COMPUTER AIDED OPTIMIZATION OF
NONLINEAR SERVOMECHANISMS
EMPLOYING A DIRECTED SEARCH OF MULTIPARAMETER
COMPONENT LIBRARIES AND STATISTICAL TOLERANCING

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
BRUCE ALLEN CHUBB
1969

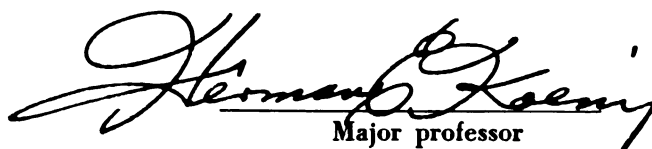This is to certify that the

thesis entitled

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of Multiparameter Component Libraries And
Statistical Tolerancing.

presented by

Bruce Allen Chubb

has been accepted towards fulfillment
of the requirements for

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ABSTRACT

COMPUTER AIDED OPTIMIZATION OF
NONLINEAR SERVOMECHANISMS EMPLOYING A
DIRECTED SEARCH OF MULTIPARAMETER COMPONENT LIBRARIES
AND STATISTICAL TOLERANCING

by Bruce Allen Chubb

Techniques are developed to automatically select from computerized
libraries the components that satisfy a given system specification at
minimum dollar cost. This is accomplished by defining an object
function to be the system cost which in turn is a function of the
probability that the design will be successful in meeting the per-
formance specification. Starting with an initial set of component
part numbers, the total system cost is minimized by iterating the
component part numbers using a directed search technique. The
manufacturing tolerances associated with the component parameters are
considered in calculating the probability of success.

Comparisons are made between the Monte Carlo and the directed
search procedure which illustrate that the directed search technique
has considerable advantage. Several examples demonstrate that such a
computer program can result in considerable cost savings.

The techniques are developed around an instrument servomechanism
as a specific example. Four component libraries are established to
list the part characteristics for the followup, amplifier,

motor-generator, a

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motor-generator, and geartrain. Combinations of up to eight pre-
assigned performance specifications in the areas of damping, accuracy,
and time response are considered. The nonlinear effects of backlash,
coulomb friction, and amplifier saturation are considered as well as
the effect of finite geartrain stiffness in evaluating the system
performance. Equations are derived for calculating, 1) the allowable
backlash without a limit cycle, 2) the nonlinear overshoot for large
step inputs, and 3) the effective bandwidth for sinusoidal inputs.

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COMPUTER AIDED OPTIMIZATION OF

NONLINEAR SERVOMECHANISMS EMPLOYING A

DIRECTED SEARCH OF MULTIPARAMETER COMPONENT LIBRARIES

AND STATISTICAL TOLERANCING

By

Bruce Allen Chubb

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1969

TO MY MOTHER AND FATHER

ACKNOWLEGEMENTS

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1. INTRODUCTION

1.1 STATEMENT OF THE PROBLEM

The system engineer operating within the framework of a typical manufacturing organization operates from the following basic information and constraints:

a) A set of customer specifications to be met,

b) A basic system configuration to be used in realizing these specifications,

c) A set of standard components that fit into this configuration.

The basic problem is to determine the collection of components that satisfies the given specification at <u>minimum</u> <u>total</u> <u>dollar</u> <u>cost</u>.

Automated techniques for selecting the optimum set of components for a system are necessitated by today's competitive market and the multitude of candidate components available. As an example, consider the problem of selecting an optimum set of components for a fixed amplifier configuration to meet a given set of customer specifications. If one extrapolates data from a 1964 survey [1], today there should be approximately 60,000 semiconductor devices manufactured and designated by part number. If one adds to this, the number of standard resistors, capacitors, transformers, etc., it becomes obvious

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that manual techniques cannot come close to yielding an optimum component selection.

The same situation exists in every area of system engineering where the configuration is "fixed" and a multitude of candidate components are available. The characteristics of these components can, of course, be stored in computer libraries by part numbers and an analysis program can be written to systematically analyze the system for any candidate set of components by merely inserting the appropriate part numbers. Such computer programs are structured so as to retrieve the data for each particular component, proceed with the various performance calculations and display the results to the designer for each set of part numbers manually selected.

The goal of this study is to go one step further and develop techniques, procedures, and programs for the effective use of computers in automating the solution to the above class of design problem.

1.2 EXAMPLE SYSTEM

The techniques presented are developed around an instrument servomechanism as a specific example. The design problem is essentially the same as that discussed in references [2,3]; however, the techniques developed are believed to be much improved.

The example instrument servomechanism consists of a follow-up device, electronic amplifier, drive motor with feedback generator, and geartrain. A pictorial diagram showing a fixed system configuration using these components is shown in Figure 1.1.

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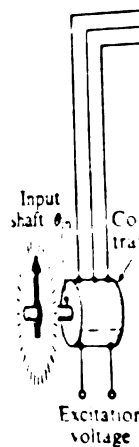
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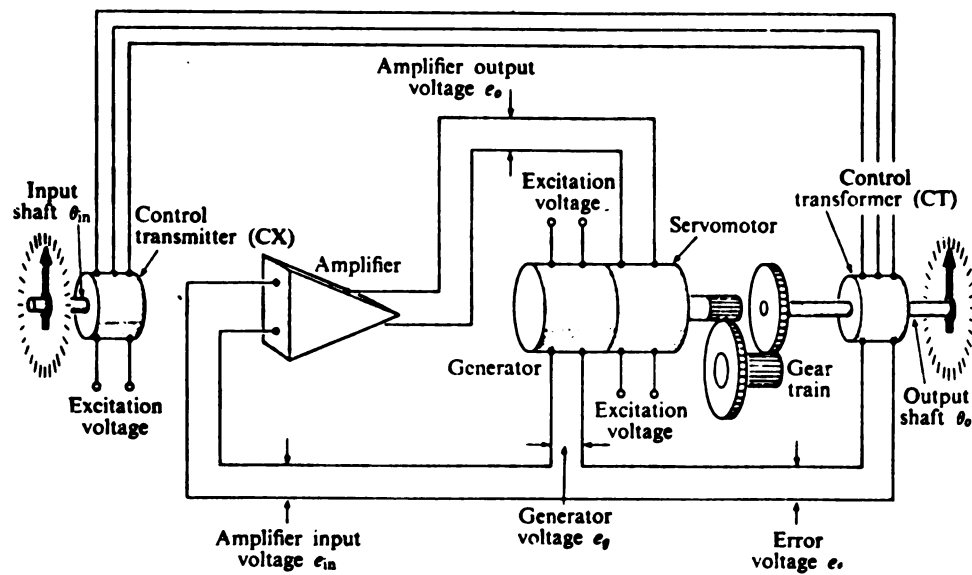
Figure 1.1. Schematic diagram of motor-generator
instrument servomechanism.

It is assumed that the design of this configuration must meet
up to eight preassigned specifications in the areas of damping,
accuracy, and time response. Four component libraries are estab-
lished to list the part characteristics as follows:

a) Follow-up - 25 part numbers

b) Amplifier - 50 part numbers

c) Motor-generator - 25 part numbers

d) Geartrain - 25 part numbers

Even though the size of each demonstration library was purposely kept
small, the number of theoretical possible candidate systems is large,
namely: $25 \times 50 \times 25 \times 25 = 781,250$.

The optimum collection of components is defined as "the one that
satisfies the given specification in a manner resulting in minimum
total cost".

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Three ancillary problems considered in the thesis have their origin in the fact that design equations did not heretofore exist for calculating the allowable backlash, large step overshoot, and bandwidth for a nonlinear instrument system. Solutions to these three problems represent significant advancements in the field of servomechanism design, and are presented as Appendices.

1.3 SURVEY OF PRESENT TECHNIQUES

Literally thousands of articles have been published which list and describe work that has been done in optimization. A few of the more comprehensive publications are listed as references [4,5,6]. Most of this work is concerned with finding that set of n parameter values, X_1, X_2, $\cdots$, X_n, which maximizes (or minimizes) a given scalar function $F(X_1, X_2, \cdots, X_n)$, subject to constraints on these parameters which limit their range to realizable values.

Three of the most popular techniques are centered about the following three basic approaches:

a) Random Experimentation

Although crude forms of this method are as old as design technology itself, the best early formal documentation as given in 1958 [7] uses repeated solution of the system design equations with random selections of the input parameters generated through Monte Carlo methods. In its simple form, a large number of computer solutions are required to achieve good results. When for reasons of computer costs, only a few runs can be justified, a partitioned

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or stratified form [8] usually provides better efficiency. In general, improved results are obtained most often if some form of strategy or learning can be employed to adjust the frequency distributions representing the parameters to be selected.

b) Steepest Ascent

This method was introduced by R. R. Brown in 1957 [9] and further improved in 1959 [10]. Today there are many computer programs available for general use which employ this technique. The Steepest Ascent methods calculate the partial derivatives $\partial F/\partial X_1$, $\partial F/\partial X_2, \cdots, \partial F/\partial X_n$ usually numerically, and then proceed along the gradient until a maximum is obtained. Since the result represents only a local or relative maximum, various starting points are used in an attempt to find the global maximum. GREAT [11] is one example of a highly effective program that is based on this method.

c) Direct Search

The Direct Search technique is attributed to Hook and Jeeves who presented the unconstrained case in 1961 [12]. This was modified in 1965 by Weisman and Wood [13] to include constraints. In direct search, the minimum is found by the sequential examination of a finite set of trial values. The result of each trial is compared with the best previous trial and the new value accepted if an improvement is observed. This series of exploratory moves, in which each

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variable is individually adjusted, is used to determine
the "best" direction for a successful move. This move, in
which all parameters are changed, is called the "pattern
move". Each pattern move is followed by a sequence of ex-
ploratory moves to revise the pattern. The sequence is re-
peated until the scalar function can be increased no further.
A good application of this type of algorithm is LOOK[14].

1.4 SCOPE OF INVESTIGATION

A logic flow diagram representing an effective design procedure
is shown in Figure 1.2. As indicated, there are two design itera-
tion loops. One loop concerns changes in the basic system configura-
tion (e.g. component interconnection) and the other changes in com-
ponent selection. Only the problems associated with automating the
component selection are considered here.

If each component could be represented by a single parameter,
the solution would be quite straightforward. One could arrange the
components in the library in ascending order of its single para-
meter and use a modified form of either the steepest ascent or
direct search method to find the optimum. However, the general
solution is far more difficult, since libraries consist of multiter-
minal components, and several parameters are required to describe
each component. These parameters are associated only with the com-
ponent part number, and there is no natural ordering between com-
ponents.

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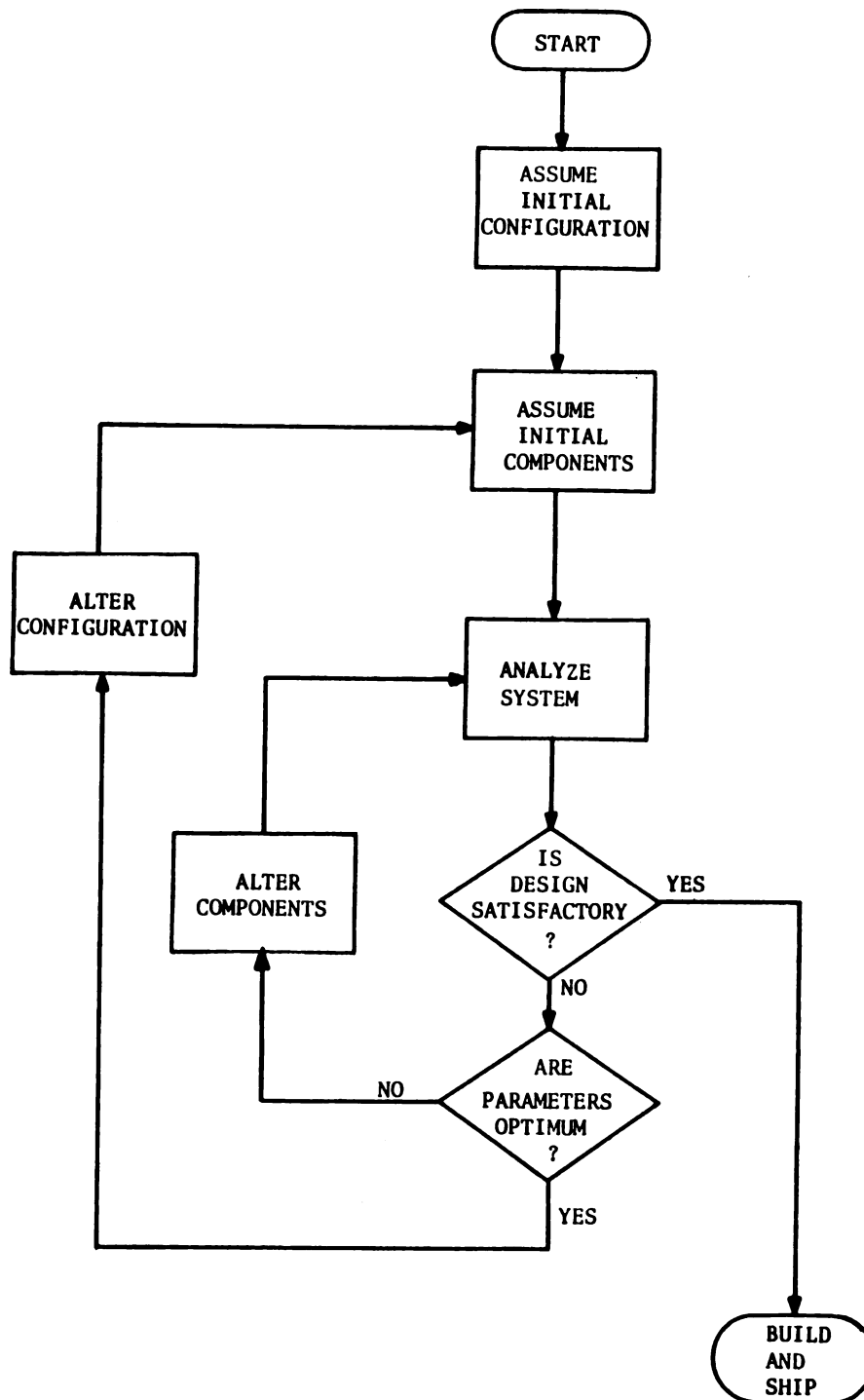
Figure 1.2. A simplified design flow diagram.

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The problem is analogous to A. M. Gleason's Search in the N-Cube [15]. Gleason states "It is entirely clear that there is no certain method of finding the maximum, short of computing the function at each point of the set in question". This exhaustive search for our particular design problem, however, is out of the question. For example, if we consider 30 seconds to be required for each analysis using a high speed machine, it would take 6510 hours of computer time to solve the problem in question. There must be a better way!

The problem can be divided into three aspects. The first aspect is concerned with developing an effective analysis program, including formulation of the necessary nonlinear state equations, codification of the system specifications, developing the required design equations and a method of handling the component parameter tolerances. This effort is presented in Section 2.

The second task is to incorporate the analysis program in the design loop by adding an optimization procedure. To this end it is necessary to formulate the object function to be minimized, set up component libraries, and to formulate the optimization strategy. This effort is presented in Section 3.

Section 4 of this thesis presents the application of the developed program to typical hardware design problems. Section 5 then presents the conclusions of the study and provides suggested guidelines for future work.

2. DEVELOPMENT OF ANALYSIS PROGRAM

2.1 BASIC APPROACH

The analysis section is the starting point of any computer-aided or automated design program. Optimization, in the design context, is derived from an efficient use of iterative analysis techniques. Devoid of a good analysis capability, the designer has nothing. Its presence provides a powerful tool in itself. In this case, however, it is simply a means to an end - Automated Design.

"But what are the requirements for an effective analysis program?" First, and primary, is the fact that it must accurately represent the hardware. This requires a significantly detailed model, including often overlooked nonlinearities, and a realistic consideration of component tolerance effects. This means that the programmer is faced with the solution of nonlinear differential equations, and that system parameters, instead of being constants, must be treated as random variables. Second the outputs of the analysis program must have a one-to-one correspondence with the list of system specifications. That is, if the customer specifies overshoot, response time, accuracy, etc., then the program must have the capability of calculating the performance characteristics in this form. Third and last, since the analysis is to be repeated many times in an iterative fashion, the solution time should be a minimum.

2.2 FORMULATION OF SYSTEM STATE EQUATIONS

A most effective method of obtaining the response of a system is
by using the state variable model [16]. Much work has been done in
the effective application of this approach to the analysis of phys-
ical systems [17,18]. Many aspects of the particular problem consid-
ered here are presented in reference [2]. However, in the interest
of continuity a limited development is repeated here.

The example system under consideration consists of a followup,
amplifier, servomotor with an integral mounted feedback generator,
geartrain, and load. The load is made up of inertia and coulomb
friction. Experience had demonstrated [2] that geartrain resilience,
along with the nonlinearities of gear backlash, amplifier saturation,
and coulomb friction, must be considered.

The state model diagram of the system is shown in Figure 2.1
and the system and component parameters are defined in Tables 2.1
and 2.2, respectively. Four state variables are required to define
the system. These are the outputs of the 4 integrators of Figure 2.1
and correspond to motor velocity, motor position, load velocity, and
load position. It should be noted that the motor velocity and posi-
tion have been reflected to equivalent values at the load (i.e. the
hardware values are simply those given by Figure 2.1 times the
gear ratio, N). The amplifier saturation is represented by an equiv-
alent torque saturation (i.e. the torque level is set at a value
equal to that of the amplifier voltage level times the product of
the motor torque gain and gear ratio).
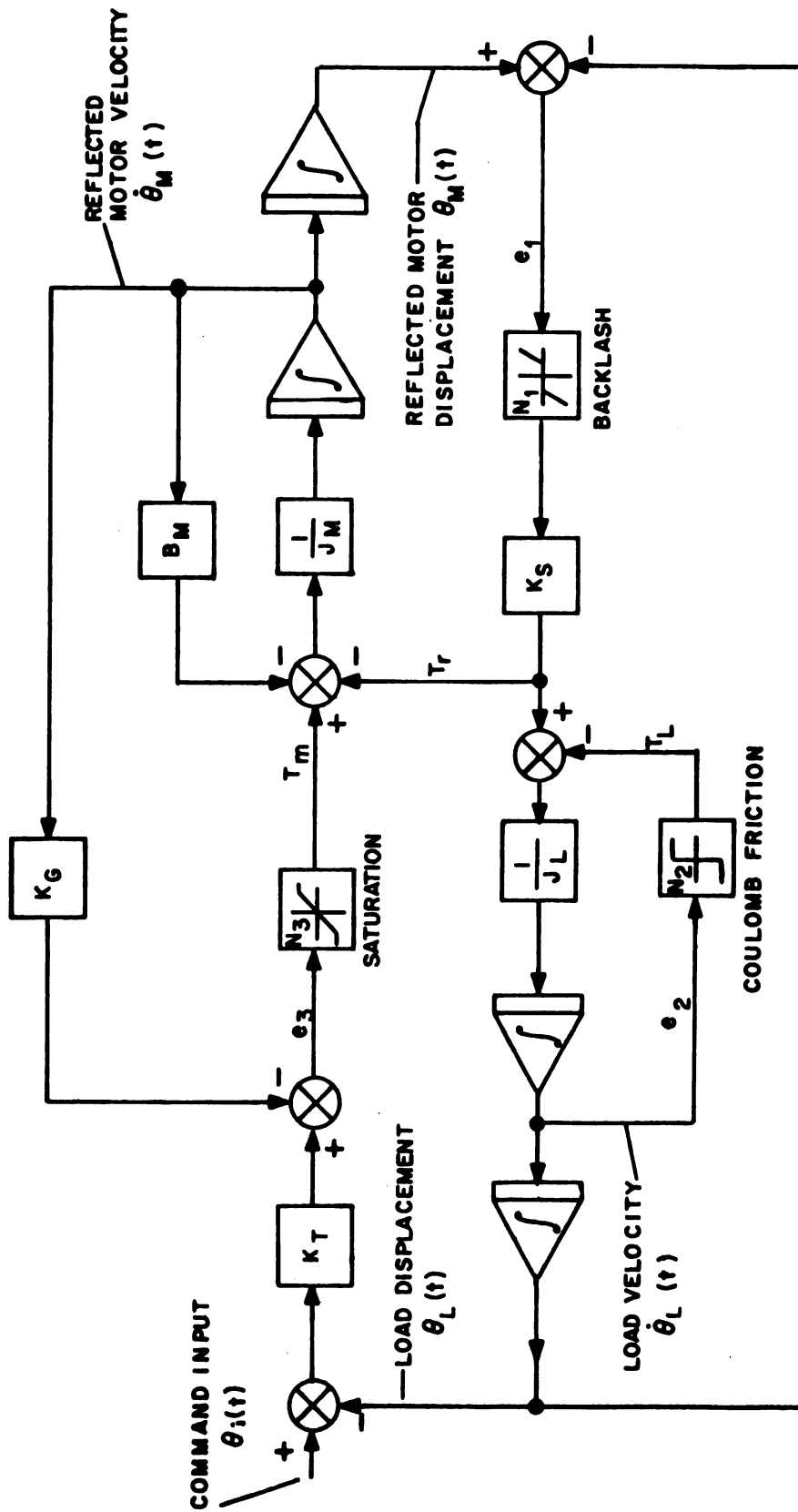
Figure 2.1. Nonlinear state model diagram for an instrument servomechanism.

Table 2.

Symbol
B_M
J_L
J_M
K_G
K_T
T_L
T_{SAT}

Table 2.

Table 2.1. System parameter definitions for state model.

Symbol	Defining Equation	Name	Units
B_M	$N^2 B_m$	Reflected motor damping	oz-in/rad/sec
J_L	$J_\ell + J_g$	Load inertia	oz-in/rad/sec^2
J_M	$N^2 J_m$	Reflected motor inertia	oz-in/rad/sec^2
K_G	$K_g K_{ag} K_m N^2$	Generator damping coefficient	oz-in/rad/sec
K_T	$K_f K_{af} K_m N$	System torque constant	oz-in/rad
T_L	$T_\ell + T_g$	Load friction	oz-in
T_{SAT}	$K_m E_{sat}$	System torque saturation	oz-in

Table 2.2. Component parameter definitions for state model.

Symbol	Definition	Units
B	Geartrain backlash	radians
B_m	Motor viscous damping	oz-in/rad/sec
E_{sat}	Amplifier saturation level	volts
J_g	Geartrain inertia	oz-in/rad/sec^2
J_ℓ	Load inertia	oz-in/rad/sec^2
J_m	Motor-generator inertia	oz-in/rad/sec^2
K_{af}	Amplifier gain to followup	volts/volt
K_{ag}	Amplifier gain to generator	volts/volt
K_f	Followup gain	volts/rad
K_g	Generator gain	volts/rad/sec
K_m	Motor torque gain	oz-in/volt
K_s	Geartrain spring constant	oz-in/rad
N	Gear ratio	
T_g	Geartrain friction	oz-in
T_ℓ	Load coulomb friction	oz-in

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N_1

$(\theta_M - \theta_L) \longrightarrow$

N_2

$\dot{\theta}_L \longrightarrow$

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N_3 SL

$K_T(\theta_i - \theta_L) - K_G \dot{\theta}_M \longrightarrow$

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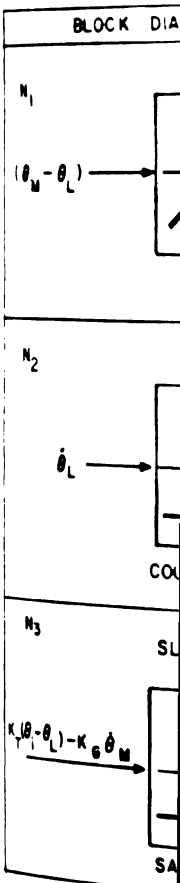
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rectly from Fig

$$\frac{d}{dt} \begin{bmatrix} \dot{\theta}_M \\ \theta_M \\ \dot{\theta}_L \\ \theta_L \end{bmatrix} =$$

The nonlinear functions representing backlash (N_1), coulomb friction (N_2), and saturation (N_3) are defined in Figure 2.2.
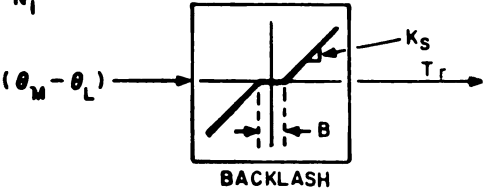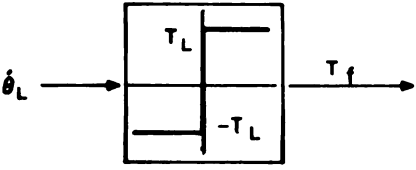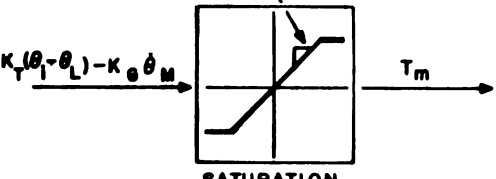
BLOCK DIAGRAM	EQUATIONS(WHERE: sgn $\theta = \theta / |\theta|$)
N_1 $(\theta_M - \theta_L)$ → [BACKLASH diagram with K_S, B] → T_r	$T_r = K_s \left[\left|\theta_M - \theta_L\right| - B\right]$ sgn $(\theta_M - \theta_L)$ for $\left|\theta_M - \theta_L\right| \geq B$ $T_r = 0$ for $\left|\theta_M - \theta_L\right| \leq B$
N_2 $\dot\theta_L$ → [COULOMB FRICTION diagram with T_L, $-T_L$] → T_f	$T_f = \left|T_L\right|$ sgn $\dot\theta_L$ for $\dot\theta_L \neq 0$ $T_f = -T_r$ for $\dot\theta_L = 0$
N_3 $K_T(\theta_1 - \theta_L) - K_e \dot\theta_M$ → [SATURATION diagram SLOPE=1] → T_m	$T_m = K_T(\theta_1 - \theta_L) - K_e \dot\theta_M$ for $\left| K_T(\theta_1 - \theta_L) - K_e \dot\theta_M \right| \leq T_{SAT}$ $T_m = T_{SAT}$ sgn $\left[K_T(\theta_1 - \theta_L) - K_e \dot\theta_M \right]$ for $\left| K_T(\theta_1 - \theta_L) - K_e \dot\theta_M \right| \geq T_{SAT}$

Figure 2.2. Nonlinear function definitions.

The nonlinear state model for the system can be obtained directly from Figure 2.1 as:

$$
\frac{d}{dt}
\begin{bmatrix}
\dot\theta_M \\
\theta_M \\
\dot\theta_L \\
\theta_L
\end{bmatrix}
=
\begin{bmatrix}
-\dfrac{B_M}{J_M} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot\theta_M \\
\theta_M \\
\dot\theta_L \\
\theta_L
\end{bmatrix}
+
\begin{bmatrix}
\dfrac{1}{J_M}\left(T_m - T_r\right) \\
0 \\
\dfrac{1}{J_L}\left(T_r - T_f\right) \\
0
\end{bmatrix}
\qquad (2.1)
$$

The cor

$N_1 = 1$ for

$N_2 = 1$ for

$$\frac{d}{dt} \begin{bmatrix} \dot{\theta}_M \\ \theta_M \\ \dot{\theta}_L \\ \theta_L \end{bmatrix}$$

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finite (i.e.

$$\frac{d}{dt} \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix}$$

where f_T is

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The corresponding linear approximation is obtained by setting

$N_1 = 1$ for zero backlash; $N_2 = 0$ for zero coulomb friction; and

$N_3 = 1$ for no amplifier saturation.

$$\frac{d}{dt}\begin{bmatrix} \dot{\theta}_M \\ \theta_M \\ \dot{\theta}_L \\ \theta_L \end{bmatrix} = \begin{bmatrix} -\frac{(B_M + K_G)}{J_M} & -\frac{K_s}{J_M} & 0 & \frac{(K_s - K_T)}{J_M} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{K_s}{J_L} & 0 & -\frac{K_s}{J_L} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_M \\ \theta_M \\ \dot{\theta}_L \\ \theta_L \end{bmatrix} + \begin{bmatrix} \frac{K_T}{J_M} \\ 0 \\ 0 \\ 0 \end{bmatrix} \theta_i$$

$$(2.2)$$

In the special case when the geartrain stiffness is considered in-finite (i.e. $K_s = \infty$) the linear state model becomes

$$\frac{d}{dt}\begin{bmatrix} \dot{\theta}_L \\ \theta_L \end{bmatrix} = \begin{bmatrix} -f_T/J_T & -K_T/J_T \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_L \\ \theta_L \end{bmatrix} + \begin{bmatrix} K_T/J_T \\ 0 \end{bmatrix} \theta_i$$

$$(2.3)$$

where f_T is the total effective viscous damping from the feedback generator and motor. This is

$$f_T = B_M + K_G \qquad (2.4)$$

and in a like manner J_T is the total system inertia given by

$$J_T = J_M + J_L \qquad (2.5)$$

Equation (2.5

$$\frac{d}{dt} \begin{bmatrix} \dot{\theta}_L \\ \\ \theta_L \end{bmatrix}$$

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Equation (2.3) may be written also in the convenient form

$$\frac{d}{dt}\begin{bmatrix}\dot{\theta}_L \\ \theta_L\end{bmatrix} = \begin{bmatrix} -2\zeta\omega_N & -\omega_N^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix}\dot{\theta}_L \\ \theta_L\end{bmatrix} + \begin{bmatrix}\omega_N^2 \\ 0\end{bmatrix}\theta_i$$

(2.6)

where ζ and ω_N are the damping ratio and natural frequency as normally defined for a second order system. For the particular case under consideration

$$\zeta = \frac{f_T}{2\sqrt{K_T J_T}}$$

(2.7)

$$\omega_N = \sqrt{\frac{K_T}{J_T}}$$

(2.8)

2.3 SYSTEM SPECIFICATIONS AND DESIGN EQUATIONS

The first step in realizing a design is to establish a thorough understanding of the set of performance specifications that the system must satisfy. The second step required is to develop a set of equations that enable one to evaluate a potential design in relation to the specifications. This section is devoted to the accomplishment of both the above tasks.

For a computer program to be effective in design, it must cover a somewhat general set of specifications. Then, for any given application, the user may choose the particular desired set and instruct the computer to ignore the others. A set of eight specifications is

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selected for the example program developed as part of this study. They are representative of those listed in numerous military and commercial specifications for such systems as manufactured by the Instrument Division of Lear Siegler, Incorporated, Grand Rapids, Michigan.

The eight specifications are now discussed one at a time, with the corresponding design equations used to evaluate a proposed design.

1) Static Accuracy

Static accuracy is unquestionably the most often specified requirement for any instrument servo. It is simply a measure of the magnitude of the error that can exist between the command input and the indicated output of the system under static conditions. Contributions to this error include followup tracking error, amplifier and generator null offsets, motor starting voltage, and gearing and load friction. By taking each of these error sources and dividing by the corresponding dc gain back to the error angle, and summing, the following equation is derived for the static accuracy (ϵ_A).

$$\epsilon_A = \theta_f + \frac{E_{an}}{K_f K_{af}} + \frac{K_{ag} E_{gn}}{K_f K_{af}} + \frac{E_s}{K_f K_{af}} + \frac{T_g + T_\ell}{K_f K_{af} K_m N}$$

(2.9)

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where

θ_f = followup tracking accuracy (rad)

E_{an} = amplifier output null voltage (volts)

E_{gn} = generator output null voltage (volts)

E_s = motor-generator no-load starting voltage (volts)

and all other notation is defined in Table 2.2.

2) Resolution

Resolution is a measure of the total dead-zone in an instrument servomechanism. It therefore represents the maximum amount that the input can be displaced without noting any motion at the output. This deadzone results from the fact that a certain amount of error must be built-up to overcome the motor starting voltage and coulomb frictions. Thus, the total deadzone or resolution (ϵ_R) is given by

$$\epsilon_R = 2 \left[\frac{E_s}{K_f K_{af}} + \frac{T_g + T_\ell}{K_f K_{af} K_m N} \right] \qquad (2.10)$$

3) Velocity Lag

Velocity lag is a measure of the servo's accuracy under constant velocity conditions. It is defined as the steady-state positional difference between the command input and the indicated output with the input rotating at a constant velocity. Since the resulting lag error is a function of the input velocity, the latter also must be specified.

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The velocity lag (ε_L) may be calculated using the equation (see reference [2])

$$\varepsilon_L = \left[\frac{N^2 \left(B_m + K_g K_{ag} K_m \right)}{K_f K_{af} K_m N} \right] \dot{\theta}_{in} + \varepsilon_A \qquad (2.11)$$

where $\dot{\theta}_{in}$ is the input velocity at which the lag error is to be measured or calculated and ε_A in the static accuracy as defined by (2.9).

4) Followup Rate

Followup rate is a measure of the maximum velocity that the servo is capable of producing. If there were no friction loading, it would be simply the motor no-load speed divided by the gear ratio. However to account for the load, one can calculate the followup rate ($\dot{\theta}_L$) using the equation: (see reference [2])

$$\dot{\theta}_L = \frac{\dot{\theta}_m}{N} \left[1 - \frac{T_g + T_\ell}{N T_s} \right] \qquad (2.12)$$

where the symbols are as defined in Table 2.2 except for the additional ones which are

$\dot{\theta}_m$ = motor no-load speed (rad/sec)

T_s = motor stall torque (oz-in)

5) Damping Ratio

Damping ratio is the most often used measure of system

stability. This is unfortunate since its definition applies only for a linear 2nd order system. However if one makes this linear approximation, then the damping ratio equation may be obtained directly in terms of the component parameters by substituting the definitions of Table 2.1 into Equation (2.7). Thus:

$$\zeta = \frac{N^2\left(B_m + K_g K_{ag} K_m\right)}{2\sqrt{K_f K_{af} K_m N\left(N^2 J_m + J_g + J_\ell\right)}} \qquad (2.13)$$

6) Null Oscillation

Null oscillations are small amplitude steady state oscillations (limit cycles) that exist about a null and are a result of backlash being present in the geartrain. A typical specification states that "no such oscillation shall exist." In Reference [2], it was established that the amount of backlash that a given design can tolerate without such a limit cycle is proportional to the amount of coulomb friction on the load side of the backlash. In this study, we shall derive the equation for the proportionality constant (derivation in Appendix A) thereby obtaining the equation for the allowable backlash as follows:

$$B(\text{allowable}) = M I N_{0<\omega<\sqrt{\frac{K_T}{J_L}}} \left\{ \frac{3.2}{\pi\left[N_1(\omega)+.4\right]\omega N_2(\omega)} \sqrt{\frac{\left(K_T - J_M\omega^2\right)^2 + \left(f_T\omega\right)^2}{\left(N_1(\omega)K_s - J_M\omega^2\right)^2 + \left(f_T\omega\right)^2}} \right\} T_L$$

$$(2.14)$$

whe

$$\mathbf{s}_1(\mathbf{e}) = \frac{\cdot \left[\left(J_{\mathbf{H}} r_L^2 r_{\mathbf{e}} \cdot \right.\right.}{}$$

$$\mathbf{s}_2(\mathbf{e}) = \frac{\cdot \left[J_{\mathbf{H}}^2 \mathbf{e}^2 \cdot J_{\mathbf{H}} \right.}{}$$

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where:

$$N_1(\omega) = \frac{-\left[\left(J_M'L_S'+J_W'T_S\right)\omega^2+\left(F_T^2K_S-J_WF_TK_S\right)\omega^2\right]+\sqrt{\left[\left(J_M'L_S'+J_W'T_S\right)\omega^2+\left(F_T^2K_S-J_WF_TK_S\right)\omega^2\right]^2+4\left[K_TK_S^2-J_TK_S^2\omega^2\right]\left[J_M^2J_L\omega^6+F_T^2J_L\omega^4\right]}}{2\left[K_TK_S^2-J_TK_S^2\omega^2\right]}$$

(2.15)

$$N_2(\omega) = \frac{-\left[J_M^2\omega^2-J_MK_T+F_T^2\right]+\sqrt{\left[J_M^2\omega^2-J_MK_T+F_T^2\right]^2-4F_T\left[J_L^2F_T\omega^2-J_LF_TK_T\right]}}{2F_T}$$

(2.16)

As long as the actual backlash B is less than B(allowable) a limit cycle will not exist.

7) Overshoot

Overshoot of the system's output to a step command input is the most often used measure of servo response. If the servo is linear and of second order, the overshoot is defined by the damping ratio (ζ) given by the equation (see Reference [2])

$$\text{Overshoot} = \left[e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right]\theta_{step} \quad \text{for } \zeta < 1$$

$$= 0 \quad \text{for } \zeta \geq 1 \tag{2.17}$$

However, because of system nonlinearities, mainly amplifier saturation and coulomb friction, the size of the actual overshoot is not proportional to the step size and is not given by a simple relationship such as (2.17). The actual overshoot could be obtained by a direct simulation of the nonlinear state model, Equation (2.1), however such a numerical solution is quite time consuming on a digital computer. For this reason, the nonlinear state equations are solved explicitly (see Appendix B for solution), thereby

enabling a much more direct calculation for the overshoot.
This is accomplished by using piecewise linear solutions
over the regions shown in the phase plane diagram of
Figure 2.3. This illustrates a response trajectory of a
typical system and the corresponding overshoot. As can be
seen, there are three regions of operation. In region 1,
the servo has negative torque saturation, while in region 2
the servo is unsaturated, and finally in region 3 there
is positive torque saturation. The solutions for the system
state vector, as derived in Appendix B for each region, are
summarized in Table 2.3. By solving the first equation at
the saturation boundary, using the result as initial condi-
tions for the appropriate second equation, and again finding
the next boundary conditions, one can proceed from boundary
point to boundary point along the trajectory until the over-
shoot is obtained. The actual logic used to obtain the
boundary conditions is summarized in Appendix B.

