A CRITICAL STUDY OF THE USE OF MATRICES IN THE ANALYSIS AND SYNTHESIS OF ELECTRICAL CIRCUITS

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A CRITICAL STUDY
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IN THE ANALYSIS AND SYNTHESIS
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TABLE OF NOTATION

$[R]$	Resistance Matrix
$[L]$	Inductance Matrix
$\left[\frac{1}{C}\right]$	Elastance Matrix
$[Z]$	Impedance Matrix
$[T]$	Reed's Symmetrical Component Matrix
$[S]$	Pipes' Symmetrical Component Matrix
$[C]$	Kron's Connection Matrix
$[P]$ & $[Q]$	Diagonalization Matrices
$[E]$	Voltage Matrix
$[I]$	Current Matrix
$[Y]$	Admittance Matrix
$[A]$, $[B]$, & $[F]$	Any Matrix
$[M]$	Characteristic Matrix of any Matrix
$[G] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$	Cascade Matrix
$[K]$	Impedance Invariant Transformation Matrix

INTRODUCTION

Matrices are becoming more and more popular as a tool for the electrical engineer. The most obvious reason for this is their ability to maintain the continuity of a complicated problem without the engineer becoming lost in a haze of computations. By indicating each step with matrix algebra, the engineer may carry the problem through to a symbolic solution and then retrace his steps making the necessary computations. This is true, for example, in cascading networks, in paralleling networks, and in the application of symmetrical components to a power system.

Matrices may also be used to advantage in finding equivalent networks on an impedance basis, that is, by maintaining the impedance invariant (unchanged). Another worth-while use is in writing the mesh or nodal equations of a network when there are a great number of mutual inductances.

If, in a given problem, one has a number of simultaneous voltage equations, the currents may be obtained by the usual process of determinants. But a more straightforward method would be with the use of matrices and the computation of the inverse of the impedance matrix. If the proper symmetry is present, an even more compact method would involve the diagonalization of the impedance matrix and the usually tedious computation of the impedance determinant would not be necessary.

The object of this thesis is to give examples and show how matrix algebra may be applied to various types of circuit problems, and to convey to the reader the continuity and compactness obtained by this method as opposed to the usual method of determinants and substitution and the loss of objective that usually follows when one makes computations as they progress in their problem.

CHAPTER I

FUNDAMENTAL DEFINITIONS and THEOREMS OF MATRICES

Since the use of matrices by electrical engineers is not widespread as yet, the logical place to begin this thesis would be with some of the fundamental definitions and theorems of matrices. This will also afford the reader the opportunity of becoming familiar with the notation used throughout the thesis.

Definition of a Matrix

It will be easier to define a matrix if an example is given. The mesh equations for the circuit in Figure I may

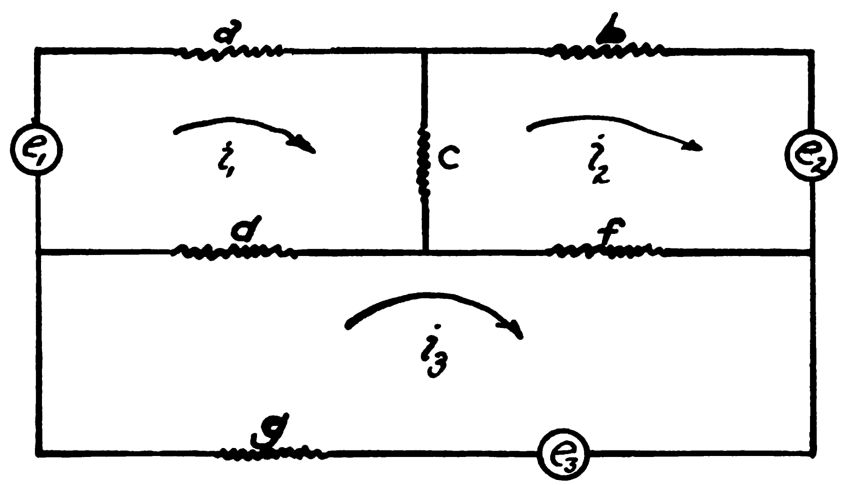

Figure 1

be written as:

$$\begin{aligned}
e_1 &= z_{11}i_1 + z_{12}i_2 + z_{13}i_3 \qquad &\text{where } z_{11} &= a + c + d \\
(1) \quad e_2 &= z_{21}i_1 + z_{22}i_2 + z_{23}i_3 & z_{22} &= b + c + f \\
e_3 &= z_{31}i_1 + z_{32}i_2 + z_{33}i_3 & z_{33} &= d + f + g
\end{aligned}$$

$$z_{12} = z_{21} = -c, \quad z_{13} = z_{31} = -d, \quad z_{23} = z_{32} = -f$$

If a proper rule for multiplying matrices is defined and followed, the equations (1) may be written as:

$$(2) \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \qquad \text{or simply} \quad [E] = [Z] \times [I]$$

where [E] is the voltage matrix; [Z] , the impedance matrix; and [I] , the current matrix. A matrix, therefore, is simply an array of numbers and is not necessarily square.

Equality of Matrices

Two matrices are equal when they cannot be distinguished from each other. Therefore, two matrices are equal, [A] = [B] , when every element of [A] equals every element of [B] , that is, $a_{ij} = b_{ij}$. (i indicates the row and j the column of the elements a and b)

The Zero and Unit Matrices

If every element of a matrix [A] is zero, ($a_{ij} = 0$) then [A] is defined as a zero matrix.

A unit matrix $[1]$ is a square matrix with all elements on the principle diagonal equal to one and all other elements equal to zero.

Multiplication of Matrices

If $[A] = [B] \times [F]$, then $a_{ij} = \sum_{k=0}^{n} b_{ik} f_{kj}$, where n is the number of columns in $[B]$ and the number of rows in $[F]$. From this definition, one can see that $[A] = [B] \times [F]$ is realizable only if the number of columns in $[B]$ is equal to the number of rows in $[F]$. It is also apparent that $\underset{(m,n)}{[B]} \times \underset{(n,q)}{[F]} = \underset{(m,q)}{[A]}$, that is, $[A]$ has the same number of rows as $[B]$ and the same number of columns as $[F]$.

If this definition is applied to the indicated multiplication in equation (2), then $[E] = [Z] \times [I]$ and $e_i = \sum_{k=1}^{3} z_{ik} i_k$.

$$e_1 = \sum_{k=1}^{3} z_{1k} i_k = z_{11} i_1 + z_{12} i_2 + z_{13} i_3$$

$$e_2 = \sum_{k=1}^{3} z_{2k} i_k = z_{21} i_1 + z_{22} i_2 + z_{23} i_3$$

$$e_3 = \sum_{k=1}^{3} z_{3k} i_k = z_{31} i_1 + z_{32} i_2 + z_{33} i_3$$

It will be noticed that this multiplication gives the desired result, that is, equations (1).

A definition of scalar multiplication is also necessary. $k\,[A] = [B]$ if $ka_{ij} = b_{ij}$. This is different than $k\,|A| = |B|$, where $|A|$ and $|B|$ are determinants. In this case only one of the row or columns of $|A|$ is multiplied by k.

The Inverse Matrix

Using equation (2) as an example, suppose that the voltages and impedances are known and that the currents are desired. Then by the usual method of determinants:

$$
\begin{aligned}
i_1 &= \frac{Z_{11}}{|Z|} e_1 + \frac{Z_{21}}{|Z|} e_2 + \frac{Z_{31}}{|Z|} e_3 \\
(3) \quad i_2 &= \frac{Z_{12}}{|Z|} e_1 + \frac{Z_{22}}{|Z|} e_2 + \frac{Z_{32}}{|Z|} e_3 \\
i_3 &= \frac{Z_{13}}{|Z|} e_1 + \frac{Z_{23}}{|Z|} e_3 + \frac{Z_{33}}{|Z|} e_3
\end{aligned}
$$

where $Z_{11}, Z_{21}, Z_{31}\ldots$ are the cofactors of $z_{11}, z_{21}, z_{31}\ldots$ in the determinant of $|Z|$; that is, Z_{ij} is the minor of z_{ij} multiplied by $(-1)^{i+j}$. Equations (3) may be written as:

$$(4) \qquad [I] = [Y] \times [E]$$

Equations (2) and (4) define the inverse. $[E] = [Z] \times [I]$ and $[I] = [Y] \times [E]$ and therefore, $[E] = [Z] \times [Y] \times [E]$ but since $[E] = [E]$, $[Z] \times [Y]$ must equal $[1]$. If the indicated multiplication is carried out, the above statement is verified. $[Y]$ is defined as the inverse of $[Z]$; that is, $[Y] = [Z]^{-1}$ or $[Z] = [Y]^{-1}$. Therefore, a matrix times its inverse is a unit matrix. $[Z] \times [Z]^{-1} = [Z]^{-1} \times [Z] = [1]$. Also, if the determinant of $[Z]$ is $|Z|$ then the determinant of $[Z]^{-1}$ is $\frac{1}{|Z|}$.

It is apparent from the above definition of the inverse that $[Z]^{-1}$ could not exist if $|Z| = 0$.

A matrix whose determinant is zero is called a singular matrix. A square matrix may or may not be singular, whereas all rectangular matrices are singular. Only non-singular matrices have an inverse and for each matrix there is only one inverse.

The transpose of $[Z]$ is defined as $[Z]_t$ and is obtained by interchanging the rows and columns of $[Z]$. Therefore, $[Z]^{-1}$ may be determined by first finding $[Z]_t$ and then replacing its elements by their cofactors over $|Z|$; that is, substitute $\frac{Z_{ij}}{|Z|}$ for z_{ij} in the transposed matrix.

Laws of Matrix Algebra

Because of the definitions of multiplication, it is, in general, not commutative; that is, $[A] \times [B] \neq [B] \times [A]$. Exceptions are $[A] \times [A]^{-1} = [A]^{-1} \times [A]$, and $[A] \times [1] = [1] \times [A]$.

In other manipulations, however, matrix algebra is similar to ordinary algebra.

(5) $[A] + [B] = [B] + [A]$

(6) $k[A] + k[B] = k([A] + [B])$

(7) $k[A] + q[A] = (k + q) \times [A]$

(8) $[F] \times [A] + [F] \times [B] = [F] \times ([A] + [B])$

(9) $[A] \times [F] + [B] \times [F] = ([A] + [B]) \times [F]$

If $[A] = [B] + [F]$ then it is necessary that $a_{ij} = b_{ij} + f_{ij}$. Caution must be used, however, in cancellation of factors. If $[A] \times [F] = [B] \times [F]$ then postmultiplying both sides by $[F]^{-1}$ gives $[A] \times [F] \times [F]^{-1} = [B] \times [F] \times [F]^{-1}$, and $[A] = [B]$. The above stipulation, however, is that $[F]^{-1}$ exists; that is, $[F]$ is a non-singular matrix. $[F]$ may not be cancelled for the case where $[A] \times [F] = [F] \times [B]$ because in general $[A] \times [F] \neq [F] \times [A]$. Also it may not be cancelled for the case where $[A] \times [F] = [B] \times [F]$ if $[F]$ is a singular matrix, because $[F]^{-1}$ is non-existent and the above reasoning could not be followed.

Two additional matrix relations are:

(10) $([A] \times [B] \times [F])_t = [F]_t \times [B]_t \times [A]_t$

(11) $([A] \times [B] \times [F])^{-1} = [F]^{-1} \times [B]^{-1} \times [A]^{-1}$

Linear Transformations

Occasionally it is desirable to interchange the rows or columns of matrices or add a row (column) to another row (column). Sometimes it is necessary to multiply a row (column) by a factor or to add to a given row (column) a certain other row (column) that has been multiplied by a factor. If the above operations are to be performed on, say the $[Z]$ matrix, where $[Z]$ is part of the equation $[E] = [Z] \times [I]$, then it is necessary to perform these operations with a matrix so that the equality of the matrix equation will not be nullified. For instance, if $[A]$ interchanges the first and second rows of $[Z]$ and $[E] = [Z] \times [I]$ then the resultant equation is obtained by premultiplying both sides by $[A]$, giving:

$$[A] \times [E] = [A] \times [Z] \times [I]$$

Each of the above-mentioned linear transform matrices are formed from the $[1]$ matrix and are, therefore, square. The following linear transform matrices are illustrated for n equal to four, but the same technique may be applied for n equal to any finite number. They are obtained as follows:

To interchange two rows <u>premultiply</u> by $[A]$ where $[A]$ is a unit matrix with the corresponding two rows interchanged.

To interchange two columns postmultiply by [A] where [A] is obtained by interchanging the two corresponding columns.

Example:

$$[A] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[A] x [B] interchanges 1st and 3rd row of [B]. [B] x [A] interchanges 1st and 3rd columns of [B] .

To add one row to another row, perform the desired operation on [1] and premultiply. To add one column to another, perform the desired operation on [1] and postmultiply.

Example:

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[A] x [B] adds 4th row of [B] to 1st. [B] x [A] adds 1st column of [B] to 4th.

To multiply a row by a factor, multiply the given row of [1] by that factor and premultiply. To multiply a column by a factor, multiply the given column of [1] by that factor and postmultiply.

Example:

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[A] x [B] multiplies 3rd row of [B] by k. [B] x [A] multiplies 3rd column of [B] by k.

If it is desirable to multiply a given row by a factor and add that row to another, perform the desired operation on [1] and premultiply. If it is desirable to multiply a column and add it to another column, perform the operation on [1] and postmultiply.

Example:

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$[A] \times [B]$ adds k times 4th row to 2nd row.
$[B] \times [A]$ adds k times 2nd column to 4th column.

The above technique will be used in the chapter on diagonalization. No unique notation was adopted for the above matrices since they will be redetermined for particular cases in Chapter V.

Cayley-Hamilton Theorem

If $[M]$ is defined as the characteristic matrix of the matrix $[A]$, then $[M] = [A] - \mu[1]$. μ is a scalar parameter and $[M]$, $[A]$, and $[1]$ are square and of the same order n. The determinant of $[M]$, $(|M|)$, is defined as the characteristic function of the matrix $[A]$ and the equation $|M| = 0$ is defined as the characteristic equation of $[A]$. The statement of the Cayley-Hamilton Theorem is that any matrix $[A]$ satisfies its own characteristic equation.

Example: $$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$[M] = [A] - \mu[1] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \mu\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1-\mu) & 2 \\ 3 & (4-\mu) \end{bmatrix}$$

Characteristic Function: $(1-\mu)(4-\mu) - 6$

Characteristic Equation: $(1-\mu)(4-\mu) - 6 = 0$

That is: $\mu^2 - 5\mu - 2 = 0$

And by the Cayley-Hamilton Theorem:

$$[A]^2 - 5\,[A] - 2\,[1] = 0$$

Where $[A]^2 = [A] \times [A]$ and $[1]$ is of the same order as $[A]$.

Therefore $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \times \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} - 5\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} - 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 0$

Carrying out the above operation indicates its correctness. The applications that follow from this theorem are very useful. Since $[A]^2 - 5\,[A] - 2\,[1] = 0$ for $[A] = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ $[A]^2 = 5\,[A] + 2\,[1]$ and multiplying through by $[A]$ gives,

$$[A]^3 = 5\,[A]^2 + 2\,[A] = 5(\;5\,[A] + 2\,[1]\;) + 2\,[A] = 27\,[A] + 10\,[1]$$

$$[A]^4 = 27\,[A]^2 + 10\,[A] = 27(\;5\,[A] + 2\,[1]\;) + 10\,[A] = 145\,[A] + 54\,[1]$$

$$[A]^5 = \ldots\ldots\ldots\ldots\ldots\ldots$$

Thus, without carrying out the matrix multiplication, $[A]^4$, $[A]^5$ .... may be determined in terms of $[A]^{n-1}$, $[A]^{n-2}$, ...$[1]$ where n is the order of the square matrix $[A]$.

