THE CONTRIBUTION OF AN ELASTIC WALL SUPPORT TO THE DEFLECTIONS OF A THIN CIRCULAR PLATE

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

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1968

This is to certify that the

thesis entitled

THE CONTRIBUTION OF AN ELASTIC
WALL SUPPORT TO THE DEFLECTIONS
OF A THIN CIRCULAR PLATE

presented by

WILLIAM V. BREWER

has been accepted towards fulfillment
of the requirements for

Doctoral degree in Mech. Engr.

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Date May 7, 1968

O-169

ABSTRACT

THE CONTRIBUTION OF AN ELASTIC
WALL SUPPORT TO THE DEFLECTIONS
OF A THIN CIRCULAR PLATE

by William V. Brewer

The axially-symmetric problem examined is as follows. A thin circular plate (or disk) is attached (bonded, glued, welded) to the wall of a cylindrical shaft (or hole) in a massive or thick walled solid. The disk and shaft share the same axis of symmetry.

The generality of the mathematical model of this problem is limited by the two component theories used in its solution: classical, three-dimensional, small-strain, theory for a homogeneous isotropic elastic solid used for the supporting wall; classical, small-strain, thin-plate theory without mid-plane forces used for the plate fixed to the supporting wall.

The other two limitations on the generality of the theory are: choice of axial-symmetry to obtain a non-plane-strain solution in two dimensions; selection of the stress profiles at the boundary shared by the two bodies. Thus the matching of the slope and displacements at this common boundary is limited to a single appropriate point.

Within the above limitations the problem is solved for all plates without holes or inclusions and having uniform thickness and material parameters. Any load $q(r)$ is admissible. The solution for the wall would allow the other plate solutions (i.e., with a hole and/or inclusions and also having variable thickness and/or material parameters) to be developed in a parallel fashion with comparative ease.

The solution allows the wall and plate to have different material parameters.

The solutions for the shear and the moment applied to the cylindrical wall are easily used together or separately to solve a variety of other problems where such a wall is similarly loaded.

Since the mathematical model for the wall is solved for all of space and since the solutions all decay with respect to increasing r and z, these solutions can be used to approximate those of objects with sufficiently thick walls having cylindrical cavities. In the case of the self-equilibrating moment, where shear is small, the walls need not be very thick.

The specific example used to illustrate the solution procedure sets the load $q(r) = q$ (a constant) and lets the plate thickness/diameter ratio be $\frac{10}{100}$, $\frac{5}{100}$, $\frac{2}{100}$. For these ratios the ratio of the elastic moduli of plate and wall is allowed to range from $E_p/E_w = 0$ to $E_p/E_w = 10$. For each of the thickness/diameter ratios, the deflections $w(r)$ based on the assumptions of a rigid wall and of a wall and plate of same material, never differ by less than 300% for any given r. Similarly the maximum radial tensile stress at the wall never varies by less than 200%. The effect of Poisson's ratio varying from .25 to .30 is relatively small but becomes more significant for thinner plates.

The deflection at the wall is small for thinner plates, while rotation at the wall is still comparatively large.

There is little reason to believe that the trends indicated by this example would change significantly for other usual loadings and geometries within the scope of the problem.

THE CONTRIBUTION OF AN ELASTIC WALL SUPPORT TO THE DEFLECTIONS OF A THIN CIRCULAR PLATE

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mechanical Engineering

1968

ACKNOWLEDGMENTS

In this work, as in all human activity, little progress would have been made without support.

Dr. R. W. Little suggested the area of study, provided resource materials, answered essential questions, gave encouragement and insisted on a thorough investigation. Without his help this work would not have been started or completed.

My thanks to the Guidance Committee, Dr. G. H. Martin, Dr. R. T. Hinkle, and Dr. E. A. Nordhaus, who gave generously of time and knowledge whenever called upon.

My thanks also to Dr. I. N. Sneddon and Dr. J. S. Frame who contributed measurably to the outcome of the work. I am indebted to the staff members of the several engineering and science departments for support of various kinds.

I am grateful especially to my wife and to several others who helped translate the manuscript into a finished product.

I would dedicate this effort to my parents who from the beginning to the end did everything they could to encourage an advanced education. Finally, my thanks to God without whom nothing would be possible.

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TABLE OF SYMBOLS

English Symbols

$A(\omega)$, $B(\omega)$ arbitrary coefficient functions of solutions to $\nabla^4 \underline{Z}(r,z)=0$

a,b radius and half-thickness of the undeformed plate

$c=2(1-\nu)$ group of constants appearing frequently

$D=\dfrac{2Eb^3}{3(1-\nu^2)}$ plate modulus

$D(x)=[x^2(K^2(x)-1)-c]$ group of terms appearing frequently in the displacement solutions

E elastic modulus in tension and compression or Young's modulus

E_p E for the circular plate

E_w E for the wall supporting the plate

$\vec{F}$ vector potential function defined by Galerkin

G Elastic modulus in shear G= the Lame' constant μ

HP= π/b or $\pi a/b$ half-period of functions to be integrated in expressions for displacements

HPS starting point for the numerical integrations expressed in half-period lengths

HPR range of the numerical integrations expressed in half-periods

$Iw(r)= \int_0^r \frac{1}{\gamma} \int_0^\gamma \delta \int_0^\delta \frac{1}{\beta} \int_0^\beta \rho q(\rho)\, d\rho\, d\beta\, d\delta\, d\gamma$ an integral appearing in the general solution for the plate displacement $w(r)$

$Iw=Iw(r)$ at $r=a$ the edge of the plate

$Iwr(r) = \frac{d}{dr}Iw(r)$

$Iwr = Iwr(r)$ at $r=a$ the edge of the plate

$Iwrr(r)=\frac{1}{r}\frac{d}{dr}(rIwr(r))$

$Iwrr = Iwrr(r)$ at $r=a$

$K_n(x)$ Modified Bessel Functions of the Second Kind and order n

$K(x)=K_o(x)/K_1(x)$ group of terms appearing frequently

$K=K(x)$ at $x=\omega a$

$M_r(r)$ moment resultant per unit circumferential length applied in the θ-direction (right-hand-rule) to the r-face of a differential element of plate material $(dr)x(rd\theta)x(2b)$

$M(\sigma)= \int_A [\sigma_r(a,z)]zdA = \int_{z=-b}^{z=b} [\sigma\cdot z]z(1\cdot dz)$ moment resultant per unit circumferential length applied to the wall

$N(x) = [1+xK(x)]$ group of terms appearing frequently in the displacement solutions

PHP partition of the half-period HP into PHP equal parts

$Q_r(r)$ shear resultant per unit circumferential length applied in the z-direction to the r-face of a differential element of plate material $(dr)x(rd\theta)x(2b)$

$Q(\tau) = \int_A [\tau_{rz}(a,z)]dA = \int_{z=-b}^{z=+b} \tau(1-\frac{z^2}{b^2})(1\cdot dz)$ shear resultant per unit circumferential length applied to the wall

$q(r)$ plate load per unit area applied in the z-direction to the z-face of a differential element of plate material $(dr)x(rd\theta)x(2b)$

$R(r)$ one of the solutions to $\nabla^2 \underline{Z}_2(r,z)=0$ where $\underline{Z}_2(r,z)=R(r)Z(z)$

$R_r=\frac{\partial R(r)}{\partial r}$

$R_{rr}=\frac{\partial^2 R(r)}{\partial r^2}$

(r,θ,z) cylindrical coordinates

$S(x) = [\frac{\sin(x)}{x} - \cos(x)]$ group of terms appearing frequently

$\vec{u}=(u_r, v_\theta, w)$ displacement vector in the elastic solid

$u_r(r,z,\sigma,\tau)$ displacement in the r-direction

$u_r(r,z)$ displacement in the r-direction

$us(r,z)=\frac{G}{\sigma} u_r(r,z,\sigma,0)$ infinite integrals

$us(z) = us(a,z)$ infinite integrals

$us = us(a,0)$ constant

$ut(r,z) = \frac{G}{\tau} u_r(r,z,0,\tau)$ infinite integrals

$ut(z) = ut(a,z)$ infinite integrals

$ut = ut(a,b)$ constant

$uzs(r,z) = \frac{G}{\sigma} \frac{\partial u_r}{\partial z} (r,z,\sigma,0)$ infinite integrals

$uzs(z) = uzs(a,z)$ infinite integrals

$uzs = uzs(a,0)$ constant

$uzt(r,z) = \frac{G}{\tau} \frac{\partial u_r}{\partial z} (r,z,0,\tau)$ infinite integrals

$uzt(z) = uzt(a,z)$ infinite integrals

$uzt = uzt\ (a,0)$ constant

$w(r)$ displacement of the plate in the z- direction

$w(r,z,\sigma,\tau)$ displacement of the elastic solid in the z-direction

$w(r,z)$ displacement of the elastic solid in the z-direction

$ws(r,z) = \frac{G}{\sigma} w(r,z,\sigma,0)$ infinite integrals

$ws(r) = ws(r,0)$ infinite integrals

$ws(z) = ws(a,z)$ infinite integrals

$ws = ws(a,0)$ constant

$wt(r,z) = \frac{G}{\tau} w(r,z,0,\tau)$ infinite integrals

$wt(r) = wt(r,0)$ infinite integrals

$wt(z) = wt(a,z)$ infinite integrals

$wt = wt(a,0)$ constant

$\underline{Z}(r,z)$ component of Galerkin's vector potential $\vec{F}$ in the z-direction....also referred to as the Love strain function

$\underline{Z}_4(r,z)$ satisfies $\nabla^4 \underline{Z}_4(r,z)=0$

$\underline{Z}_2(r,z)$ satisfies $\nabla^2 \underline{Z}_2(r,z)=0$

$Z(z)$ one of the solutions to $\nabla^2 \underline{Z}_2(r,z)=0$ where $\underline{Z}_2(r,z)=R(r)Z(z)$

$Z_z=\dfrac{\partial Z(z)}{\partial z}$

$Z_{zz}=\dfrac{\partial^2 Z(z)}{\partial z^2}$

<u>Greek Symbols</u>

ν Poisson's ratio

σ_r, σ_θ, σ_z, etc. tensile stress in the indicated directions (not partial derivatives)

σ maximum σ_r on the common boundary between the plate and wall support

τ_{rz}, $\tau_{r\theta}$, $\tau_{\theta z}$ shear stress on the faces and in the directions indicated where $\tau_{ij}=\tau_{ji}$ for all cases considered

τ maximum τ_{rz} on the common boundary between the plate and wall support

ω separation constant in the solution of $\nabla^2\underline{Z}_2(r,z)=\nabla^2R(r)Z(z)=0$ but considered a variable in the Fourier integral sine and cosine transformations

<u>Other Symbols</u>

$\nabla^2=\left(\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{\partial^2}{\partial z^2}\right)$ Laplacian operator shown for the axially symmetric case

$\nabla^4 = \nabla^2(\nabla^2)$

$$[\ ]_{xy} = \frac{\partial^2[\ ]}{\partial x \partial y} \text{ etc.}$$

I. Introduction

Design and the boundary value problem

Design may be defined as the combination of two or more known techniques or solutions in a new application to achieve a desired result. Most design problems in mechanics are in some way concerned with specifications and/or results at boundaries and therefore may also be interpreted as boundary value problems. We will combine two solutions in such a manner as to both match and influence each other at their common boundaries.

Area of interest

In the area of elastostatics the interaction of bodies having a thin cross-section with those having massive, solid or thick cross-sections is often neglected. In such cases the solid is usually assumed to be rigid and contributions to deflections therefrom are not investigated. That the solid may contribute to deflections is recognized even in cases where both bodies are made of the same material. A plane stress (strain) case of a beam intersecting a half-plane body at right angles was investigated independently by Weber [1] and Muskhelishvili [2] who employed

different assumptions. Weber modeled the problem by applying to the half-plane the linear axial tensile stress distribution of simple beam theory:

$$\sigma_x(0,y) = (\sigma_{max})\, y \qquad -b \leqslant y \leqslant b$$
$$= 0 \qquad -b > y > b$$

Muskhelishvili however applied a linear axial displacement distribution:

$$u_x(0,y) = (u_{max})\, y \qquad -b \leqslant y \leqslant b$$
$$= 0 \qquad -b > y > b$$

It is noted by O'Donnell [3], who compared these solutions, that even though the latter case results in an infinite value of stress σ_{max} at $y = \pm b$ the resultant displacements obtained due to rotation at the wall differ by only 15%. O'Donnel chose a cubic stress distribution as a compromise between the first two models and obtained results between. He also investigated the effect of shear by applying a constant shear to the wall:

$$\tau_{xy}(0,y) = \tau \text{ constant} \qquad -b \leqslant y \leqslant +b$$
$$= 0 \qquad -b > y > +b$$

O'Donnell also investigated the plane stress case experimentally. The various results and comparisons are displayed in several graphs. More recently Cook [4] generalized this problem to many evenly spaced beams intersecting a half-plane.

An investigation of a non-plane-stress problem was conducted by Brown and Hall [5]. In this case a shaft of circular cross-section intersects a half-space body at right angles. The problem is modeled by applying the axial tensile stress from simple beam theory to the half-space. Deflections are obtained both theoretically and experimentally. Shear stresses were not considered.

II. Selection and statement of the problem

We will study the contribution of an elastic wall support to the deflection of a thin plate whose edge is at every point attached (bonded, glued, welded) to the surface or cast in one piece with the surface of the wall support. See figure 1. We will want to be able to specify the material parameters of plate and wall separately.

Symmetry

The axially-symmetric problem to be examined is as follows. A thin circular disk or plate is fixed at its edge to the wall of a cylindrical hole or shaft in a solid. The disk and shaft share the same axis of symmetry.

Coordinates and geometry

Cylindrical coordinates (r, Θ, z) are used to describe the problem with independency with respect to Θ due to symmetry. The axis of symmetry $(0, z)$ is taken with "z" positive downward. The undeformed disk is described in the obvious way, occupying the space $-b \leqslant z \leqslant +b$ and $0 \leqslant r \leqslant a$ for all Θ where $a \gg b$. Similarly the undeformed solid occupies the space $-\infty < z < +\infty$ and $r > a$ where an infinite solid is chosen to further simplify the problem. The positive directions of important quantities listed in the table of symbols are shown in figure 2.