8) Bandwidth

Bandwidth is a measure of the systems ability to fol-
low sinusoidally oscillating inputs. It is normally defined
as the frequency at which the output response is attenuated
to 0.707 times the input (-3 db). For a linear second order
system this bandwidth frequency (ω_B) is given by

$$\omega_B = \omega_N \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} \qquad (2.18)$$

where ω_N is as defined in (2.8)
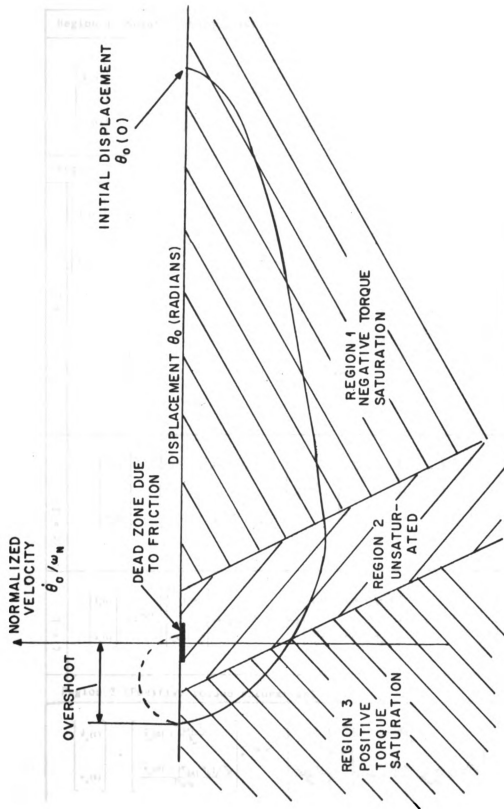
Figure 2.3. Phase plane diagram showing piecewise linear regions.

Table 2.3. State equations for calculating nonlinear overshoot.

Region 1 (Negative torque saturation)

$$\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \dot{\theta}_o(0) + \dfrac{T_{SAT}-T_L}{B_M} \\ \\ \dfrac{-\dot{\theta}_o(0) - (T_{SAT}-T_L)/B_M}{2\zeta_M\omega_N} \end{bmatrix} e^{-2\zeta_M\omega_N t} - \begin{bmatrix} 0 \\ \\ \dfrac{T_{SAT}-T_L}{B_M} \end{bmatrix} t + \begin{bmatrix} -\dfrac{T_{SAT}-T_L}{B_M} \\ \\ \theta_o(0) + \dfrac{\dot{\theta}_o(0) + (T_{SAT}-T_L)/B_M}{2\zeta_M\omega_N} \end{bmatrix}$$

Region 2 (Unsaturated)

$\zeta < 1$

$$\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)\dot{\theta}_o(0) + \omega_N^2\,\theta_o(0) - \dfrac{T_L}{J_T} \\ \\ -\dot{\theta}_o(0) - \omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)\theta_o(0) + \dfrac{T_L}{J_T\omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)} \end{bmatrix} \dfrac{e^{-\omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)t}}{2\omega_N\sqrt{\zeta^2-1}}$$

$$+ \begin{bmatrix} -\omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)\dot{\theta}_o(0) - \omega_N^2\,\theta_o(0) + \dfrac{T_L}{J_T} \\ \\ \dot{\theta}_o(0) + \omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)\theta_o(0) - \dfrac{T_L}{J_T\omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)} \end{bmatrix} \dfrac{e^{-\omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)t}}{2\omega_N\sqrt{\zeta^2-1}} + \begin{bmatrix} 0 \\ \\ \dfrac{T_L}{K_T} \end{bmatrix}$$

$\zeta = 1$

$$\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \dfrac{T_L}{J_T} - \omega_N\dot{\theta}_o(0) - \omega_N^2\theta_o(0) \\ \\ \dot{\theta}_o(0) + \omega_N\theta_o(0) - \dfrac{T_L}{\sqrt{K_T J_T}} \end{bmatrix} t e^{-\omega_N t} + \begin{bmatrix} \dot{\theta}_o(0) \\ \\ \theta_o(0) - \dfrac{T_L}{K_T} \end{bmatrix} e^{-\omega_N t} + \begin{bmatrix} 0 \\ \\ \dfrac{T_L}{K_T} \end{bmatrix}$$

$\zeta > 1$

$$\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = e^{-\omega_N\zeta t}\left\{ \begin{bmatrix} \dot{\theta}_o(0) \\ \\ \theta_o(0) - \dfrac{T_L}{K_T} \end{bmatrix} \cos\omega_N\sqrt{1-\zeta^2}\,t + \dfrac{1}{\sqrt{1-\zeta^2}}\begin{bmatrix} \dfrac{T_L}{K_T J_T} - \zeta\dot{\theta}_o(0) - \omega_N\theta_o(0) \\ \\ \dfrac{\dot{\theta}_o(0)}{\omega_N} + \zeta\theta_o(0) - \dfrac{T_L}{K_T}\zeta \end{bmatrix} \sin\omega_N\sqrt{1-\zeta^2}\,t \right\} + \begin{bmatrix} 0 \\ \\ \dfrac{T_L}{K_T} \end{bmatrix}$$

Region 3 (Positive torque saturation)

$$\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \dot{\theta}_o(0) + \dfrac{-T_{SAT}-T_L}{B_M} \\ \\ \dfrac{-\dot{\theta}_o(0) - (-T_{SAT}-T_L)/B_M}{2\zeta_M\omega_N} \end{bmatrix} e^{-2\zeta_M\omega_N t} - \begin{bmatrix} 0 \\ \\ \dfrac{-T_{SAT}-T_L}{B_M} \end{bmatrix} t + \begin{bmatrix} -\dfrac{-T_{SAT}-T_L}{B_M} \\ \\ \theta_o(0) + \dfrac{\dot{\theta}_o(0) + (-T_{SAT}-T_L)/B_M}{2\zeta_M\omega_N} \end{bmatrix}$$

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However, because of system nonlinearities; namely, saturation and coulomb friction, the actual system bandwidth is a function of the amplitude of the input sinusoid. The necessary procedure for including this nonlinear effect is developed as part of this study. The development is included as Appendix C and is based on the use of describing function approximations to obtain effective values for ω_N and ζ.

The eight system specifications that have now been described are summarized in Table 2.4. This table lists the name, symbol, and number assigned to each specification, tells whether the specification is an upper or lower bound, and the units used. In addition to specifying any desired combination of the above described eight performance requirements, the user must also define the load that the servo is to drive. For the example program developed as part of this study, the load is represented by an inertia (J_ℓ) and a nonlinear coulomb friction (T_ℓ).

Table 2.4. System specifications.

Name	Symbol	Boundary	Units
Static accuracy	S_1	upper	degrees
Resolution	S_2	upper	degrees
Velocity lag	S_3	upper	degrees
Follow-up rate	S_4	lower	deg/sec
Damping ratio	S_5	lower	-
Allowable backlash	S_6	upper	minutes
Overshoot	S_7	upper	degrees
Bandwidth	S_8	lower	hertz

The analysis problem can be now defined mathematically by letting S, Y, and X be vectors, defined in general as:

$$S = \begin{bmatrix} S_1, & S_2, & \cdots, & S_k \end{bmatrix}$$

$$Y = \begin{bmatrix} Y_1, & Y_2, & \cdots, & Y_k \end{bmatrix}$$

$$X = \begin{bmatrix} X_1, & X_2, & \cdots, & X_n \end{bmatrix} \qquad (2.19)$$

where

k = number of performance specifications

n = number of component parameters

S_i = numerical value for the ith specification as defined in Table 2.4 ($1 \leq i \leq k$)

Y_i = system performance function corresponding to i^{th} specification ($1 \leq i \leq k$)

X_j = numerical value for j^{th} component parameter ($1 \leq j \leq n$)

Thus one can write in general that

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} = \begin{bmatrix} F_1(X_1, X_2, X_3, \cdots, X_n) \\ F_2(X_1, X_2, X_3, \cdots, X_n) \\ \vdots \\ F_k(X_1, X_2, X_3, \cdots, X_n) \end{bmatrix}$$

$$(2.20)$$

It is only necessary, at this time, that the X vector contain the elements as required to calculate the system performance function vector Y . However, it is convenient to include the component costs as part of the X vector [even though they will not show up explicitly in (2.20)] since they are required to calculate the

optimization function that is introduced later. Following this practice for our particular example, k = 8 and n = 23, where the X vector is defined in Table 2.5. Likewise, the F functions are defined in Table 2.6.

Table 2.5. Component parameter definitions for library.

COMP	VAR	PARAMETER NAME	SYMBOL	UNITS
F O L L O W U P	X_1	Cost	C_f	dollars
	X_2	Gain	K_f	volts/rad
	X_3	Accuracy	θ_f	minutes
A M P L I F I E R	X_4	Cost	C_a	dollars
	X_5	Gain to Followup	K_{af}	volts/volt
	X_6	Gain to Generator	K_{ag}	volts/volt
	X_7	Output Saturation Level	E_{sat}	volts
	X_8	Output Null Voltage	E_{an}	volts
M O T O R — G E N E R A T O R	X_9	Cost	C_m	dollars
	X_{10}	Stall Torque	T_s	oz-in
	X_{11}	No-load Speed	$\dot{\theta}_m$	rpm
	X_{12}	Inertia	J_m	gm-cm^2
	X_{13}	Starting Voltage	E_s	volts
	X_{14}	Generator Gain	K_g	volts/1000 rpm
	X_{15}	Generator Null	E_{gn}	millivolts
G E A R T R A I N	X_{16}	Cost	C_g	dollars
	X_{17}	Inertia	J_g	gm-cm^2
	X_{18}	Stiffness	K_s	oz-in/rad
	X_{19}	Friction	T_g	oz-in
	X_{20}	Backlash	B	minutes
	X_{21}	Gear Ratio	N	-
L O A D	X_{22}	Inertia	J_ℓ	gm-cm^2
	X_{23}	Friction	T_ℓ	oz-in

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Table 2.6. Location of F Functions.

Function	Location
F_1	Equation (2.9)
F_2	Equation (2.10)
F_3	Equation (2.11)
F_4	Equation (2.12)
F_5	Equation (2.13)
F_6	Actual backlash (X_{20})
F_7	Table 2.3 with logic from Figure B.3
F_8	Equation (2.18) with ω_N and ζ replaced by effective values as defined in Appendix C

Thus (2.20) can be used to calculate the system performance vector (Y) given any component vector (X). By programming this equation as presented, one obtains the desired analysis program except for one deficiency. That is, due to manufacturing tolerances, the X vector varies from unit to unit, and we are interested not in a particular value of Y but what spread or limits to expect. For this reason, the next section is devoted to selecting a suitable method for determining this tolerance spread.

2.4 VARIABILITY ANALYSIS TECHNIQUES

Variability Analysis refers to the methods used to determine the ability of a system to continue to give specified performance while its component parts change value within specified limits.

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One method of insuring that a given system design meets all of the specified performance critera is to use some form of worst-case analysis. An example of this type of procedure is MANDEX which is a worst-case circuit analysis computer program [19]. Using this program the first derivative of all the output variables with respect to each of the input parameters is used to set each of the parameters to their "worst-case" tolerance extreme, so that a "worst-case" condition exists at each of the circuit outputs.

For multivariable systems, the application of the worst-case method becomes very time consuming, even when using a high speed computer, due to the multitude of possibilities that must be considered. Even if the worst-case stackup can be found, the resulting design is unrealistic since it assumes that everything is at worst-case on the same system at the same time. Using this criteria consistently throughout the whole design invariably results in component tolerance requirements that are so tight the cost is prohibitive. The resulting system is greatly overdesigned.

The application of statistical tolerance theory to iterative design problems overcomes this difficulty and provides a most realistic picture of the control system behavior to be expected in production. Statistical tolerance theory was first introduced by Shewhart in his book "Economic Control of Quality of Manufactured Products" [20]. Following this, S. S. Wilks of Princeton University in the early 1940's published two papers [21], [22] that developed the statistical foundation for tolerance theory. However these articles and those that followed [23], [24], [25] up until as late

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as 1963 concerned themselves almost universally with the design
problem of assigning tolerances to the physical dimensioning of me-
chanical parts. From a systems point of view, this case applies
only to the situation where the system function is a linear combina-
tion of the component parameters. In general, and for this example,
system performance is a complex nonlinear function of the component
parameters and the simple root-sum-square technique is not adequate.
The Monte Carlo and Moment methods developed in the last few years
may be applied to handle this problem. Both techniques are pre-
sented and the merits of each are compared by D. G. Mark [26] and
by Mark and L. H. Stember [27].

The Monte Carlo technique assumes that each component parameter
under investigation can be represented by a frequency distribution.
A system is simulated mathematically by choosing each parameter value
randomly from its frequency distribution. After parameter values for
each component in the system are selected, a solution is obtained.
Then each parameter value is again chosen as before and another solu-
tion is obtained. This sequence is repeated many times, resulting
in a tabulation of data representing the distributions of the desired
output variables. From this, the resulting mean and 3 sigma values
can be calculated.

The Moment technique makes use of an expansion of the function
about the mean parameters using a Taylor series. The higher order
terms of the series are usually neglected. This requires taking the
partial derivative of each performance variable with respect to each
component parameter. Assuming that the component performance

parameters are independent and noting that the $\partial Y_i/\partial X_j = 0$ if X_j is a component cost, the mean value of Y_i is given by the equation

$$Y_i(\text{mean}) = F_i\left[X_1(\text{mean}), X_2(\text{mean}), \cdots, X_n(\text{mean})\right] \qquad (2.21)$$

and the standard deviation of Y_i is approximated by the equation:

$$\sigma_{Y_i} = \sqrt{\left[\left(\sigma_{X_1}\right)\frac{\partial Y_i}{\partial X_1}\right]^2 + \left[\left(\sigma_{X_2}\right)\frac{\partial Y_i}{\partial X_2}\right]^2 + \cdots + \left[\left(\sigma_{X_n}\right)\frac{\partial Y_i}{\partial X_n}\right]^2}$$

$$(2.22)$$

where $i = 1, 2, \cdots, k$ and the partial derivatives are evaluated while all other parameters are held at their mean value.

Since the higher order derivatives are neglected, the Moment method prediction is considered less accurate than the Monte Carlo method, but still adequate for most purposes. The Moment method has the advantage that it provides information that is extremely useful to the designer in pinpointing sensitive areas and reducing this sensitivity to parameter variability. Because of this latter advantage and the fact that satisfactory results can be obtained with a lesser number of computer runs, the Moment method is used here.

As can be seen from (2.22), the use of the Moment method requires that we calculate the partial derivatives of each system performance function with respect to each component parameter. The matrix of these partials is the Jacobian

$$J = \frac{\partial\left(Y_1, Y_2, \cdots, Y_k\right)}{\partial\left(X_1, X_2, \cdots, X_n\right)} = \begin{bmatrix} \dfrac{\partial Y_1}{\partial X_1} & \dfrac{\partial Y_1}{\partial X_2} & \cdots & \dfrac{\partial Y_1}{\partial X_n} \\[2ex] \dfrac{\partial Y_2}{\partial X_1} & \dfrac{\partial Y_2}{\partial X_2} & \cdots & \dfrac{\partial Y_2}{\partial X_n} \\[1ex] \vdots & \vdots & & \vdots \\[1ex] \dfrac{\partial Y_k}{\partial X_1} & \dfrac{\partial Y_k}{\partial X_2} & \cdots & \dfrac{\partial Y_k}{\partial X_n} \end{bmatrix}$$

(2.23)

Approximation of these partials is easily obtained numerically by programming (2.20) and using a subroutine to make the following steps:

1) Set all the X_i's equal to their mean value $(\bar{X}_i)$, and the calculated Y vector is taken to be the mean value $\bar{Y}$.

2) X_1 is replaced by $(\bar{X}_1 + \Delta X_1)$ and the corresponding value of Y is calculated with all other X's at their mean value. From this, we obtain the first column of the Jacobian matrix using

$$\frac{\partial Y_i}{\partial X_j} \simeq \frac{Y_i - \bar{Y}_i}{\Delta X_j} \qquad \text{for} \quad i = 1, 2, \ldots, k \quad \text{and} \quad j = 1$$

3) Step 2 is repeated for each X_j for $J = 1, 2, \cdots, n$ thereby obtaining the complete Jacobian matrix.

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3. DEVELOPMENT OF COMPUTER OPTIMIZATION DESIGN PROCEDURE

3.1 BASIC APPROACH

Use of the computer-aided design procedure developed in the previous section, although many times more effective than any manual method, nevertheless represents only a passive use of the digital computer. That is, the engineer makes all the design decisions and the computer only serves as a fast calculator. The next logical step toward optimized design is to use the computer to determine how the components should be varied to converge on the desired minimum cost system.

Figure 3.1 illustrates in general, how a computer could be used in a dynamic sense. The prerequisite to design is to input the data for all components. This is accomplished by loading in the component data cards pre-punched in a prescribed format. This need be done only the first time and thereafter only if that data is to be changed, e.g., updated. These data are then stored by part number in an easily retrievable form on magnetic disk and are referred to as the "component libraries." In order to provide the mainline design program with a guide as to part number selection, some ordered array of these is desired. This is accomplished by using a "search matrix library," the precise working of which is explained later. Thus, immediately
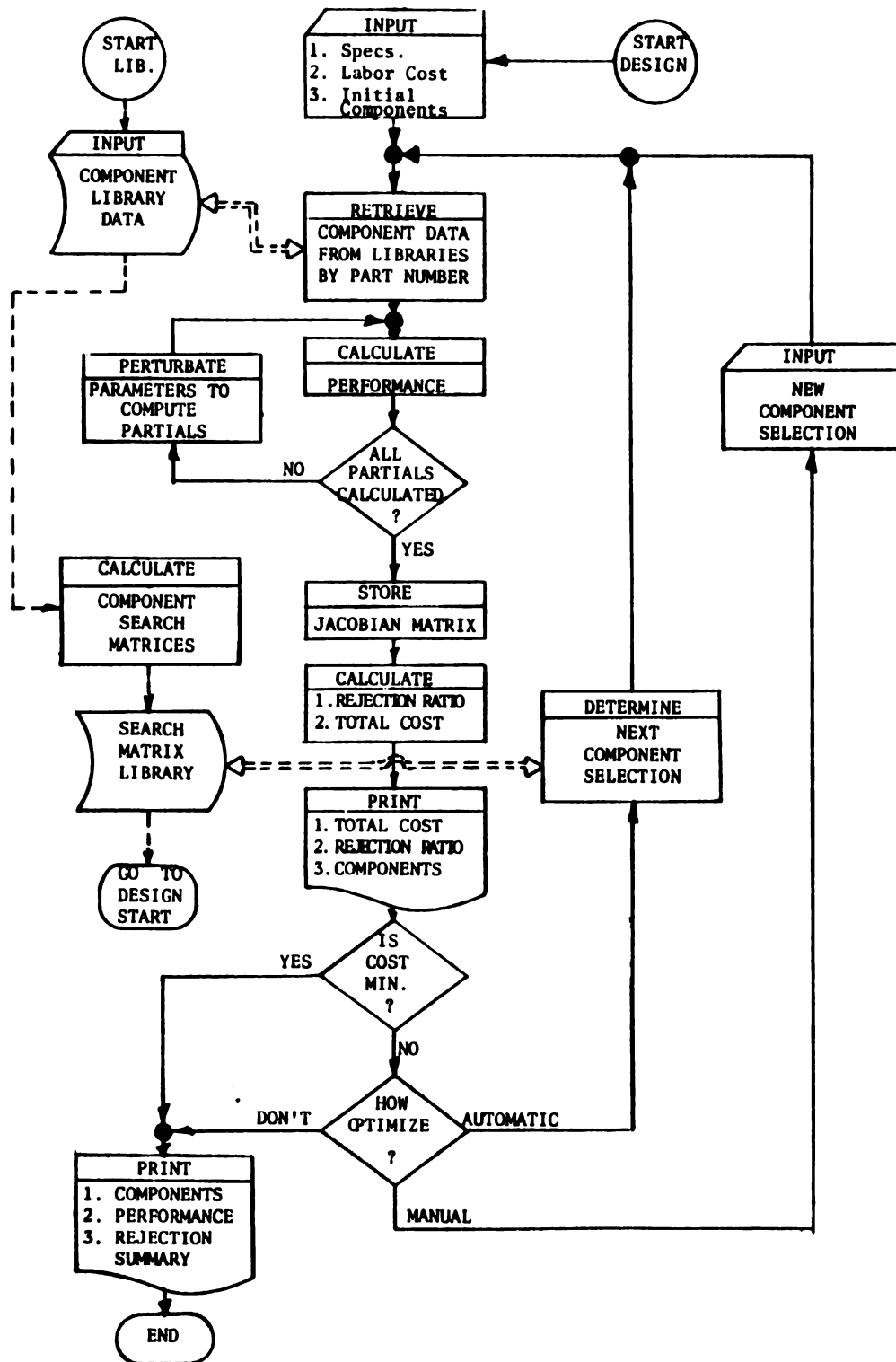
Figure 3.1. Computer aided design program flow chart.

after generation of the component libraries, the computer calculates
the component search matrices and stores these in a second block of
data -- the search matrix library. Now the program is ready to be
used. The designer inputs the system specifications, fixed produc-
tion labor cost, and any initial set of components of his choice.
The latter item could be made a random selection if desired. In
either event, the computer retrieves the component data from libra-
ries and proceeds to calculate the system performance. The component
parameters are then perturbated one at a time and the partials of
each system performance function with respect to each component para-
meter are determined. Once this is completed the partials are stored
in the form of a Jacobian matrix. The calculated performance limits
are then compared to the specification limits. The fraction of the
units produced that statistically fall outside of the specification
limits is then calculated as the "rejection ratio." From this rejection
ratio, the fixed labor cost, and the summation of the parts cost, the
total cost is calculated. A printout is then made so that the user
can follow the steps that the computer makes. Following this, some
method must be employed to determine if cost is a minimum. If it is,
then a final printout can be made. If it is not, then an option is
shown as to how one wants to optimize. This can be accomplished by
the user reading in another set of part numbers or the computer
automatically can select a set in the manner described in Section 3.5
using the search matrix library. This procedure is repeated in an
iterative manner until the optimum design is reached.

There are many associated details that are not shown in Figure 3.1. This diagram, however, gives the general outline of the procedure.

3.2 GENERATION OF OBJECT FUNCTIONS

The first question that must be answered in an optimization problem is, "What is to be optimized and what is optimum?" Often, this is not a trivial problem in itself since there are many separate and usually conflicting factors; i.e., minimum cost, maximum accuracy, small volume, best response, etc. These factors may be considered simultaneously by defining a scalar F of the form

$$F = \sum_{i=1}^{k} A_i (Y_i - D_i)^2 \tag{3.1}$$

where:

F = object function to be minimized

k = number of desired properties

A_i = weight factor selected to give the ith property the desired priority

Y_i = current value of ith property

D_i = desired value for i^{th} property

A serious difficulty inherent in this approach, however, consists in finding a set of weighting factors A_1, A_2, ..., A_k such that scaling between the various terms is properly considered in order to maintain sensitivity and obtain good convergence. Considering

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properties such as accuracy, weight, cost and response, these weight selections often become subjective in nature.

It is proposed in this thesis that an entirely different object function shall be used. It is founded on the competitive philosophy that the manufacturer wants a design that fulfills the customer requirements at minimum overall cost. With this result, he can either maximize his chances of competing or if his sale price is "fixed" he maximizes his profits. Using this minimum cost philosophy, an appropriate object function can be generated in the following manner.

The total cost to build a given number of systems is represented by the equation

$$
\begin{matrix} \text{Total} \\ \text{Cost} \end{matrix} = \begin{matrix} \text{Number} \\ \text{Built} \end{matrix} \left[\begin{matrix} \text{Labor} \\ \text{Cost} \end{matrix} + \sum \begin{matrix} \text{Component} \\ \text{Costs} \end{matrix} \right] \left[1 + \begin{matrix} \text{Overhead} \\ \text{Ratio} \end{matrix} \right]
$$

(3.2)

However, the number that must be built for a given contract is given by

$$
\begin{matrix} \text{Number} \\ \text{Built} \end{matrix} = \frac{\text{Number Required}}{\left[1 - \begin{matrix} \text{Rejection} \\ \text{Ratio} \end{matrix} \right]}
$$

(3.3)

Thus, we have for the total cost

$$
\begin{matrix} \text{Total} \\ \text{Cost} \end{matrix} = \frac{\text{Number Required}}{\left[1 - \begin{matrix} \text{Rejection} \\ \text{Ratio} \end{matrix} \right]} \left[\begin{matrix} \text{Labor} \\ \text{Cost} \end{matrix} + \sum \begin{matrix} \text{Component} \\ \text{Costs} \end{matrix} \right] \left[1 + \begin{matrix} \text{Overhead} \\ \text{Ratio} \end{matrix} \right]
$$

(3.4)

Since the number of required units and (1 + overhead ratio) are product terms which are not functions of the components, one obtains the same cost minimizing set of components using the function

$$\text{Cost} = \frac{\dfrac{\text{Labor}}{\text{Cost}} + \sum \dfrac{\text{Component}}{\text{Costs}}}{1 - \dfrac{\text{Rejection}}{\text{Ratio}}} \tag{3.5}$$

Equation (3.5) is the object function used in this thesis for what is defined later as "the fine search mode." When it is at a minimum, the desired optimum set of components has been defined. However, one problem may exist in the early portion of the iteration cycle. That is, the design can be so far away from specification that, for all practical purposes, the rejection ratio is unity, the denominator of (3.5) goes to zero, resulting in infinite cost. As long as this occurs, (3.5) has no practical value. In fact, one loses all sensitivity in calculating partials, and there is no way of telling if one design is better than another. For this reason, a "coarse search mode" is defined. Its corresponding object function is:

$$F = \sum_{i=1}^{k} A_i R_i \left(Y_i - S_i \right)^2 \tag{3.6}$$

where

F = object function to be minimized

k = number of specifications to be met

A_i = weight factor for i^{th} specification

R_i = rejection ratio for i^{th} specification

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Y_i = calculated system performance 3 sigma limit corresponding to i^{th} specification

S_i = i^{th} specification limit

It should be further noted that

$$Y_i = \overline{Y}_i - 3\sigma_{Y_i} \quad \text{if} \quad S_i \quad \text{is a lower limit, and}$$

$$Y_i = \overline{Y}_i + 3\sigma_{Y_i} \quad \text{if} \quad S_i \quad \text{is an upper limit.}$$

Since Equation (3.6) is used only in the coarse search mode, selection of the weight factors is not too critical. For this study, A_i was set at $1/S_i^2$ except for the case when S_i equals zero and then A_i was arbitrarily set equal to unity.

In the coarse search mode, cost is neglected in an attempt to determine the performance such that the rejection ratio becomes less than unity. The incorporation of the R_i term in (3.6) greatly aids in the accomplishment of this condition. First it nulls each term in the summation which represents an overdesigned condition (i.e. $R_i=0$) and secondly it applies a linearily increasing weight on the others according to their significance.

Once each of the R_i's is driven less than unity, the cost becomes finite, and the optimization process is switched from the coarse to the fine search where (3.5) is used as the object function.

3.3 CALCULATION OF REJECTION RATIO

Let $S_1, \cdots, S_k$ be the k specification limits for a given design, e.g., static accuracy, overshoot, etc. There corresponds

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then, k random variables $Y_1, \cdots, Y_k$ that represent the actual performance to be expected. Since these are a function of the n component equations, one can write as before that

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} = \begin{bmatrix} F_1 \left(X_1, \cdots, X_n \right) \\ F_2 \left(X_1, \cdots, X_n \right) \\ \vdots \\ F_k \left(X_1, \cdots, X_n \right) \end{bmatrix} \qquad (3.7)$$

Looking at small perturbations

$$\begin{bmatrix} \Delta Y_1 \\ \Delta Y_2 \\ \vdots \\ \Delta Y_k \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_n \end{bmatrix} \qquad (3.8)$$

where the $k \times n$ matrix has the general element $a_{ij} = \dfrac{\partial Y_i}{\partial X_j}$ and it therefore is identical to the Jacobian matrix (J) as defined in (2.23).

The joint density of the Y's is given by:

$$f_{Y_1, Y_2, \cdots Y_k} \left(y_1, y_2, \cdots y_k \right) = \frac{e^{-\left[(Y-\bar{Y})M_Y^{-1}(Y-\bar{Y})^T \right]}}{\left(2\pi \right)^{k/2} \sqrt{|M_Y|}}$$

$$(3.9)$$

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where:

$$(Y-\bar{Y}) = \left[(y_1-\bar{y}_1),(y_2-\bar{y}_2), \cdots, (y_k-\bar{y}_k) \right]$$

and the $(k \times k)$ covariance matrix M_Y is

$$M_Y = JM_X J^T \tag{3.10}$$

Since the component performance parameters are assumed independent and $\sigma_{X_i} = 0$ if X_i is a component cost, one can write the component covariance matrix M_X as

$$M_X = \begin{bmatrix} \sigma_{X_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{X_2}^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{X_n}^2 \end{bmatrix} \tag{3.11}$$

Since the total rejection ratio R is the probability of a design falling outside of the specification, and assuming that the specification limits are constant, it is given by

$$R = 1 - \int_{L_{11}}^{L_{12}} \int_{L_{21}}^{L_{22}} \cdots \int_{L_{k1}}^{L_{k2}} f_{Y_1,Y_2,\cdots,Y_k}\left(y_1,y_2,\cdots,y_k\right) dy_1 dy_2 \cdots dy_k \tag{3.12}$$

where:

$$\left\{ \begin{array}{l} L_{i1} = -\infty \\ L_{i2} = S_i \end{array} \right\} \quad \text{for the } i^{th} \text{ specification an upper bound}$$

$$\left\{ \begin{array}{l} L_{i1} = S_i \\ L_{i2} = \infty \end{array} \right\} \quad \text{for the } i^{th} \text{ specification a lower bound}$$

and

$f_{Y_1, Y_2, \cdots, Y_k}(y_1, y_2, \cdots, y_k)$ is given by (3.9).

In order to evaluate R using (3.12), one must evaluate the multiple integral of dimension k where $k = 8$ for the example in this study. This can be accomplished using numerical techniques [28], and [29], however, the process is very time consuming. In the interest of minimizing computer time, three alternate procedures are considered.

First one could use the upper bound on R which is simply

$$R(\text{upper bound}) = \sum_{i=1}^{k} R_i \quad \text{if} \quad \sum_{i=1}^{k} R_i < 1$$

$$= 1 \quad \text{otherwise} \tag{3.13}$$

where R_i is the individual rejection ratio corresponding to the ith specification and is calculated as

$$R_i = 1 - \frac{1}{\sqrt{2\pi\sigma_{Y_i}^2}} \int_{L_{i1}}^{L_{i2}} e^{-\frac{1}{2}\left(\frac{y-\mu_{Y_i}}{\sigma_{Y_i}}\right)^2} dy \tag{3.14}$$

where the limits of the integral are as defined for (3.12).

Equation (3.14) can be evaluated by using the standard error function

$$ERF(z) = \frac{2}{\sqrt{\pi}} \int_{o}^{z} e^{-u^2} du \qquad (3.15)$$

for the upper limit case

$$R_i = 0.5 \left[1 - ERF\left(\frac{S_i - \mu_{Y_i}}{\sqrt{2}\ \sigma_{Y_i}}\right)\right] \qquad \text{for} \quad S_i \geq \mu_{Y_i}$$

$$= 0.5 \left[1 + ERF\left(\frac{\mu_{Y_i} - S_i}{\sqrt{2}\ \sigma_{Y_i}}\right)\right] \qquad \text{for} \quad S_i < \mu_{Y_i}$$

$$(3.16)$$

and for the lower limit case

$$R_i = 0.5 \left[1 + ERF\left(\frac{S_i - \mu_{Y_i}}{\sqrt{2}\ \sigma_{Y_i}}\right)\right] \qquad \text{for} \quad S_i \geq \mu_{Y_i}$$

$$= 0.5 \left[1 - ERF\left(\frac{\mu_{Y_i} - S_i}{\sqrt{2}\ \sigma_{Y_i}}\right)\right] \qquad \text{for} \quad S_i < \mu_{Y_i}$$

$$(3.17)$$

A second possibility for approximating the total rejection ratio R is to use the lower bound given by

$$R(\text{lower bound}) = R_j \qquad\qquad (3.18)$$

where:

$$R_j \leq R_i \qquad \text{for all} \quad 1 \leq i \leq k$$

Since (3.13) represents an overdesigned case and (3.18) an underdesigned case, it would be wise to have available an approximation that lies between these extremes. A quantity which has this property is

$$R(\text{independent}) = 1 - \prod_{i=1}^{k} \left(1 - R_i\right) \qquad\qquad (3.19)$$

which is equal to the true R for the case when the Y's are independent.

For the example program, the user is given the opportunity of selecting either the R(upper bound) or R(independent) approximations. The R(lower bound), although readily available, is eliminated as a choice since it is never on the safe side.

One difficulty remains since (3.16) cannot be used to calculate the rejection ratio for the null oscillation specification. This specification that no null oscillation shall exist is converted by the computer to a specification limit on the actual backlash. This limit is not a constant but a random variable computed as described in Appendix A. Therefore, the rejection ratio must be computed by examining two frequency distributions, namely that of the allowable backlash and that of the actual backlash of the geartrain being considered. Thus a separate subroutine was written to calculate R_6 the derivation of which is explained in the remainder of this section.

For this derivation only, the random variable Y is used to represent the actual backlash and S the allowable backlash. Since both are assumed to be normally distributed their density functions are defined as

$$g_1(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2} \qquad (3.20)$$

$$g_2(s) = \frac{1}{\sqrt{2\pi\sigma_S^2}} e^{-\frac{1}{2}\left(\frac{s-\mu_S}{\sigma_S}\right)^2} \qquad (3.21)$$

and the corresponding rejection ratio is given by the probability that $Y > S$ as

$$R_6 = P(Y > S) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} g(y,s)\, ds\, dy \qquad (3.22)$$

and since Y and S are independent

$$g(y,s) = g_1(y) g_2(s) \qquad (3.23)$$

By using (3.23 in (3.22) and substituting in for $g_1(y)$ and $g_2(s)$ using (3.20) and (3.21), and simplifying, (3.22) can be written as

$$R_6 = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2} \frac{1}{\sqrt{2\pi\sigma_S^2}} \int_{-\infty}^{y} e^{-\frac{1}{2}\left(\frac{s-\mu_S}{\sigma_S}\right)^2} ds\, dy$$

$$(3.24)$$

Letting

$$A(y) = \frac{1}{\sqrt{2\pi\sigma_S^2}} \int_{-\infty}^{y} e^{-\frac{1}{2}\left(\frac{s-\mu_S}{\sigma_S}\right)^2} ds \tag{3.25}$$

where $A(y)$ can be evaluated by using the standard error function as before, one obtains

$$A(y) = 0.5 \left[1 + \text{ERF}\left(\frac{y-\mu_S}{\sqrt{2}\,\sigma_S}\right) \right] \qquad \text{for} \quad y \geq \mu_S$$

$$= 0.5 \left[1 - \text{ERF}\left(\frac{\mu_S-y}{\sqrt{2}\,\sigma_S}\right) \right] \qquad \text{for} \quad y < \mu_S \tag{3.26}$$

and since one is interested only in the region inside the 3-sigma limits, R_6 is evaluated as

$$R_6 = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \int_{\text{YMIN}}^{\text{YMAX}} A(y)\, e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} dy \tag{3.27}$$

where $A(y)$ is evaluated using (3.26) and YMAX and YMIN are taken to be $\mu_Y + 3\sigma_Y$ and $\mu_Y - 3\sigma_Y$ respectively.

3.4 OBJECT FUNCTION DERIVATIVES

It is of necessity that the partial derivatives of the object function be calculated in the steepest ascent method of optimization. If these derivatives were somehow known for the direct search technique, it would be of advantage since one could then conduct exploratory moves in descending order of importance. In our case, it would be a major task to perturbate each of the component parameters again and calculate the resulting change in the object function to obtain the partial derivatives. It is shown, however, that these can be obtained directly from the Jacobian matrix which is already available from the tolerance calculations, namely, Equation (2.23). This is accomplished in the following manner as derived first for the fine search and then for the coarse search.

The object function used in fine search, Equation (3.5), can be written as

$$C(X) = [K+f(X)][1-R(X)]^{-1} \qquad (3.28)$$

where

X =   component parameter vector $\left[X_1, X_2, \cdots, X_n \right]$

$C(X)$ =   total system cost

K =   labor cost

$R(X)$ =   rejection ratio

$f(X)$ = Σ component cost

Taking the partial derivative of C with respect to X_i

$$\frac{\partial C}{\partial X_i} = \frac{\partial f(X)}{\partial X_i}\left(1-R(X)\right)^{-1} + \left(K+f(X)\right)\left(1-R(X)\right)^{-2}\frac{\partial R}{\partial X_i} \qquad (3.29)$$

and expanding to include all X_i

$$\left[\frac{\partial C}{\partial X_1} , \frac{\partial C}{\partial X_2} , \cdots , \frac{\partial C}{\partial X_n}\right] = \frac{1}{1-R(X)}\left[\frac{\partial f}{\partial X_1} , \frac{\partial f}{\partial X_2} , \cdots , \frac{\partial f}{\partial X_n}\right]$$

$$+ \frac{K+f(X)}{\left(1-R(X)\right)^2}\left[\frac{\partial R}{\partial X_1} , \frac{\partial R}{\partial X_2} , \cdots , \frac{\partial R}{\partial X_n}\right] \tag{3.30}$$

The latter vector $(\partial R/\partial X)$ can be obtained by making use of the Jacobian defined by (2.23). Thus

$$\left[\frac{\partial R}{\partial X_1} , \frac{\partial R}{\partial X_2} , \cdots , \frac{\partial R}{\partial X_n}\right] = \left[\frac{\partial R}{\partial Y_1} , \frac{\partial R}{\partial Y_2} , \cdots , \frac{\partial R}{\partial Y_k}\right]\begin{bmatrix} \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial Y_k}{\partial X_1} & \cdots & \frac{\partial Y_k}{\partial X_n} \end{bmatrix}$$

$$\tag{3.31}$$

Substituting (3.31) into (3.30) one obtains the desired matrix equation for the fine search cost derivative vector as

$$\left[\frac{\partial C}{\partial X_1} , \frac{\partial C}{\partial X_2} , \cdots , \frac{\partial C}{\partial X_n}\right] = \frac{1}{1-R(X)}\left[\frac{\partial f}{\partial X_1} , \frac{\partial f}{\partial X_2} , \cdots , \frac{\partial f}{\partial X_k}\right]$$

$$+ \frac{K+f(X)}{\left(1-R(X)\right)^2}\left[\frac{\partial R}{\partial Y_1} , \frac{\partial R}{\partial Y_2} , \cdots , \frac{\partial R}{\partial Y_k}\right]\begin{bmatrix} \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial Y_k}{\partial X_1} & \cdots & \frac{\partial Y_k}{\partial X_n} \end{bmatrix}$$

$$\tag{3.32}$$

where:

$$\frac{\partial f}{\partial X_i} = \begin{cases} 1 & \text{if } X_i \text{ is a component cost} \\ 0 & \text{otherwise} \end{cases} \qquad (3.33)$$

and the vector

$$\frac{\partial R}{\partial Y_1} , \frac{\partial R}{\partial Y_2} , \cdots, \frac{\partial R}{Y_k}$$

is referred to as the "rejection ratio derivative vector" and given the notation $\partial R / \partial Y$.

