut it out. This work on Figure H suggests that not only was Jessica not connecting her words to the parts of Figure I, she was also unsure whether her words referred to Figure H or Figure I.

When Jessica did work on Figure I, she placed two copies of the figure together (See Figure 7.2). This move illustrated Jessica's understanding of "if you put two triangles together that's a square" (Line 555). Rather than focus on the parts of Figure I and how the two small triangles could be used to make a square, Jessica understood *triangles* to mean all of Figure I. Jessica's actions in this moment offer additional evidence that she had not yet made sense of how triangles and squares could be useful in explaining the area of Figure I. While Jessica's words in Line 555 offered a mathematically appropriate explanation for how the area of Figure I was two, Jessica's actions did not indicate that she connected these words with the pieces of Figure I or with the area of the Figure.

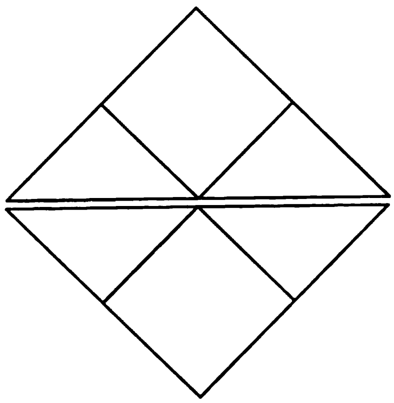

Figure 7.2 The large square constructed by Jessica from two copies of Figure I

With support and direction from the teacher, Bonita, and Minerva, Jessica was eventually able to place the pieces of Figure I so that the two triangles made a square (See Figure 7.3. I discuss this in detail in the section below on guided learning). However, she did not talk about this arrangement until 11minutes later when the teacher inquired about the work.

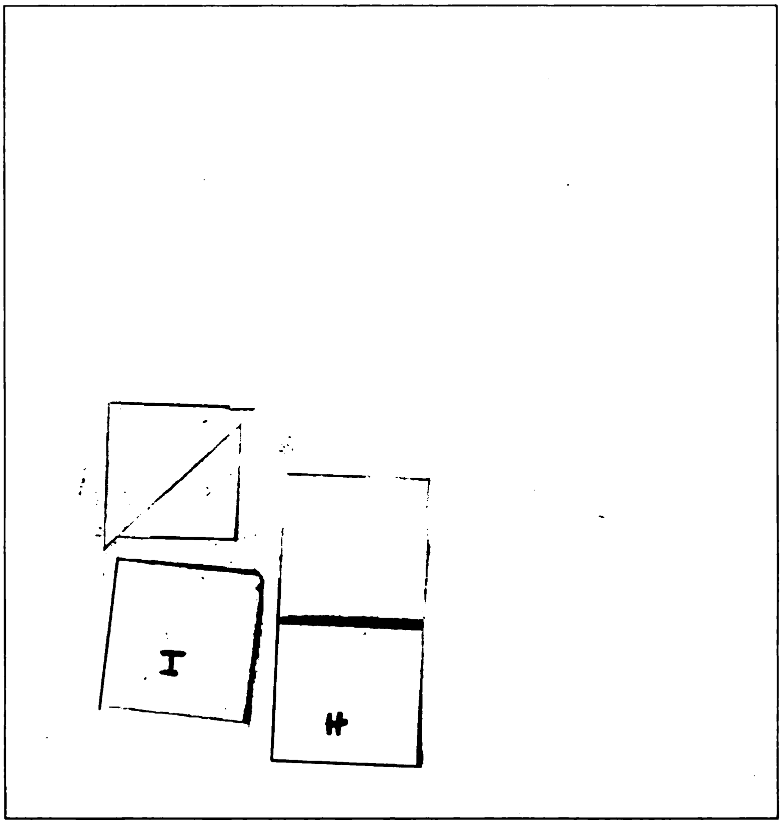

Figure 7.3 Jessica's final arrangement of Figures H and I.
This is student work. The additional copies of Figure I shown in Figure 7.1 have been removed by Jessica and Minerva.

Excerpt 5.

612	Teacher	Okay, Jessica. So how is it two? Is this how you made it?	*She points to Figure I on Jessica's paper. Jessica barely nods her head.*
		So tell me how it's two.	
613	Jessica	Because	*Jessica smiles.*
		if you put two triangles together it makes two.	*Jessica waves her scissors in the air and then points at the two triangles joined into a square on her paper.*

Unlike Jessica's earlier explanation for the area of Figure I (Line 555), it seems that her words here refer to the objects on her paper. She talks about two triangles and then points to the two triangles on her paper (Line 613). However, it is difficult to tell from this excerpt what Jessica meant when she stated, "it makes two". It is possible that the process of rearranging Figure I allowed Jessica to make sense of how the area of Figure I could be two squares. However, it is also possible that Jessica could be referring to the two triangles when she says "it makes two" or that, like in Line 555, she was uttering the words and making motions that would satisfy the teacher without really understanding how Figure I has an area of two.

One last interaction that supports the latter interpretation occurred during a subsequent lesson during the next week. (See Table 4.1 for a sequence of lessons. The case study lesson occurred on Thursday of Week 2. This subsequent lesson occurred on Tuesday of Week 3.) This is the same lesson I described during my description of Jakeel's final discourse. It involved determining the area of triangles depicted on geoboards (See Figure 5.3 and 5.4). Jessica was in a group with Jakeel and Minerva. They were working to determine the area of a figure identical to Figure I when the teacher came to the group.

Excerpt 6

426A[3]	Teacher	Okay, so you have how many squares in there Jessica?	
427A	Minerva	Two	
428A	Jessica	One	
429A	Minerva	Two	
430A	Teacher	[Okay, why do you say two?	*She points a pencil at Minerva*
431A	Jakeel	[Two	*His speech overlaps with the teacher's.*

Jessica's answer in Line 428A was mathematically accurate: There was only one square in the figure. However, as demonstrated by Minerva and Jakeel's answers, the teacher intended for students to count the two triangles as a square, for a total of two squares. Jessica's answer provides further evidence that, unlike Jakeel and Minerva, she had not yet changed her discourse to include a routine for addressing triangular spaces in figures.

[3] I have added the letter A to each of these line numbers as a reminder that this excerpt and these line numbers are not from the same lesson as the other excerpts in this chapter.

The interaction continued with Minerva and the teacher discussing how to teach Jessica. Jakeel volunteered to show his thinking.

451A	Jakeel	See these two make a square	*Jakeel turns the geoboard so he can see it. He puts his index and middle fingers on the two smaller triangles. He then turns it back to Jessica.*
452A	Minerva	Duh duh duh	*These are rude noises that seem to convey that Jessica should already understand. Minerva glances at the teacher after she makes the noises. Jakeel removes his fingers from the board and places his middle finger in the full square.*
453A	Jakeel	And so it's two squares	
454A	Teacher	Do you agree, Jessica? Do you understand really or not you're not sure?	*Jakeel continues to hold up his board*
455A	Jessica	A little bit	

Jessica's final response captures my assessment of her final discourse. While her discourse changed "a little bit" over the course of the lesson, the changes did not include links between the narrative about triangles and the area of the figure. They also did not include a mathematically appropriate routine for determining area. Instead, as demonstrated by this second lesson, it seemed that Jessica had learned the responses that met the teacher's expectations for the first lesson, but she was unable to use this discourse in subsequent lessons.

In summary, Jessica's final discourse had some features that were more mathematically desirable than her initial discourse. She was able state the appropriate area for Figure I (Excerpt 4, Line 551) and she was able state how the partial units in the Figure might be arranged so they can be counted as a whole unit (Excerpt 4, Line 555 and Excerpt 5, Line 613). However, her final discourse did not fully explain how she arrived at her area measurement nor did it explicitly state what she was measuring. In addition, Jessica seemed to be unable to demonstrate how her words were related to the picture on her paper and she was unable to use this discourse during the subsequent lesson. In contrast, in her initial discourse, Jessica's verbal discourse was tied to images on her paper. Also, she stated what object she was counting (squares, Excerpt 3,Line 398). Thus Jessica's final discourse had some changes that were more mathematically desirable and other changes that were less so.

This section has explored the change in Jessica's mathematical discourse through close attention to her mathematical talk. Before making claims about connections between Jessica's learning and the interactions of the group, I will first describe Jessica's mathematizing and identifying activities.

Learning Activity

Jessica's learning activity changed during this lesson. There was a constellation of identifying and mathematizing activities that I have labeled watchful learning. These activities transitioned to other learning activities in which Jessica's peers and the teacher were explicitly focused on Jessica's actions. By the conclusion of the lesson, Jessica again enacted watchful learning. As I did in the chapter on Jakeel, I first offer examples of watchful learning. I then describe the audience and mathematizing for watchful learning before describing the transition to guided learning. My analysis of guided learning includes an explanation of the audience and mathematizing for guided learning before describing the transition back to watchful learning.

Watchful Learning

Throughout this lesson, Jessica frequently identified as a learner. However, the activities that identified her as a learner were not explicit requests for help. Instead Jessica's learning moves were subtle and not overt. Rather than ask questions or engage her group members, Jessica watched the other students in order to determine what she should do.

The following excerpt offers an example of Jessica's watching. The excerpt comes from the beginning of the small group work. Minerva has handed out copies of the papers with Figure H and I. Jessica examined the paper and then set it in front of her and watched Minerva hand out other materials. Jessica continued to watch Minerva for 2.5 minutes. During this time, Bonita rejoined the group after a trip to the restroom.

Excerpt 7

171	Minerva	Here. (?)	*She passes out individual worksheets. Jessica writes on her sheet. Minerva starts cutting out Figure H. Bonita picks up a paper with Figures H and I.*
172	Bonita	I don't get this. We cut these things out?	*Bonita starts cutting.*
173	Minerva	Sure	*Jessica writes her name on her individual worksheet. Jessica looks at Minerva and Bonita and picks up her paper and scissors.*
174	Jessica	Why you cutting (?) huh, Bonita, huh, Bonita	*This is said like a chant and not as a question. Jessica looks at Minerva who already has one shape cut out. Jessica starts to cut. Bonita finishes cutting.*

Jessica did not start to cut out the shapes until after both Bonita and Minerva had started. In addition, before she started to work, Jessica looked at both Bonita and Minerva (Line 173). Of the three students in this group, Jessica was the student who could have started work the earliest. She had time to examine the figures while Minerva sharpened her pencil and passed out materials. She was not away in the bathroom like Bonita. She had the paper and the scissors and yet she waited until both Bonita and Minerva had started cutting before she started (Line 174). Jessica's hesitation to begin coupled with her observations of Minerva and Bonita seem to indicate that Jessica was watching Bonita and Minerva to see what to do.

This second example of watchful learning demonstrates not only how Jessica watched her peers to see what to do, but she also watched to see what activity, including what answer, was approved by the teacher. Jessica did this by carefully attending to interactions between the teacher, Bonita, and Minerva that indicated the teacher's approval of activity or answers. The following interaction occurred almost half way through the lesson. Jessica had glued Figures H and I to her paper. She was watching Minerva and the teacher. Minerva just completed her first solution and told the teacher that she was done. (Lines 258 through 261 were included in Chapter 6 as Excerpt 6).

Excerpt 8

258	Teacher	How is this? I can't tell anything about the area by this, Minerva	*She points at Minerva's paper.*
259	Minerva	I said the area's two.	*She writes "2" under the copy of Figure I on her page. Bonita cuts on scrap paper.*
260	Teacher	But how can you show me by this, how can you actually show me?	
261	Minerva	Cut it out.	*The teacher leaves the group. Jessica glues down her Figure H. Minerva picks up a paper with H and I and starts cutting. Jessica looks at Minerva. She sniffs the glue stick and twists the bottom. Bonita has small*

			pieces of paper. She puts glue on one and attaches it to her finger.
262	Bonita	Like my nail?	
263	Jessica	Gotta cut another one out? Why are you cutting another one out?	
264	Minerva	Because I'm doing something.	
265	Bonita	She's playing tic tac toe with it.	*Jessica smiles. There is a pause during which Jessica watches Bonita and Minerva.*

Jessica's questions (Line 263) indicated that she had noticed Minerva cutting out another copy of Figures H and I but wasn't sure why Minerva was doing this. Jessica did not yet start to imitate Minerva's actions. Instead, she engaged in a conversation with Bonita about her church. This conversation continued for several minutes until the teacher walked by the group. During this next excerpt, which immediately followed the conversation about Jessica's church, Jessica watched the teacher, Bonita, and Minerva.

Excerpt 9

283	Teacher	Okay. Okay, so I'm not sure what you're doing right now Bonita.	*Jessica looks at the teacher.*
284	Bonita	I'm trying to find out what to do.	*Jessica looks at Bonita.*
285	Teacher	So what you need to ask, what are you guys doing right now. Minerva what are you doing? Tell me what you're doing.	
286	Minerva	I cut the triangle apart so I could I could show how you could make it	*Jessica looks at Minerva. Minerva puts glue on the back of Figure H and glues the figure on the left middle of her paper.*
287	Teacher	Make it?	
288	Minerva	Show it's two. Two perimeters. One, two equals two.	*As Minerva counts one, two, she points first at the 1 by Figure H and then at the 1 = 2 by Figure I.*

		One and one equals two. So I just (did this?)	*Minerva puts glue on the small triangles from Figure I and glues them on the left bottom of her paper. Jessica continues to look at Minerva.*
289	Teacher	Okay. Do you understand what she just said?	*She looks at Bonita. Bonita shakes her head.*
		Okay she's not sure what you're doing so can you say it again?	*She talks to Minerva.*
290	Minerva	I'm gluing this down so (they?) can understand how it's two	*Jessica starts to cut out another copy of Figure H and I.*

During these two exchanges, Jessica paid careful attention to Minerva's activity. However, she hesitated to imitate Minerva until after the teacher had indicated her approval of Minerva's activity. This approval came in Line 289, when the teacher asked Minerva to explain what she was doing to Bonita. The approval was quickly followed by Jessica's decision to cut (Line 290). It is possible that other factors were involved in the timing of Jessica's cutting. For example, she might have been planning to cut out another copy of the figures but was distracted by the interaction with the teacher. However, the timing of her activity suggests that the decision to cut was based at least in part upon her observation of the interactions among the teacher, Minerva, and Bonita and the teacher's approval of Minerva's actions.

