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THE DEFORMATION AND STABILITY
OF A PRESSURIZED CIRCULAR TUBE
AND SPHERICAL SHELL
IN FINITE ELASTICITY AND FINITE PLASTICITY

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ABSTRACT

THE DEFORMATION AND STABILITY OF A PRESSURIZED CIRCULAR TUBE AND SPHERICAL SHELL IN FINITE ELASTICITY AND FINITE PLASTICITY

by

Dong-Teak Chung

This dissertation consists of two parts, both concerned with the investigation of problems for a cylinder and/or a sphere under conditions of finite strain. The problem considered in the first part, is carried out within the theory of finite elasticity while the problem in the second part is set within finite plasticity theory.

In part I, finite elastic deformation of hollow circular cylinders and spheres under applied uniform internal pressure is studied. Conditions for the initiation of a localized shear bifurcation are obtained. The location of this bifurcation relative to the pressure maximum is investigated. It is shown that when the ratio of the outer undeformed radius to the inner undeformed radius is larger than a critical value, the shear bifurcation occurs _before_ the pressure maximum is attained, while when this ratio is smaller than this critical value, the converse is true. The analysis is carried out for a particular _compressible_ elastic foam-rubber material (the Blatz-Ko material). The results are obtained in closed analytic form.

In part II, we carried out an explicit analysis of a bifurcation problem for a solid sphere composed of an elastic-plastic material. This problem is concerned with the bifurcation of a solid sphere under symmetric tensile load into a configuration involving an internal cavity. The analysis is carried out within the context of plasticity

theory using both finite strain J_2-flow theory and finite strain J_2-deformation theory. It is also shown that the classical infinitesimal theory of plasticity does not predict such a pheonomenon. This model may be used to describe the nucleation of a void from a pre-existing micro-void.

To my parents,
for the example and inspiration
they have provided throughout my life.

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PART I

THE FINITE DEFORMATION OF INTERNALLY PRESSURIZED HOLLOW CYLINDERS AND SPHERES FOR A CLASS OF COMPRESSIBLE ELASTIC MATERIALS

CHAPTER 1

INTRODUCTION

Recently Abeyaratne and Horgan [1] obtained an exact solution to a problem describing finite plane strain deformation of an infinite medium, composed of a certain compressible nonlinearly elastic material, the so-called Blatz-Ko material. The problem considered in [1] concerned an infinite medium with a circular cylindrical cavity under pressure loading conditions. In part I of this thesis we show that the solution technique of [1] may be applied also to the case of pressurized hollow cylinders and spheres with finite radii, and we carry out a detailed investigation of the solution to these basic problems of nonlinear elasticity.

The material considered in this study is a particular homogeneous, isotropic, compressible elastic material, namely the Blatz-Ko material. The pressurized cylinder and sphere problems in finite elasticity for incompressible materials have been considered previously by many authors (see e.g. [2,3]) and are simpler, since the incompressibility constraint immediately yields an explicit expression for the (axially symmetric) deformation field. Such simplification does not occur for compressible materials.*

The "Blatz-Ko material" is a mathematical model characterizing the constitutive behavior of a certain foam rubber-like material and was

*The pressurized sphere and cylinder problems for a class of (hypothetical) compressible materials, namely harmonic materials, have been investigated recently [4,5]. See also [3], Chapter 5, and [6].

proposed by Blatz and Ko [7] on the basis of experiments carried out by them. An extensive discussion of its properties may be found in [7] and also in the paper of Knowles and Sternberg [8]. One interesting feature of the Blatz-Ko material is that the system of partial differential (displacement) equations governing the equilibrium of a body composed of such a material may cease to be elliptic at sufficiently severe strain levels [8]. In the present work, we are interested in examining the implications of this for the pressurized cylinder and sphere problems.

It should be noted that bifurcation instabilities for pressurized cylinders and spheres for _incompressible_ elastic and elastic-plastic material response have been studied by many authors (see e.g. [3]) but we shall not be concerned with the analogs of these studies for the compressible material of concern here.

In the next Chapter, the problem of a hollow circular cylinder composed of the Blatz-Ko material subject to internal pressure is formulated. In Chapter 3 the solution to the resulting boundary value problem is obtained and features of the solution are discussed.

In Chpater 4, some illustrative examples are presented. For the case of a thin shell, an explicit relation between the deformed radius and the applied pressure is obtained and plotted in Figure 1. In particular, it is found that as the applied pressure p is increased from zero, the deformed radius increases until p reaches a maximum value. Subsequently, p decreases even though the deformed radius still increases. Such non-monotone pressure versus radius relationships are well-known in finite elasticity, particularly for _incompressible_ materials.* Numerical results pertaining to hollow cylinders of arbitrary thickness are also obtained, with corresponding pressure versus deformed radius curves shown in Figure 2. These curves exhibit a single pressure maximum. It is verified analytically in Appendix A that this occurs always for the Blatz-Ko material. It is observed that as the thickness of the cylinder increases, the maximum pressure and the corresponding deformed inner radius increases as might be expected. This is verified analytically in Appendix B. When the applied pressure is small, our results coincide with those of infinitesimal elasticity theory. In Chapter 4, we also examine the loss of ellipticity of the governing displacement equations of equilibrium at the deformation at hand. The value of the applied pressure (and the corresponding deformed inner radius) at which the cylinder first loses ellipticity is obtained. It is shown that when the ratio of the outer undeformed radius to the inner underformed radius is <u>larger</u> than a critical value, loss of ellipticity occurs <u>before</u> the pressure maximum is attained while when this ratio is smaller than this critical value, the converse is true.

In Chapter 5, the corresponding three-dimensional problem for a pressurized hollow sphere is treated. The overall behavior of the solution is similar to the two-dimensional case.

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*An extensive study of this phenomenon for incompressible materials has been carried out recently by Carroll [9]. It is of interest to note that this behavior does <u>not</u> occur in the <u>cylindrical</u> inflation of Mooney-Rivlin or neo-Hookean (incompressible) materials whereas it does for <u>spherical</u> inflation of such materials [9].

CHAPTER 2

THE PRESSURIZED HOLLOW CYLINDER; FORMULATION OF BOUNDARY-VALUE PROBLEM

Let the open region $D_0 = \{(r,\theta) | a < r < b, 0 < \theta \leq 2\pi\}$ denote the cross-section of a right circular cylinder with inner radius a, and outer radius b, in its undeformed configuration. The cylinder is subjected to an internal pressure of magnitude p. The resulting deformation is a one-to-one mapping which takes the point with polar coordinates (r,θ) in the undeformed region D_0 to the point (R,Θ) in the deformed region D. We assume that the deformation is an axisymmetric plane strain one so that

$$R = R(r) > 0, \ \Theta = \theta \text{ on } D_0, \tag{2.1}$$

where the positive function R(r) is to be determined. Unless explicitly stated otherwise R(r) is assumed to be twice continuously differentiable on $a < r < b$.

The polar components of the deformation gradient tensor $\underset{\sim}{F}$ associated with (2.1) are given by

$$F_{rr} = \dot{R}(r), \ F_{\theta\theta} = R(r)/r, \ F_{r\theta} = F_{\theta r} = 0, \tag{2.2}$$

where the dot denotes differentiation with respect to the argument. The Jacobian determinant $J = \det \underset{\sim}{F}$ is required to be positive and hence one has

$$\dot{R}(r) > 0 \quad \text{for } a < r < b. \tag{2.3}$$

The principal stretches associated with the radial deformation (2.1) are

$$\lambda_r = \dot{R}(r), \quad \lambda_\theta = R(r)/r. \tag{2.4}$$

The right Cauchy-Green deformation tensor is defined as $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$ and its fundamental scalar invariants can be taken as

$$I = \operatorname{tr}\underset{\sim}{C},\quad J = (\det \underset{\sim}{C})^{1/2} \tag{2.5}$$

so that in the present problem

$$I = \dot{R}^2 + (R/r)^2,\quad J = R\dot{R}/r. \tag{2.6}$$

Next we turn to the constitutive relation and suppose that the cylinder is composed of a Blatz-Ko material* [7]. This compressible, isotropic, elastic material is characterized, in plane strain, by the elastic potential

$$W(I,J) = \mu/2(I/J^2 + 2J - 4),\ \mu > 0, \tag{2.7}$$

representing the strain energy per unit undeformed volume. Here μ denotes the shear modulus of the material at infinitesimal deformations. The true stress tensor $\underset{\sim}{\tau}$ associated with a plane deformation is given by

$$\underset{\sim}{\tau} = (2J^{-1}\partial W/\partial I)\ \underset{\sim}{F}\underset{\sim}{F}^T + (\partial W/\partial J)\underset{\sim}{1}. \tag{2.8}$$

On substituting from (2.7) and (2.6) into (2.8) one finds that

$$\tau_{RR}(r) = \mu\left(1 - \frac{r}{R(r)\dot{R}^3(r)}\right), \tag{2.9a}$$

$$\tau_{\Theta\Theta}(r) = \mu\left(1 - \frac{r^3}{R^3(r)\dot{R}(r)}\right), \tag{2.9b}$$

$$\tau_{R\Theta} = \tau_{\Theta R} = 0,\quad a < r < b. \tag{2.9c}$$

In the absence of body forces, the equilibrium equations div $\underset{\sim}{\tau} = \underset{\sim}{0}$ in the present case reduce to the single equation

$$\frac{d}{dr}\tau_{RR} + \frac{\dot{R}}{R}(\tau_{RR} - \tau_{\Theta\Theta}) = 0,\ \text{for } a < r < b. \tag{2.10}$$

*An extensive discussion of the stress response of this material to various states of deformation may be found in [8].

This, together with (2.9), yields the following nonlinear second-order ordinary differential equation for $R(r)$:

$$3rR^3\ddot{R} - R^3\dot{R} + r^3\dot{R}^4 = 0 \quad \text{for } a < r < b. \tag{2.11}$$

The prescribed boundary conditions are

$$\tau_{RR} = -p \ \text{ at } r = a, \ \tau_{RR} = 0 \text{ at } r = b, \tag{2.12}$$

which, on using (2.9), can be written as

$$R(a)\dot{R}^3(a) = a(1 + p/\mu)^{-1}, \tag{2.13a}$$

$$R(b)\dot{R}^3(b) = b. \tag{2.13b}$$

In the next Chapter we derive an exact solution to the boundary-value problem (2.11), (2.13).