The calculation of the $\partial R / \partial Y$ vector, as required for the fine search mode, depends on the particular equation used in approximating the rejection ratio R [i.e., (3.13) or (3.19)]. Consider first the case where R is approximated by the upper bound. Since in the fine search mode

$$\sum_{i=1}^{k} R_i < 1$$

one has

$$R(\text{upper bound}) = R_1 + R_2 + \cdots + R_k \qquad (3.34)$$

and since R_j is a function of Y_i only for $i = j$

$$\frac{\partial R(\text{upper bound})}{\partial Y_i} = \frac{\partial R_i}{\partial Y_i} \qquad \text{for } i = 1, 2, \cdots, k \qquad (3.35)$$

and only the partials of the individual rejection ratios are required. This is also shown to be the case when R is approximated by using the case where the Y's are assumed independent as given by

$$R(\text{independent}) = 1 - (1-R_1)(1-R_2) \cdots (1-R_k)$$

$$(3.36)$$

Again using the fact that R_i is only a function of Y_i , one obtains

$$\frac{\partial R(\text{independent})}{\partial Y_i} = \frac{\partial R_i}{\partial Y_i} \prod_{\substack{j=1 \\ j \neq i}}^{k} (1-R_j)$$

$$(3.37)$$

The task remaining, then, is to obtain expressions for $\partial R_i / \partial Y_i$.

For the case where the specification limit is a constant, the magnitude of $\partial R_i / \partial Y_i$ is given by the Y_i density function evaluated at the point $y_i = S_i$ and the sign of $\partial R_i / \partial Y_i$ depends on whether S_i is an upper or a lower bound. That is

$$\frac{\partial R_i}{\partial Y_i} = \frac{\pm 1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{1}{2}\left[\frac{S_i - \mu_{Y_i}}{\sigma_{Y_i}}\right]^2}$$

$$(3.38)$$

where

$$S_i = i^{\text{th}} \text{ specification limit}$$

$$\mu_{Y_i} = \text{mean value of } Y_i \text{ distribution}$$

$$\sigma_{Y_i} = \text{standard deviation of } Y_i \text{ distribution}$$

and the + sign is taken if S_i is an upper limit and the - sign is taken if S_i is a lower limit.

For the case where the specification limit is not a constant but a random variable (e.g., S_6), the corresponding equation is

$$\frac{\partial R_i}{\partial Y_i} = \int_{-\infty}^{\infty} g_1(z) g_2(z) dz \tag{3.39}$$

where the density functions $g_1(\cdot)$ and $g_2(\cdot)$ are defined by (3.20) and (3.21) respectively. The solution to (3.39) is approximated in the example design program by using numerical integration over the region from $\mu_{Y_i} - 3\sigma_{Y_i}$ to $\mu_{Y_i} + 3\sigma_{Y_i}$.

In summary, Equation (3.32) gives the required partial derivatives of cost with respect to each component parameter. The necessary elements of the rejection ratio derivative vector are obtained using either (3.35) or (3.37) and with the $\partial R_i / \partial Y_i$ entries furnished by either (3.38) or (3.39) as the requirements dictate. A similar development is presented now for the coarse search mode.

The object function used for coarse search is of the form [see (3.6)]

$$F(X) = A_1 R_1(X) \left[Y_1(X) - S_1 \right]^2 + A_2 R_2(X) \left[Y_2(X) - S_2 \right]^2 + \cdots + A_k R_k(X) \left[Y_k(X) - S_k \right]^2 \tag{3.40}$$

where for the case of the example program the value used for S_6 is taken to be its mean value.

Taking the partial derivative of F with respect to X_1

$$\frac{\partial F}{\partial X_1} = 2A_1 R_1\left(Y_1 - S_1\right)\frac{\partial Y_1}{\partial X_1} + A_1\left(Y_1 - S_1\right)^2 \frac{\partial R_1}{\partial X_1}$$

$$+ 2A_2 R_2\left(Y_2 - S_2\right)\frac{\partial Y_2}{\partial X_1} + A_2\left(Y_2 - S_2\right)^2 \frac{\partial R_2}{\partial X_1}$$

$$+ \cdots + 2A_k R_k\left(Y_k - S_k\right)\frac{\partial Y_k}{\partial X_1} + A_k\left(Y_k - S_k\right)^2 \frac{\partial R_k}{\partial X_1}$$

$$(3.41)$$

Thus, in total vector form:

$$\left[\frac{\partial F}{\partial X_1}, \frac{\partial F}{\partial X_2}, \cdots, \frac{\partial F}{\partial X_n}\right] = 2 \begin{bmatrix} A_1 R_1\left(Y_1 - S_1\right) \\ A_2 R_2\left(Y_2 - S_2\right) \\ \vdots \\ A_k R_k\left(Y_k - S_k\right) \end{bmatrix}^T \begin{bmatrix} \dfrac{\partial Y_1}{\partial X_1} & \dfrac{\partial Y_1}{\partial X_2} & \cdots & \dfrac{\partial Y_1}{\partial X_n} \\ \dfrac{\partial Y_2}{\partial X_1} & \dfrac{\partial Y_2}{\partial X_2} & \cdots & \dfrac{\partial Y_2}{\partial X_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial Y_k}{\partial X_1} & \dfrac{\partial Y_k}{\partial X_2} & & \dfrac{\partial Y_k}{\partial X_n} \end{bmatrix}$$

$$+ \begin{bmatrix} A_1\left(Y_1 - S_1\right)^2 \\ A_2\left(Y_2 - S_2\right)^2 \\ \vdots \\ A_k\left(Y_k - S_k\right)^2 \end{bmatrix}^T \begin{bmatrix} \dfrac{\partial R_1}{\partial X_1} & \dfrac{\partial R_1}{\partial X_2} & \dfrac{\partial R_1}{\partial X_n} \\ \dfrac{\partial R_2}{\partial X_1} & \dfrac{\partial R_2}{\partial X_2} & \dfrac{\partial R_2}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial R_k}{\partial X_1} & \dfrac{\partial R_k}{\partial X_2} & \cdots & \dfrac{\partial R_k}{\partial X_n} \end{bmatrix}$$

$$(3.42)$$

Using the further relationship that:

$$
\begin{bmatrix}
\dfrac{\partial R_1}{\partial X_1} & \dfrac{\partial R_1}{\partial X_2} & \cdots & \dfrac{\partial R_1}{\partial X_n} \\[2ex]
\dfrac{\partial R_2}{\partial X_1} & \dfrac{\partial R_2}{\partial X_2} & \cdots & \dfrac{\partial R_2}{\partial X_n} \\[2ex]
\vdots & \vdots & & \vdots \\[2ex]
\dfrac{\partial R_k}{\partial X_1} & \dfrac{\partial R_k}{\partial X_2} & \cdots & \dfrac{\partial R_k}{\partial X_n}
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{\partial R_1}{\partial Y_1} & \dfrac{\partial R_1}{\partial Y_2} & \cdots & \dfrac{\partial R_1}{\partial Y_k} \\[2ex]
\dfrac{\partial R_2}{\partial Y_1} & \dfrac{\partial R_2}{\partial Y_2} & \cdots & \dfrac{\partial R_2}{\partial Y_k} \\[2ex]
\vdots & \vdots & & \vdots \\[2ex]
\dfrac{\partial R_k}{\partial Y_1} & \dfrac{\partial R_k}{\partial Y_2} & \cdots & \dfrac{\partial R_k}{\partial Y_k}
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial Y_1}{\partial X_1} & \dfrac{\partial Y_1}{\partial X_2} & \cdots & \dfrac{\partial Y_1}{\partial X_n} \\[2ex]
\dfrac{\partial Y_2}{\partial X_1} & \dfrac{\partial Y_2}{\partial X_2} & \cdots & \dfrac{\partial Y_2}{\partial X_n} \\[2ex]
\vdots & \vdots & & \vdots \\[2ex]
\dfrac{\partial Y_k}{\partial X_1} & \dfrac{\partial Y_k}{\partial X_2} & \cdots & \dfrac{\partial Y_k}{\partial X_n}
\end{bmatrix}
$$

$$(3.43)$$

and noting that $\dfrac{\partial R_i}{\partial Y_j} = 0$ for all $i \neq j$, and substituting (3.43) into (3.42), one obtains

$$
\begin{bmatrix} \dfrac{\partial F}{\partial X_1} , & \dfrac{\partial F}{\partial X_2} , & \cdots, & \dfrac{\partial F}{\partial X_n} \end{bmatrix}
= 2
\begin{bmatrix}
A_1\left(Y_1 - S_1\right)R_1 + \left(Y_1 - S_1\right)^2 \dfrac{\partial R_1}{\partial Y_1} \\[2ex]
A_2\left(Y_2 - S_2\right)R_2 + \left(Y_2 - S_2\right)^2 \dfrac{\partial R_2}{\partial Y_2} \\[2ex]
\vdots \\[2ex]
A_k\left(Y_k - S_k\right)R_k + \left(Y_k - S_k\right)^2 \dfrac{\partial R_k}{\partial Y_k}
\end{bmatrix}^T
\begin{bmatrix}
\dfrac{\partial Y_1}{\partial X_1} & \dfrac{\partial Y_1}{\partial X_2} & \cdots & \dfrac{\partial Y_1}{\partial X_n} \\[2ex]
\dfrac{\partial Y_2}{\partial X_1} & \dfrac{\partial Y_2}{\partial X_2} & \cdots & \dfrac{\partial Y_2}{\partial X_n} \\[2ex]
\vdots & \vdots & & \vdots \\[2ex]
\dfrac{\partial Y_k}{\partial X_1} & \dfrac{\partial Y_k}{\partial X_2} & \cdots & \dfrac{\partial Y_k}{\partial X_n}
\end{bmatrix}
$$

$$(3.44)$$

Equation (3.44) gives the desired partial derivatives of the coarse search object function with respect to each component parameter in the system. Again, like (3.32), it is in terms of the already available Jacobian matrix and no further parameter perturbations are required.

3.5 DESIGN PROGRAM STRATEGY

The design program developed as part of this study has two basic operating options — analysis and directed search. When operating with the analysis option, the four component part numbers required for each analysis may be either read in from cards or selected at random by the program. In either case, as many consecutive runs are made as requested and a final printout is provided summarizing the best design obtained. Thus the engineer can make a rapid evaluation of a selected number of designs of his choosing, or, he can perform Monte Carlo runs by letting the computer select the part numbers at random.

With the directed search option, the computer program uses the object derivatives in connection with search matrices to direct the next component selection in an attempt to reduce the object function. This process is repeated in an iterative fashion until a local minimum is obtained. Since there is no guarantee that this condition is the absolute minimum, numerous starting points are employed and the one with the lowest cost is assumed to be the best design. The starting points for each search may be specified by the user or otherwise selected at random by the program.

The generation of the search matrices is a prerequisite to a directed search. A separate search matrix is used along with each component library and their generation automatically follows each library update. These matrices consist of an ordered array of the component part numbers defined by

$$S_i = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1\ell} \\ s_{21} & s_{22} & \cdots & s_{2\ell} \\ \vdots & \vdots & & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{m\ell} \end{bmatrix} \qquad (3.45)$$

where

ℓ = the number of parameters used to describe the ith component

m = the number of part numbers for i^{th} component stored in the library

s_{nj} = a component part number for $1 \leq n \leq m$ and $1 \geq j \geq \ell$

Each column of S_i corresponds to a particular parameter of the ith component and the entries of that column consist of all the ith component part numbers arranged in ascending order of the mean value of that parameter. That is, let the jth column of S_i correspond to the kth component parameter of the X vector. Then s_{1j}, s_{2j}, $\cdots$, s_{mj} are chosen such that

$$\overline{X_k(s_{1j})} \leq \overline{X_k(s_{2j})} \leq \overline{X_k(s_{3j})} \leq \cdots \leq \overline{X_k(s_{mj})}$$

$$(3.46)$$

where

$\overline{X_k(s_{nj})}$ signifies the mean value of the component parameter X_k for the part number stored in location s_{nj}

In order to explain the strategy used by the design program to conduct a search, the following definitions are established:

search = minimization process which begins with the initial set of part numbers and ends once a local minimum is found.

base point = set of part numbers for which the object function is less than that calculated for any previous set of part numbers in a given search.

sub-search = that part of a search which takes place between successive base points.

exploratory move = a set of part numbers which are at least tentatively being considered for a system performance analysis.

failure = an exploratory move which is analyzed and the object function obtained is greater than (or equal to) that of the base point.

success = an exploratory move which is analyzed and the object function obtained is less than that of the base point.

local minimum = the object function corresponding to the base point which remains once all the exploratory moves analyzed in a given sub-search result in failure.

Thus a search is made up of many sub-searches and each of the latter are in turn made up of numerous exploratory moves. Each exploratory move consists of changing one component part number while keeping the others fixed at the base point. Once an exploratory move

results in "success," the move is defined as a new base point and a new sub-search is started. This process is repeated until all the exploratory moves of a sub-search are exhausted and no success is found. The base point for this last sub-search defines the local minimum.

The following ten steps describe the general pattern of the program's search strategy:

1) The object function being minimized is SCALAR [defined as F in Equation (3.6)] while in the coarse search mode and COST [Equation (3.5)] while in the fine search mode. The program is in the coarse search mode as long as the total rejection ratio [Equation (3.13) or (3.19)] is equal to unity, once less than unity the program switches to the fine search mode.

2) Each time a lower object function is found, the corresponding part numbers are stored as a new base point.

3) At each new base point, calculations are made to establish the object function derivative vector using Equation (3.44) for the coarse search mode and (3.32) for the fine search mode.

4) Priority and direction vectors are established as the bases for making exploratory moves. The priority vector (IPAR) consists of a re-ordering of the component parameter numbers (i.e., subscripts of the X parameter vector) such that

$$\left|\frac{\partial \text{ object}}{\partial X_{IPAR_1}}\right| \geq \left|\frac{\partial \text{ object}}{\partial X_{IPAR_2}}\right| \geq \cdots \geq \left|\frac{\partial \text{ object}}{\partial X_{IPAR_m}}\right| \qquad (3.47)$$

where m, the dimension of IPAR, equals the number of

component parameters excluding the load (e.g., with the

vector defined in Table 2.5, m = 21). The direction vector

(IDEX) is defined by

$$IDEX_{II} = \frac{\dfrac{\partial\ object}{\partial X_{II}}}{\left|\dfrac{\partial\ object}{\partial X_{II}}\right|} \qquad \text{for } 1 \leq II \leq m \qquad (3.48)$$

Thus

$$IDEX_{II} = +1 \text{ if the } II^{th} \text{ parameter should be increased}$$
$$= -1 \text{ if the } II^{th} \text{ parameter should be decreased}$$

in order to achieve a reduction in the object function.

5) A "sub-search progress number," denoted by the symbol II, is

used by the program as the subscript for the IPAR and IDEX

vectors. It is initialized equal to unity (i.e., II = 1) at

the beginning of each sub-search and incremented under pro-

gram control as the sub-search progresses. As II is

increased from 1 to m, $IPAR_{II}$ corresponds to the component

parameter numbers having decreasing sensitivity values with

respect to the object function. Likewise, $IDEX_{IPAR_{II}}$

corresponds to the desired direction the $IPAR_{II}$ parameter is

to be changed.

6) Each exploratory move is initiated by calling a subroutine,

named SEARCH, to select the new part number which is to be

investigated. This is accomplished using the statement:

$$\text{CALL SEARCH}[IDEX_{IPAR_{II}}, IPAR_{II}, IPN_{JJJ}, IBOUND]$$

where

$IDEX_{IPAR_{II}}$ = direction $IPAR_{II}$ parameter is to be changed

$IPAR_{II}$ = parameter number for change being considered

IPN_{JJJ} = present part number on entering the subroutine and on return it is the new part number to be used

IBOUND = 0 unless present part number is already at the boundary and cannot be changed further, then it is set to 1 by the subroutine

and for this example, the JJJ subscript is established from Table 2.5 as

$$JJ = 1 \text{ for } 1 \leq IPAR_{II} \leq 3$$
$$= 2 \text{ for } 4 \leq IPAR_{II} \leq 8$$
$$= 3 \text{ for } 9 \leq IPAR_{II} \leq 15$$
$$= 4 \text{ for } 16 \leq IPAR_{II} \leq 21$$

The SEARCH subroutine takes the $IPAR_{II}$ entry which corresponds to the subscript of the X vector and seeks the corresponding column of the appropriate search matrix. This column is then searched until the currently used part number is found (IPN_{JJJ}). Once this occurs the subroutine increments either down or up one location depending on whether IDEX is +1 or -1 and replaces the old part number with the new one found. If the old part number happens to be on a

boundary such that a new part number cannot be obtained, the subroutine sets IBOUND to 1 and returns with the old part number. If this occurs, no further minimization can be obtained considering the $IPAR_{II}$ parameter, therefore one returns the part numbers to the base point and increments to the next most significant parameter by increasing the sub-search progress number (II) by 1 and step 6 is repeated.

7) For each new component selected by SEARCH a library sub-routine, named LIBR, is called to retrieve the corresponding parameter data. This is accomplished by the statement

$$\text{CALL LIBR}[IPN_{JJJ},XMAX,XMIN]$$

where

IPN_{JJJ} = part number for which data is desired

XMAX = a vector containing the mean +3 sigma values for the total X parameter vector

XMIN = a vector containing the mean -3 sigma values for the total X parameter vector

The LIBR subroutine takes the part number (IPN_{JJJ}) and searches the appropriate component library, stored off-line on magnetic disk, until the part number is located. Once located its associated parameter data is read back and inserted in the proper locations of the XMAX and XMIN vector. Thus by calling the LIBR subroutine with a part number, one is able to automatically update the 3 sigma limits for the X's corresponding to that part leaving the others unchanged.

8) After the new data is obtained for the exploratory move, the program checks for the existence of two conditions before the system performance is evaluated. The first is used to control the extent that the program explores changes based on a given parameter before it moves on to the next parameter. This is accomplished by calculating a normalized distance (DIST) according to

$$DIST = \frac{XMIN_i}{XMAXS_i} \quad \text{for } IDEX_i > 0$$

$$= \frac{XMINS_i}{XMAX_i} \quad \text{for } IDEX_i < 0 \qquad (3.49)$$

where $i = IPAR_{II}$

XMAXS = a vector containing the mean +3 sigma values for the total X parameter vector for the base point.

XMINS = a vector containing the mean -3 sigma values for the total X parameter vector for the base point.

This normalized distance is then compared to a program input parameter XNN. For XNN > 1, one is assured that the $X_{IPAR_{II}}$ random variable has been varied so that its frequency distribution inside the 3 sigma limits lies outside the distribution for the corresponding base point parameter. Thus by selecting the value of XNN, the program user can control the extent to which exploratory moves are made. A value of XNN = 1.5 was found to give satisfactory results

and is used for the examples presented in this thesis. By making XNN larger one explores more possibilities at the expense of increased computer time. Thus for DIST > XNN the program returns the part numbers to the base point, increments to the next most significant parameter by incrementing the sub-search progress number by 1, and returns to step 6 above by calling SEARCH. If DIST $\leq$ XNN, the program continues to make the second check.

This second check consists of calculating the estimated change in the object function based on its first derivative vector using the equation

$$\Delta object = \sum_{i=1}^{m} \frac{\partial\ object}{\partial X_i} [XNOM_i - XNOMS_i] \qquad (3.50)$$

where XNOM and XNOMS are the mean component parameter vectors corresponding respectively to the exploratory part number vector and the base point part number vector. Since the i = $IPAR_{II}$ term in (3.50) is negative, one knows that if $\Delta object$ turns out to be positive, the summation of the changes caused by the parameters in IPN_{JJJ} other than $IPAR_{II}$ have resulted in an estimated increase in the object function. Since an increase in $\Delta object$ is undesirable, one returns to step 6 above, when $\Delta object > 0$ and calls SEARCH keeping the same sub-search progress number (II). If $\Delta object \leq 0$, a complete system performance analysis is made using the exploratory move part numbers.

9) If the exploratory move turns out to be "a success" (i.e., the object function is reduced) one returns to step 2 above and the process is repeated. If it is "a failure" (i.e., the object function is not reduced) one returns to step 6 and the next exploratory move is investigated.

10) The optimization procedure terminates once all the exploratory moves made from a given base point are completed "without success." This base point defines the local minimum.

Figure 3.2 is a simplified logic flow diagram for the total design program. For simplicity sake, only the logic fundamental to the directed search option is included. The path used to update the component libraries, and to calculate and store the search matrices is shown by the single dashed line. The linkage between the design program and the component and search matrix libraries via the above subroutines is illustrated with the double dashed lines.

In order to describe the operation of the program, the following additional program logic variables must be defined:

MODE = 1 for the first analysis using the initial part numbers for each search

= 2 for all following analyses in coarse search mode

= 3 for all following analyses in fine search mode

= 4 for final analysis of each search

ICOUNT = number of analyses that have been conducted as part of each search

ISER = search number
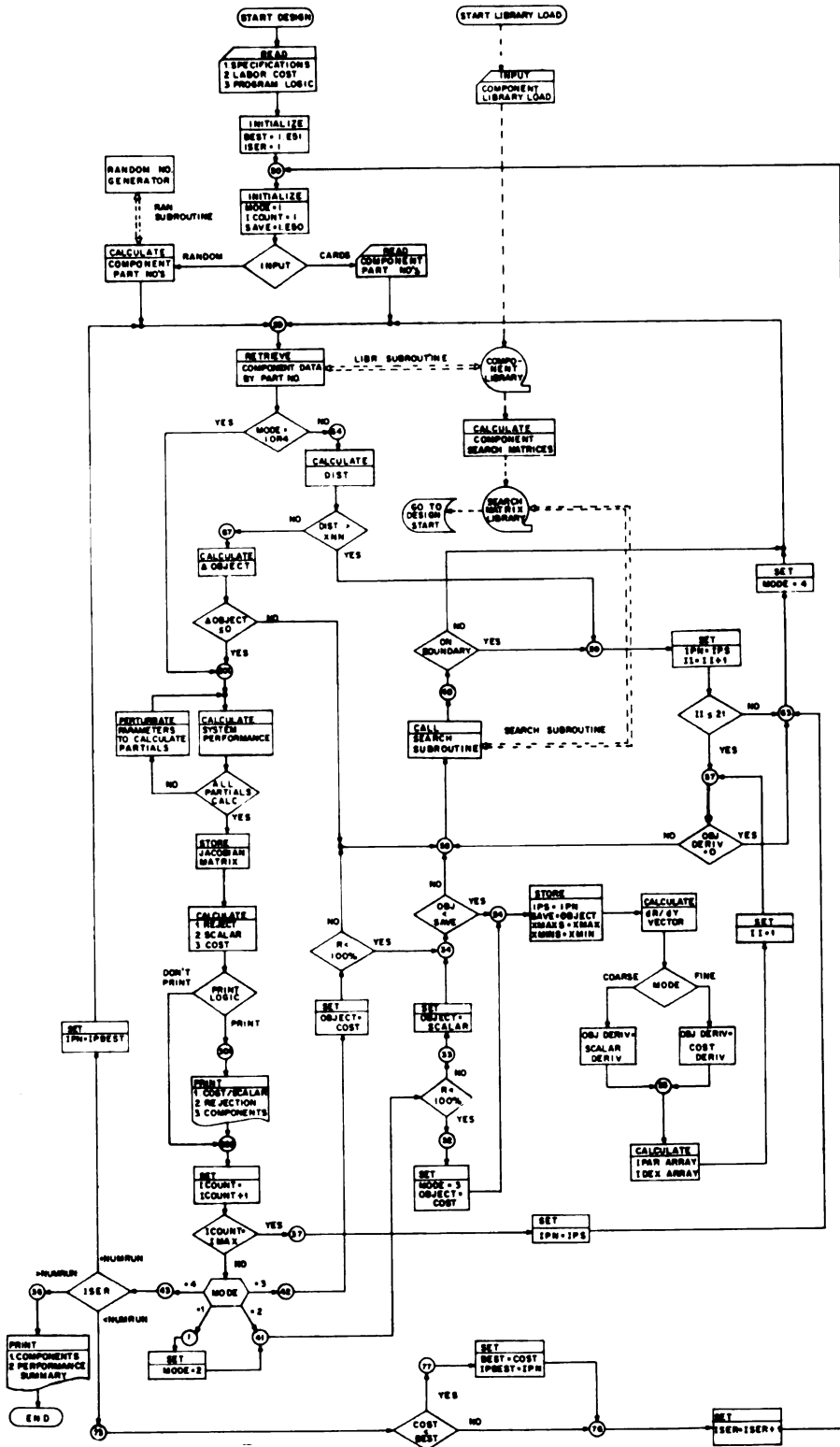
Figure 3.2.   Design program simplified logic diagram.

NUMRUN = specified number of searches to be made in an attempt
to find the best design

IMAX = a maximum allowable iterations per search

A design begins by the user inputting the system specifications,
labor cost, and the required program logic after which the program
initializes numerous parameters as shown in Figure 3.2. The program
then branches according to whether the input is to be from cards or
selected at random. Once the part numbers are obtained, the program
is at point 20 and the desired component data is retrieved from the
component library using the LIBR subroutine. Since MODE was initi-
alized = 1, the program branches to point 200 and the system
performance is calculated, the component parameters are then per-
turbated one at a time, and the system performance is re-evaluated.
This process is repeated, using the steps described in Section 2.4,
until all the entries in the Jacobian have been calculated after which
the program calculates the rejection ratio, scalar, and cost (pro-
viding the later is finite).

At this point, a decision is reached whether or not to make an
intermediate printout. In either event, ICOUNT is incremented by 1
and a check is made to see if it equals the maximum allowable value.
Assuming the answer to be no, the program goes to the mode direction
block. Since MODE = 1, it branches to point 1 and sets MODE = 2
(coarse search), branches to 41 and checks to see if the rejection
percentage is less than 100. Assuming the answer to be no, the pro-
gram stays in the coarse search mode, sets OBJECT = SCALAR and
compares OBJECT to SAVE. If OBJECT < SAVE, as it is the first

time and thereafter anytime a lower object has been found, the last
analysis is considered to be "a success" and the program branches
to point 54 in order to store the data as a new "base point." For
the cases when OBJECT > SAVE, the last analysis is considered to be
"a failure" and the program continues on with the search using the
old base point via point 58.

At point 54, the new base point is established by storing the
part numbers in a vector named IPS, the object function as SAVE, and
the mean ± 3 sigma component parameter vectors as vectors XMAXS and
XMINS respectively.

Following the establishment of each new base point, the elements
of the $\frac{dR}{dY}$ vector are calculated using (3.35) or (3.37) along with the
object derivative vector which in the coarse search mode equals the
scalar derivative given by (3.44). The priority and direction vectors
IPAR and IDEX are then established as defined by (3.47) and (3.48).
After this, the sub-search progress number (II) is initialized to
unity and the program is at point 57 of Figure 3.2 ready to begin a
sub-search by making exploratory moves to look for a smaller object
function.

At point 57, the program checks to see if $\partial object / \partial X_{IPAR_{II}} = 0$.
If it is zero for a given progress number II, the program declares the
base point to be a local minimum. This is accomplished by branching
to point 63, setting MODE = 4 and returning to point 20 to terminate
this search by repeating the best run obtained.

Consider now the case where for any given progress number II, the derivative is not zero. The program branches to point 58 and the SEARCH subroutine is called to select a new part number. Upon return from SEARCH the parameter IBOUND is examined to determine if one is at a boundary condition. If the answer is yes, no further minimization can be obtained considering the $IPAR_{II}$ parameter, therefore the program branches to point 59, returns the part numbers to the base point and increments to the next most significant parameter by increasing the progress number II by 1. If II is then greater than 21, the program has exhausted all possible parameters and therefore branches to 63 declaring the base point as the local minimum and terminates the search. If II $\leq$ 21, the program returns to point 57 and the exploratory moves are continued with the next least significant parameter.

Returning to point 40 and assuming that the new part number selected by the SEARCH subroutine is not on a boundary, the program branches to point 20 and the new component data is retrieved from the library and the program branches to point 64.

At point 64, the program strategy checks for the existence of two conditions before an analysis is made. The first is to compute DIST as given by (3.49) and compare it to the program input parameter XNN. If DIST > XNN, the program branches to point 59, the part numbers are returned to the base point and the sub-search progress number is incremented to explore based on the next parameter. If DIST < XNN, the program branches to point 67 for the second check by calculating Δobject (3.50). If Δobject $\leq$ 0, the program branches

to point 200 and an analysis is made, while if it is not, it branches to 58 and the SEARCH subroutine is called to make another part number selection.

The above described coarse search procedure is repeated until R becomes less than 100% at which time the program branches to point 32 and sets MODE = 3 (for fine search) and OBJECT = COST and continues on as before at point 54, the only difference being that the object derivative vector is taken as the cost derivative (3.32).

The program continues in the fine search mode until either the object derivative is equal to zero or all m parameters (this case m = 21) have been considered and the search is terminated via point 63. Execution of this termination is obtained by setting MODE = 4 and repeating the system analysis using the part numbers resulting in the lowest cost for this search (i.e., the base point), the latter being defined as a local minimum.

Since this time MODE = 4, the program branches to point 43 and ISER is compared to NUMRUN. As long as the number of searches is less than the number requested, the program branches to point 73 where the cost is compared to the best local minimum. If the new cost is less, the vector IPBEST is updated with the new part numbers and BEST is updated with the corresponding cost. In any event, ISER is incremented by 1 and a new search is started. This process is repeated until ISER = NUMRUN whereupon the vector IPN is set equal to IPBEST and the analysis is repeated for the best local minimum. Upon exiting at point 43 (ISER > NUMRUN), the final printout is obtained at point 36. The results obtained using the above described program are described in the next section.

4. EXAMPLE DESIGN PROBLEMS

4.1 COMPONENT LIBRARIES AND SEARCH MATRICES

The components selected to make up the libraries for this study, chosen so as to provide a broad base of design, are typical of those used throughout the servomechanism industry. The actual component parameter values used are listed in Tables 4.1 through 4.4 which consist of the "component libraries." Referring to these tables, the design problem is simply explained as "picking the one part from each table such that when combined in a system, they meet a given specification at minimum dollar cost."

Each column of the library data is labeled with the appropriate X-vector notation; i.e., X_1, X_2, ..., X_{21} each of which is assumed to be a random variable with a normal distribution defined for each component by the mean ± 3 sigma limits given by the MAX and MIN values shown. The variables X_i for i = 1, 4, 9, 16, and 21, which are the individual component costs and the gear ratio and have no manufacturing tolerance, are still treated as "random variables" but having zero variance; i.e., $XMAX_i = XMIN_i$. It should be noted that many of the numerical units of measure for the variables are purposely included as an inconsistent set (e.g., min of arc, rpm, oz-in, gm-cm^2, etc.). This is done to place them in one-to-one correspondence with what is normally given in vendor catalogs and component specification sheets.

Table 4.1. Followup library data.

PART NO.	X_1 COST DOLLARS	X_2 FOLLOWUP GAIN (VOLTS/RAD) MAX.	X_2 FOLLOWUP GAIN (VOLTS/RAD) MIN.	X_3 ACCURACY (MIN OF ARC) MAX.	X_3 ACCURACY (MIN OF ARC) MIN.
1001	300.00	23.600C	21.4000	1.0	0.0
1002	24.00	12.7C0C	1C.3000	10.0	0.0
1003	35.00	24.80CC	2C.2000	7.0	0.0
1004	200.00	0.505C	0.495C	30.0	0.0
1005	600.00	0.5025	0.4975	10.0	0.0
1006	28.00	24.8000	20.2000	15.0	0.0
1007	40.00	12.100C	1C.9000	3.0	0.0
1008	36.00	12.100C	1C.9000	7.0	0.0
1009	22.00	12.70C0	1C.3000	15.0	0.0
1010	30.00	0.5050	C.4950	120.0	0.0
1011	95.00	11.7000	11.300C	2.0	0.0
1012	90.00	0.5050	0.4950	60.0	0.0
1013	300.00	0.505C	C.495C	15.0	0.0
1014	60.00	24.800C	20.2000	3.0	0.0
1015	16.00	12.7CCC	10.300C	30.0	0.0
1016	30.00	25.9000	19.100C	10.0	0.0
1017	260.00	11.7000	11.3000	1.C	0.0
1018	150.00	23.60C0	21.4000	2.0	0.0
1019	20.00	27.000C	18.0000	30.0	0.0
1020	28.00	0.515C	C.5050	180.0	0.0
1021	26.00	5.5C00	4.5000	10.0	0.0
1022	30.00	5.25C0	4.7500	5.0	C.0
1023	20.00	5.5CC0	4.5000	15.0	0.0
1024	28.00	5.2500	4.7500	7.0	0.0
1025	18.00	5.500C	4.5000	30.0	0.0

Table 4.2. Amplifier library data.

	X VECTOR NOTATION									
	X_4	X_5		X_6		X_7		X_8		
PART NO.	COST DOLLARS	AMPLIFIER GAIN TO FOLLOWUP (VOLTS/VOLT)		AMPLIFIER GAIN TO GENERATOR (VOLTS/VOLT)		SAT. LEVEL (VOLTS)		OUTPUT NULL (VOLTS)		
		MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	
2001	20.00	12.	8.	12.	8.	20.0	16.0	1.00	0.0	
2002	75.00	55.	45.	110.	90.	25.0	17.0	0.50	0.0	
2003	85.00	1150.	850.	1150.	850.	28.0	20.0	2.00	0.0	
2004	200.00	55.	45.	11000.	9000.	28.0	20.0	3.00	0.0	
2005	140.00	11.	9.	1100.	900.	25.0	17.0	1.00	0.0	
2006	250.00	105.	95.	10500.	9500.	28.0	20.0	2.00	0.0	
2007	170.00	525C.	4750.	10.	9.	26.0	18.0	1.00	0.0	
2008	107.00	550.	450.	55.	45.	25.0	17.0	1.00	0.0	
2009	220.00	5500.	4500.	11000.	9000.	28.0	24.0	2.00	0.0	
2010	150.00	11500.	8500.	575.	425.	28.0	20.0	2.00	0.0	
2011	160.00	55.	45.	5500.	4500.	26.0	18.0	1.00	0.0	
2012	90.00	105.	95.	53.	48.	25.0	17.0	1.00	0.0	
2013	185.00	1050.	950.	5250.	4750.	28.0	20.0	1.00	0.0	
2014	180.00	105.	95.	5250.	4750.	26.0	20.0	2.00	0.0	
2015	50.00	55.	45.	11.	9.	23.0	17.0	2.00	0.0	
2016	75.00	11.	9.	110.	90.	23.0	17.0	0.50	0.0	
2017	175.00	5500.	4500.	110.	90.	28.0	20.0	1.00	0.0	
2018	220.00	1100.	900.	11000.	9000.	28.0	24.0	2.00	0.0	
2019	180.00	11000.	9000.	11.	9.	26.0	18.0	1.00	0.0	
2020	170.00	11000.	9000.	1100.	900.	28.0	24.0	2.00	0.0	
2021	200.00	550.	450.	11000.	9000.	28.0	20.0	3.00	0.0	
2022	170.00	1050.	950.	105.	95.	26.0	18.0	1.00	0.0	
2023	250.00	10.	9.	10500.	9500.	26.0	18.0	2.00	0.0	
2024	60.00	110.	90.	11.	9.	23.0	17.0	2.00	0.0	
2025	180.00	525.	475.	525.	475.	26.0	18.0	1.00	0.0	
2026	142.00	55.	45.	1100.	900.	26.0	18.0	1.00	0.0	
2027	170.00	5500.	4500.	550.	450.	28.0	20.0	1.00	0.0	
2028	130.00	1100.	900.	55.	45.	26.0	18.0	1.00	0.0	
2029	170.00	550.	450.	5500.	4500.	28.0	20.0	2.00	0.0	
2030	170.00	11000.	9000.	5500.	4500.	28.0	24.0	3.00	0.0	
2031	185.00	11000.	9000.	55.	45.	28.0	20.0	1.00	0.0	
2032	130.00	110.	90.	1100.	900.	26.0	18.0	2.00	0.0	
2033	165.00	5500.	4500.	1100.	900.	28.0	20.0	2.00	0.0	
2034	70.00	53.	48.	53.	48.	23.0	17.0	2.00	0.0	
2035	130.00	53.	48.	525.	5.	25.0	17.0	1.00	0.0	
2036	165.00	5500.	4500.	55.	45.	26.0	18.0	1.00	0.0	
2037	110.00	550.	450.	1100.	900.	28.0	20.0	3.00	0.0	
2038	105.00	550.	450.	11.	9.	25.0	17.0	1.00	0.0	
2039	160.00	11.	9.	5500.	4500.	26.0	18.0	3.00	0.0	
2040	110.00	550.	450.	110.	90.	26.0	18.0	1.00	0.0	
2041	170.00	5500.	4500.	5500.	4500.	28.0	24.0	1.00	0.0	
2042	160.00	11000.	9000.	110.	90.	28.0	20.0	2.00	0.0	
2043	85.00	11.	8.	575.	425.	25.0	17.0	2.00	0.0	
2044	100.00	110.	90.	550.	450.	26.0	18.0	2.00	0.0	
2045	160.00	1050.	950.	10.	9.	25.0	17.0	1.00	0.0	
2046	70.00	110.	90.	110.	90.	25.0	17.0	1.00	0.0	
2047	30.00	12.	8.	60.	40.	23.0	17.0	1.00	0.0	
2048	130.00	1100.	900.	550.	450.	28.0	20.0	2.00	0.0	
2049	300.00	10500.	9500.	10500.	9500.	32.0	24.0	2.00	0.0	
2050	250.00	306.	294.	306.	294.	26.0	20.0	2.00	0.0	

Table 4.3. Motor-generator library data.