Watchful learning has some activity in common with covert learning: Both kinds of learning rely on observation of others for examples of new discourse. Both emphasize the teacher as the arbiter of right and wrong answers. A key difference between the two is that as Jessica enacted watchful learning, she did not actively work to hide or deny that she was learning. She was willing to ask questions that indicated that she didn't know (See Excerpt 8, Line 263). I argue (see *Identifying the Audience* below) that her watchful learning was, in part, an outcome of Bonita and Minerva's refusal to respond to her questions and explain their activity (See for example, Excerpt 8, Line 264). In contrast, Minerva enacted covert learning through her refusal to ask questions, listen to explanations, or indicate that she didn't know. While both watchful learning and covert learning depended upon the mathematizing of the learner's peers, watchful learning was Jessica's reaction to interactions with peers while covert learning was a more proactive position in which Minerva worked to identify herself in particular ways.

The next sections will analyze the specific identifying and mathematizing moves of Jessica's watchful learner, starting with her identification of her audience.

Identifying the Audience

Toward the beginning of this lesson, Jessica made three attempts to engage Bonita and Minerva in conversation about the task. In Excerpt 7, Line 174, she spoke with Bonita about cutting the figures. In Excerpt 8, Line 263, she asked Minerva why she was cutting more figures. Finally, in Chapter 6, Excerpt 4, she attempted to explain to Minerva how two triangles make a square. Each of these examples resembled the mathematizing activity of Jakeel as he enacted engaged learning. Jessica, like Jakeel, demonstrated that she was interested in the activity of her peers and that she had mathematical ideas to contribute to the conversation. However, in each of these instances, her overtures were ignored or rebuffed. Bonita ignored Jessica's comment about cutting (Excerpt 7, Line 174). Minerva dismissed her question about why she was cutting (Excerpt 8, Line 264) and Bonita, by replying that Minerva was playing tic-tac-toe made Jessica's question seem silly and trivial (Line 265). Finally, Minerva, with her series of "Nah huhn" communicated that Jessica's attempt to explain her thinking was not important (See Chapter 6, Excerpt 4 and the accompanying discussion). Jessica's attempts to identify her peers as interested and helpful members of her audience were unsuccessful.

Instead, as the lesson continued, Jessica was more passive. Rather than position Bonita and Minerva as members of her discursive audience, Jessica positioned herself as a watchful member of their audience. Bonita and Minerva became models for her activity rather than teachers or co-learners.

The exception to this positioning was when the teacher arrived at the group. With the teacher's arrival, Jessica's interactions changed: She became a cheerleader for Minerva, building from what the teacher said to encourage her in her work. For example, the teacher was talking to Minerva about her fourth solution. The excerpt below continues Excerpt 3 from the previous chapter.

Excerpt 10

501	Teacher	I said I want you to prove how this is. I didn't say connect it. If I said connect it, I was wrong.	*Minerva groans and pulls Figure H from her paper. The square from Figure I comes up with it.*
		Just show me H on your paper. Just show me H.	
502	Minerva	I can't	
503	Teacher	Yes you can.	*Jessica is rearranging parts of Figure I.*
		Just take this off. Show me Figure H.	*The teacher points at Minerva's paper. Jessica is rearranging parts of Figure I on her desk.*
504	Jessica	Yes you can.	

On Line 504, Jessica echoed the teacher's words from Line 503, encouraging Minerva in her work. On three other occasions, Jessica supported Minerva's efforts to solve the problem (Lines 468, 472, and 515, not included in this document). Each of these four moments came while Minerva was in despair after the teacher's critique of her work (See for example, "I can't" in Line 502). Jessica's cheerleading identified Minerva as capable, but needing moral support. This cheerleading may have been a social move, enabling Jessica to contribute to the work of the group, or it may demonstrate Jessica's sympathy for Minerva. In either case, the cheerleading was consistent with Jessica's watchful learning: Minerva was a model for Jessica's mathematical work. Jessica needed Minerva to demonstrate the answer and obtain the teacher's approval so that she would know what to do.

Jessica's cheerleading also demonstrated how the teacher was an important member of Jessica's audience. Jessica was interested in supporting Minerva in getting the teacher-approved answer on her page. Jessica's attention to teacher approval indicated that Jessica identified the teacher as someone who knew the right answer and who would determine whether Jessica had the right answer. Other evidence of Jessica's interest in teacher-approved activities was evident in Excerpt 9 above, when Jessica waited until the teacher had endorsed Minerva's activity before she began to mimic Minerva. Also, Jessica only began to use Bonita and Minerva's narratives about Figures H and I after the teacher indicated her approval of their answers (I describe this moment in more detail in the section below on adoption of discourse.). Through these actions, Jessica indicated that the work she produced was primarily intended for and ultimately needed the endorsement of the teacher.

One final point I want to make about Jessica's audience members for her watchful learning discourse is that Jessica did not appear to include herself in her audience. As she enacted watchful learning, she cut and glued figures, but she did not write on her paper.

Aside from her explanation of how two triangles make squares, all of her mathematical discourse was in response to the teacher's questions. Neither her explanation to Minerva nor her responses to the teacher were directed at herself or primarily intended for herself. Jessica may have communicated with herself about mathematical ideas at times or in ways that were not visible. However, if such moments occurred, they were not a prominent part of Jessica's mathematizing. Unlike the moment in Minerva's third solution or Jakeel's engaged learning, Jessica did not seem to communicate her mathematical ideas with herself.

Mathematizing

Adoption of discourse. Jessica's final discourse had two features that were adopted from the discourse of Minerva and Bonita: the area of Figure I and the use of two triangles make a square as justification for that area (See Excerpt 4, Lines 551 and 555). Bonita was the first person to articulate these two narratives and the routine of using them to account for the area of Figure I. Toward the beginning of the work in the small group, the teacher came to the group and asked questions about the task. This excerpt was presented and discussed in the previous chapter as Excerpt 1.

Excerpt 11

199	Teacher	Okay you're going to compare their area. How could I compare it?	
200	Bonita	Both the areas, they both have two.	
201	Teacher	How do you know?	
202	Bonita	Because two triangles make a square	*Bonita points at the two triangles in Figure I.*

Bonita's statements on Lines 200 and 202 became Bonita and Minerva's routine for talking about Figure I. On two other occasions, when the teacher inquired about the area of Figure I, Bonita and Minerva responded that the area was two and then they offered an explanation that was a variation of Bonita's narrative "Because two triangles make a square" (Line 202).

Jessica adopted Bonita and Minerva's discourse after the teacher indicated her approval. This approval came more than 40 minutes into the small group time. I will first present the transcript that shows the teacher's approval of Bonita's solution. It also demonstrates Jessica's attention to this interaction.

The teacher came to the group and looked at Minerva's fourth solution, which was on a separate paper from her earlier work (See Figures 6.1 and 6.2). Minerva claimed that she had done what the teacher asked on her first paper. The teacher and the students examined Minerva's first paper (Figure 6.1).

Excerpt 12

516	Teacher	How is this two?	*The teacher talks to Minerva.*
517	Bonita	Let me see something. All you got to do is make these two.	*Bonita pulls Minerva's paper towards herself. She points to the two triangles in I.*
518	Teacher	So do that on your paper!	*Jessica looks at Bonita.*

This remark seems to indicate the teacher's approval of Bonita's solution. The conversation continued.

519	Bonita	I already did.	
520	Teacher	Where, where is it?	*Jessica stands up to look. Bonita points at her paper where she has a line connecting the two triangles in I. See Figure 7.4.*
521	Jessica	Bah, butta, spa, bah	*These seem to be celebratory sounds. There are no accompanying gestures. Jessica is continuing to look at Bonita's paper.*

When Bonita pointed to her answer on her paper (Line 520), Jessica stood up to look. In addition, she made a series of funny, celebratory noises (Line 521) that might indicate her enthusiasm that Bonita was close to having a teacher-sanctioned right answer.

522	Teacher	So you just drew a line	*Bonita nods her head. Jessica looks down at her own work.*
		So you didn't actually cut it out	*Bonita shakes her head. Jessica looks at Bonita.*
		Is that okay to do?	*Bonita shrugs her shoulders.*
		Do you think it is?	*Bonita nods her head. Jessica looks at the teacher.*
		Okay. So you showed me that. Right? Everyone in your group has to	*Jessica picks up her pieces of Figure I and places them on her paper.*

With the words, "Okay. So you showed me that" (Line 522), the teacher approved Bonita's solution. Immediately after that approval, Jessica began to copy Bonita's solution, picking up the pieces to Figure I. She looked back and forth between her paper and Bonita's paper and arranged Figure I on her paper so that it matched Bonita's. This copying suggests that Jessica was beginning to adopt Bonita's visual solution. However, Jessica only copied the position of the figure and not any of the accompanying writing. I will say more about this as I summarize Jessica's adoption of discourse.

Shortly after this moment, Minerva finished her work and the teacher told her to go get the next task (Figures J and K). Jessica celebrated the conclusion of their work on Figures H and I.

Excerpt 13

539	Jessica	We did it.	*This is said with enthusiasm. There is a pause.*
		Y'all got it right?	

Jessica's statement, "We did it" (Line 539) indicated her recognition that they finished the problem. Her question, "Y'all got it right?" (Line 539) sought confirmation that Bonita and Minerva had a correct answer. It is after this moment that Jessica first stated that the area of Figure I was two (Excerpt 4, Line 551) and gave the reason, "This is two because if you put two triangles together that's a square" (Excerpt 4, Line 555). Jessica continued to use these narratives at one other time during this lesson that she was asked about the area of Figure I (See Excerpt 5).

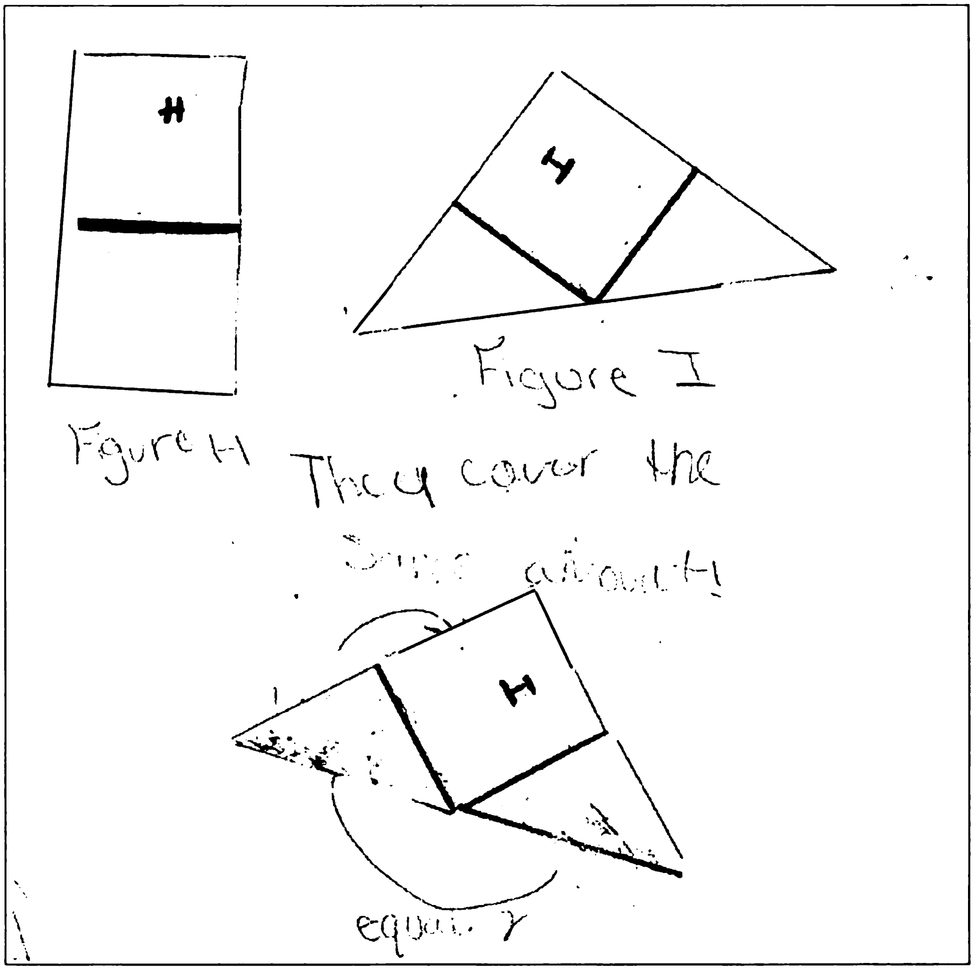

Figure 7.4 Bonita's work as approved by the teacher.
This is student work. The top two figures are labeled "Figure H" (on the left) and "Figure I" (on the right). The text in the middle reads, "They cover the same amount!". At the bottom of the page, Bonita has written, "equals 2".

As Jessica enacted watchful learner, she carefully observed the actions and interactions of Minerva and Bonita and imitated their activities, no longer using her previous discourse to respond to the area of Figure I. This imitation occurred only after the teacher had explicitly approved Minerva and Bonita's activities and discourses.

Jessica's adoption of these narratives and the routine of when to use them did not include connections to the images on the paper. Jessica could use the words in appropriate moments, but when asked to illustrate the words with the pictures on her paper, she couldn't. In Excerpt 4, when the teacher asked Jessica to show how two triangles make a square, Jessica patted her paper, but was unable to show how the figures she had glued on her paper could illustrate this narrative. Her copying of Bonita's solution (Excerpt 12) also illustrates this disconnect between words and images. Jessica adopted part of Bonita's solution, carefully arranging her Figure I, but she did not copy any of the lines or words from Bonita's paper. Jessica's adoption of discourse as she enacted watchful learning careful attention to what to adopt, but not exploration or connection from words to pictures.

It is important to note that earlier in this lesson, Jessica showed Minerva how two triangles could make a square (See Chapter 6, Excerpt 4). She had not integrated this narrative into a strategy for determining area, but she stated this narrative, seemed to understand it, could demonstrate it using all of Figure I. Therefore, part of what Jessica needed to learn about this narrative was that the small triangles in Figure I could be combined in the same way as the large triangles. At this later moment in the lesson, her adoption of the narrative about triangles and squares did not seem to be connected to either the small or the large triangles.