This theorem also indicates a method for the determination of the inverse. The following example will illustrate this method:

Let $[B] = \begin{bmatrix}6 & 2 & 1\\2 & 8 & 4\\1 & 4 & 6\end{bmatrix}$

Therefore $[M] = \begin{bmatrix}6 & 2 & 1\\2 & 8 & 4\\1 & 4 & 6\end{bmatrix} - \mu\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}(6-\mu) & 2 & 1\\2 & (8-\mu) & 4\\1 & 4 & (6-\mu)\end{bmatrix}$

$$|M| = (6-\mu)^2\,(8-\mu) + 8 + 8 - \{(8-\mu) + 4(6-\mu) + 16(6-\mu)\}$$

And $|M| = -\mu^3 + 20\mu^3 - 111\mu + 176 = 0$ (Characteristic Equation)

Therefore $[B]^3 - 20\,[B]^2 + 111\,[B] - 176\,[1] = 0$

Which gives $176\,[1] = [B]^3 - 20\,[B]^2 + 111[B]$

Multiplying through by $[B]^{-1}$ and dividing by 176 gives:

$$[B]^{-1} = 1/176\,[B]^2 - 20/176\,[B] + 111/176\,[1]$$

$$\text{And } [B]^{-1} = 1/176\left\{\begin{bmatrix}41 & 32 & 20\\ 32 & 84 & 58\\ 20 & 58 & 53\end{bmatrix} - \begin{bmatrix}120 & 40 & 20\\ 40 & 160 & 80\\ 20 & 80 & 120\end{bmatrix} + \begin{bmatrix}111 & 0 & 0\\ 0 & 111 & 0\\ 0 & 0 & 111\end{bmatrix}\right\}$$

$$[B]^{-1} = 1/176\begin{bmatrix}32 & -8 & 0\\ -8 & 35 & -22\\ 0 & -22 & 44\end{bmatrix}$$

By the method described in section 5,

$|B| = 6 \times 8 \times 6 + 2 \times 4 \times 1 - 1 \times 8 \times 1 - 2 \times 2 \times 6 - 4 \times 4 \times 6 = 176$

$$\text{And } [B]^{-1} = 1/176\begin{bmatrix}\begin{vmatrix}8 & 4\\ 4 & 6\end{vmatrix} & -\begin{vmatrix}2 & 4\\ 1 & 6\end{vmatrix} & \begin{vmatrix}2 & 8\\ 1 & 4\end{vmatrix}\\ \begin{vmatrix}2 & 1\\ 4 & 6\end{vmatrix} & \begin{vmatrix}6 & 1\\ 1 & 6\end{vmatrix} & -\begin{vmatrix}6 & 2\\ 1 & 4\end{vmatrix}\\ \begin{vmatrix}2 & 1\\ 8 & 4\end{vmatrix} & -\begin{vmatrix}6 & 1\\ 2 & 4\end{vmatrix} & \begin{vmatrix}6 & 2\\ 2 & 8\end{vmatrix}\end{bmatrix} = 1/176\begin{bmatrix}32 & -8 & 0\\ -8 & 35 & -22\\ 0 & -22 & 44\end{bmatrix}$$

It is difficult to say which method is the best. Both methods require an evaluation of a determinant of the same order as $|B|$. However, if the order of $|B|$ is five or higher, the determination of the cofactors, necessitated by the second method illustrated, becomes a difficult task. While, for the Cayley-Hamilton method, there is only the fifth order determinant to determine and the rest of the operation is matrix multiplication, addition, and subtraction. It would seem, therefore, that for matrices of fourth order or less, the method of section 5 is best and for matrices of fifth or higher, the Cayley-Hamilton method is best. If a computing machine capable of handling matrices is available, then the Cayley-Hamilton method is definitely the one to use.

CHAPTER II

FOUR TERMINAL NETWORKS

Usually, what takes place at the input and output terminals is of interest rather than what happens inside the network itself. In analyzing or synthesizing a network it may be found desirable to interconnect several four terminal boxes in various ways. The fact that matrices can be used to express any two of the four variables in terms of the other two (e_1, e_2, i_1, i_2) makes them particularly effective in handling problems of this kind. There are six different ways by which matrices may be used to express relations between the various voltages and currents.

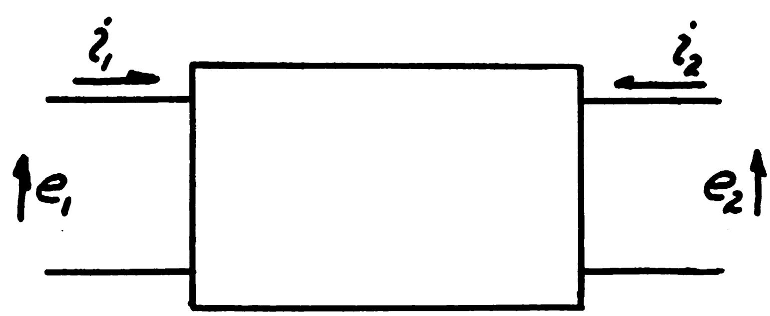

Figure 2

Referring to Figure 2, they are[1]:

(11) $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} x \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ (12) $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} x \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$

[1]E. A. Guillemin, Communication Networks, II, pp. 144.

(13) $\begin{bmatrix} i_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} x \begin{bmatrix} e_1 \\ i_2 \end{bmatrix}$ (14) $\begin{bmatrix} e_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} x \begin{bmatrix} i_1 \\ e_2 \end{bmatrix}$

(15) $\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} x \begin{bmatrix} e_2 \\ -i_2 \end{bmatrix}$ (16) $\begin{bmatrix} e_2 \\ i_2 \end{bmatrix} \begin{bmatrix} D & B \\ C & A \end{bmatrix} x \begin{bmatrix} e_1 \\ -i_1 \end{bmatrix}$

The interrelations between the various elements of the six square matrices have been derived and tabulated.[2] If one were doing many problems of this type it would be desirable to use such a table. Throughout this thesis, however, all interrelations will be derived as part of the problem.

Cascading Networks

Where several networks are to be cascaded, the type of representation to use would be that given by equation (15). For example, if in Figure 3, all of the networks are identical,

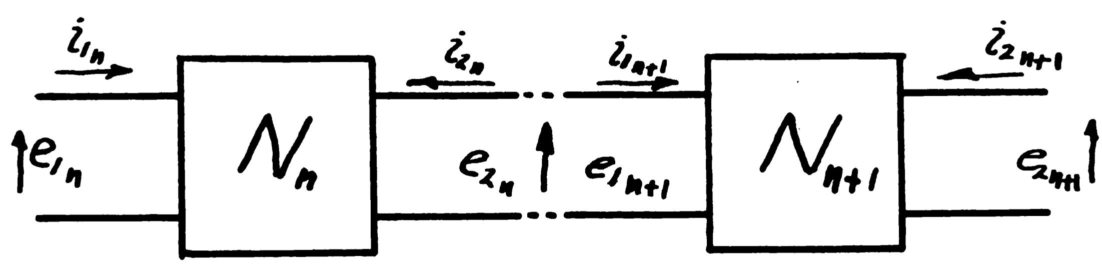

Figure 3

then the input current and voltage can be expressed in terms of the output current and voltage by this simple relation:

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[2] *Ibid.*, pp. 133-138.

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \times \begin{bmatrix} e_{2n} \\ e_{1n} \end{bmatrix}$$

The matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^n$ may be evaluated as an application of the Cayley-Hamilton theorem. Referring to Chapter I, the procedure is as follows:

Let $[G] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ Then $[M] = [G] - \mu[1]$

And $[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} (A-\mu) & B \\ C & (D-\mu) \end{bmatrix}$

Therefore $|M| = \mu^2 - (A + D)\mu + AD-BC$, but $AD-BC = 1$ (Shown later on in this section). And $|M| = 0$ is the characteristic equation, therefore, $\mu^2 - (A + D)\mu + 1 = 0$. And by the statements of the Cayley-Hamilton theorem,

$[G]^2 - k \times [G] + [1] = 0$ where $k = A + D$.

Therefore $[G]^2 = k \times [G] - [1]$

$[G]^3 = k \times [G]^2 - [G] = (k^2-1)\ [G] - k\ [1]$

$[G]^4 = (k^2-1)\ [G]^2 - k\ [G] = (k^3-2k) \times [G] - (k^2-1) \times [1]$

And if $[G]^{n-1} = p\ [G] - q\ [1]$

$[G]^n = (kp-q) \times [G] - p\ [1]$

Using the above relations for the coefficients of $[G]$ and $[1]$, $[G]^n$ can be obtained faster than the actual matrix multiplication will allow, especially if n is large and A, B, C, D are complex.

The following example will be worked out to illustrate the use of cascading networks. From the

network given, Figure 4, find the characteristic impedance looking in at terminals 1-2; that is, find R_x so that the impedance looking in at 1-2 will be R_x when 3-4 is terminated in R_x. Also, if 3-4 is the output, terminated in R_x, and 1-2 is the input, what will the attenuation be for the network?

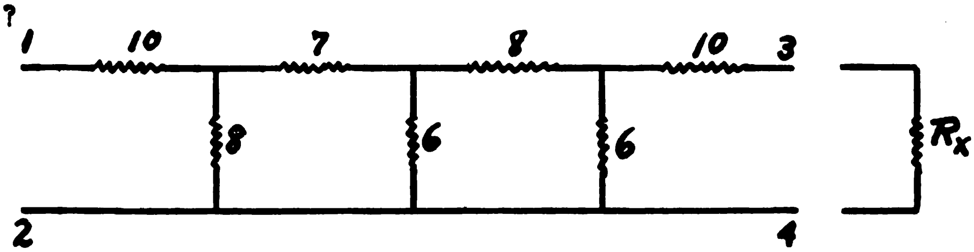

Figure 4

Solution: The network is first broken up as shown in Figure 5.

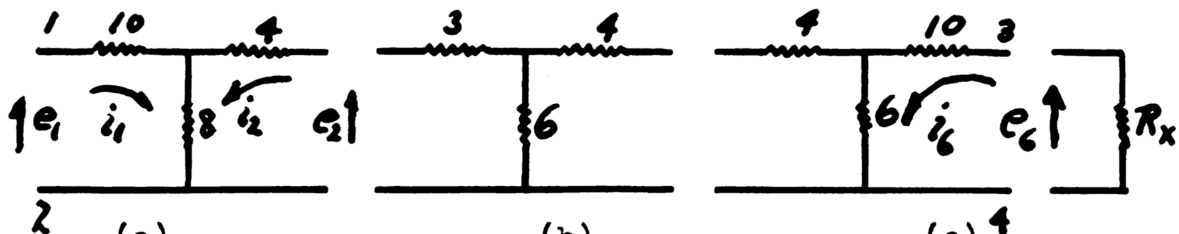

Figure 5

Using Figure 5(a) as an example:

$$(17)\quad \begin{aligned} e_1 &= (10 + 8)i_1 + 8\, i_2 = z_{11}i_1 + z_{12}i_2 \\ e_2 &= 8i_1 + (4 + 8)\, i_2 = z_{21}i_1 + z_{22}i_2 \end{aligned}$$

where $z_{11} = 18 \quad z_{12} = z_{21} = 8 \quad z_{22} = 12$

Then $$i_1 = \frac{e_2}{z_{21}} - \frac{z_{22}}{z_{21}}\, i_2\ , \quad e_1 = z_{11}\Big(\frac{e_2}{z_{21}} - \frac{z_{22}}{z_{21}}\, i_2\Big) + z_{12}i_2$$

And $$e_1 = \frac{z_{11}}{z_{21}}\, e_2 - \frac{(z_{11}z_{22} - z_{12})^2}{z_{12}}\; i_2$$

Or

$$e_1 = \frac{z_{11}}{z_{12}} e_2 - \frac{|Z|}{z_{12}} i_2 = Ae_2 - Bi_2$$

(18)

$$i_1 = \frac{1}{z_{12}} e_2 - \frac{z_{22}}{z_{12}} i_2 = Ce_2 - Di_2$$

And Therefore

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \overset{(a)}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} \times \overset{(b)}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} \times \overset{(c)}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} \times \begin{bmatrix} e_6 \\ -i_6 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 9/4 & 19 \\ 1/8 & 3/2 \end{bmatrix} \times \begin{bmatrix} 3/2 & 9 \\ 1/6 & 5/3 \end{bmatrix} \times \begin{bmatrix} 5/3 & 124/6 \\ 1/6 & 8/3 \end{bmatrix} \times \begin{bmatrix} e_6 \\ -i_6 \end{bmatrix}$$

Carrying out the indicated multiplication gives:

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 19.9 & 279 \\ 1.33 & 18.7 \end{bmatrix} \times \begin{bmatrix} e_6 \\ -i_6 \end{bmatrix}$$

And $e_1 = A\ e_6 - B\ i_6$ Where $A = 19.9$ $B = 279$

$i_1 = C\ e_6 - D\ i_6$ $C = 1.33$ $D = 18.7$

It is easily seen from equations (18), which apply to any four terminal network that has symmetry about the principle diagonal, ($[Z] = [Z]_t$), that $AD - BC = 1$. In our case $AD - BC = 372.13 - 371.07 = 1$.

If the network is terminated in R_x then $e_6 \cong -R_x i_6$

and $e_1 = A(-R_x i_6) - Bi_6 = -i_6(AR_x + B)$

$i_1 = C(-R_x i_6) - Di_6 = -i_6(CR_x + D)$

And if R_x is to be the characteristic impedance,

$$\frac{e_1}{i_1} = R_x = \frac{AR_x + B}{CR_x + D} \text{, therefore, } CR_x^2 + DR_x - AR_x - B = 0$$

And $$R_x^2 + R_x \frac{(D-A)}{C} - \frac{B}{C} = 0$$

$$R_x^2 - R_x (0.9) - 210 = 0$$

Giving $$R_x = \frac{0.9 \pm \sqrt{.81 + 840}}{2} = \frac{0.9 + 29}{2} = 14.9 \text{ ohms}$$

Also $$e_1 = Ae_6 - Bi_6 \qquad i_6 = \frac{-e_6}{R_x}$$
$$i_1 = Ce_6 - Di_6$$

Therefore $$e_1 = Ae_6 + \frac{B}{R_x} e_6 = e_6 \left(\frac{AR_x + B}{R_x} \right)$$

$$\frac{e_6}{e_1} = \frac{R_x}{AR_x + B} = \frac{14.9}{14.9 \times 19.9 + 279} = \frac{14.9}{299 + 279} = 0.0258$$

$$N = \ln\left(\frac{e_6}{e_1}\right) = \ln 0.0258 = -\ln \frac{1}{0.0258}$$

$$N = -\ln 38.8 = -3.65 \text{ nepers}$$

But since e_1 and e_6 are across the same resistance, R_x, N may be converted to decibels. The gain, therefore, equals -8.688 x 3.65 = -31.6 db.; that is, there is an attenuation of 31.6 db. for the network including R_x.

It might be suggested that one set up the network and measure Z open circuit and Z short circuit and obtain R_x by $R_x = \sqrt{Z_{oc} Z_{sc}}$. However, this could not be done in our case, because $R_x = \sqrt{Z_{oc} Z_{sc}}$ only when the network is symmetrical about a vertical line through its midpoint.

In this example all network components were resistances, however, the same technique is equally applicable when the components are complex impedances. In a later example it will be shown that matrices may be used when the Laplacian transform is involved.

There are, undoubtedly, many ways of working the above problem, but it is doubtful if there is any method more concise and straightforward than that just shown.

Paralleling Networks

In paralleling two or more networks, Figure 6,

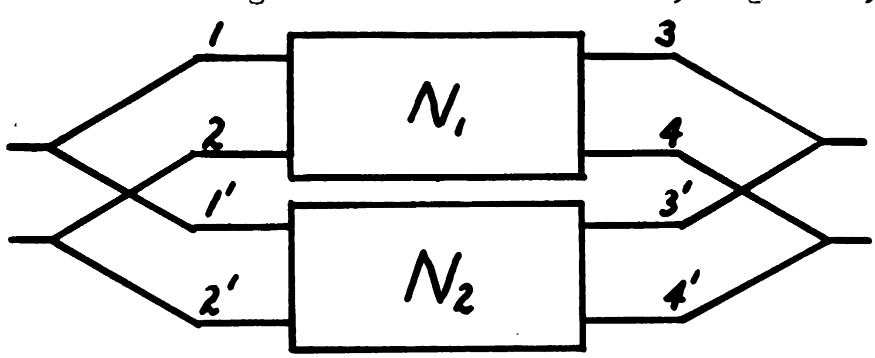

Figure 6

equations of the type given by (11) would be used. $[I]_1 = [Y]_1 \times [E]_1$ and $[I]_2 = [Y]_2 \times [E]_2$, but $[I]_T = [I]_1 + [I]_2$ and $[E]_1 = [E]_2$. Adding the two equations we have $[I]_T = [I]_1 + [I]_2 = [Y]_1 \times [E] + [Y]_2 \times [E]$ and by equation (9), therefore:

(19) $$[I]_T = ([Y]_1 + [Y]_2) \times [E]$$

In handling problems of this type with matrix algebra a great deal of caution must be used. After the connections have been made on the left, they may be made on the right providing that there is no potential difference between terminals 3 and 3' and between 4 and 4'. If potential differences exist, there will be circulating currents within the networks. The result will be that the current in at terminal 1 will not equal the current out at 2 and similarly for the other terminals of the networks. If other means cannot be used,[3] ideal transformers may be placed at one end of the network forcing the currents to be equal. If n networks are to be paralleled, the most hopeless case would require n-1 transformers. A more complete discussion of this is given in Communication Networks.[4]

The paralleling of two or more networks might be necessary in a problem in synthesis. For instance, several four terminal networks are available and a specific overall effect is desired; that is, $[I] = [Y] \times [E]$. $[E]$ contains the driving voltage e_1 and the desired output voltage e_2.

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[3] Ibid., p. 148

[4] Loc. cit.

$[Y]$ is the admittance matrix that will give this desired result with the ensuing currents $[I]$. Then by the method of combination shown in Figure 6 and equation (19), the various $[Y]$ Matrices may be added until the desired result is obtained; that is, $[Y] = [Y]_1 + [Y]_2 \ldots$ which, of course, means that $y_{ij} = y_{ij1} + y_{ij2} \ldots$. With a finite number of networks available, one would be very lucky to obtain a combination that would be exactly correct, but approximations could be obtained. And for the problem as stated, the use of matrices would lead to a solution directly.

