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THE STRESS ANALYSIS OF
HELICAL ROUND-WIRE SPRINGS

Thesis for the Degree of M. S.
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Chi-Chuan Chang
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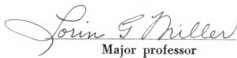
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THE STRESS ANALYSIS OF
HELICAL ROUND-WIRE SPRINGS

By

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THESIS

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List of Symbols

Symbol	Definition	Units
a ==	radius of the spring wire	inch
R ==	mean radius of the spring coil	inch
c ==	spring index ($c=R/a$)	----
p ==	load	lb
M_t ==	twisting moment ($M=PR$)	in-lb
ρ, θ, z ==	cylindrical coordinates	----
$S_{\theta\phi}$ ==	shearing stress on the plane perpendicular to θ -axis and along the direction of ϕ -axis	lb/in ²
S_s' ==	endurance limit in shear	lb/in ²
S_y' ==	yield point stress in torsion	lb/in ²
G ==	correction factor of exact theory	----
K ==	Wahl's correction factor	----
K_s ==	stress multiplication factor	----
K_c ==	stress concentration factor due to curvature	----
N ==	factor of safety	----

INTRODUCTION

Springs of most importance in machine design are helical round-wire type. They are made in a wide variety of sizes and used in tremendous quantities. Here we shall discuss the theories for stress calculation in this type of spring.

For calculating helical springs, the elementary theory as commonly given in textbooks on strength of materials or machine design is based on the assumption that the spring may be considered essentially as a straight bar under torsion. This assumption is only approximately true where the spring index is large and where the helix angle is small. Since the elementary theory does not take into account the direct shear stress and the difference in fiber length between the inside and outside of the coil which arises because of the curvature of the spring wire, considerable error will be involved if this theory is used for springs with small or moderate indexes.

In order to take care of the effects both due to the curvature of the spring wire and the direct shear stress, A. M. Wahl has introduced an approximate theory which is widely used. After Mr. Wahl's work is reviewed we shall discuss the exact theory originally developed by O. Goehner. By the theory of elasticity we can set up a set of differential equations and try to solve them under the initial and boundary conditions. Then its result is compared with that of approximate theory.

Finally the methods of evaluating working stresses are discussed, and the limitations of the formula derived are specified.

APPROXIMATE THEORY

For a stress analysis, consider a quadrant of a coil as a free body, as shown in Fig. 1. The load is assumed to act along the axis of the spring, in which case it produces a torsional moment $M_t = PR$ in the body of the wire. This same torsional moment exists in all sections of the loaded spring.

In elementary theory, the spring is considered as a straight bar under torsion. It follows that the shearing deformations and hence the shearing stress will have a linear distribution along a radius as shown in Fig. 2. The torsional resisting moment is $SJ/c = S\pi a^3/2$. Equating the applied moment to the resisting moment, we get

$$M_t = PR = S\pi a^3/2$$

solving this equation for S, we find

$$(1) \quad S = \frac{2PR}{\pi a^3}$$

Where R is the mean radius of coil, a is radius of wire, P is axial load applied.

As stated before, the stress calculated by Eq. (1) will be in considerable error for springs with small indexes for two reasons:

1. The increase in stress due to the difference in fiber length between the inside of the coil and the outside produced by wire curvature is not considered.
2. The effect of direct shear stress due to the axial load P is neglected.

In Fig. 3b, if the radial section bb' and aa' rotate through a small angle with respect to each other and about the bar axis, the inside (and shorter) fiber $a'b'$ will be subject to a much higher shearing stress because of its short length than the outside fiber ab which is longer. Moreover, the stress on the inside fiber $a'b'$ is increased because the shear stress due to the direct axial load P is added to that due to the torque moment PR at this point. In the outside fiber ab this stress is subtracted from that due to the torque moment. The result is that the stresses on the inside of the coil reach values around 2.5 times those on the outside for springs of index 3, as may be shown both by test and theory.

Considering these two effects, Wahl's approximate solution⁽¹⁾ of the problem of determining stress in springs of small index, which is sufficiently accurate for practical use can be derived as follows:

A small helix angle is assumed since this assumption is valid for nearly all practical springs. Considering an element of an axially-loaded spring with mean radius of curvature R cut by two neighboring radial cross sections aa' and bb' as shown in Fig. 4a, the forces acting on this element are resolved into a twisting moment $M_t = PR$ acting in a radial plane and a direct axial shearing force P . The stresses set up by this twisting moment are

first considered and later are superimposed on the stresses due to the direct shear load.

The shear stress S acting over small cross section area dA in Fig. 4b may be resolved into two components: $S_{\theta z}$ parallel to the axis of the spring and $S_{\theta \phi}$ perpendicular to the spring axis.

If it is assumed that the two neighboring cross sections aa' and bb' rotate relative to each other and about an axis ee' perpendicular to their surfaces and passing through their centers O , the distribution of the axial components of the stress ($S_{\theta z}$) along a transverse diameter perpendicular to the spring axis will be somewhat as shown by the shaded area of Fig. 5. Such a distribution of stress due only to a moment would not be possible since the area to the right of the center O is greater than that to the left; hence an external force would be needed to secure equilibrium. If, however, rotation occurs about some point O' , Fig. 6, which is displaced toward the axis of the spring, instead of about point O , a distribution of stress is obtained which is possible under the action of a pure moment M_t . From conditions of symmetry the transverse stress components $S_{\theta \phi}$ will be in statical equilibrium when rotation occurs about any point on the axis aa' , Fig. 4b.