Production of discourse. Toward the beginning of the lesson, Jessica used discourse that was not an echo of what others had said during the lesson. However, toward the end of the lesson, after she had begun to use the teacher-sanctioned discourse, she only used words, narratives, and routines that had been enacted by others and explicitly approved by the teacher. As I described above, her final discourse involving the area of Figure I was an adoption of what Bonita and Minerva had said.

As I noted in the section on audience, there were moments early in the lesson in which Jessica attempted to initiate conversations. For example, she explained how two triangles made a square (Chapter 6, Excerpt 4) and she asked Minerva why she was cutting (Excerpt 8, Line 263) but Minerva and Bonita rebuffed Jessica's attempts at mathematizing. These dismissals of Jessica's mathematizing not only positioned Jessica as someone who should not talk or ask about mathematical objects, they also indicated that Bonita and Minerva were not interested in mathematizing with Jessica or with each other. Exploration and discussion of mathematical discourse was not encouraged for anyone in this group. Unlike Jakeel's peers, the members of Jessica's group showed no interest in constructing an environment conducive to producing discourse. Jessica's interactions with her peers around mathematics were limited to watching.

Substantiation of narratives. As Jessica enacted watchful learning, her substantiation of narratives relied upon the teacher's indication of approval. My discussion above of her adoption of discourse demonstrated how Jessica began to use the narratives of Bonita and Minerva only after the teacher signaled that their solutions were correct. In addition, Jessica's activity as a watchful learner included no exploration or explanation of discourse (beyond her demonstration of how two triangles make a square for Minerva). In order to use logic to substantiate a narrative, Jessica would need to explore or explain the narrative. However, Jessica's activity does not provide evidence of verifying narratives except by looking for approval from the teacher.

Jessica was not alone in relying upon the teacher to substantiate narratives. In the previous chapter, I argued that Minerva relied on the teacher as she enacted covert learning. Bonita also waited for the teacher's approval before committing to a narrative. Bonita crafted two solutions to the task with Figures H and I. Her first solution was much like Minerva's first solution: She glued H and I intact on the paper and wrote, "They cover the same amount!" Much later in the lesson (while Minerva was working on her fourth solution), she glued another copy of Figure I on her paper, drew a line connecting the triangles, and wrote "equals 2". Bonita did not present this work to the teacher until after the teacher had approved Minerva's solution (See Excerpt 12). Bonita didn't explain how her work showed the right answer. Like Minerva, she stated that she had done what the teacher requested (Excerpt 12, Line 519) and then nodded and shrugged her way through the rest of the teacher's questions (Line 522), letting the teacher work her way to endorsing her work. Thus Bonita, Minerva, and Jessica depended upon the teacher to substantiate narratives.

This point is important for two reasons: First, it hints that Minerva and Bonita's refusal to mathematize with Jessica was connected, at least in part, to their reluctance to mathematize at all. While there is evidence that they were frustrated with Jessica (I discuss this below), this frustration does not appear to be the primary reason not to mathematize with her. Second, Jessica's reliance upon the teacher may have come in part from the reluctance of Bonita and Minerva to engage in mathematical conversation with her. Jessica, as I have described earlier, initiated conversations about mathematics and the tasks on several occasions. Neither Bonita nor Minerva continued those conversations. Jessica could have attempted to substantiate narratives on her own. However, it seems that it would have been challenging for Jessica to make sense of what Bonita and Minerva were doing and their narratives if they were unwilling to share their thinking with her (or with anyone else). Thus Jessica's dependence upon the teacher for substantiation is like other watchful learning activities: It was not one option of many laid at Jessica's feet. Instead, it seemed to arise through interactions with Jessica's peers.

Departure from Watchful Learning

In Chapter 5, I described how Jakeel's learning shifted during the lesson. He was initially engaging in learning. However, part way through the lesson, his activity shifted so that Rebecca was directing his learning. Jessica also experienced a shift in learning during this lesson. For most of the lesson, she participated in watchful learning. However, after the teacher approved Bonita and Minerva's solution, she realized that Jessica did not have the solution on her paper. She required Bonita and Minerva to help Jessica complete her work. As the focus of the group became Jessica and her work, the interactions in the group and Jessica's learning changed.

Jessica's shift in learning began after Bonita and Minerva had solved the task involving Figures H and I. They were preparing to start the next task when the teacher asked Jessica for her solution to the task. Jessica was able to explain that the area was two because, "if you put the two triangles together that's a square" (Excerpt 4, Line 555). However, Jessica was unable to show this on her paper (Line 558). The teacher responded:

Excerpt 14

560	Teacher	Some how I need you to show me that on your paper.	
561	Jessica	Well there ain't nobody helping me.	

With her statement on Line 561, Jessica identified herself as in need of help and identified her peers and perhaps the teacher as not being helpful. Earlier, as Jessica engaged in watchful learning, Bonita and Minerva had served as models for her learning so they were not explicitly or directly involved teaching her. Jessica's statement here seems to indicate Jessica's readiness for interactions in which others were more supportive of her learning.

The teacher accepted this invitation and required Bonita and Minerva to literally and figuratively position themselves as helpful to Jessica.

562	Teacher	Okay. So what do you need help with?	*The teacher sits down next to Jessica.*
		So Bonita, come over here please.	*Bonita stands next to Jessica's desk.*
		You know what, you need to stop it right now. I don't know what happened (last night?) but it needs to stop. (2 second pause) You are not listening to me and it needs to stop.	*The teacher talks to Bonita.*
		Come over here Minerva. Come over here Minerva. Get up and move and come over here because I want you to actually come over here. Come over here please Bonita.	*The teacher motions to Jessica's desk. Minerva gets out of her seat and stands next to her own desk.*
		Okay, how is this two?	

Minerva and Bonita's physical positioning was important to the ways in which they could identify and communicate with Jessica about her work. Bonita's position next to Jessica's desk provided her with access to Jessica's work. She could both see what Jessica was doing and could point at Jessica's work as she spoke. In contrast, Minerva stood by her own desk, diagonally across from Jessica's desk (See Figure 7.5 for their seating arrangements). From her standing position, Minerva was as far from Jessica's desk and her work as she could physically be. The difference between Minerva and Bonita's physical locations coincided with differences in the ways they identified Jessica. I will first elaborate on Minerva's identifying activity and then contrast it with Bonita's activity.

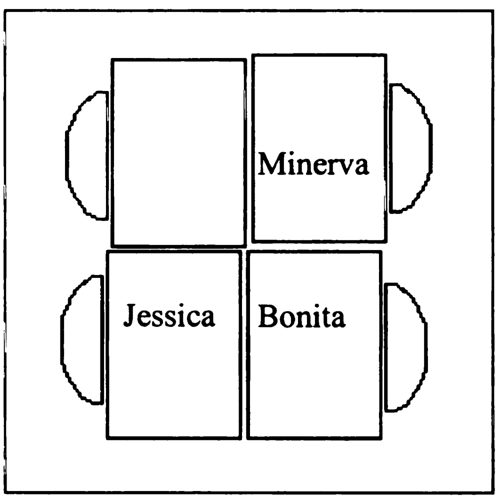

Figure 7.5 Seating arrangements during this lesson

The interaction from the episode continued as Minerva responded to the teacher's question about how the area was two.

563	Minerva	It's not.	
564	Teacher	How can she make it two?	
565	Minerva	My gosh! Look!	*Minerva sounds exasperated. She grabs her paper and holds it up.*
566	Teacher	Is that the way we teach somebody? Is that the way=	*The teacher talks to Minerva.*
567	Minerva	We've been talking about it for (2 second pause) hours.	

In this episode, Minerva continued the identification of Jessica as a watchful learner and herself as a model for Jessica. She held up her paper (Line 565) and told Jessica to "Look!" sending the message that Minerva had the answer on her paper and if Jessica just looked at Minerva's paper, she could also get the answer. Minerva's next statement, "We've been talking about it for hours" (Line 567) also reinforced the message that Jessica had access to the necessary information. Minerva was communicating that she had been available as a model for Jessica. If Jessica had attended to Minerva's modeling, she would already have the answer. With this statement, Minerva not only identified Jessica as a watchful learner, she also identified Jessica as unsuccessful in her watchful learning.

Minerva then shifted her identification of herself from model to director and of Jessica from watchful learner to directed learner. Her next several utterances were a series of commands: "Not like that" (Line 572), "Put them together!" (Line 578), "Put them together. Cut them put them together. Glue them together. Gosh." (Line 582) and "Connect it to the two squares" (Line 585). Each of these commands was directed toward Jessica, telling her what she should or should not do.

One further piece of evidence of Minerva's identifying was a statement she made to the teacher. This interaction occurred just a few moments after the interaction in the previous transcript. Minerva had just told Jessica to connect a square she was holding to the two squares:

Excerpt 15

585	Minerva	Connect it to the two squares.	*Minerva is exasperated.*
586	Teacher	You can't say it like that. That is so mean.	*She talks to Minerva.*

587	Minerva	I'm just sitting here telling her what to do.	*She responds to the teacher.*

On Line 587, Minerva summarized her activity in this episode: She was sitting and telling Jessica what to do. Jessica could be successful by merely following the orders issued by Minerva. Jessica did not need to reflect, explain, hypothesize, or describe. She just needed to follow Minerva's directions.

Jessica might have been willing to enact the directed learning Minerva proposed. However, Minerva did not provide Jessica with enough information to follow her directions. For example, Minerva, Bonita, and the teacher were trying to help Jessica understand that she needed to take the two small triangles in Figure I and put them together to make a square. Jessica had just demonstrated that she could take two copies of Figure I and put them together to make a large square (See Figure 7.2). Jessica did not yet understand that she needed to work with the smaller triangles in Figure I. Minerva directed Jessica:

Excerpt 16

578	Minerva	Put them together!	*Minerva emphasizes each syllable in each word.*
579	Jessica	I just did. You said it was wrong.	

Although Minerva attempted to tell Jessica what to do, her use of "them" (Line 578) was not specific enough to direct Jessica to work with the small triangles. Instead, Jessica protested that she had followed Minerva's directions and that Minerva was contradicting herself (Line 579).

Minerva and Jessica had the same type of misunderstanding just a few turns later.

Excerpt 17

582	Minerva	Put them together. Cut them put them together. Glue them together. Gosh.	*Jessica cuts the triangles from I.*

		Thank you	
583	Jessica	How am I going to put them together when I only got one of these.	*Jessica is holding up the square from I.*

Minerva issued a series of commands and it seemed that Jessica understood what Minerva wanted because Jessica started cutting the small triangles from Figure I. However, Jessica's question in Line 583 demonstrated that Jessica was still not interpreting "them" as the two small triangles from the figure. Instead, Jessica was focused on the square she had cut from Figure I.

These two interactions demonstrate how Minerva's attempts to position Jessica as a directed learner were unsuccessful because Minerva's mathematical discourse was not specific enough. Jessica was attending to Minerva's directions. However, because she was unable to make sense of "them", Jessica was unable to do as Minerva asked.

Instead of enacting directed learning as commanded by Minerva, Jessica enacted guided learning as supported by Bonita and the teacher. The teacher and Bonita did not give Jessica orders but instead reacted to her moves with questions and guidance that was tailored to helping Jessica bridge the gap between what she was doing and what Bonita and the teacher wanted her to do. For example, when Jessica placed the two copies of Figure I together so they made a large square (Figure 7.2), both Bonita and the teacher responded in ways that guided Jessica toward what they wanted her to do.

Excerpt 18

573	Teacher	We are comparing just this piece [Figure I] with just that piece [Figure H].	*She points at I and then at H.*
574	Bonita	How could you make a square right here?	*Bonita points at Jessica's paper.*

The teacher pointed out that Jessica needed to compare Figure H with Figure I (Line 573) and Bonita asked how Jessica could make a square (Line 574). These responses provided Jessica with information about what to compare and what she should be attempting to do without providing her with specific directions about what to do. In these responses, the teacher and Bonita identified as helpful teachers who, like Minerva, knew what Jessica should do, but unlike Minerva, expected Jessica to use their questions and feedback to make her own decisions about how to proceed toward the end they had in mind.

Throughout this episode, Jessica enacted guided learning. She did not have enough details from any one person to be directed. Instead, she needed to make sense of what each person was trying to get her to do. She enacted her interpretation of the directions from Minerva and the guiding questions and comments from Bonita and the teacher. In moments where she could not make sense of the directions or questions, she either protested ("I just did. You said it was wrong." Excerpt 16, Line 579) or asked for clarification ("How am I going to put them together when I only got one of these?" Excerpt 17, Line 583). These responses indicate that Jessica was trying to follow the path indicated by Minerva, Bonita, and the teacher.

The kind of learning Jessica enacted here was different from the directed learning enacted by Jakeel. When Jakeel was directed, he focused on following directions and not on the mathematical discourse. His mathematizing was limited to writing what Rebecca uttered without investigation or verification of the discourse. He was content to keep the mathematical discourse as a discourse-for-others rather than work to turn it into a discourse-for-himself. In contrast, in this episode, Jessica could not merely copy or repeat the mathematical discourse provided by others. Either the discourse was too vague (in the case of Minerva's commands) or it was a question or piece of information that she needed to work from. The mathematizing of others required Jessica to respond by drawing upon the mathematical discourse she owned. This mathematizing is a significant reason why Jessica was not engaging in directed learning during this episode.

Jessica was also not participating in engaged learning. When Jakeel enacted engaged learning, he performed several specific mathematizing and identifying activities. In particular, he produced his own new discourse, positioned his peer teachers as co-learners, and refused to be directed. Jessica's activity during this episode did not reflect these same features. She did not produce new mathematical discourse. Jessica had four turns during this episode. Two of these turns, Lines 579 and 583 above, reflected Jessica's confusion over what she should do. The other two turns, as demonstrated below, were short responses to questions about geometric shapes:

Excerpt 19

576	Teacher	What two shapes make a square?
577	Jessica	A triangle

Later, as Jessica continued to move shapes on her paper, she had this conversation with the teacher:

Excerpt 20

588	Teacher	Take the two triangles. ...	
		Now what is that Jessica?	*Jessica moves the two triangles so they form a square.*
589	Jessica	A square	

None of Jessica's responses show Jessica producing new mathematical discourse. Jessica also did not position her peers as co-learners. Instead of sharing ideas with Bonita and Minerva and expecting them to learn with her, Jessica turned to her peers and her teacher to help her get their answer on her paper. Jessica's activity throughout this episode demonstrated that she was working to physically construct the solution known by others. Her focus on generating what others wanted and not on using the interactions to develop her own discursive tools is a key difference between Jessica's enactment of guided learning in this episode and Jakeel's enactment of engaged learning in his group.