Component theories

The deformed solid is to be described by the classical three-dimensional elasticity theory for homogeneous isotropic solids experiencing small strains. The deformed disk is to be described by classical elastic plate theory approximating the conditions to be prescribed at the common boundaries and compatible with the resulting displacements.

Boundary conditions

On the common boundary between the plate and solid let the radial tensile stress $\sigma_r(r,z)$ and the shear on the "r" face $\tau_{rz}(r,z)$ be assumed to be $\sigma_r(r = a, -b \leqslant z \leqslant b) = \sigma \cdot z$

$$\tau_{rz}(r = a, -b \leqslant z \leqslant b) = \tau\left(1 - \frac{z^2}{b^2}\right)$$

where σ and τ are constants to be determined. Only the stress profiles are specified. It is not expected that any choice of σ and τ will produce identical displacement solutions at every point on the common boundary. If however σ and τ are determined by requiring displacements and slope be equal at the plate midplane (a,0) then displacements in each solution should be close for all coordinate values on the common boundary except perhaps near points of stress discontinuity (a,b) and (a,-b). In O'Donnel's [3] discussion of the analogous plane stress (strain) case, where

Figure 1. Thin plate with thick wall support

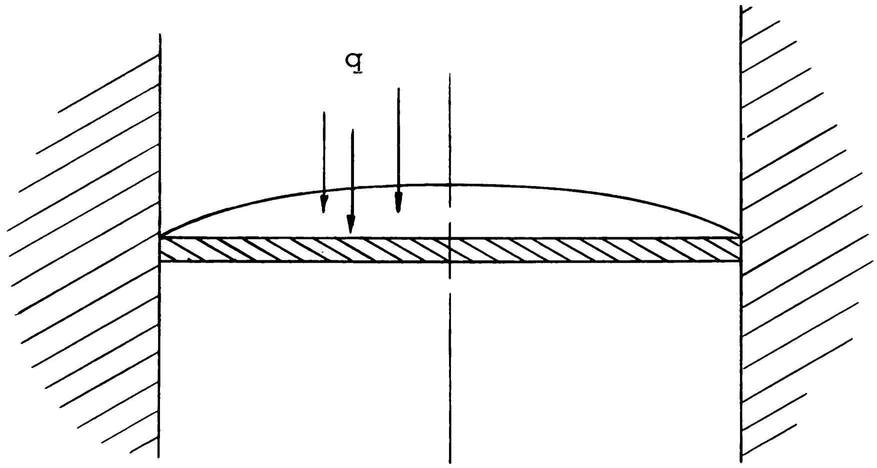
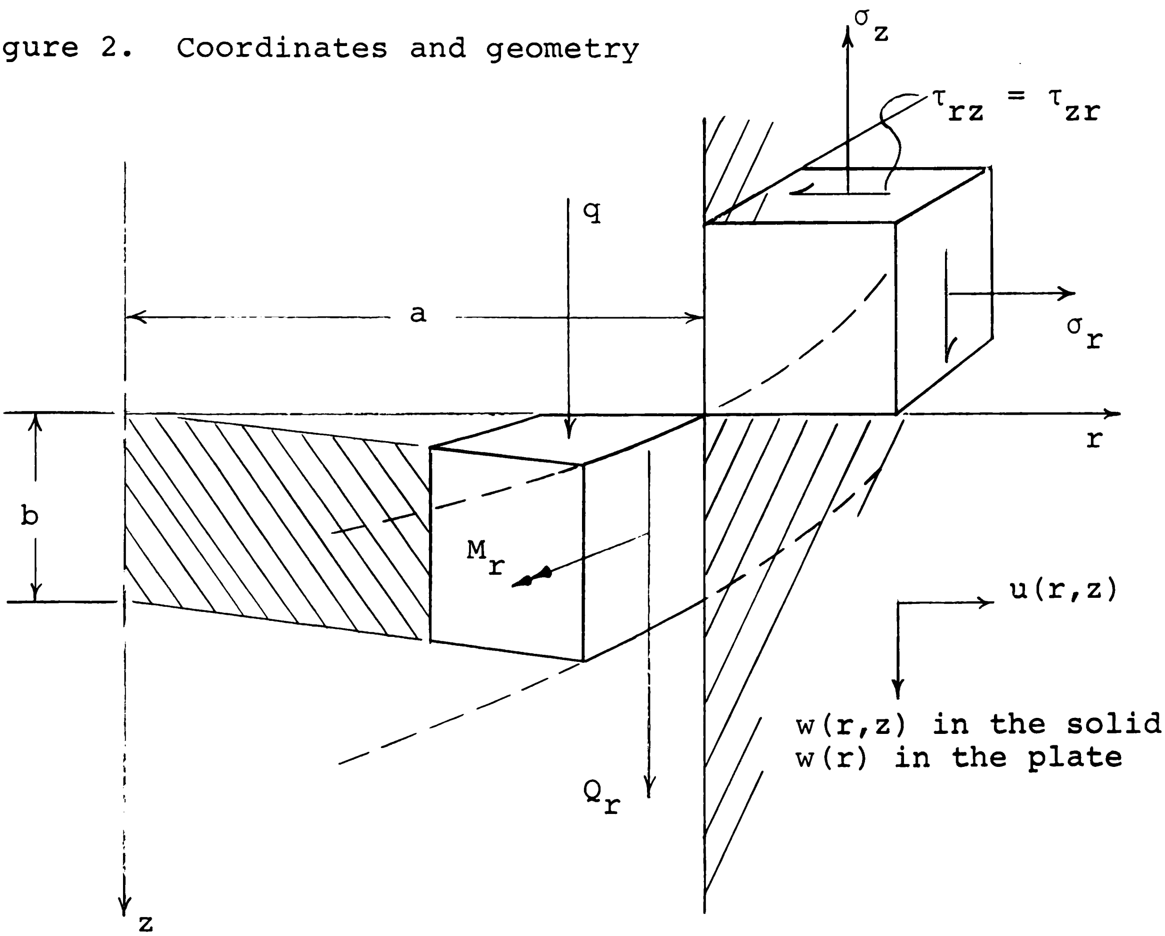


Figure 2. Coordinates and geometry



shear is neglected, he notes that if a linear displacement profile or a linear stress profile is assumed the variance of the general results from experimental evidence is approximately 15% in either case even though stress becomes infinite at (a,b) and (a,-b) for the linear displacement assumption. The stress profiles imposed are the major restrictions on the problem.

First, solutions for the wall will be found and then the requirements of the plate will be examined in Section III.

III. Mathematic development

Axially-symmetric isotropic elastic solid

In elastostatic problems without body forces the Navier equation [6, p. 88] becomes

$$[\nabla^2 + \frac{1}{1-2\nu}\nabla\ (\nabla\cdot)]\ \vec{u} = 0\ .$$

Galerkin defined a vector function

$$2G\vec{u} \equiv [2(1-\nu)\nabla^2 - \nabla(\nabla\cdot)]\vec{F}$$

such that the Navier equation reduces [6, p. 119] to

$$\nabla^4\vec{F} = 0\ .$$

In the case of symmetry with respect to the z-axis the z-component of Galerkin's vector is particularly important. Let $\vec{F} = (0,0,\underline{Z}(x,y,z))$.

If both the elastic body and its loadings are axially symmetric the $\underline{Z}$ is a function of r and z only. Let

$$\underline{Z}(x,y,z) = \underline{Z}(r,z).$$

This last function is called the Love strain function because it was developed earlier by Love [7, pp.274-277] using a different method [6, p. 130]. The displacements and stresses may be expressed in the following manner [6, pp. 129-130] as functions of $\underline{Z}(r,z)$ where $\underline{Z}(r,z)$ satisfies the biharmonic equation

$$\nabla^4 \underline{Z}(r,z) = 0$$

$$2Gw = [2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]\ \underline{Z}(r,z)$$

$$2Gv_\theta = 0$$

$$2Gu_r = \frac{-\partial^2 \underline{Z}(r,z)}{\partial r\ \partial z}$$

$$\sigma_z = \frac{\partial}{\partial z}[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]\ \underline{Z}$$

$$\sigma_r = \frac{\partial}{\partial z}\ (\nu\nabla^2 - \frac{\partial^2}{\partial r^2})\ \underline{Z}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\ (\nu\nabla^2 - \frac{1}{r}\ \frac{\partial}{\partial r})\ \underline{Z}$$

$$\tau_{zr} = \frac{\partial}{\partial r}[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]\underline{Z}$$

$$\tau_{r\theta} = 0$$

$$\tau_{\theta z} = 0$$

(3.1)

Strain function

A function $\underline{Z}_4(r,z)$ must be found such that

$\nabla^4 \underline{Z}_4(r,z) = 0$ . . . . [6, p. 122]

is satisfied.

The solution begins with

$$\nabla^2 \underline{Z}_2(r,z) = 0$$

where $\underline{Z}_2(r,z)$ is of the form $\underline{Z}_2(r,z) = R(r)Z(z)$.

In cylindrical coordinates the operator ∇^2 is

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial}{\partial r} + \frac{1}{r^2}\ \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}).$$

Then $\nabla^2(R(r)Z(z)) = R_{rr}Z + \frac{1}{r}R_r Z + RZ_{zz} = 0$

and $(R_{rr} + \frac{1}{r}R_r)Z + RZ_{zz} = 0.$

This equation is solved by separating the variables.

$$\frac{R_{rr} + \frac{1}{r}R_r}{R} = \frac{-Z_{zz}}{Z} = \omega^2$$

Choice of constant ω^2 and the minus Z_{zz} gives results suitable for the application to be made.

The resulting equations

$$\frac{-Z_{zz}}{Z} = \omega^2$$

$$\frac{R_{rr} + \frac{1}{r}R_r}{R} = \omega^2$$

become

$$Z_{zz} + \omega^2 Z = 0$$

$$r^2 R_{rr} + rR_r - r^2\omega^2 R = 0 \quad \text{respectively.}$$

They are well known and possess the solutions

$$Z = A_1 \mathrm{Sin}(\omega z) + B_1 \mathrm{Cos}(\omega z)$$

$$R = A_2 I_0(\omega r) + B_2 K_0(\omega r)$$

where A_1, A_2, B_1, and B_2 are arbitrary constants and I_0 and K_0 are Modified Bessel Functions of the first and second kind and order zero.

Boundary problem and Fourier integral transforms

From section II the conditions required on the two boundaries of the elastic solid are these:

1. The stresses must be bounded at infinity.
2. The displacement in the radial direction u_r must be zero at $(r,z) = (a,o)$ where the midplane of the plate intersects the cylindrical wall.

3. The stresses at $r = a$ on the interior surface of the hollow cylindrical shaft are

a. $\sigma_r(a,z) = \frac{\sigma}{b}z \quad -b \leqslant z \leqslant b$
$= 0 \quad -b > z > b$

b. $\tau_{rz}(a,z) = \tau[1-(\frac{z}{b})^2] \quad -b \leqslant z \leqslant b$
$= 0 \quad -b > z > b$

where σ and τ are the maximum stress values and

$$b = \frac{\text{(plate thickness "t")}}{2} \tag{3.2}$$

Consider condition (1) above. It requires that $A_2 = 0$ since $I_0 \to \infty$ as $r \to \infty$ and would lead to unbounded stress solutions at infinity. Note that solution $Z(z)$ may be separated into even and odd parts. Condition (3) above requires only the even part. Therefore let $A_1 = 0$.

The desired solution is

$$\underline{Z}_2 = R(r)Z(z) = K_0(\omega r)\text{Cos }(\omega z)$$

It can be verified by substitution that $\underline{Z}_4$ satisfies $\nabla^4\underline{Z} = 0$ when

$$\underline{Z}_4 = A\underline{Z}_2 + Br\frac{\partial}{\partial r}\underline{Z}_2$$

and since

$$\frac{\partial}{\partial r}K_0(\omega r) = -\omega K_1(\omega r)$$

then

$$\underline{Z}_4 = [AK_0(\omega r) + B\omega r K_1(\omega r)]\text{ Cos }(\omega z) \tag{3.3}$$

also satisfies $\nabla^4\underline{Z} = 0$.

If equation (3.3) is a solution, then any finite or integral sum of such functions, having different arbitrary coefficients $A(\omega)$ and $B(\omega)$, is a solution. The $A(\omega)$ and $B(\omega)$ are independent of r and z but will be considered functions of ω in the Fourier transformations to be used in the solution of the boundary problem. The desired strain function is given below.

$$\underline{Z}_4 = \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{\omega^3}[A(\omega)K_0(\omega r) + B(\omega)\ \omega r K_1(\omega r)]\mathrm{Cos}(\omega z)\ d\omega \tag{3.4}$$

Substitution of this function into equations (3.1) yields the following expressions (3.5).

$$2Gw(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{\omega}[A(\omega)K_0(\omega r)$$
$$+ B(\omega)(\omega r K_1(\omega r) - 4(1-\nu)K_0(\omega r))]\ \mathrm{Cos}(\omega z)d\omega$$

$$2Gu_r(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{-1}{\omega}[A(\omega)K_1(\omega r)$$
$$+ B(\omega)\omega r K_0(\omega r)]\ \mathrm{Sin}(\omega z)d\omega$$

At $z = 0$ the displacement in the "r" direction $u_r(r,0)$ is zero. Condition (2) is satisfied in equations (3.2).

$$2G\frac{\partial u_r(r,z)}{\partial z} = \int_0^\infty \sqrt{\frac{2}{\pi}}[A(\omega)K_1(\omega r) + B(\omega)\omega r K_0(\omega r)]\ \mathrm{Cos}\ (\omega z)d\omega$$

$$\sigma_r(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}}[A(\omega)(K_0(\omega r) + \frac{1}{\omega r}K_1(\omega r))$$
$$-B(\omega)([1-2\nu]K_0(\omega r) - \omega r K_1(\omega r))]\mathrm{Sin}(\omega z)d\omega$$

$$\tau_{rz}(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}}\ [-A(\omega)K_1(\omega r)$$
$$+ B(\omega)(2[1-\nu]K_1(\omega r) - \omega r K_0(\omega r))]\ \mathrm{Cos}(\omega z)d\omega \tag{3.5}$$

To satisfy the last boundary condition "3" (3.2) let the general expressions (3.5) for σ_r and τ_{rz} equal "3" at $r = a$.

