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SHARON K. STRICKLAND

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RHYME AND REASON:
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By

Sharon K. Strickland

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ABSTRACT

RHYME AND REASON: A RHETORICAL, GENEALOGICAL EXAMINATION OF UNDERGRADUATE MATHEMATICS

By

Sharon K. Strickland

This is a rhetorical and genealogical examination of four undergraduate mathematics courses: Real Analysis, Advanced Geometry, Complex Analysis, and Discrete Mathematics. The framework of the study is rhetorical because I read course artifacts from the perspective of literary analysis. The framework is also genealogical because I was interested in the construction of subjectivity as performed in each of the courses. To explore these issues I observed the four courses over a semester, looking at the ways students and professors interacted, how mathematics operated in the courses, and how supporting texts and documents such as textbooks, syllabi, course web pages, and course handouts contributed to the visions of teaching and mathematics presented in each course. This research presents an analysis of the construction of the subject (where the subject is undergraduate mathematics students) in these four courses to highlight the differences in pedagogical and mathematical approaches across them.

Epistemologically, the study takes a Foucaultian discourse position in which math is what math does. The approach of this study rejects a more realist stance in which ‘real’ mathematics exists beyond material practices. Instead, the study allows for multiple historical and temporal instantiations of mathematics, in which the subject of mathematics (both student and discipline) is constructed differently in each of the different settings.

This dissertation is a response to common stereotypes of mathematics courses as teacher-centered lectures, sites of content knowledge delivery rather than pedagogical experiences for students. This study takes the position that there is no pedagogy without content, and there is no content without pedagogy. The study also responds critically to assumptions that conventional lecture-based mathematics courses tend to be taught in similar ways. Even though three of the four courses examined here were lecture based, there was always more than lecture present, and even the lectures constructed different possibilities for subjectivity.

To analyze the construction of the mathematical subject, I used a four-part Foucaultian framework that asked: 1) What aspect of the students needed to change (substance), 2) How did the course invite students to take on these changes (mode), 3) How did the activities (or regimen) of the courses act to initiate that change, and 4) What a model or perfect outcome of the course might look like (telos).

This study concludes with a consideration of issues of inclusion and exclusion. It suggests that mathematics is practiced differently across multiple courses and that these differences allow for varied opportunities for students to engage with mathematics and to be excluded from it. Mathematicians and teacher educators can benefit from awareness of assumptions that shape subjectivity in mathematics courses, and such awareness can help increase access to various types of mathematical thinking.

DEDICATION

For Brad,

Who egged me on,

And for Little Juice,

Who merely threw her eggs.

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I am beyond grateful to Lynn Fendler who inspired me to take on this project and helped me immensely as I fumbled my way through the theory and writing. She had faith when I did not. She also had chocolate when I did not. Thank you Amy Parks, Steve Tuckey, Brett Merritt, and the rest of the Critical Studies group for your friendship and intellectual support all through this program. Your often irreverent and usually hysterical observations sustained my interest in post-modern thought and kept me sane. Your shared knowledge of science, mathematics, philosophy and life shaped this dissertation.

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INTRODUCTION

My plans have changed often since high school when I first declared to teachers, family and friends that I wanted to be a literary critic and thus an English major. But to their (and even my own) shock I emerged from college with majors in mathematics and classical languages. Despite my elaborate personal goals, I continued to find new interests to thwart my previous plans at each next step in my education. Shortly after college I began to teach and quickly became intrigued by the world of education. This led to a master's degree in secondary education and a minor in mathematics. Still not satisfied and wanting to learn more I began a Ph.D. program anticipating a focus on ways to make teaching mathematics better and so improve students' learning. While taking a seemingly innocent (in my opinion, at least) class during my second year I encountered new ideas that would soon shift my academic interests yet again—Foucault and postmodern thought. This in turn led to a desire to learn more philosophy in general. Fortunately, amid my many academic meanderings and seeming digressions, analysis and commentary (philosophical pursuits, both logical and rhetorical) continue to delight and so anchor much of my work.

This dissertation takes my major academic interests into consideration—mathematics as the discipline, Foucault (postmodern) and Aristotle (classical) as the thinkers supporting the theoretical aspects, and education and teaching as the major site of the research. These varied interests can combine to inform different aspects of the project, as I will elaborate.

When I began this project my primary concern was to learn more about the mathematics classes undergraduate students encounter. My overarching research questions were: *What kinds of mathematics teaching do preservice teachers witness in upper level mathematics department courses? What could preservice teachers learn about teaching in these courses?* And just as my well crafted plans for my life—to become a literary critic—changed, so did my thoughts about research and the sort of study I wanted to write.

At first I was interested in what the students in the classes were learning about teaching by taking discipline-focused, rather than education-focused, courses. Specifically, I was interested in the practices that the students in upper-level undergraduate mathematics courses might witness and therefore might (consciously or unconsciously) adopt as future practices in their classrooms. To learn more about the practices witnessed I observed four upper-level undergraduate courses that were either required by the math department major program or that were often taken by preservice teachers. Because the students in the teacher education program for preservice secondary mathematics were required to earn math majors, I had plenty of reasons to suspect that at least a few future teachers populated each of these upper-level courses. The preservice teachers were my main focus—what could they be learning in their math courses about teaching? But also, some of these students might go on to become graduate students in mathematics or related fields and so might teach undergraduates one day themselves. They too, then, might be learning from these experiences in their undergraduate courses and might apply them to their future teaching. They might be building models of practice. Finally, some other portion of the remaining students (those who were not preservice teachers or who might not become future graduate assistants, instructors, or professors of mathematics) might become parents, or community leaders, or businesswomen who come to care about education even if education is not their primary field. Their expectations of education for their future children and for students in general could be related to the experiences they witness in these upper-level undergraduate courses. I had built a whole net of causal relationships and possibilities on the backs of a few semesters of observation.

It is not that I no longer believe that these experiences have the potential for informing future teaching decisions. I still believe that what anyone experiences informs what he or she takes to be normal and abnormal, positive and negative, and desirable and undesirable. Instead, I can no longer think of the relationships as linear or as causal.

My belief in the notion that what a professor does directly influences what a student will do a year or more down the road of life is problematic for me. On the one hand such a model of the relationship makes designing a classic research study very difficult. How could my observations of classrooms speak to what the students would go onto do in their lives? With the limited time and funds of a dissertation I could not take up the longitudinal study such a model would necessitate. I would not be able to identify a cause-effect relationship. Instead I could describe the courses and hypothesize about their future possibilities. This could be done and I do think this sort of work could still inform researchers of teacher education. But more than the difficulties of research design, methods, and the limitations of conclusions from that design, my problems are primarily ones of power and knowledge.

As I mentioned before, I had taken a class in my second year of this graduate program that began to unravel some of the ways I think about power. In that course I came to read about Foucault's notion of power. That it is not a fixed resource that some wield and to which others bend. "power is neither given, nor exchanged, nor recovered, but rather exercised, and that it only exists in action" (M. Foucault, 1989/1996, p. 89, p. 89). It is not that professors have power and students do not. Professors do not do things that, voilá, determine what students learn. Everyone and everything in a course sets up a network of possibility and power relationships that change and move around. And so my change in this dissertation and my analysis of my observations was to describe the ways that the course and all of the parts and persons that make up the course come to construct a subject through discourse. In particular my analysis describes the construction of a model or model student. I have abandoned the idea that students are learning in the course in the sense that they are consuming information and experiences directed by professors. Rather, students, professors, course documents, rows, desks, chalkboards, computers, and textbooks work together to construct a model student.

The guiding question of my revised research project is: *What assumptions about students and mathematics are embedded in the pedagogical milieu of undergraduate mathematics courses?* For the purposes of analysis I have reframed my guiding question into one about the construction of the subject where the subject is the undergraduate mathematics student. In what will follow I will describe the courses and the ways in which these courses construct different model students. What sorts of things does the course assume students need to change? That is, I take a course as an opportunity to change some aspect of those in the course. What then does each course try to change?

How does the course invite students to take on these changes? What are students asked to do in the course? From the answers to these questions I then describe what a model student emerging from each course would look like, more specifically, what a model student would look like that emerged from a perfect realization of each course.

I recognize that a model student is not real. He is not the same thing as a real student that made an A in the course for example. The model student is a fiction of the course. A way of expressing what perfection would look like if it were possible. That model students are not real does not make them any less important. Take for example a Barbie doll. There is no real Barbie woman, but many people would argue that Barbie matters despite her plasticity. They would argue that Barbie's existence in the discourse still has effects on how girls and women think about perfection and themselves. Thinking about a model can also help us better understand the different values and orientations of the community that participates in that discourse. In the case of Barbie, the values and orientations and priorities would be of American culture and those who participate in that discourse. In my study the discourse is that of the undergraduate course.

This dissertation is an examination of what is said to students and about students. What is said about them out loud and what is said about them in text. What is assumed about them from course documents, patterns of interactions, and types of experiences initiated in the courses. It is not a dissertation about the students themselves, or even about the professors although each are described. This dissertation is about the stories of mathematics, teaching, and students told by the courses.

My aim in this dissertation is to speak to several ideas about undergraduate mathematics: that undergraduate mathematics courses look largely alike and share common goals about proofs; that what evolves in these courses is natural, predetermined, or expected; that because the courses are housed in the same departmental program that they assume the same things about a model student; and that what we in teacher education often think of as a dichotomy between content and pedagogy is far more interrelated.

Before delving more deeply into the details and theory behind this project let me quickly describe the work I did towards answering my original research questions. In a span of two academic years (both in the fall) I observed the four courses described in this dissertation. Each was at the 300 or 400 level, and due to the content, these were courses mostly populated with majors in the field. Of course, although students of other fields might enroll in these upper division courses, they are mostly the habitat of mathematics majors. While in these classes I took field notes focusing on teachers and students and the interplay between the two. In addition to the observational nature of the research I also collected course handouts and textbooks as well as course websites when they existed.

The artifacts gathered for my initial research questions, *What kinds of mathematics teaching do preservice teachers witness in upper level mathematics department courses? What could preservice teachers learn about teaching in these courses?* did not change, but rather my analysis and orientation to what story I would tell from the artifacts. My analytical questions were about the substance, mode, regimen, and telos of these undergraduate courses. The substance can be thought of through the question, What aspect of students (or substance) does the course target? As I will describe in more detail to come, I take a rhetorical approach to the analysis. As Aristotle described so long ago, Rhetoric is the art of persuasion. Persuasion involves change.

What aspect of the student then is the course trying to persuade students to change? The mode relates to the means by which the course enacts the persuasion. What techniques of persuasion are used? How are students invited to take up the change proposed in their substance? The regimen refers to the things students are asked or expected to do in the course. In what acts do they engage? Finally, telos is about the constructed subject, which in this study is a model student. What would a model, fictional, student who emerges from this course look like? The model student is not the best student in the course. It is not a real person at all, but an ideal, model, and perfect fabrication. In selecting these as my analytical questions I am trying to highlight the ways the courses spoke to and about students, not about the actual students themselves. Actual students were present, indeed, often 20 or more in each course. Actual students contributed to these constructions of the model, but the model student is no single one of them.

As I continue through this dissertation I address a relationship between teacher education and undergraduate mathematics, why I chose to use a postmodern analytical frame, and why these research questions can help educators think about teacher education courses for future secondary teachers.

In the first chapter I describe my project in greater detail. I begin by describing the different discourses to which this dissertation speaks: mathematics teacher education and undergraduate mathematics education. I conclude by describing my theoretical and analytical decisions.

Following this chapter there are four chapters analyzing each of the four courses, respectively. I chose to order these chapters by the course number assigned to it in the course catalog for this mathematics program for the purpose of imaging a possible sequence through the program. Of course after the first required prerequisite courses students could enroll in the later courses in a variety of orderings, but in as much as the courses are numbered in an ordering that seemed to reflect a progression from introductory to advanced, I have decided to order my chapters similarly.

Chapter two is about the Real Analysis course. Again this course was required for all majors and was considered a prerequisite to the Complex Analysis course. As will be elaborated further in chapter two, my work speaks only to the specific instance of the course observed and does not intend to generalize to other courses with the same name and number as each professor, environment, semester, and set of students constructs each course experience new again. The Real Analysis course constructed the model student as a convert to a particular vision of mathematics and the role of proof.

The third chapter describes the Advanced Geometry course, one quite popular with preservice teachers according to my general knowledge of the program and the students in it. The model student constructed in this course was a researcher who thought about proofs as final outcomes in a messy path that started with conjecturing.

The fourth chapter focuses on the Complex Analysis course, one of two 400-level courses observed. The model student constructed here was a Platonic disciple.

The fifth chapter describes the Discrete Mathematics course where the model student was constructed as a teacher.

The final chapter pulls together the themes from the four courses to describe the ways that some were similar to one another and some were very different from one another. I will argue that each course in its own way excluded types of students not in line with the model construct. I will also argue that some courses offered more accessibility, or ways that students could feel at home than others. I will also comment on the dilemma of mathematics departments that may want to either reduce or increase the diversity of visions of mathematics, teaching, and students that math programs offer.

Finally, to reiterate, this dissertation is concerned with the stories we tell about students. It is concerned particularly the stories generated by the courses, which I see as having multiple authors who exercise power differently at different times. This dissertation is not concerned with the truth of the students; who they *are.* For example, the advanced geometry chapter will speak of the substance in need of change as students' self-concepts and the complex analysis chapter's of students' discipline. This dissertation does not seek to judge whether these aspects of students' substance are indeed the best or most appropriate aspect in need of change. Readers may find that they think self-concept is a poor substance to want to change. He might think that students' skills in proofs are a better substance. Or the reader might think it a very good thing to change but may think that the mode or appeal to that change should be X rather than Y. This dissertation does not seek to say that students' self-concept *is* what is needed to change or that it *is not* what needs to change. I do not argue whether the truth of the students is this or that, or that the course got it right or wrong. This project is about the stories this particular set of courses form about students and the model students these courses construct. Simply stated this dissertation is about philosophy. In these four courses I describe I hope you will notice four different articulations of the model student and will conclude with me that answering, "what math is all about" is not quite as simple as proofs, and that philosophy matters.

CHAPTER ONE

CONTENT AND PEDAGOGY IN UNDERGRADUATE MATHEMATICS

Introduction

Teacher education for prospective secondary teachers begins with mathematically prepared students and aims to create mathematics teachers. Mathematics preparation often occurs in the mathematics departments, while education departments assume responsibility for the pedagogical aspects of teaching. However, the separation of these duties is misleading as many educators and researchers acknowledge. Teacher education courses for future mathematics teachers often incorporate mathematics preparation—particularly mathematics related to the topics and subjects common in middle and high school curriculum. Content and pedagogy merge in mathematics education courses but they also intermingle in mathematics courses housed in mathematics departments. Students of mathematics learn about pedagogy in these courses although these lessons may never be explicitly stated as goals.

A famous merging of content and pedagogy occurs in the theory of pedagogical content knowledge as articulated by Shulman (1986) where he first described pedagogical content knowledge as knowledge that "embodies the aspects of content most germane to its teachability" (p. 9) and continued to offer examples such as knowledge of the most useful metaphors and representations, and familiarity with common misconceptions about the content held by students. And while a merging of content and pedagogy, pedagogical content knowledge, a valuable and powerful way of teacher knowing, was also a separate thing from pedagogy and content. In Knowledge and

Teaching (Shulman, 1987 February), he describes seven minimum categories for teacher knowledge that could form a knowledge base for the professionalization of teaching. Three of these include pedagogical content knowledge, general pedagogical knowledge, and content knowledge. According to Shulman, the articulation of pedagogical content knowledge does not eliminate or absorb the distinction between content and pedagogy so much as offer a possibility of some overlap. Segall (2004), though, takes a different stance towards the relationship between content and pedagogy. He points out the ways in which pedagogical content knowledge as an idea has benefited teaching, but pushes on the notion that content and pedagogy retain distinct features.

> teachers' pedagogies do not initiate the pedagogical act but add further pedagogical layers to those already present in the text. In other words, the instructional or pedagogical act does not begin with teachers in classrooms, nor does the 'content act' end at the desk of the subject-area scholar. Both produce pedagogical content knowledge, that is, content that is always pedagogical and pedagogies that are always content-full. (p. 3)

Segall draws upon the work of McEwan & Bull (1991) to elaborate how all knowledge (content) is pedagogical in that it is always trying to communicate.

> there is no such thing as pure scholarship, devoid of pedagogy. The scholar is no scholar who does not engage an audience for the purpose of edifying its members...science, or any other form of scholarship for that matter, is an inherently pedagogic affair...ideas are themselves pedagogic...Explanations are not only <u>of</u> something; they are also always <u>for</u> someone. (p. 331-332, quoted by Segall)

When acting as research mathematicians, university professors are still in the role of educator with other research mathematicians acting as pupils. The mathematics research is for the benefit of other scholars and so in doing the work, the mathematician is trying to teach others about her ideas. The teaching takes place in journals, formal lectures, conferences, letters, over coffee, and in a plethora of other locations and media. The content is pedagogical. When this same mathematician enters the classroom she teaches undergraduates mathematics that at some other point in time a person wrote for the education of others (by or for Newton, Cauchy, Gauss, Euler, Pythagoras etc). Herein begins some of the layering effect, which may have started much earlier in the dissemination and dispersal of the mathematics through journals, alternate and/or amended proof, examples, textbooks, previous coursework, etc. As preservice teachers (and other for that matter) take an undergraduate mathematics course they cannot help but witness pedagogy both in the teaching of the mathematics, but also in the layers of pedagogy that worked to get the mathematics to that moment.

I want to reiterate that I am interested in studying the ways that undergraduate mathematics classes "teach" pedagogy and "speak" about students, teachers, and mathematics. Pedagogy exists in undergraduate mathematics because mathematics can be seen as pedagogical. What we teach is how we teach it. How we teach is what we teach. There is a bidirectional arrow between pedagogy and curriculum/content. I am belaboring this point because students in the courses, including undergraduate mathematics, learn both content and pedagogy[1]. To put it differently, although mathematics courses in universities are housed in mathematics departments and not in education departments, there remain things to say and research about possibilities for teacher education in these courses. Future secondary teachers take mathematics courses in these departments. They learn math. They also learn pedagogy.

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[1] I separate the terms content and pedagogy for emphasis rather than to distinguish them as two separate and unrelated aspects of teaching.

Undergraduate mathematics as site of apprenticeship of observation

At least since Lortie (1975), teacher educators have understood the power of the "apprenticeship of observation." As students, future teachers spend thirteen years in an informal, but extensive "apprenticeship" witnessing teaching from the perspective of the student. According to Lortie these experiences have a greater effect on their development as teachers than any official teacher education or alternate certification program can due in part to the disparity between the typically short duration of a certification program and the longer time spent as students in classrooms. In this view then, the teaching that students witness has the potential to inform their future teaching.

It is worth noting that Lortie's concept of an apprenticeship of observation has been found relevant to teacher education. Bird (1991) found that students in his elementary methods course (in discussions of mathematics as well as other subject lessons) expressed preferences for particular videotaped cases of teaching that were similar to the students' previous experience as students with those types of lessons. For example, after watching a direct instruction lesson students talked positively of it, but after watching an episode of Magdalene Lampert teaching mathematics, the students talked often about her needing to have told the students more. That is, she should have offered more direction and clarification of what they perceived to have been confused student ideas. Bird related these preferences and critiques of teaching to the preservice teachers' past experiences as students. Similarly, Holt-Reynolds (Fall 1991) found that all of the preservice teachers in her study consulted personal past experiences as students to make decisions about instructional strategies. "Preservice teachers frequently report actual instances from their pasts where teachers implemented activities similar to those advocated in their professional course work. They recall the effects those activities had on themselves as students. Corinne explained the process explicitly. "You just see what you liked, what got you interested as a student, if I'd want to do [this activity] as a student. Then, you correlate that to how you would teach." "(p. 5). Holt-Reynolds found that pre-service teachers not only look to their past experiences as students to offer possibilities of what they might do in classrooms, but also choose from these possibilities based on what they enjoyed. Other researchers have also found that preservice teachers consult past experiences as students in classrooms when planning to teach as well as when teaching (Holt-Reynolds, 1992; John, 1996; Johnson, 1994; Richards & Pennington, 1998)

Not only is there research that suggests past experience is at least part of the web of influences on teacher practice and theory, but also a host of teacher education initiatives take the apprenticeship of observation and a theory of cyclic teaching as the theoretical foundation of their work and recommendations for change. That is, if teachers perpetuate a certain type of teaching, and if we in teacher education want to change that teaching, we must interfere with the cycle. A widely influential publication, Everybody Counts (NRC, 1989), places the undergraduate experience of preservice teachers as a link in this model.

> Undergraduate mathematics is the linchpin for revitalization of mathematics education...Those who would teach mathematics need to learn contemporary mathematics appropriate to the grades they will teach, in a style consistent with the way in which they will be expected to teach...Since teachers teach much as they were taught, university courses for prospective teachers must exemplify the highest standards for instruction. (pp. 39, 64, and p.65 respectively)

Alan Schoenfeld in *A Sourcebook for College Mathematics* (Alan H Schoenfeld, 1990) also calls upon the notion of a cycle placing undergraduate mathematics at the pinnacle.

He states his opinion of the role of undergraduate mathematics in shaping K-12 teaching practice very clearly.

> The nation's mathematics teachers learn their mathematics from us [professors]. In our classrooms they learn much more than the subject matter. They learn how to teach (we are, after all, their last models of mathematics instruction before they go into the classroom), and they learn what it is to learn and do mathematics. If prospective teachers experience mathematics as a lively and engaging discipline, they may teach it that way. If they experience it as dead and deadly, the odds are that their students will too. Thus our instruction at the college level is a critical factor in shaping future K-12 mathematics instruction. (p. ii)

Schoenfeld is not alone among mathematicians interested in changing teaching in K-12 settings by changing the teaching in undergraduate ones. Both Cuoco (2003) and Wu (1999) encourage other mathematics professors to think carefully about their roles as teachers and the lessons about teaching and mathematics it can impart on would-be teachers. Others in mathematics education at the elementary, secondary and tertiary levels also care about changing teaching and have found a variety of ways to contribute towards such an effort.

Change in teaching

The notion that changing the teaching that preservice teachers witness will revitalize teaching is very appealing. The allure is simple if the process is not. If we in teacher education (and undergraduate mathematics!) can influence a generation or two of teachers in such a way that we disrupt current and widespread teaching practices, we will ring in a new era of better mathematics instruction. Or so the fantasy goes.

Herein lies my concern. I would like to claim that we would bring in an era of *more* and possibly *different* mathematics instruction. The idea that new means better is problematic. Through this research I do not wish to critique professors with a goal to change their teaching only because I feel that some other method is better. No doubt, at times I did think that there could be improvements in the classes I observed, but for me to dictate a type of teaching is just another way of being closed-minded. Rather than opening teaching to include a wider array of pedagogies such an exaggerated change in teaching would merely be restricting it to but another monopoly, which would have its own collection of problems. I find limited models of teaching to be an issue for concern no matter what that single view may be. To deny a best system is not to denounce organized education as an impossible undertaking. I do not want to replace one narrow teaching approach with another equally narrow one. Rather, I want to consider the value of having multiple different teaching approaches available to all mathematics teachers. In turn, different approaches in teaching can allow for student access to mathematics in more ways.

One thing that I learned quickly was that despite some prevalent biases towards mathematicians in our culture (for examples of particularly nasty impressions see Picker & Berry, 2000), these professors are human. The caricature of the mathematician is a stereotype and so are the courses they teach. From a broad perspective three of the four courses fell easily into the classification of direct teaching, but on closer inspection there were striking differences. These differences tell stories about who the teacher is, what roles students play, where these meet, and why study mathematics.

As I say above, depending on how closely you look at the classes they could be seen as similar or easily distinguishable. One common notion is that mathematicians teach undergraduate classes using a mostly lecture or direct instruction approach. In reviewing several MAA publications (e.g. MAA, 1983; NAS, 1968; Rishel, 2000; Alan H

Schoenfeld, 1990; Lynn Arthur Steen, 1989), the authors also assume this to be the case about their mathematician readership. By encouraging these professors to try new methods they are tacitly acknowledging a presumed similarity in their teaching—lecture. These assumptions might be accurate, but little research has been done on the nature of teaching in these settings (Robert & Speer, 2001). A large segment of research on undergraduate mathematics focuses on improving student understanding of undergraduate concepts, mostly in calculus (e.g. Ferrini-Mundy & Graham, 1994; Graham & Ferrini-Mundy, 1989; Lauten, Graham, & Ferrini-Mundy, 1994; Monk, 1994; Tall, 1996), or describes big issues in the teaching, learning, and context of calculus (e.g. Douglas, 1986; Alan H Schoenfeld, 1990; Lynn A. Steen, 1987). The Calculus Reform movement, I feel, directly benefited me as a student. It was not mathematics the monolith that captured my attention and formed my major, but the particular vision of mathematics I saw in Dr. Myrtle Lewin's Calculus I course my first semester in college. In no way can I imagine that I would have continued past that one required course had it not been for the work being done to reform calculus and the department. Though I do not know what those professors read or what influences acted on their teaching and curriculum decisions, much of what I read now about calculus reform resonates with my own experiences as an undergraduate. My professors at my small liberal arts women's college taught in a spirit commensurable with reform. We used the Harvard Calculus texts through the three-course sequence. We modeled functions on calculators, computers, and even carved potatoes and fruit. We worked in groups, worked on application problems, and mostly wrote pages of explanations per assignment if not technical proofs. The research by educators about student thinking and about curriculum in general helped me then and continues to do so now when I plan for teaching a course and seek out these sorts of documents to prepare.

By saying that they do not address the teaching in upper level undergraduate courses I do not mean that these calculus research projects did not influence it. The authors were quite often advocates for change after all, caring deeply about their subject and about students. What I am trying to suggest is that as a field, research in undergraduate mathematics does not often go beyond introductory courses (i.e. calculus and developmental courses such as college algebra) and nor into the teaching of upper-level courses.

Much of the work in undergraduate mathematics (and the few I offered as examples are not presumed to be the limit of research in undergraduate mathematics) takes the stance that changes can be made in courses either by better understanding of student reasoning which will lead to better instruction towards aiding that reasoning, or restructuring the environments of these classrooms to offer more interaction between students, teachers, and mathematics. Behind these concerns are assumptions about needs for some changes, which are implied directly or otherwise as changes from a direct instruction approach to a more interactive one. I point this out not because I challenge the interest in changing the teaching (I have already said a bit about that above) but to highlight a widespread sense within both the mathematics and mathematics education communities that undergraduate teaching is primarily a lecture-based environment and the subtext is that this type of teaching arrangement is at best not ideal and at worst damaging. These publications, which assume a mathematician audience, are far less likely than those assuming a K-12 mathematics teacher audience to flat out discourage direct instruction.

Lectures and mathematics education

In the K-12 realm, Chazan and Ball (July 1999) point out that admonishing mathematics teachers not to "tell" (as in lecture, for example) is too simplistic and does not consider the context of the class or learning. "As researcher-teachers, we claim that what is needed is...less embracing or rejecting of particular lessons and more effort aimed at developing understanding of and reasoning about practice." Chazan and Ball argue that describing a teacher's action as telling students something is not enough. Researchers need to understand what may have motivated that action, and what the teacher may have hoped to have achieved by it. Disregard their message for a moment and instead consider the presumed audience. Why write an article that seeks to illuminate "telling" modes of instruction in a more nuanced light if there were not a segment of the readership who believe that lecture is one-dimensional—and undesirable at that? The fact of the article's existence speaks to the status of lecture in popular education. While there might be those who do not consider lecture simply "bad", many in education are not so kind to the method.

My personal experience as a mathematics educator ranges from being a student, a high school teacher, a methods course instructor, and a researcher. In each of these contexts I have encountered other educators who claim that lecturing is bad and must be limited or done away with in education. When I was first observing mathematics classes for this project a professor of education asked me what I was working on. As soon as I told this person about my work in the undergraduate classrooms, this person made several disparaging and even vitriolic comments about mathematicians and their terrible teaching. The comments went further than the teaching and bordered on character attack. While this reaction was definitely extreme, I have encountered others who doubt that by entering these classrooms I will learn anything interesting. They tend to assume wrongly that all the professors are poor teachers—mostly due to the preferred style of lecturing in that context, and that my goal therefore is to write a dissertation that vilifies this group of teachers. My point is that although there may be favorable or more neutral positions on lecturing, the bulk of writing in education at the K-12 level and a good segment of the writing at the undergraduate level tend to prefer change away from this style of teaching. When we move beyond what is recorded in publishable spaces, the comments and opinions can become more drastic and biased against lecture and its practitioners.

In the methods courses where I teach and interact often with other methods instructors from this and other universities, there are strong assumptions that because these preservice teachers have likely seen lectures from their viewpoint as students (*a la* Lortie) we do not need to discuss this style with these future teachers. Not only is it seen as unnecessary to their education but also as potentially harmful. It is almost as if the discussion of such a thing would act as an endorsement or as legitimizing by the methods instructor. If we do not talk about it, they will not do it. While in their professional academic lives educators might hold a balanced view of lecture, this is often not shared publicly with preservice teachers. It is much more likely to be ignored as a topic altogether.

Too often the fact that many mathematics professors continue to use lecture-based (telling) styles is also used to label their teaching as bad. There is a bias in the education community that groups = reform = good whereas lecture = traditional = bad regardless of what is communicated through these styles, what motivated them, or what goals the teacher/professor had in mind. This myth hides the intricacies of what professors and teachers who primarily use lecture actually do and say in these lectures. It is unclear whether mathematicians, many of whom are educators in their own right, would agree that lecture is bad or that groups are good. They might offer a different equation. Whatever the teaching preferences of mathematicians may be, the fact that they influence so many students, and preservice teachers in particular, their teaching warrants more investigation.

I have tried to offer rationales for why undergraduate mathematics matters for teacher education by arguing that preservice teachers learn pedagogy in both settings. In researching what pedagogies preservice teachers may witness I am not trying to favor one or more styles over others. This is relevant because I have also acknowledged that many in education, a community I represent when I enter mathematics professors' classrooms, often disparage mathematicians for their teaching preferences. I have also tried to explain why undergraduate mathematics is an interesting and worthwhile site for research into teacher education, drawing upon notions of the apprenticeship of observation that has been found relevant to teacher thinking and by pointing out that pedagogy and content are not as separated as university department systems might suggest. Future teachers learn about teaching by learning mathematics.

Observing the courses

Recall the research question: *What assumptions about students, teachers, and mathematics are embedded in the pedagogical milieu of undergraduate mathematics courses?*

Because I am interested in pedagogies embedded in undergraduate classrooms as well as differences across classrooms I utilized a case-study approach where each course acts as a case of constructions of the subject where subject is an undergraduate mathematics student. There are two sites of artifact generation:

- Observations with detailed field notes of four upper-level mathematics content courses offered at a large Midwestern university
- Course documents including syllabi, schedules, textbooks, professors' websites, an article provided by one of the professors, and course assignments

The courses represent three important subfields in mathematics—algebra, analysis (2 real and complex), and geometry. I observed these classes approximately every other week during the fall semesters of 2005 or 2006 and focused on the nature of the activities during class time, the actions of the professor and students, who talked or asked questions as well as what was said or asked, and the physical environment of the class[2]. Because I was interested in more than just the professor's lecture or class notes, I rotated the focus of my observations with each visit. Some days the observations centered on the professor (including content such as theorems, examples, and pictures presented verbally and/or through written notes, comments, questions, and actions), some days on students (actions, questions, and comments), and at times on the ways that students and professors interacted to influence one another during class time. My field notes followed an approximate chronological order of the class actions. If on some given day I focused on the professor more than the students, I would record much of the work on the chalkboard

[2] The observations focused on these aspects of the course because they are features of the classroom that contribute to the discourse of the classroom. My use of the term discourse will be further discussed in the analysis section.

as well as direct quotes. On those days when I focused on students more than professors, I noted student actions, comments, and questions. At times during my observations students would approach me before and after class to share their thoughts about the professor, an assignment, or some other aspect of the course[3]. On a few occasions they even spoke to me during the official class time. Several students who sat near me would turn to me during class to show me grades they received on quizzes, homework sets, or exams. I consider these interactions as parts of my observations. Documents collected included course syllabi and schedules, assignments, textbooks and professor prepared study sheets. Some professors also utilized a course or personal web page, which served as an electronic document.

The two sites of artifact generation allowed me to explore a variety of features of the classroom the students contribute to and witness. These observations are purely phenomenological in that I focus on what I saw and heard. The observations picked up on actions and speech of the people involved in each course and also the physical environment of the classroom. Some of the documents were shared aspects of the course I would not have been able to "see" or "hear" had I acted only as an observer. Participants in the course had to seek out materials such as the textbook and the course web sites.

Finally, by generating these artifacts I was not trying to use them to pinpoint and therefore validate future claims. Instead they represented various aspects of the course to explore and might even at times seem to be contradictory. For example, a classroom scene I witness during an observation might contradict a statement made in the syllabus.

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[3] Only one course professor chose to announce or describe my presence in the class, but I did tell those students from other classes who asked what I was doing there.

Let us suppose the episode in question is that the syllabus states there will be a quiz every other week, but the observations suggest that by mid-October only one quiz has been given. In some projects this might be cause for concern. Which should be given more weight: The intentions of the course design, or the actions? How should we reconcile these differences? For the purposes of this research, coherence between what is intended, what is said about what is intended, and what is acted is not required. The phenomena are what the students experience. They see and hear and participate in these episodes during class time and so becomes a part of the rhythm of a classroom (see Hawhee, 2002 for rhythm as pedagogy). How the teacher intends an episode may tell us something important about teaching, but it does not change what the students could see. By using a variety of artifact types I am representing a range of aspects of teaching: from the physically performed to the public textbook to the between student pre-class chit-chat. They are not an attempt to pin down the truth of the teaching so much as to offer up more ways to analyze it.

Analysis and theory

The analytical tools I draw upon in this dissertation come largely from the intellectual traditions of rhetoric and continental philosophy. Some rhetoricians and philosophers that influence my work are Aristotle, Mikhail Bahktin, recently Kenneth Burke, and Michel Foucault—particularly his notions of discourse and the construction of the subject. I will address both why rhetoric, why construction of the subject, and how these relate to teaching. Before expounding on these questions, though, let me instead begin by exploring rhetoric as an intellectual activity that can complement current trends in mathematics education research.

Why rhetoric

To use a communication or rhetorical approach is not to deny other analyses as viable or useful. This is not in opposition to social science models of research or to socio-cultural or socio-linguistic approaches. It is instead complementary. It is another way of doing research—one that has long been established in academia, especially in the liberal and communication arts. Because education can be thought of in terms of communication the analyses are appropriate.