CHAPTER 3

SOLUTION OF BOUNDARY-VALUE PROBLEM

3.1 Solution

It has been shown recently in [1] that the second-order nonlinear ordinary differential equation (2.11) may be reduced to a first-order equation on making the substitution

$$t(r) = \frac{r\dot{R}(r)}{R(r)} \; (= \lambda_r/\lambda_\theta) > 0. \tag{3.1}$$

Equation (2.11) then yields

$$3r\dot{t} - t(1-t)(t^2+t+4) = 0 \quad \text{for } a < r < b, \tag{3.2}$$

where $\dot{t} = dt/dr$. It can be shown that there is no loss of generality in assuming that $t(r)$ is less than unity* for $a < r < b$. Thus it then follows from (3.2) that t increases monotonically with increasing r and so we have

$$0 < t < 1, \; dt/dr > 0 \text{ for } a < r < b. \tag{3.3}$$

Upon integrating (3.2), one finds that

$$r^8 = \frac{C\, t^6\, h(t)}{(1-t)^4 (t^2+t+4)}, \tag{3.4}$$

where $C > 0$ is a constant of integration and we have set

$$h(t) = \exp\{\frac{6}{\sqrt{15}} \tan^{-1}(\frac{2t+1}{\sqrt{15}})\} > 0. \tag{3.5}$$

On the other hand (3.1) and (3.2) also give

*The arguments given in [1] to justify this assumption carry over, with obvious modification, to the present problem. Notice that since the applied load is that of internal pressure only, one would expect $\lambda_r < 1$, $\lambda_\theta > 1$ and so one anticipates that t is indeed less than unity.

$$\frac{1}{R}\frac{dR}{dt} = \frac{3}{(1-t)(t^2+t+4)} > 0, \tag{3.6}$$

which yields

$$R^4 = \frac{D(t^2+t+4)\ h(t)}{(1-t)^2} \,. \tag{3.7}$$

Again, $D > 0$ is a constant of integration. Observe from (3.3), (3.6) that the undeformed and deformed radial coordinates (r,R) vary monotonically with t. Equations (3.4), (3.5), (3.7) provide a parametric solution to the differential equation (2.11). The range of the parameter t is

$$t_a < t < t_b, \tag{3.8}$$

where $t_a > 0$ is the value of t corresponding to $r = a$ and is to be determined from (3.4) and $t_b < 1$ is determined in an analogous fashion. The components of true stress τ_{RR}, $\tau_{\Theta\Theta}$ may also be expressed in terms of t on using (2.9), (3.1), (3.4) and (3.5). This leads to

$$\tau_{RR} = \mu[1 - \frac{(C/D^2)^{1/2}}{h^{1/2}(t)(t^2+t+4)^{3/2}}], \tag{3.9}$$

$$\tau_{\Theta\Theta} = \mu[1 - \frac{t^2\ (C/D^2)^{1/2}}{h^{1/2}(t)(t^2+t+4)^{3/2}}],\ t_a < t < t_b. \tag{3.10}$$

From the definition of t_a, t_b, it follows from (3.4) that

$$a^8 = \frac{Ct_a{}^6\ h(t_a)}{(t_a^2+t_a+4)(1-t_a)^4} \,, \tag{3.11}$$

$$b^8 = \frac{Ct_b{}^6\ h(t_b)}{(t_b^2+t_b+4)(1-t_b)^4} \,. \tag{3.12}$$

Finally the boundary conditions (2.12), in view of (3.9), may be written as

$$-p = \mu[1 - \frac{(C/D^2)^{1/2}}{h^{1/2}(t_a)\ (t_a^2+t_a+4)^{3/2}}], \tag{3.13}$$

$$0 = \mu[1 - \frac{(C/D^2)^{1/2}}{h^{1/2}(t_b)(t_b^2+t_b+4)^{3/2}}]. \tag{3.14}$$

Equations (3.11) -(3.14) consist of four equations for the four unknowns t_a, t_b, C and D. In the following section we discuss the existence of solutions to these equations.

3.2 Discussion

We eliminate the integration constant C between equations (3.11) and (3.12) and obtain

$$b^8 g(t_b) = a^8 g(t_a), \tag{3.15}$$

where g(t) is given by

$$g(t) = \frac{(t^2+t+4)(1-t)^4}{t^6\ h(t)}, \quad 0 < t < 1. \tag{3.16}$$

Also from (3.13) and (3.14), eliminating the constant C/D^2, one has

$$(1+\frac{p}{\mu})^2 = \frac{h(t_b)(t_b^2+t_b+4)^3}{h(t_a)(t_a^2+t_a+4)^3}. \tag{3.17}$$

Thus we now have two equations (3.15), (3.17) for the two unknowns t_a and t_b. The function g(t) in equation (3.16) tends to infinity as $t \to 0+$, decreases monotonically as t increases and has the value zero when $t = 1$. Thus for a given ratio of outer undeformed radius b to inner undeformed radius a, one can always express t_a in terms of t_b and vice versa. We may write

$$t_b = f(t_a, \alpha) \tag{3.18}$$

where $\alpha = b/a$ and f is an implicit function. Thus, for a given applied pressure p, if (3.17) with t_b expressed as in (3.18), can be solved for a number t_a such that $0 < t_a < 1$, then (3.18) provides a number t_b and (3.11)-(3.14) is the desired solution.

In order to verify that (3.17) can indeed be solved for an appropriate value of t_a, we consider the auxiliary function $Q(t_a)$ defined by

$$Q(t_a) \equiv \frac{h(f(t_a))\{f^2(t_a)+f(t_a)+4\}^3}{h(t_a)(t_a^2+t_a+4)^3}, \quad 0 < t_a < 1, \tag{3.19}$$

which appears on the right hand side of (3.17), when (3.18) is taken into account. For convenience here, we have written $f(t_a) \equiv f(t_a,\alpha)$. One can readily show that

$$\begin{aligned}
&\lim_{t_a \to 0+} Q(t_a) = 1, \\
&\lim_{t_a \to 0+} \frac{dQ(t_a)}{dt_a} > 0, \\
&\lim_{t_a \to 1-} Q(t_a) = 1, \\
&\lim_{t_a \to 1-} \frac{dQ(t_a)}{dt_a} < 0.
\end{aligned} \tag{3.20}$$

It follows that for $0 < t_a < 1$, there exists a maximum value for Q, with the corresponding value of p given by (3.17) as $p = p_m$. Thus, if $0 < p < p_m$, there exist at least two solutions for t_a. It is shown in Appendix A that Q has only one local maximum and so exactly two solutions for t_a exist.

Before we proceed to further examination of the foregoing solution, we return to discuss the possible alternative range of values of $t(r)$. Recall that in the preceding discussion we restricted attention to the case when $0 < t(r) < 1$ on $a < r < b$. Note that $t(r) \equiv 1$ is a solution of (3.2). However (3.1) then yields $R(r)=cr$ (c constant) which does not satisfy the boundary conditions (2.13) unless $p = 0$. On the other hand, if $t(r) = 1$ at some point in $a < r < b$, then (3.2) and a continuity argument shows that $t(r)$ must be unity on the entire range $a < r < b$. Consequently, the only case remaining to be considered is that for which $t(r) > 1$ on $a < r < b$. The preceding analysis up to equation (3.19) continues to be valid (with $1 < t_b \leq t \leq t_a$ and obvious minor modifications) and the existence of solution again hinges on the solvability of (3.19) for a root t_a, now > 1. It is easily seen that no such number exists (when $p > 0$) and thus the case when $t(r) > 1$ need no longer be considered.

Finally, we observe that the deformed inner radius $R(a)$, given by (3.7) with $t = t_a$, may be written as

$$\frac{R(a)}{a} = t_a^{-3/4}(1+p/\mu)^{-1/4}, \tag{3.21}$$

on using (3.11) and (3.13).

CHAPTER 4

RESULTS AND DISCUSSION

We now examine some features of the results derived in the previous section and consider some illustrative examples.

4.1 Thin Shell:

When the radius ratio $\alpha = b/a$ is very near to unity, it is not difficult to express the relation (3.18) between t_a and t_b explicitly. Let

$$\alpha = 1 + \varepsilon, \quad \varepsilon = (b-a)/a \quad (<<1), \tag{4.1}$$

and assume

$$t_b = t_a + \varepsilon\, A(t_a), \tag{4.2}$$

where A(t) is an, as yet, unknown function. Substituting (4.1), (4.2) into equation (3.15) and neglecting second-order terms in ε yields $A(t) = 1/3(t^2+t+4)(1-t)t$ and so (4.2) then reads

$$t_b = t_a + (\varepsilon/3)\,(t_a{}^2+t_a+4)(1-t_a)t_a, \quad 0 < t_a < 1. \tag{4.3}$$

Upon substituting from (4.3) into equation (3.17), one obtains an explicit relation between t_a and the prescribed pressure, which is given, to leading order, by

$$p/\mu = \varepsilon\,(t_a-t_a{}^3), \quad \text{for} \quad 0 < t_a < 1. \tag{4.4}$$

It is clear from (4.4) that p has only one maximum. The deformed cavity radius R(a), is given by (3.21). In the present case, where p is given by (4.4), equation (3.21) yields, to leading order,

$$\frac{R(a)}{a} = t_a^{-3/4}, \tag{4.5}$$

where the value of t_a corresponding to a prescribed pressure p is found from (4.4). A graph of the ratio of the deformed radius to undeformed radius versus pressure, according to (4.4), (4.5), is shown in Figure 1. As the pressure p is increased from zero, the deformed radius increases until p reaches a maximum value of $0.385\mu\varepsilon$, where $R(a) = 1.510a$. Subsequently p decreases even though the radius still increases. This phenomenon is well-known in finite elasticity, especially for <u>incompressible</u> materials (see e.g. Ogden [3] pp. 283-287 for the spherical thin shell and the comprehensive recent study by Carroll [9].)

4.2 <u>Thick Cylinder</u>:

For a thick-walled cylinder, the relation between t_a and t_b is implicit and thus computational work is necessary, in general to analyze the behavior of pressure versus radius. It can however be shown analytically that p has only one maximum point and this analysis is carried out in Appendix A. A graph of the ratio of the deformed radius to undeformed radius versus pressure, obtained from solving (3.15), (3.17) numerically and using (3.21), is shown in Figure 2, for different values of the radius ratio $\alpha = b/a$. As is shown in Figure 2, as the thickness of the cylinder increases, the value of the maximum pressure increases and it occurs at increasingly larger values of $R(a)/a$, as might be expected. This is verified analytically in Appendix B.

The bottom curve in Figure 2, corresponding to $\alpha = 1.05$, is seen to confirm the thin shell results discussed in section 4.1. On the other hand, as $\alpha \to \infty$ (i.e. a cavity in an infinite medium), the pressure maximum is reached asymptotically as the deformed radius tends to infinity, and has the value $p_m = 1.51677\mu$. Thus we recover the result of [1] for this case.

4.3 Linearization:

In the particular case when the applied pressure is small ($p/\mu \ll 1$), (3.19) and (3.21) show that $t_a \approx 1$, $t_b \approx 1$ and so $t(r) \approx 1$ throughout the body. Let

$$t_a = 1 - \delta_a,\ t_b = 1 - \delta_b,\ t(r) = 1 - \delta(r), \tag{4.6}$$

where the small parameters $\delta(r)$, δ_a, δ_b are unknown. Substituting (4.6) into (3.9), (3.10), (3.13), (3.14) yields, to leading order,

$$\tau_{\Theta\Theta} = \mu(\delta_b+\delta), \tag{4.7}$$

$$\tau_{RR} = \mu(\delta_b-\delta), \tag{4.8}$$

$$p/\mu = \delta_a - \delta_b. \tag{4.9}$$

Also from (3.4), (3.11), using (4.6), one obtains

$$\delta(r) = \left(\frac{a^2}{r^2}\right)\delta_a. \tag{4.10}$$

Finally the relation between δ_a and δ_b may be obtained upon substituting (4.6) into (3.15). This leads to

$$\delta_b = \delta_a\alpha^{-2},\ (\alpha=b/a). \tag{4.11}$$

The resulting linearized stress fields given by (4.7)-(4.11) yield the well known results of the infinitesimal theory of elasticity (see e.g. Timoshenko and Goodier [10], p. 71):

$$\tau_{\Theta\Theta} = \frac{pa^2}{b^2-a^2}\left(1 + \frac{b^2}{r^2}\right), \tag{4.12}$$

and

$$\tau_{RR} = \frac{pa^2}{b^2-a^2}\left(1 - \frac{b^2}{r^2}\right). \tag{4.13}$$

4.4 Loss of Ellipticity:

The pressure maxima we have encountered exhibit the unstable behavior of a pressurized tube of the particular compressible material under consideration. It is of interest to examine other possible instabilities for this problem. It is well-known that the displacement equations of equilibrium in finite elastostatics may lose ellipticity at sufficiently severe deformations. In particular, for plane strain deformations of the Blatz-Ko material, necessary and sufficient conditions (in terms of the principal stretches) for ellipticity have been obtained by Knowles and Sternberg [8]. In this section, we examine the implications of these results for the pressurized cylinder problem at hand.