PART NO.	X9	X10		X11		X12		X13		X14		X15		CONTROL
	COST DOLLARS	STALL-TORQUE (OUNCE-INCHES)		NO-LOAD SPEED (REV/MIN)		INERTIA (GM-CMSQR)		START VOLT (VOLTS)		GENERATOR GAIN (V/1000 RPM)		GENERATOR NULL (MILLIVOLTS)		VOLTAGE (RATED)
		MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	
3001	42.00	0.1300	0.1000	15700.	14300.	0.100	0.080	0.80	0.40	0.0	0.0	0.0	0.0	38.00
3002	32.00	0.2000	0.1000	8800.	7000.	1.300	0.850	2.50	1.25	0.36230	0.27630	10.00	0.0	26.00
3003	39.00	0.1300	0.0600	7000.	5000.	0.840	0.660	2.50	1.25	0.28060	0.21930	10.00	0.0	26.00
3004	41.00	0.1250	0.0850	7000.	5600.	1.150	1.050	0.70	0.35	0.38570	0.31490	15.00	0.0	26.00
3005	79.00	0.1500	0.1200	6600.	5600.	0.930	0.870	1.00	0.50	0.21940	0.16670	5.00	0.0	33.00
3006	52.00	0.8000	0.6000	8000.	6500.	1.700	1.500	0.90	0.50	0.54170	0.39720	7.00	0.0	22.00
3007	29.00	0.1700	0.1200	11000.	8300.	0.140	0.080	1.50	0.65	0.0	0.0	0.0	0.0	26.00
3008	46.00	0.1350	0.0950	7000.	5800.	1.000	0.930	0.70	0.35	0.38570	0.31340	15.00	0.0	26.00
3009	96.00	0.1500	0.1000	9200.	8000.	0.520	0.440	1.20	0.60	0.51570	0.49540	12.00	0.0	26.00
3010	54.00	0.7500	0.5000	8000.	6300.	5.900	2.000	0.90	0.45	0.65940	0.48530	15.00	0.0	26.00
3011	53.00	0.1600	0.1300	7500.	5600.	0.836	0.684	1.20	0.0	0.22000	0.16670	5.00	0.0	33.00
3012	70.00	0.2500	0.1900	7000.	5400.	0.500	0.400	1.20	0.60	0.00319	0.00236	4.80	0.0	24.00
3013	100.00	0.6800	0.4500	7900.	5600.	5.900	2.000	1.60	0.80	0.66125	0.48550	16.00	0.0	31.00
3014	52.00	0.1800	0.1000	10000.	8000.	1.100	0.900	1.50	0.75	0.57490	0.47380	15.00	0.0	26.00
3015	33.00	0.2000	0.1700	8000.	6500.	0.100	0.070	1.00	0.50	0.0	0.0	0.0	0.0	26.00
3016	35.00	0.2500	0.1200	11000.	8500.	1.800	0.850	1.00	0.50	0.36830	0.27080	40.00	0.0	23.00
3017	67.00	0.1293	0.0742	7604.	6908.	0.100	0.080	1.00	0.50	0.0	0.0	0.0	0.0	23.00
3018	75.00	0.1294	0.0743	7597.	6908.	0.100	0.080	1.00	0.50	0.0	0.0	0.0	0.0	23.00
3019	78.00	0.1280	0.0705	7592.	6917.	0.100	0.080	1.00	0.50	0.0	0.0	0.0	0.0	23.00
3020	125.00	0.1800	0.1000	11000.	9000.	1.700	1.300	1.10	0.0	0.50980	0.07900	12.00	0.0	20.00
3021	35.00	0.3300	0.2700	9000.	8500.	1.700	0.890	1.20	0.0	0.0	0.0	0.0	0.0	32.00
3022	40.00	0.2700	0.2500	9000.	8500.	0.400	0.360	1.00	0.0	0.0	0.0	0.0	0.0	40.00
3023	43.00	0.1600	0.1200	7500.	5600.	0.860	0.660	1.00	0.50	0.20000	0.08800	15.00	0.0	33.00
3024	36.00	1.6000	0.8000	24000.	20000.	7.600	1.500	2.50	1.25	0.0	0.0	0.0	0.0	28.00
3025	43.00	0.0360	0.0270	8900.	8200.	0.200	0.100	1.30	0.65	0.0	0.0	0.0	0.0	15.00

Table 4.4. Geartrain library data.

		X VECTOR NOTATION									
	X_{16}	X_{17}		X_{18}		X_{19}		X_{20}		X_{21}
PART NO.	COST DOLLARS	INERTIA (GM-CMSQR)		STIFFNESS (OUNCE-INCHES)		FRICTION (OUNCE-INCHES)		BACKLASH (MIN OF ARC)		GEAR RATIO
		MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	MAX.	MIN.	
4001	250.00	82.5	67.5	3300.	2700.	0.250	0.062	3.0	1.5	50.0
4002	50.00	69.0	37.0	2052.	1548.	0.240	0.040	15.0	7.5	50.0
4003	325.00	1040.0	960.0	20800.	19200.	0.500	0.160	3.0	1.5	200.0
4004	300.00	336.0	264.0	5750.	4250.	0.300	0.100	3.0	1.5	100.0
4005	60.00	210.0	190.0	2625.	2375.	0.400	0.250	15.0	7.5	100.0
4006	42.00	220.0	180.0	26750.	23250.	0.800	0.310	45.0	22.5	200.0
4007	49.00	222.0	198.0	12840.	11160.	1.000	0.200	30.0	15.0	200.0
4008	135.00	1575.0	1425.0	129600.	110400.	2.000	0.500	7.0	3.5	500.0
4009	34.00	60.0	52.0	2268.	1932.	0.180	0.050	45.0	22.5	75.0
4010	53.00	732.0	668.0	4200.	3800.	0.800	0.200	30.0	15.0	500.0
4011	65.00	186.0	174.0	46200.	41800.	0.600	0.150	15.0	7.5	200.0
4012	350.00	2184.0	2016.0	5400.	3600.	1.000	0.250	3.0	1.5	500.0
4013	38.00	76.0	68.0	3605.	3395.	0.200	0.070	45.0	22.5	100.0
4014	30.00	13.0	11.0	1040.	960.	0.050	0.020	45.0	22.5	50.0
4015	120.00	139.0	125.0	35400.	24600.	0.500	0.160	7.0	3.5	100.0
4016	70.00	1050.0	950.0	6890.	6110.	1.500	0.300	15.0	7.5	500.0
4017	127.00	451.0	369.0	1248.	1152.	0.300	0.070	7.0	3.5	200.0
4018	45.00	87.0	77.0	16800.	13200.	0.300	0.100	30.0	15.0	100.0
4019	112.00	113.0	107.0	84800.	75200.	0.370	0.180	7.0	3.5	75.0
4020	37.00	26.0	24.0	6540.	5460.	0.150	0.030	30.0	15.0	50.0
4021	225.00	193.0	143.0	1456.	1144.	0.150	0.050	3.0	1.5	75.0
4022	95.00	52.0	48.0	10700.	9300.	0.200	0.100	7.0	3.5	50.0
4023	41.00	86.0	78.0	1650.	1350.	0.170	0.050	30.0	15.0	75.0
4024	55.00	116.0	108.0	8560.	7440.	0.200	0.100	15.0	7.5	75.0
4025	46.00	423.0	377.0	6148.	5452.	0.900	0.290	45.0	22.5	500.0

The conversion factors required to go from the units shown to the consistent set, as shown in Table 2.2, are included as part of the computer program.

Also, with respect to the motor data, the effective values of stall torque (T_s) and no-load speed ($\dot{\theta}_m$) are functions of the voltage capability of the driving amplifier. In order to account for this effect, the values used in the program are obtained from the rated condition given in the library by the equations

$$\dot{\theta}_m(\text{effective}) = \dot{\theta}_m(\text{rated}) \frac{E_{sat}}{E_c} \qquad (4.1)$$

$$T_s(\text{effective}) = T_s(\text{rated}) \frac{E_{sat}}{E_c} \qquad (4.2)$$

where E_c is the rated control voltage of the motor and E_{sat} is the amplifier saturation level, both of which are included as part of the library data. Although (4.1) and (4.2) are not exact [see reference (2)] they are considered adequate for this study. In addition to the above, the motor torque gain (K_m) and damping coefficient (B_m) are terms that are used by the systems engineer and are required as part of this study; however, they are normally not provided directly by the vendor. They must be calculated from what is normally provided; no-load speed ($\dot{\theta}_m$), stall torque (T_s), and rated control voltage (E_c). The equations used are

$$K_m = \frac{T_s(\text{rated})}{E_c} \qquad (4.3)$$

$$B_m = \frac{T_s(\text{rated})}{\dot{\theta}_m(\text{rated})} \qquad (4.4)$$

The search matrices are generated immediately after the library
data is stored in the computer system. These search matrices are
shown as Tables 4.5 through 4.8 and consist of the component part
numbers arranged in an ordered array as previously explained in Section
3.5.

The computer program developed as part of this study has been
tested on several design problems, each employing different specification
sets. Each one has met with about equal success; however, the two
that have received the most comprehensive study are presented here as
examples and comprise the rest of this section.

4.2 FIRST DESIGN EXAMPLE

In order to demonstrate the application of the program in its
most comprehensive form, a customer requirement is assumed which
makes use of all eight specifications. The particular set is:

1. Static accuracy = 1.0 degrees

2. Resolution = 0.5 degrees

3. Velocity lag for 90 deg/sec input = 5 degrees

4. Followup rate = 90 deg/sec

5. Damping ratio = 0.3

6. Null oscillation = none allowed

7. Overshoot for 10 deg step = 2.0 degrees

8. Bandwidth for 2.0 deg peak sinusoid = 5 hertz

The assumed labor cost is $200.00.

First to be considered are the results obtained by using the above
specification set and the program operating in the Monte Carlo mode.

Table 4.5. Followup search matrix.

COST	K-F	THETA
1015	1010	1017
1025	1005	1001
1023	1004	1018
1019	1012	1011
1009	1013	1014
1002	1020	1007
1021	1024	1022
1006	1021	1024
1024	1025	1003
1020	1022	1008
1022	1023	1021
1010	1015	1005
1016	1007	1002
1003	1011	1016
1008	1009	1013
1007	1017	1006
1014	1008	1023
1012	1002	1009
1011	1014	1015
1018	1006	1025
1004	1003	1019
1017	1018	1004
1013	1016	1012
1001	1001	1010
1005	1019	1020

Table 4.6. Amplifier search matrix.

COST	K-AF	K-AG	SAT	NULL
2001	2001	2001	2001	2002
2047	2043	2045	2047	2016
2015	2047	2019	2034	2027
2024	2023	2024	2015	2036
2046	2039	2015	2024	2008
2034	2016	2038	2016	2028
2016	2005	2007	2046	2022
2002	2015	2012	2035	2012
2003	2011	2047	2005	2019
2043	2026	2008	2012	2001
2012	2034	2036	2002	2047
2044	2004	2034	2045	2013
2038	2002	2028	2043	2007
2008	2035	2031	2008	2031
2040	2032	2040	2038	2025
2037	2044	2022	2007	2046
2028	2006	2017	2028	2045
2048	2046	2002	2026	2038
2035	2014	2042	2022	2005
2032	2012	2016	2011	2040
2005	2024	2046	2040	2011
2026	2050	2035	2023	2041
2010	2029	2050	2019	2035
2011	2025	2027	2036	2017
2045	2021	2025	2044	2026
2042	2040	2048	2025	2015
2039	2038	2044	2032	2029
2036	2037	2010	2039	2014
2033	2008	2043	2014	2003
2030	2028	2026	2050	2050
2007	2013	2003	2031	2048
2029	2003	2020	2029	2024
2027	2048	2005	2027	2006
2022	2045	2037	2013	2023
2041	2022	2033	2048	2044
2020	2018	2032	2006	2010
2017	2007	2030	2003	2020
2014	2027	2029	2042	2049
2025	2041	2014	2021	2018
2019	2009	2013	2010	2009
2031	2036	2011	2037	2042
2013	2017	2041	2004	2043
2021	2033	2039	2017	2034
2004	2031	2049	2033	2033
2018	2030	2006	2030	2032
2009	2049	2023	2041	2030
2050	2042	2021	2020	2021
2006	2020	2018	2018	2039
2023	2010	2005	2009	2037
2049	2019	2004	2049	2004

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Table 4.7. Motor-generator search matrix.

COST	T-S	T-M-D	J-M	ESTART	K-G	NULL
3007	3025	3003	3015	3011	3024	3022
3002	3003	3005	3019	3022	3022	3001
3015	3019	3012	3017	3021	3007	3018
3016	3017	3004	3001	3004	3017	3007
3021	3018	3008	3018	3008	3021	3015
3024	3004	3011	3007	3020	3001	3025
3003	3001	3023	3025	3001	3015	3021
3022	3008	3013	3022	3010	3025	3024
3004	3009	3010	3012	3006	3019	3019
3001	3005	3015	3009	3015	3018	3017
3025	3014	3006	3023	3023	3012	3012
3023	3023	3018	3003	3005	3023	3011
3008	3020	3019	3011	3018	3005	3005
3014	3007	3017	3005	3017	3011	3006
3006	3011	3002	3021	3016	3003	3003
3011	3002	3025	3008	3019	3020	3002
3010	3015	3009	3014	3012	3002	3020
3017	3016	3022	3002	3009	3016	3009
3012	3012	3014	3004	3025	3008	3014
3018	3022	3007	3016	3007	3004	3023
3019	3021	3021	3020	3014	3006	3010
3005	3013	3016	3006	3013	3009	3008
3009	3010	3020	3024	3024	3014	3004
3013	3006	3001	3013	3003	3010	3013
3020	3024	3024	3010	3002	3013	3016

Table 4.8. Geartrain search matrix.

COST	J-G	K-S	T-G	B	N
4014	4014	4014	4014	4001	4001
4009	4020	4017	4020	4012	4022
4020	4022	4021	4021	4021	4020
4013	4002	4023	4023	4004	4014
4023	4009	4002	4009	4003	4002
4006	4013	4009	4013	4017	4023
4018	4001	4005	·4002	4015	4009
4025	4023	4001	4024	4008	4019
4007	4018	4013	4022	4019	4024
4002	4019	4010	4001	4022	4021
4010	4024	4012	4017	4011	4015
4024	4015	4004	4018	4024	4013
4005	4021	4025	4004	4005	4005
4011	4011	4020	4019	4002	4018
4016	4006	4016	4005	4016	4004
4022	4005	4024	4015	4007	4007
4019	4007	4022	4003	4023	4003
4015	4004	4007	4011	4020	4011
4017	4025	4018	4010	4010	4006
4008	4017	4003	4006	4018	4017
4021	4010	4006	4025	4014	4025
4001	4003	4015	4007	4013	4012
4004	4016	4011	4012	4025	4010
4003	4008	4019	4016	4006	4016
4012	4012	4008	4008	4009	4008

This is accomplished by instructing the computer to make multiple analysis runs selecting the component part numbers at random. Table 4.9 illustrates a typical program output by showing the first 50 lines of intermediate printout. Each line represents an analysis run and lists the cost (3.5), scalar (3.6), total reject (3.19), the four component part numbers used, and the individual specification rejection percentages [R_i using (3.16) or (3.17) for i = 1, $\cdots$, 8 excluding R_6 which is given by (3.27)] arranged in order as R_1, R_2, $\cdots$, R_8. For this example, the total percentage rejection was calculated using the assumption that the Y's are independent [i.e., Equation (3.19)]. As shown, 43 out of the 50 runs illustrated have 100% rejection and therefore an infinite cost (shown as **** when cost $\geq$ 1. × 10^6 dollars) while run number 20 with a cost of $663.28 and a percent rejection of 37.88 is the best of the 50. A total of 467 Monte Carlo runs where made (about 1.5 hours of computer time on an IBM 360 model 50). The lowest cost unit found was $374.02, with a zero percent rejection, using part numbers 1008, 2015, 3008, and 4006. Table 4.10 is the final output sheet obtained summarizing this design.

The results obtained using the program in the direct search mode now are illustrated in detail for two searches. The first, shown in Table 4.11, is a case where the initial guess fails completely to meet 4 out of the 8 required specifications, thus resulting in an infinite cost. Twenty-seven iterations are required by the program to minimize the scalar object function to the point where the cost becomes finite and the program switches from the course to the fine search mode. It should be noted that for this run and most subsequent computer runs,

Table 4.9.   Intermediate Monte Carlo printout
for first design example.

RUN NO.	COST	SCALAR	PERCENT REJECT	COMPONENTS SELECTED FOUP AMP MOGEN GRTH	●●●●●●●INDIVIDUAL SPECIFICATION REJECTIONS●●●●●●●● STATIC RES	LAG	FURATE	DAMP	NULL	OVER	BAND
1	●●●●●●●●●	6.041E+01	100.00	1011 2047 3005 4005	100.00 100.00 100.00	0.00	0.0	0.0	0.0	100.00	
2	●●●●●●●●●	8.508E+01	100.00	1022 2033 3025 4020	0.0 0.0 0.0	0.0	100.00	100.00	100.00	0.0	
3	●●●●●●●●●	1.856E+02	100.00	1023 2047 3017 4007	100.00 100.00 100.00	0.00	0.0	0.0	0.0	100.00	
4	●●●●●●●●●	2.361E+04	100.00	1003 2021 3002 4012	0.00 0.0 100.00	75.53	0.0	0.0	0.0	100.00	
5	●●●●●●●●●	3.780E+03	100.00	1009 2016 3010 4010	51.34 99.91 100.00	100.00	0.0	0.0	0.0	100.00	
6	●●●●●●●●●	3.783E+01	100.00	1019 2028 3017 4014	0.0 0.0 0.0	0.0	100.00	100.00	100.00	0.0	
7	●●●●●●●●●	3.457E+01	100.00	1009 2040 3018 4001	0.0 0.0 0.0	0.0	100.00	0.0	100.00	0.0	
8	●●●●●●●●●	1.047E+06	100.00	1022 2006 3004 4016	99.70 0.00 100.03	100.00	0.0	0.0	0.0	100.00	
9	●●●●●●●●●	1.409E+02	100.00	1014 2039 3024 4021	33.91 100.00 0.0	0.0	100.00	0.0	100.00	0.0	
10	●●●●●●●●●	3.451E+01	100.00	1002 2045 3015 4022	0.0 0.0 0.0	0.0	100.00	63.50	99.98	0.0	
11	●●●●●●●●●	4.944E+00	100.00	1902 2028 3018 4010	0.0 0.0 0.0	95.13	100.00	100.00	0.0	0.0	
12	●●●●●●●●●	1.636E+02	100.00	1022 2044 3016 4001	93.82 92.35 100.00	0.0	0.0	0.0	0.0	100.00	
13	●●●●●●●●●	5.751E+01	100.00	1002 2001 3020 4021	99.89 100.00 59.41	0.0	31.34	0.0	50.46	100.00	
14	●●●●●●●●●	8.530E+02	100.00	1018 2009 3009 4002	0.0 0.0 0.0	0.0	0.0	100.00	0.0	0.00	
15	●●●●●●●●●	8.438E+00	100.00	1006 2042 3016 4006	0.0 0.0 0.0	0.0	100.00	100.00	100.00	0.00	
16	●●●●●●●●●	1.186E+04	100.00	1017 2014 3002 4018	86.81 0.00 100.00	0.0	0.0	0.0	0.0	100.00	
17	●●●●●●●●●	5.040E+00	100.00	1006 2012 3005 4009	0.0 0.0 0.0	0.0	100.00	99.73	93.34	0.0	
18	37928.44	2.266E+01	98.81	1019 2048 3002 4016	0.0 0.0 0.00	77.36	0.0	61.08	0.0	86.48	
19	●●●●●●●●●	1.222E+05	100.00	1008 2004 3008 4005	99.70 67.82 100.00	0.0	0.0	0.0	0.0	100.00	
20	663.28	5.156E+00	37.88	1021 2046 3017 4007	0.0 37.88 0.0	0.00	0.00	0.0	0.0	0.0	
21	●●●●●●●●●	2.327E+02	100.00	1023 2034 3013 4012	0.65 95.52 100.03	100.00	0.0	0.0	0.0	100.00	
22	●●●●●●●●●	1.221E+01	100.00	1001 2031 3015 4017	0.0 0.0 0.0	0.00	100.00	100.00	0.0	0.0	
23	113664.19	4.129E+00	99.56	1021 2046 3017 4008	0.0 0.04 0.0	99.56	0.0	0.0	0.0	0.02	
24	●●●●●●●●●	6.974E+01	100.00	1019 2016 3024 4004	7.18 100.00 0.0	0.0	100.00	0.0	100.00	0.00	
25	●●●●●●●●●	2.174E+03	100.00	1005 2003 3014 4019	92.41 99.94 100.00	0.0	0.0	0.0	0.0	100.00	
26	●●●●●●●●●	8.110E+03	100.00	1003 2011 3004 4021	96.05 0.00 100.00	0.0	0.0	0.0	0.0	100.00	
27	●●●●●●●●●	1.001E+01	100.00	1015 2040 3013 4010	0.0 0.0 39.38	100.00	0.00	3.48	0.0	100.00	
28	●●●●●●●●●	9.324E+00	100.00	1022 2050 3023 4007	0.0 0.0 88.29	0.00	0.00	0.0	0.0	100.00	
29	●●●●●●●●●	1.464E+01	100.00	1022 2007 3003 4022	0.0 0.0 0.0	0.00	100.00	100.00	100.00	0.00	
30	●●●●●●●●●	8.316E+04	100.00	1016 2004 3009 4022	98.98 2.17 100.00	0.0	0.0	0.0	0.0	100.00	
31	●●●●●●●●●	5.791E+01	100.00	1010 2021 3018 4014	99.93 100.00 0.00	0.0	100.00	0.0	100.00	0.0	
32	●●●●●●●●●	1.227E+03	100.00	1018 2011 3020 4074	91.68 0.00 99.99	0.0	0.00	0.0	0.0	100.00	
33	3918.51	2.746E+01	83.74	1021 2008 3010 4001	0.0 0.0 0.0	0.0	0.13	0.0	83.72	0.00	
34	●●●●●●●●●	3.317E+00	100.00	1025 2012 3005 4006	0.00 45.55 99.63	1.00	0.0	0.0	0.0	100.00	
35	●●●●●●●●●	8.918E+01	100.00	1010 2009 3001 4020	66.39 0.0 0.0	0.0	100.00	0.0	100.00	0.0	
36	●●●●●●●●●	1.415E+04	100.00	1012 2032 3016 4001	99.97 100.00 100.00	0.0	0.0	0.0	0.0	100.00	
37	●●●●●●●●●	2.307E+04	100.00	1017 2018 3016 4016	51.81 0.0 100.00	0.00	0.0	0.0	0.0	100.00	
38	●●●●●●●●●	2.190E+05	100.00	1007 2023 3011 4021	99.93 100.00 100.00	0.00	0.0	0.0	0.0	100.00	
39	183335.81	1.321E+01	99.67	1018 2040 3009 4010	0.0 0.0 0.0	78.69	0.0	98.43	0.0	0.06	
40	●●●●●●●●●	5.194E+01	100.00	1006 2040 3022 4007	0.0 0.0 0.0	0.00	100.00	100.00	0.0	0.0	
41	●●●●●●●●●	1.408E+01	100.00	1003 2027 3014 4012	0.0 0.0 0.0	11.97	0.0	100.00	0.00	89.73	
42	2185.52	2.176E+01	69.71	1021 2033 3008 4021	0.0 0.0 0.0	0.0	0.0	69.71	0.00	0.0	
43	●●●●●●●●●	1.020E+01	100.00	1022 2037 3007 4011	0.0 0.0 0.0	0.0	100.00	99.70	0.00	0.0	
44	●●●●●●●●●	4.478E+01	100.00	1018 2029 3019 4022	0.0 0.0 0.0	0.0	100.00	1.12	100.00	0.0	
45	●●●●●●●●●	2.346E+02	100.00	1015 2012 3024 4022	0.00 0.0 0.0	0.0	100.00	100.00	100.00	0.0	
46	●●●●●●●●●	6.449E+00	100.00	1020 2019 3005 4003	85.41 0.0 0.03	0.07	100.00	21.71	0.26	0.00	
47	129559.62	4.246E+00	99.56	1008 2035 3017 4008	0.0 0.00 0.0	99.56	0.0	0.0	0.0	0.0	
48	●●●●●●●●●	1.436E+02	100.00	1017 2040 3024 4024	0.0 0.0 0.0	0.0	100.00	100.00	100.00	0.0	
49	●●●●●●●●●	1.003E+01	100.00	1017 2017 3018 4007	0.0 0.0 0.0	0.0	100.00	100.00	0.0	0.0	
50	●●●●●●●●●	1.186E+04	100.00	1017 2014 3002 4018	86.81 0.00 100.03	0.0	0.0	0.0	0.0	100.00	

Table 4.10. Best design obtained using Monte
Carlo for first design example.

```
                  AUTOMATED DESIGN RESEARCH PROGRAM

                         JANUARY 20, 1969

                    ****DEFINITION OF LOAD****

                                  MAX           MIN
               INERTIA (GM-CMSQR)  9.000E+02     7.000E+02
               FRICTION (OZ-IN)    8.000E-01     4.000E-01

               ****PART NUMBERS OF COMPONENTS SELECTED****

                FOLLOWUP AMPLIFIER MOTOR-GEN GEAR TRAIN
                    1008      2015      3008      4006

                       ****PERFORMANCE****

MAXIMUM      MINIMUM     SPEC LIMIT PCT REJ
4.100E+04    3.820E+04                      TOTAL INERTIA   (GM-CMSQR)
6.141E+02    4.032E+02                      TORQUE CONSTANT (OZ-IN/RAD)
1.523E+01    1.031E+01                      DAMPING COEFFICIENT (OZ-IN-SEC)
5.218E+00    4.226E+00                      NATURAL FREQUENCY (HERTZ)
4.764E-01    2.193E-01      1.000    0.0    STATIC ACCURACY (DEG)
4.649E-01    2.647E-01      0.500    0.00   RESOLUTION (DEG)
2.973E+00    2.242E+00      5.000    0.0    LAG FOR  90. DEG/SEC RAMP (DEG)
1.639E+02    1.122E+02     90.000    0.00   FOLLOWUP RATE (DEG/SEC)
4.282E-01    3.269E-01      0.300    0.00   DAMPING RATIO
4.500E+01    2.250E+01   SEE BELOW    0.0   BACKLASH (MIN)
1.712E+00    1.107E+00      2.000    0.00   OVERSHOOT FOR  10. DEG STEP (DEG)
6.478E+00    5.243E+00      5.00C    0.00   BANDWIDTH FOR   2. DEG SINE (HERTZ)

                 ALLOWABLE BACKLASH SPECIFICATION (MIN)
                        MAXIMUM = 3.861E+02
                        MINIMUM = 9.742E+01

                       ****COST SUMMARY****

            0.00  PCT REJECTION (UPPER BOUND)
            0.00  PCT REJECTION (INDEPENDENT)
            0.00  PCT REJECTION (LOWER BOUND)
          200.00  LABOR COST
          174.00  PARTS COST
          374.02  TOTAL COST (USING R-INDEPENDENT)

            E SIGNIFIES CONVENTIONAL POWER-OF-TEN NOTATION
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Table 4.11. Directed search with initial guess underdesigned for first design example.

**********BEGIN SEARCH NUMBER 1 *********

RUN NO.	COST	SCALAR	PERCENT REJECT	COMPONENTS SELECTED EQUIP	APP	MOGEN	GRTR	******INDIVIDUAL SPECIFICATION REJECTIONS********* DAMP	FURATE	STATIC	SEN	LAG	NULL	OVER	RAND	MODE
1	********	1.099E+03	100.00	1012	2012	3009	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	1
2	********	1.091E+03	100.00	1012	2012	3006	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
3	********	6.180E+02	100.00	1012	2012	3004	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
4	********	5.970E+02	100.00	1012	2012	3008	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
5	********	4.233E+02	100.00	1012	2012	3016	4006	100.00	100.00	100.00	0.30	0.0	0.0	0.0	100.00	2
7	********	3.490E+02	100.00	1012	2012	3011	4006	100.00	100.00	100.00	0.27	0.0	0.0	0.0	100.00	2
8	********	3.159E+02	100.00	1012	2012	3023	4006	100.00	100.00	100.00	0.29	0.0	0.0	0.0	100.00	2
9	********	3.523E+02	100.00	1012	2012	3007	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
11	********	1.523E+02	100.00	1012	2012	3015	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
12	********	1.087E+02	100.00	1004	2012	3015	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
14	********	1.065E+02	100.00	1004	2012	3022	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
15	********	9.452E+01	100.00	1019	2012	3022	4006	100.00	100.00	100.00	0.00	0.0	0.0	0.0	100.00	2
27	1060.21	3.320E+00	100.00	1002	2012	3022	4006	0.00	0.0	0.0	0.00	100.00	100.00	0.0	0.0	2
30	1054.86	3.466E+00	62.65	1009	2012	3022	4006	0.0	0.0	0.0	0.00	20.54	52.99	0.0	0.0	3
33	1038.79	3.532E+00	62.65	1015	2012	3022	4006	0.00	0.0	0.0	0.00	20.54	52.99	0.0	0.0	3
39	373.00	3.200E+00	62.65	1015	2002	3022	4006	0.00	0.0	0.0	0.00	20.54	52.99	0.0	0.0	3
49	366.00	2.534E+00	0.00	1015	2002	3015	4006	0.00	0.0	0.0	0.00	0.0	0.0	0.0	0.0	3
51	365.97	7.763E+00	0.00	1015	2002	3015	4013	0.00	0.00	0.0	0.00	0.0	0.0	0.0	0.0	3
58	359.91	1.617E+01	1.09	1015	2034	3015	4013	0.00	0.00	0.0	0.0	0.00	1.08	0.0	0.0	3
60	340.70	1.586E+01	0.81	1015	2015	3015	4013	0.00	0.00	0.0	0.0	0.00	0.81	0.0	0.0	3
74	340.70	1.584E+01	1.09	1015	2015	3015	4013	0.00	0.00	0.0	0.0	0.00	1.08	0.0	0.0	4

the intermediate printout is eliminated for all iterations where the scalar (cost when in fine search) is not reduced. These are considered "failure iterations" as is the case for numbers 6, 10, 13, and 16 through 26 for the course search in Table 4.11.

Once the program is in the fine search mode, the cost is minimized up to run number 60 where it is reduced from $1060.21 to $340.70. As shown, an additional 13 iterations are required, according to the termination procedure, as explained in section 3.5, in order to establish that part numbers 1015, 2015, 3015, and 4013 establish a local minimum.

Table 4.12 illustrates the results obtained from the second search. This case represents the opposite condition where the initial guess is overdesigned; i.e., all eight specifications are met to the extent that the cost is higher than required. Thus for this search, the program begins in the fine search mode and the cost is minimized. After 75 iterations, the program has reduced the per unit cost from $475.00 to $340.70 -- a savings of $134.30 per unit! This was accomplished at a computer run time of 23 minutes on the IBM 360 model 50.

By comparing the $374.02 given in Table 4.10 with the $340.70 obtained above, it is seen that the directed search provides a cost savings of $33.32 per unit over the Monte Carlo with a computer run time of only 25% of the latter.

A total of ten such searches were made at a cost of almost 5 hours of computer time. The resulting local minimums obtained and their frequency of occurance are summarized in Table 4.13.

Table 4.12. Directed search with initial guess overdesigned for first design example.

**********BEGIN SEARCH NUMBER 2 *********

RUN NO.	COST	SCALAR	PERCENT REJECT	COMPONENTS SELECTED FOUP AMP MOGEN CRTR	*******INDIVIDUAL SPECIFICATION REJECTIONS******* STATIC RES LAG FURATE DAMP NULL OVER BAND	MODE
1	475.00	9.199E+00	0.00	1003 2026 3015 4011	0.0 0.0 0.0 0.00 0.00 0.0 0.0 0.0	3
2	470.00	9.084E+00	0.00	1016 2026 3015 4011	0.0 0.0 0.0 0.00 0.00 0.0 0.0 0.0	3
16	468.00	8.980E+00	0.00	1006 2026 3015 4011	0.0 0.0 0.0 0.00 0.00 0.0 0.0 0.0	3
28	456.00	7.974E+00	0.00	1006 2035 3015 4011	0.0 0.0 0.0 0.00 0.0 0.3 0.0 0.0	3
30	452.00	7.623E+00	0.00	1002 2035 3015 4011	0.0 0.0 0.0 0.00 0.0 0.0 0.0 0.0	3
31	450.00	7.517E+00	0.00	1009 2035 3015 4011	0.0 0.17 0.0 0.00 0.0 0.0 0.0 0.0	3
32	445.74	1.527E+01	0.17	1009 2035 3015 4005	0.0 0.00 0.0 0.0 0.00 0.0 0.00 0.0	3
36	434.00	7.343E+00	0.00	1009 2035 3015 4007	0.0 0.00 0.0 0.00 0.0 0.0 0.0 0.0	3
38	430.01	1.622E+01	0.00	1009 2035 3015 4018	0.0 0.00 0.0 0.0 0.0 0.0 0.0 0.0	3
39	427.00	7.389E+00	0.00	1009 2035 3015 4006	0.0 0.00 0.0 0.00 0.00 0.0 0.0 0.0	3
41	426.45	1.645E+01	0.81	1015 2035 3015 4013	0.0 0.00 0.0 0.0 0.00 0.81 0.0 0.0	3
50	420.40	1.618E+01	0.81	1015 2002 3015 4013	0.00 0.00 0.0 0.0 0.00 0.81 0.0 0.0	3
66	365.97	1.617E+01	1.09	1015 2002 3015 4013	0.00 0.00 0.0 0.0 0.00 1.08 0.0 0.0	3
73	359.91	1.586E+01	0.81	1015 2034 3015 4013	0.00 0.00 0.0 0.0 0.00 0.81 0.0 0.0	3
75	342.70	1.584E+01	1.09	1015 2015 3015 4013	0.00 0.00 0.0 0.0 0.00 1.08 0.0 0.0	3
89	342.76	1.584E+01	1.09	1015 2015 3015 4013	0.00 0.20 0.0 0.0 0.00 1.08 0.0 0.0	4

Table 4.13. Local minimums obtained for first design example.

Number Times Occurred	System Cost	Rejection	Component Part Numbers			
			P_1	P_2	P_3	P_4
1	$634.99	19.37%	1022	2048	3002	4015
1	$475.00	0.0 %	1003	2026	3015	4011
1	$380.61	0.42%	1025	2003	3003	4020
2	$346.43	2.72%	1015	2015	3007	4006
5	$340.70	1.09%	1015	2015	3015	4013

Based on the results listed in Table 4.13, the system obtained using part numbers 1015, 2015, 3015, and 4013 is assumed to be the best design at a cost of $340.70 per unit. The final computer print-out sheet summarizing this combination is shown as Table 4.14.

4.3 SECOND DESIGN EXAMPLE

The specification set for the second example is chosen such that the computer solution time per analysis is minimized thereby enabling more example runs per dollar. This is accomplished by considering a customer requirement to consist only of the first five specifications:

1. Static accuracy = 0.35 degrees

2. Resolution = 0.3 degrees

3. Velocity lag for 300 deg/sec input = 5 degrees

4. Followup rate = 300 deg/sec

5. Damping ratio = 0.5

Since the last three specifications are not included, the calculation of Y_6, Y_7, and Y_8 as well as R_6, R_7, R_8 and S_6 can be bypassed

Table 4.14. Best design obtained using directed
search for first design example.

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                    AUTOMATED DESIGN RESEARCH PROGRAM

                          JANUARY 20, 1969

                     ****DEFINITION OF LOAD****

                                    MAX           MIN
              INERTIA (GM-CMSQR)  9.000E+02     7.000E+02
              FRICTION (OZ-IN)    8.000E-01     4.000E-01

              ****PART NUMBERS OF COMPONENTS SELECTED****

              FOLLOWUP AMPLIFIER MOTOR-GEN GEAR TRAIN
                 1015      2015      3015      4013

                        ****PERFORMANCE****

MAXIMUM     MINIMUM    SPEC LIMIT PCT REJ
1.902E+03   1.542E+03                     TOTAL INERTIA  (GM-CMSQR)
4.769E+02   3.413E+02                     TORQUE CONSTANT (OZ-IN/RAD)
2.757E+00   2.117E+00                     DAMPING COEFFICIENT (OZ-IN-SEC)
2.230E+01   1.832E+01                     NATURAL FREQUENCY (HERTZ)
8.022E-01   2.524E-01      1.000   0.00   STATIC ACCURACY (DEG)
4.494E-01   2.612E-01      0.500   0.00   RESOLUTION (DEG)
1.365E+00   7.619E-01      5.000   0.0    LAG FOR  90. DEG/SEC RAMP (DEG)
3.775E+02   2.571E+02     90.000   0.0    FOLLOWUP RATE (DEG/SEC)
4.400E-01   3.304E-01      0.300   0.00   DAMPING RATIO
4.500E+01   2.250E+01    SEE BELOW 10.85  BACKLASH (MIN)
1.240E+00   6.422E-01      2.000   0.0    OVERSHOOT FOR  10. DEG STEP (DEG)
2.599E+01   2.135E+01      5.000   0.0    BANDWIDTH FOR   2. DEG SINE (HERTZ)

              ALLOWABLE BACKLASH SPECIFICATION (MIN)
                      MAXIMUM = 6.946E+01
                      MINIMUM = 3.158E+01

                      ****COST SUMMARY****

          1.09  PCT REJECTION (UPPER BOUND)
          1.09  PCT REJECTION (INDEPENDENT)
          1.08  PCT REJECTION (LOWER BOUND)
        200.00  LABOR COST
        137.00  PARTS COST
        340.70  TOTAL COST (USING R-INDEPENDENT)

          E SIGNIFIES CONVENTIONAL POWER-OF-TEN NOTATION
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(i.e., set equal to zero). With this alteration to the program, the computer time is reduced from approximately 18.0 to 0.3 seconds per solution -- a factor of 60.

For this specification, two sets of Monte Carlo data were obtained each comprising 4000 runs. The first 50 lines of data obtained from the first set is shown as Table 4.15. As can be seen, only 4 of the 50 have a finite cost, the best being $517.00. It should be noted that the individual rejections for the last 3 specifications are zero since no specification exists. The total rejection for this example is calculated based on the upper bound approximation; i.e., Equation (3.13). Out of the total 8000 Monte Carlo runs made, which took about 40 minutes of computer time, the lowest cost design was found to be $375.00 obtained using part numbers 1006, 2003, 3002, and 4014 with a percentage rejection of zero. A summary of this combination is shown in Table 4.16.

The results obtained using the program in the direct search mode now are illustrated in detail for three searches. The first, shown in Table 4.17, illustrates the condition where the initial guess at first hand looks like a "reasonable design"; i.e., the rejection is only 0.77%. However, after 74 iterations in the direct search mode, the cost has been reduced from the original design value of $555.30 to only $374.27 -- a savings of $181.03 per unit! The computer run time was less than one minute!

Table 4.18 illustrates the opposite condition where the initial selection of part numbers yields a system that fails completely to

Table 4.15. Intermediate Monte Carlo printout for second design example.

RUN NO.	COST	SCALAR	PERCENT REJECT	FNUP	AMP	MOGEN	CRTR	STATIC	RES	LAG	FURATE	DAMP	NULL	OVER	BAND
1	*********	9.702E+01	100.00	1025	2003	3010	4019	54.80	0.0	100.00	0.00	0.00	0.0	0.0	0.0
2	*********	6.339E+00	100.00	1019	2031	3008	4010	11.62	0.0	0.0	100.00	100.00	0.0	0.0	0.0
3	*********	7.732E+00	100.00	1015	2022	3016	4022	18.59	0.0	0.0	0.0	100.00	0.0	0.0	0.0
4	*********	3.627E+02	100.00	1020	2006	3021	4018	100.00	100.00	100.00	0.00	98.99	0.0	0.0	0.0
5	*********	2.513E+03	100.00	1004	2026	3007	4012	100.00	100.00	100.00	100.00	0.0	0.0	0.0	0.0
6	*********	4.167E+01	100.00	1013	2021	3017	4024	100.00	100.00	0.00	0.00	100.00	0.0	0.0	0.0
7	*********	1.337E+07	100.00	1005	2005	3008	4023	100.00	100.00	100.00	0.71	0.0	0.0	0.0	0.0
8	*********	2.256E+03	100.00	1006	2044	3006	4008	0.15	0.0	100.00	100.00	0.0	0.0	0.0	0.0
9	*********	5.915E+00	100.00	1015	2007	3011	4023	12.26	0.0	0.0	28.32	100.00	0.0	0.0	0.0
10	*********	6.722E+00	100.00	1014	2022	3006	4010	0.0	0.0	0.0	100.00	0.0	0.0	0.0	0.0
11	*********	2.730E+03	100.00	1008	2014	3005	4024	99.48	75.41	100.00	41.01	0.0	0.0	0.0	0.0
12	682.00	7.856E+00	-0.0	1003	2020	3006	4021	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	*********	5.863E+00	100.00	1019	2040	3007	4019	16.46	0.0	0.0	0.00	100.00	0.0	0.0	0.0
14	*********	1.467E+01	100.00	1013	2033	3024	4023	0.02	0.0	0.0	0.0	100.00	0.0	0.0	0.0
15	*********	6.671E+00	100.00	1014	2010	3018	4018	0.0	0.0	0.0	0.00	100.00	0.0	0.0	0.0
16	*********	5.238E+00	100.00	1004	2028	3001	4010	89.64	56.05	100.00	100.00	0.03	0.0	0.0	0.0
17	*********	5.827E+00	100.00	1025	2007	3023	4024	13.60	0.0	0.0	32.57	100.00	0.0	0.0	0.0
18	*********	9.978E+01	100.00	1021	2002	3002	4022	100.00	100.00	100.00	0.00	0.00	0.0	0.0	0.0
19	*********	1.804E+08	100.00	1020	2004	3003	4007	99.95	100.00	100.00	100.00	0.0	0.0	0.0	0.0
20	*********	3.032E+04	100.00	1024	2023	3012	4018	99.95	100.00	100.00	13.64	0.0	0.0	0.0	0.0
21	*********	2.783E+01	100.00	1010	2007	3011	4025	98.02	0.0	53.26	100.00	100.00	0.0	0.0	0.0
22	*********	5.475E+00	100.00	1024	2045	3014	4009	0.0	0.0	0.0	0.00	100.00	0.0	0.0	0.0
23	*********	1.015E+03	100.00	1021	2035	3003	4018	100.00	100.00	99.89	95.11	0.32	0.0	0.0	0.0
24	*********	6.422E+00	100.00	1015	2042	3007	4009	11.95	0.0	0.0	0.00	100.00	0.0	0.0	0.0
25	*********	5.073E+00	100.00	1001	2024	3017	4011	0.0	0.0	0.0	100.00	100.00	0.0	0.0	0.0
26	*********	5.652E+00	100.00	1006	2025	3015	4016	0.00	0.0	0.0	100.00	100.00	0.0	0.0	0.0
27	*********	6.312E+00	100.00	1017	2045	3006	4006	0.0	0.0	0.0	100.00	100.00	0.0	0.0	0.0
28	*********	5.682E+00	100.00	1004	2030	3021	4012	18.40	0.0	0.0	100.00	100.00	0.0	0.0	0.0
29	*********	5.540E+00	99.99	1014	2010	3014	4011	0.0	0.0	0.0	99.99	0.01	0.0	0.0	0.0
30	*********	6.377E+00	100.00	1001	2027	3005	4020	0.0	0.0	0.0	0.00	100.00	0.0	0.0	0.0
31	*********	8.441E+00	100.00	1007	2050	3011	4003	0.0	0.0	100.00	100.00	0.0	0.0	0.0	0.0
32	*********	4.177E+04	100.00	1002	2014	3009	4004	99.73	2.95	100.00	0.00	0.0	0.0	0.0	0.0
33	*********	8.240E+00	100.00	1013	2028	3015	4002	99.80	100.00	0.0	0.00	100.00	0.0	0.0	0.0
34	*********	7.196E+00	100.00	1017	2007	3017	4008	0.0	0.0	0.0	100.00	100.00	0.0	0.0	0.0
35	*********	7.036E+00	100.00	1019	2019	3014	4020	11.76	0.0	0.0	0.00	100.00	0.0	0.0	0.0
36	*********	6.288E+01	100.00	1020	2031	3017	4025	99.04	0.0	0.00	100.00	100.00	0.0	0.0	0.0
37	*********	6.008E+00	100.00	1019	2018	3019	4025	12.94	0.0	0.0	100.00	100.00	0.0	0.0	0.0
38	*********	7.255E+00	100.00	1001	2029	3021	4013	0.0	0.0	0.0	0.00	100.00	0.0	0.0	0.0
39	*********	9.670E+04	100.00	1005	2001	3020	4018	100.00	100.00	100.00	42.45	0.00	0.0	0.0	0.0
40	*********	2.242E+05	100.00	1005	2002	3016	4012	100.00	100.00	100.00	100.00	0.0	0.0	0.0	0.0
41	*********	6.120E+00	100.00	1003	2038	3002	4010	0.0	0.0	0.00	100.00	100.00	0.0	0.0	0.0
42	*********	7.722E+03	100.00	1015	2026	3004	4007	99.96	98.38	100.00	100.00	0.0	0.0	0.0	0.0
43	*********	5.057E+00	100.00	1023	2042	3006	4015	0.00	0.0	0.0	0.00	100.00	0.0	0.0	0.0
44	*********	1.775E+06	100.00	1002	2004	3014	4017	99.85	97.16	100.00	99.98	0.0	0.0	0.0	0.0
45	*********	5.588E+02	100.00	1018	2026	3004	4004	93.90	3.86	100.00	57.87	0.0	0.0	0.0	0.0
46	*********	2.444E+02	100.00	1011	2034	3014	4008	34.70	96.13	100.00	100.00	0.0	0.0	0.0	0.0
47	558.00	6.434E+00	-0.0	1003	2020	3009	4024	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
48	*********	7.293E+01	100.00	1018	2047	3020	4013	100.00	100.00	99.99	2.60	4.37	0.0	0.0	0.0
49	517.00	1.296E+01	-0.0	1014	2030	3002	4024	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	*********	8.387E+00	100.00	1012	2017	3008	4020	94.17	0.00	0.0	0.00	100.00	0.0	0.0	0.0

Column groups: COMPONENTS SELECTED = FNUP AMP MOGEN CRTR; INDIVIDUAL SPECIFICATION REJECTIONS = STATIC RES LAG FURATE DAMP NULL OVER BAND.

Table 4.16. Best design obtained using Monte
Carlo for second design example.

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                 AUTOMATED DESIGN RESEARCH PROGRAM

                        JANUARY 15, 1969

                    ****DFFINITION OF LOAD****

                              MAX           MIN
         INERTIA (GM-CMSQR)   9.000E+02    7.000E+02
         FRICTION (UZ-IN)     8.000E-01    4.000E-01

         ****PART NUMBERS OF COMPONENTS SELECTED****

         FOLLOWUP AMPLIFIER MOTOR-GEN GEAR TRAIN
           1006      2003     3002      4014

                    ****PERFORMANCE****

MAXIMUM     MINIMUM    SPEC LIMIT PCT REJ
4.071E+03   2.928F+03                     TOTAL INERTIA   (GM-CMSQR)
8.953E+03   4.027E+03                     TORQUE CONSTANT (OZ-IN/RAD)
6.169E+01   2.717E+01                     DAMPING COEFFICIENT (OZ-IN-SEC)
6.845E+01   4.503E+01                     NATURAL FREQUENCY (HERTZ)
2.765E-01   2.485E-02     C.350   0.00 STATIC ACCURACY (DEG)
2.788E-02   1.364E-02     0.300   0.C  RFSOLUTION (DLG)
2.776E+00   1.633E+00     5.000   0.0  LAG FOR 300. DEG/SEC RAMP (DEG)
9.702E+02   6.192E+02   300.000   0.CO FOLLOWUP RATE (DEG/SEC)
1.592E+00   8.818E-01     0.500   0.00 DAMPING RATIO
0.0         0.0        **********  0.0  BACKLASH (MIN)
0.0         0.0        **********  0.0  OVERSHOOT FOR   0. DEG STEP (DEG)
0.0         0.0        **********  0.0  BANDWIDTH FOR   0. DEG SINE (HERTZ)


                    ****COST SUMMARY****

       0.00   PCT REJECTION (UPPER BOUND)
       0.00   PCT REJECTION (INDFPENDENT)
       0.00   PCT REJECTION (LOWER BOUND)
     200.00   LABOR COST
     175.00   PARTS COST
     375.00   TOTAL COST (USING R-UPPFR BOUND)

       E SIGNIFIES CONVENTIONAL POWER-OF-TEN NOTATION
```