Now we try to find the position of O' . Under the assumption of rotation about O' , the stress S acting on any element dA with coordinates x and y may be found. When the sections aa' and bb'

have rotated through a small angle $d\beta$ with respect to each other, the relative movement of the ends of filament dd' corresponding to dA will be $d\beta(x^2+y^2)^{1/2}$ and, since the length of dd' is $(R-r-x)d\theta$ the shearing stress S acting on this element will be

$$(2) \quad S = \frac{G d\beta \sqrt{x^2+y^2}}{(R-r-x)d\theta}$$

The axial component $S_{\theta z}$ of the stress will be

$$(3) \quad S_{\theta z} = \frac{Sx}{\sqrt{x^2+y^2}} = \frac{xGd\beta}{(R-r-x)d\theta}$$

Under the assumptions made, this distribution of stress is identical with that in a curved bar and the distribution of the stresses $S_{\theta z}$ is hyperbolic in form, Fig. 6. From curved bar theory (2) the distance y may be expressed as:

$$(4) \quad y = \frac{a^2}{4R} \left(\frac{1}{1 + \frac{a^2}{4R^2}} \right) \approx \frac{a^2}{4R}$$

The term $a^2/4R^2$ is neglected in the denominator since in practical springs a/R seldom greater than $1/3$ and hence $a^2/4R^2$ is small compared to unity. Putting y in (3),

$$(5) \quad S_{\theta z} = \frac{xGd\beta}{(R - \frac{a^2}{4R} - x)d\theta}$$

Further, it is assumed that the ordinary formula for angle of twist of circular bars will apply with sufficient accuracy for the calculation of $d\beta/d\theta$ (3). Thus

$$(6) \quad d\beta/d\theta = \frac{2MxR}{\pi a^4 G}$$

Where $M_t = PR$ Putting this in (5),

$$(7) \quad S_{\theta z} = \frac{2x(PR)R}{\pi a^3 (A - \frac{a^2}{R} - x)}$$

From this equation it is clear that the maximum value of $S_{\theta z}$ will occur when $x = A - \frac{a^2}{R}$ i.e. at point a'. Putting this value in (7) and let the spring index $R/a = c$, the stress at a' in Fig. 6 becomes

$$(8) \quad S_{a'} = \frac{2PR}{\pi a^3} \left(\frac{4c-1}{4c-4} \right)$$

As to the effect of direct shear stress due to axial load P, it appears reasonable to take for this stress that given by the theory of elasticity at the outer edges of the neutral surface of a cantilever of circular cross section loaded by a force P. This theory ⁽⁴⁾ gives a value of stress equal to $4.92P/\pi(2a)^2$ ($= \frac{2PR}{\pi a^3} \cdot \frac{2.46}{4} \cdot \frac{a}{R} = \frac{2PR}{\pi a^3} \cdot \frac{.615}{c}$). Adding this stress to that due to the twist moment PR from equation (8) the maximum stress S_{max} at a' will be expressed

$$(9) \quad S_{max} = \frac{2PR}{\pi a^3} \left(\frac{4c-1}{4c-4} + \frac{.615}{c} \right)$$

This may be written in the form

$$(10) \quad S_{max} = \frac{2PR}{\pi a^3} K$$

Where K known as Wahl's correction factor is

$$(11) \quad K = \frac{4c-1}{4c-4} + \frac{.615}{c}$$

EXACT THEORY

The approximate theory developed in the preceding section for calculating stress in helical springs of small or moderate index is sufficiently accurate for most practical purposes.

Where greater accuracy is desired, the exact method of calculation (5) developed from the theory of elasticity may be used.

If the helix angle is small, the element of the spring may be considered under pure torsion, and in cylindrical coordinates (ρ, θ, z ; Fig. 7a) All shear stress components except $S_{\theta z}$ and $S_{\phi \rho}$ may be assumed zero. Now, let us consider the equilibrium of the small element abcd a'b'c'd' as shown in Fig. 7b.

Shear force on bb'c'c is $S_{\rho\theta} [\rho d\theta \cdot dz]$

Shear force on aa'd'd is $[S_{\rho\theta} + \frac{\partial S_{\rho\theta}}{\partial \rho} d\rho] [(\rho + d\rho) d\theta \cdot dz]$

Shear force on aa'b'b is $S_{z\theta} [\frac{(\rho + d\rho)^2 d\theta}{2} - \frac{\rho^2 d\theta}{2}]$

Shear force on dd'c'c is $[S_{z\theta} + \frac{\partial S_{z\theta}}{\partial z} dz] [\frac{(\rho + d\rho)^2 d\theta}{2} - \frac{\rho^2 d\theta}{2}]$

Taking moment about z axis, we get

$$[S_{\rho\theta} + \frac{\partial S_{\rho\theta}}{\partial \rho} d\rho] [(\rho + d\rho) d\theta \cdot dz] (\rho + d\rho) - S_{\rho\theta} [\rho d\theta \cdot dz] \cdot \rho +$$

$$[S_{z\theta} + \frac{\partial S_{z\theta}}{\partial z} dz] [\frac{(\rho + d\rho)^2 d\theta}{2} - \frac{\rho^2 d\theta}{2}] [\rho + \frac{d\rho}{2}] - S_{z\theta} [\frac{(\rho + d\rho)^2 d\theta}{2} - \frac{\rho^2 d\theta}{2}] [\rho + \frac{d\rho}{2}] = 0$$

neglecting the terms involving $(d\rho)^2$ and simplifying, we have

$$\rho^2 \frac{\partial S_{\rho\theta}}{\partial \rho} d\rho d\theta dz + S_{\rho\theta} (2\rho) d\rho d\theta dz + \rho^2 \frac{\partial S_{z\theta}}{\partial z} d\rho d\theta dz = 0$$

Dividing by $\rho^2 d\rho d\theta dz$, and putting $S_{\rho\theta} = S_{\phi\rho}$, $S_{z\theta} = S_{\phi z}$ finally we have

$$(12) \quad \frac{\partial S_{\phi\rho}}{\partial \rho} + \frac{\partial S_{\phi z}}{\partial z} + \frac{2S_{\phi\rho}}{\rho} = 0$$