From equation (4.8) of [8], it follows that ellipticity holds for the Blatz-Ko material (2.7) at the axisymmetric solution (2.1) if and only if the principal stretches λ_r, λ_θ introduced in (2.4) are such that

$$2 - \sqrt{3} < t < 2 + \sqrt{3}, \quad (t = \lambda_r/\lambda_\theta). \tag{4.14}$$

Since in the present problem we have $0 < t < 1$, it follows that the right hand inequality here always holds. On the other hand, it is clear that ellipticity will be lost whenever the left inequality is violated.

In view of the monotonic increasing character of t as r increases (see (3.3)), and recalling that $0 < t_a < t < t_b < 1$, it follows that ellipticity is first lost at $r = a$ and that this occurs when $t_a = 2-\sqrt{3}$. For a given value of radius ratio $\alpha = b/a$, the corresponding value of the applied pressure, say p_e, is found from

(3.17), where the value of t_b is given by (3.15) with $t_a = 2 - \sqrt{3}$. The corresponding value of $R(a)/a$ then follows from (3.21) with $p = p_e$ and $t_a = 2 - \sqrt{3}$. In Figure 2, the pairs of values $(p_e/\mu, R(a)/a)$, for different radius ratios α, are joined by the dotted curve. There is a critical value of the radius ratio $\alpha = \alpha_c$ say, at which $p_e = p_m$. This may be found numerically from (3.17), (3.18), (3.19) with $t_a = 2 - \sqrt{3}$, and we find

$$\alpha_c \simeq 5.1. \tag{4.15}$$

Thus, for $\alpha < \alpha_c$, the load maximum is reached <u>before</u> loss of ellipticity occurs while for $\alpha > \alpha_c$, the converse is true.

In either case, <u>after</u> loss of ellipticity, the existence of the smooth solutions obtained here is still ensured. In addition, the possibility exists that non-smooth deformation fields, with discontinuous deformation gradients and stresses, might occur. The argument provided in [1] shows that axisymmetric solutions with such discontinuities do <u>not</u> exist in the present problem. Weak solutions, if they exist, must necessarily be non-axisymmetric. There is also the possibility that surface bifurcations might occur, as in [1], but we shall not pursue this issue here.

CHAPTER 5

THE PRESSURIZED HOLLOW SPHERE

In this Section, we describe briefly how the foregoing considerations can be applied to the analogous problem of an internally pressurized hollow sphere.

5.1 Formulation of Problem:

We are concerned in what follows with the pressure loading of a sphere composed of the Blatz-Ko material. Let

$$D_0 = \{(r,\theta,\phi)\,|\,a < r < b,\ 0 < \theta < 2\pi,\ 0 < \phi < \pi\},$$

denote the hollow sphere in its undeformed configuration. The sphere is subjected to an internal pressure p.

The resulting deformation is a one-to-one mapping which takes the point with spherical polar coordinates (r, θ, ϕ) to the point (R, Θ, Φ) in the deformed region D. We assume that the deformation is an axisymmetric one so that

$$R = R(r) > 0,\ \Theta = \theta \text{ and } \Phi = \phi \text{ on } D_0, \tag{5.1}$$

where R(r) is to be determined.

The polar components of the deformation gradient tensor associated with (5.1) are given by

$$\underset{\sim}{F} = \operatorname{diag}\,(\dot{R}(r),\ R(r)/r,\ R(r)/r). \tag{5.2}$$

The principal stretches associated with the radial deformation (5.1) are

$$\lambda_r = \dot{R}(r), \quad \lambda_\theta = \lambda_\phi = R(r)/r. \tag{5.3}$$

The Blatz-Ko material is characterized in the three-dimensional case, by the elastic potential (see e.g. [7])

$$W(\lambda_1,\lambda_2,\lambda_3) = \mu/2(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5). \tag{5.4}$$

The principal components of true stress $\underset{\sim}{\tau}$ are given by

$$\begin{aligned}\tau_{ii} &= \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial\lambda_i}\ \text{(no sum on i)}\\ &= \mu(1-J^{-1}\lambda_i^{-2}),\ J = \lambda_1\lambda_2\lambda_3.\end{aligned} \tag{5.5}$$

Substitution from (5.3) and (5.4) into (5.5) yields

$$\tau_{RR}(r) = \mu\left(1 - \frac{r^2}{R^2\dot{R}^3}\right), \tag{5.6a}$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \mu\left(1 - \frac{r^4}{\dot{R}R^4}\right). \tag{5.6b}$$

In the absence of body force, the equilibrium equations div $\underset{\sim}{\tau} = \underset{\sim}{0}$ reduce to

$$\frac{d}{dr}\tau_{RR} + \frac{2\dot{R}}{R}(\tau_{RR} - \tau_{\Theta\Theta}) = 0,\ \text{for } a < r < b, \tag{5.7}$$

which, by virtue of (5.6), yields the nonlinear second-order ordinary differential equation

$$3r^2R^3\ddot{R} - 2rR^3\dot{R} + 2r^4\dot{R}^4 = 0,\ \text{for } a < r < b. \tag{5.8}$$

The prescribed (pressure) boundary conditions are

$$\tau_{RR} = -p \text{ at } r = a, \tag{5.9a}$$

$$\tau_{RR} = 0 \text{ at } r = b. \tag{5.9b}$$

5.2 Solution of Boundary-Value Problem:

Just as in the two-dimensional case, equation (5.8) may be reduced to a first-order equation on making the substitution

$$t(r) = \frac{r\dot{R}(r)}{R(r)} > 0. \tag{5.10}$$

Equation (5.8) then yields

$$3r\dot{t}(r) - t(1-t)(2t^2+2t+5) = 0 \text{ for } a < r < b, \tag{5.11}$$

where $\dot{t} = dt/dr$. Again we assume that $t(r)$ is less than unity for $a < r < b$ and so deduce from (5.11) that

$$0 < t < 1, \quad dt/dr > 0 \quad \text{for } a < r < b. \tag{5.12}$$

Integration of (5.11) yields

$$r^{15} = \frac{C\, t^9 d(t)}{(2t^2+2t+5)^2(1-t)^5}, \tag{5.13}$$

where C is a positive constant of integration and the function $d(t)$ is defined by

$$d(t) = \exp\{2\tan^{-1}(\frac{2t+1}{3})\} > 0. \tag{5.14}$$

Also (5.10), (5.11) yield

$$R^6 = D\,\frac{(2t^2+2t+5)d(t)}{(1-t)^2}, \tag{5.15}$$

where $D > 0$ is a constant of integration. The range of the parameter t is

$$(0 <)\ t_a < t < t_b\ (< 1). \tag{5.16}$$

The components of true stress τ_{RR}, $\tau_{\Theta\Theta}$ may also be expressed in terms of t on using (5.6), (5.10), (5.13) and (5.14). This leads to

$$\tau_{RR} = \mu\{1 - (\frac{C^2}{D^5})^{1/6}(2t^2+2t+5)^{-3/2}d^{-1/2}(t)\}, \tag{5.17}$$

$$\tau_{\Theta\Theta} = \mu\{1 - (\frac{C^2}{D^5})^{1/6}t^2(2t^2+2t+5)^{-3/2}d^{-1/2}(t)\}, \tag{5.18}$$

for $t_a < t < t_b$.

From the definition of t_a and t_b, it follows from (5.13) that

$$a^{15} = \frac{C\ \ t_a^9\ \ d(t_a)}{(2t_a^2+2t_a+5)^2(1-t_a)^5}, \tag{5.19}$$

$$b^{15} = \frac{C\ \ t_b^9\ \ d(t_b)}{(2t_b^2+2t_b+5)^2(1-t_b)^5}. \tag{5.20}$$

Finally the boundary conditions (5.9), in view of (5.17), may be written as

$$-p = \mu\{1-(\frac{C^2}{D^5})^{1/6}(2t_a^2+2t_a+5)^{-3/2}d^{-1/2}(t_a)\}, \tag{5.21}$$

$$0 = \mu\{1-(\frac{C^2}{D^5})^{1/6}(2t_b^2+2t_b+5)^{-3/2}d^{-1/2}(t_b)\}. \tag{5.22}$$

The four equations (5.19) - (5.22) for the four unknowns t_a, t_b, C, D are analogous to equations (3.11) - (3.14) obtained in the two dimensional case. It can be verified that the considerations of section 3.2 carry over, with obvious modification, to the three-dimensional equations of concern here. In particular, the analog of (3.21) in the present case is given by

$$\frac{R(a)}{a} = t_a^{-3/5}\left(1 + \frac{p}{\mu}\right)^{-1/5}. \tag{5.23}$$

In what follows, we consider some illustrative examples analogous to those discussed in Chapter 4 for the two-dimensional problem.

5.3 Thin Shell:

When the radius ratio $\alpha(=b/a)$ is very near to unity, the explicit relation between t_a and prescribed pressure, analogous to (4.4), is given by

$$\frac{p}{\mu} = 2\varepsilon(t_a - t_a^3) + O(\varepsilon^2), \quad \varepsilon = \frac{b-a}{a} << 1. \tag{5.24}$$

Similarly, the analog of (4.5), on using (5.23),(5.24) becomes

$$\frac{R(a)}{a} = t_a^{-3/5} + O(\varepsilon). \tag{5.25}$$

A graph of the ratio of the deformed radius to undeformed radius versus pressure, according to (5.24), (5.25), is shown in Figure 3. Again we observe that the pressure first increases as the shell inflates, reaches a maximum with value $0.7698\mu\varepsilon$ where $R(a) = 1.390a$ and decreases on further inflation.

5.4 Thick Shell:

For a thick shell, numerical computation yields the graph of pressure versus deformed radius shown in Figure 4. The behavior of the spherical shell is similar to that of the cylindrical shell shown in Figure 2.