One way to investigate pedagogy is through the notion that teaching is about persuading students to agree with and thus take up a new view of some issue. A student encounters some fact, idea, concept etc. taught to them by a teacher. How can we in education get this student to take this fact, idea, concept, etc. on as truth or worth their time? We persuade them of it and this persuasion is at the heart of rhetoric. But before you cry foul that learning and scholarship is more pure than strong-arming students to some collective will, consider that there are many techniques of persuasion, not all of which involve bullying or flattery. "Rhetoric may be defined as the faculty of observing in any given case the available means of persuasion" (Aristotle, Book I: Chapter 2). Logical reasoning may be persuasive to some students and many who find their way into advanced science and mathematics find this form of persuasion very powerful indeed. For others a good allegory, personal anecdote, or retelling of an experience may be more convincing. Different disciplines often favor different means of persuasion or accept certain ones as more trustworthy than others. History, for example, pulls from a variety of sources such as personal narrative and legal documentation. Even within disciplines, different sub-fields may prefer or rely on different techniques of persuasions. A theoretical astrophysicist must use x-ray data to infer indirectly the existence of black holes, for example, whereas an applied physicist can directly test the tension of a type of chain.

Take the most "teacher-centered" form of talk as an example. A lecture is an act of speaking. It has an audience of students whom the professor tries to convince of some truth of the discipline. The mode of this persuasion in the classroom might be by appeal to the authority of text/teacher, personal experience, worked examples, or logical proof. According to Aristotle these modes of persuasion are chosen by the speaker in direct response to the speaker's sense of audience. "For of the three elements in speech-making—speaker, subject, and person addressed—it is the last one, the hearer, that determines the speech's end and object" (Book I: Chapter 3). Each student ultimately decides whether the lecture succeeds or fails. She will either take on the knowledge or not. The teacher must work to persuade (or teach) each new student and class and the pedagogies used are determined in part by what the teacher thinks will best suit the material and student. Speaker, Subject, and Audience work together.

Furthermore, if we consider Bakhtin's (1990) theory of the speech genre we can better understand lectures as responsive to student audiences. He says all speech, written and verbalized, is composed and delivered with the expectation of an audience.

> An essential (constitutive) marker of the utterance is its quality of being directed to someone, its *addressivity*...Both the composition and, particularly, the style of the utterance depend on those to whom the utterance is addressed, how the speaker (or writer) senses and imagines his addressees, and the force of their effect on the utterance. Each speech genre in each area of speech communication has its own typical conception of the addressee, and this defines it as a genre. (p.95)

A teacher will choose words, phrases, postures, props, style, and tone in expectation of an audience of students. Therefore a lecture—even if the teacher speaks the entire duration of a classroom episode, filling the class with a single yet lengthy utterance—is crafted and delivered in response to expectations of and assumptions about students. It can be rare, though, that a teacher lecturing never opens the floor to other speakers. When students speak they also do so to an audience that includes their fellow students and teacher. How they frame their speech likewise exhibits addressivity. This is but one way in which rhetoric can help me analyze these undergraduate mathematics classrooms. The rhetorical moves made by teachers and students in a classroom contribute towards the persuasiveness of the lesson and/or interaction. Analyzing these moves offers a new way to "read" teaching and thus understand the experiences of math majors who go onto become teachers themselves.

Rhetorical studies are historically linked to issues of communication through oratory and more recently (!) with the invention of printing and readily available printed text, rhetorical studies can also work with and on text. A major concern is how communication works and can thus be very pragmatic. That is, a rhetorical analysis of teaching can focus on what a lesson communicates, by what means, and for what purposes. In Aristotle's use of rhetoric, the speaker, audience, and subject work together. Although the speaker may be considered the leading character due to his obviously active role the listeners and subject are shaping what the speaker says and does. A speaker must consider the audience and its relation to the subject as well as his own to both the audience and subject. What position does the speaker take on the subject? Is the subject known to be controversial with this audience? Generally accepted? Is the speaker credible with this audience? If not (yet?), how to proceed? How does the subject suit the place, space, and time? The three make up the scene and work together in ways that make pulling them apart difficult if impossible. It may be possible to focus on one aspect for a while and hold the other two at bay, but eventually their mutual dependence on one another will emerge.

By contrast, a common model of teaching characterizes three separate but fixed bubbles possibly connected by arrows to form a triangle—the teacher, students, and curriculum/subject. In this representation their independence is held fixed and each category is stable. We know who the teacher is, the student, and the mathematics. But what if we took a position that allowed for fluidity? What could be said about teachers, students, and mathematics then?

A rhetorical analysis allows for multiple interpretations to be held simultaneously. Mathematics and action merge. Teacher as student. Students evaluate. The notion of the teacher as independent variable and student as dependent falls away. Neither would we place mathematics as an independent variable and teachers as dependent. In fact, the vocabulary "independent" does not make sense. This is not an issue of including a bidirectional arrow as sides of the familiar teaching triangle. Instead, the vertices shift, now and again collapsing upon one another to form a line or even a point before expanding. Angles are not maintained. Nor is length, or even form. Dynamic.

A rhetorical model of analysis, which traces its genealogy through the liberal arts and continental philosophy, can offer much to educational research. Again, by taking this approach I am not denying other modes of research or analysis respect. Two major strands of popular research in mathematics education are those that take the mind and thinking as primary – cognitive studies; and those that take the social aspect of learning as primary—socio-cultural. The intellectual histories of research matters as attested to by the recent discussion in *Educational Researcher* between Sfard and Juzwik concerning different traditions' uses of the terms sociocultural (as well as identity, discourse, and narrative) (see Juzwik, December 2006; A. Sfard, December 2006). During this exchange, Juzwik digs into differences in these words and traces intellectual genealogies to show the vocabulary we use in research pulls from different traditions and so imbues the research with historical meaning. Sfard, though, explains how taking a macro view can allow us to characterize large swaths of a field (e.g. participation vs. acquisition, where participation aligns broadly with sociocultural and acquisition with mind independent research) to speak about similarity in seemingly disparate traditions. I will try to take a more Sfardian approach (with bits of Juzwik for flavor) in my following discussion of sociocultural and cognitive research in mathematics education. My focal length will be long and so might sacrifice a bit of magnification, ironically, to allow for brevity.

Using chapters of *A Research Companion to Principles and Standards for School Mathematics* (Kilpatrick, Martin, & Schifter, 2003) as my macro guide, I find the following chapters:

- *Communication and Language* (Lampert & Cobb, 2003)
- *Implications for Cognitive Science Research for Mathematics Education* (Siegler, 2003)
- *Situative Research Relevant to Standards for School Mathematics* (Greeno, 2003)
- *A Sociocultural Approach to Mathematics Reform: Speaking, Inscribing, and Doing Mathematics Within Communities of Practice* (Forman, 2003)

- *Balancing the Unbalanceable: The NCTM Standards in Light of Theories of Learning Mathematics* (Anna Sfard, 2003)

In summarizing the language and communication studies, Lampert and Cobb (2003) describe the purposes of such research as work towards making communication both a goal as well as a method for learning and how this learning occurs (p. 246). Cognitive research, according to Siegler (2003), focuses on "the acquisition of particular mathematical procedures and concepts rather on broad philosophical issues..." but in so doing allows for "relatively firm conclusions…" (p. 289). Greeno (2003) describes the situative perspective as concerned with combining psychology and social science ethnography to research "how students become more successful in participating in mathematical practices and how they develop identities as mathematical knowers and learners (p. 315). Forman's (2003) chapter describes current trends in sociocultural work in mathematics education as proposing "teachers need to understand the mathematical knowledge that children bring with them to school from the practices outside of school as well as the motives, beliefs, values, norms, and goals developed as a result of those practices" (p.337).

As I read these chapters I began not only to appreciate the work of researchers acting within each tradition, reminiscing about my first readings of many of the originals cited, but also attempted a mini-Venn diagram in my mind. The overlap became too much and the task quickly abandoned! At first I found much between the situative, sociocultural and language chapters that seemed commensurable. The authors and works cited intermingled between these chapters often. I wondered whether the cognitive chapter also related to the rest and soon found connections between it and the situative through emphases on social science and psychology as research traditions. Sociocultural, as described by Forman, also has ties to psychology through interests in concepts such as belief and tracing motives. In the last chapter listed above, as well as the last of the volume, Sfard (2003) declares, "[Educational theories] thrive only in the company of other theories" (p. 355). She encourages readers to think of theories as "complementary" rather than in competition with one another.

> As long as the metaphor with which we think is not challenged by an alternative metaphor, its entailments seem natural, self-evident, and unquestionable. Only in the presence of another metaphor and another theory can we become critical towards the seemingly obvious; only when an alternative theory makes us aware of the metaphorical roots of our present beliefs are we able to open ourselves to possibilities that we could not consider before. (p. 355)

Utilizing rhetoric and the metaphor of teaching as persuasion allows me to investigate different aspects of teaching that other common research paradigms might not. Like in teaching, I advocate pluralism in research as well.

Rhetoric to Foucault

When teachers take on the role of talking to students they become an audience (and vice versa). When thinking of teaching as rhetorical, a speaker (or teacher) must know something about or have expectations about the audience (students) that influence what he chooses to say and how to say it (*a la* Bakhtin). The assumptions the teacher holds about the students' motivations, interests, needs, current knowledge, etc. influence the teacher's decisions. Know thy audience. This is what Foucault calls the construction of the subject. Here, the subject is the student/audience. For a listener (student) the speaker is important. Why should I trust this person? By their authority, their words, their actions? This is the construction of the teacher.

These constructions occur within classroom discourse, but undergraduate mathematics courses and their classrooms are not places researchers expect to find rich discussion or conversation. As will be further described in further detail in chapters of the dissertation, five of the classes I observed did not dispel this common notion but these classes did contain much more than just talk that was worthy of consideration. For example, a professor's action of looking students in the eye, the creation and distribution of a student petition for extra credit, and the voices of the textbook authors are each examples I might have missed had I concentrated solely on in-class talk or written communication, and interviews. Much, though, of the research done in mathematics education around notions of discourse does limit the definition of discourse to talk and words (Parks, 2007b). As Parks further points out many of these projects draw inspiration and justification from NCTM's *Principles and Standards* (2000) documents calling for more student engagement in classroom communication such as critiquing and creating mathematical arguments. In these uses of discourse, things beyond talk are often ignored. When other aspects of classrooms are investigated they are not usually called discourse analysis, but are given other names such as studies of *teacher beliefs*, *textbook analyses*, and *mathematical representation* to name a few.

Sfard's (2001) use of discourse does incorporate more than just talk. As she puts it she "will try to show that there is more to discourse than meets the ears…" (p. 1). A primary reason I do not start from her use of discourse is that she uses it as a theory of learning where learning mathematics is coming to terms with the mathematical discourse as used by the established mathematical community and where researchers can measure how close a student is to appropriating the discourse by reference to markers such as extension of vocabulary. I do not want to align this particular project with a theory of learning and so avoid drawing too close to her work at this time, although I find it quite interesting. Fortunately, in addition to Sfard's use of discourse has more than words, there does exist another notion of discourse that is much broader in scope than typically used in mathematics education.

By framing my guiding question, *What assumptions about students and mathematics are embedded in the pedagogical milieu of undergraduate mathematics courses?* as an investigation into the construction of the subject (e.g. the student in undergraduate mathematics) I can employ a notion of discourse formulated by the French philosopher Michel Foucault (1978/1990, 1983).

In this analytic, discourse refers to more than just talk or other verbal communication and also includes multiple actions and/or objects, which can be read and interpreted as texts (see, e.g., Michel Foucault, 1978/1990; Mills, 1997/2004). Here I consider such texts to include people's actions (students' and professors'), the physical setting, classroom objects, verbal and symbolic communication, written documents, and websites. Through these multiple texts, constituting a discourse, a subject is produced or constructed. That is, who undergraduate students are (or should be) in upper-level mathematics classes is neither a logical nor a natural phenomenon, but rather is an artifact of discourse legitimated through repetition. This type of analysis does not intend to critique current teaching practices with the goal of offering better pedagogies nor does it seek to legitimize some practices as exemplary. Instead, this analysis attempts to rethink ideas taken for granted. In this case, that undergraduate courses are not universal and timeless. The educative goals, teaching, actions, and students can differ in important ways. For example, in a 1968 publication of the National Academy of Sciences, the panel on Undergraduate Education in Mathematics reports that elementary courses in mathematics be taught in large lecture halls, not because this is best for students or most logically commensurate with learning the subject, but for "utilizing staff more efficiently" (p. 11). Contrast this with the more private tutorial systems of learning mathematics in England. These differences can be attributed to the differing histories of the subject as well as the histories of higher education in each location.

Before continuing, the idea of *constructing the subject* may need further illumination. A subject, here, does not refer to an individual, but to an abstracted category made possible to talk about through discursive constructions. For example, Popkewitz discusses as a subject *the problem-solving child* not as a specific child but as a type of child created through discourse (Popkewitz, 2004). "Neither 'adolescence' nor 'problem solvers' …are objects that you can touch, but are ways of thinking, 'seeing' and feeling about 'the things' of the world that were (and are) deemed important" (p.13).

Once *the problem-solving child* is invoked, researchers can examine the category for behaviors, tendencies, preferences, and growth. They can also create more and more fine-grained instruments and techniques for classifying "problem solvers'" thinking. In current mathematics education *the problem solver* is a particularly desirable type of student and is reinforced as a type and constructed through texts. Examples of texts might include published articles, how-to books, textbooks, and collections of problem sets. Authors can write books for teachers on how to best educate and stimulate problem solvers, how to write assessments that include good problems, as well as instructions on how to assist other children into developing as problem solvers.

Recall though, discourse as understood in this analysis includes non-written documents as well as actions, physical environments, and comments. I remember from my time as a high school teacher how I contributed to the discourse constructing the problem solver through my pedagogy. The school I worked for encouraged reform mathematics teachers (another type of subject to explore as a construction) to emphasize problem solving. In this school and district rote exercises were often considered in opposition to problem solving exercises. Both were involved in the curriculum, but problem-solving tasks were considered the better task and a priority. The school printed posters for each mathematics class that outlined how-to steps for problem solving. As a teacher I worked to get those students who were good at rote exercise to also improve their problem solving skills; to become problem solvers. The students, who were particularly skilled at problem solving tasks but not necessarily at exercises, were seen as particularly creative. I encouraged them to do their homework and improve their grades (because grades still relied on other types of tasks), but I upheld them as excellent students nonetheless and often in spite of their grades. When I attended professional development workshops we discussed problems instead of exercises. We shared stories of students who had excelled at some particularly challenging problem or of other students who had improved their problem solving skills and how we had managed this change. We learned new techniques and lesson ideas to create *problem-solving students*. I point these experiences out not to declare that problem solving skills and tasks are bad, but that through my participation in these professional development workshops, use of texts, and teaching I reinforced the problem solver as a type of human.

Moreover I included questions on my exams that were considered problem-solving questions as opposed to recall or application type questions. The textbook I used included problem-solving style questions at the end of every set of assignments and several sections of each chapter were devoted to problem solving tasks. I also assigned a 'problem of the week'. Many might think that this was good teaching and in many ways I agree. I point these out not because I am ashamed to have taught what I did, but to indicate that I was reinforcing the discourse of *the problem solver* and in particular I was prioritizing this type of student. Again subjects are not individuals. In my example offered above, many of my past students were individuals skilled at solving a variety of problems. The difference between a subject and an individual is important. I had an idealized, abstracted notion of *the problem solver*, which influenced the ways I acted on and for particular individual students. For me, *the problem solver* was desirable because they were nonconformists, used mathematics effectively in a variety of contexts, and would be able to confront life's obstacles with success. Because I wanted these things for my students, I wanted them to be *problem solvers*. By taking on this approach to my class and my students I as participating in a discourse that privileged problem-solving and constructed problem solvers as a type of person.

Furthermore Popkewitz uses the term *fabrication of human kinds* (p. 13), to emphasize these *human kinds* (a term also used by Hacking, 1995) as types are fabricated in the sense that they are both *built* and are also *fictions*. As a type of human, *the problem* *solver* is a construction, an idea, and an ideal in mathematics education. When reified, a subject can be talked about, researched, acted on, and further refined.

This extended example has focused on problem solvers as types of students constructed through various texts including policy documents, textbooks, lessons, teacher actions, and classrooms. The research for this dissertation explores the construction of *the student* in the space of four classrooms. I interpret the actions of persons, activities, documents, and dialogue in the classrooms for the purpose of describing the construction of the student and teacher through these texts in each course. Although professors and documents they create are explored as I write about students, this is not a study of only professors, but also of how the subject of *the student* is constructed in these sites. At times in the following chapters on *the student* it might seem that I am indeed focusing on the professor due to the professors' relative prominence in the classroom through their choice of texts, creation of syllabi, and talk. In three of these classes (other classes not observed may have been very different) the professors typically spoke much more often than students and acted more visibly through their position at the chalkboard. The following chapters will address construction of the student and in doing so will likely refer back and forth between students and teachers.

Furthermore, the construction of the undergraduate student is specific to each course during this semester. That is, the construction of the student in the Real Analysis course, for example, shared some features with the construction of the undergraduate student in the Discrete Mathematics course, but they also departed in some ways. This is in fact, a primary reason why I chose multiple classes—to highlight difference. Just because three of the four courses described here used a lecture-based approach does not guarantee that the construction of the student through the course is identical. It is even possible if the same professor teaching the same course in another semester a new construction of the subject might emerge due to possible differences in students enrolled, texts, syllabus changes, and assigned classroom space.

In particular, I analyzed each course across the sites of artifact generation to infer responses to the following questions[4]:

- *Substance: What Aspect of the Student is Supposed to Change?*
- *Mode: How Are Students Invited to Learn?*
- *Regimen: What Are Students Asked to Do?*
- *Telos: What Does the Model Undergraduate Mathematics Student Look Like?*

The answers depend on the respective class and in this way I am highlighting the construction of the answer to these questions through the discourse particular to each course. I am not attempting to answer these questions as generalities or recommendations for students, parents, teachers, curriculum developers, or policy makers. Instead I investigated the ways discourse in each professors' classroom constructed the answers to these questions.

Thus far I have discussed the notion of the construction of the subject (student) in great detail, but there is a third aspect of this work—the mathematics. In using a Foucaultian framework, issues of the mathematics will emerge as I describe the construction of the subject (students) but I will also devote some space to more specific discussions of the mathematics in the classrooms in the conclusion chapter.

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4 Foucault (1983) proposes these questions as a framework for exploring how subjects get constructed.

Concluding remarks

In this dissertation I speak to discourses (mathematics teacher education, mathematics education, and undergraduate mathematics) for the purpose of examining commonly held notions. These include stereotypes about mathematics teaching at the undergraduate level, namely that reform practices are better than traditional practices, and that content and pedagogy are distinguishable. I also speak about discourses in the chapters for each course. I describe how the discourses contributed to the construction of the undergraduate mathematics student. To do this I draw from postmodern and rhetorical theories and analyses.

Each course presented different pictures of a model student that might emerge from a perfect enactment of that course. We will see model students as converts, researchers, Platonists, and teachers. In these four courses alone there were four different articulations of the model student. This suggests that undergraduate courses are not as alike as imagined. In these courses there were also multiple ways that students could do mathematics partly because these courses presented different views of what it means to do mathematics and who does it. Answering, "what is math all about" is not quite as simple as "proofs". The ways that the courses structured student access to the mathematics suggested that different styles of teaching cab provide alternate and varied entry points for students. Therefore a lecture method or an inquiry method (or others) can both exclude certain types of participation and encourage others thereby excluding some students and encouraging others. Those excluded and those encouraged were not the same across the classes. A best course, one that had the best model student, best vision of mathematics, best accessibility for students, did not exist. Each had its own way. That the courses were not identical and that they were different implies that the notion of "best" may not be very helpful for mathematics education and we might instead try to think of courses as containing particular possibilities.

CHAPTER TWO

REAL ANALYSIS: CONVERTS

I have had my results for a long time: but I do not yet know how I am to arrive at them.
--Carl Gauss

Introduction

The Real Analysis course was the second of three courses required by all math majors[5] after the Calculus sequence and Linear Algebra. It was also the only Real Analysis course available to students in the semester of this study. Again, like many courses in this study, depending on a student's entry into the program, a student could have been in any year of their coursework, at or beyond the sophomore level. Given that this was a fall term, most students then would have taken a Linear Algebra course the previous spring or over the summer making them sophomores if they had tested out of the first two Calculus courses. For students who only tested out of the first Calculus or second course or neither, they would most likely be juniors when enrolled in this course. To take this class in a senior year, while possible, would have meant that a student was running close to not finishing on time. This course was a prerequisite for students electing to take Complex Analysis.

At the time that I observed this course I did not have access to records of the students enrolled in this course to compare with those accepted into the certification program. Most students intending to be student teachers would have gained acceptance to

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5 All students, that is, who were not taking the Honors program. Of the three courses—Linear Algebra, Real Analysis, and Abstract Algebra & Number Theory—only the latter two had Honors options.

the certification program in their junior year. Therefore any student who was a sophomore would not have known his or her status in the application process. Because the majority of students in the course were most likely sophomores or juniors it would have been difficult to determine who was planning to be a teacher as they would not yet be enrolled in either the mathematics methods course and/or not yet accepted into the program. There were at least a few students in the course who were awaiting their acceptance letters as evidenced by conversations I overheard. Because all preservice mathematics teachers had to also be math majors, all preservice teachers took a Real Analysis course at some time in their program.

The course met three days a week for 50 minutes each in the mathematics building in one of the smaller rooms available. The rows of desks were faced so that they pointed towards a blackboard at the front of the room with a panel of narrow windows to the left of the students. There was a computer station that linked to a projector mounted on the ceiling and an overhead projector available, but these were never used. The room was next to the stairwell and a lot of traffic from the hallways generated noise so the single door on the right side of the room was often closed during class. The 27 enrolled students took up most of the available desks. The students were mostly talkative during in-class group time and before class. While Dr. RA was lecturing they tended to only speak when they had questions or were prompted by a question. The atmosphere was generally still with bursts of talk when engaged by Dr. RA to do so. The students addressed Dr. RA by his title and last name and he addressed them by their first names. He seemed to know them well enough to comment on their interests. They seemed to only know the names of the people who sat around them and often did not engage with students not in their study group. The general atmosphere was friendly and academic.

Prelude to the analysis

Recall that throughout this chapter and the following three related to the four courses observed, I will be structuring the analysis into four parts—the substance, mode, regimen, and telos. The substance is the aspect of the student that the course design attempts to change. What aspect of the student the course design attempts to remediate, alter, or strengthen. The mode refers to the means by which the course design invited students to take on this change. The regimen relates to the things students were asked to do that would presumably effect the change. The telos is a description of what a model or perfect student might look like who emerged from the course.

Students in this Real Analysis course were positioned as residing within an intuitive epistemic world. By that I mean that the students were addressed as if they believed acquiring mathematical knowledge was a process of satisfying a personal criteria of either intuition or trust in an authority. I am not saying that they were positioned as intuitionists in the sense of the mathematical philosophy, although the assumed student position did not equate the truth of a statement with its provability. In this restricted way they could have been thought of as intuitionists. Yet they were not seen as being strictly intuitionists who also held that the law of the excluded middle, for example, was to be rejected.

The elements of the course suggested that their mathematical pasts have not trained them to be careful about what they know and take to be true in mathematics. In the absence of an authority such as a professor or textbook the students were supposed to determine what counts as knowledge—accurate and true—by referring to their intuitions or senses. The course pushed them from thinking about truth as either residing externally in authorities such as professors and textbooks, or as an internal hunch or intuition. Knowledge was not constructed as being accessible by intuitions. Rather, knowledge was derived from a process of mathematical logic and argumentation. Students were taught to develop a different epistemology: one where proof was the criterion rather than authorities or intuition. It is not exactly clear where students were presumed to have acquired this epistemology, but possibly from any prior course that was not a proof-based course, such as some Calculus courses and a lot of K-12 coursework.

Students were invited to take this different view of how one arrives at truth by appealing to their desire to participate in communities—both small group communities and the larger mathematical community beyond the classroom. Students were asked to judge the truth and falsehood of statements, craft mathematical arguments that conformed to the standards of the mathematical community, determine the relative persuasiveness of different arguments, and watch the professor do many of the above things. Overall, the model student who would emerge from this course would be a convert to a rigorous proof-based way of thinking about and participating in mathematical activities.

In that there was not a relationship between mathematics and some deeper meaning or realists or Platonic existence of ideas and that the doing of mathematics was deeply human, the course design presented a social constructionist view. The formal work of analysis was presented as a human response to the perceived flaws of the Newton and Leibniz Calculus, based on the needs of the changing community.

Substance: Epistemology

> Many math majors find these courses to be among the most challenging they take because the way of thinking may seem at first unfamiliar:
> - Computation will not be the focus; most problems will not have a number or formula as an answer.
> - You can't master this material by memorizing rote procedures.
> - Plausible reasons for expecting something is true, while to be desired, will not usually suffice. Proof will ultimately be required.

The quotation above is from the syllabus of the Real Analysis course. It expresses a view of mathematics that is utterly different from that expressed by Gauss in the epigraph of this chapter. Students will need to begin to arrive at their conclusions and stop taking them for granted. The students' epistemology, way of thinking about and determining what counts as knowledge, was the targeted substance of this Real Analysis course. As the syllabus quoted above suggests, students enter the courses beyond Calculus with a largely unquestioning orientation towards mathematics. They were assumed to think of doing mathematics as calculations related to given theorems, given either directly from the authority of a textbook or prior instructor, or from intuitions. Real Analysis both challenges the assumption of calculation and attempts to supplant it with a view of mathematics that is based in proofs.

Issues of determining knowledge precisely may have been new to these students but the formal proof approach was very much in line with the development of Real Analysis as a field as well. Much of real analysis in the 1800s to the early 1900s took the issues of knowledge, particularly related to our knowledge of real numbers, real-valued functions, and the underpinnings of calculus very seriously and hence the flourishing of precise definitions and attempts to build or describe the foundations of analysis during this era. For example, the real analysis by Dedekind, Cantor, and Heine in 1872 challenged the axiom of completeness as an axiom (something to be taken as a reasonable, intuitive starting point) and instead transformed it as a theorem of the constructability of the real numbers from rational numbers. Furthermore, real numbers were reimagined as sets of rational numbers and equivalence classes of Cauchy sequences, removing them from intuitive associations with quantity and measurement.

This course was designed so that students, like the mathematicians during the foundations era, would be provided with knowledge that they were lacking, namely a rigorous way "to know" mathematics. The course began with a statement that the students' sense of what counted as mathematical knowledge was limited. As the syllabus stated, the course was designed for students who knew a different kind of mathematics than Real Analysis. The students were positioned as needing to adopt a new way to know mathematics and so their epistemology was the target.

The textbook and the professor were both clear about this. On the professor's website there was a statement in a section called "Comments on the Course". It included this description of some goals for the students:

> To succeed in this course you will have to train yourself to think about the logical structure of the subject matter and understand the definitions of concepts and the statements and proofs of theorems. You will need to understand a collection of key examples and be able to reason about their properties. You will need to document much more of your thinking about problems than you probably have done before. While a good intuition is necessary to guide you to correct statements, just making correct statements alone will not be enough. You must be able to prove them.

In the above passage Dr. RA told students that they would need to "reason" and "document [their] thinking about problems" and acknowledged the positive role of intuition while also warning them that proof would become the standard tool for determining knowledge, specifically mathematical knowledge.

The textbook directly stated that students coming into this course were seen as lacking a rigorous epistemology. The textbook describes the challenge this way:

> instructors today are faced with the task of teaching a difficult, abstract course to a more diverse audience less familiar with the nature of axiomatic arguments." Later the author writes, "Having seen mainly graphical, numerical, or intuitive arguments, students need to learn what constitutes a rigorous mathematical proof and how to write one. (Abbott, 2001, p. v)

In both passages students are positioned as lacking a mathematically logical proof orientation towards justification of their idea and arguments. The course is designed as if students had probably encountered many graphical representations. The textbook, the syllabus, and the course website address students as if they might have worked many numerical examples of cases of an idea, and have intuitive arguments to support their thinking. They are lacking proofs. This course worked to address this cavity in students' mathematical knowledge.

The course challenged students to learn to think rigorously rather than intuitively. To accomplish this the professor did not set out to rigorously prove things the students already know or take for granted. To come to see the value of mathematical proof as a determination of knowledge, the course did not attempt to prove what seems obvious to them. Instead the focus is on using proof at times when things are not already clear or familiar. One day in late October while the class was discussing continuity, Dr. RA drew the graph of x^2 on the board. "I'm not going to try to convince you that x^2 is not continuous. You've known that it is for years. It just isn't uniformly continuous."

"As the function gets steeper, the deltas must shrink and so they are not uniform for each c_n value." Dr. RA assumed that students were already familiar with the graph of x^2 and that it was a continuous function. He did not choose to prove that it was indeed continuous because as he stated, they likely encountered this before, most likely in calculus. He chose instead to focus on the non-uniformity of that continuity, something he was assuming that they had not considered previously. After the picture argument (see Figure 2.1), Dr. RA went on to provide a formal definition for uniform continuity and a formal proof of the x^2 situation.

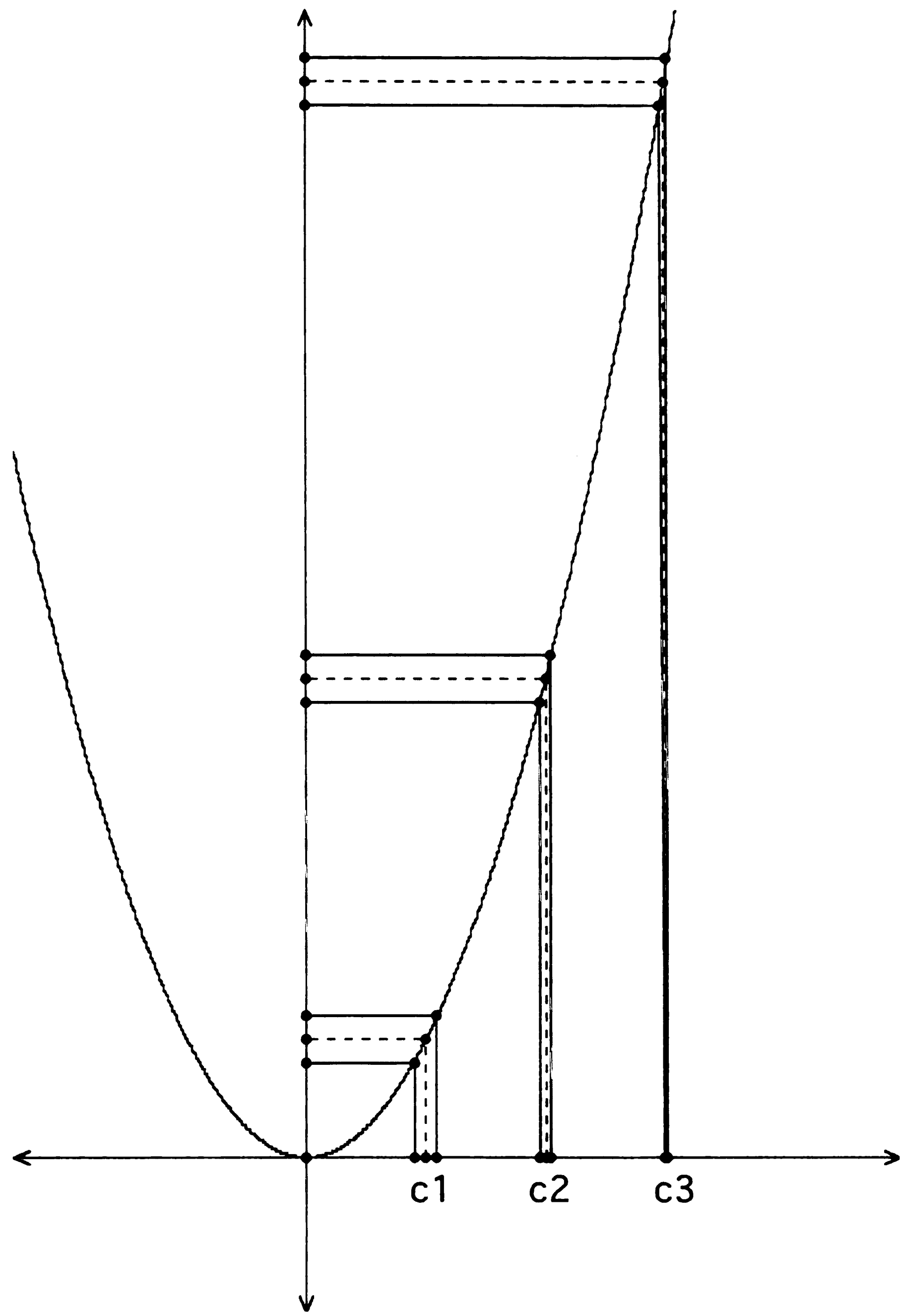

Figure 2.1: Dr. RA's sketch of x^2

This episode reveals two general pedagogic moves found frequently in Dr. RA's lectures. First, he did not spend a lot of time trying to re-examine ideas that the students already seemed to be familiar with or know. Proofs were not used to verify ideas that the students already hold to be true. Instead, proofs were a tool for dealing with unknown or unfamiliar mathematical terrain. Granted, for many mathematicians having a proof of the continuity of a familiar and assumed continuous function such as x^2 is valuable. But from a pedagogical perspective, Dr. RA did not use class time to re-prove known things. He used proof for contested ideas. To get students to value the need for a proof he used them when they were needed in the eyes of the students. Second, Dr. RA tended to start with examples (numerical sometimes and graphical in this case) and/or holistic arguments to set up the situation before moving to the formal definitions and proofs. Before providing students with the definition of a uniformly continuous function, Dr. RA discussed an example of one, and with the graph he showed what he meant by the non-uniformity of the delta neighborhood despite the uniformity of the corresponding epsilons.

To get students to change their epistemology, the course needed to negotiate a tricky position. Students must be convinced of the value of a more rigorous definition-proof system for determining knowledge, and they must also be taught how to work in this system. For students who have accepted certain mathematics as already known and true from earlier experiences (take the case of the continuity of x^2 again as an example), to provide a proof of that same knowledge might have had the effect of leading the students to perceive proof as an annoying and complicated method for restating "obvious" ideas. Students had probably encountered work in calculus courses where the continuity of certain functions was told to them. They were likely not skeptical of the source of this knowledge (a textbook or past professor). In these cases, to rehash 'known' material would not necessarily be interpreted as enlightening but possibly as redundant. Dr. RA's tendency to prove things that were not obvious suggests that he positioned students as needing to see examples of proof as necessary in their eyes, not from the larger community' eyes. This particular selection of problems set up proof as a valuable epistemic tool rather than as a redundant technique or empty exercise.