$$\left.\begin{array}{ll} -b\leqslant z\leqslant b & \frac{\sigma z}{b} \\ -b>z>b & 0 \end{array}\right\} = \int_0^\infty \sqrt{\frac{2}{\pi}}[A(\omega)(K_0(\omega a) + \frac{1}{\omega a}K_1(\omega a))$$
$$-B(\omega)([1-2\nu]K_0(\omega a)-\omega a K_1(\omega a))]\mathrm{Sin}(\omega z)d\omega$$

$$\left.\begin{array}{ll} -b\leqslant z\leqslant b & \tau(1-[\frac{z}{b}]^2) \\ -b>z>b & 0 \end{array}\right\} = \int_0^\infty \sqrt{\frac{2}{\pi}}[-A(\omega)K_1(\omega a)$$
$$+B(\omega)(2[1-\nu]K_1(\omega a)-\omega a K_0(\omega a))] \ \mathrm{Cos}(\omega z)d\omega \tag{3.6}$$

There are now two equations which may be solved for the two unknown functions $A(\omega)$ and $B(\omega)$. Equations (3.6) may be expressed in the form

$$\sigma_r(a,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty [f(\omega)] \ \mathrm{Sin}(\omega z)d\omega$$
$$\tau_{rz}(a,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty [g(\omega)] \ \mathrm{Cos}(\omega z)d\omega \tag{3.7}$$

which are already the Fourier Integral Sine and Cosine Transforms of $f(\omega)$ and $g(\omega)$. These transforms possess the property of being the same as their inverse transformation, hence if the above equations (3.6) are transformed

$$\sqrt{\frac{2}{\pi}}\int_0^\infty [\sigma_r(a,z)] \ \mathrm{Sin}(\omega z)dz = \sqrt{\frac{2}{\pi}}\int_0^\infty [\sqrt{\frac{2}{\pi}}\int_0^\infty [f(\omega)] \ \mathrm{Sin}(\omega z)d\omega]\mathrm{Sin}(\omega z)dz$$
$$= f(\omega)$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty [\tau_{rz}(a,z)]\ \mathrm{Cos}(\omega z)dz = \sqrt{\frac{2}{\pi}}\int_0^\infty [\sqrt{\frac{2}{\pi}}\int_0^\infty [g(\omega)]\ \mathrm{Cos}(\omega z)d\omega]\mathrm{Cos}(\omega z)dz$$

$$= g(\omega)$$

$A(\omega)$ and $B(\omega)$ can be determined explicitly by substituting expressions (3.2) for σ_r and τ_{rz} in the above equations.

$$\sqrt{\frac{2}{\pi}}\int_0^b [\frac{\sigma}{b}\ z]\ \mathrm{Sin}(\omega z)dz + \sqrt{\frac{2}{\pi}}\int_b^\infty [0]\mathrm{Sin}(\omega z)dz = f(\omega)$$

$$\sqrt{\frac{2}{\pi}}\int_0^b [\tau(1-(\frac{z}{b})^2)]\ \mathrm{Cos}(\omega z)dz + \sqrt{\frac{2}{\pi}}\int_b^\infty [0]\ \mathrm{Cos}(\omega z)dz = g(\omega)$$

Integration yields the following results [8,pp. 80-81]

$$\sqrt{\frac{2}{\pi}}\ \frac{\sigma}{b}\ [\frac{1}{\omega^2}\mathrm{Sin}(\omega z)-\frac{1}{\omega}z\ \mathrm{Cos}(\omega z)]_0^b + 0 = f(\omega)$$

$$\sqrt{\frac{2}{\pi}}\tau\,[\frac{1}{\omega}\ \mathrm{Sin}(\omega z)\ -\ \frac{1}{b^2}\ (\frac{2z}{\omega^2}\ \mathrm{Cos}(\omega z)\ +\{\frac{z^2}{\omega}\ -\ \frac{2}{\omega^3}\}\mathrm{Sin}(\omega z))]_0^b$$

$$+ 0 = g(\omega)$$

$$\sqrt{\frac{2}{\pi}}\ \frac{\sigma}{\omega}[\frac{\mathrm{Sin}(\omega b)}{\omega b}\ -\ \mathrm{Cos}(\omega b)]\ =\ f(\omega)$$

$$\sqrt{\frac{2}{\pi}}\ \frac{2\tau}{\omega^2 b}[\frac{\mathrm{Sin}(\omega b)}{\omega b}\ -\ \mathrm{Cos}(\omega b)]\ =\ g(\omega)$$

Groups of terms that appear frequently will be given a symbol for brevity.

Let $S(\omega b)\ =\ [\frac{\mathrm{Sin}(\omega b)}{\omega b}\ -\ \mathrm{Cos}(\omega b)]$

then

$$\sqrt{\frac{2}{\pi}}\ \frac{\sigma}{\omega}\ S(\omega b)\ =\ f(\omega)$$

$$\sqrt{\frac{2}{\pi}}\ \frac{2\tau}{\omega^2 b}\ S(\omega b)\ =\ g(\omega)$$

If the expressions (3.6) and (3.7) are used, substituting for $f(\omega)$ and $g(\omega)$, a matrix form of the result becomes

$$\begin{bmatrix} \sqrt{\frac{2}{\pi}}\,\frac{\sigma}{\omega} S(\omega b) \\ \sqrt{\frac{2}{\pi}}\,\frac{2\tau}{\omega^2 b} S(\omega b) \end{bmatrix} =$$

$$\begin{bmatrix} \{K_0(\omega a) + \frac{1}{\omega a} K_1(\omega a)\} & \{-[1-2\nu]K_0(\omega a) + \omega a K_1(\omega a)\} \\ \{-K_1(\omega a)\} & \{2[1-\nu]K_1(\omega a) - \omega a K_0(\omega a)\} \end{bmatrix} \begin{bmatrix} A(\omega) \\ B(\omega) \end{bmatrix}$$

Let $K(\omega a) = K_0(\omega a)/K_1(\omega a)$ and divide by the factors $K_1(\omega a)$, $S(\omega b)$, and $\sqrt{\frac{2}{\pi}}$, yielding the form

$$\begin{bmatrix} \frac{\sigma}{\omega} \\ \frac{2\tau}{\omega^2 b} \end{bmatrix} =$$

$$\begin{bmatrix} \{K(\omega a) + \frac{1}{\omega a}\} & \{-[1-2\nu]K(\omega a) + \omega a\} \\ \{-1\} & \{2[1-\nu] - \omega a K(\omega a)\} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\pi}{2}}\,\frac{A(\omega)K_1(\omega a)}{S(\omega b)} \\ \sqrt{\frac{\pi}{2}}\,\frac{B(\omega)K_1(\omega a)}{S(\omega b)} \end{bmatrix}$$

The functions $A(\omega)$ and $B(\omega)$ become

$$A(\omega) = \frac{\sqrt{\frac{2}{\pi}}\,S(\omega b)}{D(\omega a)K_1(\omega a)}[\sigma a(\omega a K(\omega a) - c) + \frac{2\tau a}{\omega a \omega b}(\omega^2 a^2 - (c-1)\omega a K(\omega a))]$$

$$B(\omega) = \frac{-\sqrt{\frac{2}{\pi}}S(\omega b)}{D(\omega a)K_1(\omega a)}\ [\sigma a + \frac{2\tau a}{\omega a \omega b}(1 + \omega a K(\omega a))]$$

where

$$D(\omega a) = [(\omega a)^2(K^2(\omega a) - 1) - c]$$

$$c = 2(1 - \nu)$$

(3.8)

Equations (3.5) become

$$2Gw(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{K_1(\omega r)}{\omega}\,[A(\omega)K(\omega r)$$

$$+ B(\omega)(\omega r - 2cK(\omega r))]\,\mathrm{Cos}(\omega z)\,d\omega$$

$$2Gu_r(r,z) = \int_0^\infty -\sqrt{\frac{2}{\pi}}\,\frac{K_1(\omega r)}{\omega}\,[A(\omega) + B(\omega)\omega rK(\omega r)]\,\mathrm{Sin}(\omega z)\,d\omega$$

$$2G\frac{\partial u_r}{\partial z}(r,z) = \int_0^\infty -\sqrt{\frac{2}{\pi}}\,K_1(\omega r)\,[A(\omega) + B(\omega)\omega rK(\omega r)]\,\mathrm{Cos}(\omega z)\,d\omega$$

$$\sigma_r(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}}K_1(\omega r)\,[A(\omega)(K(\omega r) + \frac{1}{\omega r})$$

$$- B(\omega)([c - 1]K(\omega r) - \omega r)]\,\mathrm{Sin}(\omega z)\,d\omega$$

$$\tau_{rz}(r,z) = \int_0^\infty \sqrt{\frac{2}{\pi}}K_1(\omega r)\,[-A(\omega) + B(\omega)(c - \omega rK(\omega r))]\,\mathrm{Cos}(\omega z)\,d\omega \tag{3.9}$$

When expressions (3.8) for $A(\omega)$ and $B(\omega)$ are substituted into equations (3.9) we obtain equations (3.10) valid everywhere in the elastic solid.

$$\frac{\pi}{a}Gw(r,z) = \int_0^\infty \frac{S(\omega b)K_1(\omega r)}{\omega D(\omega a)K_1(\omega a)}[\sigma\{(\omega aK(\omega a) + c)\ K(\omega r) \quad -\omega r\}$$

$$+ \frac{2\tau}{\omega a\omega b}\{(\omega^2a^2 - (c-1)\omega aK(\omega a))\ K(\omega r)$$

$$- (1 + \omega aK(\omega a))(\omega r - 2cK(\omega r))\}]\,\mathrm{Cos}(\omega z)\,d\omega$$

$$-\frac{\pi}{a}Gu_r(r,z) = \int_0^\infty \frac{S(\omega b)\ \ K_1(\omega r)}{\omega D(\omega a)K_1(\omega a)}[\sigma\{(\omega aK(\omega a)-c)-\omega rK(\omega r)\}$$

$$+ \frac{2\tau}{\omega a\omega b}\{(\omega^2a^2-(c-1)\omega aK(\omega a))$$

$$-(1+\omega aK(\omega a))\omega rK(\omega r)\}]\,\mathrm{Sin}(\omega z)\,d\omega$$

$$-\frac{\pi}{a}\frac{G\partial u_r}{\partial z}(r,z) = \int_0^\infty \frac{S(\omega b)K_1(\omega r)}{D(\omega a)K_1(\omega a)}[\sigma\{(\omega aK(\omega a)-c)-\omega rK(\omega r)\}$$

$$+\frac{2\tau}{\omega a\omega b}\{(\omega^2a^2-(c-1)\omega aK(\omega a))$$

$$-(1+\omega aK(\omega a))\omega rK(\omega r)\}]\mathrm{Cos}(\omega z)d\omega$$

$$\frac{\pi}{2a}\sigma_r(r,z) = \int_0^\infty \frac{S(\omega b)K_1(\omega r)}{D(\omega a)K_1(\omega a)}[\sigma\{\omega aK(\omega a)(K(\omega r) + \frac{1}{\omega r})$$

$$-(K(\omega r)+\omega r+\frac{c}{\omega r})\}$$

$$+\frac{2\tau}{\omega a\omega b}\{(\omega^2a^2-(c-1)\omega aK(\omega a))(K(\omega r)+\frac{1}{\omega r})$$

$$+(1+\omega aK(\omega a))((c-1)K(\omega r)-\omega r)\}]$$

$$\cdot\mathrm{Sin}(\omega z)d\omega$$

$$-\frac{\pi}{2a}\tau_{rz}(r,z)$$

$$= \int_0^\infty \frac{S(\omega b)K_1(\omega r)}{D(\omega a)K_1(\omega a)}[\sigma\{\omega aK(\omega a)-\omega rK(\omega r)\}$$

$$+\frac{2\tau}{\omega a\omega b}\{(\omega^2a^2-(c-1)\omega aK(\omega a))$$

$$+(1+\omega aK(\omega a))(c-\omega rK(\omega r))\}]\mathrm{Cos}(\omega z)d\omega$$

(3.10)

The expression for $w(r,z)$ above will be used to examine the decay of displacement with respect to increasing r values. This will be presented in Section IV.

Of particular importance in this problem are the values at $r = a$ where the plate is attached to the elastic solid. Let $K = K(\omega a)$. The displacement, slope, and stresses on the boundary become:

$$\frac{\pi}{a}Gw(a,z) = \int_0^\infty \frac{S(\omega b)}{\omega D(\omega a)}\Big[\frac{\sigma}{\omega a}\{\omega^2a^2K^2-\omega^2a^2+(-c+c)+c\omega aK\}$$

$$+ \frac{2\tau}{\omega^2a^2\omega b}\{\omega^2a^2K^2-\omega^2a^2+(-c+c)$$

$$+2c\omega aK+c\omega^2a^2K^2\}\Big]\text{Cos}(\omega z)d\omega$$

$$\frac{\pi}{a}Gu_r(a,z) = \int_0^\infty \frac{S(\omega b)}{\omega D(\omega a)}\Big[\sigma c + \frac{2\tau}{\omega a\omega b}\{\omega^2a^2K^2-\omega^2a^2+(-c+c)$$

$$+c\omega aK\}\Big]\text{Sin}(\omega z)d\omega$$

$$\frac{\pi}{a}G\frac{\partial u_r}{\partial z}(a,z) = \int_0^\infty \frac{S(\omega b)}{D(\omega a)}\Big[\sigma c + \frac{2\tau}{\omega a\omega b}\{\omega^2a^2K^2-\omega^2a^2+(-c+c)$$

$$+c\omega aK\}\Big]\text{Cos}(\omega z)d\omega$$

$$\frac{\pi}{2a}\sigma_r(a,z) = \int_0^\infty \frac{S(\omega b)}{D(\omega a)}\Big[\frac{\sigma}{\omega a}\{\omega^2a^2K^2-\omega^2a^2-c\}\Big]\text{Sin}(\omega z)d\omega$$

$$\frac{\pi}{2a}\tau_{rz}(a,z) = \int_0^\infty \frac{S(\omega b)}{D(\omega a)}\Big[\frac{2\tau}{\omega a\omega b}\{\omega^2a^2K^2-\omega^2a^2-c\}\Big]\text{Cos}(\omega z)d\omega$$

Let $x = \omega a$ and define $N(\omega a) = (1+\omega aK(\omega a))$,

then $dx = ad\omega$, $\omega = \frac{x}{a}$, $N(x) = 1+xK(x)$, etc.

$$Gw(a,z) = \sigma\{\frac{a}{\pi}\int_0^\infty \frac{S(\frac{b}{a}x)}{x^2}[1+c\frac{N(x)}{D(x)}]\ \text{Cos}(\frac{x}{a}z)\ dx\}$$

$$+\tau\{\frac{2}{\pi}\ \frac{a^2}{b}\int_0^\infty \frac{S(\frac{b}{a}x)}{x^4}[1+c\frac{N^2(x)}{D(x)}]\text{Cos}(\frac{x}{a}z)dx\}$$

$$Gu_r(a,z) = \sigma\{\frac{ac}{\pi}\int_0^\infty \frac{S(\frac{b}{a}x)}{xD(x)}\ \text{Sin}(\frac{x}{a}z)dx\}$$

$$+\tau\{\frac{2a^2}{\pi b}\int_0^\infty \frac{S(\frac{b}{a}x)}{x^3}[1+c\ \frac{N(x)}{D(x)}]\ \text{Sin}(\frac{x}{a}z)dx\}$$

$$G\frac{\partial u_r}{\partial z}(a,z) = \sigma\{\frac{c}{\pi}\int_0^\infty \frac{S(\frac{b}{a}x)}{D(x)} \text{Cos}(\frac{x}{a}z)dx\}$$

$$+\tau\{\frac{2a}{\pi b}\int_0^\infty \frac{S(\frac{b}{a}x)}{x^2}[1+c\frac{N(x)}{D(x)}] \text{ Cos}(\frac{x}{a}z)dx\}$$

$$\sigma_r(a,z) = \sigma\{\frac{2}{\pi}\int_0^\infty \frac{S(\frac{b}{a}x)}{x} \text{ Sin}(\frac{x}{a}z)dx\}$$

$$\tau_{rz}(a,z) = \tau\{\frac{4a}{\pi b}\int_0^\infty \frac{S(\frac{b}{a}x)}{x^2} \text{Cos}(\frac{x}{a}z)dx\} \tag{3.11}$$

The above integral equations may be used to compute displacements on the cylindrical boundary for any magnitudes of stresses σ and τ in the prescribed stress configurations. The results of such computations will be presented in Appendix I.