Table 4.17. Directed search with initial guess overdesigned for second design example.

******BEGIN SEARCH NUMBER 1 *********

RUN NO.	COST	SCALAR	PERCENT REJCT	COMPONENTS SELECTED FOUP	AMP	MOGEN	GRTR	STATIC	RES	LAG	FURATE	DAMP	NULL	OVER	BAND	MODE
1	555.30	4.431E+00	0.77	1006	2050	3016	4013	0.04	0.0	0.73	0.00	0.00	0.0	0.0	0.0	1
2	552.65	4.030E+00	0.48	1006	2050	3002	4013	0.00	0.0	0.48	0.0	0.00	0.0	0.0	0.0	3
5	547.11	4.751E+00	0.02	1006	2050	3002	4020	0.00	0.0	0.0	0.00	0.02	0.0	0.0	0.0	3
7	544.01	3.936E+00	0.00	1006	2050	3002	4009	0.00	0.3	0.0	0.0	0.00	0.0	0.0	0.0	3
8	543.10	4.821E+00	0.02	1006	2050	3002	4014	0.00	0.0	0.17	0.00	0.00	0.0	0.0	0.0	3
12	535.89	4.537E+00	0.17	1002	2050	3002	4014	0.00	0.0	0.08	0.00	0.00	0.0	0.0	0.0	3
26	466.38	4.980E+00	0.08	1002	2025	3002	4014	0.00	0.0	0.15	0.0	0.00	0.0	0.0	0.0	3
27	464.79	4.890E+00	0.17	1009	2025	3002	4014	0.02	0.0	0.19	0.0	0.0	0.0	0.0	0.0	3
36	454.88	2.802E+01	0.19	1009	2041	3002	4014	0.00	0.0	0.0	0.00	0.0	0.0	0.0	0.0	3
42	454.00	1.550E+01	0.00	1009	2030	3002	4014	0.00	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
44	449.01	5.740E+00	0.00	1009	2033	3002	4014	0.00	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
56	414.00	5.144E+00	0.00	1009	2048	3002	4014	0.00	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
74	374.27	6.973E+00	1.41	1009	2003	3002	4014	0.00	0.0	1.40	0.00	0.00	0.0	0.0	0.0	3
97	374.27	6.973E+00	1.41	1009	2003	3002	4014	0.00	0.0	1.40	0.00	0.00	0.0	0.0	0.0	4

******INDIVIDUAL SPECIFICATION REJECTIONS*********

Table 4.18. Directed search with initial guess underdesigned for second design example.

**********BEGIN SEARCH NUMBER 2 **********

RUN NO.	COST	SCALAR	PERCENT REJECT	**COMPONENTS SELECTED** FOUP	AMP	MOGEN	GRTR	******INDIVIDUAL SPECIFICATION REJECTIONS******* STATIC	RES	LAG	FURATE	DAMP	NULL	OVER	BAND	MODE
1	*********	9.722E+02	100.00	1005	2046	3015	4022	100.00	100.00	100.00	0.00	0.03	0.0	0.0	0.0	2
3	*********	7.147E+02	100.00	1005	2046	3012	4022	100.00	100.00	100.00	0.0	1.09	0.0	0.0	0.0	2
4	*********	3.042E+02	100.00	1005	2046	3021	4022	100.00	100.00	100.00	0.0	100.00	0.0	0.0	0.0	2
5	*********	8.662E+00	100.00	1021	2046	3021	4022	18.45	82.34	0.0	0.0	100.00	0.0	0.0	0.0	2
8	*********	8.618E+00	100.00	1024	2046	3021	4022	4.18	82.88	0.0	0.0	100.00	0.0	0.0	0.0	2
12	*********	8.611E+00	100.00	1022	2046	3021	4022	0.90	82.88	0.0	0.00	100.00	0.0	0.0	0.0	2
16	*********	5.330E+00	100.00	1022	2046	3021	4024	0.01	21.95	0.0	0.03	100.00	0.0	0.0	0.0	2
19	*********	4.497E+00	100.00	1022	2046	3012	4024	55.55	100.00	0.0	0.03	100.00	0.0	0.0	0.0	2
34	*********	4.334E+00	100.00	1007	2046	3012	4024	0.00	0.00	0.0	0.03	100.00	0.0	0.0	0.0	2
35	*********	3.359E+00	100.00	1007	2046	3023	4024	0.00	63.32	0.20	56.16	94.76	0.0	0.0	0.0	2
40	*********	3.291E+00	100.00	1007	2046	3005	4024	0.00	74.21	27.87	88.46	55.68	0.0	0.0	0.0	2
41	*********	3.207E+00	100.00	1007	2046	3011	4024	0.00	21.45	18.67	54.19	6.75	0.0	0.0	0.0	2
55	38261.30	3.222E+00	98.76	1011	2046	3011	4024	0.0	21.17	16.88	54.19	6.52	0.0	0.0	0.0	3
76	2989.50	3.213E+00	84.65	1011	2025	3011	4023	0.0	11.20	16.30	51.22	5.93	0.0	0.0	0.0	3
109	793.85	4.742E+00	28.32	1011	2025	3011	4020	0.0	0.0	0.03	28.32	0.0	0.0	0.0	0.0	3
119	565.04	4.254E+00	0.01	1011	2025	3002	4020	0.0	0.0	0.00	0.01	0.00	0.0	0.0	0.0	3
120	544.01	5.276E+00	0.00	1011	2025	3002	4014	0.0	0.0	0.00	0.00	0.00	0.0	0.0	0.0	3
122	537.01	5.349E+00	0.00	1011	2041	3002	4014	0.0	0.0	0.02	0.0	0.0	0.0	0.0	0.0	3
127	527.05	2.896E+01	0.01	1011	2030	3002	4014	0.0	0.0	0.01	0.00	0.0	0.0	0.0	0.0	3
133	527.00	1.536E+01	0.0	1011	2033	3002	4014	0.0	0.0	0.0	0.00	0.0	0.0	0.0	0.0	3
136	572.00	6.494E+00	0.00	1011	2048	3002	4014	0.0	0.0	0.0	0.00	0.0	0.0	0.0	0.0	3
149	487.00	5.766E+00	0.00	1011	2048	3002	4014	0.0	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
159	432.00	5.694E+00	0.00	1007	2048	3002	4014	0.0	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
162	428.00	5.450E+00	0.00	1008	2048	3002	4014	0.0	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
172	424.34	5.696E+00	1.16	1016	2048	3002	4014	0.0	0.0	0.0	0.00	1.16	0.0	0.0	0.0	3
178	424.28	5.522E+00	1.01	1206	2048	3002	4014	0.0	0.0	0.0	0.00	1.01	0.0	0.0	0.0	3
181	416.00	5.293E+00	0.00	1002	2048	3002	4014	0.00	0.0	0.0	0.00	1.01	0.0	0.0	0.0	3
184	414.00	5.144E+00	0.00	1009	2003	3002	4014	0.00	0.0	0.0	0.00	0.00	0.0	0.0	0.0	3
202	374.27	6.973E+00	1.41	1009	2003	3002	4014	0.00	0.0	1.43	0.00	0.00	0.0	0.0	0.0	3
225	374.27	6.973E+00	1.41	1009	2003	3002	4014	0.00	0.0	1.43	0.00	0.00	0.0	0.0	0.0	4

meet 3 out of the 5 specifications. It takes the program 55 iterations to obtain a finite cost and switch to fine search and then 147 more to reach a cost of $374.27 which is the same local minimum.

The third search is shown in Table 4.19 where this time the initial parts result in a design which fails completely to meet 5 out of the 6 specifications. After 55 iterations, the program has reduced the scalar from 59,610,000 to 3.396 and only one specification remains a complete failure; however, this point turns out to be a local minimum and no further reduction is obtained.

Figure 4.1 illustrates the convergence of the directed searches as compared to the results obtained from the two Monte Carlo runs by showing a plot of improved system cost versus iteration number. The two directed searches reach the minimum in 74 and 202 iterations respectively while the Monte Carlo runs employ 4000 solutions each and neither has found the minimum.

A total of 15 searches were made and the local minimums found and their frequencies are summarized in Table 4.20. Based on the results listed in Table 4.20, the system obtained using part numbers 1009, 2003, 3002, and 4014 is assumed to be the best design. The final computer printout sheet summarizing this combination is shown as Table 4.21.

Table 4.19. Directed search resulting in an unsatisfactory local minimum.

**************BEGIN SEARCH NUMBER   3   **********

RUN NO.	COST	SCALAR	PERCENT REJECT	COMPONENTS SELECTED				******INDIVIDUAL SPECIFICATION REJECTIONS********							M O D E
				FQUP	AMP	MOGEN	GRTR	STATIC RES	LAG	FURATE	DAMP	NULL	OVER	BAND	
1	**********	5.961E+07	100.00	1013	2014	3010	4017	99.91	100.00	100.00	100.00	0.0	0.0	0.0	1
2	**********	4.616E+07	100.00	1013	2014	3014	4017	99.93	100.00	100.00	100.00	0.0	0.0	0.0	2
3	**********	4.013E+07	100.00	1013	2014	3009	4017	99.94	100.00	100.00	100.00	0.0	0.0	0.0	2
5	**********	2.078E+07	100.00	1013	2014	3004	4017	99.92	100.00	100.00	78.15	0.0	0.0	0.0	2
7	**********	1.919E+07	100.00	1013	2014	3016	4017	99.89	100.00	100.00	100.00	0.0	0.0	0.0	2
8	**********	1.812E+07	100.00	1013	2014	3002	4017	99.96	100.00	100.00	100.00	0.0	0.0	0.0	2
10	**********	1.094E+07	100.00	1013	2014	3003	4017	99.97	100.00	100.00	100.00	0.0	0.0	0.0	2
11	**********	6.726E+06	100.00	1013	2014	3011	4017	99.98	100.00	100.00	100.00	0.0	0.0	0.0	2
12	**********	6.678E+06	100.00	1013	2014	3005	4017	99.98	100.00	100.00	100.00	0.0	0.0	0.0	2
13	**********	5.485E+06	100.00	1013	2014	3023	4017	99.93	100.00	100.00	100.00	0.00	0.0	0.0	2
14	**********	5.276E+06	100.00	1020	2014	3023	4017	99.95	100.00	100.00	100.00	0.00	0.0	0.0	2
15	**********	5.687E+04	100.00	1024	2014	3023	4017	99.86	99.63	100.00	100.00	0.00	0.0	0.0	2
16	**********	8.648E+01	100.00	1024	2014	3012	4017	99.82	56.60	100.00	100.00	0.0	0.0	0.0	2
17	**********	3.746E+00	100.00	1024	2014	3017	4017	58.94	98.59	0.00	100.00	69.54	0.0	0.0	2
18	**********	3.434E+00	100.00	1024	2014	3015	4017	26.65	47.15	0.08	100.00	0.00	0.0	0.0	2
19	**********	3.399E+00	100.00	1022	2014	3015	4017	14.80	47.15	0.06	100.00	0.00	0.0	0.0	2
55	**********	3.396E+00	100.00	1022	2006	3015	4017	14.80	47.15	0.06	100.00	0.00	0.0	0.0	2
156	**********	3.396E+00	100.00	1022	2006	3015	4017	14.80	47.15	0.06	100.00	0.00	0.0	0.0	4
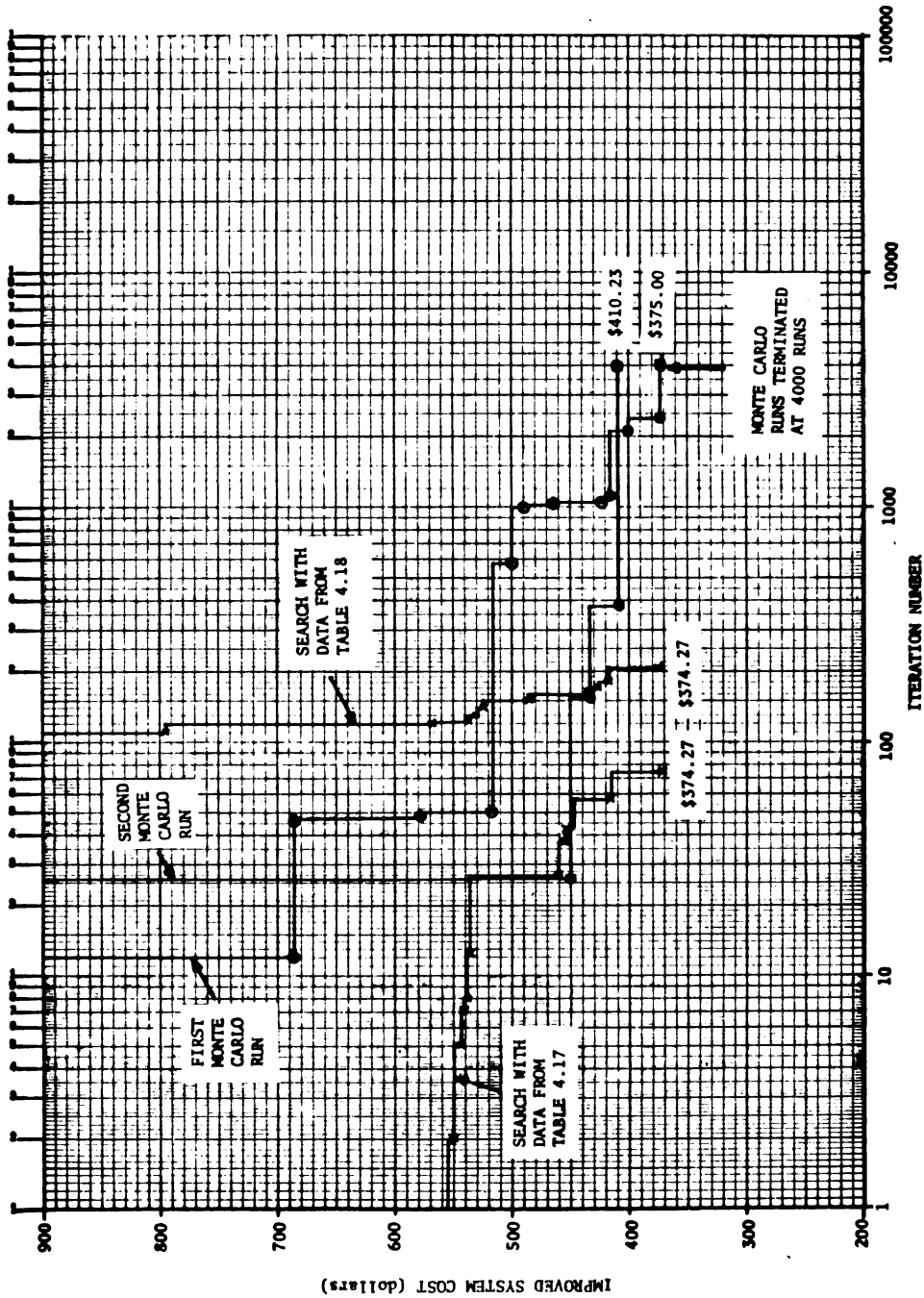
Figure 4.1. Comparison of Monte Carlo and directed search.

Table 4.21. Best design obtained using directed search for second design example.

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                    AUTOMATED DESIGN RESEARCH PROGRAM

                          JANUARY 15, 1969

                    ****DEFINITION OF LOAD****

                                      MAX            MIN
                INERTIA (GM-CMSQR)   9.00CE+02     7.000E+02
                FRICTION (OZ-IN)     8.00CE-01     4.000E-01

                ****PART NUMBERS OF COMPONENTS SELECTED****

                FOLLOWUP AMPLIFIER MOTOR-GEN GEAR TRAIN
                  1009      2003       3002      4014

                          ****PERFORMANCE****

MAXIMUM      MINIMUM      SPEC LIMIT PCT REJ
4.071E+03    2.928E+03                        TOTAL INERTIA   (GM-CMSQR)
4.578E+03    2.056E+03                        TORQUE CONSTANT (OZ-IN/RAD)
6.169E+01    2.717E+01                    .   DAMPING COEFFICIENT (OZ-IN-SEC)
4.895E+01    3.218E+01                        NATURAL FREQUENCY (HERTZ)
3.033E-01    4.712E-02       0.350     0.00   STATIC ACCURACY (DEG)
5.457E-02    2.666E-02       0.300     0.0    RESOLUTION (DEG)
5.295E+00    3.091E+0C      5.000     1.40   LAG FOR 300. DEG/SEC RAMP (DEG)
9.702E+02    6.192E+02     300.00C     0.00  FOLLOWUP RATE (DEG/SEC)
2.227E+00    1.233E+00       0.500     0.00   DAMPING RATIO
0.0          0.0          **********    0.0    BACKLASH (MIN)
0.0          0.0          **********    0.0    OVERSHOOT FOR   0. DEG STEP (DEG)
0.0          0.0          **********    0.0    BANDWIDTH FOR   0. DEG SINE (HERTZ)


                          ****COST SUMMARY****

                1.41   PCT REJECTION (UPPER BOUND)
                1.41   PCT REJECTION (INDEPENDENT)
                1.40   PCT REJECTION (LOWER BOUND)
              200.00   LABOR COST
              169.00   PARTS COST
              374.27   TOTAL COST (USING R-UPPER BOUND)