While the course was designed to get students to see the value in proofs as tools for establishing knowledge, the course also needed to deal with the epistemological problem that students take too much for granted. One common way the daily course activities attempted to change students' epistemological orientation was when the professor provided a definition and then asked students to classify things according to these definitions rather than according to their intuitions. For example, early in the semester the professor provided a definition of a *subsequence*. He then provided several examples of sequences and asked that the students decide which were subsequences and which were not. The point of the task was for the students to become accustomed to making their decisions based on the definition rather than on pictures or intuitions. The definition became the authority to check statements against.

The book troubled this use of definitions as authority a bit as it described on several occasions times in mathematics history where a definition was new, contested, or completely absent. One such case the book referenced concerned the definition of a function. The textbook provided a set-based definition of function and later in the following paragraph declared that the one provided there is “more or less the one proposed by Peter Lejune Dirichlet (1805-1859) in the 1830s” (Abbott, 2001, p. 7). By pointing out the date of the definition and its German mathematician author, “one of the leaders in the development of the rigorous approach to functions that we are about to undertake”, the text was signaling that definitions come from someone, somewhere and somewhen. Furthermore the passage described why this definition is better than others. Specifically it “allows for a much broader range of possibilities” than did function interpretations based on the idea of a formula such as $f(x) = x^2 + 1$. These prior understandings were not so much definitions of functions (as we now think of mathematical definitions) but generally accepted understandings. In this way, the textbook noted that the concept of a function existed prior to the definition but it was in defining the definition in this particular way that allowed mathematicians to formalize concepts such as continuous and differentiable functions.

By describing historical accounts of mathematicians struggling with the adequacy and accuracy of definitions, the textbook warned students of the problem of assumptions. It told the history of mathematics as an attempt to avoid these mathematical pitfalls. It also acknowledged the pragmatic aspects of mathematics history in that some definitions are based less on absolute knowledge and more on the benefits of certain definitions over others.

Before continuing through the description of the course it seems fruitful to point out how the professor's statements and activities were generally well matched with the book. On most occasions his definitions, theorems, and examples were taken directly from the textbook. Many students often followed along during the lectures by reading the book rather than taking notes. On one occasion he asked the class whether they would like to see him prove Rolle's Theorem. Some said yes. Some said no. One student responded to her neighbor, "It's already in the book isn't it?" The students were consulting their books frequently. From the students reactions in class, they seemed to have read the chapters well enough to know which theorems the book proved and which it did not. They also were asked to consult the text during class by Dr. RA. The students often referenced the textbook in class and so they very likely also read the notes on the history of the development of real analysis presented there, though I do not know what weight they gave these commentaries.

Aside from the use of definitions and their histories as warrants for claims, there were other reasons to suggest that the course worked on altering students' epistemologies about mathematical knowledge. Students were positioned as not knowing much about formal notation and proof writing. When Dr. RA wrote proofs on the board he tended to do so in full sentence and paragraph forms. To use shorthand notation would have implied that the students were comfortable reading and understanding this form of writing. Instead, he wrote nearly every word that he also spoke. This is in contrast to the other courses I observed where most spoken words in a proof were not also publicly recorded. In fact, in these other courses—with the possible exception of Advanced Geometry where pictures were also an important aspect of the work—words were rarely written at all. Take the following proof that an open set's complement is closed (which he followed with a proof that a closed set's complement is open) as an example:

> 1) Assume O is open. Let x be a limit point of O^c. By definition of limit point, every neighborhood of x intersects O^c. In other words, no neighborhood of x is contained entirely in O. Thus, because O is open, x cannot lie in O. So x is a member of O^c and so O^c contains its limit points. Therefore O^c is closed.

Writing out the words was not a reduction in rigor. Rather, students were treated as if they lacked knowledge of notation or ability to follow outlined proofs. They were treated as if they needed the fully detailed version, being positioned as unable to follow an outlined proof or a highly symbolic one. If students lack an understanding of the need for each and every logical step, why show them a proof that omits 'obvious' steps or rationales? Everything was made explicit. By including all of the words and steps in a proof, Dr. RA's pedagogy implied that students were not prepared to understand the distinction between outlined proofs and fully expounded ones. If he had only recorded outlines of proofs the students might have been able to follow along or they might not have noticed the unstated "obvious" assumptions. I do not know what the students might have been able to understand had Dr. RA only used short hand proofs. Because he did not use the abbreviated form in class, it is likely the course constructed students as needing to be shown everything explicitly. The implicit curriculum was that students who did not appreciate the logical progression of a proof might not have been able to discern which steps in an outlined proof were "obvious and therefore not included" versus what was left out because it was unrelated or truly unnecessary.

Dr RA's proofs had a more prose-like feel than did other professor's proofs I observed. He used a lot of chalk in class. If he said it, he usually also wrote it. In comparison, he did a lot of work to make the proofs he wrote readable and match spoken mathematics. But, the textbook did more. The textbook also gave a proof that an open set's complement is closed.

> Given an open set $O \subseteq \mathbf{R}$, let's first prove that O^c is a closed set. To prove O^c is closed, we need to show that it contains all of its limit points. If x is a limit point of O^c, then *every* [emphasis in original] neighborhood of x contains some point of O^c. But that is not enough to conclude that x cannot be in the open set O because $x \in O$ would imply that there exists a neighborhood $V_\varepsilon(x) \subseteq O$. Thus, $x \in O^c$ as desired. (Abbott, 2001, p.82)

Here again, is Dr. RA's proof on the same result.

> 1) Assume O is open. Let x be a limit point of O^c. By definition of limit point, every neighborhood of x intersects O^c. In other words, no neighborhood of x is contained entirely in O. Thus, because O is open, x cannot lie in O. So x is a member of O^c and so O^c contains its limit points. Therefore O^c is closed.

The textbook's proof begins similarly to Dr. RA's. Both set up the given statement or starting assumption on which the rest of the proof will rely. The textbook's version also specifies the set to which O belongs, the real numbers and it also states that the proof to follow only tackles the proof that O^c, the complement of O, is closed. Dr. RA then moves to invoke the definition of a limit point. The book by contrast foreshadows what the proof will do to prove that O^c is closed. "To prove O^c is closed, we need to show that it contains all of its limit points." Whereas Dr. RA jumped right into the proof, the textbook set up a sentence to establish the general direction for the upcoming proof. These sorts of preparatory moves suggest that the textbook author, like the professor in many of his other proofs in the course, assume that students need this sort of guidance to both understand a given proof and to begin writing their own. By providing well-formed models for students, the course is designed to construct both the students' lack of understanding of proof-writing and a change in their epistemological grasp of mathematics. Also, recall that this course was, for most students, the second course (after Linear Algebra in this program of study at this university) past the traditional three-semester Calculus sequence and so it is likely that students had had limited experience writing proofs before.

A rather cheeky MAA review of this textbook acknowledges similarly that this book offers a lot of support for new students.

> This is a dangerous book. *Understanding Analysis* is so well-written and the development of the theory so well-motivated that exposing students to it could well lead them to expect such excellence in all their textbooks. It might not be a good idea to create such expectations. You might not want to adopt this text unless you're comfortable teaching from a book in which the exposition will nearly always be clearer than your lectures. (Kennedy)

This review explicitly acknowledges that this degree of exposition is rare in textbooks and in lectures alike, going so far as to warn instructors that elect to use this text of the expectations for lectures students will come to posses. As has been mentioned, other courses observed did not provide written detailed proofs for theorems although the instructors may have verbally provided these. Other courses tended to omit the English words from proofs written on the board and only record the symbols. They would also not include phrases such as "First we need to…." or "To show X we first need to show Y." Although these sorts of anticipatory phrases were spoken in other courses, they did not get recorded publicly. In Real Analysis both the instructor, Dr. RA, and the textbook included very detailed proofs that not only did not skip "obvious" steps but that also gave commentary on the direction of the proof.

While the course was designed to change the students' criteria for the determination of knowledge, it also provided opportunities for students to learn about issues in mathematics that remain contentious and where logic and rigor can be pedagogically unhelpful. Take a definition of sets from the textbook as an example. "Intuitively speaking, a *set* is any collection of objects" (Abbott, 2001, p. 5). It goes on to provide examples of sets that are written as numbers, as words, and as symbolic algorithms. After also describing the empty set, unions, intersections, and complements, the textbook returns to the first sentence of the section on sets.

> Admittedly, there is something imprecise about the definition of a set presented at the beginning of this discussion. The defining section begins with the phrase "Intuitively speaking," which might seem an odd way to embark on a course of study that purportedly intends to supply a rigorous foundation for the theory of functions of a real variable. In some sense, however, this is unavoidable. Each repair of one level of the foundation reveals something below it in need of attention. (p. 7)

Repeatedly throughout the textbook, the author comments on the pitfalls, benefits, and implications of different definitions and issues concerning axiomatic systems in general. The conclusion of the first chapter is an epilogue that describes problems mathematicians have faced with set theory and the question of a complete axiomatic system. By striving to change students' epistemology, the book does not assert that all the troubles of the world are fixed once a rigorous definition-proof epistemology is accepted. Rather, it strives to continuously point out the problems of logic that mathematics continues to face.

> What is exactly is a real number?...the set **R** of real numbers is an extension of the rational numbers **Q** in which there are no holes or gaps…We are going to improve this definition, but as we do so, it is important to keep in mind our earlier acknowledgement that whatever precise statements we formulate will necessarily rest on other unproven assumptions or undefined terms. At some point we must draw a line and confess that this is what we have decided to accept as a reasonable place to start. (p.13)

Rather than falsely convey a sense that the epistemic system of mathematical proof is foolproof, the course included discussions of times where mathematicians have struggled to pinpoint ideas and definitions; places in history where fuzzy ideas were good enough, and times where mathematicians toiled to improve these definitions. There was nuance to this approach. The course suggested to students that rigor can be very beneficial and that much of modern mathematics depends on the foundations movement of the 18th and 19th centuries. At the same time, the course never went so far as to claim that this system is error-free or complete. For a course aimed at students who were assumed to be novices to proof and axiomatic systems, the course also treated them as if they were sophisticated enough to deal with the conflicts and contradictions of the mathematical community's preferred epistemological tools. As I will later describe, this substance is different from the Complex Analysis course's take on substance, which stressed a rational faculty in all things mathematical and personal.

Mode: Belonging

To invite students to take on a shift in their epistemology, the course appealed to a sense of belonging and membership community both in the broader field and history of mathematics and in the classroom. Despite a common stereotype that people who enjoy mathematics are not terribly social, the students in this course, as they did in the

Advanced Geometry course, worked together often. In this way, mathematics was constructed as a collective enterprise, not a solitary pursuit.

Although I was unable to be there on the first day of class, I learned several things about that first meeting. On the second day students who had not been present or enrolled for the first class were instructed to meet with Dr. RA to get placed into a study group. He explained that everyone else had already been assigned groups. These groups were to meet each Monday night and he had arranged a study room in a nearby building to use as the central location. He planned to be there and wanted everyone to attend. There would be no lectures, but he would be available to talk with groups and assist them with ideas needed to complete homework assignments or prepare for exams. These meetings fell outside the scope of my class observations but from comments made by students it seemed that they did attend these and were making use of this opportunity to get together with their professor and one another to do mathematics.

One comment that particularly suggested their interest in this community concerned what would happen on Labor Day, a Monday that should have been a study group night, but a day where no classes would be held. A student asked Dr. RA if he would still be there, suggesting an interest in attending despite the holiday. He replied that they would still meet from 7pm -9pm.

Throughout the semester students continued to talk in class about their study groups. On another day early in the semester as the class was packing up to leave, a student sitting near me turned to other students, presumably those in her study group, and asked them if they wanted to get together that night. They said that they did. "Good. I'll email you and we can set up a time." In late October I again overheard students planning a study session. A student suggested to her group mates that they get together Sunday night so that they can arrive Monday with better questions for Dr. RA. On another occasion a student walked in and told her seat neighbor, "I honestly thought about not coming because of how I feel." "I know", replied her friend, "I came for you." The norms that had begun to be established for students working together outside of class and with the professor seemed to be keeping many of the students interested in the course. Belonging to a community was a strong aspect of the class and one that provided an incentive to keep working, but the community of the study groups outside of class time was not the only means of inviting students to take on the challenges of the course. Groups were also used nearly every day of the course when I was present.

On most days Dr. RA began with a brief introduction to the ideas to be discussed that day. Sometimes he provided a new example to consider or a new definition that they would use. After this initial introduction to the day's topic he would pose a question or questions and have the students arrange themselves into their groups. After the first few days of this, students began to sit together in the class near their group members rather than taking any seat. They were deliberately arranging themselves physically to take part in the group aspect of the course. They would usually pull their desks together into a grouping and begin on the task. Some groups would have debates over the task and others seemed to tend towards working alone before checking their answers with one another. When at times the groups fell into this mode of silently working near one another rather than working together, Dr. RA would exclaim, "I want to hear you talking!" and soon they would begin to collaborate. This group time lasted anywhere from about 5-10 minutes on most occasions and on at least one day the whole period was devoted to their groups. Again, this practice emphasized that mathematics was a collective endeavor. The group work constructed mathematics not as a solitary, rational path, but as a field. Participation in the field was a necessary part of the conversion to the kind of mathematician that Real Analysis represented.

One day they were doing a review for an upcoming midterm. Dr. RA passed out a sheet of paper that was divided into three parts: 1) Definitions 2) Short answer and 3) Proofs. The definitions section included a list of twelve words of which four would appear on the test that they would need to define. Examples included "closure", "compact set", and "f(x) is uniformly continuous" For the short answer part they were given few statements that they would be asked to determine as true or false and to either provide a short explanation of their decision if true and a counter-example if false. There were twelve of these as well, and here is an example: "An open set cannot contain any isolated points." The final section contained four things to prove such as "Prove that the Cantor set C is compact." The final item on the sheet was a direction to "Construct an increasing function h: $R \rightarrow R$, whose set of discontinuity is $\{1/n : n \in N\}$."

After he passed out the sheet he said, "By all means work with your neighbors. It's not meant to be quiet." The students grouped up and began to work on the proofs mostly. "The definitions are easy. Just make up some note cards and memorize them," advised a student.

In the syllabus Dr. RA described the course as challenging because it would require students to think in ways that were different than they would have likely thought about math before. Specifically, rather than thinking of math as computation with given algorithms, they would need to think of ways to prove ideas as true or false rather than accepting them as such because an authority such as a professor or textbook said so. To invite students into this change, the course was structured to appeal to student's sense of belonging to a community and group membership. The groups in the course were quite formal compared to other courses studied. Advanced Geometry included students working together in class but they were not assigned to these groups. That course also had no formal post-class group arrangement. In Complex Analysis students were "allowed" to work together on assignments but were discouraged from doing it; in Advanced Geometry students worked together often in class but there no explicit instructions to continue doing so outside of class. Only in Real Analysis and Discrete Mathematics were students encouraged explicitly to collaborate. Dr RA went so far as to organize students to groups both in class and outside of class and to formalize the meetings of these groups.

The idea of students working together was not seen as an example of student laziness but as a mode of invitation to take on the learning of the course. This was the means of accomplishing the goal of altering their epistemology. In late October students were milling around in the hallway waiting for the previous class to exit the room. A student approached another student and asked him if he had finished a particular homework problem and if, in class, she could look it over. She quickly added, "I just want to know what is going on and I promise not to copy it." These students seemed to know that copying was taboo, but that you can learn from others without breaking that taboo. She took care to let him know her purpose for wanting to see his work. The community seemed to be seen by students as a way to connect ideas rather than as a tool to a better grade despite not being warned by Dr RA. As in-class accounts of students describing their plan to meet and the actual grouping of students in-class suggested, students were accepting this invitation to use community as a means to change.

While the groups were a formal and easily identifiable nod to community, the class banter also suggested that the course acted as a social group as well as a learning group. While it might seem a small detail that Dr. RA called on the students by name, this act was often not shared in other classes. In fact, aside from the Advanced Geometry course professors rarely if ever called on students by name. The instructors seemed to not know who was in the class at all although each course observed only had between 20 and 30 students enrolled. In Real Analysis, Dr. RA did know the students by name and used their names often. He could give papers back without needing to call on the students to identify themselves; he could call them by name when they asked a question; and he could "bless" their sneezes in a similarly personal way.

Not all work in the class was group work of groups or was cast in highly personal vocabulary referencing students name or interests. There were also more subtle instances of appeal to community. There were many times in a class period where Dr. RA would be proving some theorem at the board as students watched. At this time the students were positioned as audience members. They were not called to obviously and outwardly participate. They sat and watched. Dr. RA's pronoun choices were distinctive and constructed students as belonging to a community on these occasions. Dr RA used *we, us* or *you* as his primary pronouns. "Let me try to convince *you* with a picture." "I do want to hear *you* talking." "These are the conditions that make *us* able to do in calculus what *we* do." "How could *we* talk about this?" He did use pronouns on occasion that did not reflect a participatory assumption. "*I'm* going to do an example now." These times were fairly rare as most of his speech implied an interaction with the students through his pronoun use or eye contact and body language.

This personal relationship and subtle pronoun nods to a relationship may seem to be separate from the discourse of mathematics, but it constitutes an integral component of a mathematics course. Someone might argue that they are artifacts of pedagogy and not mathematics, yet the course design is a representation of mathematics in the world. The Real Analysis course constructed mathematics as a human, personal, and relational discipline. Mathematics was presented as something that people do together, not something that exists in the ethereal world to be pursued within an isolated mind. In this way the course represented a social constructionist view of mathematics and not a realists philosophy such as Platonism. Doing mathematics is doing what people, mathematicians in particular, do. If students wanted to belong to this group they needed to adopt the ways of thinking about mathematics that this community shares. The community aspect of the mathematics course makes mathematics seem more or less inviting to different kinds of people, and it is therefore a mathematical construct, and not just a pedagogical construct.

There were also significant amounts of time before class devoted to casual talk between the students and the professor and among the students. In other courses students might chatter before the professor arrived, but once he or she did arrive, they tended to grow quiet immediately. In contrast, when Dr. RA entered they would continue their personal conversations although usually in softer voices or they would engage him in their discussion. Sports—the university's football team and the regional baseball playoffs were favorites. Dr. RA also commented on changes he saw in them throughout the semester. For example, one day a student who had had really long hair arrived with it all cut off. Dr. RA's first comment on entering the room was to joke about the student's new look. Aside from Advanced Geometry, no other course included this sort of lighthearted chatter. Nor did those courses use student names as a part of the discourse. Although these sorts of lighthearted conversations and jokes might not seem very mathematical, they were a part of the community of mathematics present in this course.

In addition to the appeal of the community of students in this course (both in class and outside of it), an appeal to become members of a larger community of mathematics was also present. "In a recent rereading of the completed text, I was struck by how frequently I resort to historical context to motivate an idea," relates the author of the textbook (p. x). The uses of history and tales of the larger mathematical community act as enticement to the student to take on the shift in epistemology. As has been mentioned before and suggested by this passage, the textbook included many accounts of the development of real analysis in the history of mathematics. "This was not a conscious goal…" he continues, "Instead, I feel it is a reflection of a very encouraging trend in mathematics pedagogy to humanize our subject with its history" (p. x). Calling on history is calling on a shared community. Others have struggled with these ideas and so too might the students. Yet, whatever they face they are a part of something. Belonging to a community is a unifying theme of the invitation to take on a new epistemology. The grouping of students is a way to get them to take on this goal as their own. By grouping them outside of class there is both a practical resource for finding help and a shared obligation to help one another. Likewise, the in-class grouping activities create a community or a set of sub-communities that the students come to rely on. Finally, to become a part of the larger community of mathematicians is a further appeal to community.

Students were not instructed to study hard to pass or to get a good grade (and neither were they discouraged from this either). They were not invited to revel in the beauties of mathematics as they were in Advanced Geometry or to discover pleasing connections as they were in Complex Analysis. Instead they were called to join a community. Doing mathematics is to be a part of something larger than the individual. This draw of a community and a peer group acted as a motivating force. The smaller class groups and the larger mathematical context provide act as an invitation to take on the definition-proof system. It is the carrot-stick to the assumed reason they are taking this course—to shed the old ways of thinking of calculations based mathematics and begin to take on a proof based orientation. Furthermore, this group membership appeal positioned the students as capable and contributing members of the community. Students had a right to participate and a right to comment on mathematics; it was not reserved for a select few.

Regimen: Holding court

The daily activities and general atmosphere of the class has already been described in previous section, but there is still more to be said about the things students did for this course. Primarily, they were asked to determine the truth. They were asked to sit as judges, juries, lawyers, and sometimes as the general public in a mathematical court. The students were asked primarily to engage in truth activities—judge the truth of statements, be convinced of the truth of statements, argue for the truth of statements, and watch similar proceedings of truth activities.

A very common way that the in-class group work unfolded was for the students to be given a set of statements in a true/false format and to judge which were true and which were false. The second meeting of class provided an early example of this type of activity. Dr. RA instructed the students to group together and answer 1.3.9 from the textbook. It included five parts.

> a) A finite, nonempty set always contains it supremum.
> b) If $a < L$ for every element a in the set A, then $\sup A < L$.
> c) If A and B are sets with the property that $a < b$ for every $a \in A$ and every $b \in B$, then it follows that $\sup A < \inf B$.
> d) If $\sup A = s$ and $\sup B = t$, then $\sup (A + B) = s + t$. The set $A + B$ is defined a $A + B = \{a + b : a \in A \text{ and } b \in B\}$.
> e) If $\sup A \leq \sup B$, then there exists an element $b \in B$ that is an upper bound for A. (Abbott, 2001, p. 18)

Three students near me pulled their desks together and began to address each statement in turn. These students were rather quick to judge and within a few minutes had decided the order went T, F, F, T, T. Other groups around the room were having more heated debates. Whereas this collection of students near me acted as judges, other students were acting first as lawyers. They were arguing their positions against one another before passing judgment. All the while Dr. RA roamed the room and watched students, pausing to ask questions, and generally gauging the direction of the groups. After about ten minutes he called on students to offer their decisions; to have their say. Whether by purposeful selection of students or by coincidence, each person who offered a response was in turn judged to be correct by Dr. RA. This type of true/false activity represented most of the in-class group activity I observed. True/false judging was also a staple of the exams. A slight variation of this true/false work that would also appear would be like the statements below assigned in late September. Students were asked to determine if the below requests were possible and if so provide an example or if they were not possible and if so provide an explanation. They were not exactly true/false scenarios but could be created/ could not be created scenarios.

a) A Cauchy sequence that is not monotone
b) A monotone sequence that is not Cauchy
c) A Cauchy sequence with a divergent subsequence
d) An unbounded sequence containing a subsequence that is Cauchy

In addition to judging the veracity of statements, students also acted as juries within the class. When the in-class group time was not spent determining statements as true or false it was spent crafting arguments to prove some mathematical point or theorem. In mid-November, Dr RA wrote Rolle's Theorem on the board.

> Let $f : [a,b] \rightarrow \mathbf{R}$ be continuous on [a,b] and differentiable on (a,b). If f(a) = f(b) = 0, then there exists a point c in (a,b) where f'(c) = 0.

A student stopped him and asked whether f(a) and f(b) actually needed to be equal to 0. Dr RA replied that no, that condition was not needed, but it was still true. Again, this is an instance of students having rights to speak about mathematics, question it, and refine it. Dr. RA then erased the "= 0" from the initial assumption on the board to get:

> Let $f : [a,b] \rightarrow \mathbf{R}$ be continuous on [a,b] and differentiable on (a,b). If f(a) = f(b), then there exists a point c in (a,b) where f'(c) = 0.

As Dr. RA was making his adjustment to the board he was also telling the class that the proof could be found in the book for this more general case. When he turned around again two students in the front row had their books open to the proof. With a laugh, Dr. RA told them that they were cheating and that in a moment he wanted the class to prove it without using the book. The two students also laughed and closed their books. The students grouped up and began to prove the theorem. The group closest to me began by considering a pictorial case (see Figure 2.2):

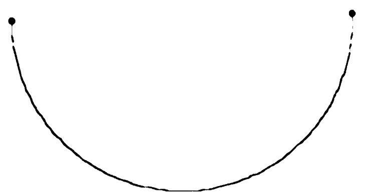

Figure 2.2: Students' sketch where f(a) and f(b) were maximums

They argued that in this case the endpoints f(a) and f(b) were the maximum points and the minimum was somewhere in the middle so there existed a $f'(c)=0$ in there (at the minimum). Their next picture was a flat line and they argued that f(a) and f(b) were both minimum and maximum points and that everywhere along the interval there were points whose derivatives were 0. See Figure 2.3

Figure 2.3: Students' sketch where f(a) = f(b)

The final pictorial case this group reasoned through was a reflection of the first and their argument was similarly reflected. If f(a) and f(b) were the minimum points then somewhere along the curve was a maximum, c, and at that point the derivative would be zero. See Figure 2.4

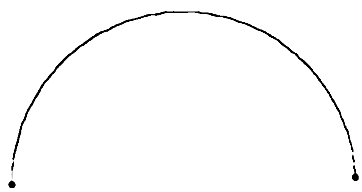

Figure 2.4: Students' sketch where f(a) and f(b) were minimums

Although this group failed to consider cases where f(a) and f(b) were neither the minimum nor the maximum, they were engaging in mathematical argumentation. This sort of in-class proving was the common practice. At these times students behaved more as lawyers than as judges. That is, they were in charge of developing the arguments to show a desired conclusion rather than to judge the validity of a statement or case. Just as tests included a judging aspect in the true/false sections, the tests also included arguing sections where students proved given theorems.

To continue the courtroom metaphor, students also acted as jury members who decided the value of an argument. On many occasions Dr. RA acted as the lawyer, proceeding carefully through an argument or proof. His proofs were very detailed as has already been noted, but they were also varied. He did not just present the statements and rationales to prove his case, but also invoked first-hand testimony and graphical evidence. On the same day that the class was proving Rolle's Theorem, Dr RA also proved the Mean Value theorem. To do so he first drew a picture of a generic function that "wobbled" between f(a) and f(b) and talked through the picture as evidence that supported his claim that there existed a point c along the interval that had the derivative

$\frac{f(b)-f(a)}{b-a}$. When he had concluded this argument he proceeded to provide a second way for the students to be convinced. He told a story about driving along the New Jersey turnpike between Philadelphia and Boston. The speed limit was 55 mph and if you got on at mile ten at 9:00 a.m. and got off at mile 50 at 9:35 a.m. you must have been speeding. There must have been a point at which you were exceeding 55 miles per hour. He concluded his argument with a formal proof of the Mean Value Theorem. As he proceeded through the layers of evidence the students sat as jury members to be convinced of his argument. He explicitly acknowledged their role by asking them twice while writing the formal proof whether they were "convinced" and one time " does everyone understand?"

That students were juries as well as lawyers and judges is important. Dr. RA did not take for granted that the students would necessarily be convinced of a mathematical argument just because he claimed it to be valid. To have assumed that sort of trust would have been to keep working within the system of thought that the course was trying to override, a system where students took ideas as truth because they were told to do so by trusted sources such as professors, teachers, and textbooks. For this course, it was important to learn that the rules of mathematical logic and argumentation are valid forms for attaining knowledge. Students were treated as if they might not yet fully believe that his arguments and logical moves are in fact "proving" the theorem and so Dr. RA needed to gauge their acceptance of his proof. Students were expected to have a say in what was convincing and what was not. They were expected to speak up when something was not persuasive or when they did not understand the proof. Doing mathematics includes being convinced.

In summary, students were expected to hold court regarding the truth and validity of mathematical knowledge. They acted as judges when they assessed the truth or falsehood of claims while working in small group sessions in class and individually on exams. They acted as lawyers when they debated one another in these small groups or when they crafted proofs or arguments of theorems. At times they were juries deciding whether Dr. RA's proofs and work at the chalkboard were convincing. In the remaining times they were the audience watching the proceedings as Dr RA performed the roles of lawyer, judge, and jury. To describe the students as either active or passive depended on the nature or genre of the class time. In one respect being a jury member is both passive and active. They typically sat still and quiet, but they were passing judgment through body language such as slumped in chairs with hands propping up heads or leaning forward with pencils in hand. Silence on their part was both active and passive. For a course that worked to alter students' sense of mathematical knowledge and how we come to know, the kinds of activities students were asked to perform constructed the learning of mathematics as a multi-dimensional practice. Students were not positioned as recipients of mathematical facts or knowledge. Students acted as lawyers, juries, and judges with respect to mathematical proofs. They practiced argument, documentation, and evaluation as daily class activities.

Telos: Converts

By describing a model student I am not describing an actual student who was found in the course. This is not a characterization of a real person, who for example, made good grades. Instead it is a description of the imaginary model—a perfect realization of the perfect course under perfect circumstances. What ideal of learning mathematics was constructed by the Real Analysis course?

A model student this course would be a convert to a community of formal proof-oriented mathematics. This model person would no longer hold that math was all about numerals and fancy calculations. Rather, in the ideal outcome of this course, students would know that math is about number systems and real functions. It would be the construction and sharing of proofs that rigorously described things most people took for granted as true. The convert would believe in these ways of thinking and doing mathematics. She would adopt proof as the criteria for knowledge and that the form of writing a proof conferred knowledge. But she would not also necessarily worry about the origins of this system. She would think that ideas/theorems needed to be grounded in proof and logic but would not question the origins of that logic. This convert would not think much about axioms and their consequences so much as use those axioms and know when others used them well.

The conversion was an epistemological one—a change in what evidence counts to create knowledge, but not whether that evidence is itself true. That is, rather than question the legitimacy of calculus, this model student only questions how we know calculus. Do we know it because we were told in Calculus I that the Mean Value Theorem is true or do we know it is true because we can prove it to be so. So long as the proof holds up to the standards of the mathematical community, it is deemed by the convert to be true. The model student is not skeptical of the system—this convert has faith in the system, but the model student is skeptical of all mathematical assertions until that knowledge has been proven, "beyond a reasonable doubt," to be of the system and by the codified rules of proving, a special genre of writing. As the textbook of this course states, "At some point we must draw a line and confess that this is what we have decided to accept as a reasonable place to start." The model student of this course does not concern himself with redrawing that line. He works to determine that once the line is drawn all further ideas and theorems conform to it.

Part of this shift in belief from thinking about math as facts and algorithms to a belief in tight definitions and logical proofs would be based on a trust in and a desire to be a part of a larger mathematical community. Proofs are to be shared and communicated with others. The convert wants to hear others' ideas and wants to share her own. As a judge she is willing to decide if arguments fall into the logical mathematical framework and is willing to have her own judged similarly by others in the same community of belief. To stretch the beliefs metaphor a bit, the model student would enjoy attending group meetings with others who shared his faith and interests. He would be happy to debate and be debated. He would also be happy to listen and observe others engaging in mathematical conversations.

Although this constructed student of Real Analysis is open to having his proofs judged and to judging others' proofs, he is not necessarily a creator of new ideas. No evidence from the class suggested that an ideal outcome would be for students to create or formulate new mathematical ideas, theorems, or concepts. Instead, this course aimed to create a model student who could operate and make decisions based on a particular code of logic and argument—not someone who could create those codes or experiment with changing them. A model student would act as a judge of mathematics but not as an artist of it. She could determine the truth or falsehood of ideas based on whether those ideas were within the bounds of mathematical logic. In the same way that a judge determines whether certain acts fall within the law or outside of that law, so too does the model student use her codes of belief to assess the worth of an argument. But this constructed students does not create new laws or codes of being. She did not push the boundaries of established mathematics but stayed within them, like staying within the Constitutional limits or within case law precedents.

In summary, the model student constructed in this Real Analysis course believes in mathematics. She believes in an organized and logical system of mathematics. She does not mind speaking her mind to question others' ideas and does not feel slighted to be questioned herself. His belief is not a gullible uninformed one. The convert is part skeptic after all and enjoys discussing the minutia of a proof, but this part skeptic is still a believer in the greater truth of mathematics. There is a line after all beyond which the believer takes as faith. Anything on this side of that line though is open for skepticism and demanding of proof. The model student is also social, reveling in conversation about challenging mathematics and observing others likewise. To be a convert is not to be naive or cowed by others. The model student is sure-minded and ready to be part of a community and group of mathematics.

This Real Analysis course positions students as being in need of an epistemic change. A model Real Analysis student no longer believes that mathematics is to be accepted at face value. Students are seen to be in need of a new way to determine the truth of mathematics beyond accepting the word of teachers and textbooks. To accomplish this change the course makes use of a community. The students are invited to participate in small in-class and outside of class communities so that they can take on the challenges of this new episteme. Community is also conceived on a larger scale—that of the wider mathematics community of mathematician and other mathematics professionals. With membership in this larger community of students as incentive, students are asked to take on the change in their knowledge system. As part of the work to create this change students are asked to act as judges, lawyers juries, and audiences of a mathematical court that rules on mathematical issues. Statements are judged, and proofs are drafted and found either persuasive or not. Procedures are observed. After all of these experiences a model student that would emerge from this real Analysis course would be a convert to a logical proof-based system of mathematics. This convert would go on to become an active member of the mathematical community.

Concluding remarks

Speaking about a model student can be a helpful way of analyzing the course as a whole. It can tell us about the assumptions the course makes about students as a category of person. What do they lack? How are they inspired? What should they do? Who should they become?

But just about anyone who has experienced classroom life and real students, whether they be young children, middle schoolers, teenagers, or college undergraduates recognizes that perfection is rarely a part of the experience. There is no perfect class, teacher, or students. Students come to classes with varied pasts, personalities, and hopes. In recognition of these realities I would like to conclude this chapter and each of the chapters describing one of the observed courses by speculating on what some real students might think about this course.

The emphasis on formal proofs rather than calculations or rote knowledge might have turned many students away just as it could have excited students with very analytical approaches to mathematics. The variety of types of tasks, though, would have provided different opportunities to do mathematics for students with varied preferences. Students who liked evaluation more than creation would have also fit in well when the class engaged in true/false type determination. More of the student in work in class tended towards this, but those who preferred writing proofs would also have had a chance to participate. There was also a place for students who liked rote learning as tests included the memorization of definitions.

Social students who enjoyed working with peers, particularly a consistent group would have enjoyed this experience. They might have liked the study sessions as well as the in-class group work not only for the mathematics but also because they could form friendships. Those who wanted more one on one time with Dr, RA had a chance for this in the after class study sessions as well. Students with a more outgoing personality would have enjoyed the before class banter with Dr. RA and felt comfortable joking with him, as the student who cut his hair was mildly teased about the dramatic change in his looks. These sorts of students would have been well served by the appeal of belonging to a community that the course design provided. Many students in the class did seem to appreciate this aspect and made comments suggesting that they enjoyed working with their peers.