When $z = 0$ in the above expressions, the quantities in braces are constants and will be designated as indicated below.

$$Gw(a,0) = \sigma\{ws\} + \tau\{wt\}$$

$$Gu_r(a,0) = \sigma\{us\} + \tau\{ut\}$$

$$G\frac{\partial u_r}{\partial z}(a,0) = \sigma\{uzs\} + \tau\{uzt\} \tag{3.12}$$

These expressions will be used in Section IV to develop the plate solution. When $z \neq o$ the corresponding expressions will be designated ws(z), wt(z), us(z), ... etc. When $r \neq a$ then designate ws(r), wt(r) or ws(r,z), wt(r,z), etc.

If the expressions for stress are rewritten as

$$\sigma_r(a,z) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\sqrt{\frac{2}{\pi}} \frac{\sigma}{\omega} (\frac{\mathrm{Sin}(\omega b)}{\omega b} - \mathrm{Cos}(\omega b))] \mathrm{Sin}(\omega z) d\omega$$

$$\tau_{rz}(a,z) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\sqrt{\frac{2}{\pi}} \frac{2}{b} \frac{\tau}{\omega^2} (\frac{\mathrm{Sin}(\omega b)}{\omega b} - \mathrm{Cos}(\omega b))] \mathrm{Cos}(\omega z) d\omega,$$

they are recognized to be the inverse transforms of the original Fourier integral transformations of the prescribed boundary stresses and are therefore equal to them by definition. These may be retained as computational checks.

The solution of the elastic solid problem is now complete. This solution was obtained by applying to the boundary two stress configurations of arbitrary magnitude. It may now be mated with any other body that would produce the same stress configurations and satisfy displacement compatibility requirements. It is this class of problems, requiring the matching of two separate solutions, that provided the motivation for this effort.

Approximate solutions can be found which satisfy stress equilibrium only in the St. Venant sense and/or are compatible only at specified points. The mating of this solid with any plate problem of the class to be described in Section IV would be an example of such an approximation.

The Plate

In order to match a plate solution to the elastic solid, the plate solution will be required to have the following properties:

a. Axial symmetry (i.e. $w=w(r)$)

b. Negligible radial displacements at the mid-plane (i.e. $z=0$)

c. Arbitrary Shear and Moment resultants at the outer edge. ($r=a$)

(C.1)

Any solution with these properties can be matched to the elastic solid at $(r,z)=(a,o)$ by solving the following boundary value problem.

Let:

plate		elastic solid	
$w(a)$	=	$w(a,o)$	(B.1)

Require the mid-plane of the plate to intersect the wall at right angles.

$\frac{\partial w(a)}{\partial r}$	=	$\frac{-\partial u(a,o)}{\partial z}$	(B.2)
$M_r(a)$	=	$M(\sigma) \equiv \int_A \sigma_r(a,z)\,z\,dA$	(B.3)
$Q_r(a)$	=	$Q(\tau) \equiv +\int_A \tau_{rz}(a,z)\,dA$	(B.4)

(such other conditions as the plate solution requires)

(B.5)

Resulting approximations to the true solution will depend, of course, on the limitations of the solid and plate theories used, and also on how well the single point matching at (a,o) compels the boundary stresses and displacements to conform to each other over the entire plate thickness

$(a,-b) \leqslant (r,z) \leqslant (a,+b)$

As $b \to o$ we would expect improved estimates, at least for values of $r >> a$ or $r << a$ since the Principle of St. Venant assures us that small differences in stress distributions are local and only the resultants are of importance elsewhere.

Any solutions satisfying conditions (C.1) and the classical fourth-order plate equation are admissible.

$$q(x,y) = [D(w_{xx} + \nu w_{yy})]_{xx} + 2[D(1-\nu)w_{xy}]_{xy}$$

$$+ [D(w_{yy} + \nu w_{xx})]_{yy}$$

where $D = \dfrac{E(x,y)\ t^3(x,y)}{12[1-\nu^2(x,y)]}$ $\quad \nu = \nu(x,y)$ etc.

(3.13)

but where condition (C.1b) disallows cases involving mid-plane forces.

For purposes of illustrating the procedure in solving this type of boundary value problem let us consider only those cases where E,t,ν are constants. The above equation becomes the familiar

$$D\nabla^4 w(x,y) = q(x,y) \quad .$$

The theory to be used in describing this plate problem is the well known "classical" theory governing small lateral deflections "w" of a thin plate as described in Section II.

The relationships required for the axisymmetric plate of uniform E,t,ν are listed on the following page.

$$-\frac{Q_r}{D} = \frac{d}{dr}\nabla^2 w$$

$$-\frac{Q_r}{D} = \frac{d}{dr}[\frac{1}{r}\ \frac{d}{dr}(\frac{rdw(r)}{dr})]$$

$$-\frac{M_r}{D} = [w_{rr}+\frac{\nu 1}{r}\ w_r]$$

$$(3.14)^1$$

where $D=\frac{2Eb^3}{3(1-\nu^2)}$ and $\nabla^2 = (\frac{d^2}{dr^2}+\frac{1}{r}\ \frac{d}{dr})$ in the axisymmetric case.

To begin the solution, forces are summed in the vertical direction for the portion of the plate from radius $\rho = \alpha$ to $\rho = \beta$.

$$\Sigma F_z = Q_\beta(2\pi\beta)+\int_\alpha^\beta q(\rho)(2\pi\rho)d\rho+Q_\alpha(2\pi\alpha)=0$$

$$Q_\beta = \frac{-1}{\beta}[\int_\alpha^\beta \rho q(\rho)d\rho+Q_\alpha\alpha]$$

$$-D\ \frac{d}{d\beta}[\frac{1}{\beta}\ \frac{d}{d\beta}(\beta\frac{dw(\beta)}{d\beta})] = -\frac{1}{\beta}[\int_\alpha^\beta \rho q(\rho)d\rho+Q_\alpha\alpha]$$

The above differential equation describes the plate solutions under consideration.

Solution of the boundary value problem when the plate has a hole, inclusions, etc. follows essentially the same procedure as below but will not be treated here.

---

[1] [9, p. 53] Note that differences in sign are due to choice of positive direction for Q_r.

Let $\alpha = 0$, then for condition B.5 the requirement will be that solutions must remain bounded at the origin and

$$D\frac{d}{d\beta}\left[\frac{1}{\beta}\frac{d}{d\beta}\left(\beta\frac{dw(\beta)}{d\beta}\right)\right] = \frac{1}{\beta}\left[\int_{\alpha=0}^{\beta}\rho q(\rho)d\rho+0\right]$$

Assuming that the necessary functions are integrable over the range $0\leqslant\rho\leqslant a$, then

$$D\frac{dw(\gamma)}{d\gamma} = \frac{1}{\gamma}\int_0^{\gamma}\delta\int_0^{\delta}\frac{1}{\beta}\int_0^{\beta}\rho q(\rho)d\rho d\beta d\delta + A\gamma + \frac{1}{\gamma}B$$

$$Dw(r) = \int_0^{r}\frac{1}{\gamma}\int_0^{\gamma}\delta\int_0^{\delta}\frac{1}{\beta}\int_0^{\beta}\rho q(\rho)d\rho d\beta d\delta d\gamma+\frac{Ar^2}{2}+B\ln(r)+C.$$

Condition B.5 requires that $w(r)$ remain bounded as $r\to 0$ which implies that $B=0$ since $\ln(r)\to\infty$ as $r\to 0$.

There are four conditions B.1 to B.4 left to satisfy and the four arbitrary quantities σ, τ, A, C to specify.

Apply B.1 $\qquad w(a) = w(a,0)$

$$\frac{1}{D}\left[\int_0^{a}\frac{1}{\gamma}\int_0^{\gamma}\delta\int_0^{\delta}\frac{1}{\beta}\int_0^{\beta}\rho q(\rho)d\rho d\beta d\delta d\gamma+\frac{Aa^2}{2}+C\right] =$$

$$\frac{1}{G}[\sigma(ws)+\tau(wt)]$$

where ws and wt are the constants in (3.12) and define

$$Iw(r)=\int_0^{r}\frac{1}{\gamma}\int_0^{\gamma}\delta\int_0^{\delta}\frac{1}{\beta}\int_0^{\beta}\rho q(\rho)\ d\rho d\beta d\delta d\gamma.$$

Then define $Iw(r)$ at $r=a$ to be Iw.

Condition B.1 becomes $Iw + \frac{Aa^2}{2} + C = \frac{D}{G}[\sigma(ws)+\tau(wt)]$

$$(3.15)$$

Apply B.2 $$\frac{dw(a)}{d\gamma} = \frac{-\partial u_r(a,0)}{\partial z}$$

$$\frac{1}{D}\left[\frac{1}{a}\int_0^a \delta \int_0^\delta \frac{1}{\beta}\int_0^\beta \rho q(\rho)\ d\rho d\beta d\delta + Aa\right] = \frac{-1}{G}[\sigma(uzs) + \tau(uzt)]$$

Where uzs and uzt are from (3.12) and define

$$Iwr(r) = \frac{1}{r}\int_0^r \delta \int_0^\delta \frac{1}{\beta}\int_0^\beta \rho q(\rho)\ d\rho d\beta d\delta.$$

Then $Iwr + Aa = \frac{-D}{G}[\sigma(uzs) + \tau(uzt)]$

$$(3.16)$$

where Iwr is defined to be $Iwr \equiv Iwr(a)$.

Apply B.3 $$M_r(a) = M(\sigma) \equiv \int_A [\sigma_r(a,z)]zdA$$

From equations (3.14)

$$-D[w_{rr}(a)+\frac{\nu}{a}w_r(a)] = \int_{z=-b}^{z=+b} [\sigma z]z(1\cdot dz)$$

$$-\left[\frac{-1}{a}Iwr+\frac{1}{a}\left(a\int_0^a \frac{1}{\beta}\int_0^\beta \rho q(\rho)d\rho d\beta\right)+A+\frac{\nu}{a}(Iwr+Aa)\right] = \frac{2}{3}b^3\sigma.$$

Define

$$Iwrr(r) = \int_0^r \frac{1}{\beta}\int_0^\beta \rho q(\rho)d\rho d\beta.$$

Then

$$(1-\nu)\frac{Iwr}{a} - (1+\nu)A-Iwrr + \frac{2}{3}b^3\sigma$$

$$(3.17)$$

where $Iwrr \equiv Iwrr(a)$.

Apply B.4 $$Q_r(a) = Q(\tau) \equiv + \int_A \tau_{rz}(a,z)dA$$

From summing forces in the "z" direction

$$\frac{-1}{a}\int_{r=0}^{r=a} rq(r)dr = + \int_{z=-b}^{z=+b} [\tau(1-\frac{z^2}{b^2})](1\cdot dz) = \frac{4b}{3}\tau$$

$$\tau = \frac{-3}{4ab}\int_o^a rq(r)dr$$

Now τ is a known constant.

Equations (3.16, 3.17) can be solved for σ and A explicitly

$$\sigma = \frac{(+2Iwr - aIwrr) + \frac{D}{G}(1+\nu)uzt\tau}{\frac{2}{3}ab^3 - \frac{D}{G}(1+\nu)uzs} \tag{3.18}$$

$$A = \frac{-(\frac{D(1-\nu)uzs}{G\quad a} + \frac{2b^3}{3})\ Iwr + \frac{D}{G}(+uzs\cdot Iwrr - \frac{2}{3}b^3 uzt\tau)}{\frac{2}{3}ab^3 - \frac{D}{G}(1+\nu)uzs} \tag{3.19}$$

From equation (3.15)

$$C = \frac{D}{G}[\sigma(ws) + \tau(wt)] - \frac{a^2A}{2} - Iw \tag{3.20}$$

Now for $0 \leqslant r \leqslant a$ in the plate

$$w(r) = \frac{1}{D}\int_o^r \frac{1}{\gamma}\int_o^\gamma \delta \int_o^\delta \frac{1}{\beta}\int_o^\beta \rho q(\rho)d\rho d\beta d\delta d\gamma + \frac{Ar^2}{D2} + \frac{C}{D}$$

$$\frac{dw(r)}{dr} = \frac{1}{Dr}\int_o^r \delta \int_o^\delta \frac{1}{\beta}\int_o^\beta \rho q(\rho)\ d\rho d\beta d\delta + \frac{Ar}{D}$$

$$M_r(r) = -D[\frac{d}{dr}(\frac{dw(r)}{dr}) + \frac{\nu}{a}\frac{dw(r)}{dr}]$$

or

$$Dw(r) = Iw(r) + \frac{Ar^2}{2} + C$$

$$D\frac{dw(r)}{dr} = Iwr(r) + Ar$$

(3.21)

Two limiting cases are of interest for purposes of comparison later

1) Rigid wall support: $G_{wall} \to \infty$

2) Simply supported plate: $G_{wall} \to 0$ and terms involving τ are neglected.