Also this equation may be written as

$$(13) \quad \frac{\partial}{\partial \rho} (\rho^2 S_{\theta\rho}) + \frac{\partial}{\partial z} (\rho^2 S_{\theta z}) = 0$$

Besides this equation of equilibrium, there are certain relations between components of strain which by Hooke's law can be changed into a set of equations involving stress components. These are known as compatibility equations⁽⁶⁾ in the theory of elasticity. Under our circumstances these are

$$(14) \quad \begin{cases} \frac{\partial^2 S_{\theta z}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial S_{\theta z}}{\partial \rho} + \frac{\partial^2 S_{\theta z}}{\partial z^2} - \frac{S_{\theta z}}{\rho^2} = 0 \\ \frac{\partial^2 S_{\theta\rho}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial S_{\theta\rho}}{\partial \rho} + \frac{\partial^2 S_{\theta\rho}}{\partial z^2} - \frac{1}{\rho^2} S_{\theta\rho} = 0 \end{cases}$$

Thus the problem reduces to the solution of Eqs. (12) and (14). Knowing from equation (13), we satisfy equation (12) by taking

$$(15) \quad S_{\theta\rho} = \frac{R^2}{\rho^2} \cdot \frac{\partial \phi}{\partial z} \quad , \quad S_{\theta z} = - \frac{R^2}{\rho^2} \cdot \frac{\partial \phi}{\partial \rho}$$

where ϕ is a stress function introduced and R the mean radius of the spring coil. Substituting (15) in Eqs. (14), we find

$$\frac{\partial}{\partial \rho} \left(\frac{\partial^2 \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{\rho} \frac{\partial \phi}{\partial \rho} \right) = 0$$

$$\frac{\partial}{\partial z} \left(\frac{\partial^2 \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{\rho} \frac{\partial \phi}{\partial \rho} \right) = 0$$

from which we conclude that the expression in the parentheses must be a constant. Denoting this constant by $-2b$, the equation for determining the stress function ϕ is

$$(16) \quad \frac{\partial^2 \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{\rho} \frac{\partial \phi}{\partial \rho} + 2b = 0$$

Let us consider now the boundary conditions for the function ϕ . From the condition that the lateral surface of the spring wire is free from external forces we conclude that at any point at the boundary (Fig. 8) the total shearing stress must be in the direction of the tangent to the boundary and its projection on the normal N to the boundary must be zero. Hence

$$S_{\theta\theta} \frac{dz}{ds} - S_{\theta z} \frac{d\rho}{ds} = 0$$

where ds is an element of the boundary. Substituting from eqs. (15), we find that

$$(17) \quad \frac{R^2}{\rho^2} \left(\frac{\partial \phi}{\partial z} \cdot \frac{dz}{ds} + \frac{\partial \phi}{\partial \rho} \cdot \frac{d\rho}{ds} \right) = 0$$

This shows that ϕ must be constant at the boundary.

It will be found advantageous to introduce new coordinates as follows (Fig. 8)

$$(18) \quad x = R - \rho, \quad z = z,$$

Referring to Fig. 8, we know that the total twisting moment acting over the spring cross section will be

$$(19) \quad -PR = \iint (S_{\theta\rho} \cdot z + S_{\theta z} \cdot x) dK dz$$

Up to this point our problem is reduced to how to solve equation (16) and ϕ satisfies both the boundary condition (17) and initial condition (19). Generally partial differential equations like eq. (16) may be solved by the separable variable method. Before doing that let us put

$$(20) \quad \phi = \psi + f(z)$$

into eq. (16), we get

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{3}{\rho} \frac{\partial \psi}{\partial \rho} + f''(z) + 2b = 0$$

If $f''(z) + 2b = 0$ then we have

$$(21) \quad f(z) = -bz^2 + cz + d$$

$$(22) \quad \frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{3}{\rho} \frac{\partial \psi}{\partial \rho} = 0$$

where c and d are arbitrary constants.

In order to solve eq. (22), let us assume

$$(23) \quad \psi = z(z) \cdot P(\rho)$$

then eq. (22) becomes

$$zP'' + z''P - \frac{3}{\rho}Pz = 0$$

$$\text{or } \frac{P''}{P} - \frac{3}{\rho} \frac{P'}{P} = -\frac{z''}{z} = \lambda^2$$

where λ^2 is constant, we have

$$(24) \quad z'' + \lambda^2 z = 0$$

$$(25) \quad P'' - \frac{3}{\rho} P' - \lambda^2 P = 0$$

The solution of eq. (24) is

$$(26) \quad z = C_1 \cos \lambda z + C_2 \sin \lambda z$$

And eq. (25) is a Bessel's equation, the first kind solution is

$$(27) \quad P = (i\lambda\rho)^2 J_2(i\lambda\rho)$$

By definition

$$(28) \quad J_2(i\lambda\rho) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!} \frac{(i\lambda\rho)^{2+2K}}{2^{2+2K} K! (2+K)!}$$

Putting (26) (27) into (23) and (23) (21) into (20) finally we have

$$(29) \quad \phi = (C_1 \cos \lambda z + C_2 \sin \lambda z) (i\lambda\rho)^2 J_2(i\lambda\rho) + bz^2 + cz + d$$

It is difficult to determine all the arbitrary constants C_1, C_2, λ ,

b, c, d of eq. (29) by both boundary and initial conditions in

general terms. Moreover, by this solution the stress will involve a power series of ρ . Where $\rho = R - x$ so it is no use in our problem practically. We had better try other method.