           E SIGNIFIES CONVENTIONAL POWER-OF-TEN NOTATION
```

Table 4.20. Local minimums obtained for second design example

Number Times Occurred	System Cost	Rejection	Component Part Numbers			
			P_1	P_2	P_3	P_4
1	∞	100%	1016	2004	3024	4025
1	∞	100%	1022	2006	3015	4017
1	$410.01	0.25%	1023	2008	3006	4014
1	$394.63	1.43%	1009	2003	3002	4002
1	$394.29	0.07%	1009	2012	3006	4014
9	$374.27	1.41%	1009	2003	3002	4014
1	Search terminated as iterations exceeded maximum allowed of 300					

The validity that the above $374.27 local minimum is also the absolute minimum can be checked, for this example, by using the procedure explained as follows. The lowest possible cost for a system made up of any collection of components is the summation of the individual component costs and the labor cost since if there are rejects, they only increase this cost. Therefore to test if a local minimum is also the absolute minimum, one need analyze only the subset of the total combinations for which

$$\text{labor cost} + \sum \text{component costs} < \text{local minimum}$$

(4.5)

If it turns out that analyzing each system in this subset results in a total system cost higher than the local minimum being investigated, the latter is the absolute minimum.

For the above $374.27 local minimum there are 17,835 combinations which satisfy (4.5). This number although large is much less than the 781,250 total possible combinations and it becomes a practical value when one considers the reduced solution time. The 17,835 combinations were therefore analyzed (at a cost of 1.7 hours of computer time compared to 74.4 hours for a complete exhaustive search) and each resulted in a total system cost > $374.27 thus proving the latter to be the absolute minimum.

It should be pointed out that the above procedure is not practical for all situations as readily apparent if one considers the $340.70 value obtained for the first design example (see Table 4.14). This time there are only 4,249 combinations which satisfy (4.5), however, with the 18 seconds required per solution a proof would take 21 hours of computer time.

5. CONCLUSIONS AND RECOMMENDATIONS

The objective of this thesis is to develop techniques to automatically select a collection of components that satisfy a given system specification at minimum dollar cost. This objective has been realized.

The techniques developed are sufficiently general to provide an effective design tool especially when there are significant production quantities and a large number of standard components available. The application of the developed technique is demonstrated by establishing libraries for electromechanical components and writing a computer program to automatically design instrument servomechanisms. The results of this effort are most rewarding. For example starting with an initial design that satisfied a customer specification at a cost of $475.00 per unit, the computer program in 23 minutes produces a modified set of part numbers that meet the same specification at a cost of $340.70 per unit. This amounts to a cost savings of $134.30 per unit, representing a 28.2% cost reduction. In a similar manner, starting with an initial design which failed completely to meet 4 out of 8 specifications, the program brings the design from the "infinite cost condition" down to the same minimum in only 60 iterations.

The optimization techniques developed here, like most others,
present no guarantee that the result obtained is an absolute minimum.
A method is presented, however, that is practical in some cases for
testing whether or not a local minimum is indeed the absolute
minimum. This is accomplished by analyzing only a subset of the total
number of possibilities which, for some cases, can be reasonably small.
The test is made for one of the design examples presented which
establishes that the minimum found by the program is truly the
absolute minimum.

Comparisons are made between designs using purely Monte Carlo
techniques and directed search. These demonstrate that the latter
offers a decided advantage in faster convergence to the lowest cost
design. The exact convergence time, of course, depends a great deal
on the closeness of the initial guess.

In meeting the primary objective, a number of other tasks are
accomplished in the area of calculating nonlinear servomechanism
performance. An equation is derived for calculating the allowable
backlash in a servomechanism without it displaying null oscillations.
Heretofore this could only be "calculated" by iterating a direct
simulation of the nonlinear state equations until the critical point
was found. This could literally take several hours of computer time
to obtain when considering nominal values. The thought of including
the component tolerance effects was, therefore, out of the question.
With the equation derived in this study, the solution is obtained,
with tolerances, in 2 seconds of computer time.

Advantages similar to the above are obtained also in the area of computing the overshoot and bandwidth for the nonlinear servo-mechanism.

An obvious next step for future work would be to extend the development presented here to include such design specifications as weight, volume, power consumption, and reliability. This would provide the system engineer with an even more effective total design capability.

For applications where the number of parts stored in a component library becomes extremely large, one should consider the use of a "working library" selected by the system engineer from a large "standard library." With this approach, an experienced user could eliminate, on an "a priori" basis, many components undesirable for a given application. Automated procedures would need to be developed to aid in sorting out the components with the desired features.

The search technique as developed in this thesis could be possibly further improved by extending the exploratory move strategy to include simultaneous multiple component changes. However, the added complexity of the control logic required might very well outweigh any advantages gained.

APPENDIX A

DERIVATION OF BACKLASH-FRICTION SLOPE EQUATION

It is well known that backlash can cause small amplitude oscillations about null. In 1947, A. Tustin [30] presented a graphical method for analyzing the stability of systems with backlash. His work was then extended by others [31], [32], [33], and [34]. However, each of these assume zero load after the backlash (i.e., the output stops instantly each time the motor reverses). For this condition, the backlash is "represented" by a simple hysteresis nonlinearity. However, it can be shown that, for the second order system with hysteresis, the sufficient condition for asymptotic stability is only that the damping ratio (ζ) be greater than one half [35]. But in actual practice [2] and [36], many systems oscillate with $\zeta > 0.5$, which can be attributed to the fact that the hysteresis characteristic does not represent the physical facts. References [37], [38], [39], and [40] attempt to circumvent the problem by considering a load which continues to move after the motor reverses. Each, however, is based on the assumption that the gearing has infinite stiffness (i.e., the impact is perfectly inelastic). However, J. Liversidge [41] in 1951 demonstrated with hardware that low gear train stiffness greatly aggravated the null oscillation problem. Later efforts [42], [43], and [44] demonstrate, at least via simulation, that for an accurate model, gear stiffness must be considered along with backlash.

C. H. Thomas [45] considered this as early as 1954 but made the assumption that there was zero damping at the load. Thus, the general problem, except for direct simulation, has remained unsolved.

In the last few years, there has been considerable discussion [46], [47], [48], [49], and [50] about extending the application of describing functions to handle two separated nonlinearities in a system. The validity of such an approach, of course, increases as the amount of integration effect between them is increased. Since at least one pure integration exists in every path between the backlash and friction blocks of Figure A.1, it was decided to follow this approach.

The null oscillation problem is formulated in the following manner. Consider the 2-space of backlash and friction and some point R that represents the numerical values of each for a particular system (see Figure A.2). From Reference [2] one knows that there

Backlash ↑

line A

● R
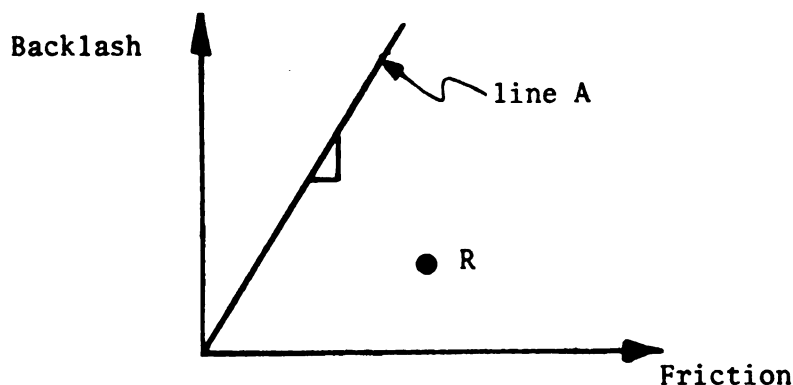
→ Friction

Figure A.2   Backlash-friction diagram
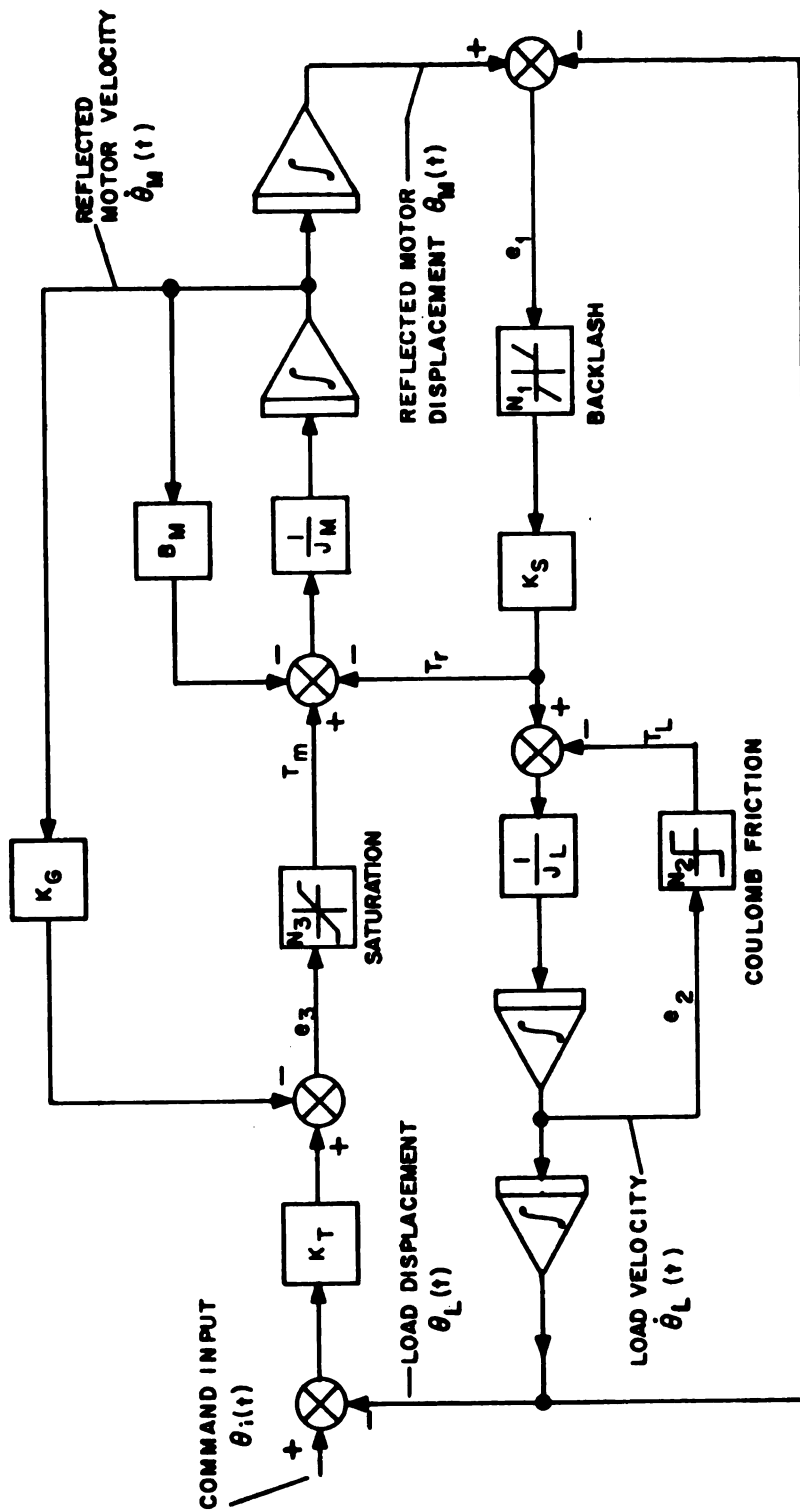
Figure A.1. Nonlinear state model diagram for an instrument servomechanism.

exists a straight line (A) through the backlash-friction space that is strictly a function of the system linear parameters. If R lies above this line, the system has a limit cycle while if R lies below this line, the system is asymptotically stable. Our problem is to find an equation for calculating the slope (m) of this line in terms of the system linear parameters. The purpose of the remaining portion of this Appendix is to accomplish this task.

Consider again the state model diagram shown in Figure A.1. Since presently only the null oscillation problem is being considered, the saturation block may be ignored (i.e., assume a gain of unity for N_3). By inserting for the backlash and coulomb friction blocks their effective gains N_1 and N_2, the autonomous state model can be written as

$$\frac{d}{dt}\begin{bmatrix} \dot{\theta}_M \\ \theta_M \\ \dot{\theta}_L \\ \theta_L \end{bmatrix} = \begin{bmatrix} -\dfrac{f_T}{J_M} & \dfrac{N_1 K_s}{J_M} & 0 & \dfrac{N_1 K_s - K_T}{J_M} \\ 1 & 0 & 0 & 0 \\ 0 & \dfrac{N_1 K_s}{J_L} & -\dfrac{N_2}{J_L} & -\dfrac{N_1 K_s}{J_L} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_M \\ \theta_M \\ \dot{\theta}_L \\ \theta_L \end{bmatrix}$$

(A.1)

The corresponding characteristic polynomial is then given by

$$D(\lambda) = \lambda^4 + \left[\frac{f_T}{J_M} + \frac{N_2}{J_L}\right]\lambda^3 + \left[\frac{f_T N_2 + J_M N_1 K_s + J_L N_1 K_s}{J_M J_L}\right]\lambda^2 + \left[\frac{f_T N_1 K_s + N_2 N_1 K_s}{J_M J_L}\right]\lambda + \left[\frac{K_T N_1 K_s}{J_M J_L}\right]$$

(A.2)

The describing function gains N_1 and N_2 are given as functions of their respective input zero-to-peak values, E_1 and E_2, by the equations

$$N_1(E_1) = 1 - \frac{2}{\pi}\left[\sin^{-1}\left(\frac{B}{2E_1}\right) + \frac{B}{2E_1}\sqrt{1 - \left(\frac{B}{2E_1}\right)^2}\right] \text{ for } E_1 \geq \frac{B}{2}$$

$$= 0 \quad \text{otherwise}$$

(A.3)

$$N_2(E_2) = \frac{4T_L}{\pi E_2} \quad \text{for all } E_2$$

(A.4)

with the corresponding range and domains

$$\frac{B}{2} \leq E_1 \leq \infty \qquad 0 \leq E_2 \leq \infty$$

(A.5)

$$0 \leq N_1 \leq 1 \qquad \infty \geq N_2 \geq 0$$

Since N_1 and N_2 are both real and frequency independent, one can separate out terms in (A.2) and write the characteristic equation in the form

$$F_4(\lambda) + N_1(E_1)F_1(\lambda) + N_2(E_2)F_2(\lambda) + N_1(E_1)N_2(E_2)F_3(\lambda) = 0$$

(A.6)

where

$$F_4(\lambda) = J_M J_L \lambda^4 + f_T J_L \lambda^3$$

$$F_3(\lambda) = K_s \lambda$$

$$F_2(\lambda) = J_M \lambda^3 + f_T \lambda^2$$

$$F_1(\lambda) = J_T K_s \lambda^2 + f_T K_s \lambda + K_T K_s \qquad (A.7)$$

Substituting $j\omega$ for λ and defining the real and imaginary parts of each coefficient

$$F_4(j\omega) = \left(J_M J_L \omega^4\right) + j\left(-f_T J_L \omega^3\right) \equiv P_4 + jQ_4$$

$$F_3(j\omega) = j\left(K_s \omega\right) \equiv P_3 + jQ_3$$

$$F_2(j\omega) = \left(-f_T \omega^2\right) + j\left(-J_M \omega^3\right) \equiv P_2 + jQ_2$$

$$F_1(j\omega) = \left(-J_T K_s \omega^2 + K_T K_s\right) + j\left(f_T K_s \omega\right) \equiv P_1 + jQ_1$$

$$(A.8)$$

Separating the real and imaginary parts of the characteristic equation

$$P_4(\omega) + N_1(E_1)P_1(\omega) + N_2(E_2)P_2(\omega) + N_1(E_1)N_2(E_2)P_3(\omega) = 0 \qquad (A.9)$$

$$Q_4(\omega) + N_1(E_1)Q_1(\omega) + N_2(E_2)Q_2(\omega) + N_1(E_1)N_2(E_2)Q_3(\omega) = 0 \qquad (A.10)$$

Equations (A.9) and (A.10) provide two of the necessary conditions for a limit cycle in terms of three unknowns N_1, N_2, and ω. Solving simultaneously the two equations for N_1 and N_2 in terms of ω and substituting in for the P's and Q's, one obtains equation (A.11) for N_1 and (A.12) for N_2, in terms of ω and the linear system parameters

$$N_1(\omega) = \frac{-\left[\left(J_M J_L K_s + J_M J_T K_s\right)\omega^4 + \left(F_T^2 K_s - J_M K_T K_s\right)\omega^2\right] + \sqrt{\left[\left(J_M J_L K_s + J_M J_T K_s\right)\omega^4 + \left(F_T^2 K_s - J_M K_T K_s\right)\omega^2\right]^2 + 4\left[K_T K_s^2 - J_T K_s^2 \omega^2\right]\left[J_M^2 J_L \omega^6 + F_T^2 J_L \omega^4\right]}}{2\left[K_T K_s^2 - J_T K_s^2 \omega^2\right]}$$

(A.11)

$$N_2(\omega) = \frac{-\left[J_M^2 \omega^2 - J_M K_T + f_T^2\right] + \sqrt{\left[J_M^2 \omega^2 - J_M K_T + f_T^2\right]^2 - 4 f_T \left[J_L^2 f_T \omega^2 - J_L f_T K_T\right]}}{2 f_T}$$

(A.12)

Equations (A.11) and (A.12) will be referred to as the frequency relationships as they give the required describing function values as a function of frequency and the linear parameters.

The third necessary condition for a limit cycle is obtained from the fact that the derivation of (A.11) and (A.12) did not place any restrictions on the amplitude requirement that must exist between E_1 and E_2, thus N_1 and N_2. From the state model diagram (Figure A.1) one can derive a transfer function relating e_1 and e_2 as

$$\frac{e_1(\lambda)}{e_2(\lambda)} = \frac{-1}{\lambda}\left[\frac{J_M \lambda^2 + f_T \lambda + K_T}{J_M \lambda^2 + f_T \lambda + N_1 K_s}\right]$$

(A.13)

Letting $\lambda = j\omega$ and by taking the magnitude of both sides and E_1 E_2 be the peak values of $e_1(j\omega)$ and $e_2(j\omega)$,

$$\frac{E_1}{E_2} = \frac{1}{\omega}\sqrt{\frac{\left(K_T - J_M \omega^2\right)^2 + \left(f_T \omega\right)^2}{\left(N_1 K_s - J_M \omega^2\right)^2 + \left(f_T \omega\right)^2}}$$

(A.14)

Solutions are now required for E_1 and E_2 in terms of N_1 and N_2. This is difficult to accomplish for E_1. Figure A.3 shows a plot of N_1 vs. $2E_1/B$ with and without the third term of
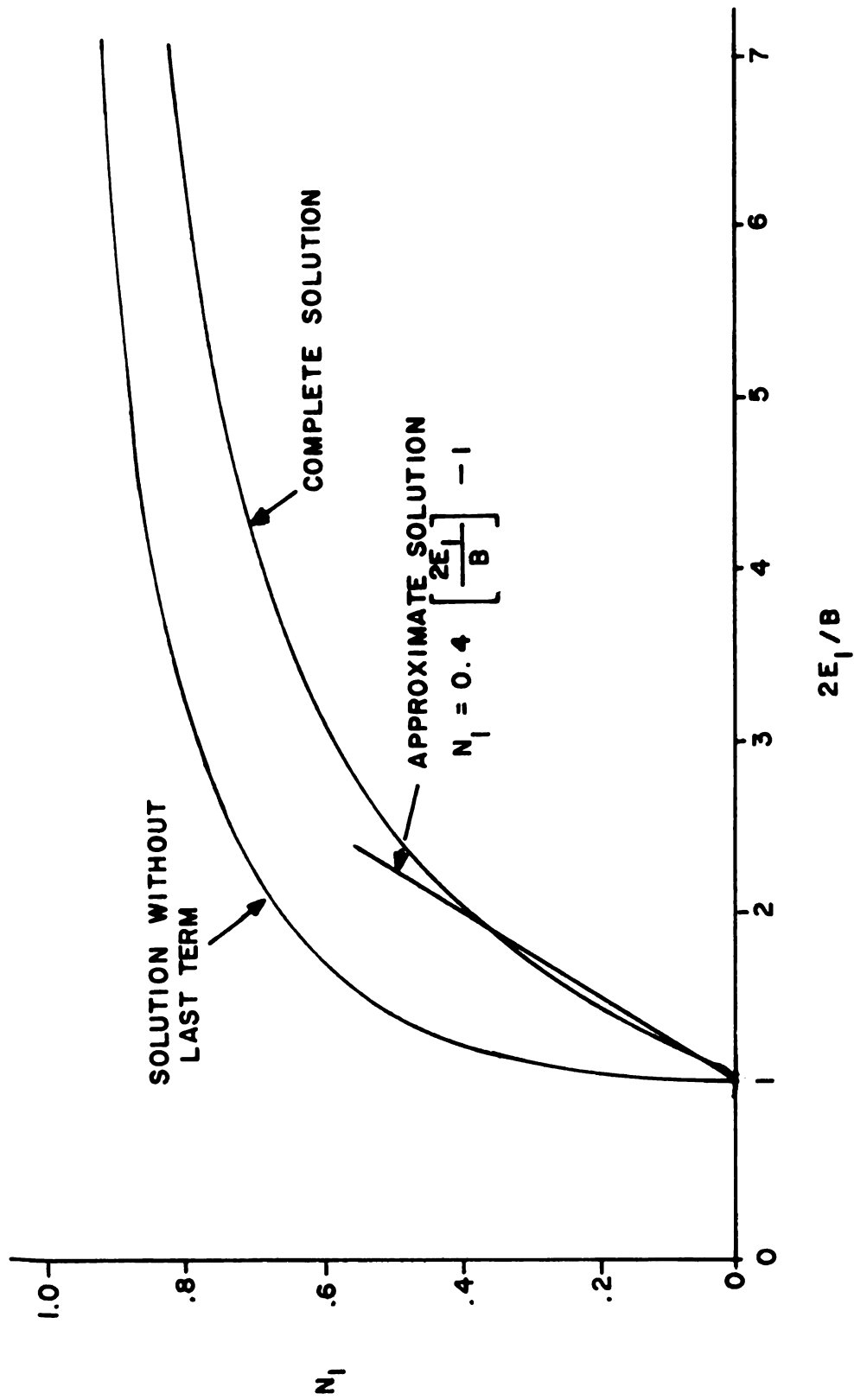
Figure A.3. Backlash describing function

Equation (A.3). This demonstrates that both terms are significant. However, in practice, N_1 is always close to zero so one could use a slope approximation through (1, 0) as

$$N_1 = 0.4 \left[\frac{2E_1}{B} - 1 \right] \tag{A.15}$$

Using this expression and solving for E_1

$$E_1 = 1.25B(N_1 + 0.4) \tag{A.16}$$

and solving (A.4) for E_2

$$E_2 = \frac{4T_L}{\pi N_2} \tag{A.17}$$

Substituting (A.16) and (A.17) into (A.14) and solving for N_2

$$N_2 \left(\omega, B \Big/ T_L \right) = \frac{3.2}{\pi \left(N_1 + .4 \right) \omega \left(B \Big/ T_L \right)} \frac{\left(K_T - J_M \omega^2 \right)^2 + \left(f_T \omega \right)^2}{\left(N_1 K_s - J_M \omega^2 \right)^2 + \left(f_T \omega \right)^2} \tag{A.18}$$

Without the slope approximation, one would be required to use the more general equation

$$N_2 \left(\omega, B \Big/ T_L \right) = \frac{4T_L}{\pi \omega E_1} \sqrt{\frac{\left(K_T - J_M \omega^2 \right)^2 + \left(f_T \omega \right)^2}{\left(N_1 K_s - J_M \omega^2 \right)^2 + \left(f_T \omega \right)}} \tag{A.19}$$

and use an iteration of the $N_1(E_1)$ equation to evaluate E_1 given N_1. Hereafter (A.19), or the approximate form (A.18), are referred to as the amplitude requirement, since it gives the required

value of N_2 as a function of the amplitude of the ratio of back-lash to friction as well as the frequency ω.

Figure (A.4) shows a plot of N_2 vs ω for a typical system using equations (A.12), and (A.18) and (A.19). The amplitude curves, from (A.18) and (A.19), coincide for each value assumed for B/T_L thus demonstrating that the slope approximation is valid. A necessary condition for a limit cycle to exist is that both the amplitude and frequency requirement be satisfied simultaneously. Thus, the possible limit cycle conditions are represented by the intersections shown in Figure A.4. The lower intersections represent the stable limit cycle condition of interest. The frequency of the limit cycle, corresponding to each of the lower intersections, will be denoted by ω_o. It is seen that as the backlash-friction ratio is decreased, ω_o decreases, which agrees with observations obtained by simulation. This phenomenon continues until a value of backlash-friction ratio is reached where the amplitude curve becomes tangent to the frequency curve. The value of the backlash friction ratio corresponding to this tangent curve is therefore the slope m of the desired backlash-friction line. Its value can be found by solving (A.18) for B/T_L and substituting in for N_2 and N_1 using (A.11) and (A.12). Thus

$$m = \underset{0<\omega<\sqrt{\frac{K_T}{J_L}}}{MIN}\left\{\frac{3.2}{\pi\left[N_1(\omega)+.4\right]\omega N_2(\omega)}\sqrt{\frac{\left(K_T-J_M\omega^2\right)^2+\left(f_T\omega\right)^2}{\left(N_1(\omega)K_s-J_M\omega^2\right)^2+\left(f_T\omega\right)^2}}\right\}$$

(A.20)
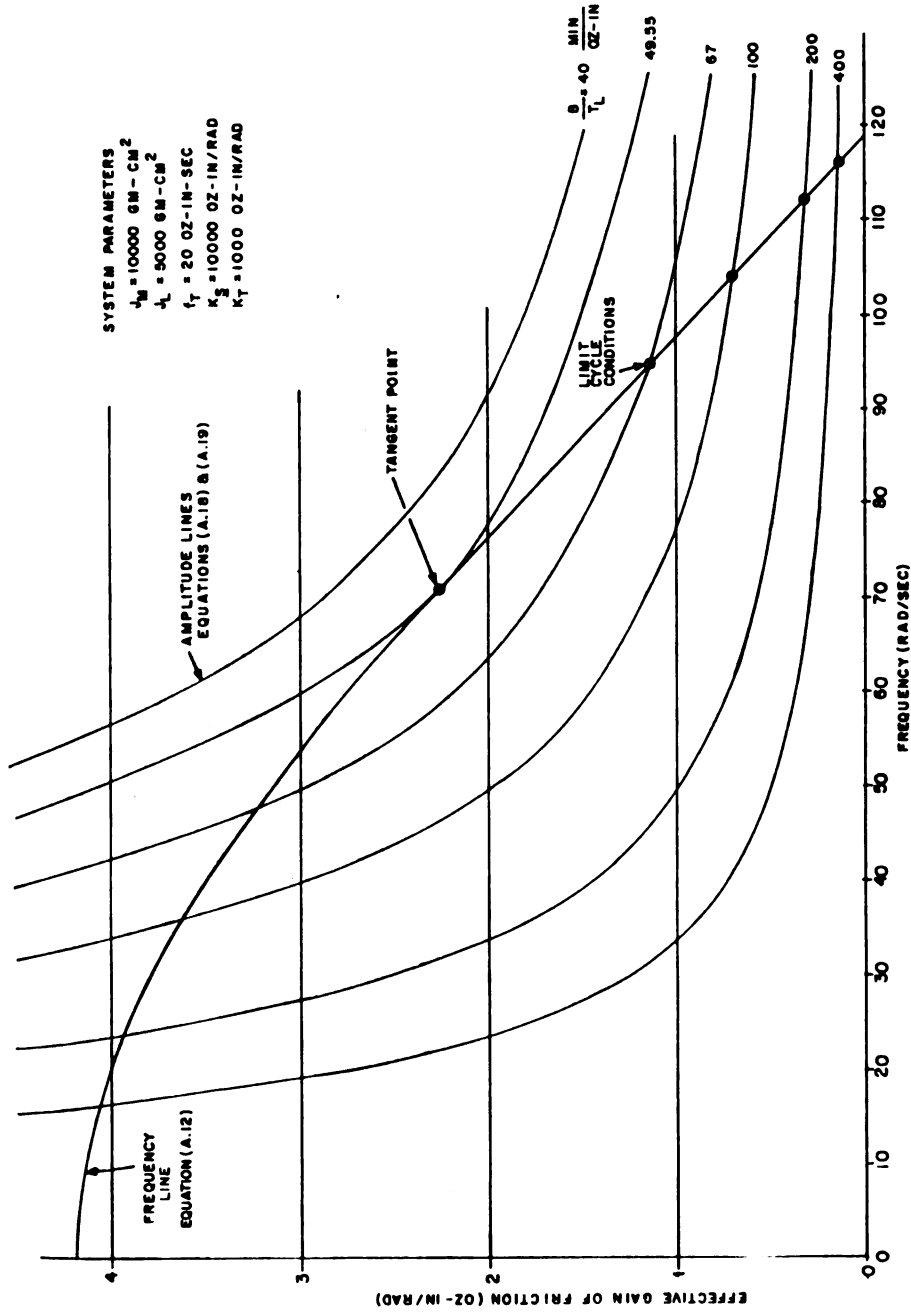
Figure A.4. Effective gain of friction vs. frequency

where $N_1(\omega)$ and $N_2(\omega)$ are defined as in (A.11) and (A.12).

Figure A.5 shows a plot of the bracketed quantity of (A.20) as a function of ω. In general the problem can be solved by using a simple optimization subroutine. The one selected is SGREAT which is the subroutine version of the program described in Reference [11].

Figure A.6 illustrates transient response curves for a typical system as obtained via a direct digital computer simulation of the nonlinear state Equation (2.1). The system parameters used are identical to those listed in Figure A.4 except that coulomb friction (T_L) was varied for the various curves. Looking at Figure A.5 one sees that for low values of friction the system displays a limit cycle. As the friction is increased, the only noticeable change is a slight decrease in the limit cycle frequency (ω_0) until finally one goes beyond the critical value and the limit cycle suddenly disappears. Qualitively the results agree exactly with what one would expect from the theory that has been developed (e.g., Figure A.4). Quantitatively, however, Figure A.6 shows that the critical value of friction is between 0.1 and 0.15 oz-in, say 0.125. The backlash value used in the simulation was 10 minutes of arc. Thus the corresponding slope of the backlash-friction curve is calculated as

$$m = \frac{B}{T_L(\text{critical})} = \frac{10}{0.125} \simeq 80 \text{ minutes/oz-in}$$

as compared to the value of 49.55 determined using (A.20). The difference is, of course, mainly attributed to the describing function approximation made in the derivation of (A.20). It can be shown, however, that neglecting the harmonics is on the safe side. That
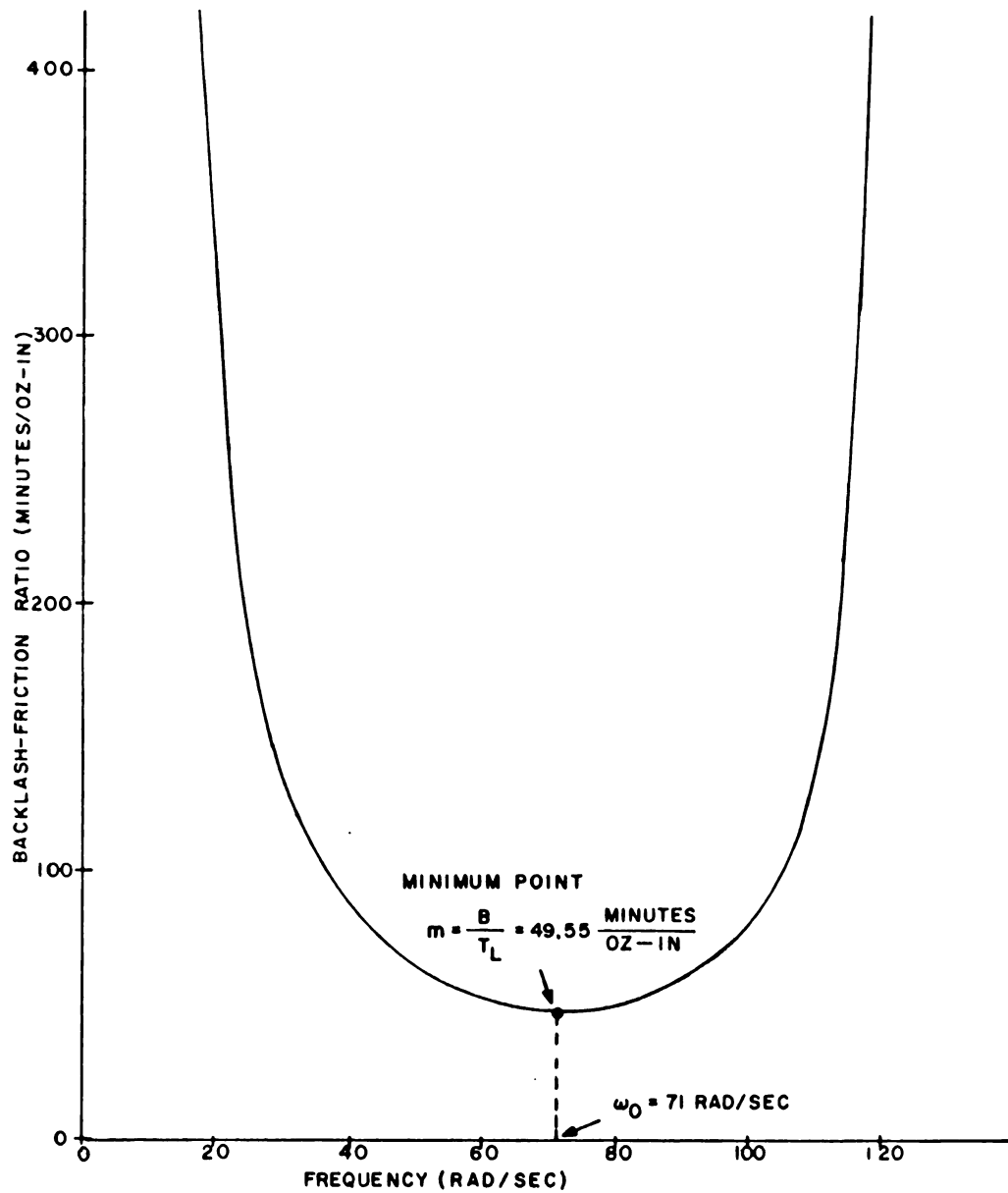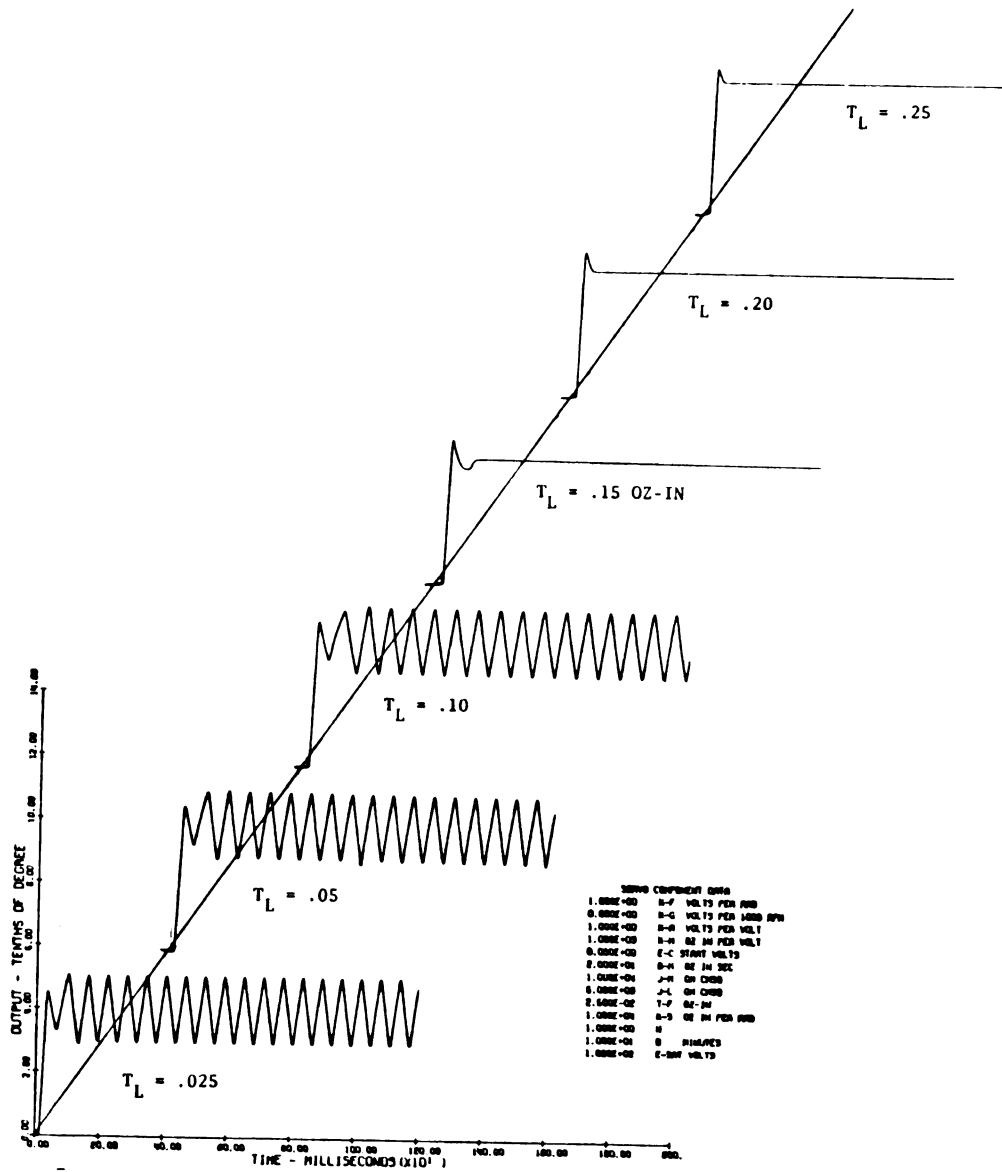
Figure A.5   Backlash-friction ratio vs. frequency

Figure A.6. System response curves for various friction values

is, if the harmonics had been included, the values of N_1 and N_2 would be somewhat higher. This would mean that the real damping effect would be greater and the backlash effect less than calculated in our derivation. Thus the amount of true allowable backlash is always greater than that calculated by (A.20), and the method used is always on the safe side.

APPENDIX B

DERIVATION OF NONLINEAR OVERSHOOT EQUATION

The purpose of this appendix is to derive the necessary equations for calculating the system overshoot for large step inputs. One way this could be accomplished would be a direct numerical integration solution of the nonlinear state model given by Equation (2.1). However, this direct simulation procedure is time consuming on a digital computer. In order to minimize the solution time, the non-linear state equations are solved explicitly using piecewise linear techniques. For simplicity sake, the system is considered to have zero backlash and infinite stiffness. This approximation is justi-fied when considering response to large inputs since these only add small oscillations about the nominal solution.

The various solutions required are visualized best by looking at the system response on the phase-plane as shown in Figure B.1. The three response curves shown were generated on an analog computer using the simulation shown in Figure B.2. Each curve is for a dif-ferent initial displacement which was selected so as to demonstrate the three different types of mode switching that can take place up to the first overshoot. Since one is concerned with the response up to the first overshoot, only the portion of the space below the abscissa is of interest. This can be divided up into three regions, in region one the drive torque is negative and saturated, and in
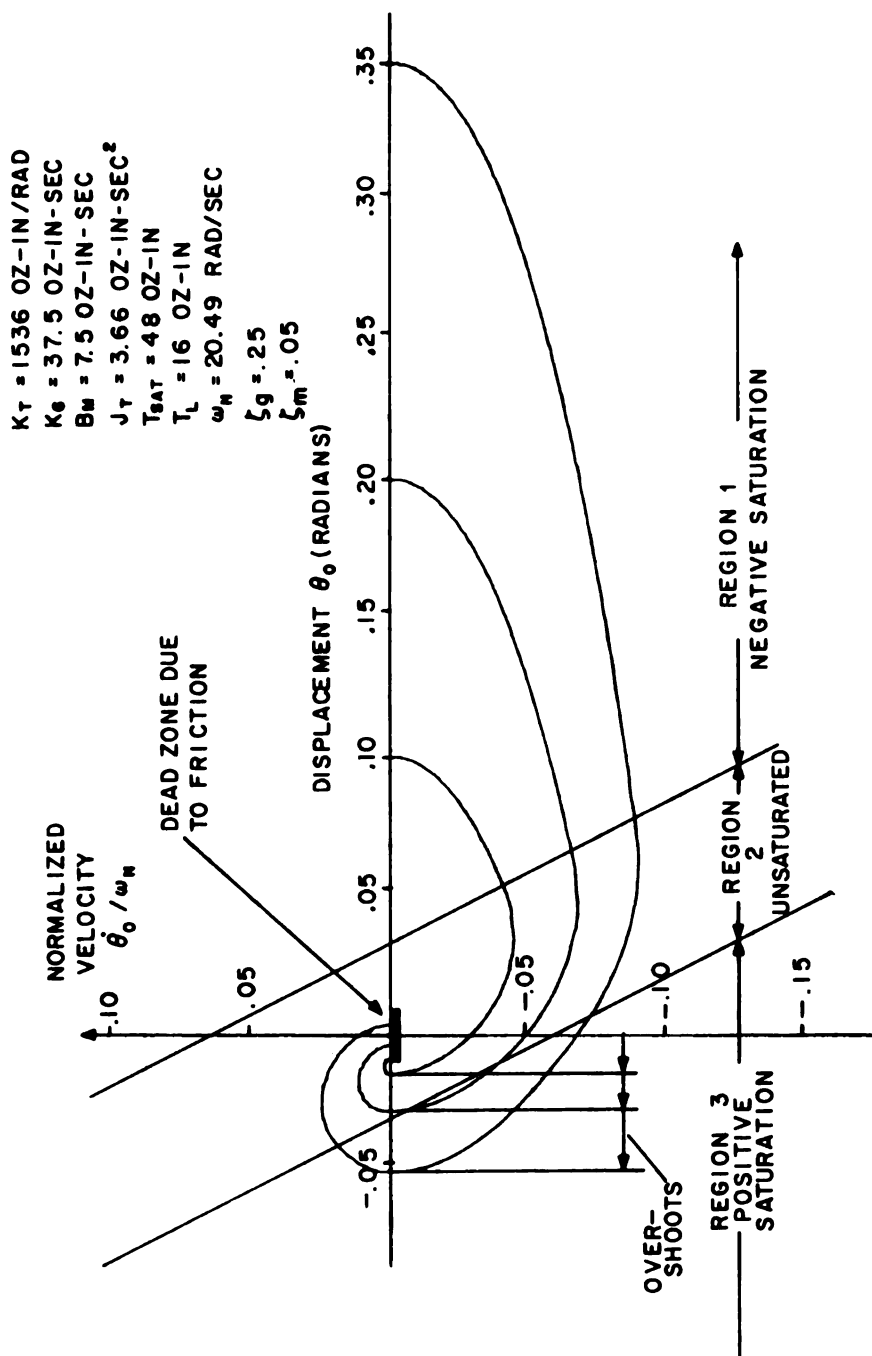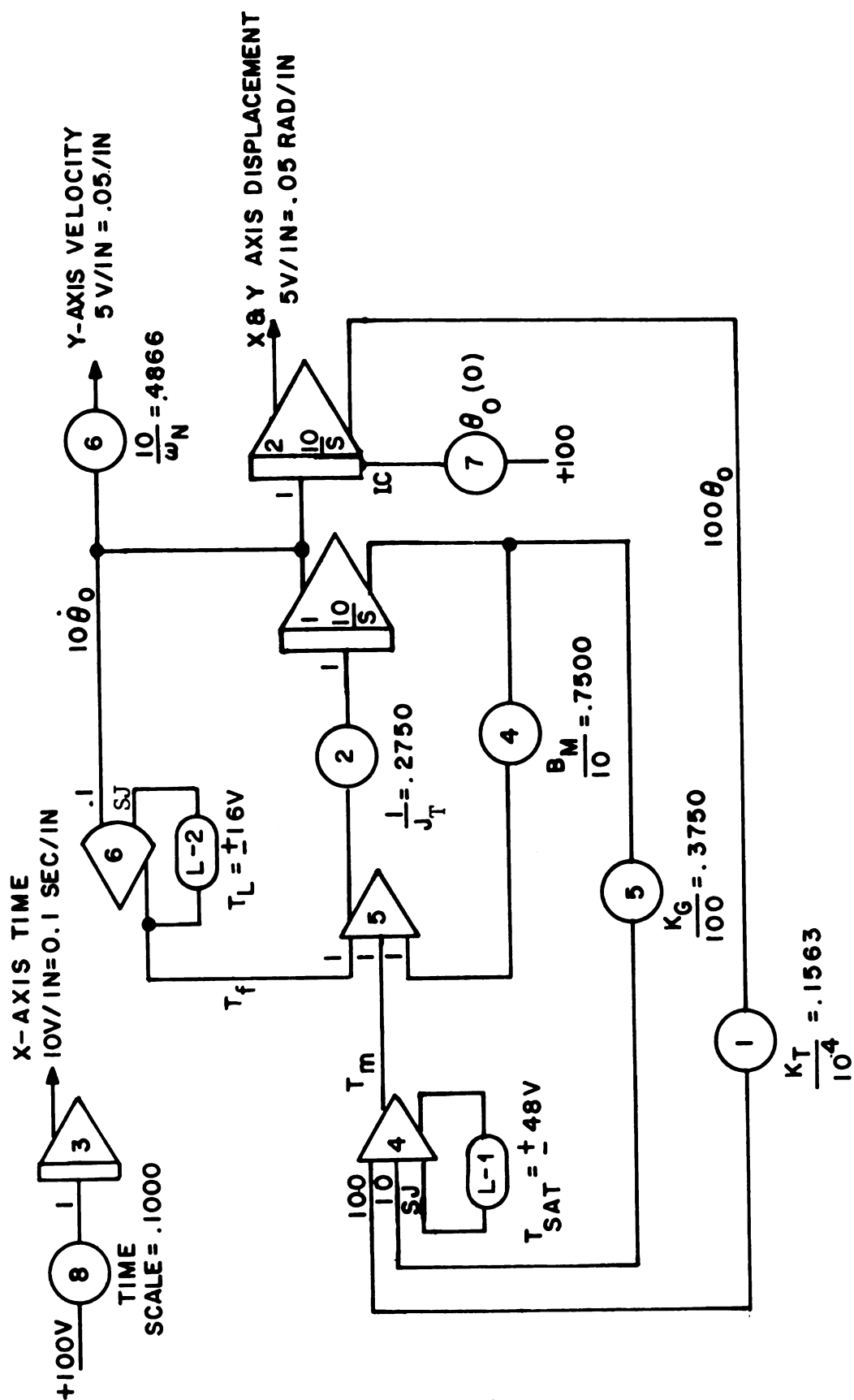
Figure B.1. System response on phase-plane diagram

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Figure B.2. Analog computer diagram used for transient analysis.

three it is positive and saturated. Region two represents the por-
tion of the space where the system is not saturated. The piecewise-
linear state equations that define the response in each region based
on the entering initial conditions are now derived. Once this is
accomplished, the switching logic required for an effective solution
is presented.

B.1 SATURATED REGION NUMBER ONE

In the saturated region number one, we have a negative driving
torque and with $\dot{\theta}_o$ also negative, the state model is:

$$
\frac{d}{dt}
\begin{bmatrix} \dot{\theta}_o \\ \\ \theta_o \end{bmatrix}
=
\begin{bmatrix} -2\zeta_M \omega_N & 0 \\ \\ 1 & 0 \end{bmatrix}
\begin{bmatrix} \dot{\theta}_o \\ \\ \theta_o \end{bmatrix}
+
\begin{bmatrix} \left(T_L - T_{SAT}\right) \big/ J_T \\ \\ 0 \end{bmatrix}
$$

(B.1)

where

$$
\zeta_M = \frac{B_M}{2\sqrt{K_T J_T}}
$$

(B.2)

$$
\omega_N = \sqrt{\frac{K_T}{J_T}}
$$

The corresponding solution is then

$$
\Theta_o(t) = e^{Pt}\,\Theta_o(0) + \int_0^t e^{P(t-\tau)}\, Q\, d\tau
$$

(B.3)

where

$$\boldsymbol{\Theta}_o(t) = \begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix}, \quad P = \begin{bmatrix} -2\zeta_M \omega_N & 0 \\ \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \dfrac{T_L - T_{SAT}}{J_T} \\ \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\Theta}_o(0) = \begin{bmatrix} \dot{\theta}_o(0) \\ \\ \theta_o(0) \end{bmatrix}$$

(B.4)

The eigenvalues are found from the characteristic polynominal

$$D(\lambda) = |\lambda U - P| = \lambda(\lambda + 2\zeta_M \omega_N)$$ (B.5)

thus

$$\lambda_1 = 0$$

$$\lambda_2 = -2\zeta_M \omega_N$$

(B.6)

The state transition matrix is given by

$$e^{Pt} = Z_{11} + Z_{21} e^{-2\zeta_M \omega_N t}$$ (B.7)

Substituting the above into (B.3) and simplifying

$$\boldsymbol{\Theta}_o(t) = Z_{21}\left[\boldsymbol{\Theta}_o(0) + \frac{Q}{2\zeta_M \omega_N}\right] e^{-2\zeta_M \omega_N t} + \left[Z_{11} Q\right] t + \left[Z_{11}\boldsymbol{\Theta}_o(0) + \frac{Z_{21} Q}{2\zeta_M \omega_N}\right]$$

(B.8)

where the constituent idempotent matrices are calculated as

$$Z_{11} = \frac{\lambda_2 U - P}{\lambda_2 - \lambda_1} = \frac{\begin{bmatrix} 0 & 0 \\ -1 & -2\zeta_M \omega_N \end{bmatrix}}{-2\zeta_M \omega_N}$$ (B.9)

$$Z_{21} = \frac{\lambda_1 U - P}{\lambda_1 - \lambda_2} = \frac{\begin{bmatrix} 2\zeta_M \omega_N & 0 \\ -1 & 0 \end{bmatrix}}{2\zeta_M \omega_N}$$

Substituting (B.9), and (B.4) into (B.8) and simplifying

$$
\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \dot{\theta}_o(0) + \dfrac{T_{SAT}-T_L}{B_M} \\ \\ \dfrac{-\dot{\theta}_o(0) - (T_{SAT}-T_L)/B_M}{2\zeta_M\omega_N} \end{bmatrix} e^{-2\zeta_M\omega_N t} - \begin{bmatrix} 0 \\ \\ \dfrac{T_{SAT}-T_L}{B_M} \end{bmatrix} t + \begin{bmatrix} -\dfrac{T_{SAT}-T_L}{B_M} \\ \\ \theta_o(0) + \dfrac{\dot{\theta}_o(0) + (T_{SAT}-T_L)/B_M}{2\zeta_M\omega_N} \end{bmatrix}
$$

$$(B.10)$$

which is the desired state equation for the saturated region
number 1.

B.2   UNSATURATED REGION NUMBER TWO

In the unsaturated region (region number 2) for $\dot{\theta}_o$ negative
we have the state model

$$
\frac{d}{dt} \begin{bmatrix} \dot{\theta}_o \\ \\ \theta_o \end{bmatrix} = \begin{bmatrix} -2\zeta\omega_N & -\omega_N^2 \\ \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_o \\ \\ \theta_o \end{bmatrix} + \begin{bmatrix} T_L/J_T \\ \\ 0 \end{bmatrix}
$$

$$(B.11)$$

The corresponding solution is again given by

$$
\Theta_o(t) = e^{Pt}\Theta_o(0) + \int_0^t e^{P(t-\tau)} Q \, d\tau
$$

$$(B.12)$$

where

$$
\Theta_o(t) = \begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix}, \quad P = \begin{bmatrix} -2\zeta\omega_N & -\omega_N^2 \\ \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \dfrac{T_L}{J_T} \\ \\ 0 \end{bmatrix}, \quad \text{and} \quad \Theta_o(0) = \begin{bmatrix} \dot{\theta}_o(0) \\ \\ \theta_o(0) \end{bmatrix}
$$

$$(B.13)$$

The eigenvalues are found from the characteristic polynominal

$$D(\lambda) = |\lambda U - P| = \lambda^2 + 2\zeta\omega_N\lambda + \omega_N^2 \qquad (B.14)$$

Thus

$$\lambda_1 = \omega_N(-\zeta + \sqrt{\zeta^2 - 1})$$

$$\qquad (B.15)$$

$$\lambda_2 = \omega_N(-\zeta - \sqrt{\zeta^2 - 1})$$

The state transition matrix is given by:

$$e^{Pt} = Z_{11} e^{\lambda_1 t} + Z_{21} e^{\lambda_2 t} , \text{ for } \lambda_1 \neq \lambda_2$$

$$\qquad (B.16)$$

$$= \left[U + Z_{12}t \right] e^{-\omega_N t} , \text{ for } \lambda_1 = \lambda_2 = -\omega_N$$

Substituting each of the above into (B.8) and simplifying

$$\theta_o(t) = Z_{11}\left[\theta_o(0) + \frac{Q}{\lambda_1}\right]e^{\lambda_1 t} + Z_{21}\left[\theta_o(0) + \frac{Q}{\lambda_2}\right]e^{\lambda_2 t} - \left[\lambda_2 Z_{11} + \lambda_1 Z_{12}\right]\frac{Q}{\lambda_1\lambda_2}$$

$$\text{for } \zeta \neq 1$$

$$\qquad (B.17)$$

and

$$= Z_{12}\left[\theta_o(0) - \frac{Q}{\omega_N}\right]te^{-\omega_N t} + \left[\theta_o(0) - \frac{Q}{\omega_N} - \frac{Z_{12}Q}{\omega_N^2}\right]e^{-\omega_N t} + \left[\omega_N U + Z_{12}\right]\frac{Q}{\omega_N^2}$$

$$\text{for } \zeta = 1$$

$$\qquad (B.18)$$

where the constituent idempotent matrices are calculated as

$$Z_{11} = \frac{\lambda_2 U - P}{\lambda_2 - \lambda_1} = \frac{\begin{bmatrix} \omega_N\left(\zeta - \sqrt{\zeta^2-1}\right) & \omega_N^2 \\ -1 & \omega_N\left(-\zeta - \sqrt{\zeta^2-1}\right) \end{bmatrix}}{-2\omega_N\sqrt{\zeta^2-1}}$$

(B.19)

$$Z_{21} = \frac{\lambda_1 U - P}{\lambda_1 - \lambda_2} = \frac{\begin{bmatrix} \omega_N\left(\zeta + \sqrt{\zeta^2-1}\right) & \omega_N^2 \\ -1 & \omega_N\left(-\zeta + \sqrt{\zeta^2-1}\right) \end{bmatrix}}{2\omega_N\sqrt{\zeta^2-1}}$$

and the constituent nilpotent as

$$Z_{12} = \left[P + \omega_N U\right]\bigg|_{\zeta=1} = \begin{bmatrix} -\omega_N & -\omega_N^2 \\ 1 & \omega_N \end{bmatrix}$$

(B.20)

Substituting (B.20), and (B.15) into (B.12) and simplifying

$$\begin{bmatrix} \dot{\theta}_o(t) \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)\dot{\theta}_o(0) + \omega_N^2\,\theta_o(0) - \dfrac{T_L}{J_T} \\ -\dot{\theta}_o(0) - \omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)\theta_o(0) + \dfrac{T_L}{J_T\omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)} \end{bmatrix} \frac{e^{-\omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)t}}{2\omega_N\sqrt{\zeta^2-1}}$$

$$+ \begin{bmatrix} -\omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)\dot{\theta}_o(0) - \omega_N^2\,\theta_o(0) + \dfrac{T_L}{J_T} \\ \dot{\theta}_o(0) + \omega_N\left(\zeta + \sqrt{\zeta^2-1}\right)\theta_o(0) - \dfrac{T_L}{J_T\omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)} \end{bmatrix} \frac{e^{-\omega_N\left(\zeta - \sqrt{\zeta^2-1}\right)t}}{2\omega_N\sqrt{\zeta^2-1}} + \begin{bmatrix} 0 \\ \dfrac{T_L}{K_T} \end{bmatrix}$$

(for $\zeta \neq 1$)

(B.21)

and

$$
\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \dfrac{T_L}{J_T} - \omega_N \dot{\theta}_o(0) - \omega_N^2 \theta_o(0) \\ \\ \dot{\theta}_o(0) + \omega_N \theta_o(0) - \dfrac{T_L}{\sqrt{K_T J_T}} \end{bmatrix} t e^{-\omega_N t} + \begin{bmatrix} \dot{\theta}_o(0) \\ \\ \theta_o(0) - \dfrac{T_L}{K_T} \end{bmatrix} e^{-\omega_N t} + \begin{bmatrix} 0 \\ \\ \dfrac{T_L}{K_T} \end{bmatrix}
$$

$$\text{for } \zeta = 1$$

$$(B.22)$$

Even though (B.21) is valid for $\zeta < 1$, it does involve complex arithmetic. A better form for this case is obtained directly from (B.21) by expressing the exponentials in terms of sine and cosine and simplifying resulting in

$$
\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = e^{-\omega_N \zeta t} \left\{ \begin{bmatrix} \dot{\theta}_o(0) \\ \\ \theta_o(0) - \dfrac{T_L}{K_T} \end{bmatrix} \cos \omega_N \sqrt{1-\zeta^2}\, t + \frac{1}{\sqrt{1-\zeta^2}} \begin{bmatrix} \dfrac{T_L}{\sqrt{K_T J_T}} - \zeta\dot{\theta}_o(0) - \omega_N \theta_o(0) \\ \\ \dfrac{\dot{\theta}_o(0)}{\omega_N} + \zeta\theta_o(0) - \dfrac{T_L \zeta}{K_T} \end{bmatrix} \sin \omega_N \sqrt{1-\zeta^2}\, t \right\} + \begin{bmatrix} 0 \\ \\ \dfrac{T_L}{K_T} \end{bmatrix}
$$

$$\text{for } \zeta < 1$$

$$(B.23)$$

The necessary state equations for the unsaturated region have thus been derived. These are (B.21) for $\zeta > 1$, (B.22) for $\zeta = 1$, and (B.23) for $\zeta < 1$.

B.3 SATURATED REGION NUMBER THREE

The only difference between saturated region number 1 and number 3 is that the driving torque is reversed. Therefore, the

state equation for this region is obtained by simply reversing the sign of T_{SAT} in (B.10). Thus for region 3 we have

$$
\begin{bmatrix} \dot{\theta}_o(t) \\ \\ \theta_o(t) \end{bmatrix} = \begin{bmatrix} \dot{\theta}_o(0) - \dfrac{T_{SAT} + T_L}{B_M} \\ \\ \dfrac{-\dot{\theta}_o(0) + \left(T_{SAT} + T_L\right)/B_M}{2\zeta_M \omega_N} \end{bmatrix} e^{-2\zeta_M \omega_N t} + \begin{bmatrix} 0 \\ \\ \dfrac{T_{SAT} + T_L}{B_M} \end{bmatrix} t + \begin{bmatrix} \dfrac{T_{SAT} + T_L}{B_M} \\ \\ \theta_o(0) + \dfrac{\dot{\theta}_o(0) - \left(T_{SAT} + T_L\right)/B_M}{2\zeta_M \omega_N} \end{bmatrix}
$$

$$(B.24)$$

B.4 SWITCHING MODE LOGIC

The remaining task is to tie together the state equations for the various regions in order to arrive at the value for the system overshoot. The procedure used is summarized by the flow chart of Figure B.3, the derivation of which can be explained in the following manner.

The first step is to examine whether the servo is initially saturated by comparing $\theta_o(0)$ to T_{SAT}/K_T. Assuming that the initial displacement is large enough to cause saturation, the next step is to solve for the intersection of the trajectory and the first saturation boundary. The boundary line defining this saturation is given by

$$
\theta_{b1} = \frac{T_{SAT}}{K_T} - \frac{2\zeta_g}{\omega_N} \dot{\theta}_o \tag{B.25}
$$

letting the $\left(T_{SAT} - T_L\right)/B_M$ term of Equation (B.10) be defined as $\dot{\theta}_s$, which is the servo followup rate, and then substituting (B.10) into (B.25):
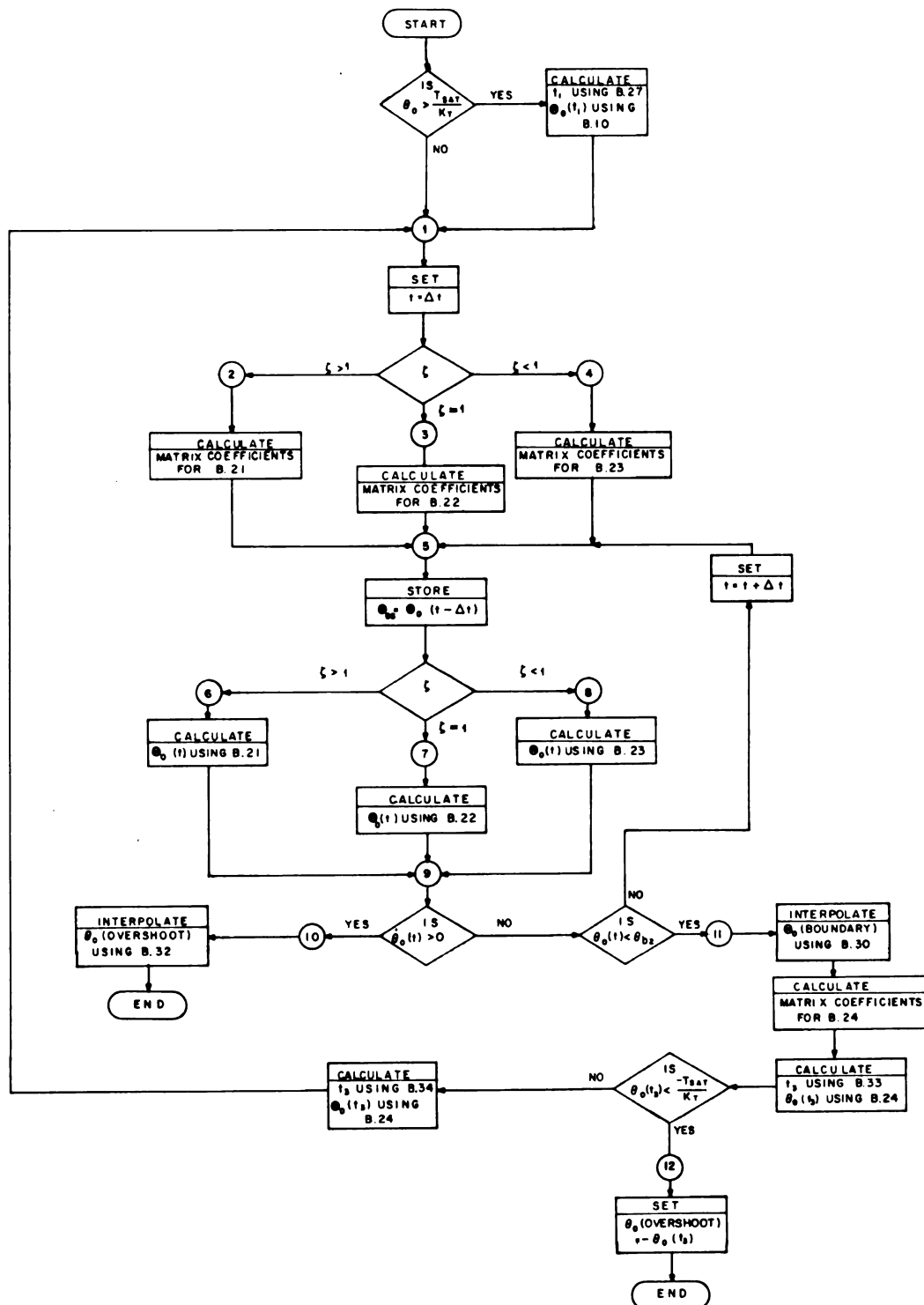
126

Figure B.3. Nonlinear overshoot logic flow diagram.

$$\left[\frac{-\dot{\theta}_o(0)-\dot{\theta}_s}{2\zeta_M\omega_N}\right]e^{-2\zeta_M\omega_N t} - \dot{\theta}_s t + \theta_o(0) + \frac{\dot{\theta}_o(0)+\dot{\theta}_s}{2\zeta_M\omega_N}$$

$$= \frac{T_{SAT}}{K_T} - \frac{2\zeta_g}{\omega_N}\left[\dot{\theta}_o(0)+\dot{\theta}_s\right]e^{-2\zeta_M\omega_N t} - \frac{2\zeta_g}{\omega_N}\dot{\theta}_s$$

(B.26)

and collecting terms

$$\left[\frac{2\zeta_g}{\omega_N} - \frac{1}{2\zeta_M\omega_N}\right]\left[\dot{\theta}_o(0)+\dot{\theta}_s\right]e^{-2\zeta_M\omega_N t_1} - \left[\dot{\theta}_s\right]t_1$$

$$+ \left\{\frac{1}{2\zeta_M\omega_N}\left[\dot{\theta}_o(0)+\dot{\theta}_s\right] + \theta_o(0) - \frac{T_{SAT}}{K_T} - \frac{2\zeta_g}{\omega_N}\dot{\theta}_s\right\} = 0$$

(B.27)

Equation (B.27) can then be solved for the time at the boundary

(t_1) by iteration. A good initial guess is

$$t_1(est.) = \frac{\theta_o(0)-T_{SAT}/K_T}{\dot{\theta}_s}$$

(B.28)

Once t_1 is found it is substituted into (B.10) to obtain the

desired boundary conditions which are the initial conditions for

region number 2. These two steps are illustrated as the first oper-

ation in Figure B.3 for the saturated case.

Region number 2 is then entered, either after the above calcu-

lations for the saturated case, or directly for the unsaturated case.

Since region number 2 is one of indecision, that is the next boundary is not known, one simply continues to solve for $\dot{\theta}_o(t)$ and $\theta_o(t)$ incrementing time in steps of Δt until one of the two boundaries is crossed. However, the particular state equation to be evaluated in region number 2 depends on the value of ζ. Thus the next step is to evaluate ζ and, depending on its value, calculate the matrix coefficients which are independent of time as required for either (B.21), (B.22), or (B.23). One is now at point 5 of Figure B.3 with $t = 0$. The previous values of the state variables are stored as Θ_{os} and depending on the value of ζ either (B.21), (B.22), or (B.23) is evaluated. It should be noted that the latter is a relatively simple calculation since the matrix coefficients are independent of t and have already been evaluated.

After point 9, one simply checks to see if either of the two possible boundary conditions have been exceeded, i.e. if $\dot{\theta}_o(t) > 0$ or if $\theta_o(t) < \theta_{b2}$ where θ_{b2} in the corresponding values for the second saturation line given by

$$\theta_{b2} = -\frac{T_{SAT}}{K_T} - \frac{2\zeta_g}{\omega_N}\dot{\theta}_o \qquad (B.29)$$

If neither of the above inequalities is satisfied then an appropriate Δt is calculated, t is incremented and one returns to point 5 and the process is repeated until an exit is obtained at either point 10 or 11.

Consider first that the exit is via 11. The corresponding situation is illustrated in Figure B.4, i.e. $\Theta_o(t)$ lies in region
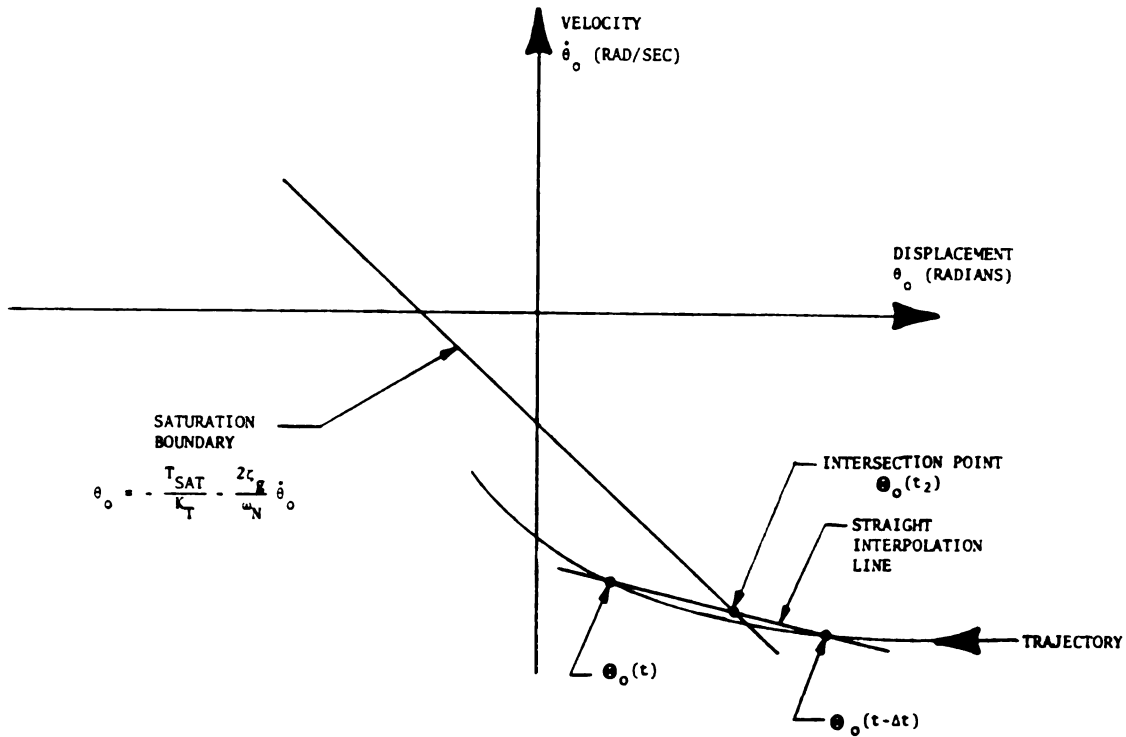
Figure B.4. Phase-plane interpolation diagram.

number 3 while $\boldsymbol{\theta}_o(t-\Delta t)$ lies in region 2. By using a straight interpolation line between these two points, the desired intersection point is readily calculated using the derived interpolation equation

$$\begin{bmatrix} \dot{\theta}_o(t_2) \\ \\ \theta_o(t_2) \end{bmatrix} = \frac{1}{2\zeta_g M + \omega_N} \begin{bmatrix} \omega_N\dot{\theta}_o(t-\Delta t) - \omega_N\theta_o(t-\Delta t) - \omega_N M T_{SAT}/K_T \\ \\ 2\zeta_g M\theta_o(t-\Delta t) - 2\zeta_g\dot{\theta}_o(t-\Delta t) - \omega_N T_{SAT}/K_T \end{bmatrix}$$

(B.30)

where M is the slope of the interpolation line given by

$$M = \frac{\dot{\theta}_o(t) - \dot{\theta}_o(t-\Delta t)}{\theta_o(t) - \theta_o(t-\Delta t)}$$

(B.31)

In a similar manner, if one exceeds the $\dot{\theta}_o = 0$ boundary (i.e. exit via point 10) the intersection point, which this time corresponds to the desired overshoot, may be evaluated using

$$\theta_o(\text{overshoot}) = \frac{\dot{\theta}_o(t-\Delta t)}{M} - \theta_o(t-\Delta t) \qquad (B.32)$$

where M is as given in (B.31).

If the intersection is on the $\dot{\theta}_o = 0$ line, one is finished; if not, then one proceeds into region number 3 using the new initial conditions as given by (B.30). Region number 3 is also one of indecision reguarding the next boundary condition. However one can readily derive which boundary applies in the following manner. If one remained in region number 3 $\dot{\theta}_o(t)$ would equal zero at some time say t_3. This value can be solved for by using the first equation of (B.24) yielding

$$t_3 = \frac{1}{2\zeta_M \omega_N} \ell n \left[1 - \frac{B_M \dot{\theta}_o(0)}{T_{SAT} + T_L} \right] \qquad (B.33)$$

and using this value in the second equation one readily obtains $\theta_o(t_3)$. If this value is less than $-T_{SAT}/K_T$, the assumption of remaining in region number 3 is valid and one ends up at point 12 of Figure B.3 with $\theta_o(\text{overshoot}) = -\theta_o(t_3)$. If not then the trajectory must enter region number 2 for the second time. An equation giving the appropriate time in region 3 can be obtained in the same manner as (B.27) by letting $\dot{\theta}_{ss} = \left(T_{SAT} + T_L\right)/B_M$ and using (B.24) instead of (B.10).

$$\left[\frac{2\zeta_g}{\omega_N} - \frac{1}{2\zeta_M\omega_N}\right]\left[\dot\theta_o(0)+\dot\theta_{ss}\right]e^{-2\zeta_M\omega_N t_3} + \dot\theta_{ss}t_3$$

$$+ \left\{\frac{1}{2\zeta_M\omega_N}\left[\dot\theta_o(0)-\dot\theta_{ss}\right] + \theta_o(0) + \frac{T_{SAT}}{K_T} + \frac{2\zeta_g}{\omega_N}\dot\theta_{ss}\right\} = 0$$

$$(B.34)$$

Once the time in region number 3 (t_3) is obtained by sloving (B.34) it is substituted into (B.24) and the initial conditions are obtained for region number 2, t is set to Δt and one returns to point 5. The procedure is thus repeated as explained before except this time a termination is reached via point 10 and Equation (B.32).

Tables B.1, B.2, and B.3 show sample outputs of a computer program that was written to perform the above described procedure. The three tabulations correspond to the three trajectories shown in Figure B.1. These trajectories are repeated as Figure B.5 with the corresponding calculated data points from the digital computer program supperimposed. Figure B.6 illustrates the same results except on the displacement vs. time plot instead of the phase-plane.

Table B.1. Nonlinear overshoot calculations
for 0.1 radian displacement.