1) Let $G \to \infty$

from (3.19)

$$A = \frac{(0 - \frac{2b^3}{3})Iwr + 0}{\frac{2}{3}ab^3 + 0} = \frac{-Iwr}{a}$$

from (3.20)

$$C = 0 - \frac{a^2}{2}\left(\frac{-Iwr}{a}\right) - Iw = \frac{a^2}{2a} Iwr - Iw$$

from (3.21)

$$Dw(r) = [Iw(r) - Iw(a)] + \frac{Iwr}{2a}(a^2 - r^2)$$

since $Iw = Iw(r)$ at $r = a$

$$D\frac{dw(r)}{dr} = Iwr(r) - \frac{r}{a}Iwr(a)$$

since $Iwr = Iwr(r)$ at $r = a$

(3.22)

2) First let $\tau = 0$ then multiply numerator and denominator of (3.18) and (3.19) by $\frac{G}{D}$ and let $G \to 0$

$$A = \frac{-\left(\frac{(1-\nu)uzs}{a} - 0\right) Iwr + (+uzs\ Iwrr + 0)}{-(1 + \nu)uzs}$$

$$= \frac{(1-\nu)Iwr}{(1+\nu)a} - \frac{Iwrr}{(1+\nu)}$$

$$C = [0 + 0] - \frac{a^2}{2}\left[\frac{(1-\nu)Iwr}{(1+\nu)a} - \frac{Iwrr}{(1+\nu)}\right] - Iw$$

from (3.21)

$$Dw(r) = Iw(r) - Iw(a) + \left[\frac{(1-\nu)Iwr}{(1+\nu)\ a} - \frac{Iwrr}{(1+\nu)}\right] \frac{(r^2-a^2)}{2}$$

$$D\,\frac{dw(r)}{dr} = Iwr(r) + \left[\frac{(1-\nu)}{(1+\nu)}\ \frac{Iwr}{a} - \frac{Iwrr}{(1+\nu)}\right] r \qquad (3.23)$$

IV. Illustrative Example

Further investigations and comparisons of interest would be awkward to carry out in general. The remaining text will deal with specific loading, geometry and material parameters.

The plate loading

Let $q(r) = q$ a constant

Recall $\tau = \frac{-3}{4ab} \int_0^a rq(r)dr$

then

$$\tau = \frac{-3}{8} \frac{a}{b} q$$

Recall

$$Iwrr(r) = \int_0^r \frac{1}{\beta} \int_0^\beta \rho q(\rho) d\rho d\beta$$

then

$$Iwrr(r) = q \int_0^r \frac{1}{\beta}(\frac{\beta^2}{2}) d\beta = \frac{qr^2}{4}$$

and $\quad Iwrr = Iwrr(a) = \frac{qa^2}{4}$

$$Iwr(r) = \frac{1}{r} \int_0^r \delta \; Iwrr(\delta) d\delta = \frac{q}{4} \frac{1}{r} \int_0^r \delta^3 d\delta = \frac{qr^3}{16}$$

and $Iwr = \frac{qa^3}{16}$

$$Iw(r) = \int_0^r Iwr(\gamma)d\gamma = \frac{q}{16}\int_0^r \gamma^3 d\gamma = \frac{q}{64}r^4$$

$$Iw = \frac{q}{64}a^4$$

With the above formuli $Dw(r)$ and $D\frac{dw(r)}{dr}$ may be found by algebraic substitution. The load q in the examples is a multiplying factor in the expressions for w(r) and is set at 100 for all numerical results. First the two limiting cases in the previous chapter will be examined.

Limiting cases

1) Rigid wall support: equations (3.22) become

$$Dw(r) = \frac{q}{64}[r^4-a^4] - \frac{qa^2}{32}(r^2-a^2) = \frac{q}{64}[r^4-2r^2a^2+a^4]$$

$$D\frac{dw(r)}{dr} = \frac{q}{16}[r^3-r\ a^2]$$

which are the classical plate equations for this problem.

2) Simply supported plate: equations (3.23) become

$$Dw(r) = \frac{q}{64}[r^4-a^4] + \frac{qa^2}{2}[\frac{(1-\nu)}{16(1+\nu)} - \frac{1}{4(1+\nu)}](r^2-a^2)$$

$$= \frac{q}{64}(r^4-a^4) - \frac{qa^2}{32}[\frac{3+\nu}{1+\nu}](r^2-a^2)$$

$$D\frac{dw(r)}{dr} = \frac{q}{16}[r^3-(\frac{3+\nu}{1+\nu})ra^2]$$

and again these are the expected solutions.

Plate geometries

For purposes of illustration three plate geometries will be used.

(a) Thickness to diameter ratio $b/a = 1/10$ will be used to explore the maximum differences between the rigid and elastic wall assumptions even though it is recognized that "thin" plate theory is probably not a very good model for this ratio.

(b) Thickness to diameter ratio $b/a = 2/100$ will be used to explore the differences between the rigid and elastic wall assumptions for a plate that "thin" plate theory should describe quite well.

(c) Thickness to diameter ratio $b/a = 5/100$.

With the geometry specified the quantities ws, wt, uzs, uzt from (3.12) can be found using the integration techniques as explained in Appendix I.

Material parameters

For the given loading and thickness to diameter ratios the ratio of the elastic moduli of the plate to that of the wall E_p/E_w will be allowed to range over eight values from zero to ten where zero represents the plate built into an inflexible wall. $E_p/E_w = \infty$ does not represent the simple support because $w(r) \neq 0$ at $r = a$.

Each of the above cases will be done for $\nu = .25$ in both wall and plate and for $\nu = .30$ in both wall and plate though ν in the wall and the plate need not be the same.

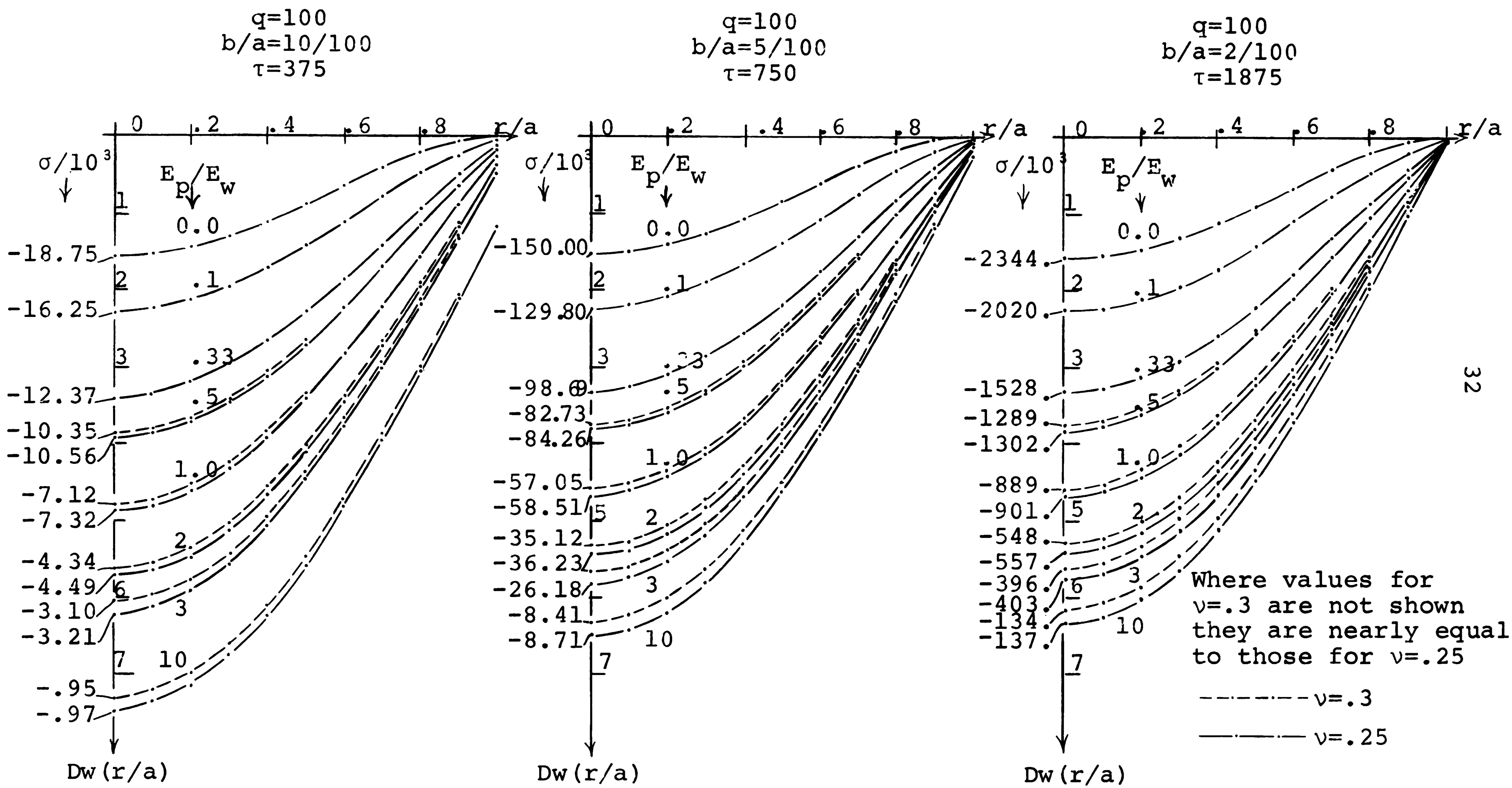
Results

The deflections Dw(r) of the midplane of the plate together with corresponding maximum stresses σ and τ at the wall are displayed in Figure 3. It is apparent that for each of these thickness/diameter ratios, the deflections w(r) based on the assumptions of a rigid wall and of a wall and plate of same material, never differ by less than 300% for any given r. Similarly the maximum radial tensile stress σ at the wall never varies by less than 200%. From a practical viewpoint the latter variation is on the safe side. The effect of Poisson's ratio varying from .25 to .30 is relatively small but becomes more significant for thinner plates.

As expected the deflection at the wall is small for thinner plates, while rotation at the wall is still comparatively large.

There is little reason to believe that the trends indicated by this example would change significantly for other usual loadings and geometries.

Figure 3. Plate displacements for the illustrative example



Other results

Also of interest is the behavior of the wall. For the case where b/a = 1/10 the displacement Dw(r,z) where z = 0, r>a is displayed in Figure 4. These results are obtained by using the technique described in Appendix I to integrate the first of the five equations (3.10). This equation is shown below.

$$G\,w(r,z) = \frac{a}{\pi}\int_0^\infty \frac{\left(\mathrm{Cos}(\omega b) - \frac{\mathrm{Sin}(\omega b)}{\omega b}\right)\dfrac{e^{-\omega r}\{e^{\omega r}K_1(\omega r)\}}{e^{-\omega a}\{e^{\omega a}K_1(\omega a)\}}}{\omega\left((\omega a)^2\left[\left(\dfrac{\{e^{\omega a}K_0(\omega a)\}}{\{e^{\omega a}K_1(\omega a)\}}\right)^2 - 1\right] - 2(1-\nu)\right)}$$

$$\cdot\left[\sigma\left\{\left(\omega a\frac{\{e^{\omega a}K_0(\omega a)\}}{\{e^{\omega a}K_1(\omega a)\}} + 2(1-\nu)\right)\left(\frac{\{e^{\omega r}K_0(\omega r)\}}{\{e^{\omega r}K_1(\omega r)\}}\right) - \omega r\right\}\right.$$

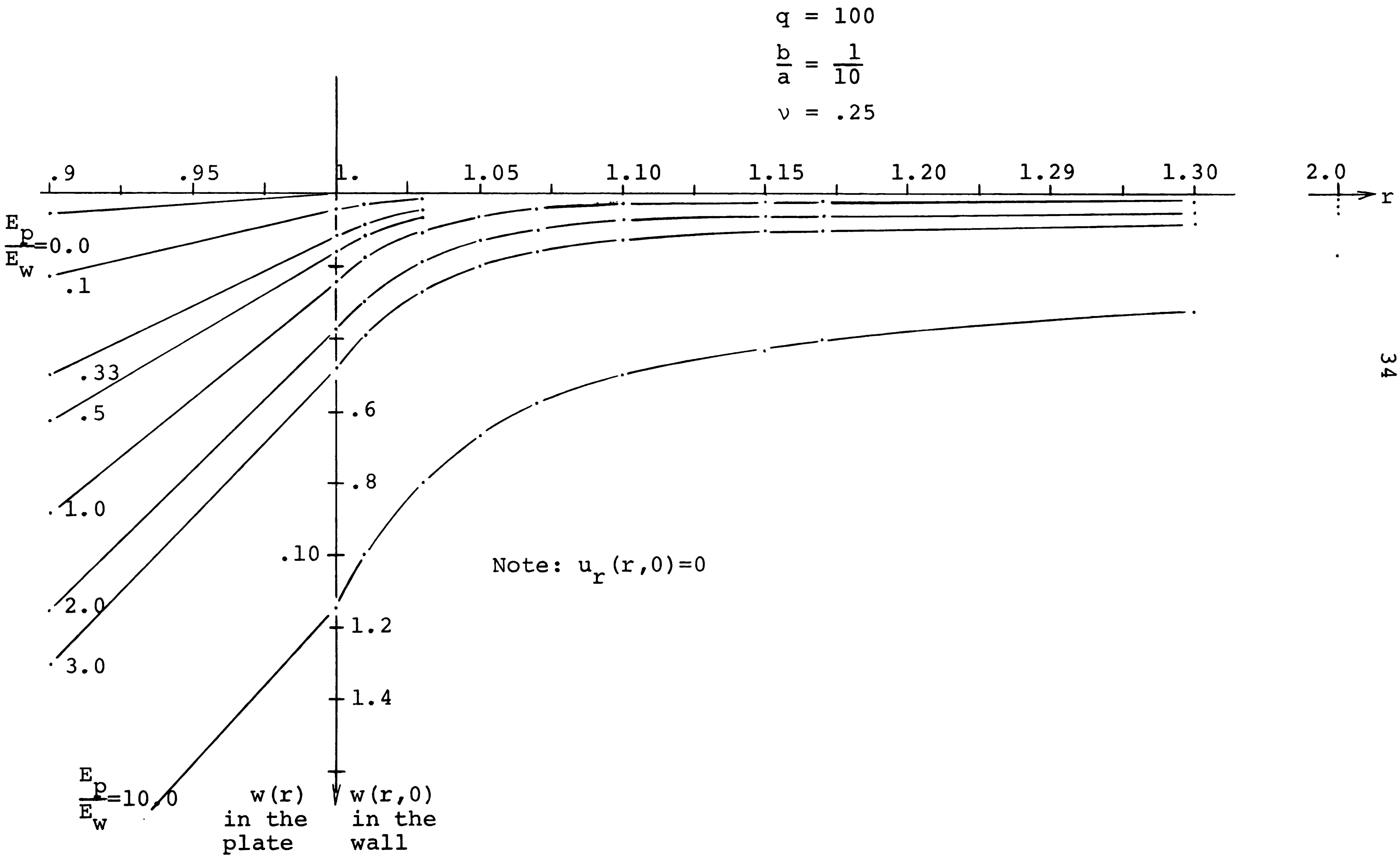
$$+ \frac{\tau}{(\omega a)(\omega b)}\left\{\left(\omega^2 a^2 - (2(1-\nu)-1)\,\omega a\frac{\{e^{\omega a}K_0(\omega a)\}}{\{e^{\omega a}K_1(\omega a)\}}\right)\frac{\{e^{\omega r}K_0(\omega r)\}}{\{e^{\omega r}K_1(\omega r)\}}\right.$$