In new coordinate system $x = R - \rho$, $z = z$ eq. (16) becomes

$$(30) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{3}{R(1-\frac{x}{R})} \frac{\partial \phi}{\partial x} + 2b = 0$$

Since in general x/R may be considered small,

$$(31) \quad \frac{1}{1-\frac{x}{R}} = 1 + \frac{x}{R} + \frac{x^2}{R^2} + \frac{x^3}{R^3} + \dots$$

Then equation (30) becomes

$$(32) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{3}{R} \left(1 + \frac{x}{R} + \frac{x^2}{R^2} + \dots\right) \frac{\partial \phi}{\partial x} + 2b = 0$$

This makes it possible to solve equation (32) by means of a series of successive approximations. We assume

$$(33) \quad \phi = \phi_0 + \phi_1 + \phi_2 + \dots$$

and determine $\phi_0, \phi_1, \phi_2, \dots$ in such a manner as to satisfy the equations

$$(34) \quad \begin{cases} \frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial z^2} + 2b = 0 \\ \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} + \frac{3}{R} \frac{\partial \phi_0}{\partial x} = 0 \\ \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} + \frac{3}{R} \frac{\partial \phi_1}{\partial x} + \frac{3}{R} \cdot \frac{x}{R} \frac{\partial \phi_0}{\partial x} = 0 \\ \dots \end{cases}$$

then, as the number of terms in the series (33) increases, the

sum of eqs. (34) approaches more and more closely eq. (32), and the series (33) approaches the exact solution for the stress function ϕ .

In the new coordinate system, (15) change to

$$(35) \quad S_{\theta\theta} = \frac{1}{(1-\frac{x}{R})^2} \frac{\partial \phi}{\partial z} , \quad S_{zz} = \frac{1}{(1-\frac{x}{R})^2} \frac{\partial \phi}{\partial x}$$

using binomial expansions

$$(36) \quad \frac{1}{(1-\frac{x}{R})^2} = (1-\frac{x}{R})^{-2} = 1 + \frac{2x}{R} + \frac{3x^2}{R^2} + \frac{4x^3}{R^3} + \dots$$

we find as the first approximation:

$$(37) \quad (S_{\theta\theta})_0 = \frac{\partial \phi_0}{\partial z} , \quad (S_{zz})_0 = \frac{\partial \phi_0}{\partial x} ,$$

For the second approximation we find from eq. (35):

$$(38) \quad \begin{cases} (S_{\theta\theta})_1 = \left[\left(1 + \frac{2x}{R}\right) \frac{\partial \phi_0}{\partial z} + \frac{\partial \phi_1}{\partial z} \right] \\ (S_{zz})_1 = \left[\left(1 + \frac{2x}{R}\right) \frac{\partial \phi_0}{\partial x} + \frac{\partial \phi_1}{\partial x} \right] \end{cases}$$

For the third approximation:

$$(39) \quad \begin{aligned} (S_{\theta\theta})_2 &= \left[\left(1 + \frac{2x}{R} + \frac{3x^2}{R^2}\right) \frac{\partial \phi_0}{\partial z} + \left(1 + \frac{2x}{R}\right) \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z} \right] \\ (S_{zz})_2 &= \left[\left(1 + \frac{2x}{R} + \frac{3x^2}{R^2}\right) \frac{\partial \phi_0}{\partial x} + \left(1 + \frac{2x}{R}\right) \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x} \right] \end{aligned}$$

If the radius of the spring wire is a , the equation of the boundary (Fig. 8) is

$$(40) \quad x^2 + z^2 - a^2 = 0$$

and the solution of the first of eqs. (34), satisfying the boundary

condition, is

$$(41) \quad \phi_0 = -\frac{b}{2}(x^2 + z^2 - a^2)$$

Substituting the above expression for ϕ_0 into the second equation of eqs. (34) we find

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} - \frac{3bx}{R} = 0$$

The solution of this equation, satisfying the condition that ϕ_1 vanishes at the boundary, is

$$(42) \quad \phi_1 = \frac{3}{8} \frac{bx}{R}(x^2 + z^2 - a^2)$$

Substituting (41) (42) in the third of Eqs. (34) we obtain

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} + \frac{3}{8} \frac{b}{R^2}(x^2 + z^2 - a^2) = 0$$

The solution of this equation satisfying the boundary condition is

$$(43) \quad \phi_2 = -\frac{b}{64R^2}(x^2 + 5z^2 - 15a^2)(x^2 + z^2 - a^2)$$

Substituting values of ϕ_0 , ϕ_1 , ϕ_2 of (41) (42) (43) in eq. (39) we can find the third approximation for the stress components

$$(44) \quad \begin{aligned} (S_{\theta\theta})_2 &= -b \left[z + \frac{5}{8} \frac{xz}{R} + \frac{x}{16R^2}(27x^2 + 5z^2 - 10a^2) \right] \\ (S_{\theta z})_2 &= -b \left[x + \frac{3}{8} \frac{x^2}{R} - \frac{3}{8R}(z^2 - a^2) + \frac{13}{16} \frac{x^3}{R^2} - \frac{9}{16} \frac{xz^2}{R^2} + \frac{1}{4} \frac{a^2 x}{R^2} \right] \end{aligned}$$

Substituting the results of (44) for the stress components into Equation (19) the corresponding torque is

$$(45) \quad PR = \frac{b\pi a^4}{2} \left(1 + \frac{3}{16} \frac{a^2}{R^2} \right)$$

From this expression we can determine constant b, and substituting it in eqs. (44), we find the stress components as functions of the applied torque PR. Along the horizontal diameter of the cross section of the spring (Fig. 8) $z=0$, $S_{\theta\theta}=0$ and from the second

of Eqs. (44), we find

$$(S_{\theta 2})_x = -b(x + \frac{7}{8} \frac{x^3}{R} + \frac{13}{16} \frac{x^5}{R^3} + \frac{3}{8} \frac{a^2}{R} + \frac{a^2 x}{4R^2})$$

For the inner point i , $x = a$ and we have

$$(S_{\theta 2})_i = -ba(1 + \frac{5}{4} \frac{a}{R} + \frac{17}{16} \frac{a^3}{R^3})$$

For the outer point o , $x = -a$, and

$$(S_{\theta 2})_o = ba(1 - \frac{5}{4} \frac{a}{R} + \frac{17}{16} \frac{a^3}{R^3})$$