```
                AUTOMATED DESIGN RESEARCH PROBLEM
                NONLINEAR OVERSHOOT CALCULATIONS
                      DECEMBER 23, 1968

                        INPUT PARAMETERS
    T-L        K-T        T-SAT      THETA-O      J-T         B-M         K-G
  (OZ-IN)   (OZ-IN/RAD)  (OZ-IN)     (RAD)    (OZ-IN-SEC2)(OZ-IN-SEC)(OZ-IN-SEC)
 1.600D+01  1.536D+03   4.800D+01  1.000D-01  3.660D+00   7.500D+00   3.750D+01

                      CALCULATED PARAMETERS
        ZETA-M       ZETA-G       ZETA      OMEGA-N    THETA-DOT-SS
                                            (RAD/SEC)   (RAD/SEC)
       5.001D-02    2.501D-01   3.001D-01  2.049D+01   4.267D+00

                      CALCULATED RESPONSE

              REGION 1-2 BOUNDARY CONDITIONS
    TIME      T-GUESS  THETA-DOT  THETA-DOT     THETA      THETA-B
   (SEC)      (SEC)    (RAD/SEC)  NORMALIZED    (RAD)       (RAD)
   0.1086     0.0161  -8.515D-01 -4.156D-02   5.204D-02   5.204D-02

                        START REGION 2
        TIME    THETA-DOT  THETA-DOT     THETA      THETA-B
       (SEC)    (RAD/SEC)  NORMALIZED    (RAD)       (RAD)
       0.0     -8.515D-01 -4.156D-02   5.204D-02  -1.046D-02
       0.0029  -8.699D-01 -4.246D-02   4.955D-02  -1.001D-02
       0.0058  -8.849D-01 -4.319D-02   4.696D-02  -9.647D-03
       0.0088  -8.963D-01 -4.375D-02   4.430D-02  -9.368D-03
       0.0119  -9.041D-01 -4.413D-02   4.157D-02  -9.177D-03
       0.0149  -9.083D-01 -4.434D-02   3.878D-02  -9.076D-03
       0.0181  -9.087D-01 -4.436D-02   3.594D-02  -9.066D-03
       0.0212  -9.053D-01 -4.419D-02   3.307D-02  -9.147D-03
       0.0244  -8.983D-01 -4.385D-02   3.017D-02  -9.319D-03
       0.0277  -8.875D-01 -4.332D-02   2.725D-02  -9.582D-03
       0.0310  -8.730D-01 -4.262D-02   2.434D-02  -9.936D-03
       0.0344  -8.548D-01 -4.173D-02   2.143D-02  -1.038D-02
       0.0378  -8.330D-01 -4.066D-02   1.853D-02  -1.091D-02
       0.0413  -8.076D-01 -3.942D-02   1.567D-02  -1.153D-02
       0.0449  -7.785D-01 -3.800D-02   1.283D-02  -1.224D-02
       0.0485  -7.458D-01 -3.640D-02   1.005D-02  -1.304D-02
       0.0523  -7.095D-01 -3.463D-02   7.319D-03  -1.393D-02
       0.0561  -6.695D-01 -3.268D-02   4.655D-03  -1.490D-02
       0.0601  -6.259D-01 -3.055D-02   2.068D-03  -1.597D-02
       0.0643  -5.784D-01 -2.824D-02  -4.295D-04  -1.713D-02
       0.0686  -5.271D-01 -2.573D-02  -2.823D-03  -1.838D-02
       0.0732  -4.717D-01 -2.302D-02  -5.094D-03  -1.973D-02
       0.0780  -4.120D-01 -2.011D-02  -7.222D-03  -2.119D-02
       0.0831  -3.476D-01 -1.697D-02  -9.176D-03  -2.276D-02
       0.0887  -2.783D-01 -1.358D-02  -1.092D-02  -2.446D-02
       0.0948  -2.033D-01 -9.924D-03  -1.240D-02  -2.629D-02
       0.1017  -1.220D-01 -5.957D-03  -1.351D-02  -2.827D-02
       0.1096  -3.371D-02 -1.646D-03  -1.412D-02  -3.043D-02
       0.1190   6.168D-02  3.011D-03  -1.398D-02  -3.276D-02

                OVERSHOOT= 1.407E-02 RADIANS
```

Table B.2. Nonlinear overshoot calculations
for 0.2 radian displacement.

```
                    AUTOMATED DESIGN RESEARCH PROBLEM
                    NONLINEAR OVERSHOOT CALCULATIONS
                          DECEMBER 23, 1968

                          INPUT PARAMETERS
   T-L        K-T       T-SAT      THETA-O      J-T         B-M         K-G
 (OZ-IN)  (OZ-IN/RAD)  (OZ-IN)     (RAD)    (OZ-IN-SEC2)(OZ-IN-SEC)(OZ-IN-SEC)
1.600D+01  1.536D+03  4.800D+01  2.000D-01  3.660D+00  7.500D+00  3.750D+01

                        CALCULATED PARAMETERS
        ZETA-M      ZETA-G       ZETA      OMEGA-N   THETA-DOT-SS
                                          (RAD/SEC)  (RAD/SEC)
        5.001D-02  2.501D-01  3.001D-01  2.049D+01  4.267D+00

                          CALCULATED RESPONSE

                    REGION 1-2 BOUNDARY CONDITIONS
   TIME      T-GUESS   THETA-DOT   THETA-DOT     THETA      THETA-B
  (SEC)      (SEC)     (RAD/SEC)   NORMALIZED    (RAD)       (RAD)
  0.1873      0.0396  -1.360D+00  -6.640D-02   6.446D-02   6.446D-02

                            START REGION 2
        TIME     THETA-DOT   THETA-DOT     THETA       THETA-B
        (SEC)    (RAD/SEC)   NORMALIZED    (RAD)        (RAD)
        0.0     -1.360D+00  -6.640D-02   6.446D-02   1.958D-03
        0.0022  -1.372D+00  -6.695D-02   6.151D-02   2.235D-03
        0.0043  -1.380D+00  -6.736D-02   5.853D-02   2.442D-03
        0.0065  -1.386D+00  -6.764D-02   5.550D-02   2.578D-03
        0.0087  -1.388D+00  -6.777D-02   5.245D-02   2.644D-03
        0.0109  -1.388D+00  -6.776D-02   4.937D-02   2.640D-03
        0.0132  -1.385D+00  -6.761D-02   4.627D-02   2.566D-03
        0.0154  -1.379D+00  -6.733D-02   4.316D-02   2.423D-03
        0.0177  -1.371D+00  -6.690D-02   4.005D-02   2.212D-03
        0.0200  -1.359D+00  -6.635D-02   3.694D-02   1.933D-03
        0.0223  -1.345D+00  -6.566D-02   3.383D-02   1.588D-03
        0.0246  -1.328D+00  -6.484D-02   3.074D-02   1.178D-03
        0.0269  -1.309D+00  -6.389D-02   2.766D-02   7.031D-04
        0.0293  -1.287D+00  -6.281D-02   2.460D-02   1.646D-04
        0.0316  -1.262D+00  -6.161D-02   2.157D-02  -4.365D-04
        0.0340  -1.235D+00  -6.028D-02   1.857D-02  -1.099D-03
        0.0365  -1.205D+00  -5.884D-02   1.561D-02  -1.823D-03
        0.0389  -1.173D+00  -5.727D-02   1.268D-02  -2.607D-03
        0.0414  -1.139D+00  -5.558D-02   9.805D-03  -3.450D-03
        0.0439  -1.102D+00  -5.378D-02   6.974D-03  -4.353D-03
        0.0465  -1.062D+00  -5.186D-02   4.197D-03  -5.313D-03
        0.0491  -1.021D+00  -4.982D-02   1.478D-03  -6.332D-03
        0.0518  -9.765D-01  -4.767D-02  -1.177D-03  -7.409D-03
        0.0545  -9.300D-01  -4.540D-02  -3.764D-03  -8.545D-03
        0.0573  -8.811D-01  -4.301D-02  -6.276D-03  -9.738D-03
        0.0601  -8.299D-01  -4.051D-02  -8.707D-03  -1.099D-02
        0.0630  -7.762D-01  -3.789D-02  -1.105D-02  -1.230D-02
        0.0660  -7.201D-01  -3.515D-02  -1.330D-02  -1.367D-02
        0.0691  -6.615D-01  -3.229D-02  -1.544D-02  -1.510D-02

                    REGION 2-3 BOUNDARY CONDITIONS
                    THETA-DOT   THETA-DOT     THETA
                    (RAD/SEC)   NORMALIZED    (RAD)
                    -6.894D-01  -3.365D-02  -1.442D-02

                    REGION 3-2 BOUNDARY CONDITIONS
   TIME      T-GUESS   THETA-DOT   THETA-DOT     THETA      THETA-B
  (SEC)      (SEC)     (RAD/SEC)   NORMALIZED    (RAD)       (RAD)
  0.0259      0.0190  -2.136D-01  -1.043D-02  -2.604D-02  -2.604D-02

                            START REGION 2
        TIME     THETA-DOT   THETA-DOT     THETA       THETA-B
        (SEC)    (RAD/SEC)   NORMALIZED    (RAD)        (RAD)
        0.0     -2.136D-01  -1.043D-02  -2.604D-02  -2.604D-02
        0.0027  -1.652D-01  -8.066D-03  -2.655D-02  -2.722D-02
        0.0056  -1.158D-01  -5.653D-03  -2.695D-02  -2.842D-02
        0.0086  -6.531D-02  -3.188D-03  -2.722D-02  -2.966D-02
        0.0117  -1.385D-02  -6.760D-04  -2.735D-02  -3.091D-02
        0.0151   3.841D-02   1.875D-03  -2.731D-02  -3.219D-02

                 OVERSHOOT= 2.734E-02 RADIANS
```

Table B.3. Nonlinear overshoot calculations
for 0.35 radian displacement.