$$\left.\left. - \left(1 + \omega a\frac{\{e^{\omega a}K_0(\omega a)\}}{\{e^{\omega a}K_1(\omega a)\}}\right)\left(\omega r - 2(2(1-\nu))\frac{\{e^{\omega r}K_0(\omega r)\}}{\{e^{\omega r}K_1(\omega r)\}}\right)\right\}\right]$$

$$\cdot\mathrm{Cos}(\omega z)\,d\omega$$

where σ and τ are obtained from the previous results and the exponentials are required in the computational algorithm for the Bessel functions. Having obtained Gw(r,0) for various r it is now necessary to apply the scaling factor

Figure 4. A sampling of wall displacements for the illustrative example



$\frac{D}{G}$ in each case to obtain $Dw(r,0)$.

$$\frac{D}{G} = \frac{\dfrac{2E_p b^3}{3(1-\nu_p^{\ 2})}}{\dfrac{E_w}{2(1+\nu_w)}} = \left(\frac{E_p}{E_w}\right) \frac{4b^3(1+\nu_w)}{3 \quad (1-\nu_p)(1+\nu_p)}$$

In this case $\nu_p = \nu_w = .25$ and $b = \frac{1}{10}$

$$\frac{D}{G} = \frac{E_p}{E_w} \frac{4\left(\frac{1}{1000}\right)}{3\left(\frac{3}{4}\right)} = \frac{16 \ E_p}{9 \cdot 10^3 E_w}$$

In a variety of related problems it would be useful to know the behavior of specified points in or on the wall responding to a unit load σ or τ. These are essentially the integrals in (3.10, 3.11, or 3.12) and were used to obtain the previous results. At the end of Appendix I, figures 5, 6, 7, 8, and 9 display these results for b/a = $\frac{1}{10}$, $\nu = .25$ together with the output stresses (3.10, 3.11) which were used as a computational check. It will be noted in Appendix I that the apparent uncertainty of convergence for some of the values is exaggerated in the display which gives the bounds on the value of the infinite integrals.

V. Summary

Limitations in theory

The generality of the mathematical model of this problem is limited by the two component theories used in its solution: classical, three-dimensional, small-strain, theory for a homogeneous isotropic elastic solid used for the supporting wall; classical, small-strain, thin-plate theory without mid-plane forces used for the plate fixed to the supporting wall.

The other two limitations on the generality of the theory are: choice of axialsymmetry to obtain a non-plane-strain solution in two dimensions; selection of the stress profiles at the boundary shared by the two bodies, thus the matching of the slope and displacements at this common boundary is limited to a single appropriate point.

Applications

Within the above limitations the problem is solved for all plates without holes or inclusions and having uniform thickness and material parameters. Any load $q(r)$ is admissible. The solution for the wall would allow the other plate solutions [i.e., with a hole and/or inclusions and

also having variable thickness and/or material parameters] to be developed in a parallel fashion with comparative ease.

The solution allows the wall and plate to have different material parameters.

The solutions for the shear and the moment applied to the cylindrical wall are easily used together or separately to solve a variety of other problems where such a wall is similarly loaded.

Since the mathematical model for the wall is solved for all of space and since the solutions all decay with respect to increasing r and z, these solutions can be used to approximate those of objects with sufficiently thick walls having cylindrical cavities. In the case of the self-equilibrating moment, where shear is small, the walls need not be very thick.

R E F E R E N C E S

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A P P E N D I X   I

APPENDIX I

Numerical Integration

The integrated quantities in equations (3.10, 3.11, or 3.12) must be computed for the illustrative example of Section IV.

Examination of the functions to be integrated reveals several things.

1. They are reasonably well behaved at the origin even though some of the component functions are not.
2. They have a periodic behavior for large values of ω.
3. They decay for large values of ω.

The partial sums of a good numerical integration procedure should therefore converge within any specified bounds on the desired value "I" for sufficiently large ω and sufficiently small increments h.

Summation formula

A suitable numerical integration procedure is obtained from the Euler-Maclaurin summation formula and is developed as follows [10].

The Euler-Maclaurin formula may be stated thus

$$C_h = h[\tfrac{1}{2}f(a) + f(a+h) + f(a+2h) + \ldots + f(b-h) + \tfrac{1}{2}f(b)]$$

$$C_h = A + b_2 h^2 D_1 + b_4 h^4 D_3 + b_6 h^6 D_5 + \ldots$$

$$\ldots + b_{2s} h^{2s} D_{2s-1} - R_{2s}(h) \qquad \text{(I-1)}$$

C_h is the approximation to A where

$$A = \int_a^b f(x)\, dx$$

$$h = \frac{b-a}{m} \qquad \text{m is an integer to be specified}$$

$$D_k = \frac{d^k f(x)}{dx^k}\Big]_a^b = f^k(b) - f^k(a).$$

$R_{2s}(h)$ is the remainder of order h^{2s+2} for small h and

$$b_0 = 1,\ b_1 = -\frac{1}{2}$$

$$\sum_{r=0}^{n-1} b_r/(n-r)! = 0 \text{ for } n \geqslant 2$$

$$b_2 = \frac{1}{12},\quad b_3 = 0,\quad b_4 = \frac{-1}{720}$$

$$b_{2k+1} = 0 \qquad k > 0.$$

"A variety of formulas for numerical integration are obtained by taking weighted averages of $C_h, C_{(2h)}, C_{(3h)}, C_{(4h)}, \ldots$ with weights chosen to eliminate certain terms in $D_1, D_3, D_5, \ldots$."

In this case the elimination of D_1 and D_3 terms keep the computation in manageable proportions.

From (I-1):

$$15C_{(h)}=15A+\frac{15}{12}(h)^2D_1-\frac{15}{720}(h)^4D_3+\frac{15}{30240}(h)^6D_5-\ldots$$

$$-6C_{(2h)}=-6A+\frac{-6}{12}(2h)^2D_1-\frac{-6}{720}(2h)^4D_3+\frac{-6}{30240}(2h)^6D_5-\ldots$$

$$+C_{(3h)}=A+\frac{1}{12}(3h)^2D_1-\frac{1}{720}(3h)^4D_3+\frac{1}{30240}(3h)^6D_5-\ldots$$

Adding these equations

$$(15C_h-6C_{2h}+C_{3h})=10A+\frac{h^6D_5}{84}+R_{2s}(h^{2s},D_{2s-1})\text{ for }s>3$$

Then $\frac{1}{10}(15C_h-6C_{2h}+C_{3h})$ is a good approximation to A if h and D_{2s-1} are sufficiently small. Note that h and D_{2s-1} are functions of intervals (b_i-a_i) for sufficiently smooth f(x) where the required infinite integral I is broken into a sum of integrals $\sum_{i=1}^{n} A_i$. Also note that the minimum value for m is six for this method and it may therefore be considered a seven point formula. To obtain this formula substitute the first of equations (I-1) into the estimate $A \simeq \frac{1}{10}(15C_h-6C_{2h}+C_{3h})$ and let m = 6.

$$15C_{(h)}=h[\tfrac{15}{2}f(a)+15f(a+h)+15f(a+2h)+15f(a+3h)+15f(a+4h)\ldots$$

$$-6C_{(2h)}=h[\tfrac{-12}{2}f(a)\qquad -12f(a+2h)\qquad -12f(a+4h)\ldots$$

$$C_{(3h)}=h[\tfrac{3}{2}f(a)\qquad \div 3f(a+3h)\qquad \ldots$$

$$A \simeq \frac{1}{10}(15C_h - 6C_{2h} + C_{3h}) = \frac{h}{10}[3f(a)+15f(a+h)+3f(a+2h)$$

$$+18f(a+3h)+3f(a+4h)+15f(a+5h)+3f(a+6h)] \qquad \text{(I-2)}$$

If $m>6$ it must be a multiple $6n$ and the formula is the same except that the terms $f(a+6ih)$ are multiplied by 6 instead of 3 when $i \neq 0$ or n. This latter form reduces redundant computation where partial sums are not required and will be referred to as (I-3).

Computation and limitations

The $f(x)$ in this computation has a decaying periodic behavior owing to the dominance of sinusoidal elements for large values of x (ω in 3.10 and 3.11). It is expected that the successive partial sums approximating "I" (i.e., $I \simeq A_1+A_2+A_3....+A_n$) will exhibit maxima and minima for intervals[1] $(\beta_n - \alpha_n)$ sufficiently small and that these values bound the desired value of infinite integrals "I." We choose to control $(\beta_n-\alpha_n)$ by partitioning the sinusoidal half-period (HP) of equations (3.10), $HP = \pi/b$, and (3.11), $HP = \pi a/b$. For $b/a = \frac{1}{10}$ and $\frac{1}{20}$ the summation began at x = HP and ran in both directions. We wish to approach limiting values of component functions $K_0(x)$ and $K_1(x)$ at $x = 0$ from the right. Summation formula (I-3) was used for x = HP summing toward

---

[1]The symbols a and b of the discussion "Summation formula" are changed to α and β to avoid confusion with plate dimensions $(r=a)$ and $(-b<z<b)$.

$x = 0$. In this transient region of f(x) it became necessary to partition the half period (PHP) into 1000 parts of length $(\beta-\alpha)$. This corresponds to an $m = 6000$ and $h = HP/6000$. Summation formula (I-2) was used from $x = HP$ summing toward $x = \infty$ because the partial sums are to be used to obtain bounds on I. A coarse partition PHP = 10 was adequate in this region. For $b/a = \frac{1}{50}$ the summation began at $x = \frac{1}{10}HP$ and for interval $0 \leqslant x \leqslant \frac{1}{10}HP$ the PHP = 10,000 was used with a PHP = 100 used for interval $\frac{1}{10} HP \leqslant x \leqslant \infty$.

The various fortran programs used are listed in Appendix II.

Note that $HP = \pi a/b$ in (3.11) is the only period (or frequency) encountered in solving the integrals for the constants (3.12) used in solving the boundary value problem at $(r,z) = (a, 0)$ but that for $z \neq 0$ the product of sinusoidal functions in f(x) produces other frequencies that increase the difficulty of controlling the bounds on any desired "I".

All component functions needed for the various f(x) were supplied by the Control Data 3600 machine with the exception of the Bessel functions. These were taken from

ANL C351
Argonne National Laboratory
3600 Library Routine

This subroutine does not give $K_n(x)$ directly but rather $e^x K_n(x)$. The machine limits the argument of e^x called by the subroutine to $x<709$. It can be seen that this restriction imposes limits on the search for improved bounds of any value "I". The asymptotic expression $K_n(x) \simeq \sqrt{\pi/2}\, e^{-x}/x$ (for all n) is limited in the same way. Ratio $K(x) = K_0(x)/K_1(x)$ becomes $K(x) \simeq 1$ using the above asymptotic expression. The Bessel functions appear in (3.12) in the ratio form exclusively but $D(x) = [x^2(K^2(x)-1)-c]$ approaches $D(\infty) \simeq [\infty^2(1-1)-c] = [\infty^2(0)-c]$ which is still indeterminant. $D(x)$ appears in each expression. Possibilities of other asymptotic expressions, computational algorithms, extrapolations, or modification of the existing subroutine have not been explored. This is partly because the expenditure of computer time required to compute to x = 709 was already sizable. The bounds obtained for the case where $b/a = \frac{1}{10}$ are displayed in figures 5,6,7,8, and 9 at the end of this appendix. As the thickness diameter ratio $b/a \to 0$ the restriction of x to less than 709 imposes a severe problem. If for example $b/a = \frac{1}{1000}$ then the half period $HP = \pi a/b = \pi 1000 = 3141.6$ which means that no bounds on any value I can be found. The values of (3.12) used in the illustrative example (Figure 3) are given in Table 1 on the following page together with the bounding values. Since the bounding values for the integrals (3.12) are the maxima and minima of a simple decaying sine curve, values between can be selected with the assurance that they are a great improvement over the gross maxima and minima and would represent the true value with less than the large apparent uncertainty. If great accuracy is required it seems

clear that improved bounds could be established without further integration.