Jessica's enactment of guided learning had some similarities to and some differences from watchful learning in terms of Jessica's identification of her audience and her mathematizing. I discuss in the next sections.

Identifying the Audience

As Jessica enacted guided learning, she identified Minerva, Bonita, and the teacher as helpful guides. She directed her discourse toward Minerva, Bonita, and the teacher. She was attempting to construct the image they had in mind. She needed their feedback on whether her actions were leading her toward the end they had in mind. She attempted to follow their directions and hints, arranging and cutting Figure I in response to their feedback. She was willing to respond to their questions (See Excerpt 19 and 20) and she held them accountable for their feedback. When they provided her with seemingly contradictory requests, she called them on it, protesting that she had done what they said (Excerpt 16, Line 579) or asking how what they wanted was possible (Excerpt 17, Line 583). She was working hard to follow their guidance: They needed to be sure it made sense.

While Jessica's identification of Minerva, Bonita, and the teacher were different as she enacted watchful learning and guided learning, her identification of herself was consistent: As in watchful learning, when Jessica enacted guided learning, she did not seem to explicitly include herself as a member of the audience. There was no moment in which she visibly (even if silently) spoke to herself or monitored her communication with others (as Jakeel did when he stopped himself before he counted all of the squares in Figure J). While it is possible that Jessica monitored her words or communicated with herself in ways that weren't visible, her lack of explicit examples indicates that she was not a primary audience for her discourse.

Mathematizing

Jessica's mathematizing as a guided learner was different from her mathematizing as a watchful learner. During the time in which she enacted guided learning, her focus, and that of her peers and the teacher, was on generating a particular arrangement of the pieces of Figure I on her paper. This focus and the shift in interactions resulted in different adoption, production, and substantiation of discourse for Jessica.

As I noted earlier, Jessica's verbal discourse as she enacted guided learning was limited to four turns. In these turns, she did not demonstrate that she adopted discourse. Instead, she named shapes (Excerpt 19, Line 577 and Excerpt 20, Line 589) or reacted to problems with directions (Excerpt 16, Line 579 and Excerpt 17, Line 583). However, the emphasis of this guided learning time was the production of a particular arrangement of Figure I on Jessica's paper and not on her verbal discourse. If Jessica had copied the arrangement from Bonita and Minerva's paper and then demonstrated that she was using the image, I could argue that she had adopted the picture as a part of her discourse. However, the interactions of guided learning were focused on Jessica's production and not adoption of the arrangement. (As I noted earlier, Minerva's contribution to Jessica's guided learning had more to do with Minerva's vague discourse than upon a desire to have Jessica work to figure out the preferred arrangement.) Jessica was to take the feedback from the group and use that to generate the desired image.

She was eventually able to do this. However, her creation of the image was highly structured and guided by Minerva, Bonita, and the teacher. As a result, Jessica's generation of the image did not seem to arise from earlier adoption of discourse, indicating that the image was not a production of discourse for Jessica. Instead, it seemed largely a result of guidance and input from the teacher, Minerva, and Bonita. While Jessica was the one who arranged the pieces, others told her when her moves were right or wrong: When Minerva and Bonita disagreed with her placement of the figure, they simultaneously exclaimed "Not like that!" (Lines 571 and 572). As Jessica got closer to constructing the desired picture, Minerva watched her and, when Jessica had the triangles together as a square, Minerva told her "You got it!" (Line 591). This feedback meant that Jessica did not have to decide whether her picture was complete or not: She could rely upon Minerva and Bonita and did not have to reason from her adopted discourse about the area is two because two triangles make a square. Further evidence that she did not produce the image by reasoning from her adopted discourse was her limited ability to connect the image and the verbal discourse when asked by the teacher (See Excerpt 5, Line 613).

The connection between the image and the verbal discourse was an important piece of Jessica's substantiation of narratives as she enacted guided learning. Jessica needed to create the picture that Bonita, Minerva, and the teacher were describing with their words. Ultimately, whether a statement reflected the state of affairs depended upon Bonita, Minerva, and the teacher's opinions. However, as Jessica worked to translate their words into a picture, she also evaluated whether their words made sense. On two occasions, Minerva told Jessica to do something that seemed problematic to Jessica: Minerva's commands contradicted Jessica's understanding of how things were. For example, Jessica had just put two copies of Figure I together and was told this was wrong when Minerva told Jessica to "put them together" (Excerpt 16, Line 578). To Jessica, this command contradicted the information she had just received about what she had done and she protested, "I just did. You said it was wrong" (Line 579). At this point, Bonita intervened and showed Jessica how she was to put two triangles together (Line 580, not included in this document). Minerva's command to put them together had many possible interpretations, one of which did not match Jessica's understanding of the state of affairs. Jessica needed Bonita to show her what was the correct interpretation. A similar sequence of events happened on the second occasion when Minerva's commands were problematic for Jessica. In Excerpt 17, Line 582, Minerva uttered a long sequence of commands, which involved doing various things to "them". Jessica was confused because she was focused on what to do with a single square and she couldn't make sense of how to put them together when she only had one (Line 583). The teacher intervened to help Jessica understand that she was supposed to be working with two triangles (Excerpt 20, Line 588). In both cases, Jessica substantiated narratives and found that they did not reflect what she thought was possible with the picture. She was then shown she was wrong in her substantiation. While she was more actively involved in substantiation when she enacted guided learning than when she enacted watchful learning, her substantiation still relied upon the evaluations of others.

Return to Watchful Learning

Jessica's enactment of watchful or guided learning largely depended upon her interactions with others. When her peers refused to mathematize with her, she enacted watchful learning. She transitioned from watchful learning to guided learning when the teacher intervened and required Bonita and Minerva to support her work with mathematical objects. She transitioned back to watchful learning when her arrangement of Figures H and I met the standards of the teacher, Bonita, and Minerva. Once Jessica had the teacher-approved arrangement on her paper, Minerva handed out the papers for the next task with Figures J and K. Jessica watched as Minerva began cutting out the figures and then began cutting her own figures. Minerva and Bonita were no longer guiding her by issuing commands or asking questions. Instead, Jessica, if she didn't already know what to do, was to learn by watching.

Autonomous Learning

As Jessica enacted both watchful learning and guided learning, there were moments in which her learning activities suggested the possibility of autonomous learning. For example, early in the lesson, Jessica explained how two triangles make a square to Minerva (Chapter 6, Excerpt 4). Jessica was sharing her understanding with another student and indicating her interest in explanations. If Minerva or Bonita had encouraged her explanation, asked her for more detail, or allowed her to wonder how her explanation was different from Bonita's (who focused on the small triangles in Figure I while Jessica focused on Figure I as a triangle), the group might have embarked on activities resembling autonomous learning. Jessica's willingness to explain and put forth her ideas was a first move toward autonomous learning. However, neither Minerva's nor Bonita's subsequent actions supported the autonomy in Jessica's first move.

Jessica provided another opportunity for autonomous learning when she asked Minerva why she was cutting out another copy of Figures H and I (Excerpt 8, Line 263). Jessica noticed that Minerva was doing something that didn't make sense to her and so she asked for an explanation. Minerva could have responded in a way that would have initiated a mathematical conversation or would have validated Jessica's curiosity. Instead, Minerva's dismissive remark ("Because I'm doing something", Line 264) sent the message that exploration of her activities was not welcome.

Instead of continuing to ask questions or offer explanations, Jessica watched her peers and adopted their words when sanctioned by the teacher. This watching, adoption without production, and substantiation by teacher sanction were not autonomous learning activities. While Jessica's initial activities during this lesson showed the potential for autonomous learning, the reaction of her peers did not support the growth of this potential.

Jessica's enactment of guided learning also seemed to have potential, but not realization, of autonomous learning activity. As in watchful learning, there were two occasions in which Jessica noticed, and commented on, a problem between the discourse of Minerva and what Jessica thought was possible with the picture (Excerpts 16 and 17). Jessica comments could have opened the possibility for conversation around what Jessica thought was problematic, what Minerva intended, and the implications for each perspective for the area of Figure I and for comparing the area of the two figures. However, in both moments when Jessica questioned Minerva's directions, she was shown (by Bonita and then the teacher) what Minerva meant in a way that indicated that Minerva's thinking is correct. While these responses were helpful and supportive of Jessica, neither response encouraged Jessica to continue her exploration of the discrepancy between her ideas and Minerva's. Instead, Bonita and the teacher indicated that Minerva was right, Jessica was wrong, and Jessica needed to work with the two smaller triangles in order to get the right solution. Jessica's enactment of guided learning, like her enactment of watchful learning, contained moments in which Jessica raised questions about problems she noticed with discourse. Like watchful learning, the response to Jessica's problems did not encourage autonomous learning and Jessica did not continue to ask questions. Thus Jessica's opportunities to engage in autonomous learning were, in part, constrained by the reactions of her peers and the teacher.

Discussion

Intertwining of Mathematizing and Identifying

Jessica's enactment of watchful learning and guided learning contain several instances in which mathematizing and identifying were intertwined. In Chapter 5, I emphasized how mathematizing identified Jakeel in different ways as he enacted engaged and directed learning. In Chapter 6, I described how Minerva resisted the identification of learner as constructed by the mathematical explanations offered by Jessica and Bonita. This case of Jessica's learning continues the theme from Chapter 6: There are multiple moments in which mathematizing fails to construct identities because of the reaction to the mathematizing.

As I described earlier, Jessica attempted to initiate mathematical conversations on three different occasions and on each of these occasions, her mathematizing met with resistance (See Excerpt 7, Excerpt 8, and Chapter 6, Excerpt 4). Jessica seemed to be identifying as someone who wanted to be engaged in learning with others. She wanted to share her ideas and talk with others about what they were doing. With her attempts at mathematizing, Jessica also identified Bonita and Minerva as interested in talking about mathematics and the task. However, both Minerva's and Bonita's replies rejected mathematizing and this identification: Minerva's "Nah huhn" dismissed the explanation. Bonita first ignored (Excerpt 7) and then downplayed (Excerpt 8) Jessica's attempts to start conversations around the task. Minerva and Bonita rejected Jessica's positioning of them and proposed instead that they not talk about mathematics (or in Minerva's case, that they not talk at all). With these rejections, Minerva and Bonita did not explicitly identify Jessica. They did not suggest that she act in a particular way. Instead, they communicated that she should *not* act in ways that required them to explicitly act as her co-learners or as her teachers. Because Minerva and Bonita were unwilling to mathematize with Jessica, Jessica was left to enact a kind of learning that did not require interactions. She could have worked independently. However, as suggested by her earlier mathematizing, Jessica seemed to prefer to learn with and from others. As a result, she enacted learning activities that allowed her to learn from her peers without requiring them to explicitly teach her. Minerva and Bonita's unwillingness to mathematize with others did not identify Jessica in particular ways, but it did restrict the ways in which she could identify.

Minerva also mathematized in ways that offered an identification of Jessica that was not taken up by Jessica. During the time in which Jessica enacted guided learning, Minerva attempted to direct her by issuing commands. These commands identified

Jessica as "under orders": Minerva was telling her what to do. Jessica did not need to ask questions or investigate. She just needed to do what Minerva commanded. However, Minerva's commands were too vague. Jessica could not make sense of how to "put them together" (Excerpt 16, Line 578 and Excerpt 17, Line 582). Jessica could not act as a directed learner. In contrast, the mathematizing of Bonita and the teacher identified Jessica as someone who could and should reason from information to figure out the answer (See Excerpt 18). They were willing to provide her with some guidance ("We are comparing just this piece with just that piece", Line 573) and ask questions ("How could you make a square right here?" Line 574). These words left enough ambiguity that Jessica was required to reason from this information and what she knew if she was going to figure out what she was supposed to do. Minerva's commands, because they were vague, also required Jessica to reason. Thus Minerva seemed to want identify Jessica as directed but her commands unintentionally supported Jessica's identification as guided.

Finally, Jessica identified herself and others such that the mathematizing in the group changed. Toward the end of the work on Figures H and I, the teacher realized that Jessica did not have a solution to the task. Jessica was able to state the area for Figure I and the narrative that two triangles made a square, but she could not show this on her paper. The teacher told her that she needed the answer on her paper and Jessica replied, "Well there ain't nobody helping me" (Excerpt 14, Line 561). With this statement, Jessica identified Bonita and Minerva as unhelpful peers and identified herself as someone who wanted help. The teacher reacted immediately and endorsed both of these identifications: She required Bonita and Minerva to get out of their seats and help Jessica and she sat down next to Jessica and began to guide her learning (Line 562). The resulting interactions involved more mathematizing than occurred in this group when the teacher was not present. While Jessica said very little, Bonita, Minerva, and the teacher watched her work and provided feedback on her actions. Thus Jessica's identification as needing help resulted in a change in mathematizing.

Development of Mathematical Discourse

Jessica's mathematical discourse changed during the course of this lesson. By the end of the lesson, she could state the area for Figure I and state a narrative that addressed the triangular spaces. She also had an arrangement of the pieces of Figure that that assembled the triangles into a square and placed the squares together as a rectangle. These changes have features of the desired mathematical discourse. However, as I discussed in the section on Jessica's final discourse, there are some aspects of these changes and Jessica's final discourse that are troubling. She was not clear how assembling the triangles into a square helped her determine the area, she did not connect her words with her picture, and she was not able to use this discourse during a subsequent lesson involving a problem with the same shape. These outcomes suggest that the activities of watchful learning and guided learning, while useful in supporting Jessica in attaining narratives and a picture, were not sufficient to help her learn the discourse so that it could be a tool in other situations. There are two points I want to make about how the mathematizing and identifying in this lesson seemed to have consequences for Jessica's final discourse.

My first point is specific to Jessica's enactment of watchful learning. As a watchful learner, Jessica had opportunities to observe the mathematical discourse of others, but she did not have opportunities to explore or practice the discourse. Bonita and

Minerva's refusal to mathematize with Jessica meant that Jessica did not ask questions, respond to others' questions, try the discourse, or receive feedback on the discourse. Jessica's work was to learn the right discourse through observation only: Others would mathematize and Jessica could use that to determine the right answer. While Jessica succeeded in mastering the right answer, her limited mathematical interactions and Bonita and Minerva's refusal to explicitly identify as teachers meant that Jessica did not have opportunities to develop the mathematical discourse as a tool for herself.