Matrices may be applied very nicely to the analysis of paralleling problems. The circuit of Figure 7(a) is

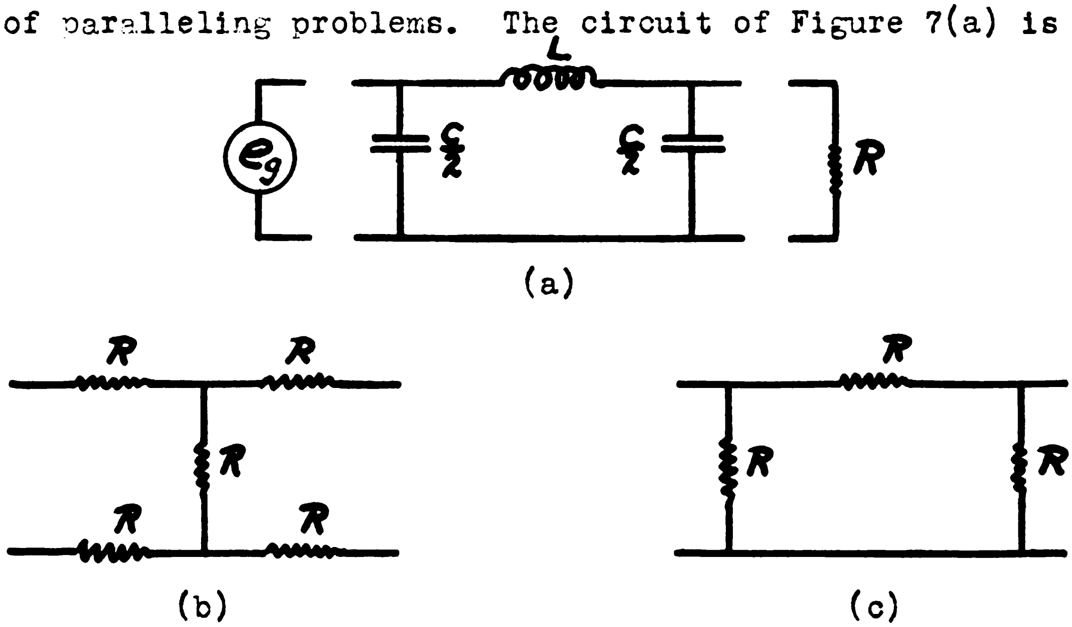


Figure 7

a one section low pas filter terminated in its characteristic impedance, R, and connected to an all frequency generator, e_g. The problem is, if the networks Figure 7(b) and (c) are paralleled to the filter itself, what will be the overall effect on the frequency characteristic at the load; that is, find $\alpha + j\beta$ where $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$. The networks in Figure 7 are redrawn and paralleled in Figure 8.

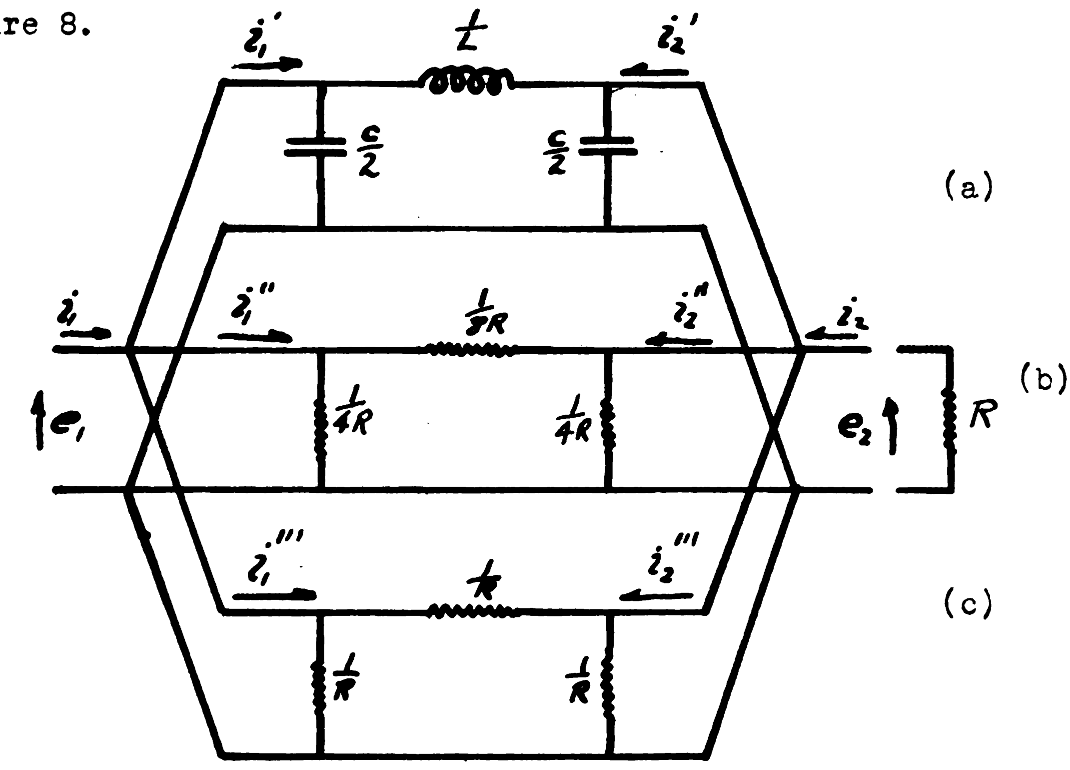


Figure 8

Part (b) of Figure 8 requires some explanation. The resistances in the lower branches were removed and placed in their respective upper branches. This is allowable since the external effects are unchanged. Then the T→π transformation was applied giving the network shown. The π form is desirable over the T because it lends itself

to the node method readily. It is possible to parallel these networks without ideal transformers since the lower conductor of all three networks contains no admittance; and therefore, the potential differences previously mentioned are zero. This was another reason for changing the network of Figure 7(b).

Figure 8(a) $\begin{cases} i_1' = \left(\dfrac{pc}{2} + \dfrac{1}{pL}\right) e_1 - \dfrac{1}{pL} e_2 \\ i_2' = -\dfrac{1}{pL} e_1 + \left(\dfrac{1}{pL} + \dfrac{pc}{2}\right) e_2 \end{cases}$ $\quad p = j\omega$

Figure 8(b) $\begin{cases} i_1'' = \left(\dfrac{1}{4R} + \dfrac{1}{8R}\right) e_1 - \dfrac{1}{8R} e_2 \\ i_2'' = -\dfrac{1}{8R} e_1 + \left(\dfrac{1}{4R} + \dfrac{1}{8R}\right) e_2 \end{cases}$

Figure 8(c) $\begin{cases} i_1''' = \left(\dfrac{1}{R} + \dfrac{1}{R}\right) e_1 - \dfrac{1}{R} e_2 \\ i_2''' = -\dfrac{1}{R} e_1 + \left(\dfrac{1}{R} + \dfrac{1}{R}\right) e_2 \end{cases}$

$$[I]' = [Y]' \times [E], \quad [I]'' = [Y]'' \times [E], \quad [Y]''' \times [E] = [I]'''$$

And $[I] = [I]' + [I]'' + [I]''' = ([Y]' + [Y]'' + [Y]''') \times [E]$

Therefore

$$(20) \qquad \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \left(pc + \dfrac{1}{pL} + \dfrac{19}{8R}\right) & -\left(\dfrac{1}{pL} + \dfrac{9}{8R}\right) \\ -\left(\dfrac{1}{pL} + \dfrac{9}{8R}\right) & \left(pc + \dfrac{1}{pL} + \dfrac{19}{8R}\right) \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

And $\quad i_1 = y_{11}e_1 + y_{12}e_2, \quad i_2 = y_{12}e_1 + y_{22}e_2$

With the termination in R, $e_2 = -R\, i_2$, and $i_2 = -\frac{e_2}{R}$

Therefore $-\frac{e_2}{R} = y_{12}e_1 + y_{22}e_2$ and $y_{12}e_1 = -e_2\left(\frac{1}{R} + y_{22}\right)$

Which gives $$\frac{e_1}{e_2} = -\frac{\left(\frac{1}{R} + y_{22}\right)}{y_{12}} = \frac{\left(\frac{1}{R} + pc + \frac{1}{pL} + \frac{19}{8R}\right)}{\left(\frac{1}{pL} + \frac{9}{8R}\right)}$$

And $$\frac{e_1}{e_2} = \frac{\left(\frac{27}{8R} + j\omega\left(c - \frac{1}{\omega^2 L}\right)\right)}{\left(\frac{9}{8R} - j\frac{1}{\omega L}\right)} = \epsilon^{\alpha + j\beta}$$

Therefore $$\epsilon^{\alpha+j\beta} = \sqrt{\frac{\left(\frac{27}{8R}\right)^2 + \omega^2\left(c - \frac{1}{\omega^2 L}\right)^2}{\left(\frac{9}{8R}\right)^2 + \left(\frac{1}{\omega L}\right)^2}}\; \epsilon^{j(\psi_1 - \psi_2)}$$

$$\psi_1 = \tan^{-1} \frac{\omega\left(c - \frac{1}{\omega^2 L}\right)8R}{27}$$

$$\psi_2 = \tan^{-1} \frac{-8R}{9\omega L}$$

$$\alpha = \ln \sqrt{\frac{\left(\frac{27}{8R}\right)^2 + \omega^2\left(c - \frac{1}{\omega^2 L}\right)^2}{\left(\frac{9}{8R}\right)^2 + \left(\frac{1}{\omega L}\right)^2}} \quad \text{And } \beta = \psi_1 - \psi_2$$

Where α is the attenuation in nepers and β is the phase shift, both being functions of frequency.

From the technique used, it is necessary that the networks (a), (b), and (c) be connected as shown in Figure 8 to give the same results as this analysis. This means that Figure 8(b) must be connected as shown, or as an equivalent T with no impedances in the lower branches.

It is the ability of matrices to maintain the continuity of the problem that lends itself so nicely to this solution.

Series Networks

When two or more networks are to be placed in series, Figure 9, equations of the type given by (12) would be used.

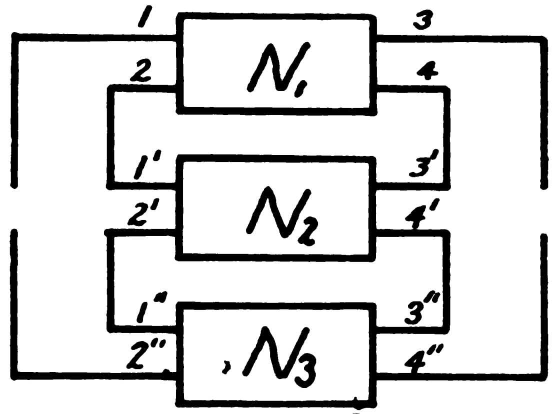


Figure 9

$[E]_1 = [Z]_1 \times [I]_1$, $[E]_2 = [Z]_2 \times [I]_2$, and $[E]_3 = [Z]_3 \times [I]_3$, but $[I] = [I]_1 = [I]_2 = [I]_3$ and $[E] = [E]_1 + [E]_2 + [E]_3$ provided that the precautions referred to in the previous section are taken; that is, the current in at terminal 1 is equal to the current out at terminal 2 etc. Under these conditions $[E] = ([Z]_1 + [Z]_2 + [Z]_3) \times [I]$.

Matrices apply themselves to problems involving series connections just as nicely as they do to problems involving cascade and parallel connections.

A good illustration would be an extension of the problem of the previous section. Referring to Figure 7(a) suppose the filter network was matched to the generator originally; that is, between the filter section and the generator there is a resistance R equal to $\sqrt{L/C}$. We have already determined the effect of paralleling the networks Figure 7(b) and (c) to the filter section itself and then connecting them to e_g; we would now like to know what the overall attenuation would be if the resistance R was placed between the combination network and the generator. Naturally, the whole problem could be redone, but using the series connection technique we can utilize most of the previous work. The procedure can best be illustrated by referring to Figure 10. The step from Figure 10(a) to (b)

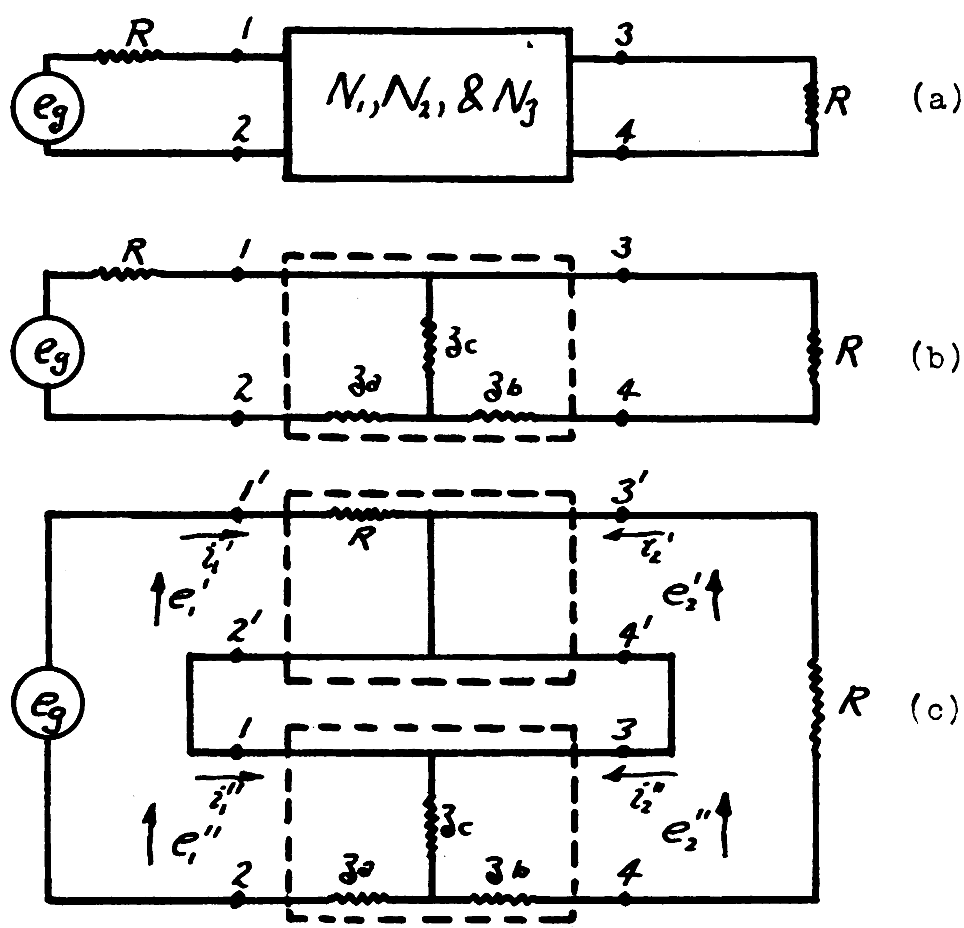


Figure 10

is possible because all four terminal, bilateral, π networks have an equivalent T. Figure 10(c) is electrically the same as Figure 10(b) and if terminals 3 and 4' were disconnected there would be no potential difference between them; therefore, (a), (b), and (c) are equivalent networks and the matrix method may be used.

Solution:

Figure 10(c) (top) $\quad e_1' = Ri_1' + 0i_2' \quad e_2' = 0i_1' + 0i_2' \quad$ and $\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} x \begin{bmatrix} i_1' \\ i_2' \end{bmatrix}$

Equations (20) may be used for Figure 10(c) (bottom)

$$\begin{bmatrix} i_1'' \\ i_2'' \end{bmatrix} = \begin{bmatrix} (pc + \frac{1}{pL} + \frac{19}{8R}) & -(\frac{1}{pL} + \frac{9}{8R}) \\ -(\frac{1}{pL} + \frac{9}{8R}) & (pc + \frac{1}{pL} + \frac{19}{8R}) \end{bmatrix} x \begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix}$$

$$\begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix} = \begin{bmatrix} (pc + \frac{1}{pL} + \frac{19}{8R}) & -(\frac{1}{pL} + \frac{9}{8R}) \\ -(\frac{1}{pL} + \frac{9}{8R}) & (pc + \frac{1}{pL} + \frac{19}{8R}) \end{bmatrix}^{-1} x \begin{bmatrix} i_1'' \\ i_2'' \end{bmatrix}$$

and, therefore, our equations for the two subnetworks are:

$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} x \begin{bmatrix} i_1' \\ i_2' \end{bmatrix}$$

And

$$\begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} x \begin{bmatrix} i_1'' \\ i_2'' \end{bmatrix}$$

Where $z_{11} = z_{22} = \dfrac{2.375\,R + j\omega(L - \frac{1}{\omega C})}{6.375\,R - \omega^2 LC + j2.5\omega R\,(1.9c - \frac{1}{\omega^2 L})}$

And $\quad z_{12} = z_{21} = \dfrac{1.125\ R - j\ \frac{1}{\omega C}}{6.375 - \omega^2 LC + j2.5\omega R(1.9C - \frac{1}{\omega^2 L})}$

Adding the two above equations gives:

$$\begin{bmatrix} e_1' + e_1'' \\ e_2' + e_2'' \end{bmatrix} = \begin{bmatrix} (z_{11} + R) & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

But $e_1' + e_1'' = e_g$ and $e_2' = 0$, also, $e_2'' = e_2$

Therefore $\quad e_g = (z_{11} + R)i_1 + z_{12}i_2$

$$e_2 = z_{21}i_1 + z_{22}i_2$$

Since $\quad e_2 = -Ri_2 \quad$ or $\quad i_2 = -\dfrac{e_2}{R}$

And $\quad e_2 = z_{21}i_1 - \dfrac{z_{22}e_2}{R} \qquad$ And $\quad i_1 = e_2\left(\dfrac{R + z_{22}}{z_{21}R}\right)$

$$e_g = \frac{(z_{11} + R)(z_{22} + R)}{z_{21}R} - \frac{z_{12}}{R}\ e_2$$

$$\frac{e_g}{e_2} = \frac{(z_{11} + R)(z_{22} + R)}{z_{21}\ R} - \frac{z_{12}}{R}$$

$$\frac{e_g}{e_2} = \frac{z_{11}z_{22} + Rz_{11} + Rz_{22} + R^2 - z_{12}^2}{z_{21}R}$$

$$\frac{e_g}{e_2} = \frac{z_{11}(z_{11} + 2R) + R^2 - z_{12}^2}{z_{12}R}$$

And $\quad \dfrac{e_g}{e_2} = \dfrac{x + jy}{u + jv}$

Where $x = 118\,R^2 + \omega^4 L^3 C - 50.56\,\omega^2 L^2 - \dfrac{13.25}{\omega^2 C^2}$

$y = 43.66\,R\omega L - 63.5\dfrac{R}{\omega C} - 11.5\,\omega^3 R\,L^2 C$

$u = 11.9\,R^2 - 1.125\,\omega^2 R^2 LC - 2.5\,\dfrac{R^2}{\omega^2 LC}$

$v = 6.34\,\omega RL - 9.18\dfrac{R}{\omega C}$

But $\dfrac{e_g}{e_2} = \epsilon^{\alpha + j\beta}$

Therefore

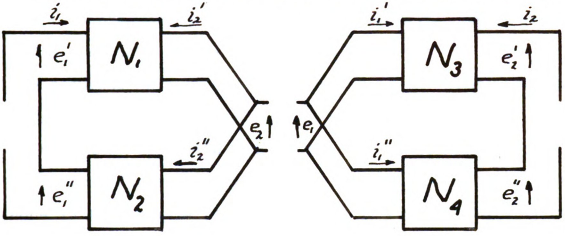
$$\alpha + j\beta = \ln\left(\frac{x + jy}{u + jv}\right) = \frac{1}{2}\ln\left(\frac{x^2 + y^2}{u^2 + v^2}\right) + j\left(\tan^{-1}\frac{y}{x} - \tan^{-1}\frac{v}{u}\right)$$

And $\alpha = \frac{1}{2}\ln\left(\dfrac{x^2 + y^2}{u^2 + v^2}\right)$, and $\beta = \tan^{-1}\dfrac{y}{x} - \tan^{-1}\dfrac{v}{u}$

Although the solution to this problem is not a simple one, it wouldn't be less complicated if it were worked by some other means. It is doubtful if there is a more direct route to the solution than that afforded by the use of matrices as illustrated above.