During the in-class group time many groups were very animated and talkative. But there were also groups that did not function very much like a group, but rather a set of individuals who moved their desks closer together. The social aspects of the class were not so appealing to these students. They preferred to work alone and then share and check their answers as a group rather than actually proceeding through the tasks together. On one day when the class was reviewing and preparing for a test by working on a set of tasks, one student was irritated that Dr. RA was not leading the review. "This is stupid. I should have stayed home. I expected he was going to go over something." The implication was that without Dr. RA's commentary or lecture this was pointless and that the group work was not at all attractive. For a while she and another student discussed their applications to the teacher education program She ended up leaving class before the period ended.

The social aspects of the class did not seem to be helpful for everyone. In another group the students seemed lost frequently and unable to make progress on tasks. It was not that they did not work together or try the problems. Rather the collective did not seem to be able to support one another. This group frequently called Dr. RA over to help them.

Both of these two cases, the girl who left class and the stalled out group, might have preferred a more direct instruction approach. It may have been that each held the assumptions about mathematics the course was trying to change—that the professor, an authority was needed to tell them how to proceed. Yet there was a different flavor to these two cases. In the first example of the student irritated by a group review worksheet, she did not express a need for Dr. RA's assistance. She implied that she could have done the work at home by herself. It was that the model of mathematics in the class did not match hers—a vision of a math classroom where the teacher talked and students took notes. That she needed him in the sense of his help was not expressed, but that it was a habit and an expectation about a normal classroom. The other example of the group that called Dr. RA over frequently seemed to be of an opinion that they needed him and that without him they could not make progress. Possibly direct instruction would have been more pedagogically appropriate for them or possibly it was the close interaction with him that was wanted. Either way, the group time was not a positive time for them. Students who did prefer lecture styles still had that available as class time alternated between two types of interactions—group and lecture.

Just because a student might have preferred either group or lecture (or both or neither), does not mean that they also enjoyed the removal of calculations from the work. As my analysis described, students were positioned as thinking that knowledge is the result of executing some calculation type activity. Expressing your knowledge for these students is executing those calculations well. Their skills would not have been valued and they would not have been asked for. For student that did have this image of mathematics and knowledge, this course would have been very abstract and possibly very annoying. They might not have seen it as mathematics at all but maybe as tedious or nonsensical. For others, the emphasis on proof was very positive. Take for example an exchange between two students, "I'm at a place where I actually feel like I know what I'm doing, which is a weird idea. I'm really excited about it." "That's where I was last week."

This course had space for students who enjoyed social and/or individual mathematics. In terms of how they do math, there were at least two possibilities: lecture and group work. Again, with tasks, there was multiple ways for students to express what they could do and liked to do: memorization, evaluation of arguments and statements, and the construction of proofs. But when it came to a vision of what mathematics the course design modeled there was only one option. Students who did not see proofs as a part of doing mathematics would have very likely had negative or at least not so positive experiences. But, others, like the students quoted above, found a new way to think about math and kind of liked it.

CHAPTER THREE

ADVANCED GEOMETRY: FUTURE RESEARCHERS

In mathematics the art of proposing a question must be held of higher value than solving it.
--George Cantor

Introduction

The Advanced Geometry course I observed was one of three courses in a set where math majors had to select one as required by their plan of study. The other two options in the set were an Axiomatic Geometry course and an Ordinary Differential Equations course. They could have taken multiple courses from the set, but these would count as electives in the major. Students planning to also earn a teacher certification in secondary mathematics were restricted in their options by being required to take one of the two geometry selections. Therefore, preservice teachers could be assumed to be largely present in these two courses. Furthermore, students who were not in the teacher certification program (or were not planning to be) were required to take the Ordinary Differential Equations course. This means that the only way students not intending to be teachers would be in this course would be if they were taking it as an elective. Everyone taking it as a requirement would be in (or planning to be in) the certification program.

As a 300 level course students could have been taking it anytime after they took the Calculus sequence and Linear Algebra, the prerequisite for nearly all courses at and beyond the 300 level. Therefore these are the only guarantees for students' prior coursework. Depending on a students' particular entry into the mathematics program a student could have taken this course in their sophomore, junior, or senior years. This made determining who in the class was intending to be a future teacher difficult as students typically entered the teacher certification program sometime in their junior year. Inferring from other research related to this course and from comments made by students in the course the semester observed, I would estimate that about 50% of the 24 enrolled students were either already in the certification program or had intentions to be so in the future. Across the math department, the course was often spoken about as appropriate for future teachers because of the geometry content—a major high school and middle school curriculum strand.

The class met in a computer lab in the education building, the only course observed that did not meet in the mathematics building. Another unique feature of the course was that there were two instructors. One was the professor of record as listed on the university's schedule of courses and the other was a person with particular expertise in the geometry software, Geometer's Sketchpad (Jackiw, 1991), which the course used each day including exams. The secondary instructor was not present at each class meeting, but the primary instructor was for each day I observed.

The room was arranged so that the computers faced the perimeter walls on all four sides and the projector was directed at a spot over the computers at the front. The instructors operated a laptop from the interior of the room but nearer the back (seats were available behind them although students rarely chose those) and had a good view of each student's or pair's screen. When the class engaged in whole class activities students either faced the screen at the front or towards the middle to look at the instructors. The general atmosphere of the class was informal and a bit noisy as most students spoke often either in whole-class discussion or in small groups or pairs. Considerable portions of class time were spent in these smaller group situations with many people talking at once all over the room. As the semester progressed most people, instructors included, seemed to come to know most others by first name and the students referred to their instructors by first name as well. The general feeling was informal but still academic.

Prelude to the analysis

The Advanced Geometry course constructed students as future researchers by creating a mini research experience that simulated some aspects of a research mathematician's habits such as investigation, pondering, conjecturing, selecting conjectures or ideas to pursue or reject, and proving. Unlike other courses observed in this study, Advanced Geometry was the only one where students constantly initiated the questions under investigation. Although there was a lot of comparative freedom in these explorations, the two course instructors, Dr. AG and Mr. SP, worked to provide a structure for the investigations to keep the class focused towards similar topics of geometry. Within these ever changing boundaries between freedom and limitation, students were asked to think less of themselves as students and more of themselves as novice research mathematicians. Students' self-concept –from student to novice mathematician – was the focus of the course. It was the substance, the aspect of the student that the course design worked to alter.

To further the student-as-research-mathematician construct, the course mobilized aesthetics as the invitation (the mode) to fuel a desire to understand already established mathematics and create new mathematics. Appeals to wonder about interesting, surprising, beautiful, and unexpected relationships and possibilities were frequent.

Students were asked to play in this course. Play was the regimen, or the theme relating the activities students were asked to do. In particular, they were asked to play games of "I Spy" and "What If." That is, they looked for things of interest to pursue. These might have been patterns or anomalies in a geometric scenario. They were also asked to imagine mathematics if certain ideas were removed or altered. In the parts of class devoted to non-Euclidean geometries this "What If" types of activities were central. "What if the parallel postulate were ignored?" "What if I tried to make a kite on the Poincáre disc?"

The course's work to change students' self concept, the use of aesthetics to invite students to make this change, and the conflation of doing math (working) with play suggest that a model student (telos) that would emerge from this course would be a novice research mathematician. Particularly, someone who is interested in creating new ideas with a focus on the aesthetics of doing so, who can articulate them to others in a style consistent with the greater mathematics community, who can defend them, and who takes pleasure in all of the above.

Substance: Self-concept

Students' self-concept of their own identity was the primary site of intervention in this course. The course worked to move students from seeing themselves as students to conceptualizing themselves as novice research mathematicians. That is, rather than act in a conventional "student" way where they sat, listened, took notes, and mostly remained quiet, the students in this course acted (or progressively did so throughout the semester) in ways more closely resembling a novice research mathematician. Obviously, they did not posses Ph.D.s in mathematics, but they were performing as researchers within the space of this class. They explored ideas, debated conjectures, offered critiques of one another and of the instructors, and proved results.

One important way that these students were moving away from a traditional "student" concept is through the role of questioning in the course. Students authored many of the questions under investigation in the class, they directed their questions (and expectations of a competent response) to other students, and they questioned ideas proposed by the instructors. By taking responsibility for posing the problems and conjectures and expecting their classmates to respond to their questions, they were moving towards a self-concept of a mathematician—someone who not only studies and knows established mathematics but also someone who creates it.

One example that illustrates the ways that students were seen as authors of mathematics occurred in early November. A student suggested that the class try to construct a square in taxicab geometry. Dr. AG suggests that they first construct one in Euclidean and then use that to help them create the square in taxicab. After the students alternated turns giving directions to Dr. AG for the Euclidean construction, three students began tag-teaming what to do for the taxicab one. After each student offered a "next-step", Dr. AG rephrased her suggestions. "Jamie wants us to…"; "Sasha says…"; "Have I done it right, Terri?" Both Dr. AG and Mr. SP constantly insert students' names into their comments. By referring back to the students there is a continual reaffirmation that these ideas are student ideas. Carla then spoke up, "I'm going to disagree with you," referring to a comment made by Mr. SP that a square in both Euclidean and taxicab geometry act the same. Dr. AG tells the class, "That is an interesting idea." After she explains that in Euclidean the circle that defines the construction of the square is everywhere symmetrical, but in taxicab it is not and therefore limits the square's construction. Several students as well as Mr. SP get involved in a conversation about the symmetry of the objects in both geometries. Finally Mr. SP asks another student, Aaron, can you answer Carla's question more effectively than I can?"

In this episode many students offered ideas, they spoke directly to one another as well as to the instructors. One also disagreed with the instructor and her idea drives the conversation for about ten minutes. When Mr. SP was unsuccessful explaining his point, he deferred to another student hoping that Aaron might be more successful. By deferring to other students, and by taking up student disagreement seriously, the instructor was signaling that the students had responsibility for the direction of the discussion. They were not to just sit and listen in the ways that a common image of a student might.

Although the differences between the ways that students interacted with other students in this class, and particularly in regards to questioning, were striking there were other ways that the course acted on students' self-concept as novice research mathematicians. By deemphasizing the instructor's role and providing many opportunities for students to interact with one another the course laid a foundation for students to seek out resources in addition to the instructors for assistance and ideas. When students could begin to act without always relying on Dr AG or Mr. SP's immediate guidance, they could begin to move away from acting as students who were reacting to the instructors. To shed a self-concept of the traditional student was not enough, but an important step. The course also worked to get them to take on a self-concept of novice research mathematician.

The primary way that this was enacted was by having students choose the questions that drove the curriculum. Students in Advanced Geometry were asked to create conjectures—to question the geometric scenario under investigation and to describe it in terms of a possible relationship or relationships. The first Friday of the semester provides a simple example. After the announcements were over Dr. AG directed the class to work for ten minutes on Chapter 1: Activity 5 from the textbook's CD supplement. It states:

> If one side of a triangle is extended at one vertex, the angle it creates with the other side at that vertex is called an *exterior angle*. Construct an example of this in Sketchpad. Measure the exterior angle. Compare this measurement with each of the interior angles at the other two vertices. What do you observe? Express your observation as a conjecture. Is your conjecture still true if you vary the triangle? (Reynolds & Fenton, 2006b)

The students began working in small groups, pairs, or a few as individuals. The two instructors as well as myself circulated around the room watching students work and pausing to answer technical questions about the software program and the task in general. After the exploration time drew to a close, the second instructor asked the students to share out their conjectures. One group said that the exterior angle is less than or equal to the sum of the other two. The instructor typed this into a textbox in the software that projects onto a large screen at the front of the classroom. A few more groups chimed in that the first group's conjecture is false—the exterior angle is exactly equal to the sum of the other two. Another group conjectured that the exterior angle is larger than any other interior angle. The lead instructor initiated a discussion of vocabulary such as "other two", remote, and non-adjacent angles. When all of the conjectures offered had been written on the screen the students were challenged to prove or disprove the conjectures under consideration. One student tackled the conjecture that the exterior angle will have a measure greater than either of the non-adjacent interior angles, but his proof was based on the idea that a triangle's interior angles sums to 180 degrees. Dr. AG recorded the proof he outlined but challenged the class to prove that a triangle's interior angles actually do sum to 180 degrees. If his prove relies on this, they should be able to demonstrate it. She next walked them through a proof, pausing to prompt students to offer the "next step." A student interrupted this exchange. He amends the proof about exterior angles, strengthening it to say that the exterior angle is equal to the sum of the two remote angles and proceeded to provide a proof, again relying on the 180 sum. Dr. AG then told the class that they have the mathematical tools to prove the 180 degree sum conjecture, which was abandoned to follow the amended exterior angle proof. The class continues onto the next activity, but ended before completing it. They were told that they would return to it next week after the Labor Day holiday.

This day described above resembled most other days of the semester. The class typically ran on a schedule of brief announcements, Dr. AG posed a situation, student pairs or small groups created it and explored it either on paper or by software, students publicly offered conjectures, then proved one or more of the conjectures.

By posing their own conjectures students had the opportunity to author the mathematics under consideration, but it is important to notice that the instructors set boundaries in which the class could explore. Rather than tell the students to "come up with some conjectures" in a completely free environment, the instructors give a specific context and direct the conjectures towards those relating exterior angles to remote interior ones. This allows space for the students to move away from traditional roles of sitting and listening to a lecture, for example, and instead create the topics to discuss. This is a move that opens the possibility for their self-concept to change to a novice researcher. Because they are constrained by the specific conjecture scenario the students are not exploring anything and everything about exterior angles. The direction to look at the relationship between interior and exterior angles provides scaffolding and limitations, hence the use of the adjective 'novice.' They remain students in a classroom context with instructors and course material. There is, though, a move towards allowing more authorship by the students, which in turn allows for the possibility of their self-concept to alter.

As the class came to be more habituated to the practice of creating their conjectures, the instructors gave more open-ended scenarios. One example of a very open-ended task was the major assignment, The Book, worth 30% of their course grade. In this assignment students were asked to create 3 conjectures and their proofs. They were guided to consider modifying or extending a task explored in class or in homework. They could also depart from class-based tasks for their Book entries. This assignment exemplifies many features of the course, some to be discussed in later sections. For now it is enough to notice how the students choose what to include in their books. The students get to choose how close they want to stick to topics from the course or how far they want to reach beyond the course topics. By providing this range of possibility the course make space for the students to alter their self-concept from student to novice mathematician. Just how far the students choose to progress away from a traditional student concept and towards a novice research mathematician one is up to each person.

Because students offered their own conjectures rather than the professors offering them (in class and in The Book assignment), it was sometimes very unclear whether a conjecture was true or false. That is, false conjectures made their way onto the projected screen by the typing of one of the two instructors. Ideas were not censored at this stage. To move towards a self-concept of a novice mathematician, students not only needed to create conjectures and mathematical ideas, they also needed to argue against them or prove them. The students, prompted at first by the instructors and progressively throughout the semester by other students, demand proofs in part because they have no other way to know the truth of the statement. That is, they create the conjectures rather than the professor stating them as theorems to be proven and so *need* a proof in a way that a student who is told to prove a theorem does not need the proof. In the former method, the proofs are necessitated by the uncertain nature of the conjecture. Granted, many students were satisfied with "seeing" the results on the computer program and accepted these as proofs (especially early in the semester), but the instructors and other students continually pushed on these empirical observations.

In this Advanced Geometry course, it was the responsibility of the class to debate conjectures by refuting them, disproving them, or proving them. In this way the Advanced Geometry course shares features with the Real Analysis course. Students here also act as lawyers, juries, and judges. But to steal, the judicial metaphor, students here also wrote the theorems and might be thought of as members of congress. That students directed the content set it apart from other courses. In this course students directed the content, within the boundaries provided by the instructors and textbook—as well boundaries imposed by student ideas.

By making student curiosity central the instructors could not have complete control over the content although they directed it through the scenarios they dictated as starting points for the student investigations. The pedagogy of eliciting student-driven content created mathematical uncertainty that later needed resolution. Student formulated idea were often non-rigorous in part because they were initial ideas and brainstorming, but as the class period continued (or the week depending on time issues) the instructor(s) and other students demanded that the conjectures be restated with more technical vocabulary and proven if possible.

The course syllabus speaks of mathematics in terms of topics initially and later in terms of investigation and processes. The first paragraph declares:

> This course will begin with the geometry of the ancient Greeks, and will progress through to more modern topics in geometry inspired by those ancient roots. In the spirit of a mathematics defined first by its generating tools (the ancient compass and straightedge) we will use in this course The Geometer's Sketchpad, a powerful modern visualization tool for expressing and analyzing mathematical relationships. The topics will include:
> • Classical Euclidean plane geometry,
> • Modern Euclidean geometry,
> • Transformational geometry,
> • Projective geometry and the complex plane
> • Hyperbolic geometry

This list of topics tells us that the construction of the subject here in this course needs to learn some topics, but does not quite say what these will be. The last four bullets refer to non-Euclidean geometries. By including different geometries the course design is exposing students to new ideas (new compared to the Euclidean they likely know from high school). Furthermore, these geometries often contradict one another. A theorem in one will not hold in another. Like the Discrete Mathematics course, this one is interested in expanding students' views of what counts as mathematics. It is a contrast to the Complex Analysis course where students are exposed to a unitary and coherent vision of mathematics.

The syllabus later describes other aspects of the course:

> In this class, you are being invited to explore geometry through computer-based investigations, to make observations and conjectures about what you see, and then to develop proofs to support or refute your conjectures. You will be "constructing" many of the theorems and ideas we encounter, both in class and through your homework. While we will be learning about many results in geometry, some discovered long ago and some discovered more recently, you will also be expected to initiate your own explorations as well.

The first section quoted refers to topics to be explored in the course and makes explicit mention of the ways that modern mathematics have been inspired by Greek mathematics, including both topics and tools or processes as modern inspirations of ancient techniques. There are topics and techniques or processes at play in the course. The processes seem to be given greater weight during the class observations. Although students were invited to learn theorems and definitions, it is the processes of exploring, conjecturing and proving these theorems that take up most of the class time and conversation. Because the Advanced Geometry students were as active as they were in creating the conjectures, refutations, and proofs, they were acting less like traditional students and more like contributing members of the mathematical community. Students were treated as if they were capable of making decisions about which mathematics to pursue, capable of creating their own ideas, and capable of verifying them.

Mode: Aesthetics

If the primary aspect of the student this course works on is self-concept from traditional student or observer of mathematics to novice researcher or creator of mathematics, the mode of inviting students to take up a new self-concept here is by invoking and highlighting aesthetic elements of mathematics.

In the above example from the first week of class the students work to create conjectures and proofs from a situation involving interior and exterior angles of triangles—one that they likely had encountered before in high school geometry. As the semester continued the students began to work on mathematics not typical of high school curriculum. In mid-October Dr. AG emailed a Sketchpad sketch to everyone before class with instructions to download it for class discussion during the next meeting. It looked like the one below in Figure 3.1.

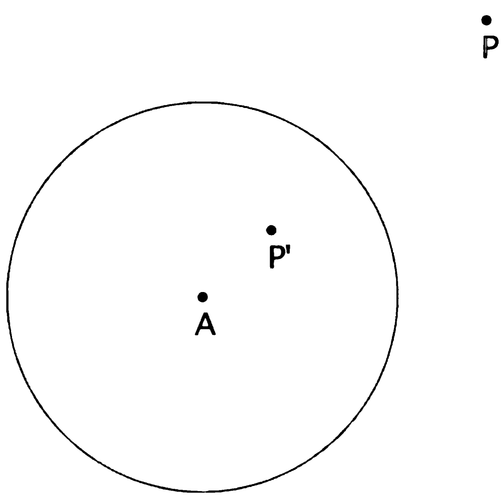

Figure 3.1: Dr. AG's emailed sketch

As class began and before either instructor gave any directions about the sketch students downloaded it and started dragging different points around to see what would happen when changes were made in the position of P, P' and the size of circle A. About halfway through the semester the routine had become established that when given a sketch or situation their job was to start asking questions about what changed and what stayed constant. On this day the professor instructed them to focus specifically on the relationship between P and P'. Developing curiosity is a central part of the course but curiosity is a way into aesthetics, the means of inviting students to participate. By providing opportunities to wonder—to be interested in, to doubt, and to speculate—and time to follow that curiosity, the course attempts to strengthen students' aesthetic relationship with geometry and mathematics.

When given the opportunity to explore the students could take up issues they found interesting that satisfied a personal aesthetic. Some students seem to revel in strange behaviors of seemingly normal objects—points P and P' in this case. Others took up the aesthetic by methodically analyzing the behavior of the points. For the first group of students the fun seemed to be the unexpected itself, for others the fun was to explain the unexpected. There was space for students to follow their own personal sense of wonder and aesthetic. Again, as will be examined in the chapter related to Complex Analysis students are not invited to take on the challenges of the course by presenting a well-ordered and structured sense of beauty—an aesthetic that values structure—but instead one that had room for considering structures as aesthetically pleasing and/or for considering strangeness beautiful.

As Dr. AG wandered the room she stopped to work with a student who had an idea but was unsure how to use the program to investigate it. Dr AG walked her through the technical details but did not confirm or deny her conjecture, leaving her to keep exploring. By refraining from judging the accuracy or merit of the idea, Dr AG kept wonder available. The student did not request validation of her idea, only asking for technical help in exploring it. She needed support with the program and was given it, but her own sense of wonder and possibility kept her investigating her mathematical idea. Another student was also working alone at a computer but instead of being excited by the seemingly erratic behavior of P and P', she seemed annoyed by it. Her aesthetic was to know what was happening in a linear and predictable way. She preferred to know why they were behaving as they were. Her annoyance revealed an aesthetic response, but one of frustration rather than joy. Some students wanted more order in their aesthetic and seemed to reject wonder for the sake of wonder. They wanted answers. These students would possibly have preferred the well-ordered rationality of the Complex Analysis course over this more open environment.

After about ten minutes students shared these observations at the request of Dr AG:

> 1) A, P and P' are on the same line.
> 2) P and P' are "sort of" a reflection through the circle but distance is not maintained.
> 3) As P gets really far away P' approaches A but never gets there.
> 4) When you bring P inside the circle, P' goes outside of it.
> 5) If you draw a straight line with P, P' will trace a circle.

As the students offer these ideas Dr AG represents them on the screen by tracing out what the students say. As she performs the 5th idea, a student exclaims, "That's Cool!" and Dr AG asks her to say that louder so that the rest of the class can hear. She wants them to know that mathematics is a place for coolness, here an expression of an aesthetic response. Mathematical ideas can tap into feelings. As the class continues Dr AG gives a mini lecture on the definition of an inversion, finally describing to the relief of the annoyed student what transformation was at work in this sketch, and what a locus is and how to use Sketchpad to create one.

After the lecture they were instructed to create loci for different positions of a line under inversion. One student I approached noticed that a line inverts to another line when the original line is on the circle. She seemed both fascinated and confused by her result considering that most of the positions of the line results in an inversion that created a circle. She spent a lot of time repeating her work and kept getting the same result. I left her as she began to try to understand why this happens. She was visibly frustrated but did not stop because of her emotional response. Instead it seemed to sustain her engagement. We can wonder about beautiful and amazing things, but we can also wonder about upsetting and confusing things. Harmony can be motivating but so can discord be a driving force. Both forms of aesthetic possibility coexisted in this Advanced Geometry course. Tapping into feelings of pleasure, annoyance, pride, and confusion were techniques for inviting students to take on the work of authoring conjectures and explaining or (dis)proving them.

A day in mid-September serves as another example of students using the vocabulary of aesthetics as personal interest, particularly interest arising from surprise that the same things kept happening. One student pulled the second instructor over and shows him that in his sketch, whenever he connects the midpoints of the triangles' sides he gets a new triangle that shares the same centroid with the previous and original triangle. If he repeats the process of creating new triangles based on the new midpoints, he continues to find smaller and smaller triangles that continue to share the same centroid. “This is interesting,” the student declares. The student was articulating that unexpected patterns were a type of aesthetic that he found important and that was moving him towards thinking of himself (his self-concept) as a person who could describe patterns and investigate them.

The class meetings devoted to hyperbolic geometry provided many opportunities to exploit the unexpected as an invitation to do mathematics. In late October Dr. AG and Mr. SP were leading a discussion of triangles in hyperbolic geometry. Dr. AG asked if it was possible to have two non-congruent similar triangles in hyperbolic geometry. A student cautiously replied, “Maybe,” bellying her uncertainty in this geometry. She went on to explain that she thought you would need to alter your understanding of similarity as understood in Euclidean space and that the trouble she was having was in thinking about the nature of comparing angles between triangles in hyperbolic space. Other students seemed to share her uncertainty. No one was rushing to share ideas. Dr. AG posed a new question. “Will the triangle congruence theorems work here?” Again, the students seem both intrigued by the question but hesitant to respond. One finally suggested that they would need to determine if the congruence theorems relied on the parallel postulate. If so, then, no, those theorems would not hold in hyperbolic geometry. With these introductory questions, Dr. AG and Mr. SP set up the students to think about the differences between Euclidean space and hyperbolic and the underlying assumptions of related theorems in the two geometries. Many of the students seemed to be rather perplexed but intensely interested. Furrowed brows and students leaning far out of their seats towards the projection screen filled the room.

On that same day Mr. SP instructed the students to take five minutes and construct a square on the Poincaré disc. When called together again a student provides this set of steps: Construct a segment; construct the circles with the respective segment endpoints as center and point; construct the line between the intersections of the two circles; construct the segments between the four points. Below in Figure 3.2 is an illustration of the final square when using this construction technique.

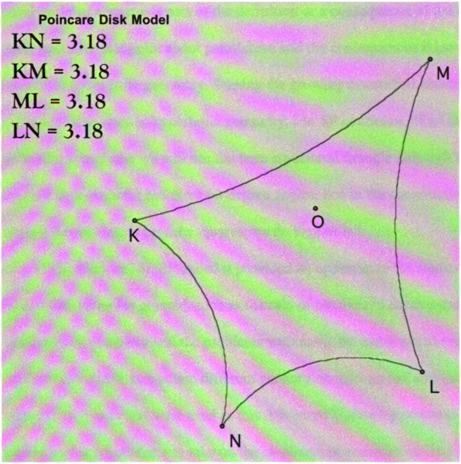

Figure 3.2: Student's hyperbolic square

The class embarked on a discussion of how this strange figure was a square because each side was the same length, but that it did not look like a square because the angles are not 90 degrees each. Mr. SP introduced the distinction between a regular 4-gon and a square. He commented that in Euclidean they are the same idea.

Before the students could stew on that idea for very long a second student offered another construction to consider. In this construction the square did have 90-degree angles, but the sides were not equilateral. Mr. SP posed the dilemma: Do the students prefer their squares to have 90-degree angles or four equal sides? A student asks if parallelism of a square is part of the definition or a consequence of the definition. The class discusses this point about parallelism and the consequences it has on constructing squares in hyperbolic geometry. Sensing the growing discomfort about the possible non-existence of squares in hyperbolic geometry, Mr. SP continues, "Part of the disquiet may be that in hyperbolic we still use the term equilateral triangle although no equilateral triangle in hyperbolic has three 60-degree angles. But in the construction, we do not exploit the 60-degree idea. In squares, we do exploit it."

Disquiet, as Mr. SP called it provided an opportunity for students to consider ideas often taken for granted, such as the role of parallels in the construction of a square in Euclidean geometry and the problems with using the same words such as square and equilateral triangle across two different geometries. Discomfort and even horror are also aesthetic elements and were used in this Advanced Geometry course to provide opportunities for students to take on the challenge of constructing new self-concepts—from a passive, note-taking student to an active participant in creating mathematics.

Complex Analysis, included elements of aesthetics but horror, disquiet, and discomfort were not seen as positive. They were to be avoided.

Students were curious is this class and they also found one another's ideas "cool," but they found their own ideas both interesting and possibly troubling. Some students seemed very annoyed by the whole process at times. These students might have been more comfortable with the use of aesthetics found in the Complex Analysis course where order, structure, and rationality were constructed as beautiful and inspiring. Mathematics in this class is seen as having the ability to create emotional responses. Very often these responses (coolness, interest, confusion) motivate further work. Not every student, though, found these more negative aesthetics pleasing. Both excitement and discomfort can lead to further interest and mathematics. By feeling excited or troubled they can channel emotions towards curiosity in understanding why. The course worked to get students to move away from traditional student roles and towards novice researcher roles. Perturbation and excitement were some of the aesthetic responses used as invitations to take on the new self-concept.

The Advanced Geometry course also drew on social aspects of doing mathematics as did the Discrete Mathematics and Real Analysis courses. In both Discrete Mathematics and Real Analysis the students were invited to belong to a community. Part of that invitation was to highlight explicitly the values of the larger mathematical community. Advanced Geometry did this at times, but not as frequently. Group membership was not a mode in this course, but there was a strong sense of communal responsibility for the doing of mathematics. All three of these courses constructed a vision of mathematics as about people working together. They had strong ties to social constructionism and humanist philosophies of mathematics.

Regimen: Play

The students in this Advanced Geometry course were asked to play with ideas. As much in the above sections suggest, students did a lot in this course. They talked, asked questions, conjectured, proved, wrote and thought. In doing much of this they were playing. Students played with their ideas by testing them out, trying this and that, changing their mind, and revising their conjectures. Sometimes they played alone and at other times with classmates and the instructors. There were many opportunities to be wrong and to try ideas out in multiple ways.

The use of the word play might seem out of place for the dual reasons that this is a university course, not a kindergarten, and that it is mathematics, not recess. Yet play, as a description of the activity of the course, is not far-fetched. In a section of the textbook titled, "To the Student," we see this description of the course:

> Playing can be a gateway to new ideas. In this course, we ask you to play with geometric figures, to explore their properties, and to observe relationships and interactions among those figures. As you play you will be asked to make conjectures about what you see happening. (Reynolds & Fenton, 2006a, p. xix)

Furthermore, the passage says, "Playing can be a gateway to new ideas." The focus is on the play—the explorations, visualizing, conjecturing, curiosity. By allowing play to be a theme of the activities students had space to come to new ideas. Recall that most class days included the instructor posing a set of geometric figures and asking students to create conjectures based on the objects in the scenario. Take this episode as an example that came at the end of a class period. Dr. AG had triangle ABC projecting from her laptop onto the large screen in the front of the computer lab. Inside of the triangle were four points labeled Mo, Bo, Jo, and Lo. She asked the class, "How could you figure out by dragging, what Lo, Bo, Mo, and Jo, are? What triangle centers are they?" A long conversation ensued. Students instructed Dr. AG to drag point B and watched the points move around the screen, then they asked her to drag A, and C likewise. Several rounds of such dragging continued, but by the end of class the students could only discover that Jo was the circumcenter. The next day they began playing again on this task.

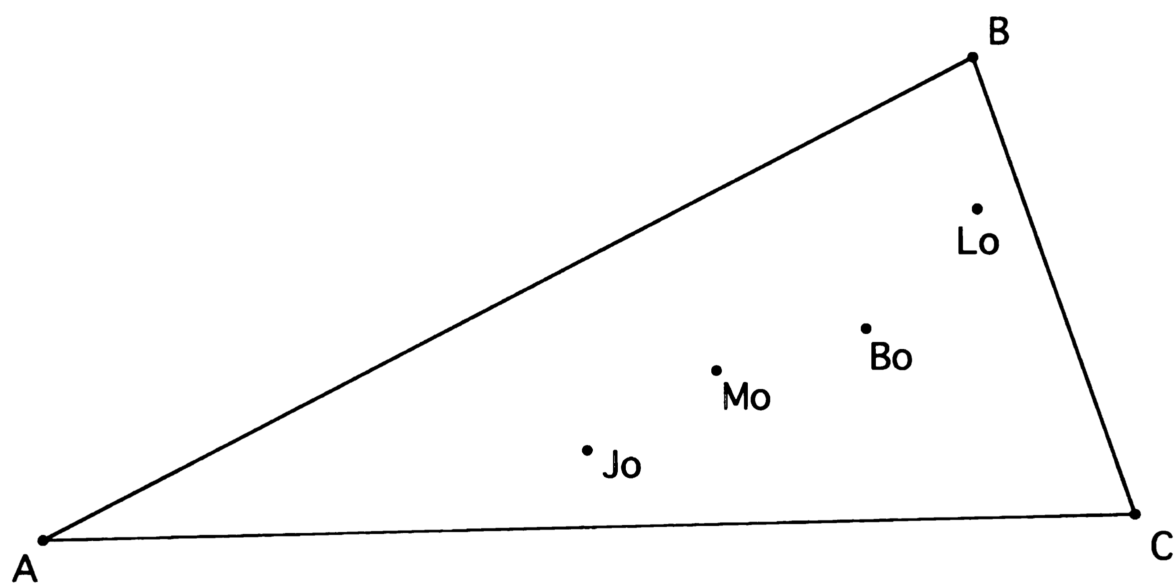

Figure 3.3: Dr. AG's triangle

Dr. AG asked, "Any ideas on how we can find these points?" but no students answered. She probed again, "Can we decide what Lo can't be?" A student suggested, "Centroid" and Dr. AG asked why, but again no one responded so again she probed. "What is the centroid?" Finally, someone spoke, but rather than answer her questions the student told Dr. AG to "Move B closer to C. I want to see what happens." She did and the student followed up by describing what was happening as Dr. AG moved B closer to C. "As this happens Mo seems to be the center of mass." Dr. AG replied, "What else can we learn?" A second student replied, "Bo may be the incenter because the incenter is the center of the incircle so Bo has to stay inside." "Great argument," Dr. AG praised. Soon a rush of ideas were shared:

> "Well, when Lo is inside all the rest are as well."
> "It [Lo] is inside when all the angles are acute."
> "The altitude one is always inside when the angles are acute."
> "It would be nice if all the centers met at a point for an equilateral triangle."
> "What would happen if it were isosceles?"
> "Three of them are always on a line."

In this episode the students were playing with the four given 'mystery' points. Unlike most of these sorts of exploration activities, Dr. AG controlled the triangle and the students gave her instructions on how to move it. They were not playing with their own computers but were—through her—playing with the ideas. Their work was very similar to a young child playing with the world. They watched to see if things changed, what changed, and when they changed. They watched to see what stayed the same, and when. They instructed Dr. AG to move things around to test out their conjectures. The task was a big game of "What if" and "I spy" combined into a math lesson.

What if angle BCA was roughly 90 degrees? I spy Jo coinciding with point C.

What if angle BAC was roughly 90 degrees? I spy Jo coinciding with point A.

What if angle ABC was roughly 90 degrees? I spy Jo coinciding with point B.

By making the activity an opportunity for play, the students were able to observe the actions of the points and begin to combine their observations into conjectures about the properties of the triangle centers. Play was the gateway to new ideas.