Second, as a watchful learner and as a guided learner, Jessica's goal was to attain the right answer, either the right words (watchful learning) or the right picture (guided learning). The focus on attaining the right answer meant that Jessica (and her peers) did not work toward what was mathematically logical and did not build from what she knew. Instead, she watched to see or listened to figure out the right answer. Furthermore, when Jessica enacted guided learner, she only worked (and was only supported in working) on getting the right picture: There was no attempt by Jessica, her peers, or the teacher to connect the picture to the original task or the narratives about area. This focus on getting the right answer suggests that Jessica, Bonita, Minerva, and the teacher understood learning in terms of a metaphor of acquisition (Sfard, 1998). Rather than emphasize what Jessica was doing during the lesson, everyone (including Jessica) emphasized whether Jessica could recite the right answer and had the right picture. This emphasis on what Jessica had at the conclusion of the task rather than upon what she did as she worked on the task helped Jessica develop her mathematical discourse as a set of answers to tasks, but not as a tool for problem solving.

Autonomous Learning

The lens of autonomous learning offers additional insights into connections between Jessica's learning activities and the changes in her discourse. I defined autonomous learning as the constellation of mathematizing and identifying activities that demonstrate curiosity about what others say or what was true. In order to demonstrate curiosity, Jessica needed to investigate what others said and pose problems based upon her observations. As I have emphasized earlier, there were moments in which she seemed curious, but her curiosity was not encouraged by her peers. While Jessica was interested in what others said, for most of the lesson, she did not visibly investigate the discourse of others or pose problems about her observations. Her interest in others' discourse was, by the end, an interest in whether their discourse was the right discourse. While enacting guided learning, she protested and asked questions when she did not understand, but again, her curiosity and problem posing was dismissed as Bonita and the teacher guided her to the right answer.

Jessica's activities as a watchful learner, aside from her initial explanation and attempts to start conversations, did not resemble the activities of autonomous learning. As a watchful learner, Jessica's audience was ultimately the teacher who could approve her work: She was not a member of her audience. She adopted discourse but did not produce discourse and she relied upon the teacher to substantiate her discourse. Jessica's watchful learning activities did not resemble that of autonomous learning and the outcomes of her learning were also not the outcomes of autonomous learning. In particular, Jessica's final discourse was a discourse-for-others and not a discourse-for-herself. According to Sfard (2008), discourse-for-others is "discourse in which one engages only with those for whom this discourse makes sense and for the sake of communication with these other people (as opposed to practicing this discourse in self-communication)" (p. 297). Jessica only mathematized when asked by the teacher and she repeated the teacher-approved discourse used by Bonita and Minerva. When asked by the teacher, Jessica could not explain the narratives she uttered or link them to the work on her paper. Jessica's final discourse was not one that she used to communicate with herself. Nor was it one that she probed and evaluated. Jessica's final discourse was used to communicate to the teacher that she had the right answer and not that she understood the mathematics of the task: It was a discourse-for-others.

Jessica's final discourse might have been different if she had been encouraged to continue the curiosity she demonstrated early in the lesson. Dewey defined curiosity as "interest in *problems* provoked by the observation of things" (p. 33, italics in original, 1910). Jessica displayed such an interest when she asked Minerva why she was cutting (Excerpt 8, Line 263). If Bonita and Minerva had responded to this overture and Jessica's other attempts at mathematical conversation with interest in her explanations and mathematical responses to her questions, Jessica might have continued to be curious about the task. Through exchanges with Bonita and Minerva and continuing observation, she might have explored their discourse and used it to communicate with herself. Instead, Jessica's preference for learning with others coupled with Bonita and Minerva's preference not to overtly teach or learn meant that Jessica primarily watched and adopted discourse and did not pose or explore problems on her own.

Jessica's enactment of guided learning also lacked many features of autonomous learning. However, guided learning initially seemed to be very similar to autonomous learning. Bonita, Minerva, and the teacher did not tell Jessica what to do, how to construct the final arrangement, or what the final arrangement was (Minerva attempted to more direct but her discourse was too vague to make clear what Jessica should do). Jessica was not expected to copy someone's work. Instead, Bonita, Minerva, and the teacher provided Jessica with information and clues and Jessica was to reason from those clues. This reasoning could be seen as an autonomous learning activity because Jessica was expected to do intellectual work. This expectation matches Warfield, Wood, and Lehman's definition of autonomy as the ability to solve mathematical problems without being shown the solution by another person (Warfield *et al.*, 2005). This definition suggests that the key feature of autonomy is the lack of input from others. It implies that a teacher interested in promoting autonomy should help his/her students become more independent in their work. However, Jessica's learning suggests that working independently is not a necessary or sufficient criterion for autonomous learning. Even if Jessica had been able to construct the desired arrangement of Figure I without input from her peers or the teacher, this activity could fail to be autonomous if her purpose was limited to constructing the right arrangement.

Autonomous learning is not merely about requiring students to reason on their own and it does not have the right answer as its end goal. Autonomous learning involves investigation of problems the student has posed based upon observations. This does not mean that the student won't construct the right answer, but it does mean that the goal of the student's work is solving problems related to discourse. Arriving at the right answer is just one of the outcomes of this problem-solving work. Jessica's work to construct the right arrangement of pieces did not involve her in a problem arising from her observations. Instead, the problem she needed to solve was figuring out what Bonita, Minerva, and the teacher had in mind as they guided her toward the desired arrangement of Figure I. Jessica was not identified (and did not identify herself) as someone who was critically evaluating discourse. Instead, she was someone who had failed to arrange Figure I on her page and who needed to craft that arrangement. While the teacher and Bonita (and, inadvertently, Minerva) had the best intentions as they guided (rather than directed!) Jessica toward the desired arrangement, this guiding learning, because it had the right answer as an end goal, could not provide the problem-posing environment necessary for autonomous learning.

The activities of guided learning also failed to resemble autonomous learning in other ways. Jessica identified her audience as helpful guides who were monitoring her work. They were not co-learners, fellow explorers, or teachers interested in explanations which might be the case in autonomous learning. Also, an autonomous learner includes him/herself in their audience: Jessica did not seem to have herself as a member of her audience. Also, the mathematizing of guided learning differed from that of autonomous learning. Both kinds of learning emphasize production of discourse, but Jessica was expected to produce a particular image by following hints rather than producing discourse by investing and exploring narratives and images. Jessica was also not expected to adopt the discourse used to guide her. She needed to follow the guidance, but there were no opportunities for her to demonstrate its use. Finally, Jessica's substantiation of discourse during guided learning relied upon others. She asked questions or commented when the clues she was given did not match her picture. However, she did not explore and was not encouraged to explore her questions or conflicts. Instead, she was guided toward seeing what she should be thinking.

As in watchful learning, the activities that contributed to getting the right image did not support Jessica in communicating with herself about the image. She crafted the image that others wanted. As a result, the arrangement of pieces was discourse-for-others and not discourse-for-herself. This became evident as she was later asked to show how the arrangement showed Figure I as two (Excerpt 5). Jessica's gestures indicated that her verbal explanation of "two triangles together" makes two (Line 613) might not have been connected to the arrangement on her paper and to the area of Figure I. Also, her inability to use the discourse in the subsequent lesson provides further evidence that the discourse was not a discourse-for-herself.

As Jessica enacted both watchful learning and guided learning, there were moments in which she showed curiosity and posed problems based upon her observations. However, her curiosity was not supported by her peers or by the end goal of reaching the right answer. Jessica could have attempted to explore the discourse on her own or she could have persisted in asking mathematical questions of her peers, but she seemed to want to learn with or from her peers and she was responsive to their refusal to answer her questions or discuss mathematics with her. In the end, very few of Jessica's learning activities were autonomous learning activities and her final discourse was a discourse-for-others. Perhaps if she had persisted and had been allowed to persist in her curiosity her final mathematical discourse would have been a more mathematically appropriate discourse-for-herself.

Reconceptualizing Jessica as a Struggling Student

The careful study of Jessica's activities enabled by this data suggests the need to take a closer look at the notion of "struggling student". I initially selected Jessica as a focal student because I was interested in what I had framed as her struggles to learn mathematics: I had noticed that she copied from her neighbors during math lessons. Furthermore, her peers complained that she played too much. I had assumed that copying and playing were strategies to avoid engaging in intellectual work involving mathematics. However, copying and playing are not truly struggles to learn. Instead, struggling to learn would mean the learner was confronting challenges in investigating and exploring discourse. Thus the label of "struggling student" is problematic because as a euphemism for "failing student" it hides the ways in which students may not be struggling to learn at all because they have given up and are copying from their neighbors or because they do not have opportunities or encouragement to investigate discourses. In addition, the use of struggle to label failing students implies that learning should not be a struggle. It reinforces the notion of the need to already know examined in the previous chapter: Minerva, as she enacted covert learner, did not want to be seen as struggling to know and yet the struggle to know should be a central focus of activity in schools.

Having just argued that I should not have originally labeled Jessica as a struggling student, I will now argue that struggling student is an appropriate label for her at times because it captures her efforts to engage with discourse. During this case lesson, Jessica demonstrated that she was interested in discussing mathematics and she had questions about her work and the work of others. At moments, her mathematizing and identifying indicated that she was curious about mathematical discourse. However, her peers were unwilling to accept her mathematical overtures and Jessica seemed unwilling to risk the social consequences of insisting upon a math-talk agenda. Thus Jessica did struggle to learn. However, her struggles to learn mathematics were not about what we would want her to struggle with (the discourse) but were more about her peers' unwillingness to mathematize or to identify as co-learners or teachers.

This conclusion represented a shift from my original thinking. It was only possible through examination of the whole lesson: If I had only sampled parts of the lesson, I might not have seen Jessica's attempts to engage her peers and might have assumed that her watchful learning reflected her reluctance to mathematize. Instead, detailed examination of the whole case demonstrates that Jessica did struggle to learn mathematics, but that her struggles focused on the engagement of her peers and not on the kind of struggle we might want to encourage in schools: struggles to investigate discourses.

CHAPTER 8

DISCUSSION & CONCLUSION

In Chapter 1, I present a vignette that described how Dquan acquired the answer to a mathematics problem by copying from his neighbor. Dquan's copying was not a unique or isolated event: I have seen students borrow answers from others on numerous occasions, including in my own third and fourth grade classrooms. My research goal was to make sense of students' activities as they engaged in learning activities, including the activity of copying from others. In Chapter 2, I elaborated a theoretical framework that described how learning mathematics could be seen as the interplay of mathematizing and identifying activities. Sfard's commognitive framework (2008) provided the foundation for describing how mathematizing, identifying, and learning could be defined and examined through discourse, with the outcome of learning defined as a change in discourse. I also constructed a theory of autonomous learning that considered what mathematizing and identifying activities might lead to mathematically desirable changes in discourse. These frameworks refined my questions about student activity and learning: I looked at the effects of the activities of mathematizing and identifying on each other and on the development of mathematics discourse. I also considered what advantage autonomous learning offered for the development of mathematical discourse.

In Chapter 3, I explained my research methods, outlining my data selection, collection, and analysis process. I described the classroom, teacher, and students involved in the study; how I chose which lessons to analyze, and how I examined discourse to make claims about mathematizing, identifying, autonomy, and learning outcomes. The fourth chapter provides background on the lessons and introduces the three findings chapters. Throughout the findings chapters, I illustrated how my theoretical framework and analytic tools provided insights into learning activities, suggesting how student activity constituted and was connected to interactions with peers and with the teacher and how those activities related to changes in mathematical discourse.

Each findings chapter offered a discussion of one student's learning activities including a comparison of those activities to my theoretical construction of the activities of autonomous learning. In this chapter, I summarize the claims in the three findings chapters and look across all three to draw other conclusions about students' activities and changes in their discourse. I will first specifically address my two research questions and then I will discuss other claims that arise from my examination of the data.

Mathematizing, Identifying, and Learning Outcomes

My first research question asked what effect the activities of mathematizing and identifying have on one another and on the development of mathematics discourse. As I elaborated in each findings chapter, mathematizing and identifying are intertwined such that sometimes they are the same activity while at other times the activity of one is tied to activity for the other. Several studies that have explored the connection between student's mathematical activity and their identities haven taken a larger scale view of identity, looking at multiple individuals across classrooms or at individuals across an extended period of time. They have looked at how students come to see themselves as math people (Boaler & Greeno, 2000), how the intersection between pedagogy and identity supported students (Jilk, 2007), how identities and stories about the future are connected to engagement with mathematics (Sfard & Prusak, 2005). A smaller scale view offers the possibility of adding complexity to the constructions of identity already elaborated by these researchers. This smaller scale view can capture the ways in which identities might be more fleeting or fluid (for example, Jessica's few moments as an explainer of mathematics toward the beginning of her lesson). It also provides a glimpse at mechanisms of identity construction (for example, how Minerva learns while conveying that she already knows) and how interactions between students construct identifications. These cases of micro-analysis also show connections between learning activities and learning outcomes, suggesting what kinds of mathematizing and identifying activities might be more conducive to mathematical learning.

In each findings chapter, I discussed how mathematizing was also identifying. For example, as Rebecca explained to Jakeel how she counted the half squares in Figure J (Chapter 5, Excerpt 8), she identified herself as knowledgeable and as interested in helping Jakeel. She also identified Jakeel as capable of understanding and acting on her explanation. In addition to examining in detail how mathematizing identified others, close analysis of the interactions also showed that all three cases contain examples of how students rejected the identifications constructed by others through mathematizing. For example, Rebecca repeatedly attempted to direct Jakeel by telling him what to write. While he did eventually allow her to tell him what to do, he initially rejected her directions (3 times), telling her that he knew what to do (Chapter 5, Excerpt 7). In the case of Minerva's learning, Minerva undermined the explanations of Jessica and Bonita, disagreeing with their identification of her as not understanding how two triangles make a square (Chapter 6, Excerpt 4). Finally, in the case of Jessica's learning, she attempted to mathematize with Minerva and Bonita, identifying them as helpful, but they refused her overtures (Chapter 7, Excerpts 7 and 8). These moments of disagreeing with identifications provide glimpses into the identities students were trying to communicate and provide insights about student activity that might not be available if I had constructed identity as a more static feature or as only visible across a larger span of time.