Series Parallel and Parallel Series Networks

Sometimes it is desirable to connect the networks in parallel on the left and in series on the right or vice versa. Or, if a network is given, it is sometimes desirable to picture it in this way to facilitate the solution. Equations (14) would be used for a series-parallel connection and equations (13) would be used for a parallel-series connection. Referring to Figure 11(a), the equations for



(a) (b)

Figure 11

N_1 are: $$\begin{bmatrix} e_1 \\ i_2' \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} \times \begin{bmatrix} i_1 \\ e_2 \end{bmatrix}$$

And for N_2: $$\begin{bmatrix} e_1'' \\ i_2'' \end{bmatrix} = \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} \times \begin{bmatrix} i_1 \\ e_2 \end{bmatrix}$$

Adding the two sets of equations gives:

$$\begin{bmatrix} e_1' + e_1'' \\ i_2' + i_2'' \end{bmatrix} = \begin{bmatrix} (h_{11}' + h_{11}'') & (h_{12}' + h_{12}'') \\ (h_{21}' + h_{21}'') & (h_{22}' + h_{22}'') \end{bmatrix} \times \begin{bmatrix} i_1 \\ e_2 \end{bmatrix}$$

And the resulting equations for Figure 11(b) would be:

$$\begin{bmatrix} i_1' + i_1'' \\ e_2' + e_2'' \end{bmatrix} = \begin{bmatrix} (g_{11}' + g_{11}'') & (g_{12}' + g_{12}'') \\ (g_{21}' + g_{21}'') & (g_{22}' + g_{22}'') \end{bmatrix} \times \begin{bmatrix} e_1 \\ i_2 \end{bmatrix}$$

When applying matrices to these types of networks, the same precautions must be heeded as for the parallel and series connections of the last two sections. After the connections have been made on one side of the networks, there must be no potential differences between the terminals to be connected on the other side. If there is, then an ideal transformer must be used or other steps taken, as was done in the example used to illustrate the parallel connection. This cannot be over emphasized. The use of matrices will give erroneous results if the current into a network, on a given side, is not equal to the current out on that same side.

The following example will be used to illustrate how matrices may be applied to the series-parallel connection.

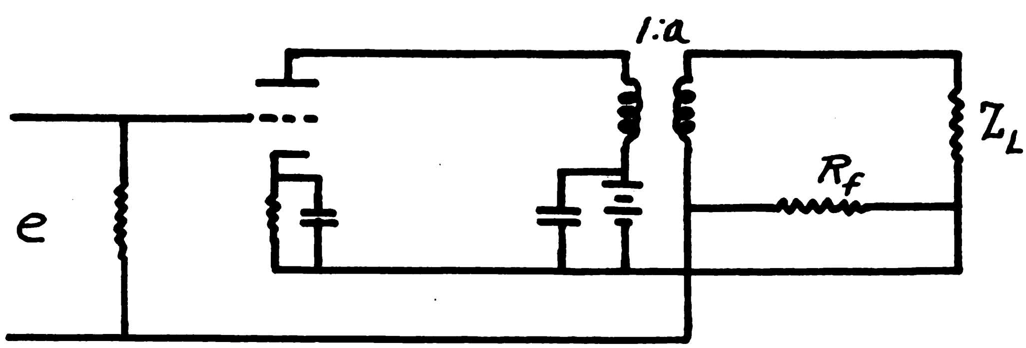


Figure 12

In Figure 12 is a diagram of a single stage triode amplifier, with negative feedback, coupled to its load, Z_L, through a transformer. The problem is to find the gain of the overall network.

Before starting the problem, however, something should be said about the application of matrices to tube circuits in general. As a rule it is possible to obtain a $[Z]$ or a $[Y]$ for a given tube circuit. Once this is done the other forms (page 10) may be obtained, and the operations of cascading, paralleling, etc. may be carried out providing the previously outlined rules are not broken.[5] However, if the analysis is to be on the basis of an equivalent circuit, the use of matrices is not a short-cut. To find the $[Z]$ and $[Y]$ is somewhat of an ordeal compared to the techniques for solution found in any text on tube circuits.[6] If the solution is to take into consideration the interelectrode capacitances, and is to cover the entire frequency range, say for a square wave input, then the authors Gardner and Barnes outline a nice technique[7] using the laplace transform. The following solution will illustrate why the use of matrices is not always the best procedure for a unilateral network.

The equivalent circuit of Figure 12 is given in Figure 13. Matrices may be used since the transformer insures that the current into either network will equal the current out

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[5] Ibid., p. 148.

[6] S. Seely, Electron-tube Circuits, pp.85-86.

[7] M. F. Gardner and J. L. Barnes, Transients in Linear Systems, pp. 180-186.

on a given side. For the bottom box of Figure 13(b), the equations are:

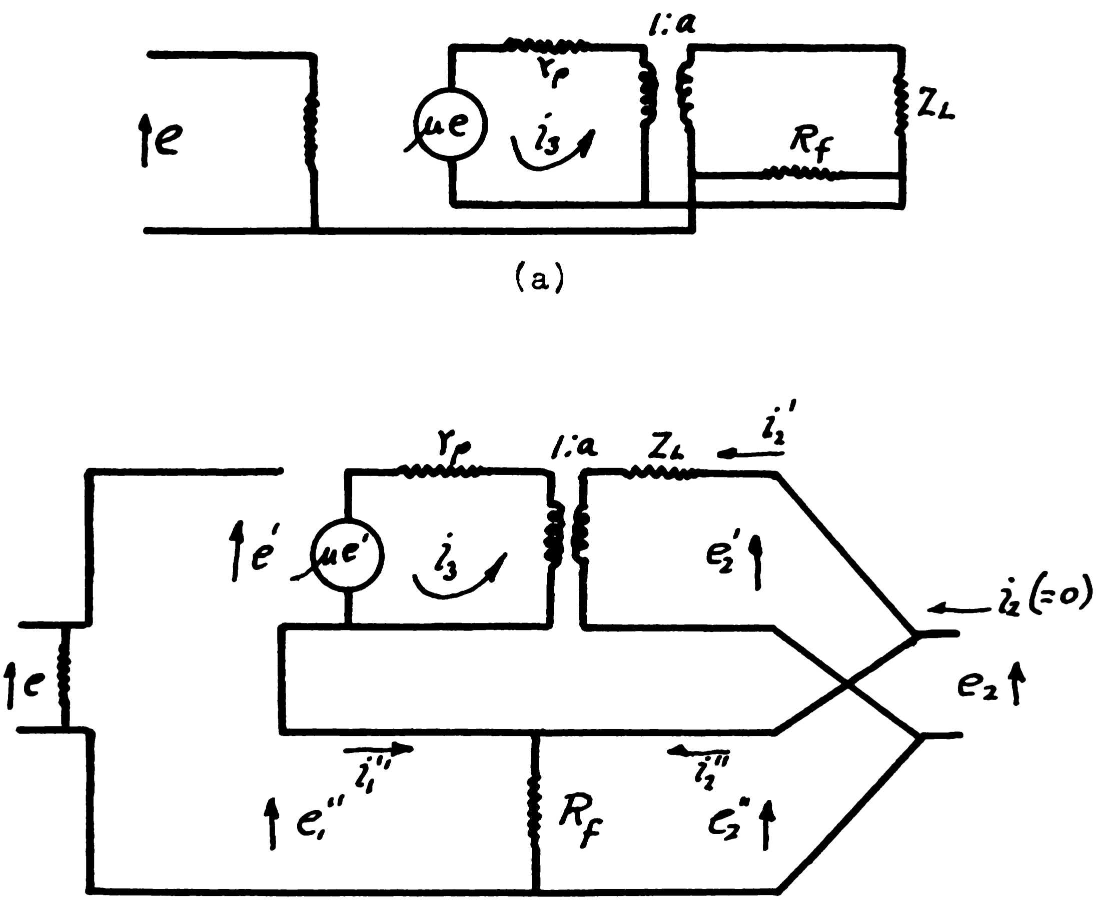
$$e_1'' = R_f i_2'' + 0 i_1''$$
$$e_2'' = R_f i_2'' + 0 i_1''$$

And

$$e_1'' = 0 i_1'' + e_2''$$
$$i_2'' = 0 i_1'' + (1/R_f) e_2''$$

Therefore

$$\begin{bmatrix} e_1'' \\ i_2'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{R_f} \end{bmatrix} \times \begin{bmatrix} i_1'' \\ e_2'' \end{bmatrix}$$



(a)

(b)

Figure 13

But the equations for the top box are not so easily obtained. They may be found as follows:

$$a(\mu e_1' - i_3 r_p) = e_2' - Z_L i_2'$$

And $$\mu e_1' = \frac{e_2'}{a} - \frac{Z_L i_2'}{a} + i_3 r_p$$

Since, in tube circuit analysis of this kind it is usually assumed that the transformer is ideal, $i_3 = -a i_2'$.

Therefore $$\mu e_1' = \frac{e_2'}{a} - \frac{Z_L}{a} i_2' - r_p a i_2'$$

Or $$e_1' = \frac{e_2'}{\mu a} - \left(\frac{Z_L + r_p a^2}{\mu a}\right) i_2'$$

But $$e_2' = e_2'' = e$$

And $$i_2 = i_2' + i_2'' = 0$$

Therefore $i_2' = -i_2''$ and $R_f i_2'' = e_2'' = e_2'$

And therefore $e_2' = -R_f i_2'$ and $i_2' = \dfrac{-e_2'}{R_f}$

This gives $$e_1' = \frac{e_2'}{\mu a} + \left(\frac{Z_L + r_p a^2}{\mu a}\right) \frac{e_2'}{R_f}$$

Therefore $$e_1' = e_2' \left(\frac{R_f + Z_L + r_p a^2}{\mu a R_f}\right)$$

Or $$e_1' = 0 i_1' + \left(\frac{R_f + Z_L + r_p a^2}{\mu a R_f}\right) e_2'$$

$$i_2' = 0 i_1' - \frac{1}{R_f} e_2'$$

The two matrices to be used, therefore, are:

$$\begin{bmatrix} e_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{R_f + Z_L + r_p a^2}{\mu a' R_f} \\ 0 & -1/R_f \end{bmatrix} \times \begin{bmatrix} i_1' \\ e_2' \end{bmatrix}$$

And $$\begin{bmatrix} e_1'' \\ i_1'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1/R_f \end{bmatrix} \times \begin{bmatrix} i_1'' \\ e_2'' \end{bmatrix}$$

Since $e_1' + e_1'' = e$, and $i_2' + i_2'' = 0$, adding the two matrices gives:

$$\begin{bmatrix} e \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{R_f + Z_L + r_p a^2}{\mu a R_f} + 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} i_1 \\ e_2 \end{bmatrix}$$

Therefore $$e = \frac{e_2}{R_f} \left(\frac{aR_f + R_f + Z_L + r_p a^2}{\mu a} \right) \times \frac{Z_L}{Z_L}$$

But $$\frac{e_2}{R_f} = -i_2'$$

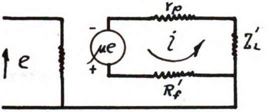
And therefore $$e = -i_2 Z_L \left(\frac{aR_f - R_f + Z_L + r_p a^2}{\mu a Z_L} \right)$$

And gain $$\frac{i_2' Z_L}{e} = \frac{\mu a Z_L}{\mu a R_f + R_f + Z_L + r_p a^2}$$

Or gain =

$$\frac{\mu a \dfrac{Z_L}{a^2}}{\mu a \dfrac{R_f}{a^2} + \dfrac{R_f}{a^2} + \dfrac{Z_L}{a^2} + r_p} = \frac{\mu a Z_L'}{r_p + (\mu a + 1) R_p' + Z_L'}$$

The usual technique would be as follows:[8]



$$Z_L' = \frac{Z_L}{a^2}$$

$$R_f' = \frac{R_f}{a^2}$$

Where Z_L' and R_f' are Z_L and R_f reflected back across the ideal transformer.

$$\mu e_1 = (r_p + Z_L' + R_f')i$$

$$e_1 = \frac{(r_p + Z_L' + R_f')\,i}{\mu} = e - iaR_f'$$

Therefore $$e = \left(\frac{r_p + Z_L' + R_f'}{\mu} + aR_f'\right)i$$

$$e = \frac{r_p + Z_L' + R_f' + \quad aR_f')}{\mu}\;\frac{iZ_L'}{Z_L'} \times \frac{a}{a}$$

And Therefore $$\frac{iZ_L'a}{e} = \frac{\mu a Z_L'}{r_p + (\mu a + 1)R_f' + Z_L'}$$

An even simpler technique is as follows:

$$K_T = \frac{K}{1-\beta K}$$ where K = nominal gain

β = feedback ratio

$$K = \frac{iaZ_L'}{e_1}$$ and $$i = \frac{\mu e_1}{r_p + R_f' + Z_L'}$$

---

[8]Seely, op. cit., pp. 85-86.

Therefore $K = \dfrac{\mu a Z_L'}{r_p + R_f' + Z_L'}$ and $\beta = \dfrac{-R_f}{Z_L} = \dfrac{-R_f'}{Z_L'}$

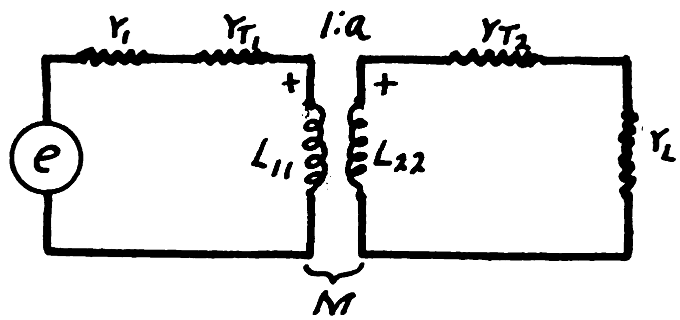
And Therefore $K_T = \dfrac{\mu a Z_L'}{r_p + (\mu a + 1)R_f' + Z_L'}$

The purpose of this illustration is not to discredit the use of matrices for the series-parallel connection, but to point out the difficulty of using matrices when a unilateral circuit is involved, and still have an example illustrating the technique of visualizing a network as being composed of two networks connected in series on one end and in parallel on the other.

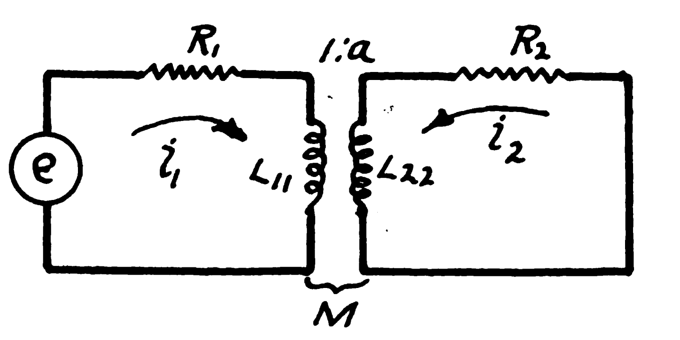
Transformer Analysis

An ideal transformer has been referred to throughout this chapter. In analyzing the feedback amplifier, an ideal transformer was assumed and in forcing the current into a given network to be equal to the current out on a given end, it was suggested that an ideal transformer might be used. An ideal transformer is one that has neither leakage nor losses and has infinite inductances on the primary and secondary sides. It is therefore impossible to attain. The losses can usually be kept within reason, but in air core transformers or when the ratio of an iron core transformer is high, it is difficult to keep the flux leakage low.