Using (45) for b , the value of these stresses becomes

$$(S_{\theta 2})_i = -\frac{2PR}{\pi a^3} \cdot \frac{1 + \frac{5}{4} \frac{a}{R} + \frac{17}{16} \frac{a^3}{R^3}}{1 + \frac{2}{16} \frac{a^3}{R^3}} = -\frac{2PR}{\pi a^3} (1 + \frac{5}{4} \frac{a}{R} + \frac{7}{8} \frac{a^3}{R^3})$$

$$(S_{\theta 2})_o = \frac{2PR}{\pi a^3} (1 - \frac{5}{4} \frac{a}{R} + \frac{7}{8} \frac{a^3}{R^3})$$

Calculation of further approximations shows that the final expression for the greatest shearing stress can be put in the form⁽⁷⁾

$$(46) \quad (S_{\theta 2})_{max} = -\frac{2PR}{\pi a^3} \cdot \frac{\frac{1}{1-\frac{a}{R}} + \frac{1}{4} \frac{a}{R} + \frac{1}{16} (\frac{a}{R})^2}{1 + \frac{3}{16} \cdot \frac{(\frac{a}{R})^2}{1 - (\frac{a}{R})^2}}$$

Substituting spring index $C = R/a$ this becomes

$$(47) \quad (S_{\theta 2})_{max} = -\frac{2PR}{\pi a^3} \cdot \frac{(\frac{C}{C-1} + \frac{1}{4C} + \frac{1}{16C^2})}{(1 + \frac{3}{16} \frac{1}{C^2-1})}$$

This may be written in the form

$$(48) \quad (S_{\theta 2})_{max} = \frac{2PR}{\pi a^3} G$$

Where the correction factor G is

Call No.	Date
Copies	
Vols	
Order	
In	
Jud	
Fun	

Signature
Name
Address

Total _____

Total _____

These
Spring (Mech)
Sheng (Chi-Cheng)
The stress analysis
of helical spring

$$(49) \quad G = \frac{\frac{C}{C-1} + \frac{1}{4C} + \frac{1}{16C^2}}{1 + \frac{3}{16} \cdot \frac{1}{C^2-1}}$$

By the expressions (11) and (49) we can calculate the correction factors of the two theories and plot curves against spring index C. From both the Table I and the curves in Fig. 12, we can conclude that;

- (1) The value of K is larger than the corresponding value of G. So approximate theory is on the safe side.
- (2) K and G are so close that at C as small as 2 their difference is only 2%.
- (3) The approximate theory is simpler and accurate enough for the practical use.

WORKING STRESSES

For the purposes of working stress evaluation, spring applications may be grouped into two fundamental classes as follows:

(1) Statically-loaded helical spring; it may be defined as one subject to a constant load or to a load repeated but a relatively few times during the life of the spring. These include safety-valve springs and springs in mechanisms which operate only occasionally.

(2) Fatigue or repeated loading helical spring; for example, automotive valve springs are usually subject to a load (or stress) which varies from a minimum value to a maximum. Now we shall treat these two kinds of springs separately.

In the design of springs subject to static loading it is suggested⁽⁸⁾ that for the usual spring material which has some ductility, stress concentration effects such as those due to curvature may be neglected, since the localized peak stresses could be relieved by plastic flow as a consequence of the material ductility.

To calculate the stress in the spring under static-load condition by neglecting stress concentration due to wire curvature, the procedure is as follows: Assuming a helical spring of mean coil radius R , and wire radius a , under static axial load P , the torsion moment at any radial section of the spring will be equal to PR while the direct shear will be equal to P . In neglecting curvature effect of wire the shear stress S , due to moment PR alone will be

that given by the usual formula. Thus

$$S_1 = \frac{2PR}{\pi a^3}$$

On this stress must be superimposed the shear stress S_2 due to the direct shear load P , which for our purposes may be considered uniformly distributed because of neglecting stress concentration and is

$$S_2 = \frac{P}{\pi a^2} = \frac{2PR}{\pi a^3} \cdot \frac{a}{2R} = \frac{2PR}{\pi a^3} \cdot \frac{.5}{C}$$

So the resultant shear stress will be the sum of S_1 , S_2

$$S = S_1 + S_2 = \frac{2PR}{\pi a^3} \left(1 + \frac{.5}{C}\right)$$

This equation may be written

$$(50) \quad S = \frac{2PR}{\pi a^3} K_s$$

Where $K_s = 1 + \frac{.5}{C}$ Known as a shear-stress multiplication factor.

Because the Wahl factor K takes care of both the effect of direct shear and that of wire curvature, we can write $K = K_c \cdot K_s$ where K_c accounts mainly for the effects of curvature and K_s accounts for the direct shear as mentioned above. Then eq. (50) may be written

$$(51) \quad S = \frac{2PR}{\pi a^3} K / K_c$$

$$\text{and } (52) \quad K_c = \frac{K}{K_s} = \left(\frac{4C-1}{4C-4} + \frac{.615}{C} \right) / \left(1 + \frac{.5}{C} \right)$$

Fatigue or repeated loading springs are subjected to a continuous cyclic stress between the minimum stress $S_{\min}$ and the

maximum value S_{max} as shown in Fig. 9. This is equivalent to a static or mean stress S_m equal to half the sum of maximum and minimum stresses on which is superimposed a variable stress S_v . The variable stress is equal to half the difference between S_{max} and S_{min} , the proper algebraic sign being considered.

In calculating the static or mean stress S_m , the consensus of opinion at present is that stress concentration effects due to wire curvature may be neglected for ductile materials. This is consistent with neglecting stress concentration effects where static loads only are involved. In figuring the variable component S_v , however, stress concentration may not be neglected.