```
                   AUTOMATED DESIGN RESEARCH PROBLEM
                   NONLINEAR OVERSHOOT CALCULATIONS
                        DECEMBER 23, 1968

                         INPUT PARAMETERS
   T-L        K-T        T-SAT      THETA-O      J-T        B-M        K-G
 (OZ-IN)  (OZ-IN/RAD)   (OZ-IN)     (RAD)    (OZ-IN-SEC2)(OZ-IN-SEC)(OZ-IN-SEC)
 1.600D+01  1.536D+03  4.800D+01  3.500D-01   3.660D+00  7.500D+00  3.750D+01

                       CALCULATED PARAMETERS
        ZETA-M     ZETA-G      ZETA      OMEGA-N    THETA-DOT-SS
                                        (RAD/SEC)    (RAD/SEC)
       5.001D-02  2.501D-01  3.001D-01  2.049D+01   4.267D+00

                       CALCULATED RESPONSE

                   REGION 1-2 BOUNDARY CONDITIONS
     TIME      T-GUESS  THETA-DOT  THETA-DOT    THETA      THETA-B
     (SEC)     (SEC)    (RAD/SEC)  NORMALIZED   (RAD)       (RAD)
     0.2737    0.0747  -1.832D+00 -8.942D-02  7.597D-02   7.597D-02

                          START REGION 2
        TIME      THETA-DOT  THETA-DOT    THETA      THETA-B
       (SEC)      (RAD/SEC)  NORMALIZED   (RAD)       (RAD)
       0.0       -1.832D+00 -8.942D-02  7.597D-02  1.347D-02
       0.0017    -1.839D+00 -8.976D-02  7.293D-02  1.365D-02
       0.0033    -1.844D+00 -9.000D-02  6.987D-02  1.377D-02
       0.0050    -1.847D+00 -9.014D-02  6.679D-02  1.383D-02
       0.0067    -1.847D+00 -9.016D-02  6.370D-02  1.384D-02
       0.0083    -1.845D+00 -9.008D-02  6.059D-02  1.380D-02
       0.0100    -1.842D+00 -8.989D-02  5.748D-02  1.371D-02
       0.0117    -1.836D+00 -8.960D-02  5.436D-02  1.356D-02
       0.0134    -1.828D+00 -8.921D-02  5.124D-02  1.337D-02
       0.0151    -1.817D+00 -8.871D-02  4.812D-02  1.312D-02
       0.0169    -1.805D+00 -8.812D-02  4.501D-02  1.282D-02
       0.0186    -1.791D+00 -8.742D-02  4.191D-02  1.247D-02
       0.0203    -1.775D+00 -8.663D-02  3.882D-02  1.208D-02
       0.0221    -1.757D+00 -8.575D-02  3.575D-02  1.164D-02
       0.0238    -1.737D+00 -8.477D-02  3.269D-02  1.115D-02
       0.0256    -1.714D+00 -8.369D-02  2.966D-02  1.061D-02
       0.0273    -1.691D+00 -8.253D-02  2.664D-02  1.002D-02
       0.0291    -1.665D+00 -8.127D-02  2.366D-02  9.397D-03
       0.0309    -1.637D+00 -7.992D-02  2.070D-02  8.724D-03
       0.0327    -1.608D+00 -7.849D-02  1.776D-02  8.007D-03
       0.0345    -1.577D+00 -7.697D-02  1.487D-02  7.247D-03
       0.0364    -1.544D+00 -7.537D-02  1.200D-02  6.444D-03
       0.0382    -1.509D+00 -7.368D-02  9.174D-03  5.598D-03
       0.0401    -1.473D+00 -7.190D-02  6.386D-03  4.711D-03
       0.0420    -1.435D+00 -7.004D-02  3.640D-03  3.782D-03

                   REGION 2-3 BOUNDARY CONDITIONS
                   THETA-DOT  THETA-DOT    THETA
                   (RAD/SEC)  NORMALIZED   (RAD)
                  -1.438D+00 -7.019D-02  3.854D-03

                  OVERSHOOT= 4.936E-02 RADIANS
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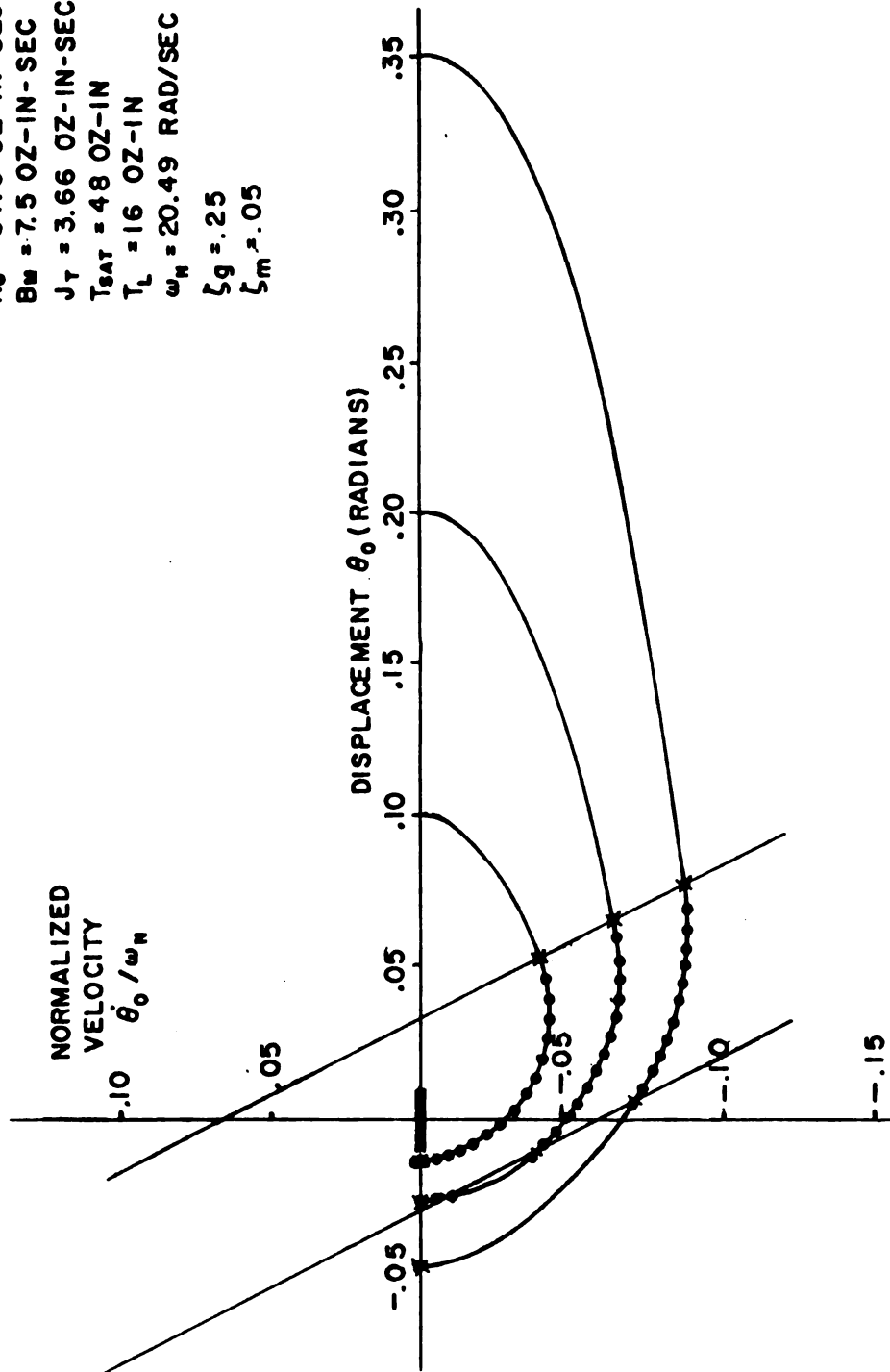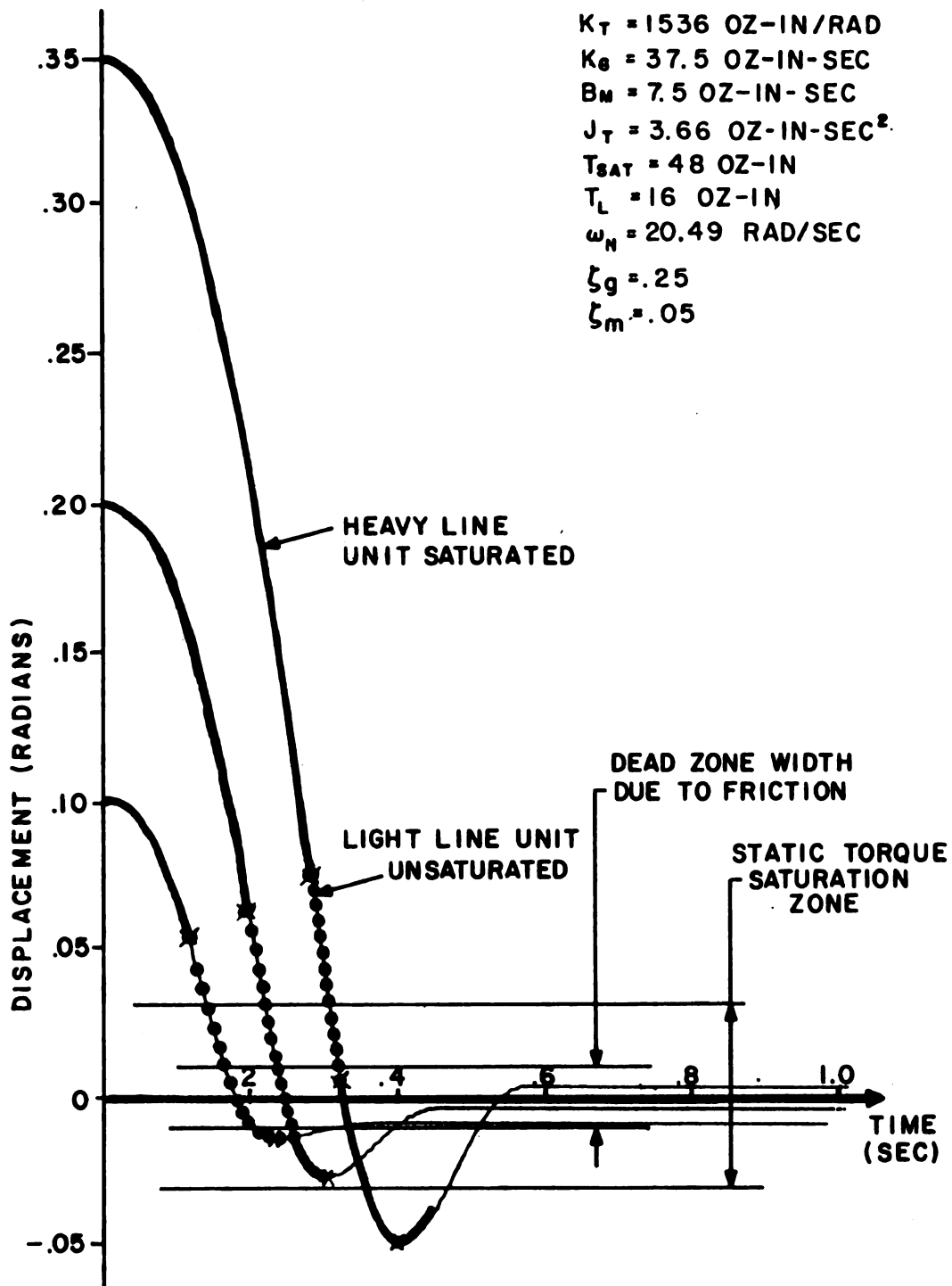
Figure B.5. System response on phase-plane diagram.

Figure B.6. Typical system response curves.

APPENDIX C

DERIVATION OF NONLINEAR BANDWIDTH EQUATION

The purpose of this appendix is to derive the necessary equations for calculating the system bandwidth. Often, this is "accomplished" by using the linear closed loop transfer function, with s replaced by $j\omega$, and calculating the frequency at which $\left| \theta_o(j\omega) / \theta_i(j\omega) \right|$ is down 3 db. However in the real world, one rarely realizes this value. This discrepancy is mainly a result of amplifier saturation, which in turn causes the actual bandwidth to be a function of input level. The true affects of saturation, as well as coulomb friction, backlash and finite stiffness, could be accounted for by direct simulation of the nonlinear state equations presented in Section 2.2. However, including this simulation in an iteration loop (as necessary to find $\omega = \omega_B$ such that the response is -3 db) is very time consuming. This is due to the fact that for each iteration one must wait for steady-state conditions before an evaluation can be made. In order to minimize the solution time, an algebraic equation for the bandwidth frequency is derived including the effects of saturation and coulomb friction. This is accomplished by using describing function approximations. For simplicity sake, the system is considered to have zero backlash and infinite stiffness. This approximation is justified since the displacements are considered to be large compared to any deflections that may exist in the gear train.

The bandwidth frequency for the linear second order system

is found by taking the closed loop transfer function

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{\omega_N{}^2}{s^2 + 2\zeta\omega_N s + \omega_N{}^2} \qquad (C.1)$$

replacing s by $j\omega$ and setting the magnitude equal to 0.707 for

$\omega = \omega_B$

$$\left| \frac{\omega_N{}^2}{\left(\omega_N{}^2 - \omega_B{}^2\right) + j\left(2\zeta\omega_N\omega_B\right)} \right| = 0.707 \qquad (C.2)$$

and solving for the bandwidth frequency (ω_B)

$$\omega_B = \omega_N \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} \qquad (C.3)$$

This equation can be extended to the desired nonlinear case using

describing functions, by replacing ω_N and ζ by their correspond-

ing effective values $\omega_N{}'$ and ζ' given by

$$\omega_N{}' = \sqrt{\frac{K_f K_a{}' K_m}{J_T}} \qquad (C.4)$$

$$\zeta' = \frac{B_M + K_G K_a{}' / K_a + N_2}{2\sqrt{K_f K_a{}' K_M J_T}} \qquad (C.5)$$

where K_a' and N_2 are the effective gain values for the amplifier and coulomb friction elements as given by their respective describing functions. For the friction element

$$N_2 = \frac{4T_L}{\pi \dot{\theta}_o(\text{peak})} \tag{C.6}$$

and letting N_3 be the describing function for the amplifier saturation

$$N_3 = \frac{2K_a}{\pi}\left[\psi + \frac{\sin(2\psi)}{2}\right] \quad \text{for} \quad E_i(\text{peak}) \geq E_{sat}K_a \tag{C.7}$$

$$= K_a \quad \text{otherwise}$$

where

$$\psi = \text{arc sin}\left[\frac{E_{sat}}{K_a E_i(\text{peak})}\right] \tag{C.8}$$

Thus N_3 is a function of its respective input $E_i(\text{peak})$ and likewise N_2 is a function of $\dot{\theta}_o(\text{peak})$. One can obtain $E_i(\text{peak})$ for any ω from the transfer function

$$\frac{E_i(s)}{\theta_i(s)} = K_f\left[\frac{s^2 + 2\zeta_{ML}'\omega_N's}{s^2 + 2\zeta'\omega_N's + \omega_N'^2}\right] \tag{C.9}$$

where

$$\zeta_{ML}' = \frac{B_M + N_2}{2\sqrt{K_f K_a' K_M J_T}} \tag{C.10}$$

by replacing s with jω and solving for E_i(peak)

$$E_i\text{(peak)} = K_f \sqrt{\frac{\omega^4 + \left(2\zeta_{ML}'\omega_N'\omega\right)^2}{\left(\omega_N'^2 - \omega^2\right)^2 + \left(2\zeta'\omega_N'\omega\right)^2}} \; \theta_i\text{(peak)}$$

(C.11)

Since the output amplitude is 0.707 of the input amplitude

$$\dot\theta_o\text{(peak)} = 0.707 \; \omega\theta_i\text{(peak)}$$

(C.12)

and thus

$$N_2 = \frac{4T_L}{0.707 \; \pi\omega\theta_i\text{(peak)}}$$

(C.13)

The necessary relationships have now been derived and ω_B given by

$$\omega_B = \omega_N' \sqrt{1 - 2\zeta'^2 + \sqrt{2 - 4\zeta'^2 + 4\zeta'^4}}$$

(C.14)

is calculated using the iteration scheme illustrated in Figure C.1. This procedure consists of one iteration inside another. The outer loop is used to adjust ω until it equals ω_B as given by (C.14) to the desired accuracy $(\delta\omega_B)$, while the inner loop adjusts, for each value of ω , K_a' so that it equals its describing function value (N_3) as given by (C.7) to the desired accuracy (δK_a).

Tables C.1 and C.2 show sample outputs from a computer program that was written to perform the above described procedure. The only difference between the two tables is that Table C.2 is for the zero friction case. Each table consists of a tabulation of the system
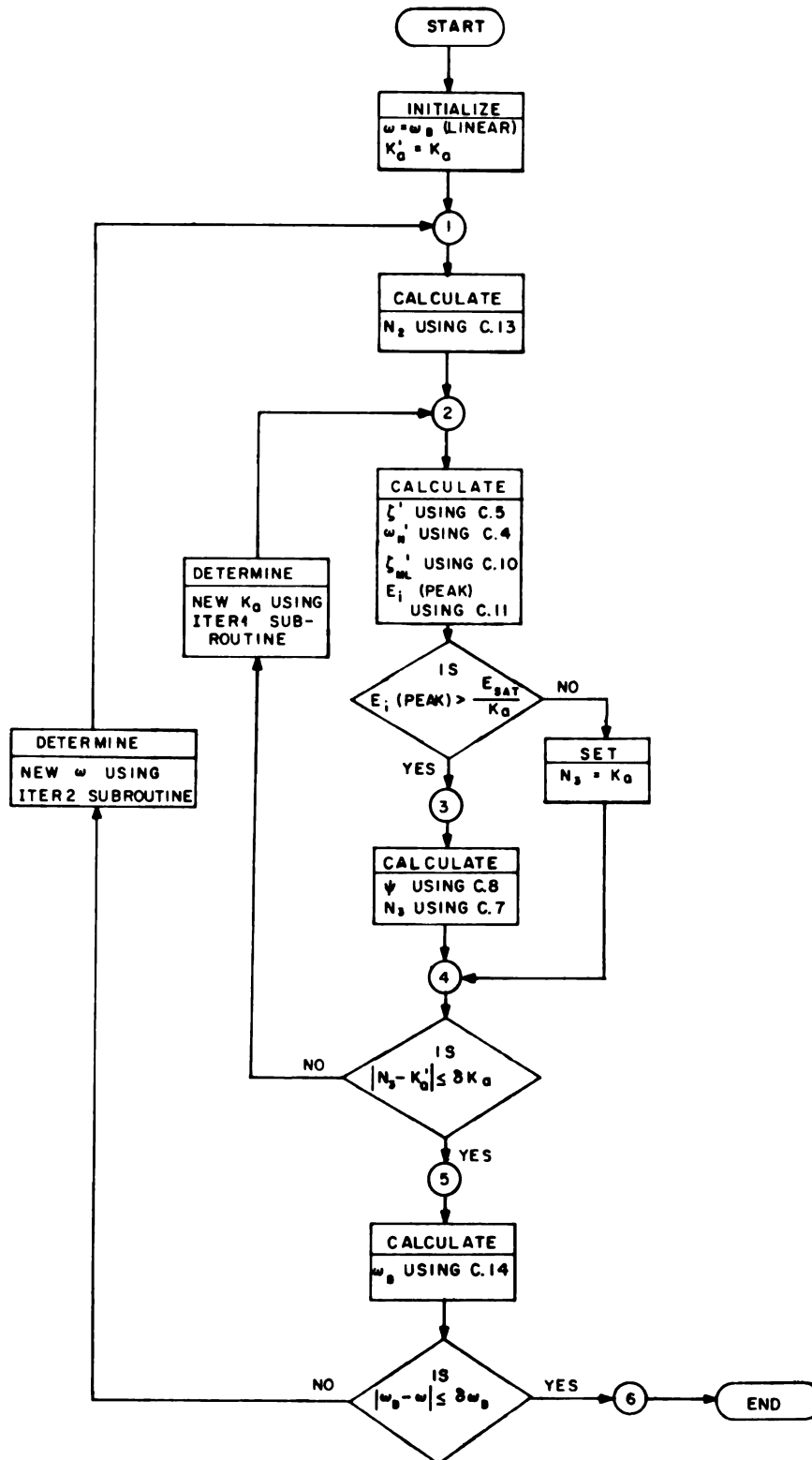
Figure C.1. Nonlinear bandwidth flow diagram.

Table C.1. Nonlinear bandwidth calculations with
friction and amplifier saturation.

```
            AUTOMATED DESIGN RESEARCH PROBLEM
            NONLINEAR BANDWIDTH CALCULATIONS
                   JANUARY 15, 1969

                    INPUT PARAMETERS
         K-F         K-A        K-M         K-G
      (VOLT/RAD)    (V/V)    (OZ-IN/V)   (OZ-IN-SEC)
      1.460E+01   1.420E+02  1.350E+00   5.000E+01


  B-M          J-T         T-L       E-SAT        DELTA
OZ-IN-SEC)(OZ-IN-SEC2)  (OZ-IN)    (VOLTS)
 3.000E+00   1.500E-01  1.000E+00  2.050E+01   1.000E-03

            CALCULATED LINEAR PARAMETERS
         W-N         ZETA        W-B       THETA-SAT
      (RAD/SEC)                (RAD/SEC)     (DEG)
      1.366E+02   1.293E+00  6.174E+01   3.731E+00

NONLINEAR BANDWIDTH VS THETA-IN CALCULATIONS

       THETA       W-B         N-3         N-2
       (DEG)     (RAD/SEC)    (V/V)    (OZ-IN-SEC)
      5.000E-01  5.665E+01  1.420E+02   3.638E+00
      1.000E+00  5.919E+01  1.420E+02   1.742E+00
      1.500E+00  6.004E+01  1.420E+02   1.145E+00
      2.000E+00  6.046E+01  1.420E+02   8.528E-01
      2.500E+00  6.072E+01  1.420E+02   6.793E-01
      3.000E+00  6.089E+01  1.420E+02   5.645E-01
      3.500E+00  6.101E+01  1.420E+02   4.829E-01
      4.000E+00  6.129E+01  1.360E+02   4.207E-01
      4.500E+00  6.278E+01  3.928E+01   3.645E-01
      5.000E+00  5.955E+01  2.899E+01   3.464E-01
      5.500E+00  5.665E+01  2.382E+01   3.311E-01
      6.000E+00  5.411E+01  2.047E+01   3.178E-01
      6.500E+00  5.185E+01  1.805E+01   3.061E-01
      7.000E+00  4.982E+01  1.620E+01   2.957E-01
      7.500E+00  4.803E+01  1.474E+01   2.865E-01
      8.000E+00  4.635E+01  1.351E+01   2.781E-01
      8.500E+00  4.482E+01  1.249E+01   2.706E-01
      9.000E+00  4.349E+01  1.166E+01   2.646E-01
      9.500E+00  4.220E+01  1.091E+01   2.581E-01
      1.000E+01  4.101E+01  1.026E+01   2.522E-01
      1.050E+01  3.991E+01  9.684E+00   2.467E-01
      1.100E+01  3.887E+01  9.175E+00   2.416E-01
      1.150E+01  3.791E+01  8.720E+00   2.369E-01
      1.200E+01  3.700E+01  8.312E+00   2.325E-01
      1.250E+01  3.615E+01  7.941E+00   2.283E-01
      1.300E+01  3.532E+01  7.594E+00   2.240E-01
      1.350E+01  3.457E+01  7.292E+00   2.206E-01
      1.400E+01  3.386E+01  7.013E+00   2.173E-01
      1.450E+01  3.318E+01  6.756E+00   2.142E-01
      1.500E+01  3.253E+01  6.518E+00   2.112E-01
      1.550E+01  3.192E+01  6.298E+00   2.084E-01
      1.600E+01  3.138E+01  6.110E+00   2.057E-01
      1.650E+01  3.080E+01  5.912E+00   2.031E-01
      1.700E+01  3.024E+01  5.727E+00   2.007E-01
      1.750E+01  2.971E+01  5.554E+00   1.984E-01
      1.800E+01  2.920E+01  5.393E+00   1.961E-01
```

Table C.2. Nonlinear bandwidth calculations
for zero friction case.

```
           AUTOMATED DESIGN RESEARCH PROBLEM
           NONLINEAR BANDWIDTH CALCULATIONS
                  JANUARY 15, 1969

                   INPUT PARAMETERS
         K-F          K-A          K-M          K-G
      (VOLT/RAD)      (V/V)     (OZ-IN/V)    (OZ-IN-SEC)
      1.460E+01    1.420E+02    1.350E+00    5.000E+01


    B-M          J-T          T-L          F-SAT        DELTA
 OZ-IN-SEC)(OZ-IN-SEC2)  (OZ-IN)        (VOLTS)
   3.000E+00    1.500E-01   0.0            2.C50E+01   1.000E-03


           CALCULATED LINEAR PARAMETERS
         W-N          ZETA         W-B        THETA-SAT
      (RAD/SEC)                  (RAD/SEC)      (DEG)
      1.366E+02    1.293E+0C    6.174E+01    3.731E+00


 NONLINEAR BANDWIDTH VS THETA-IN CALCULATIONS


        THETA         W-B          N-3          N-2
        (DEG)      (RAD/SEC)      (V/V)     (OZ-IN-SEC)
      5.000E-01    6.174E+01    1.420E+02    0.0
      1.000E+00    6.174E+01    1.420E+C2    0.0
      1.500E+00    6.174E+01    1.420E+C2    0.0
      2.000E+00    6.174E+01    1.420E+C2    0.0
      2.500E+00    6.174E+01    1.420E+C2    0.0
      3.000E+00    6.174E+01    1.420E+C2    0.0
      3.500E+00    6.174E+01    1.420E+C2    0.0
      4.000E+00    6.201E+01    1.347E+C2    0.0
      4.500E+00    6.326E+01    3.449E+C1    0.0
      5.000E+00    5.996E+01    2.663E+01    0.0
      5.500E+00    5.706E+01    2.225E+C1    0.0
      6.000E+00    5.450E+01    1.929E+C1    0.0
      6.500E+00    5.222E+01    1.710E+01    0.0
      7.000E+00    5.019E+01    1.539E+01    0.0
      7.500E+00    4.842E+01    1.407E+C1    0.0
      8.000E+00    4.675E+01    1.294E+C1    0.0
      8.500E+00    4.524E+01    1.199E+01    0.0
      9.000E+00    4.384E+C1    1.118E+C1    0.0
      9.500E+00    4.256E+C1    1.048E+C1    0.0
      1.000E+01    4.137E+C1    9.865E+00    0.0
      1.050E+01    4.027E+01    9.323E+C0    0.0
      1.100E+01    3.924E+01    8.843E+00    0.0
      1.150E+01    3.827E+C1    8.408E+C0    0.0
      1.200E+01    3.737E+C1    8.023E+C0    0.0
      1.250E+01    3.652E+C1    7.672E+00    0.0
      1.300E+01    3.572E+C1    7.353E+00    0.0
      1.350E+01    3.500E+01    7.076E+00    0.0
      1.400E+01    3.426E+01    6.800E+C0    0.0
      1.450E+01    3.356E+01    6.546E+00    0.0
      1.500E+01    3.290E+C1    6.312E+00    0.0
      1.550E+01    3.232E+C1    6.115E+00    0.0
      1.600E+01    3.172E+C1    5.914E+00    0.0
      1.650E+01    3.110E+01    5.712E+00    0.0
      1.700E+01    3.060E+01    5.551E+00    0.0
      1.750E+01    3.011E+C1    5.399E+00    0.0
      1.800E+01    2.956E+01    5.233E+00    0.0
```

bandwidth as a function of peak magnitude of the input sine wave. There is little difference between the two tabulations thereby demonstrating that amplifier saturation is the dominate nonlinearity except for small input levels. At small input levels (less than about 4 degrees) the zero friction case is equivalent to the linear case (no saturation) and the bandwidth equals the linear value of 61.74 rad/sec. For the case with friction the bandwidth falls off for both high and low values of input amplitude, as would be expected. The results obtained are plotted as Figure C.2.

An analog computer was used to check the validity of the describing function approximations used to represent the two nonlinearities. Figure C.3 documents the simulation used. The analog computer was operated by setting potentiometers 5 and 6 to the corresponding values of ω_B and θ_i(peak) as tabulated by the digital program and making Lissajous diagrams in predrawn boxes with the height (output) equal to 0.707 times the base (input). As demonstrated by Figure C.4 the analog computer responses are almost exactly tangent to all sides, thereby demonstrating that the describing function assumption provides an effective model of the system.
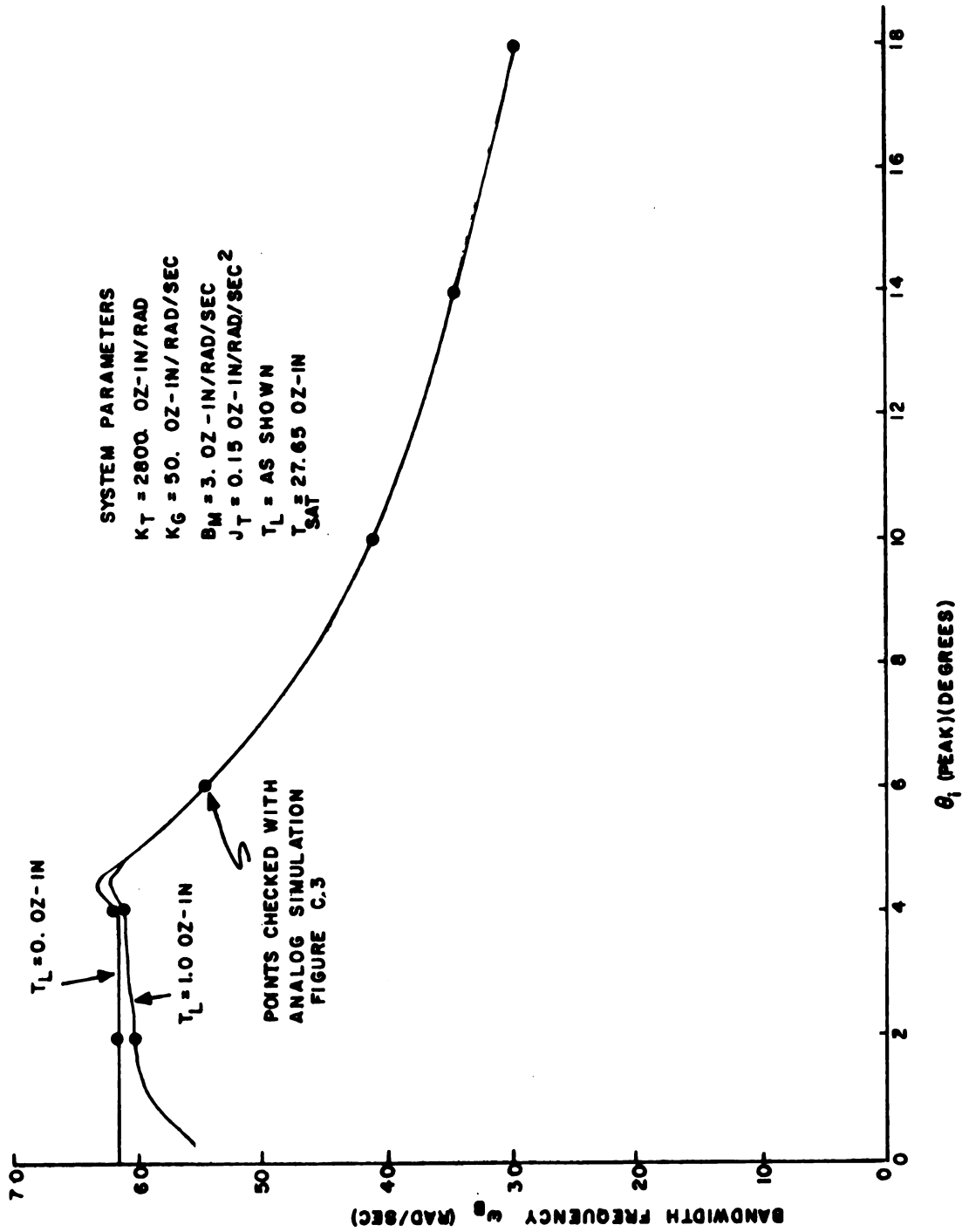
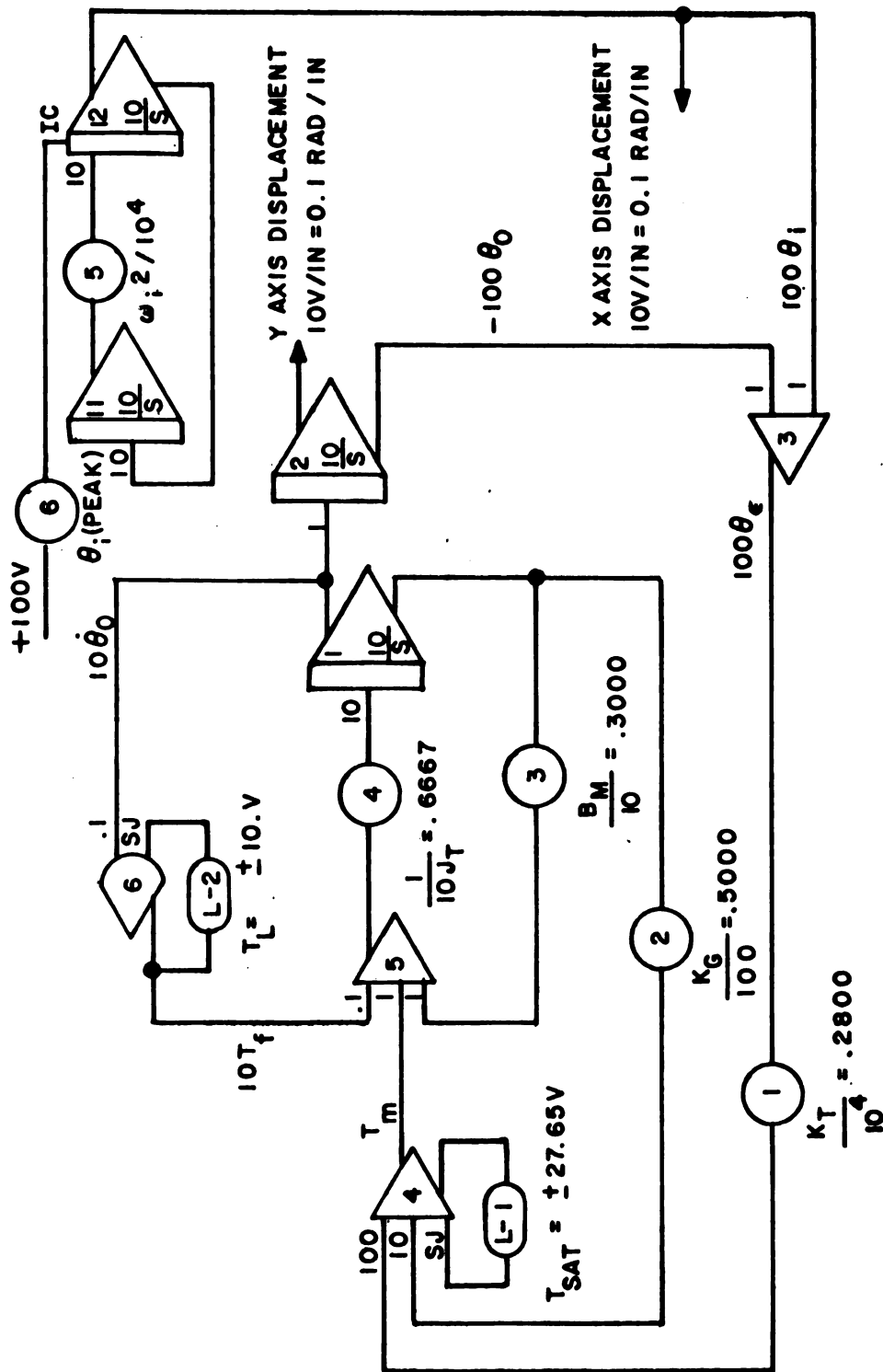Figure C.2. Effective bandwidth as a function of input zero-to-peak amplitude.

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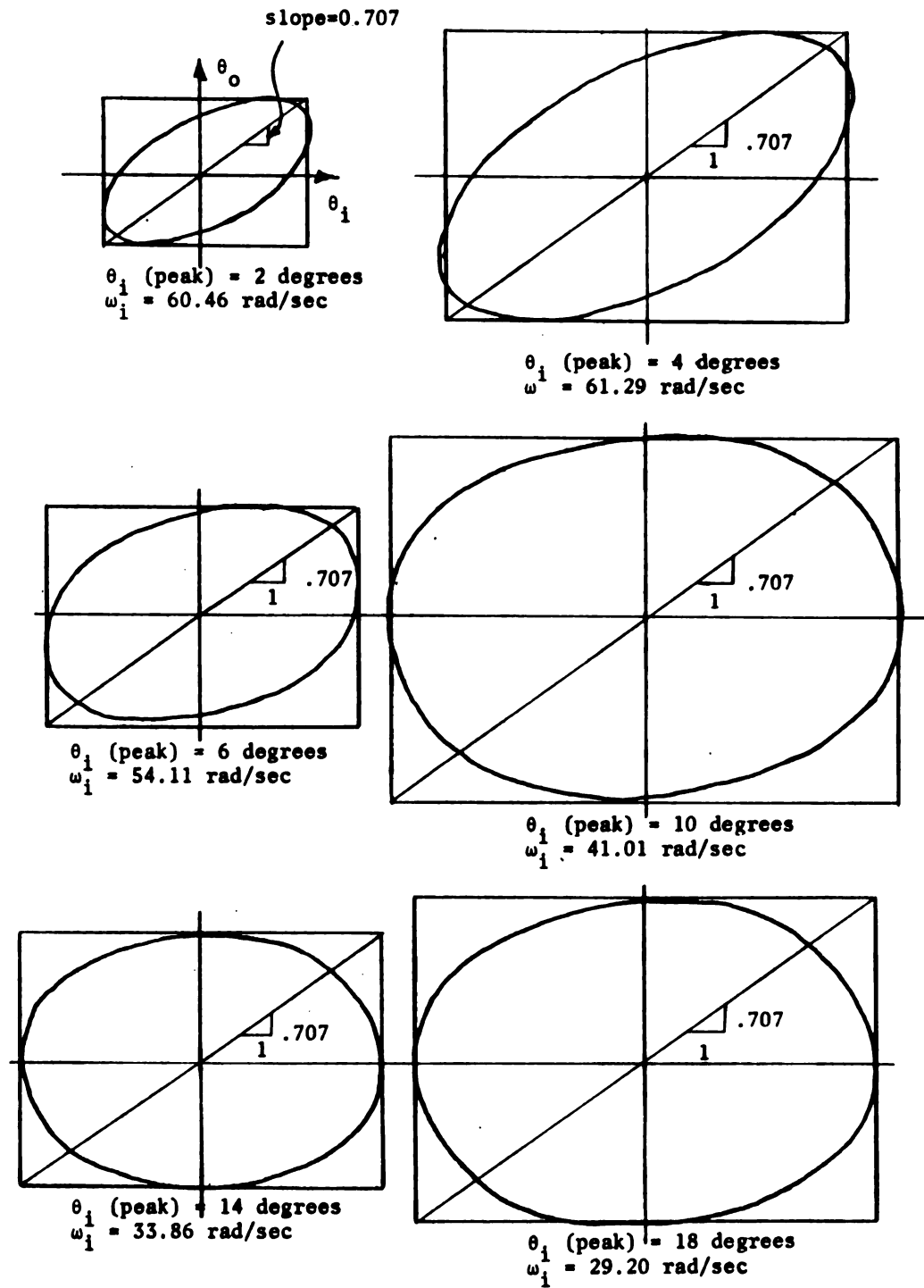Figure C.3. Analog computer diagram used for bandwidth analysis.

Figure C.4. Lissajous diagrams obtained from analog simulation.

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