In network analysis or synthesis involving transformers, the work is considerably simpler if an ideal transformer can be assumed. In this section, therefore, we will determine, with the use of matrices, the conditions when an ideal transformer may be assumed and will derive an equivalent circuit for the case when an ideal transformer may not be assumed. Figure 14(a) is a typical transformer circuit and its equivalent is Figure 14(b). This second circuit will be analyzed to determine when the assumption of an ideal transformer is practical.



(a)



(b)

Figure 14

$$e = (R_1 + j\omega L_{11})i_1 + j\omega M i_2$$

$$0 = j\omega M i_1 + (R_2 + j\omega L_{22})i_2$$

$$\begin{bmatrix} e \\ 0 \end{bmatrix} = \begin{bmatrix} (R_1 + j\omega L_{11}) & j\omega M \\ j\omega M & (R_2 + j\omega L_{22}) \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} (R_1 + j\omega L_{11}) & j\omega M \\ j\omega M & (R_2 + j\omega L_{22}) \end{bmatrix}^{-1} \times \begin{bmatrix} e \\ 0 \end{bmatrix}$$

Therefore $i_1 = \dfrac{e(R_2 + j\omega L_{22})}{(R_1 + j\omega L_{11})(R_2 + j\omega L_{22}) - (j\omega M)^2}$

And $\dfrac{e}{i_1} = Z = \dfrac{R_1R_2 + j\omega(R_1L_2 + R_2L_1) - \omega^2(L_{11}L_{22} - M^2)}{R_2 + j\omega L_{22}}$

If $(R_1L_{22} + R_{22}L_{11}) \gg R_1R_2 - \omega^2(L_{11}L_{22} - M^2)$

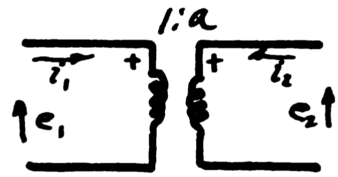
And if $\omega L_{22} \gg R_2$ and $\omega L_{11} \gg R_1$

Then $Z = \dfrac{(R_1L_{22} + R_2L_{11})}{L_{22}} = R_1 + R_2 \dfrac{L_{11}}{L_{22}}$

But $\dfrac{L_{11}}{1} = \dfrac{L_{22}}{a^2}$ And $\dfrac{L_{11}}{L_{22}} = \dfrac{1}{a^2}$

Which gives $Z = R_1 + \dfrac{R_2}{a^2} = R_1 + R_2'$

But R_2' is simply the secondary load with losses reflected across the transformer. If the above underlined conditions exist, as they often do with iron core transformers, then the transformer of Figure 14 may be represented as in the following diagram.



$$e_1 = \frac{e_2}{a} + 0i_2 \qquad \begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & a \end{bmatrix} \times \begin{bmatrix} e_2 \\ -i_2 \end{bmatrix}$$

$$i_1 = 0e_2 - ai_2$$

When the leakage is not neglegible; that is, $R_1R_2 - \omega^2(L_{11}L_{22} - M^2)$ is not $\ll \omega(R_1L_{22} + R_2L_{11})$ then an equivalent circuit for the transformer is useful.

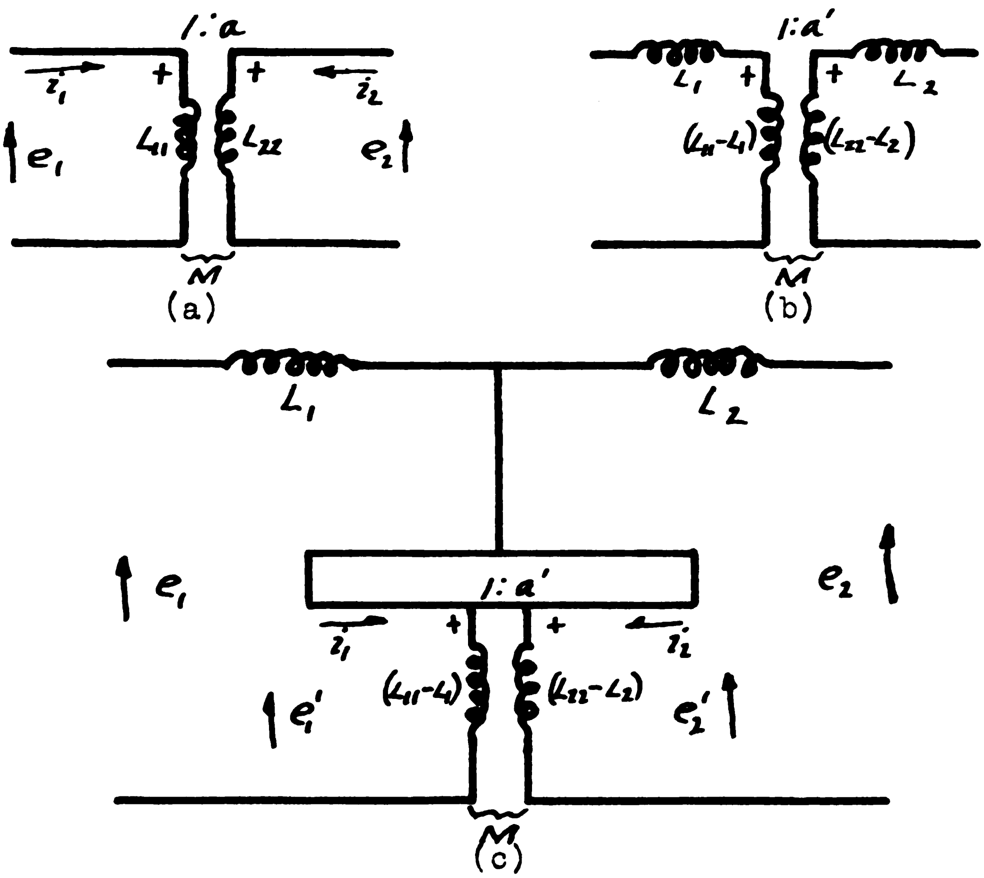


Figure 15

The voltage equations for Figure 15(a) are:

$$e_1 = j\omega L_{11}i_1 + j\omega M i_2 \qquad \text{and} \qquad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = j\omega \begin{bmatrix} L_{11} & M \\ M & L_{22} \end{bmatrix} x \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$
$$e_2 = j\omega M i_1 + j\omega L_{22}i_2$$

Therefore $[L] = \begin{bmatrix} L_{11} & M \\ M & L_{22} \end{bmatrix} = \begin{bmatrix} (L_{11}-L_1) & M \\ M & (L_{22}-L_2) \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$

And $\quad [L] = [L]' + [L]''$

Figure 15(b), therefore, is the equivalent of Figure 15(a).

$$a^2 = \frac{L_{22}}{L_{11}} \quad , \text{ and } k = \frac{M}{\sqrt{L_{11}L_{22}}}$$

For the leakage to be zero, k must equal 1, that is, $L_{11}L_{22} - M^2 = 0$, $(|L| = 0)$. But the reason for this analysis is because $|L| \neq 0$. By choosing L_1, and L_2, however, $|L'|$ can be forced to zero giving $(L_{11}-L_1)x(L_{22}-L_2) - M^2 = 0$. k' for the transformer in Figure 15(b) is, therefore, 1, and

$$(21) \qquad a' = \sqrt{\frac{(L_{22} - L_2)}{(L_{11} - L_1)}} = \frac{M}{L_{11}-L_1} = \frac{L_{22} - L_2}{M} .$$

Figure 15(b) is verified.

Since $[L] = [L]' + [L]''$, two impedance matrices are being added and, therefore, two voltage matrices are being added. The indication is that two networks are being placed in series and, therefore, Figure 15(c) is the equivalent of (a) and (b). The next step is to analyze the bottom box of Figure 15(c), the equations of which are:

$$(22)\qquad \begin{aligned} e_1' &= j\omega\{(L_{11}-L_1)i_1 + Mi_2\} \\ e_2' &= j\omega\{Mi_1 + (L_{22}-L_2)i_1\} \end{aligned}$$

And
$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = j\omega \begin{bmatrix} (L_{11}-L_1) & M \\ M & (L_{22}-L_2) \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

But $M = a'(L_{11}-L_1)$, and $(L_{22}-L_2) = a'^2\,(L_{11}-L_1)$

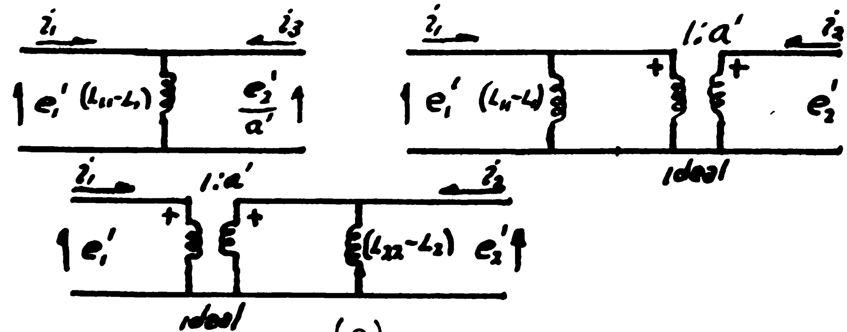
Therefore
$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = j\omega \begin{bmatrix} (L_{11}-L_1) & a'i_2(L_{11}-L_1) \\ a'(L_{11}-L_1) & a'^2(L_{11}-L_1) \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

And
$$\begin{aligned} e_1' &= j\omega\{(L_{11}-L_1)i_1 + a'i_2(L_{11}-L_1)\} \\ e_2' &= j\omega\{a'(L_{11}-L_1)i_1 + a'^2(L_{11}-L_1)i_1\} = a'e_1' \end{aligned}$$

And since $k' = 1$, $a'i_2 = i_3$

$$(23)\qquad \begin{aligned} e_1' &= j\omega(L_{11}-L_1)i_1 + j\omega(L_{11}-L_1)i_3 \\ e_2' &= a'e_1 \end{aligned}$$

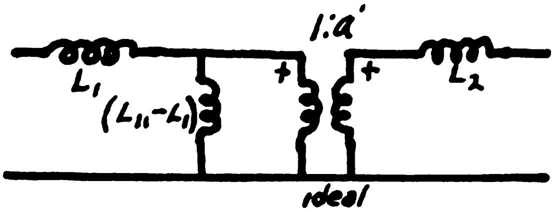
Equations (23) indicate that the bottom box of Figure 15(c) could be redrawn as shown in Figure 16(a) or (b). Figure 16(c) could be obtained from equations (22) in the same manner.



(a) (b) (c)

Figure 16

By substituting Figure 16(b) for the bottom box of Figure 15(c) the following equivalent circuit is obtained.



But since k' = 1 and, therefore, $(L_{11}-L_1)(L_{22}-L_2) - M^2 = 0$

Then $L_{11}L_{22}-L_1L_{22}-L_2L_{11}+L_1L_2-M^2 = 0$

And $L_1 = \dfrac{|L| - L_2L_{11}}{L_{11}-L_1}$ also $L_2 = \dfrac{|L| - L_1L_{22}}{L_{11}-L_1}$

This means that either L_1 or L_2 is arbitrary. To simplify the equivalent circuit, choose $L_2 = 0$.

Therefore $L_1 = \dfrac{|L|}{L_{22}} = \dfrac{L_{11}L_{22} - M^2}{L_{22}} = L_{11} - \dfrac{M^2}{L_{22}}$

But $K^2 = \dfrac{M^2}{L_{11}L_{22}}$ and therefore $L_1 = L_{11}(1-K^2)$

And $a' = \sqrt{\dfrac{L_{22} - L_2}{L_{11}-L_1}} = \sqrt{\dfrac{L_{22}}{\cancel{L_{11}}-\cancel{L_{11}}+L_{11}K^2}} = \dfrac{1}{K}\sqrt{\dfrac{L_{22}}{L_{11}}} = \dfrac{a}{K}$

From the previous analysis a very useful equivalent circuit is given in Figure 17.

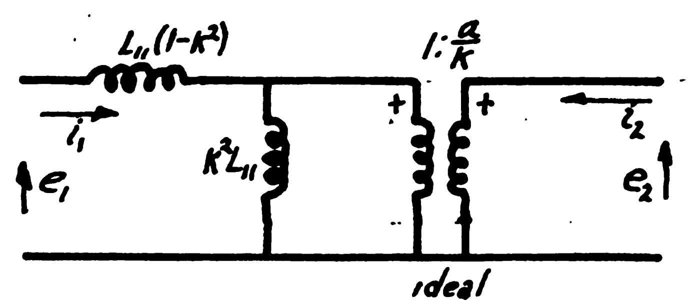


Figure 17

In most cases the inductance $L_{11}(1-K^2)$ can be assimulated into another inductance of the network. However, if it cannot be taken care of in this fashion, the overall equations can be obtained readily by the method of cascading, giving:

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{aK} & j\omega a L_{11}(1-K^2) \\ \frac{1}{j\omega a K L_{11}} & \frac{a}{K} \end{bmatrix} \times \begin{bmatrix} e_2 \\ -i_2 \end{bmatrix}$$

Where L_{11} = primary inductance

L_{22} = secondary inductance

M = mutual inductance

$$a = \sqrt{\frac{L_{22}}{L_{11}}} \qquad K = \frac{M}{\sqrt{L_{11}L_{22}}}$$

Equivalence of the T, π, Bridged T, and Symmetrical Lattice Structures

The $T \rightarrow \pi$ and $\pi \rightarrow T$ transformations are common knowledge to all electrical engineers and the transformation equations can be found in several texts. With the use of matrices, the derivation of these equations is straightforward and concise. Equations (24a) refer to Figure 18(a) and equations (24b) refer to Figure 18(b).

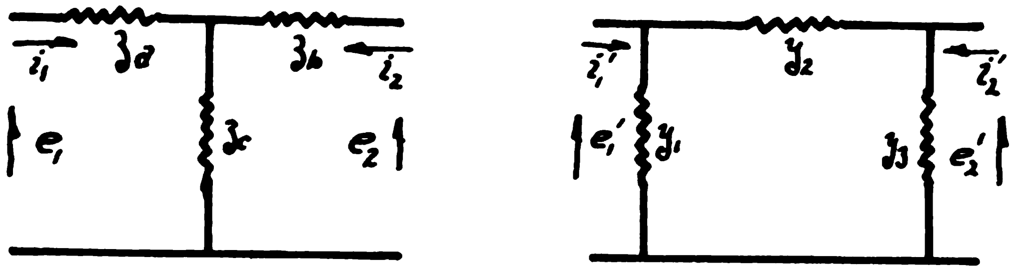


Figure 18

(24a) $$e_1 = (z_a + z_c)i_1 + z_c i_2 \qquad e_2 = z_c i_2 + (z_b + z_c)i_2 \quad \text{and} \quad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Where $z_{11} = z_a + z_c$, $z_{22} = z_b + z_c$, and $z_{12} = z_{21} = z_c$

(24b) $$i_1' = (y_1 + y_2)e_1' - y_2 e_2' \qquad i_2' = -y_2 e_1' + (y_2 + y_3)e_2' \quad \text{and} \quad \begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \times \begin{bmatrix} e_1' \\ e_2' \end{bmatrix}$$

Where $y_{11} = y_1 + y_2$, $y_{22} = y_2 + y_3$, and $y_{12} = y_{21} = -y_2$

Therefore $$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = [Y]^{-1} \times \begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} \frac{y_{22}}{|Y|} & \frac{-y_{21}}{|Y|} \\ \frac{-y_{12}}{|Y|} & \frac{y_{11}}{|Y|} \end{bmatrix} \times \begin{bmatrix} i_1' \\ I_2' \end{bmatrix}$$

But since the networks are to be equivalent:

$$e_1 = e_1', \; e_2 = e_2', \; i_1 = i_1', \; i_2 = i_2'$$

Therefore $[Z] = [Y]^{-1}$ and $z_{11} = \frac{y_{22}}{|Y|}$, $z_{22} = \frac{y_{11}}{|Y|}$, $z_{12} = \frac{-y_{21}}{|Y|}$

$$|Y| = y_{11}y_{22} - (y_{12})^2 = (y_1 + y_2)(y_2 + y_3) - y_2^2 = y_1y_2 + y_1y_3 + y_2y_3$$

Let $\Delta = y_1y_2 + y_1y_3 + y_2y_3 = \frac{1}{z_1z_2} + \frac{1}{z_1z_3} + \frac{1}{z_2z_3}$

Therefore $\Delta = \dfrac{z_3 + z_2 + z_1}{z_1 z_2 z_3}$ and $\dfrac{1}{\Delta} = \dfrac{z_1 z_2 z_3}{z_1 + z_2 + z_3}$

Also $z_{11} = z_a + z_c = \dfrac{y_{22}}{\Delta} = \dfrac{y_2 + y_3}{\Delta}$, $z_{22} = z_b + z_c = \dfrac{y_1 + y_2}{\Delta}$

And $z_{12} = z_c = \dfrac{y_2}{\Delta}$

$$z_a = \frac{y_3}{\Delta} = \frac{z_1 z_2 z_3}{z_1 + z_2 + z_3} \times \frac{1}{z_3} = \frac{z_1 z_2}{z_1 + z_2 + z_3}$$

(25) $$z_b = \frac{y_1}{\Delta} = \frac{z_1 z_2 z_3}{z_1 + z_2 + z_3} \times \frac{1}{z_1} = \frac{z_2 z_3}{z_1 + z_2 + z_3}$$

$$z_c = \frac{y_2}{\Delta} = \frac{z_1 z_2 z_3}{z_1 + z_2 + z_3} \times \frac{1}{z_2} = \frac{z_1 z_3}{z_1 + z_2 + z_3}$$