Some evidence in support of this method lies in certain fatigue tests⁽⁹⁾ on notched bars under combined static and variable stress. As shown in Fig. 10, the mean or static stress represented by the dot and dash line is not diminished by the stress-concentration effect, while the variable stress represented by the vertical distance between either the full lines or the dashed lines is diminished in a more or less constant ratio by the stress concentration effect of the notch. Moreover, the fact that all spring stresses under fatigue load calculated by means of the Wahl factor K are too high is shown by the results of a series of carefully made fatigue tests on small helical springs of different indexes carried out by Zimmerli⁽¹⁰⁾.

If Wahl's formula with correction factor K is used in figuring S_{max} and S_{min} the static component of stress S_m , when figured by neglecting stress concentration effects due to curvature, then becomes

$$(53) \quad S_m' = \frac{S_m}{K_c} = \frac{S_{max} + S_{min}}{2 K_c}$$

And the variable components of stress S_v is

$$(54) \quad S_v = \frac{S_{max} - S_{min}}{2}$$

In accordance with the previous discussion and the Soderberg method⁽⁸⁾ that the relation between the limiting value of the static and variable stress components at failure follows a linear law, we can evaluate working stress in helical springs under variable loading as follows:

Referring to Fig. 11, the dashed line shows a typical experimental curve of failure for materials under a combination of static and variable stress. The ordinates represent values of variable stress which will just cause failure when superimposed in the static stresses shown by the abscissas.

We represent the shear endurance limit in a zero to maximum stress range by S_n' . Then for this case (0 to maximum stress) the mean stress S_m and the variable component S_v are both equal to $S_{max}/2$. If we locate a point B on the curve such that the mean component $S_m = S_{max}/2$ and the variable component $S_v = S_{max}/2$

then the point **B** represents the condition where failure will occur for a repeated stress from 0 to S_{max} . With the point **B** located, we can approximate the experimental curve as in Soderberg's diagram by the straight line **BA**, where **A** is located by the yield-point stress of the material in torsion S'_y . Again, dividing S'_y and $S'_n/2$ by the factor of safety (N) defines the line **CD**, such that any point **G** on this line represents a safe combination of mean and variable stresses. From the similar triangles **QGD** and **RBA**, we get

$$\frac{S'_y/N - S_m/k_c}{S'_y - S'_n/2} = \frac{S_v}{S'_n/2}$$

from which

$$(55) \quad \frac{1}{N} = \frac{S_m}{k_c S'_y} + \frac{S_v(2S'_y - S'_n)}{S'_y \cdot S'_n}$$

$$\text{or } (56) \quad \frac{S'_y}{N} = \frac{S_m}{k_c} + S_v \left(\frac{2S'_y}{S'_n} - 1 \right)$$

The eq. (55) or eq. (56) may be used in design or to check for the factor of safety N if we keep in mind the following limitations:

(1) In deriving Wahl's formula we assume that the helix angle of the spring is small, maximum stress is below the elastic limit, and effects of eccentricity of loading due to end turns are neglected.

(2) Data concerning the endurance limit and yield point stress of spring materials are not plentiful, and those data available

from various experimenters are not in good agreement. Because they are influenced by many factors such as surface condition, shot blasting, overstressing, corrosion, and wire size, etc., we should choose them carefully, otherwise a larger factor of safety may be used.

(3) For higher temperatures, the effects of creep or relaxation must be considered, but not many data are available for springs under such conditions⁽¹¹⁾.

(4) Because some materials are not fully sensitive to stress concentration the full stress-concentration effect corresponding to the curvature correction factor K does not always occur even for fatigue loading. In other words, when such materials are tested by means of specimens having notches, holes, or fillets, the fatigue strength reduction produced by the presence of such "stress raisers" is not as great as that to be expected based on theoretical stress-concentration factors. This so called "notch sensitivity" of material is still unknown to metallurgists. Until we can get more such informations about materials, we may introduce other factors to rationalize our design procedure farther.

Example: The force on an automobile valve spring of 27/32 in. inside diameter is 52 lb. when the valve is closed; and 140 lb. when the valve is open. Determine the diameter of wire if the factor of safety is 1.2.

Solution: Choose valve-spring wire of $S_y = 98,000 \text{ lb/in}^2$
 and $S_n = 70,000 \text{ lb/in}^2$. The mean load is $F_m = \frac{(52+140)}{2} = 96 \text{ lb}$
 and the variable component is then $F_v = \frac{140-52}{2} = 44 \text{ lb}$.
 From eq. (56) we have

$$\frac{98,000}{1.2} = \frac{2 \times 96 R}{\pi d^3} K_s + \frac{2 \times 44 R}{\pi d^3} K \left(2 \frac{98,000}{70,000} - 1 \right)$$

or

$$(57) \quad 81600 = \frac{192 R}{\pi d^3} K_s + \frac{88 R}{\pi d^3} K \times 1.8$$

Assume $a = 0.094"$, then $R = 0.094 + \frac{1}{2} \cdot \frac{27}{32} = 0.094 + 0.422 = 0.516"$
 $c = R/a = .516/.094 = 5.5$

From Fig. 12 as $c = 5.5$, $K_s = 1.09$, $K = 1.281$. The right side of
 Eq. (57) becomes

$$\frac{192 \times .516}{3.14 \times .094^3} \times 1.09 + \frac{88 \times .516}{3.14 \times .094^3} \times 1.281 \times 1.8 = 41400 + 40200 = 81600 \text{ lb/in}^2$$

hence diameter of wire = 0.188"