In other courses observed, students were usually told what mathematics was important and needed to be proven. In both Real Analysis (during lecture time) and Complex Analysis, the instructor would give a definition and then follow up with an example, and then a theorem related to that definition. Under that model, an instructor would have likely provided a definition for the orthocenter as the intersection of the altitudes and then would have posed a theorem such as "For right triangles, the orthocenter lies on the vertex of the right angle." The instructor would then have likely led the students through a proof. Instead, Dr. AG posed the task as a mystery game. By doing so, the students observed the behaviors of the points for various types of triangles, right, acute, obtuse, isosceles, equilateral, and scalene and used these observations to deduce which center was which point. Sketchpad became a toy in the game—a toy that maintained the geometric relationships inherent in the points' constructions. By playing the game students could formulate and formalize their ideas.

As evidenced by many of the above examples from the class meetings and textbook, students are asked to play both by posing and answering questions related to "What if" and "I spy". The aesthetics of curiosity, wonder, and meaning are exploited to draw students into these tasks.

Below are some questions and ideas projected onto the screen as students enter class one day late in the semester.

> Which triangle centers work in hyperbolic geometry and why?
> When will regular polygons tile the plane?

Which Euclidean theorems hold in hyperbolic geometry?
What does inversion look like in Taxicab Geometry?
How to solve CC.
How to solve CCC[6].
What does Napoleon's Theorem look like in taxicab geometry?
Constructing a hyperbola using power of a point.
How do polygons behave under inversion?
How to translate or reflect on the Poincaré disk.
Creating a new metric.
Finding the perimeter of an ellipse.
Construct hyperbolic tools from scratch.
The 'carpet' problem extended.
What properties of cyclic quadrilaterals hold in hyperbolic geometry?

These questions were selections taken directly from students' second entries in the project, *The Book*, an assignment inspired by Erdös' famous idea that God, or in his vernacular The Supreme Fascist, hoarded the world's most elegant proofs in this tome. In this assignment for the Advanced Geometry course students were asked to pay attention to ideas of personal interest and develop a conjecture/proof or mini project of the person's own design to include in his or her personal *Book.* There were three rounds of submission with the final and third round to be worth 20% (slightly more than a single test) and to include revised versions of the earlier rounds along with a new entry. The syllabus describes it thus:

> When the great mathematician Paul Erdös wanted to express particular appreciation of a mathematical result (a proof, theorem, or idea), he would exclaim, 'This is one from the Book!' Over the course of the semester, you will submit at least three geometric results. The following kinds of work would be suitable for inclusion:
> • A variation on a theorem or result we cover in class that you discover while constructing, exploring or playing.
> • A discovery based on one of the homework assignments.
> • A new way to visualize or prove a result covered in class or in the

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6 This is in the context of Apollonius' Problem—when given some combination of three objects, point, line, and circle, find the circle(s) that are tangent to each. CCC refers to three given circles and to solve it means to create a method for determining the circle(s) tangent to the three arbitrary given circles.

homework
• An illuminating collation or integration of different ideas, proofs or results from class

> Your aim should always be to attempt to explain what you know about a certain phenomenon and why or where it occurs. Proof is one language for making such statements, but you may find others. You are encouraged to work with others in the class during your investigations but you should each submit your own presentation of the Book entry. The following criteria will be used to assess this work: (a) originality and interest of findings (b) presentation of findings, including clarity and articulacy. Feel free to ask me if you're not sure whether something is suitable for inclusion.

The book entries quoted above that were projected onto the screen suggest that to be a student in the course, specifically a graded student, meant that you must attempt personally interesting and meaningful mathematics. To begin to have a sense of personal interest in mathematics reflects the themes of changing self-concept to an identity of a research mathematician with aesthetic sensibilities. The entries were self selected and so could be about nearly any topic related to geometry although they tended to be extensions of course topics. They were also teasingly similar to math papers, meaning that if these things had never been published before, it would not be hard to imagine a paper titled, "On Creating a New Metric." To be a student was to create new (at least for the student) mathematics and to prove or otherwise justify it. This major assignment also reflected the theme of play. The Book was a place for students to play with ideas related to class ideas and topics but out side of or extensions of class topics.

Very often in class students came up with ideas or shared observations not quite aligned with the investigations' focus that day. When these were shared Dr. AG would often direct the student to consider playing with these ideas further in the Book rather than in class. One day Mr. SP decided to stop class to make an announcement to the everyone that reminded them that although lots of them are coming up with interesting observations worthy of further study, these were departing from a focus on the 1:2 distance ratio involved in the task. He suggests that they put those off-topic observations aside for now and consider them instead for the Book.

When students play with mathematics, even when asked to focus in a certain direction, their ideas could jump out of the space of the current investigation. The Book servds as a place to hold these explorations. While they might not be related right now, there is a set place, an assignment, where these extra ideas and explorations found a home that remained within the space of the course. Students did a lot of conjecturing. Some of these fit the goals of the class meeting. Others did not, but instructors and students also did not discard them. Students could choose to return to these ideas as part of their Book assignment. Therefore play was both a part of the day to day activities but it was also a major assignment—one that served the purpose of continuing student play outside of class and that can also respect play that is unrelated to the core tasks of the given class period.

In addition to using play to further conjecturing and proving, students were also asked to explain their ideas both in small group and pair situations and in whole class arrangements. In other courses observed the students were also asked to share ideas during class time, but in Advanced Geometry, there seemed to be more participation by more students—even most. During the second week of class, pretty early in the semester, the class observation revealed a high level of student-to-student and student-to-instructor interaction. What follows are some abbreviated notes on the interactions that day.

Dr. AG called on two students to walk the class through an explanation of the previous class meeting's discussion. They sparred as a verbal tag team, alternating back and forth. A third student then commented on their idea. Then a fourth. Dr. AG next called on three more students, who each responded with an observation. Dr. AG moved away from collecting observations and asked the class if they could justify any or all of these observations. Two more students did so. Ten minutes into class and nine different students had spoken, either to offer an observation, a verification, or a comment. Next the class turned to small group work and everyone was talking to at least someone else in the class. Students asked one another questions such as, "What do we know about…." And "How can we use that?" Some students got up and walked over to other groups to watch them work or to ask another question. Dr AG and I were also moving around the room interacting with students when asked to do so. Students were pointing to their screens, gesturing with their hands, and taking notes both on their computers and in notebooks. At one point the laptop that was projecting the task onto the large screen went blank—it had timed out. A student looked up, seemed to notice this and that Dr. AG was busy with a group. He walked up to the laptop to wiggle his finger on the touchpad, reactivated the screen and returned back to his group to continue playing. When the class returned to a whole class discussion Dr. AG asked a student who had not yet contributed to the public conversation to walk them through her proof. This student had used different notation than was currently projecting on the large screen so Dr. AG adjusted her sketch to match the students'. This student continued her explanation, but made a mistake. The class waited while she collected her thoughts. When ready she began again. By the end of the class thirteen students had spoken in the whole class format—some short answers or comments and some long descriptions or proofs—and everyone had contributed something at some point in class. Play took many forms and everyone played in some way—alone, in groups, and in whole-class discussion.

Some aspects of the course were impossible to observe in class such as studying for exams, doing homework and working on their Books. Some examples I've already discussed point to the role of the Book assignment as an important part of the course in which students participate. The Book was worth 20% of their grade and was worth more than any other single assignment. There were three in-class exams each worth 15% and homework as an assignment group was worth 35%. This information comes from the syllabus and hardly any class talk revolved around grades although non-Book assignments were discussed every few observations (the Book was mentioned much more frequently as a place to house ideas not suited to the class' focus on a given day). Although the assignments were not discussed often in class, they did reflect a solitary aspect to serve as a small counterpoint to all of the collective play. In the syllabus, students were "encouraged' to work together on the Book. But whereas in class there was a strong bias towards group work (students did work as individuals from time to time), the homework assignments and Book were places where students might have continued to play together or they could have chosen to work as individuals. Exams, though, provided an exclusively solitary aspect to the course.

Class meetings just before an exam were used as time to discuss ideas relevant to preparation for the exam, but also as time to keep doing math. On the day just before the 2^{nd} exam and after a quick lecture on creating tessellations in hyperbolic geometry by Mr. SP, Dr. AG signals a switch in class activities to exam preparation. The first question a student asks is about tessellations in Euclidean—whether they are possible or not. Mr. SP quickly reminds the student of previous lessons on them and provides an example of the classroom cinder block walls. The next student question, "Can we go over the construction of a square in Taxicab?" opens up a long discussion of math that is related to previous class discussions and activities but that is also new. Dr. AG verifies that the class has not yet done squares in Taxicab, "so we can definitely do that now." These two questions and the instructors' replies suggest that student questions about previously covered material is not suited to exam review time, but that questions that are slightly new are worthy of discussion. The former would have been review. The latter is extended play. That is, exam preparation is not so much a review of old material but a chance to play with past ideas modified. The class pursues the construction of a square for most of the rest of the period and many different students get involved offering suggestions and disagreeing on whether squares in Euclidean and squares in Taxicab are "essentially the same."

After a few quick questions of the type, "Will X be on the exam?" Dr. AG wrapped up the hour by telling them a few details about the availability of Sketchpad for the exam (yes, they can use it; yes, the Poincaré disk will be available; no, they'll need to make their own tools rather than use saved ones; and yes, she'll send them the sketches they need by this afternoon) and instructed them to skip around as they take the exam—do not necessarily work in the written order of questions. "Hardness is subjective," she says and she does not arrange the questions assuming a progression of difficulty. Furthermore, she suggests, they should try to respond to each question if only in part because "everyone likely has something to say about each question." In other words, students should not work through the exam in a linear fashion but should play around. They should skip here and jump there, but try to play at least a bit on every question.

Students did many things in class and outside of class. They did things with fellow students, alone, and with the instructors. Play tied these varied activities together. Play was a way to have students doing mathematics with a focus on curiosity and wonder. The aesthetics of the environment sustained student engagement in this play. Through all of these experiences students were provided opportunities to reconsider their selves as research mathematicians.

Telos: Researchers

To describe the model student that would emerge from this course is not to describe a flesh and blood real student from the course who was considered a good student. It is to describe the imaginary. The constructed student is not a real person. He and she are figments, fictions. The person the course may dream about producing. What does this imagined person look like in this course? If the course were to have its way, who would emerge from this experience?

The constructed student emerging from this Advanced Geometry course would always be curious and endlessly fascinated with geometry. Life after this class would be a constant hunt for interesting ideas and coordinating explanations. She would make every observation into an opportunity to think about geometry. He would consider the assumptions undergirding each conjecture. This imagined student would ask what, when, and why. The constructed student was an apprentice research mathematician. This course works hard to train students in the pre-proving aspects of mathematics—the preliminary necessities of a researcher's work—the conjecturing, selecting, and eliminating of mathematical ideas.

An apprentice research mathematician would not have the worries of tenure and promotion. She would not need to teach courses or attend department meetings. This researcher would only ever need to think about mathematics, but mostly geometry. He would have long conversations with colleagues and mentors. In these chats they would debate ideas and offer counterarguments. And when presented a counterargument to her own idea, she would gracefully and gratefully accept it. This Advanced Geometry course would have prepared him well for these episodes of social mathematics. The idealized student would play well with others. Her experiences with her classmates and instructors would have offered her experience in articulating her own ideas and listening acutely to others'. When he felt the sting of a rejected idea he would remember that mathematics was not a competition, but would nevertheless swell with pride when his ideas were appreciated by others.

This future researcher, though, would continue to appreciate guidance from peers and mentors. As much as the social gatherings were fun, they were also important for structuring her thinking. Rather than create ideas from scratch the constructed student of Advanced Geometry depends on the social interaction with others to spur inquiry. He prefers it when someone offers a starting place. Like in her preparation in Advanced Geometry where the instructors set the task into motion, the model student likewise needs a good scenario. Once the student has this beginning point, she is capable of exploring it and even running off in new directions.

The model student would work seamlessly between her computer and paper, valuing the benefits of both mediums for recording and exploring mathematics. This apprentice research mathematician would spend his evenings tweaking the proofs bandied about during his usual mathematics social hour. One of her favorite hobbies would be to test out new ideas in different geometries. To do mathematics would be to pose questions about mathematical objects and how these questions might be answered in different systems and under differing constraints and assumptions.

A favorite mode of inquiry for this constructed student would be to draw and sketch. He would spend quite a bit of time, in fact, on creating orderly sketches and would make liberal use of color and line thickness to highlight aspects of the rendering. The drawing would sometimes become very cluttered and would require purging and a fresh start. Nevertheless, the sketching itself was an act of visible and physical thinking.

All of these sketches would be important for weeding out the obvious counterexamples quickly and locating the extreme cases to be further investigated and sorted. All of this would not be dreary work though, no matter how consuming. It would instead be playful. She would delight in both the mystery and in the solution; in the camaraderie and in the solitude of her ideas.

The constructed student would have been prepared well for this future by this Advanced Geometry course. The daily class meetings, tasks within class, and assignments constructed a model student who took up ideas for the pleasure of doing so; who enjoyed social mathematics more than solitary mathematics; who strove for new ideas and their explanations; and who emerged from this course no longer thinking of herself as a student of mathematics but as a person ready to begin creating her own.

The course worked to move students away from seeing themselves as students who watched mathematics into a self-concept of a capable, though far from perfect, mathematician. They were asked to create their own ideas and conjecture to explore, albeit within a narrow window defined by the instructor's and the department's selection and arrangement of content. To take up this new self-concept the course encouraged students to embrace a personal aesthetic of mathematics. They were asked to find interesting relationships, with 'interesting' being highly personal. Unexpected and confusing elements were not ignored but were likewise embraced as spaces for fascination to grow. They were given opportunities to think of mathematics as a place for the oxymoronic serious-play. It was serious in that the mathematics studied aligned with accepted modes of proof. The class did not make up rules for proving and reasoning. The mathematics explored was not frivolous but consisted of ideas common to other courses of similar title. Yet it was also playful. Rather than take notes and listen, the students drew pictures and shared ideas. They conferred with classmates and their instructors. They messed around making errors and they took wrong or far too circuitous paths. This messy mathematics, though, was important to the course. To construct a model student who as an apprentice researcher the course needed to allow space for students to create their own ideas and face their own mistakes.

Concluding remarks

Again, I want to conclude by describing some real students in the course and what sorts of things I heard them say about the class. When I cannot offer quotes from students or examples of observations I will speculate from the position of a teacher educator on the sorts of things students might say about the course if prompted to do so.

As I circulated around the room during the first week one student pulled me over to offer some insight. He tells me that the students' and instructors' conjectures and language are not very rigorous and that Sketchpad is a tool to "observe a collection of instances and is not 100% general." By his tone I interpret that this is not a satisfactory way of doing mathematics in his opinion. He even seems to be warning me. The atmosphere of the class encourages students to form their own opinions and ideas and this student wants me to be aware of his. This student saw himself as knowledgeable about mathematics to a degree that he could articulate what aspects of the subject were present or absent in the discourse. To this student, the students' and instructors' vocabularies were not formal enough. That a major tool of the program was not a tool of generality but of empiricism seemed to disturb him. This course was not functioning as he might have preferred. His vision of a mathematics as a process from messy and uncertain to maybe clean and known (after all many of the students' conjectures never panned out) did not seem to match his own. It is likely he would have enjoyed the Complex Analysis or Real Analysis courses better for their more formal approach in terms of definitions and generality.

Other students also seemed to not care for the collective nature of the mathematics. Several students would mostly sit by themselves and maybe scoot over to confer with a neighbor every now and again. Some students would immediately enter class and sit in a seat to share a computer station with someone else. Different means of participating allowed students to find the one that best fit their needs. Although students had choice in whether they worked alone all of the time, some of the time, or never they did not have a choice in whether they were a part of the larger class discussions. Dr. AG called on students to offer thoughts when they had not contributed in a while. Some students spoke out less frequently than others but to some extent they were required to participate in this activity.

Furthermore, that students had to be a part of the full class conversations also meant that they were excluded from activities such as lecture. It was mostly absent in the course whereas other classes used this style of delivery frequently. Dr. AG and Mr. SP did provide instruction, definitions, summaries and other sorts of lecture-like activities, but these were rare and usually brief. Students who might have wanted for explicit instruction more frequently (or exclusively) did not have access to this type of participation.

Other students seemed to really enjoy the class. Many students came into the classroom and immediately began to work on their sketches. Most students tended to share ideas during class discussions without being prompted to do so. They sat forward in their seats and leaned into the discussions, a sign I took for interest and enthusiasm. No one personally came up to me to share positive comments about the course but they would in class say things about finding certain ideas interesting, fun, and cool.

Inferring from most class sessions the students found the course to be an interesting one, but I do not want to leave the impression that they all did as the examples previously offered provide cases of students who were explicit in their negative opinions of the course. Many students did seem to find the spirit of the class inviting, but not all did. This vision of mathematics was not to everyone's taste.

CHAPTER FOUR

COMPLEX ANALYSIS: PLATONIC DISCIPLES

...from the time of Kepler to that of Newton, and from Newton to Hartley, not only all things in external nature, but the subtlest mysteries of life and organization, and even of the intellect and moral being, were conjured within the magic circle of mathematical formulae.
--Samuel Taylor Coleridge

Introduction

This Complex Analysis course met three days a week for 50- minute sessions. It was a 400 level course and tended to be populated by seniors although juniors could also be enrolled. Sophomores would have been very unlikely given that the course had Real Analysis as a prerequisite. Therefore students would also have had taken the Calculus sequence and Linear Algebra as well. As other courses in this study, the Complex Analysis course was one of a set of options students could choose to fulfill a requirement, an analysis stand in this case. Other options in the set were Analysis II, Ordinary Differential Equations II, and Partial Differential Equations. Again, students could take more from the set to count as electives.

Seven of the 23 enrolled students were also taking the methods course for secondary teachers through the College of Education. Others could have been planning to become secondary teachers but not yet taking the methods course although this would only have been the junior or possibly sophomore students. Of these seven students four were also simultaneously taking the Discrete Mathematics course.

The class met in one of the larger rooms in the mathematics building. Like other courses the noise in the hallway could become quite loud, but no one ever chose to close one of the two classroom doors unless it had been closed at the start of class. Like the other rooms, the windows were to students left and the doors to their right. The rows of desks faced a large chalkboard where Dr. CA stood. The computer station and projectors were not used.

During class the students were very quiet compared to other courses and only a few students tended to speak either to ask questions or offer comments. These students tended to sit near the front of the room and were the ones Dr. CA would call by name. Students never spoke to one another during class and only seemed to know a few others in the room. After class the atmosphere lightened and students spoke about their assignments and approached Dr. CA to ask questions.

Prelude to the analysis

The Complex Analysis course generally presented a view of mathematics that was unified and linear—a progression of steps from one idea to the next. Students were positioned by the course as in need of developing a fully rationalized existence (the substance). The course was designed to help students improve the way that they think and structure ideas to view mathematics as rational and ordered; and they were expected to reform the ways that they physically studied and behaved to be more orderly themselves. To entice students to take up these changes, the course design mobilized an interest in orderliness (the mode). If students wanted to be orderly with neat and tidy proofs and theorems they should also rationalize their lives fully. A structured and mostly predictable life was presented in the course to invite students to learn to value the rationality of mathematics not just so that they can know mathematics more completely, but so that their lives will be more rational as well. To help students work towards this goal the course design asked students to follow the rules and stay the course. The regimen was for them to follow the class rules for things from homework labeling to not being disruptive in class. They also followed the model of Dr. CA who led by example, walking step by step through well-ordered and tidy proofs. A model student (telos) who was a perfect model of the perfect realization of this course would become fully rationalized individual in their academic work and as well as in their physical and behavioral lives.

Substance: Well-ordered rationality

The aspect of the student that this Complex Analysis course worked to change was their irrational existence. Students were very often positioned as sloppy in both their thoughts and study habits—neither did they think carefully about ideas and nor did they apply discipline to their studying or class work. To combat their disorderly tendencies in both mind and body the course design highlighted connections between mathematical ideas, content domains and techniques of proving theorems. Once students could begin to notice these connections presumably they would begin to hunt for them on their own. They would seek out the unity, order, and rationality of mathematics across domains and within them. To change students' haphazard behaviors, the course design targeted their study habits and stressed the rules that students needed to follow in their behaviors. As for study habits, the course instituted strict policies and practices to regulate attendance, behavior during class, and outside of class preparation such as homework and studying for exams. Mathematics, then, is an instance of mental discipline and mathematical ability is an indicator of the quality of a student's mental discipline.

The course website and syllabus outlined a string of topics (mostly types of functions and their integrals) to be discussed throughout the semester. Course CA's syllabus offered one other piece of information about the course content, which is the first direct statement to the student: "You'll need a good working knowledge of Calculus, especially material on (1) infinite series, and (2) functions of several variables—their continuity, differentiation, and how to integrate them along curves. If your background in these areas is weak, you must be prepared to review when the need arises." Immediately students are positioned as persons with certain content knowledge and skills while simultaneously acknowledging that students do not always have these backgrounds at the skill level the course demands. Students who are in need of remediation in any area are directed to attend office hours or seek help outside class. Discipline in their study habits and preparation are expected. Furthermore, that students were directed to draw upon these backgrounds from other courses suggests that the course will use student knowledge of calculus regularly. That past material would be an important prerequisite for the work of the course was made very explicit in the Complex Analysis course whereas in others it was briefly commented on, or hardly mentioned. This course differed from others with its focus on a linear and ordered progression of mathematics. Calculus was not just a prerequisite it was a part of the linear progression of mathematics that this course highlighted.

Having a good background in Calculus initially seems like a commonplace requirement in an upper-level mathematics course given that it is a prerequisite to nearly all courses at and beyond the 200 level, but for Dr. CA, this is essential in other ways. Inferring from my class observations, his course would extend Calculus concepts and techniques to reveal connections within mathematics the students had not appreciated previously. The course was designed as if students had not learned enough in Calculus and Real Analysis. Furthermore, the course design suggested that what they did learn in previous courses was not well integrated into a larger framework of the field of mathematics. Students were treated as if they had done just enough to get by in their other courses and did not have the mental discipline to seek connections across courses or even within them. The official course documents emphasized math topics, but the course itself included additional implicit and explicit goals. The students were expected to strengthen or form mental connections between the new topics of complex analysis and the previously experienced topics from real analysis and calculus. They should begin to see the unity and coherence of mathematics. Seeking out connections between mathematical ideas was practiced and highlighted throughout the course. Students were positioned as needing help in disciplining their minds towards an orientation that seeks connections and does not just take topics as discrete ideas. They were asked to work hard to integrate what they experienced in complex analysis with what they had already experienced. Other courses in this study also had the prerequisite of Calculus, but in several of these it was not explicitly called upon. The Advanced Geometry and Discrete Mathematics courses promoted diversity of experience in mathematics whereas the Complex Analysis course highlighted a progression of necessary concepts, definitions, and theorems that built off of one another to form a unified whole.

Dr. CA emphasized connections between prior course content, current course topics, and general techniques and reasoning in mathematics. For example, in one of the first classes of the semester he was solving a question a student raised concerning a homework problem. He stated that $|z-a| = |z-b|$ (where z are in the set of complex numbers) is the set of numbers equidistant from both a and b. To illuminate this, he drew a picture of a point *a* and a point *b* with a segment connecting them. Next he sketched the perpendicular bisector (see Figure 4.1) to indicate that the set of numbers, or points along this bisector, are equidistant from *a* and *b*.

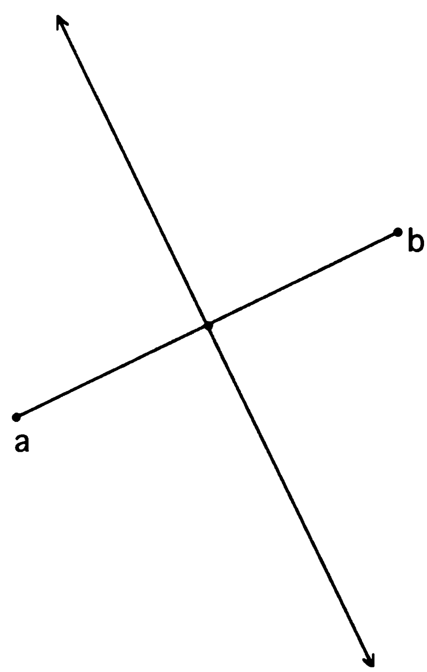

Figure 4.1: Dr. CA's illustration of the points equidistant to a and b

He stated, "We should always be thinking in terms of geometry," but continued by explaining that although this geometric representation is "best" it does contain "at least one significant problem": it is a picture and not a proof. Because of this problem he then proceeded to provide an analytical approach to the solution. Intuitive geometric and formal analytic representations are offered as complementary techniques. In this course sketching leads to understanding and deducing leads to certainty. Body and mind contribute to changing students' connections among mathematical ideas, but despite his claim that the geometric one was "best," he continued to provide a better approach--analysis. It is the more disciplined, rigorous, and rational approach. The geometric analysis might be helpful but it is sloppy and needs improvement. While explaining that points equidistant from *a* and *b* lie along the perpendicular bisector, Dr CA was connecting geometric representations with formal ones. There is a nuanced relationship here. One form is easily understood and a good model to share with students, but it is not enough. In this course, if students were to use only the geometric models, they would not grow to become more mathematically disciplined. They need both to be fully realized. Calling the geometric model best seemed to mean a best first step in the context of his overall point. The geometric understandings are ordered to the left of analytic ones on a line of improving mathematical precision.

In another classroom episode Dr. CA brought in colored chalk to help students, "get a feeling for what this exponential *[e^z]* does, what it does to points in the plane. Try to understand e^z through pictures by first examining e^2 in the real plane." Through these episodes several elements of the curriculum and pedagogy of this course can be interpreted. The pedagogy conveyed concern that students connect geometric and analytic methods as two viable means to first understand and then prove a result. By asking them to first examine e^2, the problem was reduced to the real plane before returning to the complex. Geometric approaches are complementary to formal ones. Students were first asked to feel to learn and only afterwards attempt an approach consistent with mathematical logic. This tension between the two continually resurfaces in class. Formal approaches are preferred, and this was one of the ways students were treated as if they were not quite ready for them. The course design functioned to bridge or connect what students were assumed to already understand and what they need to develop mathematically.

Consider the following episode from much later in the semester. The topic was Green's Theorem, which he described as, "A topic done in Calculus III and quickly forgotten." After the quick aside referencing students' tendencies to forget what they have previously been told, he continued by describing how a positively oriented domain means that any time you "walk" on the boundary you keep your left hand towards the domain, "typically counter-clockwise". "This isn't really a rigorous definition, but without advanced topology we can't do much better for now." As he drew a stick figure walking the boundary with orange chalk he explained that to create a formal definition in mathematics is to "manipulate symbols and hope everything is OK." The pedagogy of this course went back and forth between acknowledging the current preparedness of the students and teaching with their needs in mind—that they forget theorems and don't know advanced topology—and moving them forward. There is also a strong nod towards formalism in the comment above. Mathematics is a symbol game without meaning, but as we will see he also expressed Platonic views.

In course CA students were expected to learn the topics outlined in the syllabus and in the department's course website, but it was also important for students to learn that previous course content (such as Calculus I, II, III, and Real Analysis) was going to be expanded in the complex plane. One of the learning goals was for students to see the continuity among all mathematics courses. Moreover, topics learned in Complex Analysis could be returned to with new and deeper understandings after taking other upper-level or graduate courses. Successive courses illuminate new connections. Connections between topics and techniques (geometric and analytic) combine to provide understanding and rigor. Geometric and holistic arguments have a place in the course, but they are seen as stepping stones to a more disciplined and rigorous approach. The course both acknowledged that students are not ready for a fully rigorous approach but also that they need to begin working with more rigor. They needed to begin to see mathematics as linear and that learning mathematics is also progresses linearly and step by step from informal and geometric arguments to formal analysis and proof.

The textbook acted as another text (here a literal one) for interpretation. The explanatory sections and examples in the text often utilized geometric representations in conjunction with algebraic equations. These explanations and examples also pointed out places where the complex plane could be understood through examining connections and analogies with the real plane. The section titled, "A Note to the Student," explains, "The presentation has been molded by my belief that what you have already studied in calculus can be successfully applied to learning complex variables, which at its basic level is just the calculus of complex-valued functions. Where there are strong, and even obvious, analogies between the new material and calculus, I have pointed them out and arranged the presentation to emphasize these analogies" (Fisher, 1990, p. xi). Notice that students even need assistance noticing the obvious analogies. The textbook positions mathematics as well ordered and linear by highlighting the ways that what students have learned before will be called upon to learn the new things of this course. One cannot proceed to the next step without the necessary prior experiences.

The syllabus also pointed out behavioral expectations for students that most other courses did not include. For example the syllabus told the students that the professor will "assign problems each day, and will expect you to work on them so that we can discuss difficulties that may arise in class the next class." Although Complex Analysis was definitely not the only the course to assign problems each class, it was the only one where an explicit expectation that students would work on them was noted. In the other classes it was an unstated assumption that students would work on the problems. By being explicit, Complex Analysis was signaling an assumption about students' work habits. It was not taken for granted that students would work on the problems. This directive in the syllabus is a way of constructing students as lacking mental discipline. If left to their own choices, they would not do the problems. I am not claiming here that students in the other classes did their work just as well as they did in Complex Analysis nor am I claiming that if had Dr. CA not included these expectations in the syllabus that the students would have worked more or less hard. The major point is that in making it explicit we can discern a certain set of assumptions about students in this course that are different from the assumptions about students' work habits in the other courses. To have disciplined work habits was to be desired in the course and this was made explicit here. Students are positioned as needing step- by- step guidance in how they prepare for class.

There are other behaviors that the syllabus is explicit about reforming. "Repeated disruptive classroom behavior can lead to failure in the course or dismissal from the class." Again, Complex Analysis was the only course whose syllabus mentioned disruptive behavior in any way. To make a point of the consequences of poor behavior is to make visible an assumption that such behavior is likely enough to need mention in the syllabus, and more interestingly it is in the sub-heading on grading policies. Mental behavior such as performance on homework and exams is an aspect of the grade and so is physical behavior. In the same section on grading, the syllabus also alerts students to the impact that "exceptional effort, positive contributions to the classroom experience, improvement over time...[ellipsis in original]" can have on their grades. Students are told that these positive behaviors might be rewarded in the form of "raising your preliminary grade...". Disciplined behaviors that are constructed as positive will be rewarded but "negative factors such as lack of effort, declining performance, or disruptive behavior can lower it." Grades do not just depend on growth in tests and homework but also in behaviors. Students' minds and bodies must become more ordered and rational.

These behavioral expectations construct mathematics as linear and step by step, and that the learning of mathematics is likewise linear and step by step. Unlike in Real Analysis where mathematics is a social enterprise, in Complex Analysis mathematics is like a mountain to be climbed: you cannot reach the top without proceeding step by step from the bottom. Complex Analysis is not constructed as a separate or unique sub-field of mathematics. Rather, it is one step along the way of an orderly and linear progression of understanding. Along this linear path, every individual must travel alone.

The dual issue of mental and physical aspects of the discipline also appears in comments from the syllabus on grading policies. "In order to get credit for a problem set, you must be present—both physically and mentally—for the entire class at which it is due." That is, a student must be present on the day that the homework is due and cannot turn it in *in absentia*. Students must be present physically, but to be mentally present is also a requirement and suggests that a person could be physically in class but not mentally in class. Possibilities that might fall into this category include daydreaming, reading the newspaper, or sleeping. Yet these too are mental activities so to not be present mentally is presumably to not be thinking about the mathematics of the course. The course wants to limit off-mathematical physical and mental behavior and makes to do otherwise into gradable offences. Again, no other courses made mention of mental presence in class. This aspect of discipline was unique to Complex Analysis. By not taking students' behaviors (physical and mental) for granted, the course is suggesting that students tend to behave otherwise and need reforming in the form of management. They need well-defined steps and definitions. Being present is defined as being mentally and physically in class. The behavioral guidelines mimic the mathematics of the course. To improve in either you need to proceed along an ordered and explicitly defined path.

Students are positioned as lacking organized and structured models of mathematics; they do not see the "obvious" analogies and connections between the courses they have already had and this one. They are also seen as needing reforming in their physical behaviors that are also connected to study habits like paying attention in class and being physically and mentally present. Students, if they want to be mathematicians, are also seen as needing explicit instruction regarding issues of disruptive behaviors. To progress through mathematics as a student, he must also progress through a set of rationally defined behaviors.

The course's approach to remedy the physical and mental sloppiness is to highlight connections between content areas and between techniques (geometric and analytical arguments). There are also explicit guidelines for behaviors in class and outside of class. Professor CA, in many ways, resembles a reformist. He points out that students have not learned important information from previous courses and seeks to provide ways to (re)connect them to that material as well as introducing them to other connections between geometric and analytic approaches to mathematics. Students' past experiences have been flawed, and the course design seeks to remedy this as best it can. Overall, a well-ordered rationality is at the heart of what needs to change for these students in this complex analysis course.

Mode: Pleasure in orderliness

How does Professor CA invite students to take up disciplined work habits, both mental and behavioral? Through this course, students are constructed as in need of more structured habits in how they attempt (or did not attempt) to make connections between ideas and how they behave in class and outside of class. By highlighting connections when they appear, Professor CA's pedagogy was directed toward a desire for orderliness as a reason for making changes in student orientations towards mathematics. Having experienced the pleasure of an ordered environment, students would hopefully then seek out further connections within content and therefore more orderliness, a self-sustaining cycle. In addition to the pleasure of mathematical ideas students construct a different way to think about pleasure—the enjoyment of getting good grades on tests and homework sets brought about by taking on an ordered and rational approach to mathematics and their habits. The unity of mathematics was constructed as highly desirable and the recognition of the orderliness of that unity was the highest goal of mathematical learning.

After proving a tidy result related to a particularly complicated integral Dr. CA narrated, "You expect some spirit to be overlooking this whole process and that allows us to do this. To really understand the miracle you need to take further courses."

Dr. CA often explained formal and rigorous definitions and theorems with their accompanying proofs in terms of a mystery or as magical, but instead of finding these qualities distasteful, it is while he talks about these issues that he is most animated. His eyes sparkle and his voice and pace quicken. He turns to the class, abandoning the board for a moment and lets the chalk fall to his hip. In these moments when he is describing mysteries and miracles in mathematics and how fun it is to revel in them and to explain them, he looks students in the eye. The pleasure of mathematics is to be in awe of unexpected connections, not in divergent unrelated phenomena. These unexpected connections are byproducts of the unity of mathematics. Recall that earlier I described an episode where Dr. CA seemed to express formalist philosophy of mathematics. In this episode a Platonic philosophy is described. Math is sublime and out there somewhere. It is exciting to stumble upon a problem where all seemed chaotic (like the very messy integral) only to find it was *already* structured and fit nicely into the existing mathematical framework. Seeking out the elusive connections between mathematical topics, techniques, and ideas is presented as exciting and as a reason to do the work of the course. When something seems unorganized and irrational it is just that the connections have not yet been revealed. Finding these is desirable as positioned by the course design.