The $T \rightarrow \pi$ transforms are derived below:

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{z_{22}}{|Z|} & \dfrac{-z_{21}}{|Z|} \\ \dfrac{-z_{12}}{|Z|} & \dfrac{z_{11}}{|Z|} \end{bmatrix}$$

Let $\nabla = |Z| = z_{11} z_{22} - z_{12}^2 = (z_a + z_c)(z_b + z_c) - z_c^2 =$

$$z_a z_b + z_a z_c + z_b z_c$$

Therefore $y_{11} = y_1 + y_2 = \dfrac{z_{22}}{\nabla} = \dfrac{z_b + a_c}{\nabla}$, $y_{22} = y_2 + y_3 = \dfrac{z_a + z_c}{\nabla}$

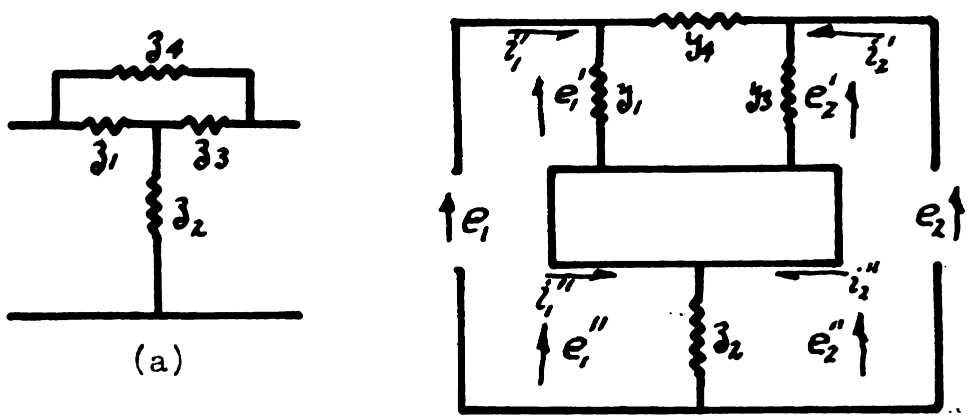
$$-y_{12} = y_2 = \frac{z_{21}}{\nabla} = \frac{z_c}{\nabla}$$

$$y_1 = \frac{z_b}{\nabla} \text{ and } z_1 = \frac{\nabla}{z_b} = \frac{z_a z_b + z_a z_c + z_b z_c}{z_b}$$

$$(26) \quad y_2 = \frac{z_c}{\nabla} \text{ and } z_2 = \frac{\nabla}{z_c} = \frac{z_a z_b + z_a z_c + z_b z_c}{z_c}$$

$$y_3 = \frac{z_a}{\nabla} \text{ and } z_3 = \frac{\nabla}{z_a} = \frac{z_a z_b + z_a z_c + z_b z_c}{z_a}$$

The bridged **T** is a rather common structure, and it will be analyzed with the object of obtaining its equivalent T. The bridged T of Figure 19(a) is redrawn in Figure 19(b), and their equations follow.



(a) (b)

Figure 19

Bottom box Figure 19(b)
$$\begin{cases} e_1'' = z_2 i_1'' + z_2 i_2'' \\ e_2'' = z_2 i_1'' + z_2 i_2'' \end{cases} \text{ and } \begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix} = \begin{bmatrix} z_2 & z_2 \\ z_2 & z_2 \end{bmatrix} \times \begin{bmatrix} i_1'' \\ i_2'' \end{bmatrix}$$

Top box Figure 19(b)
$$\begin{cases} i_1' = (y_1 + y_4) e_1' - y_4 e_2' \\ i_2' = -y_4 e_1' + (y_3 + y_4) e_2' \end{cases} \text{ and } \begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} (y_1 + y_4) & -y_4 \\ -y_4 & (y_3 + y_4) \end{bmatrix} \times \begin{bmatrix} e_1' \\ e_2' \end{bmatrix}$$

Therefore

$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} \frac{y_3 + y_4}{|Y|} & \frac{y_4}{|Y|} \\ \frac{y_4}{|Y|} & \frac{y_1 + y_4}{|Y|} \end{bmatrix} x \begin{bmatrix} i_1' \\ i_2' \end{bmatrix}$$

But since $e_1 = e_1' + e_1''$, $e_2 = e_2' + e_2''$

And $i_1 = i_1' = i_1''$, $i_2 = i_2' = i_2''$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{y_3 + y_4}{|Y|} + z_2 & \frac{y_4}{|Y|} + z_2 \\ \frac{y_4}{|Y|} + z_2 & \frac{y_1 + y_4}{|Y|} + z_2 \end{bmatrix} x \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Referring to Figure 18(a) and equations (24a)

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (z_a + z_c) & z_c \\ z_c & (z_b + z_c) \end{bmatrix} x \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Since the two networks are to be identical

$$z_a + z_c = \frac{y_3 + y_4}{y_1 y_3 + y_1 y_4 + y_3 y_4} + z_2 , \quad z_c = \frac{y_4}{y_1 y_3 + y_1 y_4 + y_3 y_4} + z_2$$

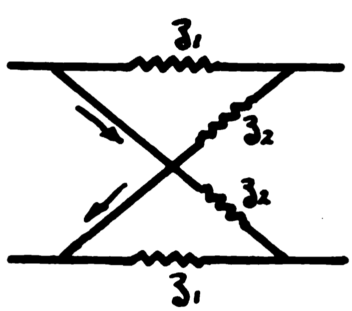
$$z_b + z_c = \frac{y_1 + y_4}{y_1 y_3 + y_1 y_4 + y_3 y_4} + z_2$$

$$(27) \begin{cases} z_a = \dfrac{y_3}{y_1 y_3 + y_1 y_4 + y_3 y_4} = \dfrac{\frac{1}{z_3}}{\frac{z_4 + z_3 + z_1}{z_1 z_3 z_4}} = \dfrac{z_1 z_4}{z_1 + z_3 + z_4} \\ \\ z_b = \dfrac{y_1}{y_1 y_3 + y_1 y_4 + y_3 y_4} = \dfrac{\frac{1}{z_1}}{\frac{z_4 + z_3 + z_1}{z_1 z_3 z_4}} = \dfrac{z_3 z_4}{z_1 + z_3 + z_4} \end{cases}$$

$$(27) \qquad z_c = \frac{y_4}{y_1y_3+y_1y_4+y_3y_4} = \frac{\frac{1}{z_4}}{\frac{z_4+z_3+z_1}{z_1z_3z_4}} = \frac{z_1z_3+(z_1+z_3+z_4)z_2}{z_1+z_3+z_4}$$

Equations(27) determine the relations between the three elements of the equivalent T in terms of the four elements of the bridged T. Using the T$\rightarrow\pi$ transformation, the equivalent π could be obtained from the equivalent T, or the bridged T (Figure 19(a)) could be treated as two networks in parallel and the equivalent π obtained directly.

The symmetrical lattice or bridge, Figure 20(a), is another common structure and may be represented as in



(a)

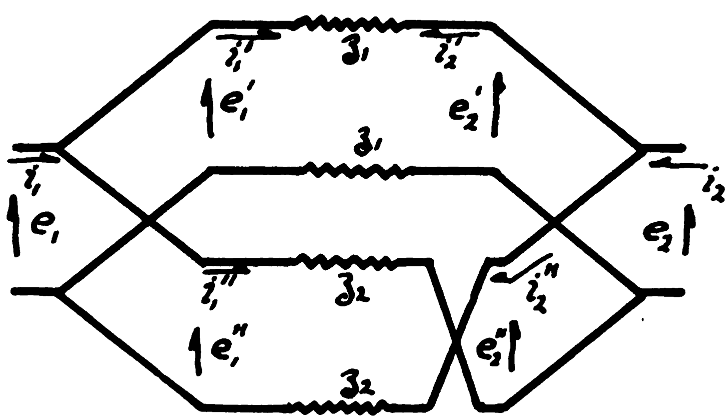


Figure 20 (b)

Figure 20(b). This is possible because the currents indicated by the arrows in Figure 20(a) are equal.

Top box $\quad i_1 = \frac{y_1}{2} e_1' - \frac{y_1}{2} e_2'$

Figure 20(b) $\quad i_2 = \frac{-y_1}{2} e_1' - \frac{y_1}{2} e_2'$

and $\begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} \frac{y_1}{2} & \frac{-y_1}{2} \\ \frac{-y_1}{2} & \frac{y_1}{2} \end{bmatrix} x \begin{bmatrix} e_1' \\ e_2' \end{bmatrix}$

Bottom box Figure 20(b)

$$i_1'' = \frac{y_2}{2}e_1'' + \frac{y_2}{2}e_2''$$
$$i_2'' = \frac{y_2}{2}e_1'' + \frac{y_2}{2}e_2''$$

and

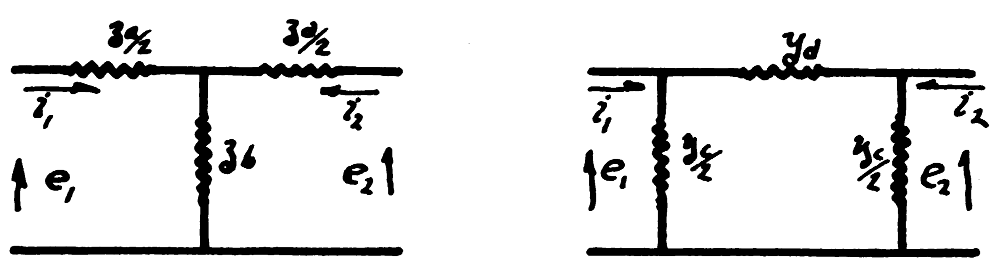
$$\begin{bmatrix} i_1'' \\ i_2'' \end{bmatrix} = \begin{bmatrix} \frac{y_2}{2} & \frac{y_2}{2} \\ \frac{y_2}{2} & \frac{y_2}{2} \end{bmatrix} \times \begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix}$$

Since the two networks are in parallel, $i_1 = i_1' + i_1''$, $i_2 = i_2' + i_2''$, $e_1 = e_1' = e_1''$, and $e_2 = e_2' = e_2''$

Therefore
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{y_1+y_2}{2} & \frac{y_2-y_1}{2} \\ \frac{y_2-y_1}{2} & \frac{y_1+y_2}{2} \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

And
$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{y_1+y_2}{2} & \frac{y_2-y_1}{2} \\ \frac{y_2-y_1}{2} & \frac{y_1+y_2}{2} \end{bmatrix}^{-1} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{z_1+z_2}{2} & \frac{z_2-z_1}{2} \\ \frac{z_2-z_1}{2} & \frac{z_1+z_2}{2} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$



(a) (b)

Figure 21

The equations for the symmetrical T and π are:

Figure 21(a)

$$e_1 = \frac{z_a+2z_b}{2}\, i_1 + z_b i_2$$

$$e_2 = z_b i_1 + \frac{z_a+2z_b}{2}\, i_2$$

And

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{z_a+2z_b}{2} & z_b \\ z_b & \frac{z_a+2z_b}{2} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Figure 21(b)

$$i_1 = \frac{y_c+2y_d}{2}\, e_1 - y_d e_2$$

$$i_2 = -y_d e_1 + \frac{y_c+2y_d}{2}\, e_2$$

And

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{y_c+2y_d}{2} & -y_d \\ -y_d & \frac{y_c+2y_d}{2} \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

For the lattice $\rightarrow$ T transformation, $[Z]_L = [Z]_T$ and for the lattice $\rightarrow \pi$ transformation, $[Y]_L = [Y]_\pi$.

Therefore

$$\frac{z_a}{2} + z_b = \frac{z_1+z_2}{2}, \quad z_b = \frac{z_2-z_1}{2},$$

$$\frac{y_c}{2} + y_d = \frac{y_1+y_2}{2}, \quad y_d = \frac{y_1-y_2}{2}$$

And lattice $\rightarrow$ T gives the following conversions:

$$z_a = 2z_1, \text{ and } z_b = \frac{z_2 - z_1}{2}$$

The T $\rightarrow$ lattice conversions are:

$$z_1 = \frac{z_a}{2}, \text{ and } z_2 = \frac{z_a}{2} + 2z_b$$

(28)
The lattice $\rightarrow \pi$ are:

$$y_c = 2y_2, \text{ and } y_d = \frac{y_1 - y_2}{2}$$

And the $\pi \rightarrow$ lattice are:

$$y_1 = \frac{y_c}{2} + 2y_d, \text{ and } y_2 = \frac{y_c}{2}$$

The above analysis of the T, π, bridged T, and symmetrical lattice networks, with matrices, has established directly a relation between them. Equations(27) show that the bridged T has a unique equivalent T whereas a given T may have several equivalent bridged T's. Also equations (28) show that any symmetrical T or π may be represented as a symmetrical lattice, but that a symmetrical lattice may not always be represented by a T or π (negative impedances).

Three Basic Matrices

A review of this chapter will show that the three basic matrices of four terminal networks are the impedance, admittance, and cascade matrices. Then the

determinant of the impedance matrix is zero, it does not have an equivalent admittance matrix and when the determinant of the admittance matrix is zero, it does not have an equivalent impedance matrix. However, both of these matrices do have an equivalent cascade matrix, but when C of the cascade matrix is zero, no equivalent impedance matrix exists, and when B is zero, no equivalent admittance matrix exists. These three matrices are, therefore, the fundamental matrices, and from them the other three forms can be obtained (equations 13, 14, and 16).

It appears, therefore, that any four terminal network can be handled with the impedance, admittance, and cascade matrices and the matrices obtained through the manipulation of these matrices. It must be kept in mind, however, that to use matrix algebra on a four terminal network, the current in on a given end must equal the current out on that end. This sometimes requires the insertion of an ideal transformer to make the mathematics valid. If the transformer is inserted for purposes of analysis, then it must be present in the actual circuit or the analysis will be false.

The circuit of Figure 22(a) will be used to illustrate the matrix analysis of a four terminal network. This circuit may be redrawn as shown in Figure 22(b). The only difference between the two circuits is that a 1:1 ideal

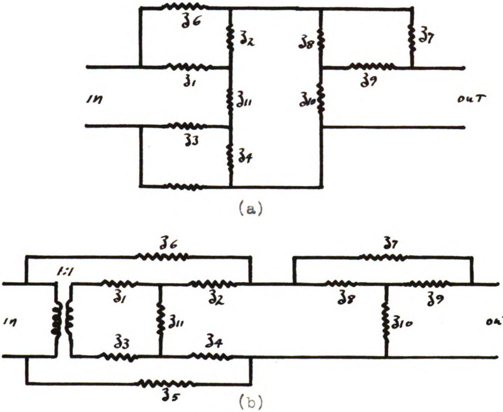


Figure 22

transformer has been inserted in part (b). However, this does alter the network, and, therefore, the transformer must appear in the final circuit. The circuit of Figure 22(b) may be broken up as shown in Figure 23.

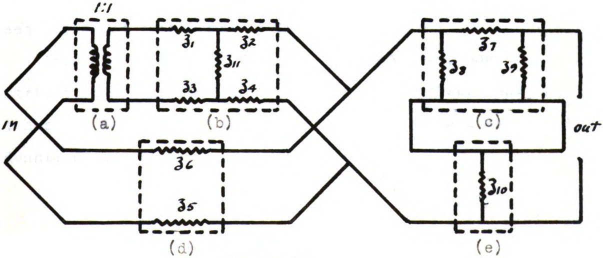


Figure 23

This chapter has shown how the subnetworks of Figure 23 may be combined, with matrices, to give one matrix for the overall network. It may be done in the following steps:

First--Multiply the cascade matrix of (a) times that of (b).
Second--Obtain the equivalent admittance matrix for the (a) and (b) combination.
Third--Add this admittance matrix to the admittance matrix of (d).
Fourth--Obtain the equivalent cascade matrix from the third step for the overall (a), (b), (d) combination.
Fifth--Add the impedance matrices of (c) and (e).
Sixth--From step 5, obtain the equivalent cascade matrices for the (c), (e) combination.
Seventh--Multiply the cascade matrix of the (a) (b) (d) combination times the cascade matrix of the (c) (e) combination.

The last step will give the cascade matrix for the overall network and from it the impedance or admittance matrices may be obtained.

There are a large number of examples in this chapter, illustrating the procedures of cascade, parallel, and series connections, therefore, the solution of this problem has been indicated rather than actually carried out.

This chapter has shown, with illustrations, how matrices may be applied to four terminal networks. And it has pointed out some of the limitations as well as advantages of matrices when so used.

CHAPTER III

TWO TERMINAL NETWORKS

A large number of networks contain only one voltage and may, therefore, be considered as two terminal networks. Matrices, here as always, are a decided factor in maintaining the continuity of the problem. There are many other advantages to be had by using matrices, however. The rest of this chapter will be used to illustrate some of these advantages.