This close examination also shows the social work accomplished through mathematizing. Historically, mathematics has been viewed, even celebrated, as acultural: 1 + 1 = 2 around the world. More recently, this view of mathematics has changed and some mathematics educators recognize the ways in which mathematics is socially constructed, cultural knowledge (Bishop, 1988). However, mathematical discourse maintains some of the sense of aculturalism through its objectified, impersonal sentence structure. Sfard (2008) notes that alienation is one outcome of the objectification that occurs in mathematical discourses. The sense of human agency is removed through the use of the passive voice and use of mathematical nouns as subjects in sentences. These cultural conventions create the impression that mathematical utterances are entirely about mathematical content and free of messages about the speaker and his/her audience even as they simultaneously identify the speaker and the audience. For example, when Rebecca states, "H and I cover the same amount. Figure I has one square and two half squares that equals two squares." (Chapter 5), she does not mention herself, Daren, or Jakeel. However, with this statement, she identifies herself as knowledgeable and begins to position Daren and Jakeel as needing to copy from her. As I began this study, I partitioned student activity into mathematizing activities or identifying activities. However I soon realized that students were sending messages and accomplishing social work with talk that seemed to be only about mathematical objects. The close examination of student's mathematical utterances and the reactions of other students made it clear that mathematizing activities were also identifying activities.

Identifying activities can also have implications for mathematizing, even when they do not specifically address mathematical objects. Each of the cases illustrates this. In Chapter 5, Rebecca identified Jakeel as playing (Excerpt 11) and implied that he needed to write the words she dictated, suggesting that Jakeel's mathematizing should be limited to recording her words. Minerva's sequence of "Nah-huhn"s (Chapter 6, Excerpt 4) identified her as not interested in mathematical explanations (even as she timed her utterances so that she could hear the explanation). Finally, in Chapter 7, Jessica identified Bonita and Minerva as not helping her (Excerpt 14), prompting the sequence of exchanges in which they guided her learning. In addition to demonstrating a link between identifying and mathematizing, this study also demonstrates the ways in which identifying activities affect multiple individuals. For example, when Jessica identified as needing help (Chapter 7, Excerpt 14), her request changed the mathematizing for everyone in this group. Immediately following the request, Bonita, Minerva, and the teacher began to ask Jessica about and provide guidance for the mathematical task.

Finally, this study examines how mathematizing and identifying are involved in the development of mathematics discourse. As students identified themselves and others, they suggested and enacted different mathematizing activities. The mathematizing activities provided or prevented opportunities to explore and practice discourse, which seemed to be linked to changes in discourse. For example, as Jakeel enacted engaged learning, he identified as interested in exploring discourse. He asked questions and worked to change his discourse. In the end, the change in his mathematical discourse was the most desirable of the three students. In contrast, Minerva's enactment of covert learning prevented her from asking questions or investigating other's mathematical discourses. She seemed to want to appear as knowledgeable and did not mathematize with Bonita or Jessica unless required to do so by the teacher. In the end, the change in her mathematical discourse contained some mathematically undesirable features.

Across the five kinds of learning, the mathematizing and identifying that seemed to be linked to the most desirable change in discourse were ones in which the learner identified and was identified as capable of exploring other's ideas and of working from those ideas to his/her own ideas. In contrast, the learning outcomes that were the least mathematically desirable were those in which the mathematizing focused on right answers and the learner relied upon the teacher or peers to approve their work. This conclusion supports Ben-Zvi and Sfard's learning-teaching agreement (2007). Ben-Zvi and Sfard argue that changing the learner's discourse arises from an unwritten agreement about how the learning is to happen. They specify three requirements for positive learning outcomes: The learner and his/her teacher (including peer teachers) must agree on whose discourse is to be learned, on their respective roles of learner and teacher, and on how the change will proceed. Ben-Zvi and Sfard emphasize that the learner cannot passively adopt the discourse of the teacher but must rationalize and critically examine the discourse. Jakeel's enactment of engaged learner demonstrates this. He did not merely adopt the discourse of Rebecca and Daren, but asked questions and produced his own discourse based upon their discourse. In contrast, as Jessica enacted watchful learning, she adopted the narratives of the available discourse but did not examine, rationalize, or work to connect them to the pictures of the figures.

This study also shows the challenges learners face in taking on the examination of others' discourses. First, this examination requires that learners acknowledge that they don't already know the discourse. As I discussed in Chapter 6, this identification as not-yet- knowing runs contrary to what schools teach implicitly about what it means to be a good student: When work is assigned, good students are supposed to already know how to do it. Minerva's enactment of covert learning displays this tension. Her presentation as knowledgeable means that she cannot ask questions or examine others' discourses. If learners are going to engage in thoughtful exploration of others' discourses, then they must be willing to identify in ways that may not be valued in many classrooms. A second challenge of examining others' discourses is obtaining the support of peers. Jessica's enactment of watchful learning and Jakeel's enactment of directed learning demonstrate that peers may be reluctant to engage with the learner in examining their discourse. Minerva and Bonita refused to mathematize with Jessica and Rebecca, in many instances, pushed Jakeel to copy from her rather than to ask her questions or explore her discourse. In order for learners to enact the learning-teaching agreement, they need more than the willingness to take on critical examination of others' discourse; they need teachers (especially peer teachers) who are willing to support them in examining the teachers' discourses.

Autonomous Learning

My second research question asks about the advantage of autonomous learning for the development of mathematical discourse. As a reminder, autonomous learning is the constellation of identifying and mathematizing activities that reflect curiosity about what others think/say and what seems to be true. My study shows that autonomous learning can be a useful lens for evaluating learning activities and outcomes. Examining the similarities between autonomous learning and each of the five kinds of learning shows that the most autonomous learning was also the learning with the most mathematically desirable change in discourse. The lens of autonomous learning offers reasons for this outcome. I elaborate each of these points after I introduce a table that summarizes this comparison.

Table 8.1 shows the mathematizing and identifying activities of autonomous learning across each of the five kinds of learning I describe in this study. I list the kinds of learning horizontally. The vertical axis lists each of the mathematizing and identifying activities of autonomous learning: the learner's identification of his/her audience, whether the learner includes him/herself in his/her audience, adoption of discourse, production of discourse, and substantiation of discourse. The *yes* or *no* in each cell indicates whether the activities of the kind of learning match that of autonomous learning. The table also includes the discursive outcome of the learning: whether the discourse is for the learner, others or neither.

Table 8.1 Mathematizing and identifying activities of autonomous learning for each kind of learning

		Kinds of Learning				
		Engaged	Directed	Covert	Watchful	Guided
Autonomous Learning Activities	Identifying the Audience	Yes	No	No	No	No
	Self as Audience	Yes	No	Yes, limited	No	No
	Adoption of Discourse	Yes	No	Yes	Yes	No
	Production of Discourse	Yes	No	Yes	No	Yes
	Substantiation of Narratives	Yes	No	No	No	No
	Discourse for whom	Himself	Others	Own	Others	Others

In my discussion of mathematizing and identifying above, I claimed that Jakeel's enactment of engaged learning produced the most mathematically desirable discourse. As the table demonstrates, his engaged learning also had mathematizing and identifying activities that most closely resembled autonomous learning. He identified his audience as evaluators and explainers, supporting him in his exploration and elaboration of their discourse. He included himself as a member of his audience, which meant that he was working to make sense of the discourse for himself. He adopted the discourse of his peers and built from that discourse to produce his own discourse (the counting and pointing routine). He substantiated his new discourse himself. By the end of the lesson, his final discourse was one that he could use to communicate with himself and with others (discourse-for-himself).

These autonomous learning activities suggest reasons why Jakeel's enactment of engaged learning may have resulted in the most mathematically desirable discourse. He worked to make sense of and then use the mathematically appropriate discourse of his peers. He then verified that their discourse and his new discourse reflected the state of affairs. He engaged with discourse in order to change his discourse so that it was an accurate tool for counting figures. Jakeel could have asked Rebecca to show him how to count to get the right count of 8. Then he could have mimicked what she did. Instead, his activities reflected curiosity about the discrepancy between what he counted and what he knew to be the correct count and an interest in crafting a change to his discourse such that it counted 8.

In contrast to engaged learning, the other kinds of learning did not demonstrate as many activities in common with autonomous learning. Directed, covert, watchful, and guided learners did not identify their audience as explainers or co-learners. For the most part, the learner did not communicate with him/herself (with covert learning as a minor exception). When the discourse of others was adopted, the learner did not use it to produce new discourses, so he/she did not explore the implications of the adopted discourse (see watchful learning). When the learner produced discourse, he/she did not build from or in reaction to adopted discourse (see covert learning). Instead, the new discourse was not probed or investigated using the logic of other discourses. Finally, each of the four other kinds of learning relied upon the teacher or another student to substantiate the discourse: The learners did not use their logic to determine whether the discourse described the state of affairs. In summary, as learners enacted directed, covert, watchful, or guided learning, they did not attempt to engage mathematical discourse in ways that probed what others had said or whether the discourse seemed to be true. The discourse was not the object of activity.

The Importance of the Problem and the Problem of the Teacher- Approved Answer

The advantages of autonomous learning for the development of mathematical discourse might be linked to the problem the learner worked to solve. As I noted above, as Jakeel enacted engaged learning, he was working to solve a problem of his own construction: how to alter his counting and pointing so that he could count to what he knew was the true answer. In contrast, for each of the other kinds of learning, the problem faced by the learner was determining the teacher-approved answer. As Minerva enacted covert learning, she was working to generate an answer to the task that was acceptable to the teacher. As Jakeel enacted directed learner, he was working to write what Rebecca dictated in order to get what he was supposed to have on his paper.

Jessica's enactment of watchful learning focused on finding the teacher-approved answer as articulated by Minerva and Bonita. Finally, Jessica's problem as she enacted guided learning was attaining the right arrangement of pieces on her paper. For each of these kinds of learning, the problem of getting the teacher-approved answer did not focus the learner on exploring or evaluating discourse for truthfulness.

Focusing on the problem of getting the teacher-approved answer provides limited opportunities to enact the mathematizing and identifying of autonomous learning. I will elaborate two reasons for this. First, autonomous learning emphasizes exploration of discourses. As Jakeel enacted engaged learning, he worked to change his discourse. He listened to and incorporated Rebecca's explanation and he experimented with his discourse until he was satisfied. In contrast, as the students in this study enacted directed, covert, watchful, and guided learning, they did not investigate discourses. The students watched or listened to others to determine the correct answer, but they did not ask for or offer explanations. They did not question whether something was true, but instead relied upon the teacher or another student to substantiate or correct their work. As students focused on the teacher-approved answer, it limited their mathematizing so that they were not engaging in the autonomous activity of exploring discourses.

A second contradiction between autonomous learning and seeking teacher-approved answers is the reliance upon others to evaluate the truth of discourses and to pose problems. Piaget (1932/1960) described perils of reliance upon others, what he called heteronomy. When a child obeys others, he/she does not seek truth or question what others say. Instead, he/she interprets the directions and follows the directions using his/her own perspective. In order to decenter or become less egocentric (and more autonomous), the child must not blindly obey, but instead must determine truth for him/herself. This requires examining and trying on others' perspectives in order to realize the multiple ways in which a situation might be viewed. The students in my study are in similar situations to Piaget's child. As they focused on teacher-approved answers, they followed the directions and cues of others and did not seek truth for themselves. They did not use the task as an opportunity to explore other perspectives or decenter. While in some instances they were able to adopt the narratives of others (for example, Jessica's watchful learning), they were unable to use these narratives as a tool for themselves. Thus, while the discourse of the learners changed, their reliance upon others to indicate the teacher-approved answer seemed to result in changes that were not as mathematically desirable or as useful to the learner as the changes in the discourse of the most autonomous learner (Jakeel as he enacted engaged learning).

The quest for the teacher-approved answer also meant that students did not pose problems for themselves. They accepted the problem presented by the teacher. Brown and Walter (1990) suggest that problem posing is central to learning mathematics. Not only does the act of posing a problem allow the learner to decenter, it also contributes to the sense of that the learner is in charge. The learner is not a passive recipient of orders and directions: He/she chooses what to learn. This identifying matches the identifying of autonomous learning. As the learner determines what questions to ask, he/she positions him/herself as curious, as someone who investigates discourses, and as a member of his/her own audience. As learners pose their own problems based upon their observations of mathematical discourse, they may engage in more autonomous investigation of discourse, which could contribute to the development of the learner's mathematical discourse.

My data offer illustrations of this claim. As Jakeel enacted engaged learning, he pursued the problem of changing his discourse. The teacher initiated the problem by asking Jakeel to count the area of Figures K and J. However, the teacher did not point out that Jakeel's count of Figure J was problematic: Jakeel realized this as he counted and he began to work on changing his count without prompting from the teacher. Rebecca and Daren supported Jakeel in his identifying and mathematizing. They offered explanations but not solutions, allowing Jakeel to invent his solution. As Jakeel pursued this problem he posed to himself, he investigated his mathematical discourse. His peers were supports, but did not tell him what do what or whether his investigation was done. In the end, he changed his discourse so that it was more mathematically desirable. Furthermore this change in discourse was not temporary: Jakeel used this same discourse in a subsequent lesson on area.

In contrast, when Jakeel and the other students enacted learning activities that focused on the right answer, they identified as more reliant upon others and less in charge of their learning. They needed the teacher and their peers to approve their work. Their teacher and peers concurred with this identification and served as judges of whether students were done. The learners also relied upon their peers to supply them with the teacher-approved answer, which limited their mathematizing. For example, Jessica initially identified as a problem poser: She asked Minerva why she was cutting another copy of the figures (Chapter 7, Excerpt 8). However, Minerva and Bonita rejected Jessica's attempt to position herself as responsible for investigating what didn't make sense and to position Minerva as someone who would be helpful. Rather than insist on pursuing her question, she became more passive and accepted the problem of getting the right solution to the teacher's task. As she watched Bonita and Minerva, she looked for the teacher-approved answer and did not ask questions or explore ideas. She did not indicate that she was in charge of her learning and she did not choose what to change about her discourse. As a watchful learner, Jessica's mathematizing was limited: She did not have the opportunity to explore Bonita and Minerva's discourse and they did not explain their thinking. In the end, the change in Jessica's mathematical discourse was not as desirable or as useful to her as the change in Jakeel's discourse. As Jessica's watchful learning illustrates, as students enacted kinds of learning that focused on the teacher-approved answer instead of on posing their own problems, they relied more on others, which resulted in different mathematizing and different interactions with peers and the teacher. In the end, the changes in their mathematical discourse were not as mathematically desirable and, as I will discuss below, were not tools they could use to generate or solve future mathematical tasks.