On many occasions Dr. CA would finish some proof at the board and comment on a pleasurable reaction to that result. On one occasion he even referred to a proof as "really quite cute" with a big grin. Sometimes the pleasure was to be in the simplicity of the result despite the intricacies in the proof. $e^{i\pi} = -1$ was a notable example. Sometimes the pleasure was that a complicated earlier proof and result could be re-used reinforcing the linear progression of mathematics. For example, after Dr. CA worked for quite some time to establish Green's Theorem for a triangle, he pointed out the joy to be had in noting that once the case of the triangle was established, many other regions could also fall under the theorem by dividing up the region into a collection of triangles. That is, any region that could be shown to be a region composed of multiple triangles, could also be shown to fall under Green's Theorem. An example of pleasure to be had in direct connections between functions of real variables and those of complex variables was that several rules the students might take for granted for real-valued functions are still applicable for complex-valued functions. $e^{z+w} = e^{z}e^{w}$ still held, for example despite the introduction of the imaginary parts of the functions z and w. At other times the pleasure was to be found in the implications of a theorem. The Cauchy-Riemann Equations were one example of this type. Dr. CA pointed out with several examples how these equations could be helpful in finding u when v is known or finding v when u was known. Finding unexpected simplicity, transferable ideas (from the real plane to the complex), and utility were pleasurable.

There was a duality to Dr. CA's talk about the mysterious and sublime aspects of mathematics. On the one hand it could sustain his interest and fascination with his work, but on the other hand he diminished the mystery of mathematics by explaining that further course taking would explain away any sense of wonder. What is wondrous now might not be so wondrous after more hard work. It is and always has been coherent, but we as humans might not always be in a position yet to see the coherence. Recall that he told students that without advanced study in topology some current ideas taken for granted could become illuminated. The mysteries can be understood with more hard work and study. Topics learned at higher levels explain assumptions taken for granted previously. But still, there remains a sense of wonder. Calling on a spirit to explain the mathematics can also be read to imply that he considers mathematics to be connected, not because mathematicians designed it to be, but because it will and always has been. The sense of inevitability pervades much of this discourse. Students grow by seeking out explanations to a priori knowledge and doing this is pleasurable. Both the hunt for connections and capture of them are thrilling.

The textbook is often different from the professor's lessons in respect to pleasure and mystery. Whereas Dr. CA made mention of the "amazing" or "exciting" results often in his classroom talk, only the slightest hints of this theme are found in the textbook, which is mostly restricted to providing definitions, theorems, proofs, examples, and exercises and has very little commentary. One comment on the aesthetics of mathematics comes from the section, "Preface to the First Edition": "Complex variables…is a mathematical structure of enormous beauty and elegance" (Fisher, 1990, p. x) Notice that by invoking the word structure a particular orientation towards mathematics is noted. It is organized and structured, not diverse and haphazard. This short line in the textbook also makes a claim that to study complex analysis is to participate in the beauty of the structure. Much later in the textbook, in an appendix "Locating the Zeros of a Polynomial" there is a commentary on finding the zeros of polynomials of 5^{th} degree or higher.

> The reader may well draw the conclusion that the zeros of any polynomial may eventually be found by a succession of steps that work for the quadratic, cubic, and quartic cases—that is, by a sequence of substitutions and extraction of roots. Quite surprisingly, this is not the case if the degree is five or more. In fact, one of the more profound mathematical results of the early nineteenth century was that there is no formula of this type that will solve every fifth-(or higher) degree polynomial. (Fisher, 1990, p. 386)

This paragraph makes explicit that there is surprise in mathematics. Patterns do not always follow through in the ways that might become expected. There is an implicit expectation that the reader or student is seeking out these patterns and connections and that surprise is an emotional response to mathematics. In this case, surprise that a pattern does not hold. Furthermore, the passage points out that the theorem that says that not all roots of fifth or higher degree polynomials can be located using a formula is profound. There are aspects of mathematics that are profound and non-profound, surprising and expected. The invitation of the course to get students to take on disciplined ways of thinking about mathematics wants them to not rely only on their expectations and assumed patterns, but to test those and be emotionally invested when they come up short just as they are pleased when patterns hold. On the other hand, just because mathematicians discovered that the pattern did not hold they also sought a proof that it did not hold, not just that they seemed to not hold. Even when there are apparent anomalies, there are also ways to codify those and bring them into the rational structure through proof. All mathematics is organized into the structure even if we humans at times have to work to find out the limits of the structure.

Both Dr. CA and the textbook position students as if they could be inspired by the pleasure of finding connections within seemingly unconnected ideas, patterns in what might have been chaos, and/or surprise when ideas thought to connect in fact do not, or patterns that should have been were by proof not. The non-existence of a pattern was still structured. To use surprise and pleasure—both aesthetic emotional reactions—as an invitation to take on disciplined activity here looked quite different than the aesthetics deployed to invite students in the

Advanced Geometry course. In Complex Analysis students were often directed or explicitly told what was interesting, surprising, elegant, and/or beautiful.

What was striking in the comparison between Advanced Geometry and Complex Analysis was the rarity of student comments about what is interesting, elegant, or beautiful in the latter course. In Advanced Geometry students often expressed an interest in following up certain ideas, commented on the "coolness" of an observation, and came to surprising conclusions of their own as evidenced by their frustrations when expected phenomena did not materialize or when an unexpected pattern did. In Complex Analysis these comments about aesthetics were pronounced by the professor and the textbook, but not by students. This is not to suggest that students preferred Advanced Geometry to Complex Analysis. In fact, the two courses did not have students in common that I could ascertain and no one ever made a comment to compare this course to another. In fact, students in Complex Analysis did make comments about enjoying the course as a whole, but they did not make comments about finding the mathematics particularly interesting or surprising or some other aesthetic element. This tells us that in the Complex Analysis course the aesthetic qualities of mathematics were not for students to determine but to be told. The beauty of mathematics resided in mathematics itself, and was not determined by humans. To come to a well-ordered view of mathematics was in part to take on a particular notion of beauty—that is structure or orderliness.

Student questions or comments during the class time never touched on the pleasure of finding or describing connections between complex analysis, real analysis and/or calculus. Instead, their talk about pleasure involved the happiness of getting good grades. This was pleasure in following the behavioral steps for studying outlined in the syllabus. A group of students who sat a few seats behind me often expressed excitement about problems they had solved correctly. A common interaction might begin by one telling the others how s/he had figured out the solution to some homework problem with what appeared to be pride. The others would ask more about this solution or share a modified version that he had used. The student from this group that sat nearest to me would frequently share that he enjoyed the course very much and often wanted me to read his solutions. These examples of excitement were not so much about the theorem or the connection(s) to other ideas so much as joy of finding the correct answer to a test or homework question. This differs from the professor's pleasure in mathematics in that his joy was often about the simplicity or unexpectedness of a result. Where Dr. CA's emphasis on pleasure was related to the doing of mathematics—the fruits of disciplined inquiry, the students' take on pleasure was related to the fruits of behavioral discipline and study habits that resulted in high grades. Pleasure was present as a reason to take on the course but the emphasis differed between that related to the larger field (Dr. CA's and the textbook's) and that related to individual success (the students').

Although the students' primary expressions of pleasure were related to grades, Dr. CA also commented on grades as important aspects of the course. Throughout the semester on days when he returns a graded test, Professor CA also drew on the board a frequency table detailing how many students made which scores, and how many students were currently making what cumulative grade. For example, one day the table indicated that nine students were making 4.0's, 11 were making 3.0's, two were making 2.0's and another two were making 1.0's. The table never indicated a particular student's grades, but if a student knew that she had made a 3.0, then she might infer that she was performing similar to most of her fellow classmates.

Students also spoke about their grades often. The one student who talked to me before class would also show me his homework and test grades. These were usually quite high, some even being above 100%. When he did miss a problem he would complain to his friends (and they to him) and they would pass around their graded papers for each to critique or admire.

The pleasure of doing well on tests seemed to be a big concern for these students and others in the course. They would ask the professor for review problems to practice and for study sheets to examine before upcoming test days. The five tests counted for 500 of the total 600 points considered in the final grade with the final 100 points reserved for homework scores. On some occasions when Professor CA described a theorem and its proof he would point out for the students that he likes to use this theorem on tests or the final and therefore students need to study it. What was done in class related in an ordered and connected fashion to the tasks on the exams. Also, that the grades were based on a cumulative points model provides a strong analogy to the construction of mathematics in the course. As a student passed through homework and exams, he or she accumulated points. Each new assignment provided another step-by-step way to reach the top of the mountain where 600 points was seen as perfect. The course policies on homework were also fully rationalized. Not only the mathematics, but also the learning of mathematics conformed to a rationally unified coherent system.

In addition to homework sets created from class lecture material and the textbook, Professor CA also used some extra-credit problems, which relied on the Maple software program. In the syllabus he pointed out that these extra credit problems could significantly alter the students' homework average and thus final average. One reading of this practice invited students to use the program by offering a better grade as the reward.

In summary, the course worked to create students who were fully rationalized in many aspects of their life. Like mathematics, they needed a larger organizing structure that could coordinate their mental activities and behavioral activities. To invite them to take on these challenges the pleasure of order-- exemplified by finding connections in mathematics --was highlighted by Dr. CA and the textbook. Recall that he described further course-taking as a means to better understanding of ideas encountered in previous courses. That is, by taking complex analysis, students could appreciate calculus and real analysis in new ways. These sorts of comments offered reasons to learn new topics, theorems and techniques. Pleasure in structure was the aesthetic element explicitly called on in the textbook and in Dr. CA's commentaries in class, but students also often expressed their pleasure in getting good grades.

To invite students to make changes in their habits—to become more ordered and rational in their study habits, orientations towards content areas, and behaviors—the course highlighted the pleasure of mathematics. To have a fully rationalized mind and body is constructed as pleasurable and therefore a reason to strive for the change the course hopes to initiate.

Regimen: Following rules and staying in line

The course set up an expectation that students needed to become more disciplined in their work in mathematics—seeking out connections in ideas, being attentive and well-behaved students, and transitioning from geometric arguments to analytic arguments. To appeal to students to make these changes the pleasures of hard work are made explicit. These are pleasures in the grades earned for doing well and the pleasures of connections both expected and unexpected within complex analysis and across complex analysis real analysis, and calculus. What then are students asked to do? What did the course construct as the regimen to achieve these changes?

The major thing students were asked to do for the course was to follow the rules of the course and follow the modeling of the instructor. Very often comments in class directed students to alter some behavior or to be on their watch for undesired behaviors. That the class was rather rule-oriented is not to suggest that Dr. CA came across as mean or uncaring. Many students expressed a fondness for him and the course. This Complex Analysis course asked students to become better disciplined by expecting more disciplined behaviors. They were also expected to follow along with Dr. CA's lectures. To become more disciplined themselves they were to follow the example of Dr. CA.

Through the observations of the class, it was apparent that homework discussion was an important activity. Each day, if not a test, began with time for students to ask questions about the homework sets. This could take anywhere from 5 minutes to 20 minutes of a 50 minute class period. The previous class meeting Dr. CA would have assigned a set of questions for students to prepare for the following class and he would later (every following Monday) collect this work to grade. In early September Dr. CA described the homework rules for the next Monday's turn-in date (because of Labor Day this Monday was not a turn-in date and the schedule was not yet as organized as it would later become). He explained that he will select two problems from each of the smaller sets to grade in detail and he will spot-check the rest. "Make an effort to write up all the problems and make it clear which are which. I'll only spend a minute looking for it before it's a zero." Students were instructed directly to not only do the homework but to also follow the guidelines from the syllabus that outlines: "Problem sets must be written up neatly and logically, with appropriate explanation provided." Students were very directly asked by Dr. CA to follow the rules for turning in homework.

There were other rules about homework related to the class time spent addressing it. At the start of almost every day of the class, Dr. CA asked for students to tell him which questions they had found troubling or difficult from the previous selection. At first students would tell him a problem number rather than a question but after repeated queries on his part for a question students began to adhere to this unspoken rule. Rather than tell him the problem number that was difficult, they should ask him an actual question. When students did ask him questions rather than assume that this time was for him to work their homework problems, he gave their questions considerable attention. One day a student shared with him a problem she was having with a homework assignment. In particular she was having trouble understanding how two problems from different but consecutive sections were related. She described how they seemed similar to her but she was confused by what each was asking. Rather than address the two exact questions that were the target of her confusion, Dr. CA talked about the general case the incorporated both questions. He spent about ten minutes describing the general case that would not only answer her questions but also a whole class of related problem types. He drew pictures and worked back and forth between geometric explanations and more symbolic ones. When he was done he turned back to the class and asked her if he had answered her question. She responded that he had. Although the start of the homework question involved a little bit of discussion between the student and Dr. CA once he began on the problem he worked on his own. The discussion turned quickly into a mini-lecture where students were expected to follow along with his explanation. Most examples from homework time followed this routine. A student would initiate a question and there would be a few exchanges between the student and Dr. CA before he began to lecture on the problem. As he did this, students would be writing in their notebooks presumably following along with his usually lengthy responses.

When students were not able to formulate their questions well—when they did not follow the rules for asking about homework, Dr. CA would not provide a lecture on the subject. In mid-October a student asked about an integral from the homework set. "What part is the real part?" To which Dr. CA responded, "I don't know what your question means." The student responded, "I guess I don't either." After this statement Dr. CA moved on to another student. To follow the rules of the homework discussions were important and the consequences were to not be assisted. A student was expected to be able to produce a coherent expression of his confusion. Even confusion needed to be ordered.

Following along as Dr. CA did mathematics was another important aspect of the class as a whole and not just related to his mini-lectures during homework time. The majority of non-homework time was new lecture time. During this segment of class Dr. CA provided definitions, theorems, proofs, examples, and/or applications. The rules that students were to follow were simple. "Don't zone out," as he instructed one day. He often told them that he wanted them to "really pay attention." The rule to follow was to follow along carefully and stay in line; he made this expectation quite explicit on several occasions.

Students seemed to comply with both. Aside from one occasion when a student was reprimanded for tapping his pencil students hardly moved in class. They never left their chairs for any reason. Even when there were loud noises in the hallway that would have prompted students in many other classes observed to get up to shut the door, no student moved to do so in this course. Similarly, in other courses students would throw away tissues or excuse themselves into the hallway, but not in Complex Analysis. They did not read their textbooks or other books. They did not nod off or slump in their chairs as happened in a few of the courses from time to time. Students were performing to the expectations of the syllabus. They were behaving well by following the rules by staying in line at least in their physical responses.

During lecture time Dr. CA often seemed rushed to finish an idea before the period ended. In other classes observed the professor usually announced that the class would finish whatever idea on the next day. In this Complex Analysis course, though, students were instructed to follow along but just more quickly. Dr. CA's words and writing were both very fast. Even on days when he did not make outward comments about being rushed, he still seemed hurried.

One day Dr. CA asked the class how much time was left and quickly answered himself that there were 10 minutes left. "Good because now I can prove [Casorti-Weierstrass Theorem]." A student yawned and was quickly encouraged to "Hang in there. This proof is worth it!" Fifteen minutes later and five minutes past class he announces that he wants to show them another theorem but warns them that if they leave they won't have access to this information for their homework. He also warned them that if anyone missed the next class (the Wednesday before Thanksgiving) that they "owed him" but he did not explain this further. Following along in this episode includes following a very fast lecture of a complicated proof that took fifteen minutes even in its rushed form. Following also includes following the rules for being physically and mentally present both for an in-session university calendar date and for post-class lectures. That is they were being expected to be there on a day that school was actually in session even though many students were positioned as planning to be absent. They were also asked to stay late as Dr. CA finished a proof and even later as he began a new idea. Students were expected to stay in line with the rules of the class.

When the course asked students to follow along with lectures they were not always asked to stay late. On Halloween Dr. CA looked at his watch and turned to the class. "Quickly, before the hour closes. Let's give a proof of Cauchy's Formula," as he promptly turned towards the board. Fourteen minutes later Dr. CA again turned back to face the class, who had been following along as indicated by outward physical signs, and announced with what seemed to be glee, "Four minutes to spare!" Following along and doing so while Dr. CA did mathematics quickly was a common experience. There was a flurry of chalk and notebooks on most days of the course. Students were expected to follow the path of the mathematics as Dr CA performed it at the chalkboard.

That Dr. CA often seemed rushed to complete his class suggests that he had ordered the lectures to begin and end with certain ideas. To have not finished a proof before class would have set the ordered course off track. A slip in time on a given day would set the class behind for each next new day. Since the topics were well ordered in a step-by-step path, skipping one was not an option. Time became very important. Students were expected to follow along because doing so meant that they could follow the progression of the mathematics to a final conclusion. If they missed a day they missed a step. If they missed a step they could not arrive logically at the conclusion. Students therefore needed to be present and paying attention or following along with what Dr. CA was doing at the chalkboard.

Possibly because of this rushed state, students were explicitly discouraged from asking questions during lecture time that commenced once homework time had ended. New rules applied for this segment of time. Whereas Professor CA provided detailed explanations for homework questions, often solving the entire problem for the class and providing a mini-lecture on the general case, when students asked questions about the lecture he would provide terse responses or state that he did not have time to go over it. He would instruct them to visit him in office hours or after class, which many students did as evidenced by the clusters of students who approached him after class.

These clusters of students were fairly common. On most days students congregated around a large desk at the front of the room where he usually stood collecting his notes and colored bits of chalk. On at least a few days some of the students who formed these post-class discussions followed him from the room. The "following" was not reserved for in-class only. There were several days during the semester where I wondered if students had followed him into class from somewhere else as they entered the room quite close on his heels. Just as students tended to gather around him after class, many also did so before class began. It was entirely likely that some students had indeed followed him to class given that office hours coincided with the hour immediately before class began and given that student comments before and after class suggested that many of them made use of the opportunity for help from Dr. CA.

That students followed Dr. CA in and out of class is a possible indication of their following a suggestion from the syllabus that was not quite a rule. The syllabus states that if students were having problems with their homework sets they were invited to discuss them during office hours and later in a section on collaboration students were warned about the penalties of cheating off of other students although collaboration was technically permitted. "You may work with others on the homework problems, *but you must acknowledge their contributions,* and your final write-up must be your own." Although there was no rule prohibiting working with other students the choice between a welcome invitation to work with Dr. CA or a tolerated but possibly penalized collaboration with a fellow student might have been very easy for some students to make and might account for the clusters of students seeking Dr. CA's council before and after class. That students sought out the professor for help was an indirect way of following two different expectations. The first was that students should come to a rationalized and ordered way of thinking about mathematics. The second was that they should not rely too much on the assistance of other students in the course. To accomplish both students would often seek out the professor for assistance.

Overall in the lecturers students watched Dr. CA point out connections in mathematics and how ideas fit together or built off of one another. They did not do the finding. He did. On the homework assignments students largely used the insights gained from the connections described by Dr. CA, but again they did not find them. Students were constructed as not yet ready to be full mathematicians, people who found the connections, themselves for they still had much to learn.

To follow the rules of the course was often to follow the modeling of Dr. CA. These rules were quite explicit in both the syllabus and in class comments made by Dr. CA. Students' behaviors also suggested that they understood the expectations and that they were following them. To become fully rationalized students of mathematics, these students were expected to perform within a linear step by step course. They should follow the steps of the mathematics as they watched Dr. CA solve homework problems and give lectures. They should ask their questions about homework within a well-organized framework. Everything they did needed to align with a structured approach to the course.

Telos: Platonic disciples

What model student would emerge from this complex analysis course? Again the model student is not an actual person and model does not refer to a student who happened to make good grades or some other evidence of an actual student who might be deemed a good student. The model student is constructed. He is fabricated and he is a fabrication.

He would be a person who would change from a lazy, ill-mannered, uncouth way of life into a hard-working, polished, and articulate member of society. The model student would see the errors of her ways and seek out the council of someone who was a dynamic and inspiring force. The model student would be inspired both by the pleasure of leading a disciplined life and would be interested in the worldly rewards of such hard work. The constructed model student of this complex analysis course would be a Platonic disciple.

As the course goal was to create more disciplined students in their ways of thinking about mathematics – the sublime connections, the mysterious breaks in pattern and the well-ordered structures between and across mathematical ideas—the model student would take up pursuit of both the mysteries and the explanations. The model student would also become reformed with respect to behaviors and codes of living such as living up to obligations like time commitments, giving attention and respect to others, behaving respectfully towards others, and generally being what most of our current American society considers polite. When respectful of others, the model student would also be respected and given attention and care.

The model student as Platonic disciple would have come forth from the experience of this class appreciating the pleasures and rewards of mathematics. These pleasures would include the happiness and surprise of both the expected and unexpected as well as seeking out answers to math's riddles. The duality of reveling in the sublime and other worldly, and in the explanation would be a part of the model student's attitudes towards mathematics. It is out there to be discovered. In addition to the pleasures of the mind the model student would also find pleasure in external rewards and others' approval. Of particular interest to the disciple would be the recognition granted by a person of high regard and esteem. Just as this Complex Analysis course contained aspects of non-rewarded personal pleasures and individual rewards in the form of grades so too would the model student seek both pleasures. Math would be as the textbook described, "a structure of immense beauty" (Fisher, 1990, p. x) and a field where personal contributions were also valued in addition to more aesthetic elements.

The disciple would conduct his life by following the codes of conduct as described by the mathematical community and persons of high status in that community. They in turn were following the model of the rationality and structure discovered in mathematics. This mathematics was of a realist and Platonic philosophy. The model student would follow the rules as written down and explicitly stated. He would also follow the model and guidance of those in higher standing in the mathematical community who also led lives by the rules. These models for life would be dynamic and encouraging but also strict and demanding. The model student would seek this sort of leader out. He would attend this person's public seminars and lectures but would also attend smaller study sessions and counsel.

The model student is not an unthinking and blind follower, but questions ideas and speaks his mind. At the same time, the model student does not seek discord by asking these questions. He wants to better understand and so the questions are seen as contributing to his overall betterment. The questions the model student asks are also not about posing new knowledge but are about better understanding the given knowledge of the community. In time and after much hard work the disciple might become a leader in the mathematical community but currently is positioned in that community as something of a novice interested in learning more about the codes of conduct for life and mathematics.

The model student wants to know more and pays very careful attention when others who do know more than he are speaking. The model student seeks a more disciplined life with the metaphysical and physical rewards of it by following the codes of conduct and the examples of others. The model student is a Platonic disciple of mathematics.

Concluding remarks

As with the preceding chapter I will conclude by considering what real rather than ideal or model students might say about this course and what their opinions of it might be. In some cases I have examples of real students who made comments relating to their likes and dislikes. At other times I get to speculate.

Students who found pleasure in order and routine would have enjoyed this Complex Analysis experience. This type of student would not have minded the rules and would likely have appreciated the structure and explicit set of expectations. Students who had had positive experiences in Calculus and Real Analysis would also have found this course interesting. They would have been able to use their past knowledge well and find a place for integrating it with the new ideas of complex numbers and functions. Students who thought of success in terms of getting good grades and who appreciated public acknowledgment would also find a home here. Some students appreciate fully rationalized grading policies, and those students would have found satisfaction in this course. Many students in the course seemed to like the course very much.

As has been mentioned a group of students who sat near me seemed excited to share their graded assignments and compare scores as well as the solutions they wrote to get these scores. Another student, one of those also enrolled in the secondary math methods course, told me that her experience was so positive that she had decided to leave the teaching program so that she could continue into graduate school in mathematics. She explained that she was torn a bit. If she continued in the certification program she would have needed to spend the next year interning in a school, but she was eager to go to graduate school instead. In the end she decided that delaying teaching rather than delaying graduate school would be a better option for her. This student described Dr. CA as encouraging and inspiring. She said that she'd never really thought about graduate school before this class, but that she liked what the class did so much that she wanted to continue to have more of these experiences. She and the other students around me enjoyed this class and the vision of mathematics it represented.

Many students never talked to me at all or expressed opinions about the course that I could hear at least in my time in this class. They, too, may have enjoyed it. Maybe they did not. Because I cannot offer particular examples of these students I will try to describe the sorts of things students who might not have liked the class may have given as reasons for thinking this way.

A student who was not interested in connections between mathematical ideas may not have found these things inspiring at all, as the course nevertheless constructed. If finding links between courses and ideas does not match a person's sense of mathematics, this course would not have seemed very interesting. Or if a student preferred to think of mathematics as disconnected and isolated units, he might not have valued the emphasis on the unity of mathematics.

Students who preferred to work in teams with others may have found the individual nature of the course limiting. Those who enjoyed solving more open problems without well-defined paths toward solutions would have likely found the explicit discussion of method and proof restrictive. Similarly students who appreciated real world concepts and tangible ideas might have missed that in this class. Students who were not linear thinkers may have had trouble coordinating their preferences with the step by step approach offered in this Complex Analysis class. The structured approach to classroom behaviors might have had given some students trouble staying seated all class; those that liked to do work while listening to music, or in silence (as the frequently open doors made for loud disturbances in the hallways often) could have also not done well in this environment.

That the course constructed a model student is not to suggest that every student in the course left the semester as that model. Some may have moved towards it like the student who wanted to continue her progression up the mountain of mathematics by entering into graduate study in math. Others may have been simply happy that the experience was behind them. Some may even have rejected the notion of this model altogether. The Complex Analysis course constructed a particular model and students who found that model appealing were served well by it, but the diversity of humans suggests that not everyone could have.

By pointing out that some students might not have liked the experience I am not attempting to devalue the positive experiences of others. Nor am I trying to argue that the class should become more inclusive of students with different orientations and preferences. I do not hold that this model student or this vision of mathematics found in Complex Analysis is a bad or a good thing. It is one of several orientations to mathematics and the model student that found expression in my study.

CHAPTER FIVE

DISCRETE MATHEMATICS: TEACHERS

Everything that is written merely to please the author is worthless.
--Blaise Pascal

Introduction

The Discrete Mathematics class was a 400 level course that met three days a week for 50-minute periods. It was an option in a set of five selective courses in the algebra strand. Students could also have chosen Abstract Algebra II, Linear Algebra II, Topics in Number Theory, or Honors Algebra I to fulfill this requirement. Students could also take this course to fulfill an elective if they had chosen one of the other four to meet the requirement. Students taking this course tended to be juniors or seniors although the prerequisites only stipulated that they must have had Linear Algebra first and so would have had the calculus sequence as well. A sophomore could have been enrolled but this would have been unlikely considering the program requirements and the schedule of courses.

At the time I observed this class I had access to the number of students enrolled in this course and simultaneously in the methods course for preservice secondary teachers. Eight of the 24 students were jointly enrolled in these courses. Therefore at least eight were seniors and were intending to be teachers. It is possible that others were in the certification program but not yet enrolled in the methods course.

Like most courses taught in the mathematics building, this one was arranged with the desks facing a large chalkboard in the front of the room with the row of windows to the students' left. There were no computer stations or projectors available. This was one of the larger rooms and had two doors on the right side; one in the back and one in the front. Students entered through both doors and would occasionally exit through the back door if they needed to leave before class ended. This room was on the ground floor near the restrooms and therefore like the other classes described, tended to get lots of noise from the hallway. Dr. DM usually shut the front door once class began and a student would do the same for the back door. Once the doors were closed the room was still and generally quiet.

The general atmosphere was pleasant as the windows were usually open and, being on the ground floor were larger than in other rooms, let in a nice breeze. Dr. DM usually addressed the class from a table at the front of the room or at the chalkboard and did not engage in conversations with the students before class as some of the instructors did. Groups of students knew one another and would chat before class and would help one another during class. Dr. DM addressed the students by their first names and they addressed him by his title and last name. This class tended to be more reserved than the Advanced Geometry and the Real Analysis courses although they often would offer ideas and ask questions in class.

Prelude to the analysis

The Discrete Mathematics course constructed the undergraduate mathematics student as needing a different self-concept (the substance). The course constructed students as needing a shift from a conception of themselves as calculators to one of problem solvers able to articulate pedagogically appropriate explanations. Here, the notion that students do not recall or understand theorems, techniques or other content from previous courses is not the main concern, as it was in Complex Analysis. Nor was it that their belief system about knowledge needed changing like in Real Analysis. Rather, students have not been exposed to "real" mathematics before this course. Real, here, looked like problem solving and explaining. They may have been bright students who have managed to navigate the college experience with success so far—they are in fact in a 400-level course in what is viewed by most Americans to be a very tough subject, but their ways of thinking about what it means to do math have been flawed. Their success has been false and they need to alter their definition of mathematics. Whatever they may have learned previously, it has not helped them lean to problem solve and explain their solutions, which they should learn if they want to actually be members of the greater mathematics community rather than students of formulas. Belonging to the community of professionals who work in mathematics as well as membership in a classroom community is set up as an appeal (mode) to get students to themselves and their work differently. They need to not think of themselves as students concerned with calculating solutions but instead as students interested in explaining solutions. To effect this change, the course deemphasizes formal definitions and formulas and focuses instead on problem solving and proving where proof is a tool of explanation. To accomplish this, the students are expected to engage in a regimen of explaining themselves—their confusions, their ideas, and their solutions to homework assignments, quizzes, and tests. Explaining here means that students are rendering mathematical ideas into pedagogical forms. Mathematics is explaining mathematics to others. The model student that would emerge from a perfect realization of this Discrete Mathematics course would be a teacher.

Substance: Self-concept

As in the Advanced Geometry course, students in this Discrete Mathematics course were constructed as in need of a new self-concept. But whereas the Advanced Geometry course worked to change students' concept from traditional student to novice researcher, students in this course were treated as if they needed a change from students-as-memorizers and human calculators to students-as-problem-solvers. They were not positioned as needing to improve their capacity to memorize definitions, theorems, and proofs although that was certainly not discouraged. The course did include instances of definitions, theorems, and lots of proofs, but whereas in some classes it seemed that students were trying to memorize proofs as given by the professors of the courses, in Discrete Mathematics the memorization was not emphasized by either the professor or the students. To "do a proof" was not to memorize a proof or replicate it later. Instead it was to use proofs as a form of explanation and a problem-solving tool. They also were not seen as needing to get better at their calculation skills, but again this was also not discouraged. The primary new self-concept targeted was to mathematize problems into theorems and then prove them. This work was for the purpose of changing students' self concept about what it meant to be a student of mathematics. They should shed their views of mathematics as memorization and calculations and instead take on a new self-concept of a student who views mathematics as about solving problems and proving solutions.

Again, a distinction between the Advanced Geometry and this course might be helpful. In the geometry course the self-concept change was from student to researcher. In Discrete Mathematics it was from one type of student to another type of student. They were still positioned primarily as students who did very conventional student-like things such as watch, listen, take notes, and offer up ideas in class. They did not author their own conjectures as students in the geometry course did. That distinction is important to the way that the self-concept target differed in each class. In geometry students decided, within a given framework, what issues and mathematical questions they would pursue. In the discrete course, the professor provided the questions that would be investigated and solved. Who authored the questions differed between the courses—students or professors.

Although the targeted substance of the Discrete Mathematics course was from students' self-concept from a one type of student to another type of student, the differences between these two types was still important. The course treated the students entering the course as if they valued calculations and memorization of formulas, definitions, theorems, and possibly proofs. The self-concept the course was designed to instill is one where the students see themselves as problem solvers who use techniques of proving to tackle both real-world and abstract problems. Proofs were presented as a way to solve a problem and as a pedagogical tool, and not as an extraneous add-on to a theorem.

The syllabus described the course as one that could conceivably have been students' first course in undergraduate mathematics and so there are no prerequisites[7] or expectations for students to utilize some previous course's material. Whatever they may have learned prior to this course is not seen as particularly valuable. That is, whatever sense of mathematics the students come to the course with is not seen as necessary. In fact, it is viewed as rather limiting and an impediment to getting them to adopt a new self-concept of what it means to study and be a student of mathematics. Compare this to the Complex Analysis course where students' past courses were positioned as very

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[7] The university has a collection of prerequisites for the course, but the syllabus denies that they are truly required background.

important and absolutely necessary. Here mathematics was not viewed as a continually progressing and building upon itself. Rather, it was discrete. Other than the above statement, the syllabus does not describe the course material or content focus in much detail. There is no list of topics to be covered, only a suggestion that certain chapters will be discussed although many sections within those chapters may be omitted and other topics added by the professor during lectures. The syllabus does not emphasize content in the way that others did. By not including a list of topics the syllabus sends a message that mathematics as topics and well-defined content is not the way of this course. They will not progress linearly through a pre-set group of definitions, formulas, theorems, and proofs. The syllabus works against such a notion of mathematics.

The textbook instructed the students that the course would not attempt to cover topics related to discrete mathematics in depth and that such a goal would be "ill-defined and impossible" (Lovasz, Pelikan, & Vesztergombi, 2003, p. vi). Instead, the course would discuss a few topics and their related theorems and proofs. The book stated that calculus might not be the best introductory course for students of undergraduate mathematics. The authors suggested that calculus has been an important development in mathematics, is worthy of study, and is appropriate (eventually) for future mathematicians, engineers, and computer scientists, but may not be the best way to introduce students to what mathematics as a field is "all about" (p. v).

Both the syllabus and the textbook agreed that what students encounter in other courses might not be helpful to establishing what they suggest are appropriate self-concepts as students of mathematics. The textbook calls calculus "very technical" (Lovasz et al., 2003, p. v) and went on to explain that to start a program of mathematical study with all of the technical ins and outs found in calculus and analysis misses the "feeling" (p.v) for mathematics and sets up a too narrow view of what it means to be a student of mathematics. This course was designed to widen that view.

What then is mathematics "all about" in this course? What will this course try to change about their self-concepts? The textbook's first chapter begins by positioning students of mathematics as problem solvers. The characters in the first problem, Alice, Bob, Carl, Diane, Eve, Frank, and George are considering how many total handshakes will have occurred after everyone shakes hands at a party. The solution and proof emerge through conversation amongst one another and take wrong turns along the way, as new ideas are suggested, rejected, or accepted. For example, in the handshake problem Bob observes that he shook six hands, inspiring Carl to reason that because they each shook 6 people's hands and there are seven of them present then the total number is 6 times 7 equals 42. Diane informs Carl that his number seems far too high and has used poor logic because he shaking her hand and she shaking his is not two shakes, but one. Eve steps in and declares that they should then divide 42 by 2 to account for the double counting. No one argues with her and so the matter is settled.