Synthesis of Equivalent Networks

Occasionally it is desirable that the contours of a given network be changed and that the input impedance be maintained invariant. For instance, suppose that $[E] = [Z] \times [I]$ represents the given network and that $[E]' = [Z]' \times [I]'$ represents the desired network, then a transformation matrix $[K]$ may be used to transform $[Z]$ into $[Z]'$ and maintain $z_{input} = e_i/i_i$, invariant. The development is as follows (assuming the voltage to be in mesh 1):

$$\begin{bmatrix} e_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ i_n \end{bmatrix}, \quad [E] = [Z] \times [I]$$

$[E] = [Z] \times [I]$ is the matrix equation of the given network, and $[E]' = [Z]' \times [I]'$ is of the desired network. $[K]$ must transform $[Z]$ to $[Z]'$ and still allow $e_1/i_1 = e_1'/i_1'$ (input impedance invariant). It has been found that $[K]$ may be used in the following manner, $[K]_t \times [Z] \times [K] = [Z]'$. $[K]_t$ premultiplies $[Z]$ for no other reason than that the end result justifies it. Equality must be maintained on both sides of the equation $[E] = [Z] \times [I]$ at all times. So if both sides are premultiplied by $[K]_t$, then $[K]_t \times [E] = [K]_t \times [Z] \times [I]$. Also, $[K] \times [K]^{-1}$ may be inserted between $[Z]$ and $[I]$ since $[K] \times [K]^{-1} = [1]$. Therefore,

$$\underbrace{[K]_t \times [E]}_{[E]'} = \underbrace{[K]_t \times [Z] \times [K]}_{[Z]'} \times \underbrace{[K]^{-1} \times [I]}_{[I]'}$$

Where $[K] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$ and $[K]_t = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \vdots & & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix}$

if $k_{11} = (-1)^{i+j}$ x minor of

And $|K| =$ Determinant of $[K]$

Then $[I]' = \begin{bmatrix} \frac{k_{11}}{|K|} & \frac{k_{21}}{|K|} & \cdots & \frac{k_{n1}}{|K|} \\ \frac{k_{12}}{|K|} & \frac{k_{22}}{|K|} & \cdots & \frac{k_{n2}}{|K|} \\ \cdot & \cdot & & \cdot \\ \frac{k_{1n}}{|K|} & \frac{k_{2n}}{|K|} & & \frac{k_{nn}}{|K|} \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ i_n \end{bmatrix} = \begin{bmatrix} i_1' \\ i_2' \\ \cdot \\ i_n' \end{bmatrix}$

And therefore $i_1' = \frac{k_{11}}{|K|} i_1 + \frac{k_{21}}{|K|} i_2 + \ldots + \frac{k_{n1}}{|K|} i_n$

Also

$$[E]^1 = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & & \alpha_{n2} \\ \vdots & & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix} \times \begin{bmatrix} e_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

And $e_1' = e_1 + 0 + 0$.

But it is necessary that $\frac{e_1}{i_1} = \frac{e_1'}{i_1'}$; that is, the transformation matrix $[K]$ does not change the input impedance. A simple way to satisfy this condition would be to choose $\alpha_{11} = 1$ and $\alpha_{12} = \alpha_{13} = \ldots. = \alpha_{1n} = 0$, and therefore, $k_{21} = k_{31} \ldots. = k_{n1} = 0$ giving $e_1' = e_1$ and $i_1' = i_1$.

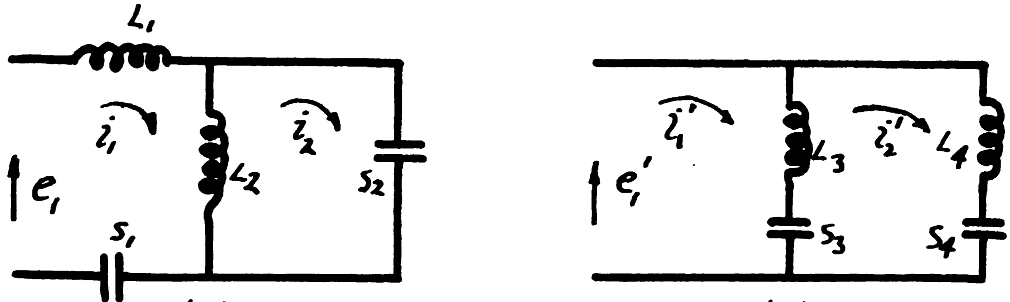
Therefore

$$[K] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$$

The above derivation stipulates that $[K]^{-1}$ exists and, therefore, that $|K| \neq 0$. This requires that $[K]$ be a non-singular matrix and, therefore, must be square.

The next step is to determine the α's. This may best be shown with the following example. Figure 24 (a) is a

$L_1 = 0.05$ H, $S_1 = 40{,}000$ D, $L_2 = 0.10$ H, $S_2 = 100{,}000$ D



(a) (b)

Figure 24

network that has the desired input impedance, but not the desired circuit configuration. The problem is to find the parameters of Figure 24(b) such that $i_1' = i_1$ and $e_1' = e_1$.

Solution:
$$e_1 = \left\{ j\omega(L_1+L_2) + \frac{S_1}{j\omega} \right\} i_1 - (j\omega L_2) i_2$$
$$0 = -(j\omega L_2) i_1 + \left(j\omega L_2 + \frac{S_2}{j\omega}\right) i_2$$

Therefore
$$\begin{bmatrix} e_1 \\ 0 \end{bmatrix} = \left\{ j\omega \begin{bmatrix} (L_1+L_2) & -L_2 \\ -L_2 & L_2 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \right\} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Also
$$e_1' = \left\{ j\omega L_3 + \frac{S_3}{j\omega} \right\} i_1' - \left\{ j\omega L_3 + \frac{S_3}{j\omega} \right\} i_2'$$
$$0 = -\left\{ j\omega L_3 + \frac{S_3}{j\omega} \right\} i_1 + \left\{ j\omega(L_3+L_4) + \frac{(S_3+S_4)}{j\omega} \right\} i_2'$$

If $[K] = \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix}$, then $i_1 = i_1'$, $e_1 = e_1'$, and $z = z'$ (input impedances)

Therefore $\begin{bmatrix} L_3 & -L_3 \\ -L_3 & (L_3+L_4) \end{bmatrix} = 10^{-2} \times \begin{bmatrix} 1 & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix} \times \begin{bmatrix} 15 & -10 \\ -10 & 10 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix} =$

$$10^{-2} \times \begin{bmatrix} (15-20\alpha_1 + 10\alpha_1^2) & (10\alpha_1\alpha_2 - 10\alpha_2) \\ (10\alpha_1\alpha_2 - 10\alpha_2) & (10\alpha_2^2) \end{bmatrix}$$

And $\begin{bmatrix} S_3 & -S_3 \\ -S_3 & (S_3+S_4) \end{bmatrix} = 10^4 \times \begin{bmatrix} 1 & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix} \times \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix} =$

$$10^4 \times \begin{bmatrix} (4 + 10\alpha_1^2) & (10\alpha_1\alpha_2) \\ (10\alpha_1\alpha_2) & (10\alpha_2^2) \end{bmatrix}$$

Since the first row elements of the two matrices on the left, are negatives of each other:

(29) $4 + 10\alpha_1^2 = -10\alpha_1\alpha_2$ and $15-20\alpha_1 + 10\alpha_1^2 = -10\alpha_2(\alpha_1 - 1)$

Solving these simultaneously:

$$\alpha_1 = -0.288 \text{ and } \alpha_2 = 1.376$$

Therefore $L_3 = 10^{-2}(15-20\alpha_1 + 10\alpha_1^2) = 21.589 \times 10^{-2} = 0.216$ henries

$L_4 = 10^{-2}(10\alpha_2^2 - (15-20\alpha_1 + 10\alpha_1^2)) = 0.065$ henries

$S_3 = 10^4(4+10\alpha_1^2) = 10^4(4+.829) = 48{,}290$ darafs

$S_4 = 10^4(10\alpha_2^2 - (4+10\alpha_1^2)) = 232{,}610$ darafs

Foster's Reactance Theorem states that the two networks of Figure 24 are potentially equivalent. By a similar method Cauer's two potentially equivalent networks may be found. The limitations of this technique are in finding

the α's. If resistances were introduced into the two mesh network, there would, in general, be three independent equations of the type(29) to be solved simultaneously for the two α's. Of course, this is not possible. If the solution had been in terms of z's there would have been only one equation and again the α's could not have been found. However, if there had been resistances associated with each inductance and the R/L ratio was constant throughout the network, then a three parameter solution could have been found. For a three mesh network there would have been six α's and for a four mesh, twelve α's. The limitations are, therefore, quite great. For a two mesh, however, this method is faster than Foster's or Cauer's.

Change of Reference Frame

The following technique has been called by Le Corbeiller, The Kron Mesh Method[9], since it has been developed and used extensively by Kron.

In networks containing several meshes and a large number of mutual impedances, it is often times difficult to obtain the mesh impedance matrix $[Z]'$ by writing the maxwell mesh equations. In the simple bridge, for instance,

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[9] Le Corbeiller, _Matrix Analysis of Electric Networks_, pp. 34-44.

there is a possibility of fifteen mutuals. One can visualize the task of even attempting to write the maxwell mesh equations for this case. The Kron Mesh Method eliminates this sort of confusion and allows one to obtain the most complex mesh impedance matrix by a purely mechanical process.

A simple proof of this technique is given by Le Corbeiller, and it takes about ten pages of his book "Matrix Analysis of Electrical Circuits."[10] Since most engineers are interested in method rather than proof, only the method of application will be presented here. It must also be said that this method is not limited to two terminal networks, but that any finite numbers of voltages may be present, providing that they are either all d.c. or all a.c. and of the same frequency.

The method is as follows:

For each branch draw an arrow representing the current in that branch and choose the current direction the same as the voltage in that branch. Next, arbitrarily choose the mesh currents and indicate their directions with arrows. For an n branch network there will be n equations expressing the n branch currents in terms of the mesh currents. From these equations the matrix equation may be written by the usual method, $[I] = [C] \times [I]$.

---

[10] *Loc. Cit.*

$[I]$ is the branch current matrix, $[I]'$ is the mesh current matrix, and $[C]$ is the connection matrix between the two.

If there is a voltage in each branch or only one voltage present in the whole network the branch voltage matrix may be written as $[E]$ with sufficient zeros to give the column matrix n rows (n is number of branches). Each mesh voltage, in general, will contain more than one branch voltage and may be represented as $[E]'$. The "method" then says that $[E]' = [C]_t \times [E]$

The power of the technique will now be presented. The branch impedance matrix $[Z]$ is obtained in a very simple manner. If there are n branches, then $[Z]$ will be square and of order n. It is composed simply by placing the n branch impedances on the principle diagonal and the z_{ij} mutual impedances are placed in the ij positions of the matrix. The maxwell impedance matrix $[Z]'$ is then obtained as $[C]_t \times [Z] \times [C] = [Z]'$.

The mesh equation may now be written as $[E]' = [Z]' \times [I]'$ where $[I]'$ are the unknown maxwell mesh currents. Therefore, $[I]' = [Z]^{-1} \times [E]'$. After $[I]'$ is obtained the branch currents can be obtained from our original equation, $[I] = [C] \times [I]'$.

It must be cautioned, at this point, that $[I] \neq [Z]^{-1} \times [E]$. Recalling how the $[Z]$ matrix (branch) was written, the equation $[I] = [Z]^{-1} \times [E]$ would give branch currents as though each branch impedance and voltage were shorted on themselves. For a two terminal network (one voltage), it is obvious that i_2, i_3...and i_n, are not zero, but $[I] = [Z]^{-1} \times [E]$ would give every current but i_1 as zero. And i_1 would equal e_1/Z_{11} which is incorrect.

Since the circuit in Figure 25 is of a network that is not flat, the mesh equations are difficult to obtain. This circuit will be used to illustrate the Kron Mesh Method.

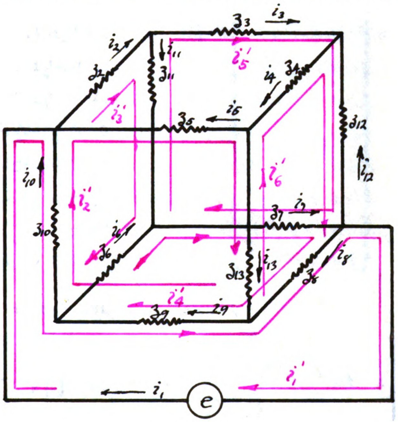


Figure 25

In Figure 25 is the classical cube problem. The problem is to find the input impedance across the diagonal. There are 8 nodes and 1 subnetwork, therefore, 8-1 = 7 independent node pairs. Also, there are 13 branches and, therefore,

13-7 =6 independent meshes. The chosen mesh currents are indicated in red while the branch currents are in black.

$$
\begin{array}{lllllll}
i_1 = & i_1' & 0 & 0 & 0 & 0 & 0 \\
i_2 = & 0 & 0 & i_3' & 0 & 0 & 0 \\
i_3 = & 0 & 0 & 0 & 0 & i_5' & 0 \\
i_4 = & 0 & 0 & 0 & 0 & 0 & -i_6' \\
i_5 = & 0 & -i_2' & 0 & 0 & 0 & 0 \\
i_6 = & 0 & 0 & -i_3' & +i_4' & 0 & 0 \\
i_7 = & 0 & 0 & 0 & i_4' & -i_5' & 0 \\
i_8 = & -i_1' & 0 & 0 & i_4' & 0 & +i_6' \\
i_9 = & -i_1' & +i_2' & 0 & +i_4' & 0 & 0 \\
i_{10} = & -i_1' & +i_2' & +i_3' & 0 & 0 & 0 \\
i_{11} = & 0 & 0 & i_3' & 0 & -i_5' & 0 \\
i_{12} = & 0 & 0 & 0 & 0 & -i_5' & 0 \\
i_{13} = & 0 & i_2' & 0 & 0 & 0 & -i_6'
\end{array}
\qquad
[I] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 1 \\
-1 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 & 0 & -1
\end{bmatrix} \times [I]'
$$

Where the 13 x 6 matrix is the $[C]$.

$$
[E]' = [C]_t \times [E] \quad \text{where } [E] = \begin{bmatrix} e_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0_{13} \end{bmatrix} \quad \text{Therefore } [E]' = \begin{bmatrix} e_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0_6 \end{bmatrix}
$$

$$
[Z] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0_{13} \\
0 & z_2 & 0 & 0 & 0 & & & & \cdot \\
0 & 0 & z_3 & 0 & 0 & & & & \cdot \\
0 & 0 & 0 & z_4 & 0 & & & & \cdot \\
0 & 0 & 0 & 0 & z_5 & & & & \cdot \\
\cdot & & & & & \cdot & & & \cdot \\
\cdot & & & & & & \cdot & & \cdot \\
0_{13} & \cdot & \cdot & \cdot & \cdot & & & \cdot & z_{13}
\end{bmatrix}
$$

$[Z]' = [C]_t \times [Z] \times [C]$ and carrying out the indicated operation gives:

$$[Z]' = \begin{bmatrix} (z_8+z_9+z_{10}) & (-z_9-z_{10}) & (-z_{10}) & (-z_8-z_9) & (0) & (-z_8) \\ (-z_9-z_{10}) & (z_5+z_9+z_{10}+z_{13}) & (z_{10}) & (z_9) & (0) & (-z_{13}) \\ (-z_{10}) & (z_{10}) & (z_2+z_6+z_{10}+z_{11}) & (-z_6) & (-z_{11}) & (0) \\ (-z_8-z_9) & (z_9) & (-z_6) & (z_6+z_7+z_8+z_9) & (-z_7) & (z_8) \\ (0) & (0) & (-z_{11}) & (-z_7) & (z_3+z_7+z_{11}+z_{12}) & (z_{12}) \\ (-z_8) & (-z_{13}) & (0) & (z_8) & (z_{12}) & (z_4+z_8+z_{12}+z_{13}) \end{bmatrix}$$

A two terminal network is being used to illustrate Kron's technique, because it works very nicely here. Since in this problem the interest is in the input impedance only and, therefore, in i_1' only; it would be necessary to find $|Z|'$ if the method of determinants was used. The following method will illustrate how i_1' may be found without solving for the determinant of a 6 x 6 matrix:[11]

Let $[Z]' = \begin{bmatrix} [Z]_1 & [Z]_2 \\ [Z]_3 & [Z]_4 \end{bmatrix}$ Where $[Z]_1$, $[Z]_2$, $[Z]_3$, and $[Z]_4$ are 3 x 3 matrices.