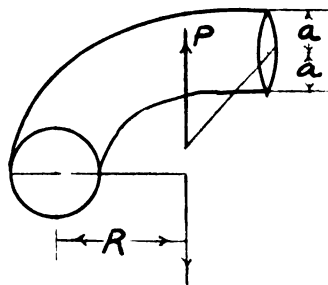


fig. 1

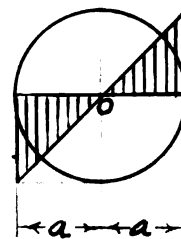


fig. 2

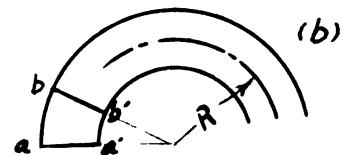
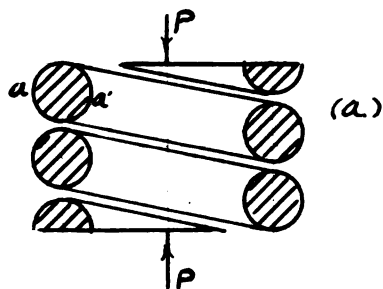


fig. 3

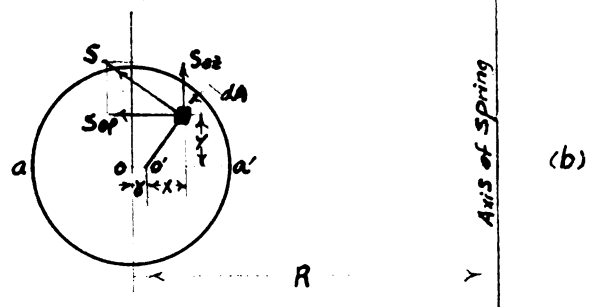
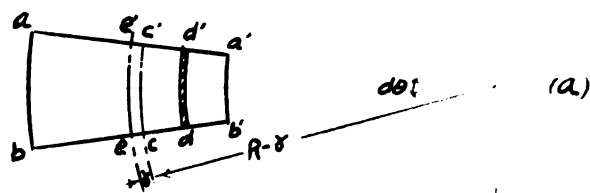


fig. 4

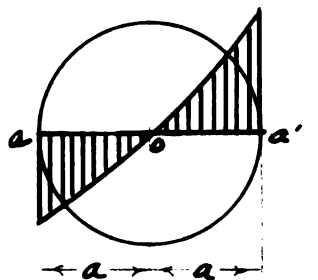


fig.5

Axis of Spring

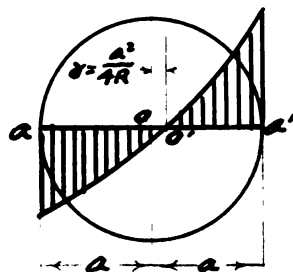


fig.6

Axis of Spring

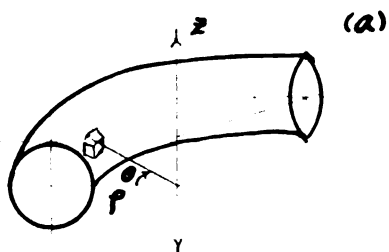
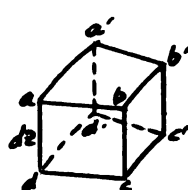


fig.7



(b)

do f

o'
o

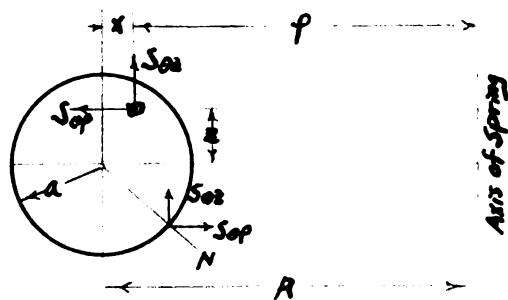


fig.8

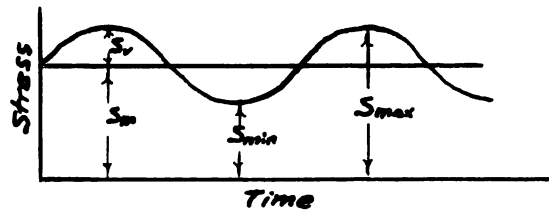


fig. 9

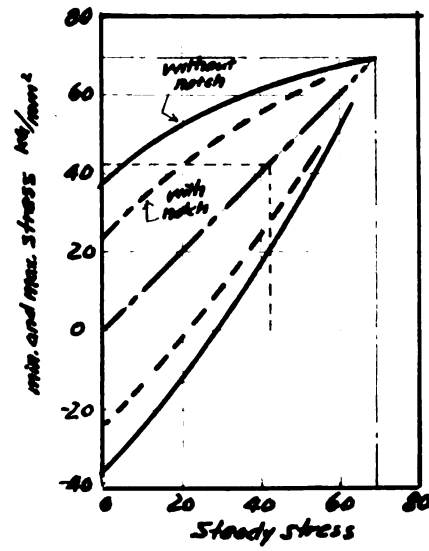


fig. 10

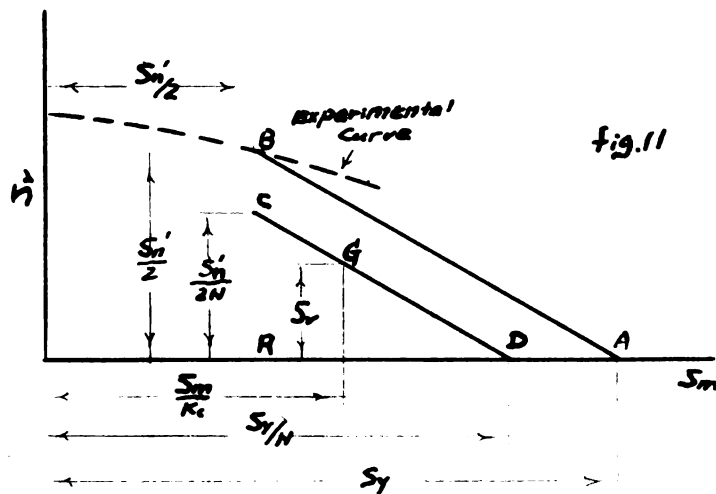


fig. 11

$$G = \frac{\frac{C}{C-1} + \frac{1}{4C} + \frac{1}{16C^2}}{1 + \frac{3}{16} \cdot \frac{1}{C^2-1}}$$