In this example, the introductory problem, there is no definition or theorem given. There is no summary of the chapter, which is titled *Let's Count*. The action begins straightaway with Alice inviting her friends to her party. Furthermore, this introductory problem takes up 13 lines of text, barely a quarter of a page, eight lines of which are dialogue. Mathematics then is someone being curious, asking a question, posing a problem. To answer the question another person starts by making a personal observation. This, then, is followed by some other person generalizing from that observation. Next, this hypothesis is critiqued based on feeling, a sense of what is reasonable, and logical. Finally, a fifth person resolves the conflict and amends the hypothesis in consideration of the critique.

This approach is interesting because it seems to disagree with many other accounts of mathematics. First, there are people and they are in a conversation. People talk, and this talk generates questions, ideas, critiques, and resolution. Sometimes the ideas are experiential observations. Other times they are plausible and/or logical arguments. At other times the talk is based on feelings such as when Diane suggests that Carl's number seems too high. She does not know the answer and so cannot compare his number to the correct one, but it just does not feel right. People and language are important in mathematics. Thinking about mathematics requires observations, feelings, and logic. The resolution of the problem is another statement (Eve's declaration to divide 42 by 2 and thus there were 21 handshakes)—one that is not challenged by contradictory observations or feelings, and which stands up to every guest's sense of logic.

Although the textbook begins to wean the reader off Alice and her friends quickly by inserting a new character, the third-person omniscient narrators (a.k.a. textbook authors), the course continues to emphasize the contextual nature of mathematics as people solving problems. That is, that problems are best understood in a human context, even if at times a bit bizarre. Consider for example, this episode from the class observations, which involves a wonderfully absurd problem scenario.

The professor was standing in front of the six rows of student desks, not all of which were filled. Those rows closest to the two doors connecting the room to the hall were taken up completely as were the front desks. He paced back and forth with chalk in his hand with the textbook on the table in the front center of the room. "We will discuss the Pigeonhole Principle." The fluorescent lights create bright white glares on the dark green and dusty chalkboard. He drew two squares on the board followed by a gap before drawing a third. He told the class to assume that there were N boxes drawn. He then said that the Pigeonhole Principle states that if we have N+1 objects to place in N boxes, then at least one box will have at least two objects in it. "Let us count the number of hair-strands for every person in New York city. Claim—At least two persons have the same number of hair-strands."

The lesson continued, but before we follow the lecture, instead consider the characteristics of the question under investigation in an upper-level undergraduate mathematics course. The textbook began by stating that one of the goals of the course is to examine the sorts of problems a mathematician might consider. Granted, not every problem is supposed to be of interest to working mathematicians, but this one would allow us to infer some decidedly juicy things about their preoccupations. If this is the view of mathematics and what a student of mathematics should be thinking about, it is quite different from a memorization and calculation based approach. Here math is presented as a bit quirky by focusing on the number of hairs on people's heads.

Like many problems in mathematics courses where contexts are provided, the context can still be a bit fantastic. That is, yes, New Yorkers are humans, hair is a real thing and it is countable, but to use this problem to motivate a reason to study or learn a new technique is interesting. It is grounded in reality in the sense that these objects are tangible, but just what constitutes a realistic problem is less definable. Mathematics educators talk about a *real problem* as one that people care about, find relevant, or

otherwise intriguing. What may be a problem for Alpha may be no problem for Beta; but what *is* compelling about this task is the ease of understanding the claim being made—At least two persons in New York have the same number of hairs on their respective heads. Simple to understand. Either every person has a unique number of hairs, or at least two heads are adorned by the same number of hairs.

The simplicity of understanding the problems[8] in discrete mathematics plays a role in the expectations for students' self concept shifts. In this course, students need to prove things, but the professor points out that he thinks proofs should be introduced into the curriculum not as an "artificial trick" but as a technique, the only technique for solving a problem. Proof as a way of thinking through a solution. This requires a problem that can be easily understood and thought through logically. Recall that the textbook described calculus and analysis as too technical for students to get a "feeling" for what math is "all about." Many of the problems investigated in this Discrete Mathematics course do not require really technical vocabularies or definitions to initially comprehend and ponder. One should not waste time in understanding the problem[9] if that time could better be used in formulating the solution. When the problem can be easily understood without the student needing to be trained in a lot of technical matters then different strategies for thinking through the problem can be the focus. To move away from a technical view of math as formulas, definitions, procedures and more to memorize, this course enveloped students in problems that are simple to articulate but often more tricky

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[8] Not necessarily simple to solve.

[9] Many mathematics education researchers and mathematicians (e.g., Polya, 1957; A. H. Schoenfeld, 1992) agree that understanding the problem at hand is extremely important and sufficient time should be given to understanding the problem before solving it. In Dr. DM's class, the problems do tend to be fairly simple to initially understand.

to solve.

Students were constructed as needing to view themselves as people who study problems and how to mathematize real-world scenarios. They need to be able to mathematize before they can solve or prove (explain) the solutions using mathematics. Let us return to New Yorkers and their tresses. Where we left off, the professor had just stated the claim under investigation, that at least two New Yorkers share the same number of hairs on their heads. He next asserted that this is actually not a math problem at all because "neither hairs nor New Yorkers are mathematical objects." It is only an excuse to talk about some mathematics. The task has to be mathematized, or abstracted away from a problem about actual living breathing New Yorkers. After the class had discussed why at least two New Yorkers have the same number of hair strands[10], Professor DM says that this problem about New Yorkers is ridiculous. We do not know who New Yorkers are exactly or how many there are, no one will actually count their hairs, and absolutely no self-respecting New Yorker would let us put him or her in a box to make a point. The set up is crazy and so is the solution method, but if we pull away from the New Yorkers, hairs, and boxes and think instead about the Pigeonhole Principle, this is the real point. If you have more objects (mathematical or not) than categories, then at least one category must have two objects in it. Although Dr. DM made a distinction between real objects and mathematical objects in this example, students were still positioned as needing to think about math as including the step of mathematizing. When they think of math as too abstract right from the start—when they jump to formulas and

[10] After a slight aside on the vagueness of the idea of a *New Yorker*, he states that there are over 10 million of them, but it is a scientific observation that no person has more than 500,000 strands of hair so there are 500,001 boxes if we include baldness. By the pigeonhole principle if we tried to place the 10 million New Yorkers in boxes representing the number of hairs they have, then 10 million far exceeds 500,001 and so at least two must have the same number of hair strands.

definitions and uncontextualized theorems they are seen as excluding much of the work of being a student of mathematics. Their self-concepts need this mathematizing aspect (re)inserted after so much prior coursework that has ignored it to the detriment of students' sense of mathematics as a field.

There was evidence from the class that students really did enjoy calculations and formal definitions and vocabulary. In early September a student is at the board showing his work on a counting problem—"What is the number of pairings (in all of the senses as above [referencing an above situation with Alice and her party friends]) in a party of 10?" (Lovasz, Pelikan, & Vesztergombi, 2003, p. 4)—and got to a point where he had $\frac{10*9*8*7*6}{2^5} = 5*9*7*3$ and began to calculate this to get a single number solution. Dr. DM stopped him and told the class, "There is no need to compute numerical answers." To continue would be an exercise in multiplication. The problem had already been solved in Dr. DM's estimation and did not require further work despite the students' desire to keep going to a final conclusion with no operation signs. Such a move by Dr. DM was a direct intervention against students' sense that mathematics is the doing of calculations and that their sense of self as students of mathematics was tied to how well they could calculate numerical results.

Soon after the above incident a student asked how the problem just worked was different from the handshake problem. Dr. DM began to describe the differences in the problems and their solution methods using casual vocabulary and the difference between a pair and a pairing. She asks further, "Is there a formal definition?" referring to the definition of pairs. Dr. DM seemed a bit taken aback and stated a bit incredulously, "You want a formal definition?" But rather than resist this request, he offered one and wrote it on the board as he also said it aloud:

> Let A be a finite set. A pair is a two element subset of A.
> $\{a,b\}$ $a,b \in A$, $a \neq b$.
>
> A pairing of A (which exists only if $|A|$ is even) is a set of pairs, $|A|/2$, such that their union is A.
> Ex: $A = \{a,b,c,d\}$ gives us $\{\{a,b\}, \{c,d\}\}$

He then turned the conversation quickly from the formal definitions just given and suggested that they instead think of the difference between pairs and pairings graphically and drew a picture (see Figure 5.1).

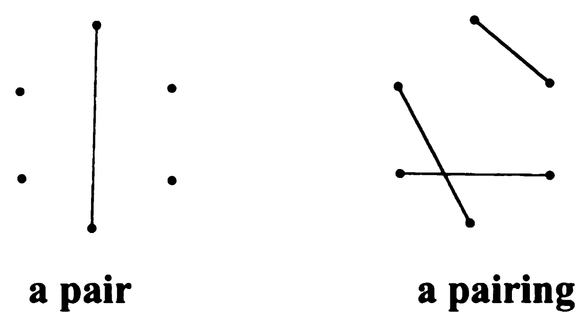

Figure 5.1: Dr. DM's pair and pairing illustrations

Although Dr. DM did indeed provide students with the formal definition when requested (and is able to conjure it very quickly suggesting that he knows this well) he tried to downplay the formal definition by offering instead that they consider a graphical representation to help illuminate the distinction between a pair and a pairing. Contrast this with Real Analysis and Complex Analysis where providing definitions was one of the first things the instructors did before transitioning into a new topic. Altogether this work on the pairing problem indicates that Dr. DM did not encourage students to focus on performing calculations or memorizing definitions. He conveyed that they should be able to articulate a solution and posses general understandings about ideas—not necessarily technical details. To do otherwise would be to lose sight of the forest for the sake of the mathematical trees. The trees may be wonderful but they are not everything.

In another example of the course's stance on calculations, Dr. DM had been answering a student's question about difficulty she was having with a homework problem. In the end Dr. DM ended up with $100 \log_2 10$. He turns back to the class and asks if someone would get out their "gadget" so that the class can get a sense of the scale on this number. He goes on to add, "I'd never expect you to calculate that alone. Well, maybe I would have 400 years ago." The use of the gadget or calculator was a tool to help everyone get a feel for the size of the answer, not to determine the answer. That was already done. It was $100 \log_2 10$. Furthermore, his comment also suggested that anyone who would busy themselves with tedious calculations, especially without the assistance of a gadget, would be centuries behind the times.

Throughout the course problems were posed, mathematized, and then solved through proof. Primarily the course treated students as if they had come into undergraduate mathematics with prior experiences that emphasized formulas and calculations over proofs (if proofs at all) enter mathematics majors with no idea what sort of work, problems, and techniques mathematicians really use. Furthermore, the course conveyed that by taking a Calculus sequence in their first two years as undergraduates, these notions are only reinforced. They had been stuck for several semesters in a study of only a few trees when they could have been admiring the forest. Students should preferably learn to dismiss their notations of mathematics as formulas and calculations, or at least expand them to include a much wider range of problem types. Students were expected to alter their self-concept to include seeing themselves as students of a much bigger problem-solving field.

Mode: Belonging

The Discrete Mathematics course shared elements of mode with the Real Analysis course despite Real Analysis's relationship to calculus, which the syllabus and textbook in Discrete Mathematics suggest provided students a view of mathematics considered too narrow. Both courses, though, invoked a sense of belonging to a community as an invitation to students to take up the respective changes each course desires for students. To get students to view themselves as students of a much larger field that is far more than memorizing and calculating, the Discrete Mathematics course invited students to belong to the greater mathematical community.

One way that Dr. DM set up the issue of belonging to a community was by setting boundaries by saying what it did not value. In that way, the values of the community were communicated indirectly. In problem sessions, such as that of the New Yorkers discussed previously, Dr. DM pointed out small details of what mathematics *is* compared to what it *is not*. In this case, the New Yorkers were not mathematical, but they were an excuse to do mathematics. That phrase foreshadowed other distinctions about what math is and what it is not; and who does it and who does not. Particularly, doing mathematics and belonging to the math community is set up in opposition to doing science and belonging to that community. One class in late October, Dr. DM is asked by a student to share the proof of a problem in the text. He says he wants to do it both graphically and by induction[11]. He begins with the graphical version and draws a sequence of numbers forming a Pascal's triangle. He shows, by a series of examples, a property of the triangle before saying, "Now we'll do it formally by induction. Well, that was by induction as well, but informally." This phrase also sets up a distinction to come about the difference between mathematical (formal) induction and scientific (informal) induction. Once he completes the mathematical induction proof he turns to the class and describes how the Pascal's triangle example was "something different." It was scientific induction, which he explains as seeing that a pattern holds for some finite number of cases before making a conclusion that it must always hold true.

> This is the real danger in using scientific induction in mathematics—even in science! [Laughs.] An experiment might hold up 10^{10} times but does not hold on the next term. We need a proof.

Through these seeming asides, he described the nature of mathematics as distinguished by proofs from other fields including the sciences. He invited the students to study these proof techniques so that they too could see the differences between mathematics and everything else. Until they could appreciate this distinction, they would not understand the expanse and boundaries of mathematics, which is what the course set as a goal, namely that they would identify as a part of who they were.

On the one hand the course wants to widen what is presumed to be a limited view of the discipline. But to know the fullness of mathematics also requires knowledge of the boundaries. To conceive of everything as mathematics would be also to conceive of nothing as mathematics. The word as a label would become superfluous and unnecessary. Placing boundaries is needed here. To get students to take on this wider view was also to

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[11] The mathematical proof technique.

get them to recognize some limits for the community. Science is an example given on the other side of the line from mathematics. The difference offered is that science is satisfied with a large body of cases whereas those who belong to the community of mathematics do not trust even a really big set of cases and desire a proof.

The issue of proof as a defining feature of the community was key in this Discrete Mathematics course. "It is important to realize that there is no mathematics without *proofs*." (Lovasz et al., 2003, p. vi) The textbook's comment echoed the sentiment that to avoid the case where mathematics is so broad that it does not exist, we need some feature by which to know it when we see it. It is proofs. That the community of mathematicians care about proof is important. To work on students' sense of self as problem solvers where proof is a technique of explanation, the course mobilizes the values of the larger community to entice students to take on new sets of personal orientations towards math. If the larger community values proofs over calculation and memorization work, then maybe these students should as well.

Proofs played a big role in class time. Usually each day began with students asking questions related to homework questions that were usually of the form "Prove that…" During this time Dr. DM would answer questions usually by doing a proof on the board. On a few occasions he would ask a student to come up to the board and show what they were attempting or what they were finding a struggle. That students were asked to come forward publicly with their proofs was a nod towards belonging to a community on a smaller scale. Dr. DM was not solely responsible for public talk and ideas although he did seem to bear the brunt of the load. When Dr. DM or some student was not proving some theorem or case, Dr DM was usually setting up a problem scenario (such as the

New Yorkers, or Alice and her friends, or rabbits, or castles) and then posing a question related to the scenario. Next he would begin to work on a proof that would resolve the issue. On some occasions students helped along as he proceeded through the proof. They might call out next steps or stop him to ask questions or clarification. They participated in this community by coming to the board, asking questions, advancing the proofs, and by helping one another. Many times throughout the semester students would lean over and work with other students even while Dr. DM was working through something at the board. They would lean over to ask one another questions, point out things Dr. DM was writing, and stay after class to continue their work. Unlike in the Complex Analysis course where students would stay late to talk to Dr. CA, in this course they most likely stayed late to talk with one another. As one student told me one day before class started, "There just aren't a lot of math classes at [this university] that let you work with your friends," and went on to add that he preferred working with his friend Amanda. In this course to belong to a community of mathematics (macro and micro) was to work together. Doing math was not seen as a solitary path. In this example we see students taking up this invitation to be a part of the community. The student quoted above was explicitly commenting on a feature of this class that was appealing and that he did not see in many other courses; he could work with a friend and this collaboration was welcome.

One day in late October as several students were taking notes Amanda had her notebook paper out in a tidy stack and at neat right angles to her desk but she wasn't writing on it with any one of her three mechanical pencils also lined up ready to work. Instead she was looking back and forth from her textbook to the board. She glanced up, looked through her book, and then looked back again at the board. Every now and again she flipped pages as she passed from staring at the board to her textbook and back. Finally, she asked Dr. DM a question and seemed satisfied with the response. But soon after she asked her friend, Derrick, what page Dr. DM was working from and he leaned over and pointed it out for her. She again went through her book-to-board glance dance. She then turned to Derrick again and they carried out a whispered conversation. When that ended she began to take some notes, but stopped soon after and put her hand up as if to ask a question. Then back down again. Then up again. She seemed conflicted. Dr. DM asked her if she had a question. They then engaged in a few exchanges, but neither seemed to be understanding the other. Finally, Derrick told her that they could talk after class. They did. All of this confusion it seems was the result of Dr. DM and Amanda not sharing a picture of the problem, but assuming that they did. He had drawn on the board a bird's eye view of a staircase whereas she had assumed that the staircase discussed in the textbook would have a side-on view (it was not drawn in the textbook). Coordinating her idea of the staircase with his impeded her ability to comprehend the accompanying notations and theorem and a fellow student took the time to acknowledge her need for help and to act on it. But with the assistance of a friend in the class who took on the work of helping her she was able to make sense of her confusion. Again, students were taking responsibility for one another and were taking up not only the invitation to belong, but particularly they were helping one another by explaining ideas and not stating answers. The way that the above student helped Amanda was by explaining the mathematical features of her model of the staircase and Dr. DM's. He was solving a problem of mathematics and of pedagogy.

That students worked together was not unique to this course, but what seemed different in Discrete Mathematics was that working together was not seen as a separate type of class time. For example of contrast, in Advanced Geometry the students worked together almost all of each class, but there were two basic types of class time—students working in small groups or students and instructors holding group conversations. As another contrast, in Real Analysis students were expected to work together during the portions of class when told to do so either in small grouping-class sessions or in the once a week study sessions. In Discrete Mathematics, however, the students were much more flexible between whole-class time, professor lecture time, and student work time. In fact, those distinctions do not really make sense for that class. Any moment it seemed could turn into a small pair activity, or into a whole class conversation, or into a professor lecture. The ways of belonging in this community were their own among the courses observed. Belonging to the community did not mean that a student always needed to work with a partner or group as they mostly did on the Advanced Geometry course. Part of the appeal of this type of belonging was that students could create the ways they wanted to participate. In Real Analysis Dr. RA largely defined the times when students would switch from observer to actor. In Discrete Mathematics there were no assumed best times for particular ways of belonging. That is a student could observe one minute and then choose to work quietly with a neighbor and possibly switch back again as best fit their interests.

The ways that students worked together, with the professor, and alone reflected the variety of preferences one might have about work in social settings. Someone might want to be alone all the time, or only work with one other person in particular. She might want to always work with others and prefer doing it with just about anyone. The syllabus did offer some guidance on student collaboration. "While the discussions of the material and homework assignments with your fellow students are encouraged, you must prepare your homework papers alone." Discussing material and homework then could be done alone or with others and collaboration was actually encouraged rather than being permitted as it was in the Complex Analysis course. It was only the writing up of the homework solutions that was dictated to be a solitary activity.

Furthermore, an interesting bit of evidence that suggests that the course values collaborative activity but without exactly defining who should take on which roles is on the first page of the syllabus. There is a large sketch of three cartoon turtles working together on a math problem. One turtle stands at a chalkboard and holds the chalk. He or she is looking back at the other two, seemingly for confirmation. This turtle leans on the desk of the third turtle and has his or her left leg casually crossed over the right ankle. A second stands near the first scratching his or her head. A third turtle sits at a nearby table drinking a cup of steaming coffee as a large question mark hovers above his or her head. The setting might be a classroom, but there is only one table in view and it is slightly larger than an individual classroom desk and has an unattached chair. In this scene it is unclear whether any turtle is taking on the role of "teacher." The first holds the chalk, but the second stands nearby as if monitoring the work closely, and the third could be interpreted as evaluating the work of the first and is finding it lacking. This scene is just below the course title and instructor contact information. The syllabus sends a pictorial message to students that collaboration is encouraged but is free to interpretation regarding who teaches, who learns, and who evaluates.

Belonging to a community was a way the course invited students to take on new self concepts about mathematics and students were fairly free to choose how they wanted to be in this community regarding their participation in class and in their discussions about course material and homework assignments. On the other hand, they were presented with a particular view of the field as understood by others outside of their class and according to the view the criterion for membership or belonging was to think of mathematics as about the study of and construction of proofs. Belonging to a community on the small classroom scale was fairly open to student preference, but if students wanted to belong to the larger community they would need to take on new views about math that better matched those already members of the larger mathematics community.

Regimen: Pedagogically effective prose

Students were asked to do many things in this course and this regimen falls under a theme. The students were asked to explain. Undergraduate math courses are known for their emphasis on proof and this course was no exception. In several ways this course shared a lot with both the Advanced Geometry course and the Real Analysis one. That it targeted the students' self-concepts about math it has ties to the Advanced Geometry course. Yet in that course the emphasis on proofs was tied more closely to conjecturing statements that would later become proofs if possible. In that it worked on changing conceptions about what it means to do math—particularly away from a calculations and memorization orientation there are echoes of the Real Analysis course. The Real Analysis course, though, worked on students' epistemologies—how they decide what is known. This was done through getting students to judge the truth of statements, the persuasiveness of a proof, and by having students write proofs to given theorems. In the

Discrete Mathematics course the emphasis on proofs was in getting students to see the explanative power of proofs. It was not so much that students needed to learn that proofs determine knowledge, but how proofs can explain solutions and provide insight. Mimicking this orientation towards proving, students were asked to explain in many facets of their course experience using the discursive features of proof.

Recall that the textbook declared that there is no mathematics without proofs. Throughout the textbook proofs are given as explanations of results more than as validation of them. In the first chapter the textbook gives a theorem: "A set with n elements has 2^n subsets" (Lovasz et al., 2003, p. 10). The text goes on to provide two arguments for the theorem. The first is a tree diagram that shows how to select the possible subsets of a three-element set and goes on to describe this diagram and how to modify it for an n-element set. The second proof shows how to enumerate subsets or label subsets with a number so that a person could speak about the 53^{rd} subset, for example. After showing two such ways to make lists of subsets the textbook declares that there is a problem with this method. What if there were ten elements and you were asked to name the 233^{rd} subset. Creating a list of them all would be rather tedious and so a system of counting using binary notation is introduced. From this explanation of the use of binary labeling an argument is forwarded to declare that for n elements there must be 2^n subsets. After the two proofs are given the textbook makes a remark.

> You may have wondered why we needed two proofs. Certainly not because a single proof would not have given enough confidence in the truth of the statement!...The answer is that every proof reveals much more that just the bare fact stated in the theorem, and this revelation may be more valuable than the theorem itself. For example, the first proof given above introduced the idea of breaking down the selection of a subset into

> independent decisions and the representation of this idea by a 'decision tree'; we will use this idea repeatedly. The second proof introduced the idea of enumerating these subsets...we also saw an important method of counting: We established a correspondence between the objects we wanted to count (the subsets) and some other kinds of objects that we can count easily (the numbers 0, 1, ..., $2^n - 1$). (Lovasz et al., 2003, p. 10)

Proofs are explanations as much as they are validations and possibly more so. When students did the work of the course they were not just proving in the sense of verifying and making a claim about knowledge. They were explaining ideas to others and themselves. When submitting these homework sets, the syllabus clearly states:

> ...you will be required to write your homework and exam papers in full and coherent sentences. Even if the solution of a problem consists mostly of computations, you will be required to provide explanations of these computations, like you will explain them to a friend... Papers consisting of only formulas without any explanations will get **0 credit**. You are advised to prepare a handwritten draft before typing, and to clarify your ideas during the preparation of the final version.

The expectation for students as they did their homework and constructed their proofs was to explain their own mathematical thoughts in ordinary English sentences. The word "explain" or some form of it is invoked three times. Doing the work of the class meant to explain. On the day Dr. DM returned students' first quizzes, he again pointed out that students needed to explain their work. "Formulas and just answers are not enough to get full credit." Furthermore, these explanations needed to be in "full and coherent sentences", or in other words, in prose.

In late October Dr. DM begins a unit on graph theory by telling the class about graphs and how they can be helpful visual representations of things that are hard to put into words or are too cumbersome for other representations. He draws a simple 4-node graph. He also gives a lot of examples, referring again back to Alice and her handshake party. He tells the class that he could give a formal definition of a graph but he really prefers it if they "approach it with pictures. Informal language and the language of graph theory are the best explanation."

Students also demanded pedagogical explanations from Dr. DM and the textbook assumes that they will likewise expect full explanations from it as well. One day Dr. announced his plans to present two proofs for a particular theorem related to graph theory; the first will use induction and the second will use double counting towards the solution. He began the first one by using induction on the number of edges in the graph. Once he completed this, a student asked him if this proof based on induction of the number of edges was the only induction proof. Was it possible to do induction on the vertices? The instructor replied that it was possible but that it was tricky, and then seemed to be about to move on past her question. He paused though and announced that he would do it her way. The student wanted to know more about induction in graph theory and whether induction by edges was the only way, and if so why and if not, what would it look like with induction on the vertices.

The instructor's decision to follow her question was an instance of a student having expectations for him and his explanations. That he tended to explain ideas through his proofs was not just him modeling what he wanted students to do. It was also an example of how a course constructs ideas, not just the professor. Students contribute to the constructions and expectations as well. In this case they demanded more from him. After he finished her proof and the second one (now third) that he had originally planned for he commented, "A theorem does not need three proofs. One is enough to be convincing, but these are illustrative for us as they shed light on different methods."

The textbook also assumed that students would demand better proofs and explanations. These explanations should not just be about the truth of the statement but should be helpful to the reader. In the chapter on Fibonacci numbers the textbook presents a theorem for a formula to give an nth Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$ (Lovasz et al., 2003, p. 71)

Here is the accompanying proof:

"It is straightforward to check that this formula gives the right value for n = 0, 1, and then one can prove its validity for all n by induction." (p.72). This is followed by an exercise to do so. Just after the declaration that the reader should carry forth this induction proof the textbook continues.

> Do you feel cheated by this proof? You should; while it is, of course, logically correct what we did, one would like to see more: How can one arrive at such a formula? What should we try to get a similar formula if we face a similar, but different, recurrence? So let us forget Theorem 4.3.1 for a while and let us try to find a formula for F_n from 'scratch.' (p. 72)

The induction proof would verify the formula but it would not be an explanation of where the formula came from or be helpful if students faced similar tasks in the future. An induction proof would have been cheating them. They should expect better explanations from others just as they were required to deliver better explanations to Dr. DM and the rest of the class.

The professor and the textbook wants to explain ideas to students, wants them to expect more from them, and in turn wants them to explain ideas back to him on their assignments. They also need to be able to explain their ideas to one another. As has been mentioned, students were asked to come to the board to show what progress they had been making on a problem or to come show the class what struggles they were facing. Explaining was not reserved for student to professor alone and nor was it only for use when you were sure of the result. Explaining could happen even when a student was lost.

At times Professor DM's responses to student questions were brief for yes/no type questions, and at other times he engaged the student to better understand the question. Consider the following exchange from the second week of the semester just after Professor DM asked for any questions about the homework assignment from the students.

> Student: "1.8.8 (referring to an assignment number from the textbook)."
> Professor DM: "So what's the question?"
> Student: "What's my question?"
> Professor DM: "Yeah."
> Student: "I did it but I don't know if it's right."
> Professor DM: "Yes. (laughs) What is not clear about it?"
> Student: "The overcounting is not clear"

This exchange reveals some interesting expectations for him from the students and for the students from him. He expected them to ask questions, but there was more. They were expected to formulate the question in terms of pedagogically legible English prose—to explain what it was they needed help solving or thinking about and to provide their work thus far. By contrast, the students expected him to begin solving the problem when they offered a problem number. For the first few weeks this pattern continued. Students offered problem numbers instead of questions and he kept asking for a question or had them come forward to show their work. As the semester progressed students began to ask questions differently. Instead of only offering the problem number they began to follow the problem number with an explanation of the difficulty they were facing.

That part of explaining included getting lost or muddled was reiterated in the textbook. The authors advised the readers to be aware that learning mathematics requires “dirtying your hands” (Lovasz et al., 2003, p. vi). The metaphor suggests an active role and possibly also some unpleasantness. To be able to explain something you need to try it out and possibly fail from time to time. The nod to the role of failure might be a reason Dr. DM dropped the lowest quiz grade and on a few occasions threw out a whole set of quizzes and homework telling the class that he was not happy with using them as measures of what they could do. This example also provides insight into the substance targeted for change by the course requirements. The students’ self-concepts about mathematics could include errors. Students of mathematics can make mistakes; they are not always going to arrive at correct answers. Dr. DM’s grading policies reflected this aspect of the substance.

The issue of grading and assessment in his course included a very interesting idea related to another way that explaining emerged. Students in this class (and in others) talked to me every now and again about the course and shared opinions on it and so forth, usually in the form of pre-class chit-chat. One day a student was telling me about the grading policies. Before I share the students’ take on grading let me offer what the syllabus proclaims. “Your grade will be based on your homework, quizzes, and tests, with approximately 1/3 of the grade determined by each of them.” According to this student and with several nearby nodding in agreement, the unstated method for determining grades was that near the end of the course Dr. DM would tell each person what grade they would receive and then it was up to the students to set up a private meeting to put their thinking into pedagogically effective, persuasive, prose. They should argue why (if they felt it worthwhile) their grade should be altered. With no confirmation of this by the professor I can’t declare whether this was the case or if it was a rumor passed around the class. On the other hand I was able to determine by observation that neither of the guidelines in the syllabus about grading quizzes or homeworks was enforced. Regardless, that students held this idea speaks to the ways that a culture of explaining had formed. Everything, it seemed, revolved around making your thinking pedagogical through explanations. It also speaks to the role of community in the class that grades were possibly (or at least thought to be) negotiable in a face-to-face manner. Students saw themselves as full members with rights and responsibilities for determining their final grades.

Students also found other ways to explain themselves. Amanda told me one day of her plans to petition Dr. DM for extra credit opportunities, as she walked around the room before class asking students to sign a paper. He had previously told the class that there would be no extra credit. It seemed he had recently told Amanda that if she could get everyone in the class to agree to want extra credit then he would create something. She must have persevered and collected everyone's signature because the next week Dr. DM announced two options for extra credit. Amanda, in this example, had presented her concerns to Dr. DM and he had been moved by her response to his concerns that the whole class needed to in agreement. Her use of a petition was pedagogically effective for him.

The first option offered by Professor DM was to submit an additional homework set to replace a previous lower score. The second is to present a 25-minute (half the period) lesson. To make room for these lessons Professor DM reserved three class dates for a total of six possible presenters. Students were instructed that if they wanted to present they needed to meet him to jointly agree on a lesson date, topic and for help in preparing the lesson. Only four students decided to pursue this option[12]. During these four student lessons, students presented and proved problems/theorems with the usual commentary and questions by fellow students while Professor DM sat in a desk in the front corner of the room, adding comments and questions as well.

This extra credit option was an option about explaining. Students were to take on the role of the instructor and explain ideas to their fellow students for half of a class period. It was not explaining on their homework with words and maybe some symbols and pictures. It was not explaining a quick idea while seated at their desks. It was to explain on a grander scale. They would teach. Furthermore, to make space for students to teach the class Dr. DM had to forfeit three days of time he would have largely controlled. He would have chosen the topics, the order, the examples. Instead, getting students to explain and teach the class was important enough that not only did they get the days but they also, in consultation with him, chose the topics and what the explanations and teaching would look like.

The expected regimen for students was to craft explanations. They explained themselves through their assignments and proofs. They explained their ideas to the class by calling out suggestions, coming to the board, and working together during lectures. Dr. DM and the textbook also engaged in explaining through the text as presented in the book and on the chalkboard and through Dr. DM's in-class commentaries. Students' explanations also factored into their grades. They made up the bulk of their homework, quizzes, and tests. They were also important enough to the students that they understood

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12 It is unclear how many decided to accept option 1.

their final grades to in part be determined by their abilities to articulate their concerns about their final grades.

As in Complex Analysis the vision of mathematics as linear and rational extended into their ways of behaving in class and on assignments, this course's view of mathematics as pedagogical explanations also extended into their behaviors. Students were expected to explain themselves when confused, explain themselves on graded assignments, and possibly explain their arguments for a different final course grade. And finally, students were given the opportunity to take their explanations to the whole class by teaching.

Telos: Teachers

The model student is not satisfied with learning formulas and computing answers. This person wants to be able to explain why things work and seeks these answers through proof. These proofs need to be both convincing and illustrative. A proof that only answers whether something is true or false is not complete because it should also explain why in such as manner that it can be understood by someone else. The model student works towards creating proofs and listening to others' proofs. Explaining her ideas and expecting the same from others. The model student would not be satisfied with either being given only an answer or submitting only an answer. The constructed model would demand more of others: Would want explanations, fully articulated. The model student would be a teacher.

A model student emerging from this course would have a very broad view of the field. This person would have had the experience of calculus courses and analysis courses and so would be versed in the technical details of that type of mathematics but would have added to his repertoire knowledge of other fields from computer science, the math of games, social math, and topics of debate (Is the Four-Color Conjecture proven?) and those from the typical canon of mathematical history (Gauss's counting strategies, Fibonacci numbers). The model student would have previously exhibited mastery of calculations and memorizations but can now explain ideas and proofs from a wide selection of sub-fields—some considered firmly in the field and some on the fringes. Having a broad view can be helpful for a teacher who needs to be able to draw from a variety of experiences and ideas for the purpose of making a pedagogically effective explanation. The explanations must be for someone and so must take into consideration what that person will likely find helpful, but the teacher must also be able to draw accurately from her knowledge of the field to craft her explanations.

The bulk of the regimen section highlights skills a teacher would be expected to have. The course design emphasized proofs as explanations. These explanations were not to uncover hidden meanings, and not only to verify. They were explanations for someone and so they needed to move between the abstracted mathematics and the reality of the reader or student. Proofs as explanations were pedagogical. A model student would be able to render mathematics into understandable prose while still remaining faithful to the mathematics. This sort of work is the work of a teacher. Teachers try to make ideas understandable to others by recasting ideas into language that is accessible to students. The Discrete Mathematics course provided many experiences that would have served as teacher preparation. Students in the course were expected to explain themselves to the professor and also to other students in the course. A model student would be able to alter explanations for different audiences. As constructed in this course a model student would be able to translate their problem-solving for people who might not need a lot of support (like when real students of the course wrote up their homework proofs) and for people who needed more support (like when real students in the course taught one another formally as teachers-for-the-day and in small pairs). Teachers likewise need to be able to offer explanations that are appropriate for people with diverse needs and degrees of support.

Because the course design also expected students to expect better explanations from others, the model student/teacher emerging from the course would be also expect future students to be able to explain themselves in a well-articulated and coherent manner. The teacher would need to be able to understand a student who was struggling with an idea and would at times need that student to articulate his confusion in such a way as to assist the teacher. This model student-as-teacher would also expect effective explanations from students in their homework and assignments. The real students in the Discrete Mathematics course did this by expecting well-crafted explanations from Dr. DM. This would serve the model student-as-teacher by helping him better understand what the student could do/explain and what the student had not yet been able to explain. The quality of the explanation would be the key to the students' learning and the teacher's response.