5.5 Linearization:

For small pressure ($p/\mu << 1$), we find again that $t_a \simeq 1$, $t_b \simeq 1$ and so $t(r) \simeq 1$ throughout the body. Let

$$t_a = 1 - \delta_a, \; t_b = 1 - \delta_b, \; t(r) = 1 - \delta(r) \tag{5.26}$$

where the small parameters δ_a, δ_b and $\delta(r)$ are unknown. We find that

$$\tau_{\Theta\Theta} = 2\mu/3(\delta+2\delta_b), \tag{5.27a}$$

$$\tau_{RR} = 4\mu/3(\delta_b-\delta), \tag{5.27b}$$

$$p = 4\mu/3(\delta_a-\delta_b), \tag{5.27c}$$

$$\delta(r) = (a^3/r^3)\delta_a, \tag{5.27d}$$

$$\delta_b = \alpha^{-3}\,\delta_a,\ \alpha = b/a. \tag{5.27e}$$

The resulting linearized stress fields given by (5.27) again yield the well known results of the infinitesimal theory of elasticity (see e.g. Timoshenko and Goodier [10], p. 395),

$$\tau_{RR} = \frac{pa^3}{b^3-a^3}\left(1-\frac{b^3}{r^3}\right), \tag{5.28a}$$

$$\tau_{\Theta\Theta} = \frac{pa^3}{2(b^3-a^3)}\left(2+\frac{b^3}{r^3}\right). \tag{5.28b}$$

5.6 Loss of Ellipticity:

In the three-dimensional case also, for the Blatz-Ko material, necessary and sufficient conditions for ellipticity of the displacement equations of equilibrium have been obtained by Knowles and Sternberg [8]. Thus from equation (3.1) of [8] it follows that ellipticity holds for the Blatz-Ko material (5.4) at the axisymmetric solution (5.1) if and only if the principal stretches λ_r, λ_θ, λ_ϕ introduced in (5.3) are such that

$$2-\sqrt{3} < \lambda_r/\lambda_\theta < 2+\sqrt{3}, \tag{5.29a}$$

$$2-\sqrt{3} < \lambda_r/\lambda_\phi < 2+\sqrt{3}, \tag{5.29b}$$

$$2-\sqrt{3} < \lambda_\phi/\lambda_\theta < 2+\sqrt{3}. \tag{5.29c}$$

Since in this problem we have $\lambda_\theta = \lambda_\phi$, (5.29c) is automatically satisfied and (5.29a), (5.29b) are equivalent to

$$2 - \sqrt{3} < t < 2 + \sqrt{3}, \quad t = \lambda_r/\lambda_\theta = \lambda_r/\lambda_\phi. \tag{5.30}$$

The inequality on the right in (5.30) always holds (see (5.12)) and, by virtue of the monotone increasing character of t as r increases (see (5.12)), ellipticity is first lost at $r = a$ and this occurs when $t_a = 2-\sqrt{3}$. For a given value of the radius ratio $\alpha = b/a$, the corresponding value of the applied pressure, say p_e, is found in a similar manner to the two-dimensional case on using (5.19)-(5.22). The corresponding value of $R(a)/a$ then follows from (5.23) with $p = p_e$ and $t_a = 2-\sqrt{3}$. In Figure 4, the pairs of values $(p_e/\mu, R(a)/a)$ for different radius ratios α are joined by the dotted curve. As in the case of the two-dimensional problem, there is a critical value of the radius ratio $\alpha = \alpha_c$ at which $p_e = p_m$. This may be found numerically and we find that

$$\alpha_c \simeq 3.25.$$

Thus for $\alpha < \alpha_c$, the pressure maximum is reached <u>before</u> loss of ellipticity while for $\alpha > \alpha_c$, the converse is true. Finally, <u>after</u> loss of ellipticity, the non-existence of non-smooth axisymmetric solutions may be demonstrated by an obvious modification of the argument given in [1]. The possibility of surface bifurcations also exist, but we shall not pursue this here.

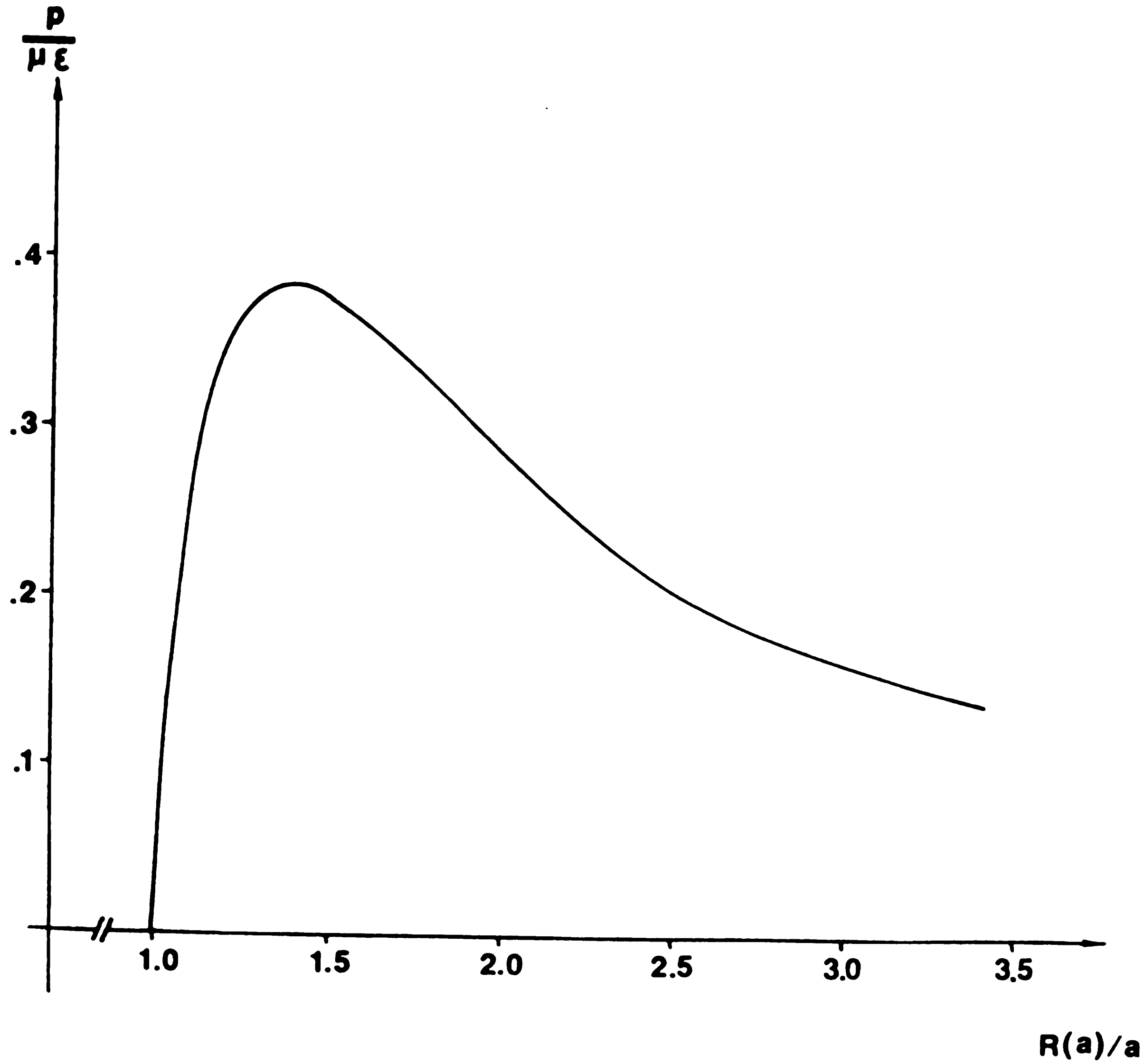

Figure 1. Applied pressure versus ratio of deformed to undeformed radius for a thin cylindrical shell.

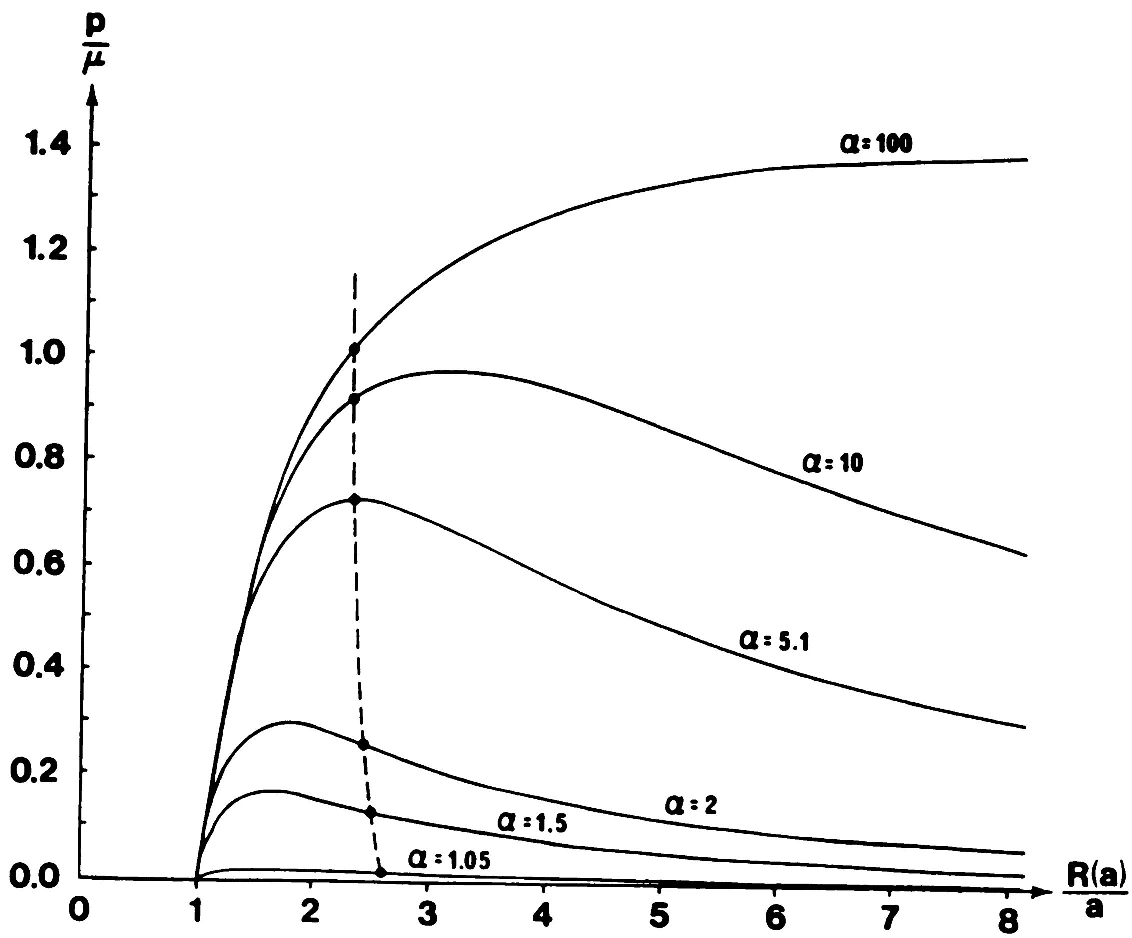

Figure 2. Applied pressure versus ratio of deformed to undeformed radius for hollow cylinders for different radius ratio $\alpha = b/a$. The dotted curve connects the points $(p_e/\mu, R(a)/a)$, where p_e denotes the pressure at which ellipticity is lost