This study is not the first to note that student learning is less desirable when students focus on getting the teacher-approved answer. However, this study elaborates two points about students and teacher-approved answers. First, students are not always focused on the teacher-approved answer. Jakeel, Jessica, and Bonita illustrate this point. Jakeel switched between engaged learning, which was not focused on teacher-approved answers, to directed learning, which was entirely focused on the answer. At the beginning of the lesson, Jessica and Bonita both enthusiastically explained how two triangles made a square. Later in the lesson they were both focused on the teacher-approved answer.

These shifts between mathematizing for the teacher-approved answer and mathematizing to explore and explain suggest that shifting students from a focus on teacher-approved answers to a focus on exploring discourse may require as little as providing situations which support students in the mathematizing and identifying activities they already know.

A second point is that providing students with the right answer might create an opportunity for mathematical exploration and thereby promote rather than discourage autonomous learning. Jakeel knew that the squares in Figure J should total 8: Rebecca had already counted both J and K and arrived at 8 for both figures. Jakeel counted 8 when he counted Figure K and he expected Figure J to have the same count. Jakeel was not working to figure out the teacher-approved answer, he was working to figure out what he needed to do differently to arrive at the count he knew to be correct. Confirmation of the right answer might have helped Jessica as well. Creating the final arrangement of the pieces for Figure I did not help Jessica understand the area of the figure. If she had been shown this arrangement and then asked to explain how it related to the original figure and the area of the original figure, she might have connected the picture to the discourse and explored the discourse in ways that could have made it a tool for herself. Teachers and mathematics educators have sometimes dealt with students' focus on the teacher-approved answer by refusing to disclose their answer, believing that if they don't emphasize the answer, students will need to trust their own logic and problem solving skills rather than the authority of the teacher or the textbook. However, learning mathematics involves developing a mathematically desirable discourse. Having the right answer might provide students a means of evaluating and learning to trust their logic and discourse (as it did for Jakeel) and might raise questions and curiosity about mathematizing rather than foreclosing opportunities to think.

This discussion suggests that problem posing is entangled with identifying and mathematizing activities and that it is a critical activity for autonomous learning and perhaps for the development of mathematically desirable discourses. In particular, it seems that focusing on the teacher-approved answer prevents the identifying and mathematizing activities of autonomous learning. Many mathematics educators have lamented the negative consequences of student's attention to right answers (e.g. Bishop, 1991; Brown & Walter, 1990). What this study contributes is analysis of the connection between seeking teacher-approved answers, identifying, mathematizing, and changes in discourse.

Discourse for Whom?

The lens of autonomous learning highlights variations in the discursive outcomes of learning activities. The five kinds of learning activities resulted in three different kinds of discourse: discourse-for-others, discourse-for-the-learner, and the learner's own discourse. In my discussion above of Jakeel's enactment of engaged learner, I described how engaged learning resulted in a discourse-for-the-learner (a discourse-for-himself). Ben-Zvi and Sfard (2007) describe a discourse-for-the-learner as the result of the learner's efforts to critically examine others' discourse. As a result of this examination, the learner can use the discourse to communicate with him/herself as well as with others using the discourse. They can also use the discourse to solve their own problems (Sfard, 2008). These are the outcomes of autonomous learning. Jakeel's final discourse, resulting from his enactment of engaged learning, met these criteria.

In contrast, a discourse-for-others is not a tool for the learner, but instead is used in ritualized communication with others to whom the discourse makes sense (Sfard, 2008). Jessica's enactment of watchful learning and guided learning produced such a discourse, as did Jakeel's enactment of directed learning. Finally, as I described in Chapter 6, Minerva's covert learning resulted in her own discourse. It had peculiar routines and narratives that did not match the discourse of anyone else.

These students also had differences in interactions with peers as they enacted these different kinds of learning. As Jakeel enacted engaged learning, he was the most interactive. Rebecca and Daren offered evaluations and explanations. Jakeel responded and contributed his ideas. Jessica's enactment of watchful learning was somewhat less interactive. While she carefully watched Bonita and Minerva, she had limited opportunities to talk about mathematics with them. At the opposite end of the continuum from Jakeel's engaged learning was Minerva's covert learning. Minerva actively worked to limit interactions with her peers. One well-discussed example is her dismissal of Jessica's question about what she was doing.

This continuum of interactions correlates with the spectrum of discursive outcomes, with the discourses connected to more interactions being more mathematically desirable and more useful to the learner. Piaget offers an explanation for this. He (1932/1960) suggested that learning with others was essential to autonomy: By co-operating with others, children learn to decenter, or take on others' perspectives, making their own thinking less egocentric (Kamii, 1994). As students decenter and become less egocentric, they are able to critically examine their own thinking and to use the ideas of others. Thus as Jakeel enacted engaged learning, he considered the ideas of Rebecca and

Daren and received their input on his ideas. His final discourse was one that he could use to successfully communicate with himself and others. It was a discourse-for-himself.

Jessica's one-sided interactions as a watchful learner allowed her to take on the discourse of others but not to use or critically evaluate it. She developed a discourse-for-others. Finally, as Minerva enacted covert learning, she did not explain her ideas and she limited the explanations of other students. She had few opportunities to decenter and critically examine others' discourses. Her final discourse was meaningful to her, but not to others. She developed her own discourse. Piaget's theory suggests that if Minerva had been able to identify as wanting to learn from/with others, she might have engaged in mathematizing with others that explored discourses and provided her with the opportunities to develop her discourse as a discourse-for-herself.

This examination of Minerva's independent work and development of discourse bears some similarities to Stanley Erlwanger's (1973) well-known study of Benny's mathematical learning. Benny's sixth grade mathematics curriculum was Individually Prescribed Instruction (IPI) Mathematics. Students worked individually through the curriculum, solving practice exercises. When a student felt he/she had mastered the material, he/she could take the accompanying test. If 80% of his/her answers were correct, the students could proceed to the next material. By the standards of this curriculum, Benny was a successful student. He had completed more units than many other students in his class. He worked independently and when he experienced difficulty with material, he was able to determine the pattern for correct answers and change his answers. However, Erlwanger found that Benny's explanations of mathematical concepts were not mathematically appropriate. Benny had generated some interesting and highly unconventional routines for a number of mathematical processes. Erlwanger suggested that Benny's troubling success in this program was a result of the program's focus on right answers rather than on the individual's process. An examination of Benny's situation through an autonomy lens would concur that the focus on right answers is problematic (as I discussed in the section above), but an autonomy lens also suggests that the individualized nature of the curriculum was also problematic. The curriculum was not structured to allow students to mathematize with each other and they only mathematized with the teacher if they couldn't get right answers. Benny did not discuss mathematics with others. He had no opportunities to evaluate other's discourse or examine other perspectives. He did not decenter and, like Minerva, he developed his own mathematical discourse. Erlwanger's case of Benny contributes to the evidence from Minerva's covert learning that autonomous learning is not synonymous with independent work and that student's learning outcomes can be more problematic when they don't engage the discourses of others.

Linking Discursive Features

One final feature of autonomy that played a role in each case was connections between the verbal discourse and images drawn on paper. Students' adoption, production, and substantiation of discourse is enabled by their use of visual mediators to support their use of words. Sfard (2008) notes how one student was able to use the visual image of a table to support himself in solving a function problem even as he confused the words *slope* and *intercept.* Sfard also notes that the visual image can serve as a powerful tool for supporting students in generating new narratives. Jakeel's engaged learning illustrates this. As Jakeel enacted engaged learning, he seemed to focus on Rebecca and Daren's gestures and pictures, linking that visual to his own words, narratives, and routines. In contrast, as he enacted directed learning, he only focused on words, writing what Rebecca dictated without creating connections to images. Minerva and Jessica also demonstrated limited connections between words and images. As Minerva enacted covert learning, she worked hard on generating the picture desired by the teacher. However, she rarely gestured as she spoke and she frequently used pronouns that made the visual mediator for her words unclear. Finally, as Jessica enacted watchful learning, she mastered the narratives of the teacher-approved answer, but did not construct the matching image. Her guided learning resulted in a teacher-approved picture, but did not provide her with an opportunity to connect the picture to her words. With the exception of Jakeel's engaged learning, there were few strong links between what students said and what they indicated visually on paper. Even Jakeel's engaged learning, which did link his words with his pointing, might have resulted in more mathematically desirable discourse if Jakeel had attended more to Rebecca and Daren's use of words (such as half). These findings lead to a conjecture that is worth future exploration. They suggest that students' engagement in the autonomous learning activities of adopting, producing and substantiating discourse is facilitated by their work to link words with visual mediators.

The possible need for this connection suggests that teachers should attend carefully to both students' use of words (such as when they might use pronouns instead of nouns or fail to use nouns at all and only use counting words) and to how they link those words with visual images. Teachers might also present students with tasks that explicitly require students to connect words and images. One activity that has been used successfully in classrooms is a task in which students work in groups to demonstrate four different ways to solve a multi-digit multiplication problem. Students must then make connections across the different solutions. As students link solutions, they discover how a solution that uses an algorithm can be represented as an array, groupings of objects, or repeated additions. As students verbalize the connections they see, they necessarily use words (although they could still use vague or inappropriate words if the teacher is inattentive) to describe mathematical processes and visual representations. As they produce narratives and substantiate them as they make these links. While it is possible for students to engage in this task in unproductive ways, the task illustrates a problem that specifically promotes autonomous activities.

In my discussions at the end of each chapter, I described how my examination of the mathematizing and identifying activities enacted by the students added detail to a description of autonomous learning. Jakeel's engaged learning illustrated how students could engage in autonomous learning while learning from others. Minerva's covert learning demonstrated that enactment of identities of successful students (independence, persistence, creativity) does not necessarily result in autonomous learning. The case of covert learning also suggests the value in identifying as not knowing. Jessica's watchful learning shows how interactions among students can limit student's attempts at autonomy. Finally, the work of her peers and her teacher during guided learning illustrates how an emphasis on the teacher-approved answer can prevent students from engaging in autonomous learning even when they are given the responsibility for reasoning to that answer. Using the lens of autonomy to examine the mathematizing and identifying in each of these cases has resulted in an elaboration of autonomous learning that seems useful in evaluating and encouraging student learning.

Assumptions and Contributions to Research on Identity

In my theoretical framework, I constructed a theory of identity that described how identity could seem to be static but could also be fluid. I wanted to be able to account for how the same individual might respond in different ways to very similar circumstances. For example, Jakeel shifted from engaged learning to directed learning and back to engaged learning. Why was it that Rebecca was eventually successful in getting him to copy from her and then why did he return to engaged learning? My notion of identity as fluid allows this transition. Jakeel initially identified as capable and knowledgeable. However, when he had the wrong arrangement of pieces on his paper, Rebecca firmly asserted, based upon the incorrect solution Jakeel had created, that he was not knowledgeable. Rebecca identified Jakeel in a different way and he acquiesced. Later, when the teacher asked him to show her how to count Figure J, he realized that his counting routine was problematic. He also worked to try to change his counting and Rebecca supported him in this, identifying him as capable of changing his counting.

Jakeel's transitions in identifying activity illustrate the ways in which individuals can foreground and background different identifications. Acting as knowledgeable or as in need of direction were not new identifying activities for Jakeel. My data provide evidence that he had enacted engaged learning and directed learning in the past. He mobilized those identifications and stories as situations changed. Jessica's shift between watchful learning and guided learning reflected this same mobilization of identifications. When Jessica's attempts at questioning failed, she became a learner who watched. However, she later claimed that nobody was helping her (Chapter 7, Excerpt 14, Line 561) and identified as a learner who needed more explicit help. Jessica and Jakeel identified and acted differently at different times sometimes placing one identification forward and sometime another depending upon circumstances.

This notion of how identifying activity foregrounds and backgrounds different identifying stories has two important implications for mathematics educators and researchers. First, teachers and researchers (and students) should not assume that individual students are only one particular kind of learner. As teachers identify students in particular ways, especially if those ways are negative, they might not notice or provide opportunities for students to enact other more positive identities. They may also interpret student activities through those negative identifications such that they might respond differently to same question from different students. This more fluid notion of identity also provides opportunities for researchers to examine identity at higher resolution. Rather than account for an individual's activity over an extended period of time, researchers can examine how identity unfolds in a moment. This also opens the possibility for examining how interactions foreground certain identities. For example, Rebecca was able to identify Jakeel as needing to copy by telling him that he didn't do what he was supposed to do. She could have told him that he needed to rethink his answer or she could have asked him to explain his thinking. Instead, she insisted he didn't know what to do.

A second implication of this notion of foregrounding and backgrounding identities involves identities that such as race, ethnicity, gender, and class. These social identities are frequently intertwined with academic identities. For example, for some women, at some moments, their gender is salient in their mathematical activity as they worry about whether others think they are as capable as the man sitting next to them in math class. This worry may prevent them from asking questions or engaging in conversations about mathematics. However, at other times, gender may be backgrounded. For example, a woman who is identified as good at explaining a problem may foreground this academic identification and background her gender. Thus it is important to recognize how students might foreground or background these social identities, sometimes allowing them to constrain or enable their mathematical activity.

In the three cases I present, I did not provide examples of student's identification as black, white, boy, or girl because my data for this lesson included only one instance in which students were engaged in mathematics and specifically invoked a racial or gender identity: Toward the beginning of the small group time, Jakeel told the teacher that he had not yet "cutted" out Figure H (Line 212). Daren teased him about his use of "cutted" and Rebecca smiled. Jakeel turned to Rebecca, "(I see you Rebecca you smile) what's so funny? A black brother can't talk?" (Line 216). As this example illustrates, students' identifying activities may specifically acknowledge race. In this moment, Jakeel foregrounded his racial identity. However, *he* did not tie it to his mathematizing, but instead seemed to use it to engage Rebecca (and not Daren, who was the one who was most active in teasing Jakeel). I do not mean to imply that Jakeel was not indicating his race through his use of "cutted", but that until Daren teased him, he seemed to be backgrounding his racial identity.