All of these things could be argued to fall within the domain of a teacher. A teacher needs to know the canon of typical math courses such as calculus and analysis, but also needs to know more about math as it is experienced by non-mathematicians. The teacher should have interesting problems and ideas on hand to entice unenthusiastic students she may encounter. The course's use of discrete topics from many areas of mathematics combined with the assumed prior experiences constitute a very full repertoire of experiences and knowledge from which the teacher may need to draw. This model student would particularly know how to cast mathematics into real-world or at least near-real-world experiences and applications for the hopeful benefit of future students. Having a big picture view could help the model student select tasks and topics well suited to the interests of particular students.

Overall, the experiences, skills, and orientations to mathematics that the model student would have gained through the course would reflect the skills of a knowledgeable yet flexible teacher concerned with sharing well crafted explanations and hearing students learn to do likewise.

Concluding remarks

As with the other chapters I will close by speculating, from the perspective of a teacher educator, what real students in the course may have found promising in the course as well as what they may have found frustrating. Speculating on the experiences of real students offers another way to think about a course. A model student can help us imagine what perfection could look like, but perfection is hardly a human characteristic. By considering real students we can see how students sharing features with the model construction may have been more at home in the course and how those with different sorts of mathematical dispositions may have been excluded.

The targeted substance was students' self-concepts. Those who had found visions of mathematics as calculations based comforting would have felt uneasy in the course. I provided an example where a student was cut off from calculating his answer to single numerical expression. Students who enjoyed this sort of mathematics and saw themselves as good at it may have found the lack of this type of work as well as the explicit and implicit instructions to cease these activities, as limiting their ability to participate. They might not have wanted to change their self-concept and may have rejected a new one that seemed too much of a mismatch. Others may have felt that the view of mathematics as problem solving and explaining was liberating.

On the other hand, the course was not too abstract. Application-style problems that mimicked or at least came close to real world scenarios were abundant. Dr. DM, the textbook, and students abstracted away from these scenarios to solve them, but the problems were not presented as abstractions. Students that enjoyed being able to work with contextual problems would have enjoyed these opportunities. And again, students who preferred working with mathematics typically referred to as "pure" mathematics might have considered these problems to be too applied and not mathematical enough. They may have thought of the many problems of the partygoers (Alice and friends) as juvenile and tiresome.

Others, like Amanda, who asked for more formal definitions may have described the class as not rigorous enough and relying far too often on assumed shared interpretations. Others might have found the intuitive and graphical definitions as helpful and relatable.

Another feature of the course that students could have taken issue with was the requirement to explain. Students that preferred giving an answer and getting one might have been disappointed that they had to also provide explanations. Students with this viewpoint would have had trouble on the homework assignments, quizzes, tests, in class lecture and homework discussions. As for the possibly negotiated final course grade, several students seemed to find that appalling. When the student was telling me about this rumored grading scheme he seemed upset by it and others nearby also seemed shocked that a course grade could be argued rather than earned or given. These students did not like this possibility too much although one student to chimed in that it was a good thing in her opinion because you'd only ever argue your grade up and not down. That is, it couldn't hurt. Those that didn't like the idea, though, seemed upset that the grading policy was too ill defined and they preferred a more fully rationalized structure.

Derrick commented on the ways that this course encouraged group work and that he liked this feature. Those who preferred working with others had chances to do this. They could collaborate outside of class or hold mini lessons together in class. Students that wanted to work alone also had this opportunity as group work was not required and was not a formal aspect of the course. Across this dimension, students could elect to participate as isolated or as gregarious as desired. Belonging to the community was very open to choice and so most students should have been able to find some means of participation that was satisfactory.

This study was not intended to answer the question, "Who liked the course and why." I am speculating here from the viewpoint of a math teacher educator. But just as a model student would be a realization of the perfect enactment of this course, the image of the model student constructed by the course also provides us with insight into the perfectly alienated student. Real students probably fall somewhere in between these two extremes and might hold positions that look like the model student in one aspect and like the anti-student in others. Like in the other courses described, some students may find this Discrete Mathematics course to be a perfect or near-perfect match to their interests whereas others find it disagreeable. Whatever possibilities exist for students to embrace mathematics in a course also sets up boundaries for others.

CHAPTER SIX

WHAT MATHEMATICS IS ABOUT

We also want them to know what mathematics is about
and how mathematics is done.
--Hung-Hsi Wu

One of the most reliable finding from research on
teaching and learning is that students learn what they
are given opportunities to learn.
--James Hiebert

While working on an assignment for a course one year I had the opportunity to interview a very distinguished mathematician who is also a well-known mathematics educator. My project partner and I were excited by the chance to ask him about the different philosophies of mathematics, particularly Platonism, intuitionism, humanism, constructivism, and formalism. We had planned a lot of follow up questions to our first. I cannot recall exactly how we phrased it but it was not much more elegant than, "Tell us about your take on the philosophies of math and their relationship to mathematics education." Again I do not recall the exact response but it was not more complicated than, "I don't have a philosophy of mathematics. I just do it." We were fairly stunned. There went the interview! I even recall being concerned about what we'd put in our report. At first glance it seemed we had nothing to work with. But in retrospect I recognize that was an important moment that contributed to the first stirrings of this dissertation.

That we got such a response is not entirely shocking, really. I've heard many similar reports in my encounters with students, parents, teacher educators, mathematics educators, and mathematicians. Math is often seen as just something you do or teach. You might even enjoy doing it or teaching it or both. You might not. Math, though, just is. Math teaching also just is.

Oh, yes, there are people who do think a lot about mathematics and their teaching. This interview included one such person. He has thought far more about mathematics and the teaching of it than I surely have. Yet, still, there remains a common notion that mathematics is a single thing and that singular and unified nature is outside of philosophy. Teaching, too, is commonly viewed as independent of philosophy (despite the many requests I obliged to send my teaching philosophy to search committee chairs). It is a practice, a real world, atheoretical thing. As a person deeply interested in philosophies this piques me to put it bluntly.

This dissertation is part of my attempt to investigate some of these commonly accepted notions about mathematics, mathematics education and particularly undergraduate courses. Many people think of undergraduate mathematics classrooms as mostly similar with different topics. For example, a Differential Equations course would look roughly similar to a Number Theory course except the topics, definitions, theorems and proofs would differ. The stereotypical model is that a professor stands at a chalkboard and delivers information which students record in their notebooks. There would be homework, quizzes, and tests. Students sit in rows. So forth. The point of the class would be for students to learn the important theorems, ideas, and techniques of that topic. This stereotype is communicated often in the ways that math educators talk about math courses. We refer to courses as about something, about the topics typical to that course. When we ask one another to tell us what they teach we usually tell one another the course title or the topics of the course. But these topics are only part of the curriculum.

Undergraduate mathematics, as Hung-Hsi Wu refers to in the quote above (1998), also presents a vision of mathematics, an implicit curriculum, a construction of the subject that addresses philosophical issues about the discipline. What is it all about? How is it done? The answers to these questions are often assumed to be the same just as the surface features are so assumed. Math is about proofs. Yet again a simple statement hides a lot of diversity. The example courses I have presented here illuminate just some of the possibilities. Proofs are things to judge and critique; they determine knowledge. Proofs are codified ways of describing phenomena for a pedagogical purpose; proofs are the end product of conjecturing; proofs are the building blocks of a coherent and unified field. These are not just differences of emphasis. They are differences of philosophy. They are differences of mathematics.

"Social constructivism describes mathematics as a set of socially situated discursive practices" (Ernest, 1998, p. 263). In many ways Ernest's articulation of social constructivism as a mathematical philosophy resembles the Foucaultian perspective undergirding my analysis. Both seem to be saying that math is as math does. In my analysis of these courses math can be social, individual, Platonic, concerned with truth, and concerned with beauty among others. I did not try to argue that one course's vision was the correct one, for example, but described the ways that mathematics was done in multiple contexts as multiple possibilities for students to potentially adopt. "Persons always learn or use mathematics within a range of such contexts and are multiply positioned in them. Mathematical knowledge itself is recontextualized, reproduced, and operationalized within multiple contexts." (p. 263) Ernest's statement is very much aligned with my position. In this project I examined four courses to highlight the ways that mathematics operated across different classroom contexts at a particular time in history. I take the position that there is no one unitary mathematics and that there are multiple ways that mathematics is performed, specifically in classrooms for this project. This is a Foucaltian analysis in that I am not trying to argue for a particular truth, but am exploring how visions of certain truths of mathematics operate and exist as possibilities.

On the other hand, I, like Ernest, am not trying to develop or propose a particular philosophy of mathematics. He lays out several features of a mathematical philosophy and then sets about evaluating his take on social constructivism with respect to these criteria. In many ways he is trying to argue for a particular philosophy. In particular he is arguing against what he terms absolutist (Platonism, for example) rather than fallibilist philosophies of mathematics (humanism, for example). Similarly, Hersh (1997)uses historic examples to argue that Platonism is not well-matched to the daily lives of mathematicians or to the historical accounts of it. Although I am sympathetic to both author's work and positions, this dissertation is not an argument against a particular philosophy of mathematics. I do not present these accounts of the four courses for the purpose of claiming that certain philosophies are false. In fact, this dissertation is not about the truth or falsehood of ideas. To take Platonism as an example (because it is frequently the philosophy most under attack by philosophers of mathematics), I recognize that elements of Platonism exist in the discourse of mathematics classrooms, particularly in the Complex Analysis course. Its presence in the discourse is not something I necessarily am trying to eliminate. In fact, its presence gives rise to the ability to speak against it (as Ernest (1998), Hersh and Davis (1980), and Hersh (1997) do). Platonism makes counter-Platonic arguments possible and thinkable. By describing the courses in this dissertation I am not trying to argue for or against a philosophy. I am not trying to establish the truth of the discipline as socially constructed, for example (e.g. Ernest) and so am not mobilizing my varied examples for that purpose. For me, the multiple visions of mathematics do not form an argument for rejecting a particular vision, but instead provide a way to think of mathematical discourse as containing all of these contradictory positions and visions. That they are contradictory is not problematic for me as they may be for philosophers interested in a solitary criterion-based truth.

In The Mathematical Experience (1980), Davis and Hersh explain their basic conclusion, "It is reasonable to propose a different task for mathematical philosophy, not to seek indubitable truth, but to give an account of mathematical knowledge as it really is—fallible, corrigible, tentative, evolving, as is every other kind of human knowledge" (p. 406). Although they propose a "different task" "not to seek indubitable truth" they go on to suggest that their account is what "mathematical knowledge really is." This sort of rejection of truth and then a reclaiming of it is what I have tried to avoid in this dissertation. Although these mathematician philosophers share many of my interests and positions, this dissertation assumes a multiplicity of truths to describe the practices of four courses as instances of some of the possibilities in mathematics, rather than an assumption of a truth and the corresponding hunt for supporting multiple practices.

In the chapters devoted to each course I attempted to describe the construction of the subject that these courses developed. By looking at the construction of the subject rather than the learning of students I have tried to recast the issues of undergraduate education into issues of philosophy rather than psychology or even pedagogy. I believe that this approach to research offers distinct advantages for educators. It allows us to see the courses as more than students' test scores or instructors' teaching practices. It does not assume that there is a right way to teach or a right single thing to learn. When we can see the possibility of more than one best way (whether that be best lecture, best curriculum, best groupwork, best assessments) we can imagine constructions that we had not yet been able to articulate.

For now, though, let us return to the constructions that I did describe in this dissertation and allow me to summarize those descriptions here.

The Real Analysis course constructed a model student as a convert—someone who thought of mathematics as a way of knowing, of securing beliefs to evidence. That evidence was the proof. Students were positioned as needing to alter their epistemologies away from trusting in authorities such as professors and authors and away from their intuitions. Their belief and trust needed to shift towards proof as a technique for ascertaining knowledge. To get students to place their trust elsewhere the course drew upon community. It called the larger community of mathematicians throughout history to provide testimony for the new belief system. It also called on the community of the class as a tool. On both scales, the macro field and the micro classroom, belonging to these communities was seen as a desirable enticement for the students to proceed through the course and effect epistemological change. Students were asked to act as a mathematical court. They judged, crafted, and watched proofs. Mathematics was something to believe in as much as it was something to do. When they crafted their proofs they were performing the skills of proof-writing. They were following the norms and codes of the

genre, proof-writing. In this course students were not asked to examine the logic and the assumptions of that logic and axiomatic system. They were not asked to question whether or how the proof as a written collection of symbols and signs conferred meaning and truth. They were asked to draw a line in the sand and believe everything proven from that point forward. That this genre, this way of writing, conferred truth was an act of faith.

The Complex Analysis course presented a similarly unquestioning view of mathematics as well as a model student that presents analogies with religious vocabulary. In this course students were constructed as Platonic disciples. In Complex Analysis the substance of the course was in students crafting well-ordered and rational mathematical lives. Real Analysis only dealt with students minds and beliefs, whereas Complex Analysis targeted their bodies as well. Rational habits of mind and body were important. They needed to see the coherence and unity in mathematics and take up that coherence in their physical habits as well. Every aspect of their selves should be ordered and orderly. To motivate this change, this course appealed to a particular aesthetic of mathematics—the structured and connected aspects. Finding connections was pleasurable. Students were asked to follow the linear structure of mathematics, follow the rules, and follow the model of the instructor. They were not asked to deviate from the path or find their own way. They were provided with explicit step-by-step approaches to mathematics as well as their behaviors. Furthermore they were expected to stay in these lines. That I have described the model student as a disciple includes an aspect of belief. There was an implicit sense that in following these rules for their lives and the linear structure of mathematics, they would believe that these were good and pleasing ways to proceed. But where the Real Analysis students acted as judges and lawyers, there was an element of decision-making. The Real Analysis students also participated in a much more collaborative community. They were participants in the process. By contrast the Complex Analysis students followed and did not participate in those ways. The model student followed the teaching of mathematics and the mentor. That this view of mathematics encompassed a Platonic ideal of mathematics—it is discovered—made them disciples of this particular vision of mathematics. In Real Analysis the convert has learned to trust in the formalistic nature of proof—proofs determine what we know. When we write proofs we are making decisions about knowledge. The Platonic disciple uncovers knowledge through proof, but does not craft it. Finally, that the Complex Analysis course included aspects of living, the disciple also takes on a way of being, a worldview not just a way of thinking about mathematics.

The Discrete Mathematics course constructed the model student as a teacher—someone who could render proofs into a pedagogical artifact. Proofs are ways of explaining for the education of others and the self. The substance of the student that needed reformation was their self-concept as a student. Rather than think of their roles as students of mathematics in terms of calculations and memorization of formulas and techniques, as the course assumed they did, they were to think of themselves as problem solvers who could also translate those solutions into pedagogical forms. They needed to see themselves as mediators between the problem and some audience or reader. The mode of the course, like the Real Analysis course, was belonging. The course design invoked membership to a larger community as a reason to take up the new self-concept. And again, with ties to Real Analysis, there was an aspect of belonging to the classroom community. This course asked students to explain mathematics. By explain I mean they

were asked to put their mathematical ideas and arguments into a language and structure pedagogically appropriate for the audience. Mathematics was a communicative exercise. They needed to use their explanations to help others and themselves learn something—not only verify it. They were not asked to study a unified set of topics. In fact, the topics were discrete and the students jumped around from topic to topic. Mathematics was this and it was that—a grab bag of ideas. What held the course together was the emphasis on communicating the mathematics of the problems. They were asked to teach by rendering their thoughts publicly. Even when unsure they were asked to craft their confusion into a form that could teach the professor and other classes what was on the student's mind. The model student was a teacher. Mathematics was something to translate into pedagogically effective prose.

Like the Discrete Mathematics course, the substance targeted in Advanced Geometry was students' self-concepts. But instead of seeing themselves as problem solvers who could translate their work into pedagogical terms, this course worked on students' identities as researchers. Aesthetics, like in Complex Analysis, was the mode. In Advanced Geometry though, there was no single aesthetic. Students could determine for themselves what was interesting and pleasing. It could have been structure, form, surprise, patterns, connections, or deviations. Students were asked to play. In class they were given limits on the direction of their play when the instructors presented a particular diagram, for example. For their Book assignment, they had more freedom of choice. Within the wide or narrow space they decided what ideas were of interest to play around with and what their conjectures would be. That personal aesthetics drove what they chose to investigate was an important contrast with the Complex Analysis course where the aesthetic of unity was the only option. Students in Advanced Geometry not only conjectured things that turned out to be true but also things that turned out to be dead ends. They played to create conjectures. In this way the model student was constructed as a researcher. Researchers have limits and restrictions in their work, but they also have choice about what to pursue and what to abandon. That the Advanced Geometry course was interested in play in the form of "I Spy" and "What If" reflects a researcher's work of noticing patterns and discontinuities and pursuing conjectures and ultimately, and hopefully, theorems.

Table 6.1: Summary of analysis

	Real Analysis	**Advanced Geometry**	**Complex Analysis**	**Discrete Mathematics**
Substance	Epistemology	Self-Concept	Well-Ordered Rationality	Self-Concept
Mode	Belonging	Aesthetics	Pleasure in Orderliness	Belonging
Regimen	Holding Court	Play	Following Rules	Crafting Pedagogically Effective Prose
Telos	Converts	Apprentice Researcher	Platonic Disciples	Teacher

After summarizing these courses (and see Table 6.1 for a tabular summary) I now want to address several themes across them:

- Who does mathematics?
- What sorts of students might be excluded from the course objectives, based on their respective model constructions?
- What degrees of freedom were present in each course design? That is, how much space was allowed for a diversity of students to participate?

Wu's comment quoted above construes that there is one version of what mathematics is all about and how is it done. I hope that my analysis of these four courses has shown that several visions of mathematics are at least possible and in these what mathematics is (or rather could be) all about and how it is (or can be) done look quite distinctive. But to Wu's two-part comment I would like to add another. Who does mathematics?

Who does mathematics?

The Real Analysis and Discrete Mathematics courses presented visions of mathematics as communities and used membership in this community as an invitation to students to delve into the course. A sense belonging and inclusion constituted the modes. But these two courses also had students engage in communities within the classrooms, as did Advanced Geometry. Real Analysis, Discrete Mathematics, and Advanced Geometry did not just tell students about the wonderful collections of mathematician professors and researchers that thought in X ways or believed Y ideas or did Z things. Their invocations were not "trust me, this is how it is out there" sorts of arguments—the ones that ask students to suspend belief and delay gratification. Instead, they included elements of modeling this X, Y, and Z in the classroom. In these courses students could do mathematics. What that mathematics was, though differed, and so the ways that students did math looked dissimilar.

In Advanced Geometry students did mathematics by working with a partner or group (and alone in their Books) to observe phenomena such as patterns or variation. They did mathematics by telling other people about their ideas whether true or false. They did mathematics by refuting one another's ideas or assisting each other by refining or proving the other's conjecture. To engage in mathematics was to engage in social contact. The instructors were responsible for setting up the context and scenario and students were expected to pick up from there.

Real Analysis also expected students to be actors in the mathematical court—judges, lawyers, and jury. Math was an activity of judging the validity of ideas and constructing proofs performed in a social setting. To evaluating a proof's ability to convince the student was to evaluate another student's or Dr. RA's work. The proof may have been constructed in isolation at home or far off in the textbook author's office, but in class and within group sessions students, the proof was judged by peers or Dr. RA. Mathematics required a form of interaction and interrelatedness. Work happed on the individual level, but at some point it must pass through a social phase and so students must play some role in this cycle. The students were put in positions to judge and be judged.

To do math in Discrete Mathematics was to interpret problems into mathematical language for the purpose of solving them and then translating them back into a pedagogical form. Students were asked to do both of these things in the course. In their homework and quizzes and tests they mathematized to get a solution and then they were required to explain it back again to Dr. DM. To fail to do either of these parts impacted their grades. In that math was conceived in this way students did mathematics. Some students also taught half-class periods. They did these things in front of their peers. So to when they would come to the board to show their ideas or articulate their confusions.

Complex Analysis offers a strong counterpoint to the other courses in relation to "Who does math?" In CA, doing math was about uncovering connections and finding order. For the most part students watched Dr. CA do this and describe it. They did do homework sets and exams, but the homework tasks were often either calculations or simplifying expressions. Students were practicing the knowledge Dr. CA or the textbook shared, but they were not finding the connections. They were using them, but never finding them. Exam problems did have proofs but these were usually exact reproductions of homework questions or proofs offered in class by Dr. CA. In that students followed the well-defined sets of behavioral expectations, they were doing a part of mathematics. They were ordering their lives to match the structure of mathematics.

Describing the possible ways that students in these courses did mathematics—and 'doing' looked different in them—is one way to see the courses as open. We can also look at the courses as closed by examining what sorts of students and activities were not encouraged. Who would be excluded from the course? What follows are mostly speculations about the sorts of students the class did not construct as models and not necessarily examples of particular individuals.

Who was excluded from the course?

Advanced Geometry was not a very conducive place for students who preferred structured lessons with a definition, examples, theorem, then proof sort of path. Students who preferred a direct path to an idea would not have likely described the class as their favorite undergraduate experience. Mathematics was not presented as neat and orderly; as a progression through ideas. It was messy and spun off at tangents. Students were not expected to necessarily follow ideas in a linear fashion. Someone who wanted a set of directions or held a "just tell me so I can get on with it" sort of approach would have been stuck in a sea of ideas that were not obviously connected at times. Students who felt comfortable in a traditional lecture and note-taking environment may have been very unsettled by the open environment. Students who preferred to work on well-defined problems with a structured approach towards known conclusions were not the sorts of students who may have found the course exciting, but possibly may have found it too diffuse and too chaotic.

The Real Analysis course also excluded students. Those who wanted to know why proofs as a form of writing led to knowledge would not have their questions answered. Those wanting to know more about the systems of logic and axioms that mark a proof from another form of argument would have been disappointed. Someone who hated group work would not have liked the formal requirements to work together. These students would have wanted class to "get on with it" during the group aspects in class. A student that was good at the tacit logical procedures but who was not good at communicating his or her ideas would have had some difficulty with the expectations to share ideas in a group setting and write them up in assignments. On the other hand students who did enjoy these aspects may not have liked the other parts of class that did follow a set path of definition, examples, theorem, and proof. They may not have liked the traditional classroom feel of these times, wanting instead more of the time to follows the small group work model where they played with ideas rather than watching Dr. RA do so. Overall students with a desire for more than the foundational theorems of analysis, who wanted to learn more about the warrants for proof as a measure of knowledge would have been excluded in this aspect. Others preferring one or the other of the two types of class interaction (lectures, groups) might not have enjoyed the back and forth between the two and would have wanted the class to stick with whatever their preference was.

Students who were in the Complex Analysis course that did not like the rules or the ordered paths would have been excluded. They would have wanted more explorations off the track presented by the course. Nonlinear thinkers would not have had many opportunities to depart from the direction the course took. Students who did not like to remain still and quiet, who enjoyed working with others and wanted it to be encouraged rather than allowed might have not felt comfortable. Students who wanted their questions about the lecture answered in class or ones who wanted a less hurried pace would have been excluded. An excluded student might also not have an aesthetic of mathematics that highlighted structure and order but possibly diversity and novelty.

In Discrete Mathematics an excluded student would not be good communicators. They would not find the requirements to explain themselves in effective prose easy. Students who liked grades calculated according to the syllabus might have been troubled by the possibility that it might not be. Those who wanted a syllabus that outlined a set of topics and possibly a schedule for the course might have felt lost at regarding the direction and purpose of the course. Students with a vision of mathematics as interrelated and unified might not have seen the diversity of problem types as interesting but possibly as incoherent and disorganized. Those that preferred formal definitions and had an orientation towards proof as verification rather than explanation and articulation would likely not have found the Discrete Mathematics course very open to their visions of mathematics.

By listing the possible preferences of students that may have been excluded in each course I am not saying that there was one real life, non-model, excluded student that found a particular course distasteful on each of these aspects, but rather that in different ways and at different times and to different degrees some students may not have been included in the discourse. That is, a student may have not felt comfortable with the conjecturing aspect of the Advanced Geometry but might have felt more comfortable with the proving. Likewise, a student in the Complex Analysis course may have appreciated the linear and organized lectures, but not have liked the emphases on behavioral rules. Or in each of these, vice versa. This sets me up to now explore the degrees of accessibility in each course. How much room was available in each course for students to still participate despite having opposing views on mathematics or preferences of classroom environment?

How was the course accessible?

That courses constructed a model student has been analyzed and speculations about students who may have been excluded from each course have also been offered. Now I want to consider the degrees of accessibility in the courses. How narrow or open was the course to students of different orientations and preferences?

In Real Analysis students of different preferences about classroom environment could find somewhere where they felt comfortable. Students could feel at home in the group work or in the lectures. That Dr. RA used full sentences in his proofs may have been very accessible to some students. They could feel welcomed by the use of first names and casual pre-class talk. The might like the idea of becoming part of a community. They could have appreciated the references to history of real analysis and they could have simply liked analysis and thinking of mathematics and proofs as providing justification for knowledge. That also could have felt that their contributions were welcome when they determined the truth and falsehood of statements, crafted proofs, and critiqued them.

A student who wanted to feel at home learning the Discrete Mathematics could find comfort in the individual nature of the lectures or in the opportunities to quietly pair up with another student during the lecture, as both were available options. A student could also feel supported by the ways Dr. DM responded to student needs by altering his lesson plans to support their needs, how he answered their questions whether they were asked during the homework discussions or in lecture, and how he responded to requests for extra credit. A student with strong communication skills could appreciate the opportunity to show his abilities of explaining mathematics in his assignments and possibly by teaching the whole class. Students could also feel supported in how he dropped low scores and entire quiz sets. They could also feel good about being invited into a larger community. They could have enjoyed the problems and the contextual and application types of them offered in the textbook and in lectures and assignments.

Students in the Complex Analysis course could have found the rhythm of rules and structure of the classroom comforting. A student could enjoy the straightforward exposition of the lectures. That Dr. CA spent a lot of time doing homework problems might have been very helpful for a student. The model of mathematics as linear and

ordered could have been very accessible for students; that he highlighted connections between what they had learned before with what they were learning now could have been quite illuminating. Students could have felt inspired by the references to mathematics as mysterious and pleasurable.

Students in the Advanced Geometry course could have found a home in the opportunities to work alone, in a small group, and as a whole class. They could have enjoyed the use of technology. Students also might have liked being able to create their own paths to explore and/or the encouragement to prove them as well. That there were two instructors might have been very supportive. A student could have felt pride at having her ideas honored publicly in the whole class discussions. Being allowed the space to determine their own criteria for beauty could have been novel. That a major course grade was not a test in the sense of a sit down exam but rather a cumulative collection of their ideas might have been refreshing. A student could find the opportunities to do non-Euclidean work to be liberating.

That the courses provided many ways for students to find a home is exciting. There was likely something accessible to every student in each course. Although no one student was enrolled in each of the courses, I imagine that if a student were to have experienced each that the student, no matter his preferences or personal philosophies or attitudes towards mathematics could have found some aspect of each course accessible. The diversity of approaches to mathematics, pedagogy, and philosophy in the courses speaks to the troubles of philosophy of mathematics. What "math is about" is too narrow a question and presumes a single answer, but these courses offer many possible interpretations.

How did the courses speak to teacher education?

As I began this dissertation with questions about what models of teaching preservice teachers are exposed to in their undergraduate courses, I am happy to return to that question, albeit modified. Instead of asking what they learned or how they might go onto teach future students I want to ask what we in mathematics education can learn about the preservice teachers we encounter in light of this dissertation.

On the one hand I hope that teacher education is excited by these courses. If we could imagine a preservice teacher enrolled in each of these exact course observed, as I am not implying all complex analysis courses, for example, look alike, that student would have a wealth of experiences that we could draw from in a teacher education course. Just about any method suggested in a typical methods course was here. There was small groups, technology, lecture, whole class conversation, student presentations, pair-share, inquiry, problem solving, proving, connections, communication, representation, communities of practice, and alternative assessments. I may even have missed some! From a teacher education perspective we can be happy to see that this hypothetical student has experienced first hand so many things we value for K-12 classrooms. As Deborah Schifter (1986) noted teachers need to learn in the same ways that we hope they will also teach. That is, they need experiences as students in mathematics classrooms (or professional development) that mimics the pedagogies and views of mathematics that we teacher educators want them to go onto teach. There are a lot of things in my list above to suggest that my hypothetical preservice teacher is getting these experiences in at least some of their courses. When we look across this hypothetical teacher's course load we can see that this preservice teacher has been taught in ways that the mathematics education community typically desires or has at least been exposed to different parts in different classrooms.

Yet we also know that many of the non-hypothetical, non-model, utterly real preservice teachers we find in teacher education courses seem to be clueless about the methods we try to teach. To our eyes it is as if this is brand new stuff and even a tad suspect. Why then don't they recognize the methods when we talk about them or use them in our own classes?

My study was in no way designed to answer that question but I do have some thoughts about it. We largely assume that they have not had these experiences and act accordingly. Our discourse in our field and in our classrooms does not consider their mathematical preparation as sites of pedagogical possibility. In all of my work I have never seen or taught a methods class that asks students to seriously consider their experiences as undergraduate mathematics students in terms of pedagogy. The mathematics classes they take are assumed by us to be classes about content and so contribute to their content knowledge, which we distinguish from pedagogical knowledge.

I hope that through this dissertation you, the reader, has noticed how philosophy of mathematics, actions of mathematics, formulations of students needs, invitations and motivations for learning all are present in undergraduate classrooms and that to draw nice walls between pedagogy and content is artificial. Did Dr. RA use groups because that is an effective teaching tool or because it reflected his collaborative and community-based vision of mathematics? Did Dr. CA highlight connections between mathematical strands because that reflects his deep content knowledge of calculus and analysis or because connections are an effective pedagogical tool? Did Dr. AG use inquiry because it promotes student engagement or because it reflects an important quality of a mathematician?

There is of course, Shulman's (2004) theory of pedagogical content knowledge that views the two as merged knowledge bases, but Segall (2003, p. 10) presents a nice argument that the two categories are conflated, not overlapping. Because we, and I include myself here, in teacher education think of pedagogy and content as separate but possibly related, we may be missing out on opportunities to get our preservice teacher to analyze their undergraduate experiences in terms of pedagogical/mathematical moves in such a way that can be fruitful to teacher education.

On the other hand, there are a lot of things in these courses that we may not want our students to go on to mimic. Yet, as Hiebert (NCTM, 2000) notes, students learn what they have the opportunity to learn. They have been and will be learning things all of the time that might not be advocated by reforms to mathematics education. Yet, again, as Hiebert notes, students learn what they have the opportunity to learn. If we do not provide opportunities for them to learn from their mathematical experiences in departments of mathematics they may not be able to coordinate experiences from their undergraduate courses with the sorts of experiences we want them to produce for students.

In the sections above on who is excluded and what aspects of the courses were accessible we could see that any course could exclude and include. I hope that you will not think of X course as good and Y course as bad. Each included ways for students to be at home in the course and each included ways to be exclusive. Each presented mathematics and students roles in mathematics differently. The diversity of these courses allowed for multiple entry points for different students.

I want to return again to the list generated a few paragraphs above when I declared that practices that teacher education tends to value were present in these courses. Here they are again: groups, technology, lecture, whole class conversation, student presentations, pair-share, inquiry, problem solving, proving, connections, communication, representation, communities of practice, and alternative assessments. This list includes many of the sorts of activities mathematics teacher education holds as best practices. Our discourse and the methods courses we design promote these visions of not just pedagogy, but of mathematics as well by argument of their conflation. But like the undergraduate mathematics courses observed in this study, our courses too can serve to exclude some students and provide access for others. Our discourse excludes preservice teachers who find inquiry activities chaotic and those who prefer solitary thinking. We are more accessible to students who enjoy describing their thoughts, and giving presentations. Our discourse limits student engagement in some ways and promotes others.

Just as the discourses that constitute undergraduate mathematics is more diverse than many in teacher education assumes, our discourse in mathematics teacher education is more narrow than we like to believe. Through our preferred pedagogies we also offer a vision of mathematics and this vision may at times compete with those of our preservice teachers. By setting up a system of explicit and implicit best practices we are telling them that our vision is not only better but best. Because our preservice teacher students are deeply invested in mathematics this could be coming across as a personal attack.

So we are faced with a dilemma. On the one hand the things mathematics teacher education cares about have been shown to be present in the union of the four courses I studied. The possibility that our preservice teachers witness these practices and have learned mathematics from them should excite us. We may be drawn to highlighting these practices through work in our courses that draws students to share and reflect on these experiences. But when we call on our students to do this sharing and reflecting we are reinscribing a particular discourse that by omission excludes some students. In trying to make use of the knowledge that preservice teachers have had the sorts of experiences we want to promote, we may be setting up an opposition between the mathematics that math teacher education values and some of the mathematics that mathematics the disciplinary field values. We are asking our preservice teachers to choose.

Parks (2007a) suggests several ways that mathematics teacher education might begin to open up the discourse around good teaching: "giving assignments that recognized mathematical knowledge and teaching practice as well as self-analysis, doing research about teacher education in ways that framed practices not as *best*, but as possible, and explicitly working to expand – in teaching and in writing – visions of good teaching and acceptable beliefs" (p. 170). I hope that her suggestions are as powerful to you as they are to me, and I hope that through this dissertation I have managed to do the last two reasonably well.

On the one hand Hiebert (yes, one last time) might be a good summary of this dissertation. *One of the most reliable finding from research on teaching and learning is that students learn what they are given opportunities to learn.* The experiences across these classrooms provided different opportunities for students. On the other, I have tried to cast this dissertation as a construction of the subject rather than as a description of what students learned. By framing this as a construction of the subject I can call discourse into play. By doing so I hope I have illuminated the ways that instructors, textbooks, documents, and philosophies of mathematics as well as students work together to create a discourse and a classroom experience both in undergraduate mathematics and in mathematics teacher education. These discourses do not always agree but there is more overlap than we might have supposed.

When I very first started thinking about the ideas that inspired me, that ultimately found their expression here—now in a very different form than first imagined—I intended to document the philosophies of mathematics present in undergraduate mathematics courses and those found in secondary methods courses. My big plan was to show how those philosophies found in the mathematics courses infected the preservice teachers in such a way that they were poisoning our preservice teachers away from the better philosophy! How things change, huh? Like Gauss, I had had my conclusions and only needed to arrive at them. Through this dissertation I have learned to take a more generous view of undergraduate mathematics courses and inversely, a more critical view of mathematics teacher education. This is not because I have learned that the teacher education courses are the ones doing the poisoning, but because it is too easy find blame in others and perfection in yourself.

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