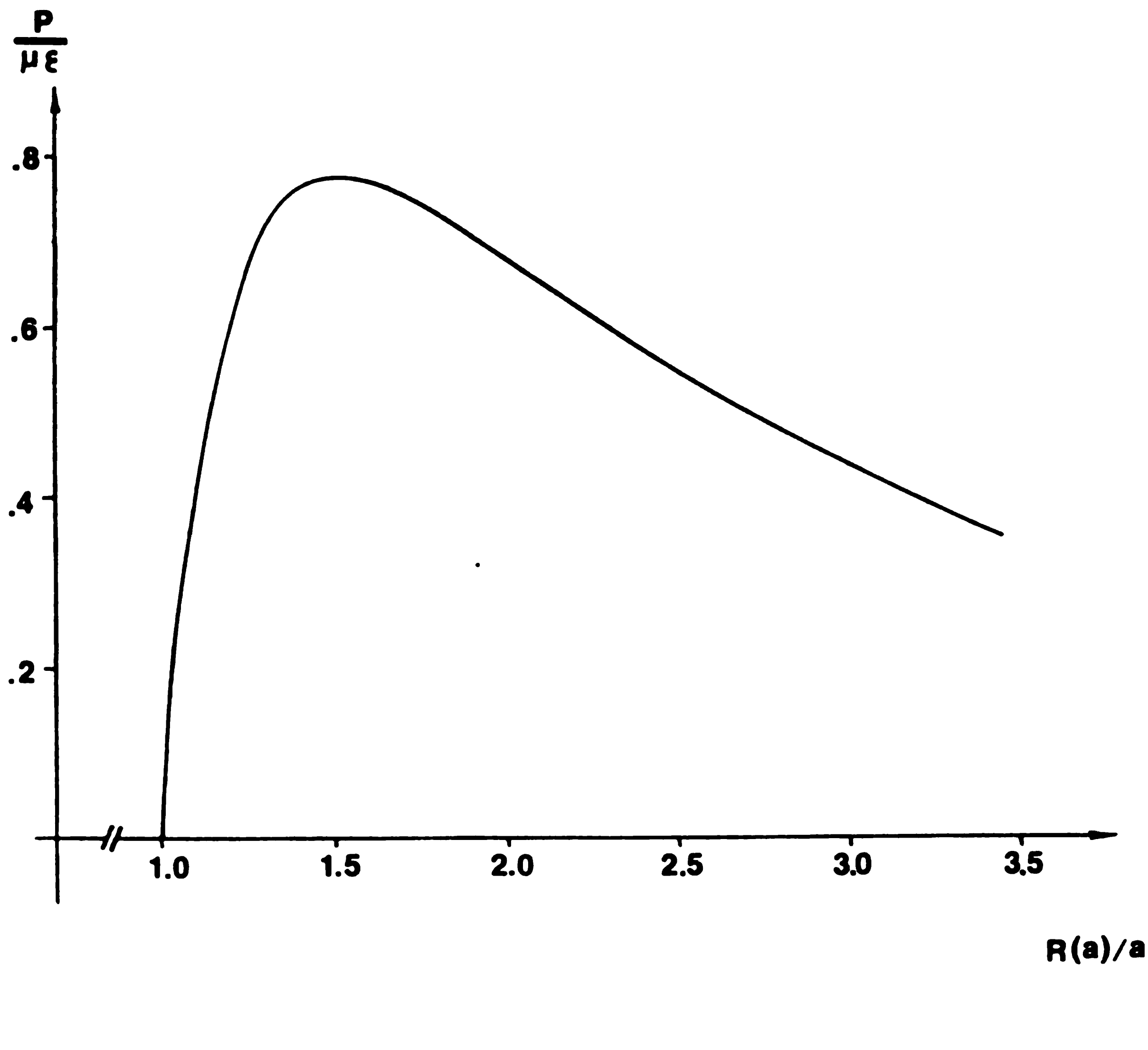

Figure 3. Applied pressure versus ratio of deformed to undeformed radius for a thin spherical shell.

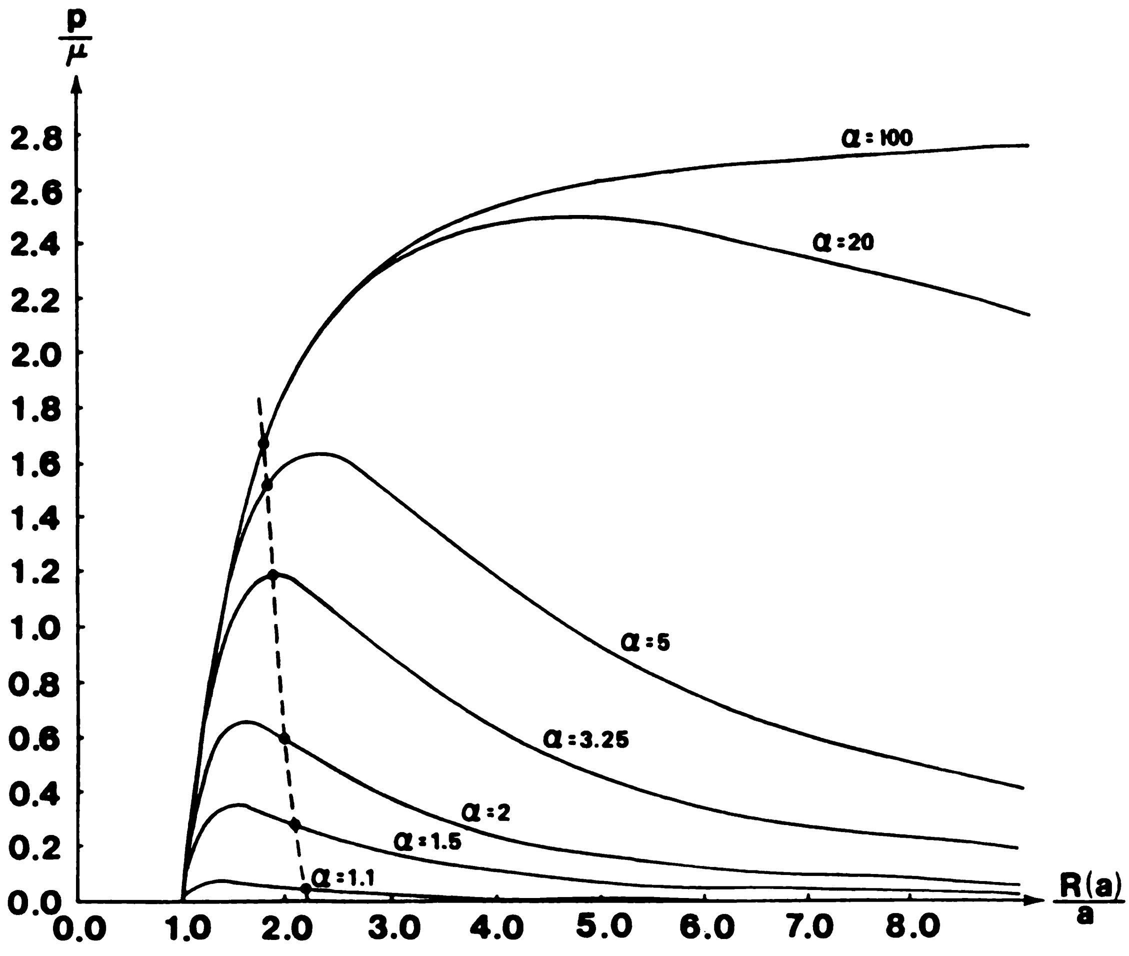

Figure 4. Applied pressure versus ratio of deformed to undeformed radius for hollow spheres for different radius ratios $\alpha = b/a$. The dotted curve connects the points $(p_e/\mu, R(a)/a)$, where p_e denotes the pressure at which ellipticity is lost.

APPENDICES

APPENDIX A:

Here we present the details of the calculation showing that $Q(t_a)$ in equation (3.19) has only one maximum for $0 < t_a < 1$.

We write

$$Q(t) \equiv Q(t,\alpha) = \frac{h(f(t))(f^2(t)+f(t)+4)^3}{h(t)(t^2+t+4)^3}, \tag{A.1}$$

using t instead of t_a for simplicity. For later purposes, it is also convenient to emphasize here that Q depends on the radius ratio $\alpha = b/a$ (see 3.18). We recall that (3.20) shows that $Q(t,\alpha)$ has at least one maximum on $0 < t < 1$. Now

$$\frac{\partial Q(t,\alpha)}{\partial t} = \frac{6(t^2+t+4)^2(f^2+f+4)^2 \cdot h \cdot h(f)}{\{H(t)\}^2} R(t,\alpha), \tag{A.2}$$

where

$$H(t) = h(t)(t^2+t+4)^3, \tag{A.3}$$

and

$$R(t,\alpha) = \{(t^2+t+4)(f+1)\frac{\partial f}{\partial t} - (f^2+f+4)(t+1)\}. \tag{A.4}$$

Observe from (A.2) that $R(t,\alpha) = 0$ is equivalent to

$$\frac{\partial Q(t,\alpha)}{\partial t} = 0.$$

Thus at a maximum point for $Q(t,\alpha)$, say at $t=\tilde{t}(\alpha)$, it follows that

$$R(\tilde{t},\alpha) = 0. \tag{A.5}$$

Now we want to show that only one such point $t = \tilde{t}(\alpha)$ exists for which $0 < t < 1$. From (A.2) upon differentiating again, we find

$$\frac{\partial^2 Q}{\partial t^2} = \frac{6}{\{H(t)\}^2}[2(t^2+t+4)(2t+1)(f^2+f+4)^2 \cdot h \cdot h(f)R(t,\alpha)$$

$$+ (t^2+t+4)^2 \cdot 2 \cdot (f^2+f+4)(2f+1) \cdot f' \cdot h \cdot h(f)R(t,\alpha)$$

$$+ (t^2+t+4)^2(f^2+f+4)^2 \cdot h' \cdot h(f)R(t,\alpha)$$

$$+ (t^2+t+4)^2(f^2+f+4)^2 \cdot h \cdot h'(f) \cdot f' \cdot R(t,\alpha) \qquad \text{(A.6)}$$

$$+ (t^2+t+4)^2(f^2+f+4)^2 \cdot h \cdot h'(f)\frac{\partial R(t,\alpha)}{\partial t}]$$

$$- \frac{72}{\{H(t)\}^2}(t^2+t+4)(t+1)(f^2+f+4)^2 \cdot h \cdot h(f)R(t,\alpha).$$

Using (A.5) we find

$$\left.\frac{\partial^2 Q}{\partial t^2}\right|_{t=\hat{t}} = \frac{6}{\{H(t)\}^2}(t^2+t+4)^2(f^2+f+4)^2 \cdot h \cdot h(f)\left.\frac{\partial R(t,\alpha)}{\partial t}\right|_{t=\hat{t}}. \qquad \text{(A.7)}$$

Differentiating (A.4) with respect to t and using (A.5) we find

$$\left.\frac{\partial R(t,\alpha)}{\partial t}\right|_{t=\hat{t}(\alpha)} = 4f'\left(\frac{t+1}{f+1} - \frac{f+1}{t+1}\right) + f''(t^2+t+4)(f+1). \qquad \text{(A.8)}$$

We now observe from equation (3.15), (3.18) that

$$\frac{\partial f(t,\alpha)}{\partial t} > 0, \qquad \text{(A.9)}$$

$$\frac{\partial^2 f(t,\alpha)}{\partial t^2} < 0, \qquad \text{(A.10)}$$

$$t < f(t,\alpha). \qquad \text{(A.11)}$$

Thus $\left.\frac{\partial R(t,\alpha)}{\partial t}\right|_{t=\hat{t}(\alpha)} < 0$ for $0 < \hat{t} < 1$, and so from (A.7), we deduce that

$$\left.\frac{\partial^2 Q}{\partial t^2}\right|_{t=\hat{t}(\alpha)} < 0 \quad \text{for } 0 < \hat{t} < 1.$$

Therefore, we conclude that $Q(t_a,\alpha)$ has only one maximum for $0 < t_a < 1$.

APPENDIX B

In Appendix A, we have shown that $Q(t_a,\alpha)$ in equation (3.19) has only one maximum for $0 < t_a < 1$, and the corresponding value of t_a is denoted by $\tilde{t}(\alpha)$. In this Appendix, we analyze the behavior of $\tilde{t}(\alpha)$ as α varies and thereby verify the remark made in Chapter 4.2 regarding the value and location of the pressure maximum p_m as the radius ratio $\alpha = b/a$ increases.