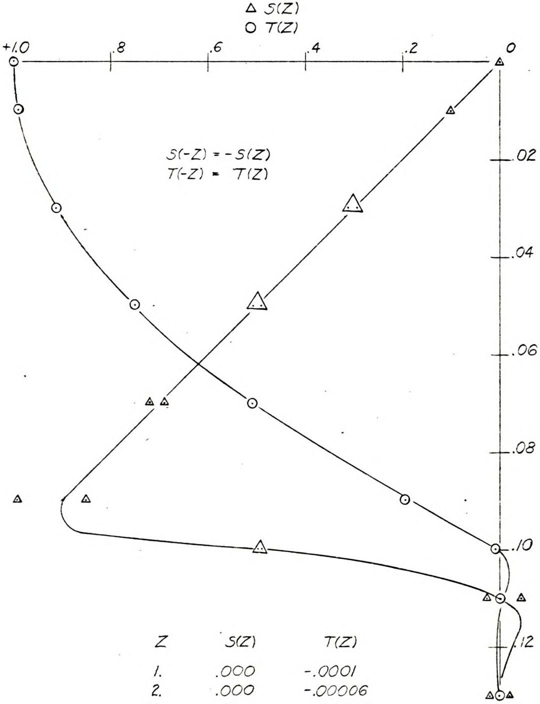
TABLE 2.--Results of the numerical integrations used in the illustrative example

b/a	ν	ws	wt
.1	.25	-.0116	-.14157
.1	.3	-.0096	-.13575
.05	.25	-.0060	-.0815
.05	.3	-.0049	-.0779
.02	.25	-.0025	-.0386
.02	.3	-.0020	-.0367

		uzs			uzt
		upper bound	value used	lower bound	
.1	.25	-.464	-.457	-.450	-.232
.1	.3	-.436	-.430	-.423	-.192
.05	.25	-.480	-.466	-.450	-.241
.05	.3	-.450	-.436	-.423	-.197
.02	.25	-.52	-.48	-.44	-.25
.02	.3	-.48	-.44	-.41	-.20

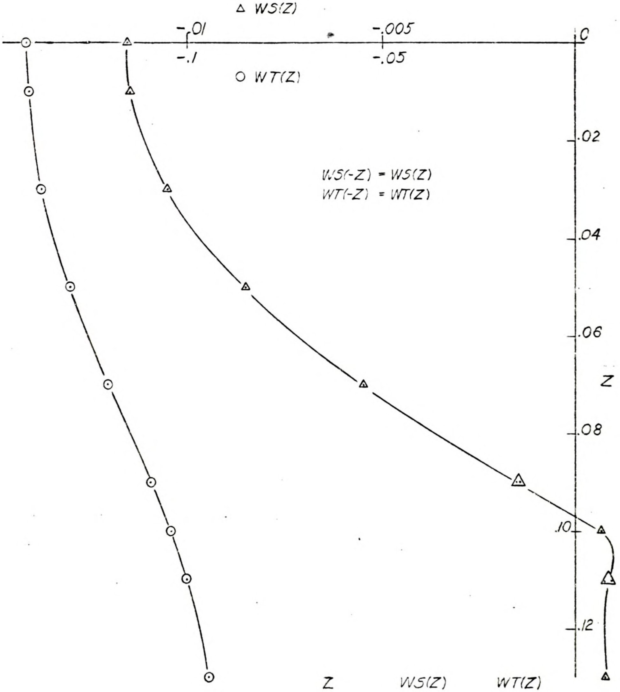
Values of the upper and lower bounds agree to the number of places shown unless otherwise indicated.

FIGURE 5



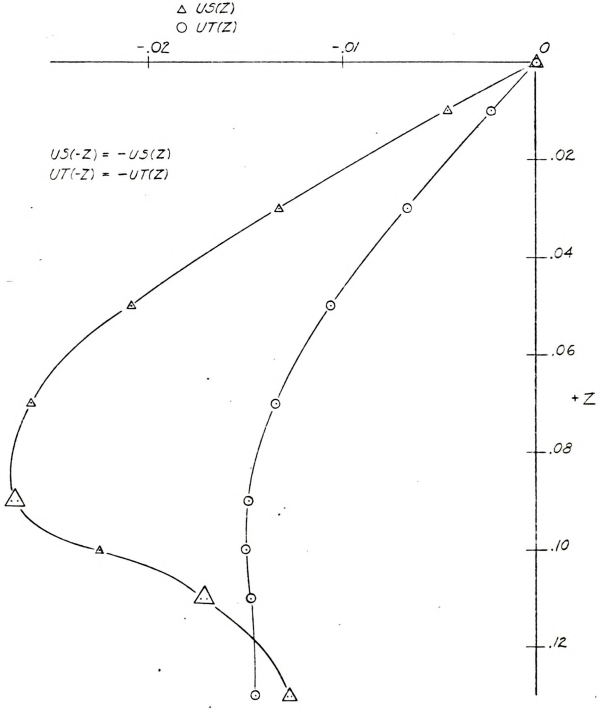
Z	S(Z)	T(Z)
1.	.000	-.0001
2.	.000	-.00006
10.	-.01	-.00023

FIGURE 6



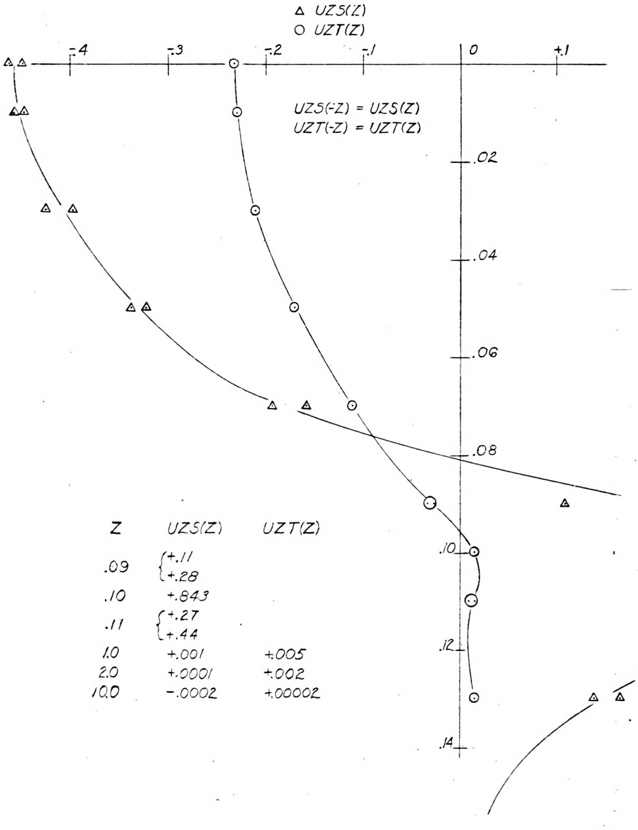
Z	WS(Z)	WT(Z)
1.	+.00012	-.0359
2.	+.0001	-.0226
10.	-.000001	-.0057

FIGURE 7



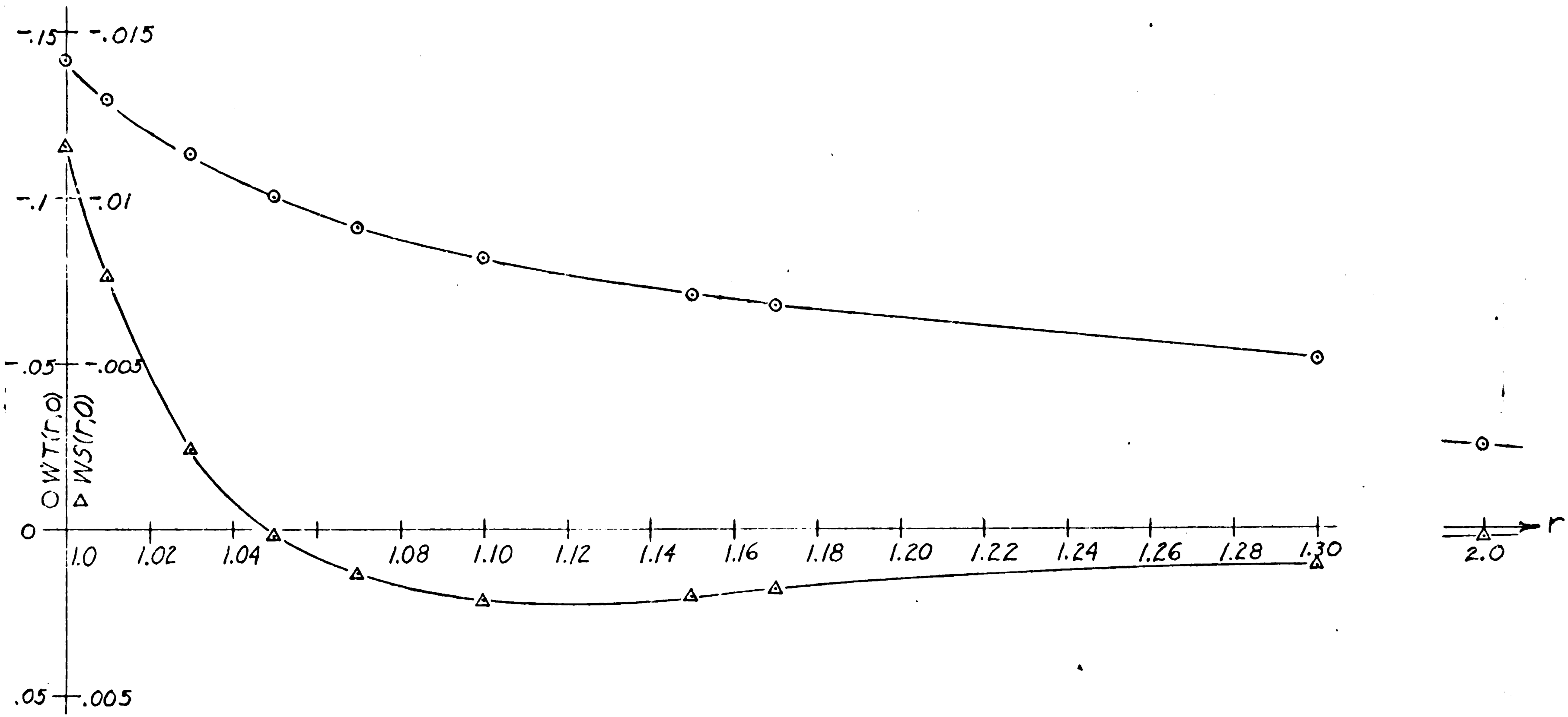
Z	US(Z)	UT(Z)
1.	-.00058	-.00677
2.	-.00011	-.0035
10.	+.0003	-.00023

FIGURE 8



Z	UZS(Z)	UZT(Z)
.09	{+.11, +.28}	
.10	+.843	
.11	{+.27, +.44}	
1.0	+.001	+.005
2.0	+.0001	+.002
10.0	-.0002	+.00002

FIGURE 9


-.15
-.015
-.1
-.01
-.05
-.005
0
.05
.005
○ WT(r,0)
△ WS(r,0)
1.0
1.02
1.04
1.08
1.10
1.12
1.14
1.16
1.18
1.20
1.22
1.24
1.26
1.28
1.30
2.0
r

A P P E N D I X   II

.3A NUMERICAL INTEGRATION FORMULA I-3 COMPUTES EQUATIONS 3.11 FOR $HP \geq X > 0$ 11/06/67

Poisson's ratio

plate thickness initial values

plate radius

partition of the half period

half period range

half period start

```
      PROGRAM    S T W U UZ
C     FRAMES  7 PT
      DIMENSION A(1000)
      READ 100,HPS,HPR,PHP,AR,BT.  S,T,WS,WT,U S,U T,UZS,UZT,  PR,Z
  100 FORMAT(5F10.0)
      R=BT/AR
      C=2.-2.*PR
      PI=3.14159265358
      X=HPS*PI/R
      H=PI/(R*PHP)
      K=HPR*PHP
      H06=H/6.
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S0=S
      T0=T
      WS0=WS
      WT0=WT
      U S0=U S
      U T0=U T
      UZS0=UZS
      UZT0=UZT
      DO 900 N=1,K
      X=X-H06
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S1=S1+S
      T1=T1+T
      WS1=WS1+WS
      WT1=WT1+WT
      U S1=U S1+U S
      U T1=U T1+U T
      UZS1=UZS1+UZS
      UZT1=UZT1+UZT
      X=X-H06
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S2=S2+S
      T2=T2+T
      WS2=WS2+WS
      WT2=WT2+WT
      U S2=U S2+U S
      U T2=U T2+U T
      UZS2=UZS2+UZS
      UZT2=UZT2+UZT
      X=X-H06
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S3=S3+S
      T3=T3+T
      WS3=WS3+WS
      WT3=WT3+WT
      U S3=U S3+U S
      U T3=U T3+U T
      UZS3=UZS3+UZS
      UZT3=UZT3+UZT
      X=X-H06
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S2=S2+S
      T2=T2+T
```

```
      WS2=WS2+WS
      WT2=WT2+WT
      U S2=U S2+U S
      U T2=U T2+U T
      UZS2=UZS2+UZS
      UZT2=UZT2+UZT
      X=X-H06
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S1=S1+S
      T1=T1+T
      WS1=WS1+WS
      WT1=WT1+WT
      U S1=U S1+U S
      U T1=U T1+U T
      UZS1=UZS1+UZS
      UZT1=UZT1+UZT
      X=X-H06
      CALL        FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S4=S4+S
      T4=T4+T
      WS4=WS4+WS
      WT4=WT4+WT
      U S4=U S4+U S
      U T4=U T4+U T
      UZS4=UZS4+UZS
  900 UZT4=UZT4+UZT
      S1=S1*15.
      T1=T1*15.
      WS1=WS1*15.
      WT1=WT1*15.
      U S1=U S1*15.
      U T1=U T1*15.
      UZS1=UZS1*15.
      UZT1=UZT1*15.
        S2=(  S2+  S0)*3.
        T2=(  T2+  T0)*3.
       WS2=( WS2+ WS0)*3.
       WT2=( WT2+ WT0)*3.
      U S2=(U S2+U S0)*3.
      U T2=(U T2+U T0)*3.
      UZS2=(UZS2+UZS0)*3.
      UZT2=(UZT2+UZT0)*3.
      S3=S3*18.
      T3=T3*18.
      WS3=WS3*18.
      WT3=WT3*18.
      U S3=U S3*18.
      U T3=U T3*18.
      UZS3=UZS3*18.
      UZT3=UZT3*18.
        S4=  S4*6.-3.*  S
        T4=  T4*6.-3.*  T
       WS4= WS4*6.-3.* WS
       WT4= WT4*6.-3.* WT
      U S4=U S4*6.-3.*U S
      U T4=U T4*6.-3.*U T
      UZS4=UZS4*6.-3.*UZS
      UZT4=UZT4*6.-3.*UZT
      F=H06/(PI*10.)
      F1=2.*F
      F8=F1/R
      F2=2.*F8
      F3=AR*F
      F4=AR*F8
      F5=C*F3
      F6=F4
      F7=C*F
         S=F1*( S1+  S2+  S3+  S4)
         T=F2*( T1+  T2+  T3+  T4)
        WS=F3*(WS1+ WS2+ WS3+ WS4)
        WT=F4*(WT1+ WT2+ WT3+ WT4)
      U S=F5*(U S1+U S2+U S3+U S4)
      U T=F6*(U T1+U T2+U T3+U T4)
      UZS=F7*(UZS1+UZS2+UZS3+UZS4)
      UZT=F8*(UZT1+UZT2+UZT3+UZT4)
```

```
      PRINT 600, N,X,    S,T,WS,WT,U S,U T,UZS,UZT,    M
  600 FORMAT(3H0N=1I2,3X,2HX=1F10,6,3X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1F10,6,2X,
     C1I2)
      END
```