$$K_s = 1 + \frac{5}{C}$$

$$K = \frac{4C-1}{4C-4} + \frac{.615}{C}$$

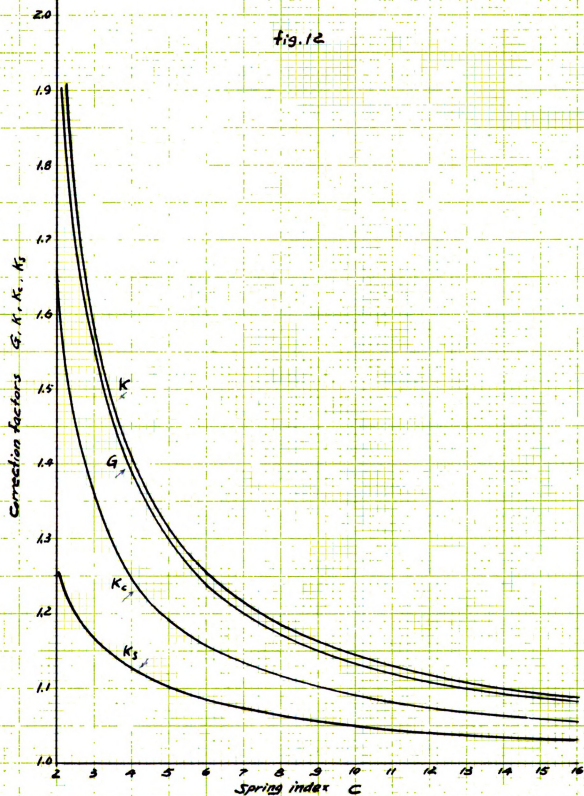
$$K_c = K / K_s$$

C	4C	$\frac{1}{4C}$	$16C^2$	$\frac{1}{16C^2}$	$\frac{3}{16C^2-16}$	$\frac{5}{C}$	G	K	K _s	K _c
2	8	.125	64	.016	.062	.250	2.016	2.057	1.250	1.645
3	12	.083	144	.007	.023	.167	1.554	1.580	1.167	1.353
4	16	.063	256	.004	.012	.125	1.373	1.403	1.125	1.248
5	20	.050	400	.003	.008	.100	1.292	1.310	1.100	1.191
6	24	.042	576	.002	.005	.083	1.237	1.252	1.083	1.157
7	28	.036	784	.001	.004	.071	1.198	1.213	1.071	1.132
8	32	.031	1024		.003	.062	1.170	1.183	1.062	1.113
9	36	.028	1269		.002	.055	1.150	1.161	1.055	1.100
10	40	.025	1800		.002	.050	1.133	1.144	1.050	1.090
11	44	.023	1936		.002	.045	1.120	1.130	1.045	1.080
12	48	.021	2304		.001	.041	1.109	1.119	1.041	1.073
13	52	.019	2704		.001	.038	1.100	1.109	1.038	1.068
14	56	.018	3136		.001	.035	1.093	1.101	1.035	1.063
15	60	.017	3600			.033	1.088	1.094	1.033	1.058
16	64	.016	4096			.031	1.082	1.088	1.031	1.054

Table I

Correction factors of
helical round-wire springs

fig. 12

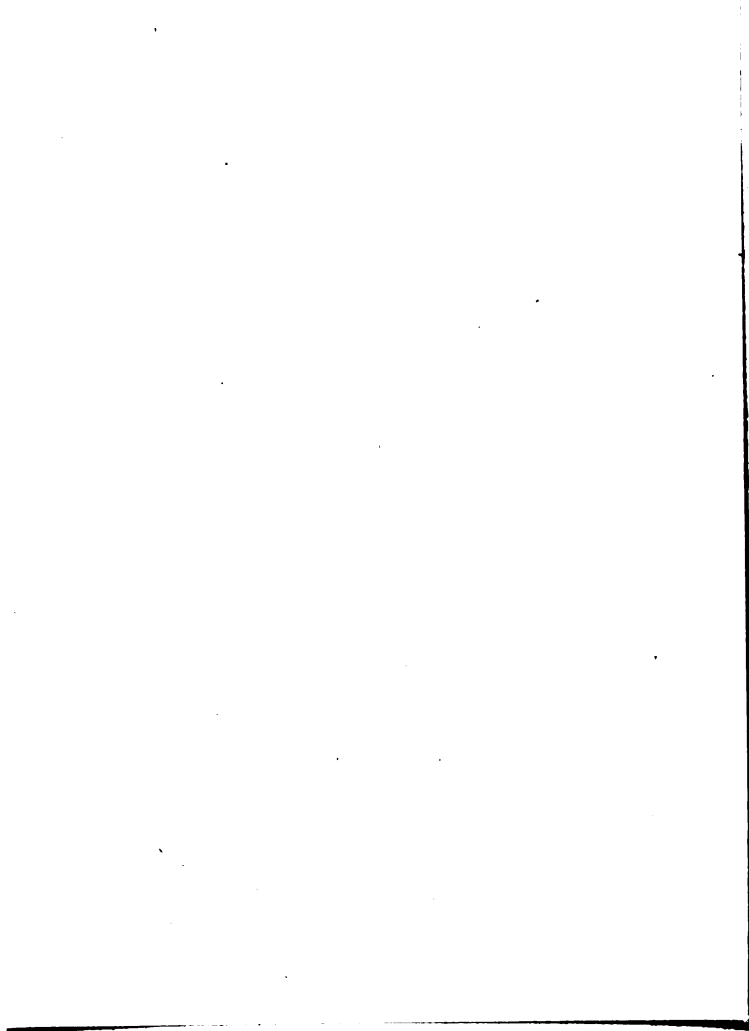


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