As we have shown in Appendix A, at the maximum point for $Q(t_a,\alpha)$ we have

$$R(\tilde{t}(\alpha),\alpha) = 0 \tag{B.1}$$

where $R(t,\alpha)$ is defined by (A.4). Differentiating (B.1) with respect to α yields

$$\frac{dR}{d\alpha} = \frac{\partial R}{\partial \hat{t}}\frac{d\hat{t}}{d\alpha} + \frac{\partial R}{\partial \alpha} = 0. \tag{B.2}$$

We have shown in Appendix A that

$$\frac{\partial R}{\partial \hat{t}} < 0 \quad \text{for} \quad 0 < \hat{t} < 1.$$

Observe from (B.2) that $\frac{\partial R}{\partial \alpha}$ has the same sign as $\frac{d\tilde{t}}{d\alpha}$. Now from (A.4) and (3.18), we have

$$\frac{\partial R}{\partial \alpha} = -192\,\frac{\alpha^{-1}(1-f)^2(t^2+t+4)(\partial f/\partial t)g(f)}{f^8 h(f)(f^2+f+4)(\partial g/\partial f)^2}(3f^4+3f^3+11f^2-f-4), \tag{B.3}$$

where we use t,f instead of $\tilde{t}(\alpha)$, $f(\tilde{t}(\alpha),\alpha) \equiv \tilde{f}(\alpha)$ for simplicity. The last term on the right hand side of (B.3) vanishes when $f = 1/\sqrt{3}$. Thus from (B.2) we see that

$$\frac{d\hat{t}(\alpha)}{d\alpha} = 0 \quad \text{when } \alpha = \alpha^*, \tag{B.4}$$

where α^* is the value of α for which

$$\overset{*}{f} \equiv f(\hat{t}(\overset{*}{\alpha}),\overset{*}{\alpha}) = 1/\sqrt{3}. \tag{B.5}$$

Thus recalling (A.9), we conclude that

$$\text{if } \; 0 < \hat{f} < \overset{*}{f} \; \text{ then } \frac{d\hat{t}}{d\alpha} > 0,$$

$$\text{if } \; \hat{f} = \overset{*}{f} \qquad \text{ then } \frac{d\hat{t}}{d\alpha} = 0, \tag{B.6}$$

$$\text{and if } \; \overset{*}{f} < \hat{f} < 1 \; \text{ then } \frac{d\hat{t}}{d\alpha} < 0.$$

To analyze the behavior as a function of α of the point $(\hat{t}(\alpha),\hat{f}(\alpha))$ corresponding to the maximum value of $Q(t_a,\alpha)$ in the $(t_a,f(t_a))$-plane, it is necessary to find the behavior of $d\hat{f}/d\alpha$ also. If we treat f in (3.18) as the independent variable in (3.19), we find that

$$\frac{\partial Q(f,\alpha)}{\partial f} = 0 \quad \text{if} \quad S(\hat{f}(\alpha),\alpha) = 0, \tag{B.7}$$

where

$$S(f(\alpha),\alpha) = (t^2+t+4)(f+1) - (f^2+f+4)(t+1)t'(f). \tag{B.8}$$

Here

$$t'(f) = \frac{\partial t(f,\alpha)}{\partial f} .$$

Just as in the derivation of (B.4), (B.5), one can show that

$$\frac{d\hat{f}(\alpha)}{d\alpha} = 0 \quad \text{when } \alpha = \alpha^{**}, \tag{B.9}$$

where α^{**} is the value of α for which

$$\overset{*}{t} \equiv t(\hat{f}(\alpha^{**}),\alpha^{**}) = 1/\sqrt{3}. \tag{B.10}$$

Thus we obtain a result analogous to (B.6), namely that

$$\text{if} \quad 0 < \hat{t} < \overset{*}{t} \quad \text{then} \ \frac{d\hat{f}}{d\alpha} > 0,$$

$$\text{if} \quad \hat{t} = \overset{*}{t} \quad \text{then} \ \frac{d\hat{f}}{d\alpha} = 0, \tag{B.11}$$

$$\text{and if} \quad \overset{*}{t} < \hat{t} < 1 \quad \text{then} \ \frac{d\hat{f}}{d\alpha} > 0.$$

To complete our analysis, we need the following two auxiliary results.

Lemma 1

Let $f(x)$ be a continuously differentiable function on $[1,\infty)$. Let k be some fixed number. Furthermore, suppose that f and k are such that $f'(x) < 0$ whenever $f(x) < k$. If there exists an $x_0 \in [1,\infty)$ such that $f(x_0) < k$ then

$f(x) < k$ and $f'(x) < 0$ on $[x_0,\infty)$.

Proof:

If $f(x_0) < k$ then

$$\left.\frac{df}{dx}\right|_{x=x_0} < 0 \text{ by hypothesis.}$$

Thus

$f(x) < f(x_0)$ for $x_0 < x < x_0 + \varepsilon$ and so $f(x) < k$ for $x_0 < x < x_0 + \varepsilon$.

Thus

$$\left.\frac{df}{dx}\right|_{x=x_0+\varepsilon} < 0.$$

Lemma 2

Let $f(x)$ be a continuously differentiable function on $[1,\infty)$. Let k be some fixed number. Furthermore, suppose that f and k are such that $f'(x) > 0$ whenever $f(x) > k$. If there exists an $x_0 \in [1,\infty)$ such that $f(x_0) > k$ then

$f(x) > k$ and $f'(x) > 0$ on $[x_0,\infty)$.

The proof of Lemma 2 is similar to that of Lemma 1 and will be omitted.

Let

$$\alpha = 1 + \varepsilon + \varepsilon^2/2^{*} \tag{B.12}$$

and assume

$$f(t) = t + \varepsilon A(t) + \varepsilon^2 B(t)/2, \tag{B.13}$$

where A(t), B(t) are unknown functions. Proceeding as in the analysis sketched in Chapter 4.1, one can show that

$$\hat{t} = 1/\sqrt{3} - 0.2\varepsilon, \tag{B.14}$$

$$\hat{f} = 1/\sqrt{3} + 0.2\varepsilon. \tag{B.15}$$

Thus, when the radius ratio α (=b/a) is near to unity we see from (B.14), (B.15) that

$$0 < \hat{t} < \overset{*}{t}, \quad \overset{*}{f} < \hat{f} < 1 \quad \text{for } \alpha = 1 + \varepsilon + \varepsilon^2/2. \tag{B.16}$$

It would follow from (B.6), (B.11) that

$$\frac{d\hat{t}}{d\alpha} < 0 \text{ and } \frac{d\hat{f}}{d\alpha} > 0 \quad \text{for } \alpha = 1 + \varepsilon + \varepsilon^2/2. \tag{B.17}$$

Then, by Lemma 1 and Lemma 2, we can conclude that

$$\frac{d\hat{t}}{d\alpha} < 0 \text{ and } 0 < \hat{t} < \overset{*}{t} \quad \text{for } 1 < \alpha < \infty, \tag{B.18}$$

and

$$\frac{d\hat{f}}{d\alpha} > 0 \text{ and } \overset{*}{f} < \hat{f} < 1 \quad \text{for } 1 < \alpha < \infty, \tag{B.19}$$

which establishes the desired monotonicity results.

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* When $\alpha = 1 + \varepsilon$ then $\hat{t} = \hat{f} = 1/\sqrt{3}$ which is the intermediate case in (B.6), (B.11).

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PART II

A BIFURCATION PROBLEM IN FINITE PLASTICITY RELATED TO VOID NUCLEATION

CHAPTER 1

INTRODUCTION

In a recent paper [1], Ball has made an extensive study of a class of singular problems for the equations of nonlinear elasticity, in which a spherical cavity forms at the center of a ball placed in tension. He showed that the existence of such solutions depends on the growth properties of the stored energy function W for large strains and the singular solution bifurcates from a trivial (homogeneous) solution at a critical value of the surface loading or displacement at which the trivial solution becomes unstable under appropriate hypotheses. An alternative physical interpretation for such problems (for a solid circular cylinder composed of a particular nonlinearly elastic material) in terms of the growth of pre-existing micro-void is given in [2]. The purpose of the present study is to analyse the corresponding bifurcation problem within the context of plasticity theory.

In the next Chapter we consider an incompressible solid sphere under symmetric, monotonic increasing, tensile dead load p. The constitutive relation describing the material behavior is taken to be a generalization of J_2-flow theory to finite deformations. One solution to this problem, for all values of p, corresponds to a homogeneous state in which the sphere remains undeformed but stressed. However for a certain critical range of p, one has in addition, a second possible configuration involving an internal spherical cavity. An explicit expression for the critical load p_{cr} at which the cavity is initiated is obtained (see Eq. (2.14)). It is important to note that this critical load is given automatically by the analysis. The issues of stability and post-bifurcation behavior are discussed in Chapter 5.

In Chapter 3, we treat the aforementioned problem using J_2-deformation theory. This is simply a special case of finite elasticity for a particular incompressible material. It is found that the final results are identical with those in Chapter 2.

It is worth pointing out that the bifurcation considered here is inherently associated with the kinematic nonlinearity. When the present problem is examined using classical infinitesimal strain plasticity theory as we do in Chapter 4, one finds that bifurcation is <u>not</u> predicted at any finite load.

CHAPTER 2

FINITE STRAIN FORMULATION AND SOLUTION; J_2-FLOW THEORY

Consider a solid sphere of radius A, subjected to a monotonically increasing radial tension (dead load), p(t), applied to its surface R = A. In view of symmetry, the resulting deformation of the sphere is described by

$$r = r(R,t),\ \theta = \Theta \text{ and } \phi = \Phi,\ r(0+,t) > 0, \tag{2.1*}$$

where (r, θ, ϕ) are the current spherical polar coordinates of the point which, in the undeformed configuration, was located at (R, Θ, Φ). If the material is assumed to be incompressible, the deformation gradient $\underset{\sim}{F}$ obeys det $\underset{\sim}{F} = 1$. For the deformation (2.1), this implies $r^2 \partial r/\partial R = R^2$, which when integrated gives

$$r = r(R,t) = \{R^3 + c^3(t)\}^{1/3},\ c(t) > 0, \tag{2.2}$$

where c(t) is to be determined. If it is found that c(t) = 0, (2.2) implies that the body remains a solid sphere in the current configuration. On the other hand, if c(t) is found to be positive (i.e. r(0+,t) > 0), there is a cavity of radius c centered at the origin in the current configuration.

From (2.1) and (2.2), the non-vanishing components of the Eulerian strain-rate tensor are found to be

$$D_r = -2\dot{r}/r,\ D_\theta = D_\phi = \dot{r}/r, \tag{2.3}$$

*Note that the convention used here for undeformed and deformed coordinates is opposite to that employed in Part I.

where the dot denotes the Lagrangian time derivative. In view of symmetry, and assuming the material to be isotropic, the non-zero components of the (Cauchy) true stress tensor are the radial stress $\sigma_r(r,t)$ and the hoop stresses $\sigma_\theta(r,t) = \sigma_\phi(r,t)$. The prescribed dead load boundary condition on the surface of the sphere requires that

$$\sigma_r(a, t) = p(t)(A/a)^2, \tag{2.4}$$

where $a = r(A,t) = (A^3 + c^3)^{1/3}$ represents the deformed outer radius.