I could have examined my data through a lens that specifically examined discourse for features (such as the word "cutted") that would indicate race, gender, ethnicity or class. However, in addition to the lack of data indicating foreground of these social identities, I also lacked data that determine what it might mean to an individual student to identify using one of these social categories. Different students might draw upon different narratives of race, ethnicity, gender, or class to guide their identifying activities. Thus, what one student might see as acting black might be understood in a very different way by another student. Jilk's (2007) research on Latina immigrants supports this claim: She described both the diverse identities and the diversity in narratives associated with one identity label used by her participants. In order to analyze my classroom data for these features I would need to collect that that would allow me to specifically notice how each student narrates these identities.

It is important to recognize how students and others foreground race, gender, ethnicity and class in order to avoid situations in which these identities are foregrounded in ways that limit mathematizing. For example, McGlone and Aronson's (2006) studies of stereotype threat suggest that asking students to identify their race and gender at the beginning of a standardized test may foreground these identities in ways that affect the student's performance on the test. I suggest that we need to carefully examine learning situations to see whether and how students are foregrounding race, whether there are other factors that are bring race to the front for students, and how that foregrounding affects their mathematizing and learning outcomes.

Different Kinds of Learning

One significant contribution of this study is the description of different kinds of learning. These kinds of learning are useful because they offer vocabulary for describing learners beyond the broader categories of smart/struggling or math person/not a math person. These categories run the risk of identifying students in ways that make the labels seem like inherent traits of students. Smart becomes who the student is, rather than a word describing the student's activities (Sfard & Prusak, 2005). By describing learning in terms of student activity, we avoid labels like smart, slow, struggling and instead focuses on how student activity (including interactions with others) connects to changes in mathematical discourse. Identifications become transient outcomes of identifying activity rather than predictions of necessary future activity. By focusing closely on and categorizing learning activity, this study can propose changes in activity that might lead to changes in learning outcomes.

One limitation of my construction of kinds of learning is that it came from close observation of a few students. I cannot make claims that other students will enact the same cluster of identifying and mathematizing activities. Indeed, the complexity of individuals and interactions strongly suggests that a careful analysis of many students in many learning situations would identify many different kinds of learning.

More investigation across more individuals and more activities may reveal broader categories of kinds of learning. For example, my analysis could be used to sort student activity into autonomous and nonautonomous learning. These categories could be helpful in focusing on how student activity supports the development of desirable mathematical discourses.

Reform and Traditional Mathematics Pedagogies

This discussion would be incomplete if I failed to address the implications of this study for reform mathematics pedagogy and for the debate between reformers and advocates of more traditional pedagogies. I recognize that the terms *reform* and *traditional* may be used in different contexts to emphasize different mathematical tasks.

As I use these terms, I draw upon Suzanne Wilson's (2003) distinction between the pedagogies. She describes reform math pedagogies as

> based on a Dewey-ian conception of the child and curriculum as two sides of the same coin. This position places emphasis on process as well as content, on the child as well as the curriculum. It also places emphasis on students and teachers working together, and on teaching "higher-order thinking" or "conceptual understanding" as well as the basics (pp. 17-18).

Traditional math pedagogies use "more teacher-dominated instruction, more focus on skills and mastery of the basics, and more emphasis on a "canon" of legitimate knowledge" (p. 17).

This study addresses two concerns of reform pedagogies: student interactions and autonomy. Reform pedagogies make extensive use of student talk. Ostensibly, interactions among students promote understanding of mathematics as students propose and argue about mathematical ideas, reason through conjectures and alternative solutions, and evaluate the thinking of others (National Council of Teachers of Mathematics, 1991, 2000). However, students don't always learn mathematics when they work with and talk with others: The theoretical promise of interaction does not consistently result in the desired outcomes. Determining what factors might be implicated in the success or failure learning through interactions requires a framework that illuminates the complex cognitive, affective, social, historical, and cultural elements that might come into play in an interaction. Using discourse to consider mathematizing, identifying, and learning outcomes provides a lens for coordinating the study of these complex factors. For example, the lens allows examination of how a student in an interaction might simultaneously communicate social information, mathematical content, and emotion.

The findings in this study confirm that interactions are implicated in learning. Students who interacted more had more desirable changes in their mathematical discourses. However, some interactions were also problematic: Students sometimes foreclosed the mathematical conversations of others by refusing to participate or by insisting on directing the learner's activity. These findings are not new to the field: Many researchers have examined interactions among students and teacher's struggles to promote productive interactions (e.g. Cobb *et al.*, 1997; Cohen, 1994; Hufferd-Ackles *et al.*, 2004; Lampert, 2001). Some of this research has adopted the stance that students need to be taught how to interact in productive ways. For example, Yackel and Cobb (1996) suggest that students can (and should) be taught how to evaluate each other's mathematical contributions. While teaching students specific ways of mathematizing and identifying surely leads to desirable changes in mathematical discourse, my study suggests that we might first consider encouraging the mathematizing and identifying activities that students already bring to mathematics lessons. For example, each student in these two groups demonstrated productive mathematizing and identifying activities: Jessica explained her thinking and asked questions, Rebecca and Daren explained their ideas in ways that promoted Jakeel's thinking, Jakeel asked questions and explored discourse, Minerva examined her third solution and changed her writing, and Bonita explained her thinking. Perhaps a first step in promoting a reform classroom is helping teachers and students recognize ways in which they already act and interact that support learning mathematics.

That last sentence was easy to write and given my time spent in analyzing this classroom data, I might feel comfortable noting productive actions and interactions.

However, as many very competent teachers have demonstrated, making sense of what kinds of activities to promote in classrooms can be a difficult task. What can be especially confusing for teachers enacting reform pedagogies is their role in encouraging the development of autonomy in their students. Many mathematics educators have noted that autonomy is a central goal of mathematics education (Ben-Zvi & Sfard, 2007; Kamii, 1994; National Council of Teachers of Mathematics, 2000; Warfield *et al.*, 2005; Yackel & Cobb, 1996). However, as I have noted earlier, what constitutes autonomous learning has been poorly defined. It is frequently discussed in ways that connect it to student independence. Some teachers interpret this construction of autonomy to mean that they should design lessons in which students work to discover math without teacher guidance. Learning without the teacher becomes possible when students learn from each other in small groups. Thus the reform notions of student interactions and small group work lead to an enactment of autonomy as students learning without the teacher. However, as Ben-Zvi and Sfard (2007) note, because mathematics is socially constructed, there are some aspects that can only be learned from someone who is already fluent in the desired mathematical discourse. Teachers must be involved in modeling the expert discourse students should adopt.

This study proposes modifications to the conception of autonomy as independence and elaborates what student interactions (including interactions with the teacher) are essential to autonomous learning. As I noted in my analysis of Minerva's covert learning, autonomous learners are not learning by themselves. Nor are they focused exclusively on finding the teacher-approved answer. Instead, students enact autonomous learning when they choose to work on a problem arising from their observations. Jakeel demonstrated this as he worked to change his counting. He did not do this work without support: However, the support was not a demonstration of what he needed to do nor did Rebecca coach Jakeel toward one specific way in which he should change his gestures. Instead, Rebecca provided discourse Jakeel could adopt and she gave him feedback on his work. In addition to providing examples of autonomous activity, this study has elaborated a definition of autonomy and a list of mathematizing and identifying activities that might help teachers conceptualize, promote, and analyze autonomous learning activities.

This study's emphasis on student interactions (along with the researcher's own predilections) indicates a bias toward reform pedagogies over traditional pedagogies. It addresses some of the concerns proponents of traditional pedagogies voice about reform pedagogies. For example, some traditionalists are concerned that reform math requires students to discover mathematical concepts without guidance from the teacher (Mathematically Correct, 2005). I have shown that guidance from the teacher is essential to learning mathematical discourse. However, it is also essential that students pose their own problems about mathematical discourse rather than only seeking teacher-approved answers on teacher-assigned tasks.

This study also suggests that reform pedagogy can be hard to enact: not all student interactions are productive and not all enactments of autonomy have supported student learning. Furthermore, it is possible for students in traditional classrooms to act autonomously. They might be curious about the discourse the teacher is using. They might wonder about and investigate patterns across multiple problems they've worked. However, traditional pedagogies often fail to encourage such curiosity and to promote interactions that encourage students to decenter. If learning mathematics requires students to develop a more mathematically desirable discourse, students must be allowed and encouraged to engage in mathematizing and identifying activities that feature examination of mathematical discourse. This means that the traditional focus on the mathematics canon is appropriate as is the reform focus on interaction and autonomy.

Beyond Area and Beyond Math

Because this is a case study of student learning across a lesson, the scope of the lesson content is necessarily narrow. My interest in mathematics education has led me to focus on mathematics. However, my focus in this study on the mathematical concept of area resulted from the possibilities presented by the data I collected, rather than from a essential connection between my theoretical framework and area. Focusing on area was useful because it necessarily contains a visual component, which allowed me to see how students were (or were not) making connections between their words and the possible images. However, my theoretical framework is not tied to any specific math content. Presumably, I could have conducted my analysis on a lesson from one of the other math units (geometric shapes or probability) from which I collected data and potentially find connections among mathematizing, identifying, and learning outcomes.

There is also nothing in my theoretical framework that limits it to exploration of learning mathematics. While the framework is founded on research describing mathematical contexts, the assumptions underlying the framework could apply to other content areas. For example, the notion that the outcome of learning is a change in discourse and that the activity of learning arises from discourse about content and learners could apply in science or history classrooms. Indeed, connections between discourse about the content and discourse about the learners might be more explicit in content that focuses on human thought and culture literacy or history. For example, Wortham's study of social identification and academic learning (2006) examined a class that was a joint English and history class. Wortham examined how the social categories available in the curriculum were used to identify students. Wortham used a larger time scale than I used in my study: He considered learning and identifying across a year and considered how students learned content through their use of the content to identify other students. He also had a different construction of learning: the use of "new combinations of resources to represent or react productively to subject matter across events" (p. 105). It would be possible to examine Wortham's data through my theoretical framework to see what connections might be found across discourse about humanities, identifying, and outcomes of learning on a smaller time scale.

Limitations and Future Research

I designed this project as a case study in order to explore what seemed to be complex connections between mathematizing, identifying, and changes in discourse. As I described in my methods chapter, this study design allowed detailed probing of student activity, but it only allowed detailed probing of the activities of three students during one lesson. Certainly there is more that could be learned through additional case studies, studies that consider a range of mathematical content, and studies that connect these theories to larger populations. In the paragraphs below, I describe three specific limitations of this study and how they could be addressed by future research.

First, this study lacked examination of highly successful learning activities: None of the three cases presented learning outcomes that entirely matched the desired learning outcomes. While these three cases were interesting and helpful in examining connections across mathematizing, identifying, and change in discourse, their stories could be enhanced by comparison to a situation in which the student's learning activities resulted in a more mathematically desirable change in discourse. For example, during the lesson with Jakeel, Rebecca and Daren, I noted that Daren had figured out that the triangular spaces were half of the square spaces. This seemed to be a significant (and notable to the students) change in discourse. It would be interesting to see what activities surrounded this change. However, Rebecca and Daren were not focal students and I did not collect data on them during the previous lesson. Examining a few lessons with more successful outcomes could enrich my findings.

Another limitation of this study was the narrow time. Studying one lesson allowed me to notice and investigate detail. However, it leaves unexamined a question I have about my framework. Both Nasir (2002) and Wenger (1998) suggest that learning involves a change in identity. This seems quite reasonable given my framework: If learning is a change in discourse and identity is discursively constructed, then one outcome of learning could be a change in discourse. I hoped to examine this in my data, but I was persuaded that one lesson was too short a time to expect students to construct new stories about themselves and others. While I did see changes in identifying activity, I was persuaded that these changes reflected foregrounding and backgrounding of identities and not construction of new identities. It would add complexity to the framework of mathematizing, identifying, and change in discourse to examine changes in discourse that reflect new stories about individuals. This study would require data collection over a more extended time frame in order to capture the perhaps more gradual process of students writing new or modifying existing stories about themselves.

Finally, in this chapter and in the findings chapters, I make several suggestions to teachers about what they might do differently to encourage autonomous learning. These suggestions are based upon my three detailed but limited cases, which means both that they may not work in other situations and that much could be learned by examining the implementation of these suggestions. I propose that a design-based research project (Design-Based Research Collective, 2003) would provide an opportunity to enact these suggestions, study the outcomes, and modify both the theory and the teaching suggestions. Engaging in a research cycle that allows implementation, investigation, and modification would help strengthen the connection between my theoretical framework and practice.

Conclusion

This study raises questions about how and whether the kinds of learning I elaborated might be enacted by different students in a different setting and about how my suggestions for teachers might affect student interactions and activity. However, the study has also answered questions I've had about making sense of student activities and interactions. I especially wanted a tool that would help me account for differences in student activity such as Dquan's copying and his later investigation of mathematical operations. Examination of discourse for mathematizing, identifying, and learning outcomes (changes in discourse), I was able to suggest ways in which the intertwining of mathematizing and identifying link to student activity and changes in discourse. I was also interested in examining the differences in moments when students might choose to copy or when they might choose to examine discourse in order to evaluate what student activities and interactions might encourage mathematically desirable changes in discourse. The development of autonomous learning and its constellation of identifying and mathematizing activities provided a tool for highlighting more productive student activity and suggesting what activity and interaction to encourage during mathematics lessons.

This study could be seen through a pessimistic light: The students demonstrate changes in discourse, but none of the students demonstrates all of the features of the mathematically desirable discourse. Each of the students enacts one or more kinds of learning that is far removed from autonomous learning. However, I see cause for optimism: The lens of mathematizing and identifying and the special case of that lens, autonomous learning, allows teachers and researchers to see how undesirable learning outcomes are not inherent traits of students but instead arise from a predominance of nonautonomous learning activities and interactions. Each student also demonstrated autonomous learning activities, which suggests that success in learning mathematics does not require overcoming cognitive deficits or teaching students entirely new learning skills. Instead, we might find that by seeking, encouraging, and designing for moments in which students display curiosity, we can support students in enacting more activities that connect to mathematically desirable changes in their discourse.

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