N5,3A 11/06/67

```
      SUBROUTINE FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      DIMENSION A(1000)
      RX=R*X
      X2=X*X
      X3=X*X2
      X4=X*X3
      ZXOA=Z*X/AR
      CZ=COSF(ZXOA)
      SZ=SINF(ZXOA)
      FS=SINF(RX)/RX-COSF(RX)
      V=0.
      N=1
      CALL BESK(X,V,N,A,M)
      FK=A(1)/A(2)
      FA=1.+X*FK
      FB=X2*(FK**2-1.)-C
      CAOB=C*FA/FB
      SCZ=FS*CZ
      SSZ=FS*SZ
      S = S SZ/X
      T =SCZ/X2
      WS =T  *(1.+CAOB)
      WT =SCZ*(1.+CAOB*FA)/X4
      US=S/FB
      UT=SSZ*(1.+CAOB)/X3
      UZS =  SCZ/FB
      UZT =WS
      RETURN
      END
```

D
*BINARY DECK*** Binary Deck → BESK(X,V,N,A,M)
N,5,00,900,7,M

FTN5.3A *NUMERICAL INTEGRATION FORMULA I-2 COMPUTES* 11/06/67

```
      PROGRAM B S T W U UZ     EQUATIONS 3.11 FOR  HP≤X<∞
C     FRAMES  7 PT
      DIMENSION A(1000)
      READ 100,HPS,HPR,PHP,AR,BT,  S,T,WS,WT,U S,U T,UZS,UZT,  PR,Z
  100 FORMAT(5F10.0)
      R=BT/AR
      C=2.-2.*PR
      PI=3.14159265358
      X=HPS*PI/R
      H=PI/(R*PHP)
      K=HPR*PHP
      H06=H/6.
      F=H06/(PI*10.)
      F1=2.*F
      F8=F1/R
      F2=2.*F8
      F3=AR*F
      F4=AR*F8
      F5=C*F3
      F6=F4
      F7=C*F
      CALL      FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      DO 900 N=1,K
      S2=   S
      T2=   T
      WS2=    WS
      WT2=    WT
      U S2=     U S
      U T2=     U T
      UZS2=     UZS
      UZT2=     UZT
      X=X+H06
      CALL      FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S1=   S
      T1=   T
      WS1=    WS
      WT1=    WT
      U S1=     U S
      U T1=     U T
      UZS1=     UZS
      UZT1=     UZT
      X=X+H06
      CALL      FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S2=S2+S
      T2=T2+T
      WS2=WS2+WS
      WT2=WT2+WT
      U S2=U S2+U S
      U T2=U T2+U T
      UZS2=UZS2+UZS
      UZT2=UZT2+UZT
      X=X+H06
      CALL      FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S3=   S
      T3=   T
      WS3=    WS
```

```
      WT3=WT
      U S3=      U S
      U T3=      U T
      UZS3=      UZS
      UZT3=      UZT
      X=X+H06
      CALL       FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S2=S2+S
      T2=T2+T
      WS2=WS2+WS
      WT2=WT2+WT
      U S2=U S2+U S
      U T2=U T2+U T
      UZS2=UZS2+UZS
      UZT2=UZT2+UZT
      X=X+H06
      CALL       FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S1=S1+S
      T1=T1+T
      WS1=WS1+WS
      WT1=WT1+WT
      U S1=U S1+U S
      U T1=U T1+U T
      UZS1=UZS1+UZS
      UZT1=UZT1+UZT
      X=X+H06
      CALL       FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      S2=S2+S
      T2=T2+T
      WS2=WS2+WS
      WT2=WT2+WT
      U S2=U S2+U S
      U T2=U T2+U T
      UZS2=UZS2+UZS
      UZT2=UZT2+UZT
      S1=S1*15.
      T1=T1*15.
      WS1=WS1*15.
      WT1=WT1*15.
      U S1=U S1*15.
      U T1=U T1*15.
      UZS1=UZS1*15.
      UZT1=UZT1*15.
        S2=(  S2        )*3.
        T2=(  T2        )*3.
       WS2=( WS2        )*3.
       WT2=( WT2        )*3.
      U S2=(U S2        )*3.
      U T2=(U T2        )*3.
      UZS2=(UZS2        )*3.
      UZT2=(UZT2        )*3.
      S3=S3*18.
      T3=T3*18.
      WS3=WS3*18.
      WT3=WT3*18.
      U S3=U S3*18.
```

```
      U T3=U T3*18.
      UZS3=UZS3*18.
      UZT3=UZT3*18.
      P S=F1*( S1+  S2+  S3)+PS
      P T=F2*( T1+  T2+  T3)+PT
      PWS=F3*(WS1+ WS2+ WS3)+PWS
      PWT=F4*(WT1+ WT2+ WT3)+PWT
      PUS=F5*(U S1+U S2+U S3)+PUS
      PUT=F6*(U T1+U T2+U T3)+PUT
      PUZS=  (UZS1+UZS2+UZS3)*F7+PUZS
      PUZT=  (UZT1+UZT2+UZT3)*F8+PUZT
  900 PRINT 600,N,X,  PS,PT,PWS, PWT,PUS,PUT,PUZS,PUZT,   M
  600 FORMAT(3HON=1I2,3X,2HX=1F10.6,3X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1I2)
      END
```



```
      SUBROUTINE FNC (C,R,AR, Z,  H,X,  S,T,WS,WT,U S,U T,UZS,UZT,  A,M)
      DIMENSION A(1000)
      RX=R*X
      X2=X*X
      X3=X*X2
      X4=X*X3
      ZXOA=Z*X/AR
      CZ=COSF(ZXOA)
      SZ=SINF(ZXOA)
      FS=SINF(RX)/RX-COSF(RX)
      V=0.
      N=1
      CALL BESK(X,V,N,A,M)
      FK=A(1)/A(2)
      FA=1.+X*FK
      FB=X2*(FK**2-1.)-C
      CAOB=C*FA/FB
      SCZ=FS*CZ
      SSZ=FS*SZ
      S = S SZ/X
      T =SCZ/X2
      WS =T  *(1.+CAOB)
      WT =SCZ*(1.+CAOB*FA)/X4
      US=S/FB
      UT=SSZ*(1.+CAOB)/X3
      UZS =  SCZ/FB
      UZT =WS
      RETURN
      END
LOAD
***BINARY DECK***
RUN,1,30,900,7,M
```

Binary Deck → BESK(X, V, N, A, M)

FTN5.3A *NUMERICAL INTEGRATION FORMULA I-3* 01/09/68

```
      PROGRAM WAFORAZ   COMPUTES THE 1st OF 3.10
C     FRAMES  7 PT
      DIMENSION A(1000)                 FOR  HF ≥ X > 0
      READ 100,HPS,HPR,PHP,AR,BT,       WS,WT,                    PR,Z,R
  100 FORMAT(5F10.0)
      C=2.-2.*PR
      PI=3.14159265358
      X=HPS*PI/BT
      H=PI/(BT*PHP)
      K=HPR*PHP
      HO6=H/6.
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS0=WS
      WT0=WT
      DO 900 N=1,K
      X=X-HO6
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS1=WS1+WS
      WT1=WT1+WT
      X=X-HO6
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS2=WS2+WS
      WT2=WT2+WT
      X=X-HO6
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS3=WS3+WS
      WT3=WT3+WT
      X=X-HO6
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS2=WS2+WS
      WT2=WT2+WT
      X=X-HO6
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS1=WS1+WS
      WT1=WT1+WT
      X=X-HO6
      CALL        FNC (C,R,AR,BT,Z,    X,    WS,WT,      A,M)
      WS4=WS4+WS
  900 WT4=WT4+WT
      WS1=WS1*15.
      WT1=WT1*15.
       WS2=( WS2+ WS0)*3.
       WT2=( WT2+ WT0)*3.
      WS3=WS3*18.
      WT3=WT3*18.
       WS4= WS4*6.-3.* WS
       WT4= WT4*6.-3.* WT
      F=HO6/(PI*10.)
      F3=AR*F
      F4=2.*F/BT
       WS=F3*(WS1+ WS2+ WS3+ WS4)
       WT=F4*(WT1+ WT2+ WT3+ WT4)
      PRINT 600, N,X,  WS,WT,     M
  600 FORMAT(3H0N=1I2,3X,2HX=1F10.6,3X,
     C1F10.6,2X,
     C1F10.5,2X,
     C1I2)
      END
```

```
      SUBROUTINE FNC (C,R,AR,BT,Z,   X,   WS,WT,     A,M)
      DIMENSION A(1000)
      RX=R*X
      X2=X*X
      AX=AR*X
      AX2=AX*AX
      BX=BT*X
      ZX=Z*X
      CZ=COSF(ZX  )
      FS=SINF(BX)/BX-COSF(BX)
      V=0.
      N=1
      CALL BESK(AX,V,N,A,M)
      AK=A(1)/A(2)
      A2=A(2)
      AXK=AX*AK
      AA=AXK+C
      AB=AX2+(1.-C)*AXK
      AC=1.+AXK
      AD=AX2+(AK**2-1.)-C
      CALL BESK(RX,V,N,A,M)
      RK=A(1)/A(2)
      R2=A(2)
      RAK=(R2/A2)*(EXPF(AX)/EXPF(RX))
      RC=RX-2.*C+RK
      SCX=FS*CZ*RAK/(X*AD)
      SCX3=SCX/X2
      WS=SCX*(AA*RK-RX)
      WT=SCX3*(AB*RK-AC*RC)
      RETURN
      END
LOAD
***BINARY DECK***
RUN,5.00,900,7,M
```

Binary Deck → BESK

```
      PROGRAM WAFORAZ  COMPUTES THE 1st OF 3.10
C     FRAMES  7 PT
      DIMENSION A(1000)              FOR  HP≤X<∞
      READ 100,HPS,HPR,PHP,AR,BT,       WS,WT,                PR,Z,R
  100 FORMAT(5F10.0)
      C=2.-2.*PR
      PI=3.14159265358
      X=HPS*PI/BT
      H=PI/(BT*PHP)
      K=HPR*PHP
      HO6=H/6.
      F=HO6/(PI*10.)
      F3=AR*F
      F4=2.*F/BT
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      DO 900 N=1,K
      WS2=WS
      WT2=WT
      X=X+HO6
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      WS1=WS
      WT1=WT
      X=X+HO6
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      WS2=WS2+WS
      WT2=WT2+WT
      X=X+HO6
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      WS3=WS
      WT3=WT
      X=X+HO6
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      WS2=WS2+WS
      WT2=WT2+WT
      X=X+HO6
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      WS1=WS1+WS
      WT1=WT1+WT
      X=X+HO6
      CALL        FNC (C,R,AR,BT,Z,   X.   WS,WT,      A,M)
      WS2=WS2+WS
      WT2=WT2+WT
      WS1=WS1*15.
      WT1=WT1*15.
      WS2=WS2*3.
      WT2=WT2*3.
      WS3=WS3*18.
      WT3=WT3*18.
      PWS=F3*(WS1+ WS2+ WS3)+PWS
      PWT=F4*(WT1+ WT2+ WT3)+PWT
  900 PRINT 600, N,X, PWS,PWT,    M
  600 FORMAT(3H0N=1I2,3X,2HX=1F10.6,3X,
     C1F10.6,2X,
     C1F10.6,2X,
     C1I2)
      END
```

Subroutine FNC on page 61

FTN5.3A COMPUTATION OF W(R) AND W(R,O) 03/19/68

```
      PROGRAM WAFORGA
      DIMENSION   WSGA(100),WTGA(100),RGA(100)
      READ 100,A,B,          Q  ,      WS,WT,    UZS,UZT,  AJ,AK,AI
  100 FORMAT(5F10.0)
      J=AJ
      K=AK
      I=AI
      DO  90 N=1,I
   90 READ 100,RGA(N),WSGA(N),WTGA(N)
      DO 901 M=1,J
      READ 100,PRW,PRP,EW,EP
      POW=EP/EW
      T=-3,*A*Q/(8,*B)
      A2=A*A
      A3=A*A2
      A4=A*A3
      B2=B*B
      B3=B*B2
      WAA=Q*A2/4,
      WA =Q*A3/16,
      W  =Q*A4/64,
      G=EW/(2,*2,*PRW)
      D=EP*2,*B3/(3,-3,*(PRP**2))
      DG=D/G
      P1=1,+PRP
      DGP1=DG*P1
      UZTT=UZT*T
      C=2,*A*B3/3,
      S=(2,*WA-A*WAA+DGP1*UZTT)/(C-DGP1*UZS)
      CA=(WA+DG*(S*UZS+UZTT))/(-A)
      CC=DG*(S*WS+T*WT)-A2*CA/2,-W
      PRINT 600,POW ,PRW,PRP,S,T,A,B
  600 FORMAT(
     C1F10,4,2X,
     C1F10,4,2X,
     C1F10,4,2X,
     C1F10,0,2X,
     C1F10,4,2X,
     C1F10,4,2X,
     C1F10,4)
      R=0,
      H=A/(AK-1,)
      DO 900 N=1,K
      R2=R*R
      R3=R*R2
      R4=R*R3
      DW    =Q*R4/64,+CA*R2/2,+CC
      PRINT 600,R,DW
  900 R=R+H
      DO 901 N=1,I
      DW=DG*(S*WSGA(N)+T*WTGA(N))
  901 PRINT 600,RGA(N),DW
      END
RUN, .30,900,7,M
```