The constitutive relation for the elastic-plastic material is taken to be a generalization of J_2-flow theory to finite deformations, (see e.g. [3]):

$$\underset{\sim}{D} = \frac{3}{2E}\overset{\nabla}{\underset{\sim}{S}} + \Lambda \frac{3}{2\sigma_e} \dot{\varepsilon}_p(\sigma_e)\underset{\sim}{S} \, . \tag{2.5}$$

Here $\underset{\sim}{S}$ is the deviatoric Cauchy stress; σ_e is the effective Cauchy stress; Λ is a loading coefficient; $\varepsilon_p(\cdot)$ is a given constitutive function representing the effective plastic logarithmic strain. The Jaumann (co-rotational) rate of the Cauchy stress deviator is denoted by $\overset{\nabla}{\underset{\sim}{S}}$, so that $\overset{\nabla}{\underset{\sim}{S}} = \dot{\underset{\sim}{S}} - \underset{\sim}{\Omega}\underset{\sim}{S} + \underset{\sim}{S}\underset{\sim}{\Omega}$ where $\underset{\sim}{\Omega}$ is the spin-tensor. In the case of uni-axial tension, the relation between true stress σ and the logarithmic strain ε, in monotonic loading, can be obtained from (2.5) as $\varepsilon = \hat{\varepsilon}(\sigma) \equiv \sigma/E + \varepsilon_p(\sigma)$. We assume this relation to be invertible so that we may write the <u>stress-strain relation in uni-axial tension</u> as either

$$\sigma = \hat{\sigma}(\varepsilon) \text{ or } \varepsilon = \hat{\varepsilon}(\sigma). \tag{2.6}$$

In the present problem, $\underset{\sim}{\Omega}$ vanishes, and thus $\overset{\nabla}{\underset{\sim}{S}} = \dot{\underset{\sim}{S}}$. Also, $\sigma_e = \sigma_\theta - \sigma_r$. Equation (2.5) and (2.6), under conditions of loading ($\Lambda = 1$), yield

$$\dot{\hat{\varepsilon}}(\sigma_e) = 2\dot{r}/r, \quad \sigma_e = \sigma_\theta - \sigma_r. \tag{2.7}$$

On using (2.1), we may integrate (2.7) with respect to the parameter t to obtain $\hat{\varepsilon}(\sigma_e) = 2\ell n(r/R)$. Using (2.6) to invert this gives

$$\sigma_e = \hat{\sigma}\{2\ell n(r/R)\}. \tag{2.8}$$

Finally, in the absence of body forces, the equilibrium equations reduce to the single equation

$$\frac{\partial \sigma_r}{\partial r} - 2\frac{\sigma_e}{r} = 0. \tag{2.9}$$

Thus, <u>the problem to be solved is the following</u>: We wish to find $\sigma_r(r,t)$ and $c(t) \geq 0$ such that the field equations (2.2), (2.8), (2.9) and the boundary condition (2.4) hold*. In addition, if $c(t) > 0$ it is also required that

$$\sigma_r(c,t) = 0. \tag{2.10}$$

This stipulates that when a hole appears at the origin, it must be traction-free.

First, it is readily shown that, for all values of $p > 0$, one solution to the foregoing problem is

$$\sigma_r(r,t) = p(t), \; c(t) = 0. \tag{2.11}$$

This corresponds to a homogeneous state of deformation $r = \hat{r}(R,t) = R$, with resulting stresses $\sigma_r = \sigma_\theta = \sigma_\phi = p(t)$.

Next we seek a solution for which $c(t) > 0$. Combining (2.2), (2.8) and (2.9), integrating with respect to r, using boundary condition (2.4), and employing a change of variables yields

*The remaining physical quantities can be immediately found thereafter from (2.2), (2.9) and $\sigma_\theta = \sigma_\phi = \sigma_e + \sigma_r$.

$$\sigma_r(r,t) = P\left(\frac{A}{a}\right)^2 - \int_{2\ell n(a/A)}^{2\ell n(r/R)} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2)-1}\, d\varepsilon,\ R=(r^3-c^3)^{1/3}. \quad (2.12)$$

On enforcing the remaining boundary condition (2.10), one is led to

$$p = \left(\frac{a}{A}\right)^2 \int_{2\ell n(\frac{a}{A})}^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2)-1}\, d\varepsilon,\ a = (A^3 + c^3)^{1/3}. \quad (2.13)$$

Thus, if for a given value of p, (2.13) can be solved for a positive root c, then this c together with (2.12), provides a solution to the problem at hand.

It is readily shown that under usual constitutive conditions, [$\hat{\sigma}(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$, $\hat{\sigma}(\varepsilon) = O(\varepsilon^n)$ as $\varepsilon \to \infty$, $n > 0$], the integral in (2.13) is bounded for all $a > A$. Therefore, there does exist a value of pressure p(>0) corresponding to each $c > 0$. A schematic graph of p vs. c is shown in Figure 5. The critical load p_{cr} at which the cavity is initiated is found by letting $c \to 0+$ in (2.12), i.e.

$$p_{cr} = \int_0^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2)-1}\, d\varepsilon. \quad (2.14)$$

As noted previously, the integral in (2.14) is bounded and so, the cavity is initiated at a finite value of load.

One may also study this problem using the finite strain version of J_2-deformation theory (e.g. [4]). This will be done in the next Chapter.

CHAPTER 3

FINITE STRAIN FORMULATION AND SOLUTION; J_2-DEFORMATION THEORY

In this Chapter, we treat the problem at hand using a finite strain version of J_2-deformation theory (e.g. Hutchinson and Neale [4]). This is simply a special case of finite elasticity for a particular incompressible material.

Consider a solid sphere of radius A, subjected to a radial tension (dead load), p, applied to its surface R = A. The resulting deformation is a mapping which takes the point (R,Θ,Φ) to the point (r,θ,ϕ). We assume that the deformation is axisymmetric one so that $\theta = \Theta, \phi = \Phi$ and $r = r(R)$. In order to avoid interpenetration, it is required that

$$r = r(R) > 0 \quad \text{on } 0 < R < A,\ r(0) > 0. \tag{3.1}$$

Observe that if $r(0) > 0$, the deformation is not one-to-one at the origin. If the material is assumed to be incompressible, one readily finds that

$$r = r(R) = (R^3 + c^3)^{1/3},\ c > 0, \tag{3.2}$$

where the constant c is to be determined.

The polar components of the deformation gradient tensor $\underset{\sim}{F}$ associated with the radial deformation (3.2) are given by

$$\underset{\sim}{F} = \operatorname{diag}\left(\left(\frac{R}{r(R)}\right)^2, \frac{r(R)}{R}, \frac{r(R)}{R}\right). \tag{3.3}$$

The corresponding principal stretches are

$$\lambda_\theta = \lambda_\phi = r(R)/R = \lambda,\ \lambda_r = \lambda_\theta^{-2} = \lambda^{-2} \tag{3.4}$$

In view of symmetry, the non-zero components of the true stress tensor are the radial stress $\sigma_r(r)$ and the hoop stresses $\sigma_\theta(r) = \sigma_\phi(r)$. The prescribed dead load boundary condition of the surface requires that

$$\sigma_r(a) = p(A/a)^2, \tag{3.5}$$

where $a = r(A) = (A^3 + c^3)^{1/3}$ represents the deformed outer radius.

We now turn to the constitutive relation for a general isotropic incompressible elastic material characterized by an elastic potential $W(\lambda_1, \lambda_2, \lambda_3)$ representing the strain energy per unit undeformed volume. The components of the principal true stress $\underset{\sim}{\sigma}$ are given by

$$\sigma_{ii} = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3} \frac{\partial W}{\partial \lambda_i} - p, \text{ (no sum on i)}. \tag{3.6}$$

For all $\lambda > 0$ it is convenient to define $\hat{W}(\lambda)$ by

$$\hat{W}(\lambda) \equiv W(\lambda^{-2}, \lambda, \lambda), \ \lambda > 0. \tag{3.7}$$

From (3.4), (3.6) and (3.7), one can derive the following:

$$\sigma_r - \sigma_\theta = -\frac{r}{2R} \hat{W}'\left(\frac{r}{R}\right). \tag{3.8}$$

In the absence of body force, the equilibrium equations div $\underset{\sim}{\sigma} = \underset{\sim}{0}$ reduce to

$$\frac{d}{dr}\sigma_r + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0, \quad c < r < a. \tag{3.9}$$

On using (3.6) and (3.7), (3.8) becomes

$$\frac{d}{dr}\sigma_r - \frac{1}{R} \hat{W}'\left(\frac{r}{R}\right) = 0. \tag{3.10}$$

<u>Thus, the problem to be solved is the following</u>: We wish to find $\sigma_r(r)$ for $r \in [c,a]$ and $c > 0$ such that the field equations (3.2), (3.10) and boundary condition (3.5) hold. In addition, if $c > 0$ it is also required that

$$\sigma_r(c) = 0 \text{ (i.e. } \sigma_r\big|_{R=0} = 0\text{)}. \tag{3.11}$$

This requires that when a hole appears at the origin, it must be traction-free.

First, it is readily shown that, for all values of $p > 0$, one solution to the foregoing problem is

$$\sigma_r(r) = p, \; c = 0. \tag{3.12}$$

This corresponds to a homogeneous state of deformation $r = \hat{r}(R) = R$.

Next, we seek a solution for which $c > 0$. Integrating (3.10) with respect to r, using boundary conditions (3.5) and employing a change of variable $\xi = r/R$ yield

$$\sigma_r(r) = p(\frac{A}{a})^2 - \int_{\frac{r}{R}}^{\frac{a}{A}} \frac{\hat{W}'(\xi)}{\xi^3-1} d\xi, \; R = (r^3-c^3)^{1/3}. \tag{3.13}$$

On enforcing the remaining boundary condition (3.11) yields

$$p = (\frac{a}{A})^2 \int_{\frac{a}{A}}^{\infty} \frac{\hat{W}'(\xi)}{\xi^3-1} d\xi, \; a = (A^3+ c^3)^{1/3}. \tag{3.14}$$

Thus, if for a given value of p, (3.14) can be solved for a positive root c, then c together with (3.14), provides a solution to the problem at hand.

As we mentioned before, J_2-deformation theory is a special case of finite elasticity, the class of incompressible materials being now characterized by the particular energy function of the form

$$W(\lambda_1, \lambda_2, \lambda_3) = \overset{\circ}{W}(\varepsilon_e), \tag{3.15}$$

where

$$\varepsilon_e = [(2/3)\{(\ell n\lambda_1)^2+(\ell n\lambda_2)^2+(\ell n\lambda_3)^2\}]^{1/2}. \tag{3.16}$$

When $\lambda_1 = \lambda^{-2}$, $\lambda_2=\lambda_3=\lambda$, as is the case here, (3.16) gives $\lambda = \exp(\varepsilon_e/2)$. In the case of uni-axial tension, the relation between true stress σ and the logarithmic strain ε can be obtained, i.e.

$$\hat{\sigma}(\varepsilon) = \mathring{W}'(\varepsilon). \tag{3.17}$$

Employing a change of variable, $\varepsilon = 2\ell n\xi$, and using (3.17), (3.14) readily becomes

$$p = \left(\frac{a}{A}\right)^2 \int_{2\ell n\left(\frac{a}{A}\right)}^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\exp\left(\frac{3\varepsilon}{2}\right)-1}\, d\varepsilon. \tag{3.18}$$

Equation (3.18) is seen to be identical to (2.13).

CHAPTER 4

INFINITESIMAL STRAIN FORMULATION AND SOLUTION

In this Chapter, we consider briefly the infinitesimal strain analog of the problem just treated. In view of symmetry, the displacement field has the form

$$u_R = u(R,t), \; u_\Theta = u_\Phi = 0. \tag{4.1}$$

The non-vanishing components of the infinitesimal strain tensor are

$$\varepsilon_R = \frac{\partial u}{\partial R}, \quad \varepsilon_\Theta = \varepsilon_\Phi = \frac{u}{R}. \tag{4.2}$$

Incompressibility in the present case requires

$$\varepsilon_R + \varepsilon_\Theta + \varepsilon_\Phi = 0. \tag{4.3}$$

Thus, (4.2) and (4.3) lead to a simple ordinary differential equation for u whose general solution is

$$u(R,t) = (a(t)-A)A^2/R^2, \tag{4.4}$$

where $a(t)$ represents the deformed outer radius of the sphere which is to be determined. If it is found that $a(t) = A$, then $u(0+,t) = 0$ i.e., the sphere remains solid in the current configuration (in fact, in this case $u(R,t) \equiv 0$). On the other hand, if it is found that $a(t) \neq A$ ($a(t) > A$) then $u(0+,t) = \infty$ and the body has ruptured at the origin.

The non-zero stress components are the radial stress σ_R and the hoop stresses σ_Θ, σ_Φ ($\sigma_\Theta = \sigma_\Phi$). The prescribed tensile load boundary condition of the surface of the sphere requires that

$$\sigma_R(A,t) = p(t), \; p(t) > 0. \tag{4.5}$$

The constitutive relation for the elastic-plastic material (at infinitesimal deformations) is taken to be the classical Prandtl-Reuss equation with a Von Mises (J_2) yield criterion; i.e.

$$\dot{\underset{\sim}{\varepsilon}} = \frac{3}{2E}\dot{\underset{\sim}{S}} + \Lambda\frac{3\dot{\varepsilon}_p(\sigma_e)}{2\sigma_e}\underset{\sim}{S} \, . \tag{4.6}$$

Here $\underset{\sim}{S}$ is the deviatoric stress; σ_e is the equivalent stress, $\sigma_e=\{3/2\ tr(\underset{\sim}{S}^2)\}^{1/2}$; Λ is a loading coefficient. The constitutive function $\varepsilon_p(\sigma_e)$ is, of course, given (and gives the value of the corresponding "effective plastic strain").

In the case of uni-axial tension, the relation between stress and strain in loading can be obtained from (4.6) as

$$\varepsilon = \frac{\sigma}{E} + \varepsilon_p(\sigma) \equiv \hat{\varepsilon}(\sigma). \tag{4.7}$$

The response function may be inverted to give

$$\sigma = \hat{\sigma}(\varepsilon). \tag{4.8}$$

Returning to the problem at hand, from (4.1) and (4.2), the non-zero components of strain-rate are

$$\dot{\varepsilon}_R = -2\frac{\dot{a}A^2}{R^3}, \ \dot{\varepsilon}_\Theta = \dot{\varepsilon}_\Phi = \frac{\dot{a}A^2}{R^3}, \tag{4.9}$$

where the dot denotes differentiation with respect to t. Equation (4.6) and (4.7), under conditions of loading (Λ=1), yield

$$\dot{\hat{\varepsilon}}(\sigma_e) = 2\,\frac{\dot{a}A^2}{R^3}\,, \ \sigma_e = \sigma_\Theta - \sigma_R. \tag{4.10}$$

Integrating (4.10) with respect to t and using a = A when t = 0 gives $\hat{\varepsilon}(\sigma_e) = 2u/R$. Using (4.8) to invert this gives

$$\sigma_e = \hat{\sigma}\left(\frac{2u}{R}\right). \tag{4.11}$$

Finally, in the absence of body forces, the equilibrium equations reduce to the single equation

$$\frac{\partial\sigma_R}{\partial R} - 2\,\frac{\sigma_e}{R} = 0. \tag{4.12}$$

<u>Thus, the problem to be solved is the following</u>: We wish to find $\sigma_R(R,t)$ and $a(t) > A$ such that the field equations (4.1), (4.11), (4.12) and the boundary condition (4.5) hold. In addition, if $a(t) \neq A$ it is also required that

$$\sigma_R(0+,t) = 0. \tag{4.13}$$

This requires that when a hole appears at the origin, it must be traction free.

First, it is readily shown that, for all value of $p > 0$, one solution is

$$\sigma_R(R,t) = p(t),\ a(t) = A. \tag{4.14}$$

This corresponds to a homogeneous state of deformation with resulting stresses

$$\sigma_R = \sigma_\Theta = \sigma_\Phi = p(t).$$

Next we seek a solution for which $a(t) > A$. Integrating (4.12) with respect to R from R = R to R = A, using the boundary condition (4.5) yields

$$\int_{\sigma_R}^{p} d\sigma_R = 2\int_R^A \frac{\hat{\sigma}\left(\frac{2u}{R}\right)}{R}\,dR,\quad 0 < R < A. \tag{4.15}$$

Employing a change of variable $\varepsilon = 2u/R$ now gives

$$\sigma_R(R,t) = p - \frac{2}{3}\int_{2(\frac{a}{A}-1)}^{2(a-A)A^2/R^3} \frac{\hat{\sigma}(\varepsilon)}{\varepsilon}d\varepsilon, \quad 0 < R < A. \tag{4.16}$$

In order to enforce the boundary condition (4.13) we calculate $\sigma_R(0+,t)$ from (4.16). Recalling that $a \neq A$ in the present case, this gives

$$\sigma_R(0+,t) = p - \frac{2}{3}\int_{2(\frac{a}{A}-1)}^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\varepsilon}d\varepsilon. \tag{4.17}$$

However for a general strain hardening material, the integral in (4.17) is readily shown to be <u>unbounded</u>. This implies that it is impossible to satisfy the boundary condition (4.13) at any finite value of load. Thus, at all finite value of p, bifurcation is <u>not</u> predicted by infinitesimal theory.

Another way to attack this problem is to follow the procedure in [2] and to first solve the problem for a sphere with a pre-existing spherical cavity at its center, and then, to let the radius of the cavity tend to zero. For all values of $p < p_{cr}$ the solution tends to the homogeneous solution in this limit while for $p > p_{cr}$ the solution tends to another (non-homogeneous) state. It is found that

$$p_{cr} = \frac{2}{3}\int_0^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\varepsilon}d\varepsilon. \tag{4.18}$$

Note from (4.17) that by enforcing the boundary condition (4.13) and letting $a = A$, we may formally recover (4.18). Alternatively, note that (4.18) may also be formally derived from the finite strain formula for p_{cr}, (2.14), by replacing the exponential in (2.14) by the first two terms in its Taylor expansion about $\varepsilon = 0$.

CHAPTER 5

DISCUSSION

Here we discuss further the bifurcation predicted by the finite strain formulation of Chapters 2 and 3.

For all values of the prescribed radial dead load p one possible configuration is that in which the sphere remains solid (see (2.11)). On the other hand, for a certain range of p one has, in addition, a second possible configuration involving an internal spherical cavity. Equation (2.14) gives the critical value of the load, p_{cr}, at which a cavity may initiate.

It is necessary to examine the stability of these two possible configurations in order to determine whether the homogeneous solution will in fact bifurcate, at $p = p_{cr}$, into the one involving a cavity. Figure 5 shows, schematically, a graph of the cavity radius c versus the applied load p. The bold horizontal line coinciding with the positive p-axis corresponds to the homogeneous solution. The curves emanating from $(p_{cr}, 0)$ correspond to a bifurcated solution involving a cavity. Presumably, if such a curve comes off to the right, the bifurcated solution is locally stable and so the sphere would indeed develop an internal cavity at $p = p_{cr}$. Conversely, if bifurcation to the left occurs, the solution is locally unstable and the sphere remains solid.

On using a Taylor expansion near p_{cr}, (2.13) yields

$$p = \hat{p}(c) = p_{cr} + \frac{2}{3}\,\frac{c^3}{A^3}\,(p_{cr} - \frac{2E}{3}) + o(c^3) \quad \text{as } c \to 0, \qquad (5.1)$$

where $E = \hat{\sigma}'(0)$ is the Young's modulus. Thus when $p_{cr} > 2E/3$, the slope of the curve at $(p_{cr}, 0)$ is positive and so bifurcation to the right occurs. On the other hand, when $p_{cr} < 2E/3$, the slope is negative. Consequently a void will actually appear at $p = p_{cr}$ only if p_{cr} is greater than $2E/3$.

Of course, the load level at which stable bifurcation is predicted here is unreasonably large. This feature is commonly encountered in bifurcation analyses employing classical flow theories of plasticity. It may be possible to use more elaborate constitutive models or to include the effect of a pre-existing stress concentrator (such as an inclusion) to obtain more realistic values for the critical load.

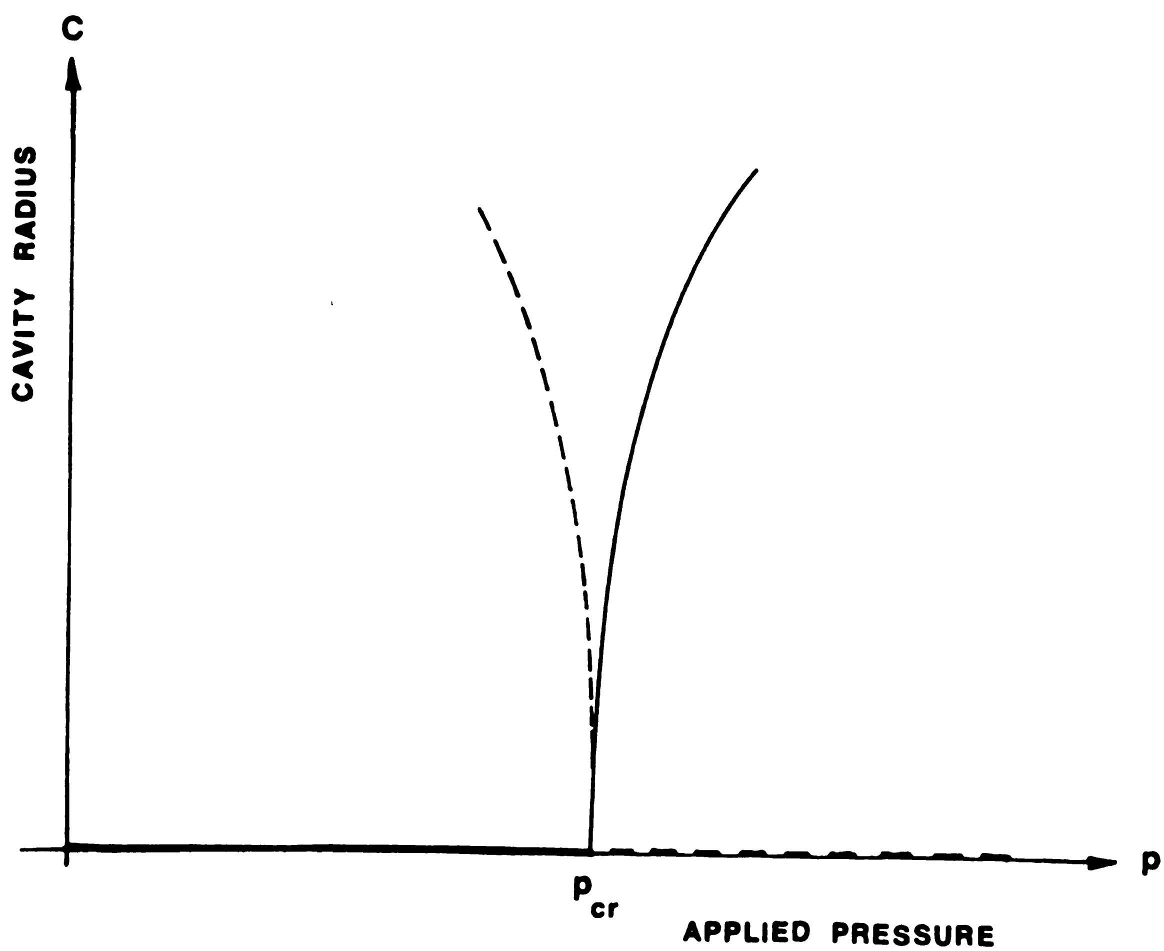

Figure 5. Schematic graph showing variation of cavity radius c versus applied load p.

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