Also let $[E]' = \begin{bmatrix} [E]_1 \\ [E]_2 \end{bmatrix}$ Where $[E]_1 = \begin{bmatrix} e \\ 0 \\ 0 \end{bmatrix}$ and $[E]_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Also let $[I]' = \begin{bmatrix} [I]_1 \\ [I]_2 \end{bmatrix}$ Where $[I]_1 = \begin{bmatrix} i_1' \\ i_3' \\ i_3' \end{bmatrix}$ and $[I]_2 = \begin{bmatrix} i_4' \\ i_5' \\ i_6' \end{bmatrix}$

---

11 Gabriel Kron, Tensor Analysis, pp.21-22.

Therefore $[E]_1 = [Z]_1 \times [I]_1 + [Z]_2 \times [I]_2$ and $[E]_2 = [Z]_3 \times [I]_1 + [Z]_4 \times [I]_2$. Eliminating $[I]_2$, the following equation is found:

$$[E]_1 - [Z]_2 \times [Z]_4^{-1} \times [E]_2 = ([Z]_1 - [Z]_2 \times [Z]_4^{-1} \times [Z]_3) \times [I]_1$$

But $[E]_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and therefore,

$$[I]_1 = ([Z]_1 - [Z]_2 \times [Z]_4^{-1} \times [Z]_3)^{-1} \times [E]_1$$

In using this technique, it is assumed that it is easier to find the inverse of a 3 x 3 twice, than it is to find the inverse of a 6 x 6 once. The method will be illustrated by carrying the previous problem to conclusion. If we let all of the branch impedances equal one ohm, then:

$$[Z]' = \left[\begin{array}{ccc|ccc} 3 & -2 & -1 & -2 & 0 & -1 \\ -2 & 4 & 1 & 1 & 0 & -1 \\ -1 & 1 & 4 & -1 & -1 & 0 \\ \hline -2 & 1 & -1 & 4 & -1 & 1 \\ 0 & 0 & -1 & -1 & 4 & 1 \\ -1 & -1 & 0 & 1 & 1 & 4 \end{array}\right] = \begin{bmatrix} [Z]_1 & [Z]_2 \\ [Z]_3 & [Z]_4 \end{bmatrix}$$

If we let $[Z^*] = [Z]_1 - [Z]_2 \times [Z]_4^{-1} \times [Z]_3$

Then $[Z^*] = \begin{bmatrix} 3 & -2 & -1 \\ -2 & -4 & 1 \\ -1 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 1.1 & -0.4 & 0.6 \\ -0.4 & 0.8 & -0.6 \\ 0.6 & -0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} 1.9 & -1.6 & -1.6 \\ -1.6 & 3.2 & 1.6 \\ -1.6 & 1.6 & 3.2 \end{bmatrix}$

And $[Z^*]^{-1} = \frac{1}{6.4} \begin{bmatrix} 7.68 & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$

$$\text{Therefore} \begin{bmatrix} i_1' \\ i_2' \\ i_3' \end{bmatrix} = \frac{1}{8.4} \begin{bmatrix} 7.68 & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \times \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}$$

and $i_1' = \dfrac{7.68\ e}{8.4}$, $z_{input} = \dfrac{8.4}{7.68} = \dfrac{5}{6} = 0.8333$ ohms

This illustration was used to show how the Kron Mesh Method could be applied to advantage in determining the mesh impedance matrix of a complex circuit and to show how a desired current might be found without finding the determinant of a large $[Z]'$, providing the other currents are not desired. The two terminal illustration presented here is not meant to imply that this technique is limited to circuits containing one voltage. It has been derived for an n mesh network containing n voltages. The two terminal problem presented is, more or less, a classical one, and was used to simplify the calculations and still present the method.

Matrix Parameter Representation

Matrix parameter representation is a method of representing a circuit in terms of its resistance, inductance, and elastance (reciprocal capacitance) matrices. For instance, if it is necessary to give a person information on a two terminal network, it is much more compact to simply give him three matrices and let him draw the circuit. The method can best be illustrated with an example.

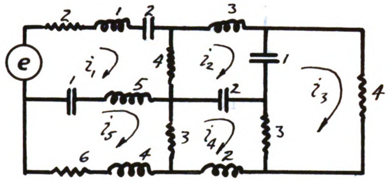


Figure 26

The parameter values chosen in Figure 26 are highly improbable, but serve for purposes of illustration. The values opposite the condensers are values of elastance (1/c). This method applies for passive (two terminal) networks only and it is assumed that the voltage is a component of mesh 1 only.

The mesh equations are:

$$e = (6+j\omega 6+3/j\omega)i_1 - 4i_2 + 0i_3 + 0i_4 - (j\omega 5 + 1/j\omega)i_5$$

$$0 = -4i_1 + (4+j\omega 3+3/j\omega)i_2 - (1/j\omega)i_3 - (2/j\omega)i_4 + 0i_5$$

$$0 = 0i_1 - (1/j\omega)i_2 + (7+1/j\omega)i_3 - 3i_4 + 0i_5$$

$$0 = 0i_1 - (2/j\omega)i_2 - 3i_3 + (6+j\omega 2+2/j\omega)i_4 - 3i_5$$

$$0 = -(j\omega 5+1/j\omega)i_1 + 0i_2 + 0i_3 - 3i_4 + (9+j\omega 9+1/j\omega)i_5$$

The matrix equation $[E] = [Z] \times [I]$ is, therefore:

$$\begin{bmatrix} e \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (6+j\omega 6+3/j\omega) & -4 & 0 & 0 & -(j\omega 5-1/j\omega) \\ -4 & (4+j\omega 3+3/j\omega) & (-1/j\omega) & (-2/j\omega) & 0 \\ 0 & (-1/j\omega) & (7+1/j\omega) & (-3) & 0 \\ 0 & (-2/j\omega) & (-3) & (6+j\omega 2+2/j\omega) & (-3) \\ -(j\omega 5+1/j\omega) & 0 & 0 & (-3) & (9+j\omega 9+1/j\omega) \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

However, the impedance matrix may be divided as follows:

$$[Z] = \begin{bmatrix} (6) & (-4) & 0 & 0 & 0 \\ (-4) & (4) & 0 & 0 & 0 \\ 0 & 0 & (7) & (-3) & 0 \\ 0 & 0 & (-3) & (6) & (-3) \\ 0 & 0 & 0 & (-3) & (9) \end{bmatrix} + j\omega \begin{bmatrix} (6) & 0 & 0 & 0 & (-5) \\ 0 & (3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (2) & 0 \\ (-5) & 0 & 0 & 0 & (9) \end{bmatrix}$$

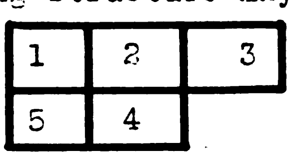
$$+ \frac{1}{j\omega} \begin{bmatrix} (3) & 0 & 0 & 0 & (-1) \\ 0 & (3) & (-1) & (-2) & 0 \\ 0 & (-1) & (1) & 0 & 0 \\ 0 & (-2) & 0 & (2) & 0 \\ (-1) & 0 & 0 & 0 & (+1) \end{bmatrix}$$

That is, $[Z] = [R] + j\omega[L] + (1/j\omega)\ [1/C]$

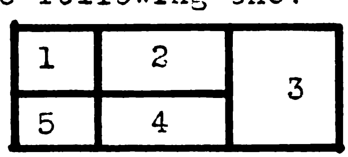
It is quite apparent that the method of parameter representation is much simpler than the actual circuit.

To draw the circuit from its parameter matrices is a simple matter. It must be remembered that the voltage is in mesh 1, only. The method for obtaining the circuit from the parameter matrices will now be illustrated.

The first rows of the three matrices indicates that there is no mutual between meshes 1 and 3, and 1 and 4. The second row indicates that there is no mutual between 2 and 5. The third row indicates none between 3 and 5, and of course, 3 and 1. With this information the following structure may be drawn in this order:



But there are mutuals between 3 and 4, therefore, the structure must be changed to the following one:



After the structure of the network is obtained, it is a simple matter to insert the R, L, and 1/C, values giving the circuit of Figure 26.

Impedance Level Change

There are an infinite number of networks having the same input impedance as a given network. And, for purposes of economy, any one of these may be more desirable than the given network. Or, it may be that the given network has an inductance or capacitance that cannot be obtained. In any event, the given network can be changed to another one having the same configuration and input impedance by a very simple method. The basis for this operation will now be derived. The following equations represent any two terminal network.

$$
\begin{aligned}
e_1 &= z_{11}i_1 + z_{12}i_2 + \dots z_{1n}i_n \\
0 &= z_{21}i_1 + z_{22}i_2 + \dots z_{2n}i_n \\
&\vdots \\
0 &= z_{n1}i_1 + z_{n2}i_2 \dots z_{nn}i_n
\end{aligned}
$$

And $i_1 = \dfrac{Z_{11}}{|Z|} e_1$ where Z_{11} = minor of z_{11}

Therefore $\dfrac{e_1}{i_1} = z_{input} = \dfrac{|Z|}{Z_{11}}$

Now, if any one of the columns or rows of the above equations, except column one or row one, is multiplied by N, then the new determinant $|Z|'$ will equal $N|Z|$ (chapter I), and the new minor Z_{11}' will equal N Z_{11}, since Z_{11} is the determinant $|Z|$ with column one and row one deleted. If all of the columns and rows are multiplied by N(except column one and row one), the new determinant $|Z|'$ will equal $N^{2n}|Z|$ and Z_{11}' will equal $N^{2n}Z_{11}$. It is easy to see, therefore, that regardless of how many rows or columns are multiplied by some finite number, or numbers, these numbers may be factored out giving the same relation between $|Z|'$ and $|Z|$ as between Z_{11}' and Z_{11}. The result is that $z_{input} = e_1/i_1 = N|Z| / NZ_{11}$ is unchanged. Referring to the impedance matrix of the previous section, the following illustration will be given.

$$[Z] = \begin{bmatrix} (6) & (-4) & 0 & 0 & 0 \\ (-4) & (4) & 0 & 0 & 0 \\ 0 & 0 & (7) & (-3) & 0 \\ 0 & 0 & (-3) & (6) & (-3) \\ 0 & 0 & 0 & (-3) & (9) \end{bmatrix} + j\omega \begin{bmatrix} (6) & 0 & 0 & 0 & (-5) \\ 0 & (3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (2) & 0 \\ (-5) & 0 & 0 & 0 & (9) \end{bmatrix}$$

$$+ \frac{1}{j\omega} \begin{bmatrix} (3) & 0 & 0 & 0 & (-1) \\ 0 & (3) & (-1) & (-2) & 0 \\ 0 & (-1) & (1) & 0 & 0 \\ 0 & (-2) & 0 & 2 & 0 \\ (-1) & 0 & 0 & 0 & (1) \end{bmatrix}$$

Multiplying the last column by 0.6 gives:

$$[Z]' = \begin{bmatrix} (6) & (-4) & 0 & 0 & 0 \\ (-4) & (4) & 0 & 0 & 0 \\ 0 & 0 & (7) & (8) & 0 \\ 0 & 0 & (-3) & (6) & (-1.8) \\ 0 & 0 & 0 & (-3) & (5.4) \end{bmatrix} + j\omega \begin{bmatrix} (6) & 0 & 0 & 0 & (-3) \\ 0 & (3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (2) & 0 \\ (-5) & 0 & 0 & 0 & (5.4) \end{bmatrix}$$

$$+ \frac{1}{j\omega} \begin{bmatrix} (3) & 0 & 0 & 0 & (-0.6) \\ 0 & (3) & (-1) & (-2) & 0 \\ 0 & (-1) & (1) & 0 & 0 \\ 0 & (-2) & 0 & (2) & 0 \\ (-1) & 0 & 0 & 0 & (-1) \end{bmatrix}$$

and because of the previous derivation, [Z] and [Z]' have the same inout impedances.

Care must be used, however, or a network may be made physically unrealizable. For instance, if the second column is multiplied by 10, the resistance and elastance matrices are unobtainable. That is, there is more impedance in the mutual than there is in the mesh. This, of course, is not possible. The method might be used, however, to obtain a physically realizable network from one that is not.

This chapter has explained and illustrated some of the two terminal network analysis and synthesis possible with the use of matrices.

CHAPTER IV

SYMMETRICAL COMPONENTS

Symmetrical components are defined by means of linear algebraic equations. Because of this fact the details of handling these equations can, and often do, obscure the end results. Matrices, because of their compactness, can be applied to symmetrical components with the result that the problem is always in sight rather than hidden among the algebraic manipulations. Mr. Reed makes this statement in advocating the teaching of symmetrical components with matrices, "Keeping all the details in their proper places... by the usual methods is such a tremendous task that rarely, if ever, are all these details given."[12] The rest of this chapter will be used to outline a procedure for the applications of matrices to symmetrical components, and a couple of examples will be included to illustrate their use.

The defining equations will be stated in terms of the current, but will apply equally well to voltages. Also, it will be assumed that an A, B, C, sequence is present, since a given sequence can be represented as an A, B, C, sequence and used in these equations.

---

[12] Myril B. Reed, Alternating-Current Circuit Theory, p.481.

Let

i_a, i_b, i_c, be the unbalanced three phase currents.

i_{a1} i_{b1} i_{c1} be the positive sequence of balanced currents.

i_{a2} i_{b2} i_{c2} be the negative sequence of balanced currents.

i_{a0} i_{b0} i_{c0} be the zero sequence currents.

Then $i_{a0} = \dfrac{i_a + i_b + i_c}{3}$ where $a = \epsilon^{j120^\circ}$

$1 + a + a^2 = 0$

$i_{a1} = \dfrac{i_a + a i_b + a^2 i_c}{3}$ and $a^3 = 1$

$i_{a2} = \dfrac{i_a + a^2 i_b + a i_c}{3}$

Therefore
$$\begin{bmatrix} i_{a0} \\ i_{a1} \\ i_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \times \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

That is, $[I_a]_s = [T] \times [I]$, and $[I] = [T]^{-1} \times [I_a]_s$

Where $[I_a]_s = \begin{bmatrix} i_{a0} \\ i_{a1} \\ i_{a2} \end{bmatrix}$, $[T] = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$

$[I] = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$, $[T]^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$

But
$$\begin{aligned} i_{b0} &= i_{a0} + 0 + 0 \\ i_{b1} &= 0 + a^2 i_{a1} + 0 \\ i_{b2} &= 0 + 0 + a i_{a2} \end{aligned}$$

And $i_{c_0} = i_{b_0} + 0 + 0$

$i_{c_1} = 0 + a^2 i_{b_1} + 0$

$i_{c_2} = 0 + 0 + a i_{b_2}$

Therefore $$[I_b]_s = \begin{bmatrix} i_{b_0} \\ i_{b_1} \\ i_{b_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} \times \begin{bmatrix} i_{a_0} \\ i_{a_1} \\ i_{a_2} \end{bmatrix}$$

And $[I_b]_s = [T]_r \times [I_a]_s$

Also $[I_c]_s = [T]_r \times [I_b]_s = [T]_r \times [T]_r \times [I_a]_s$

But $[I_a]_s = [T]_r \times [I_c]_s = [T]_r^{\,3} \times [I_a]_s$

And $[T]_r^{\,3} = [1]$ therefore $[T]_r^{-1} = [T]_r^{\,2}$

With the two transformation matrices, $[T]$ and $[T]_r$, the symmetrical components of any one of the phase currents may be found from the three unbalanced phase currents, or the three unbalanced phase currents may be found from the symmetrical components of any one of the phase currents. The following example will illustrate this statement.

Referring to Figure 27 find the symmetrical components of the line currents in terms of the symmetrical components of the phase currents.

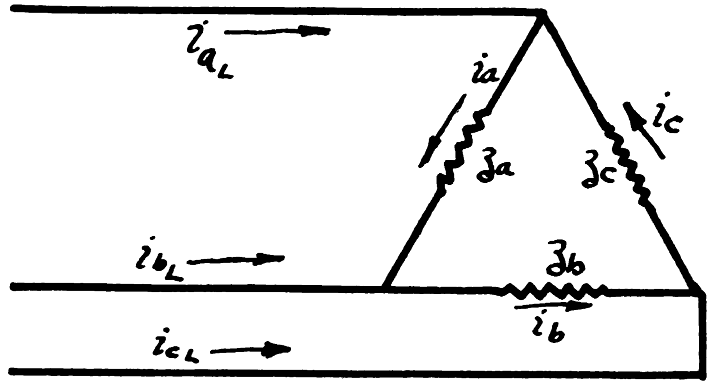


Figure 27

Solution:

$$i_{a_L} = i_a + 0 - i_c \qquad \text{Therefore } [I_L] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \times [I]$$

$$i_{b_L} = -i_a + i_b + 0$$

$$i_{c_L} = 0 - i_b + i_c \qquad \text{That is, } [I_L] = [A] \times [I]$$

But $[I_{a_L}]_s = [T] \times [I_L]$

And $[I_a]_s = [T] \times [I]$ or $[I] = [T]^{-1} \times [I_a]_s$

Therefore $[I_{a_L}]_s = [T] \times [A] \times [T]^{-1} \times [I_a]_s$

That is, $$\begin{bmatrix} i_{a0_L} \\ i_{a1_L} \\ i_{a2_L} \end{bmatrix} = [T] \times [A] \times [T]^{-1} \times \begin{bmatrix} i_{a0} \\ i_{a1} \\ i_{a2} \end{bmatrix}$$

And $$\begin{bmatrix} i_{a0_L} \\ i_{a1_L} \\ i_{a2_L} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-a) & 0 \\ 0 & 0 & (1-a)^2 \end{bmatrix} \times \begin{bmatrix} i_{a0} \\ i_{a1} \\ i_{a2} \end{bmatrix}$$

Therefore $i_{a0_L} = 0$, $i_{a1_L} = \sqrt{3}\,\epsilon^{-j30^\circ} i_{a1}$, $i_{a2_L} = \sqrt{3}\,\epsilon^{j30^\circ} i_{a2}$

The above results show that the zero sequence current in line a is zero (the zero sequence of line current is always zero for a three wire system, unless grounded). They also show that the relation between the positive sequence current in line a and the positive sequence current in phase a is the same as the relation between line and phase current in a balanced load. For the negative sequence the line current leads the phase current. The relations between the symmetrical components of line b and the components of phase b may be established as follows:

$$\left[I_{b_L}\right]_s = [T]_r \times \left[I_{a_L}\right]_s \quad \text{And} \quad \left[I_b\right]_s = [T]_r \times \left[I_a\right]_s$$

Therefore $$\left[I_{b_L}\right]_s = [T]_r \times \sqrt{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon^{-j30^\circ} & 0 \\ 0 & 0 & \epsilon^{j30^\circ} \end{bmatrix} \times [T]_r^{-1} \times \left[I_b\right]_s$$

And $$\left[I_{b_L}\right]_s = \sqrt{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon^{-j30^\circ} & 0 \\ 0 & 0 & \epsilon^{j30^\circ} \end{bmatrix} \times \left[I_b\right]_s$$

These results show that the relations between the line components and phase components are the same for each phase.

The purpose of the next example is to further illustrate the use of matrices, but more important, to illuminate a possible technique for the solution of circuit problems that have special symmetry. The technique of diagonalization is being referred to, and will be presented in the following chapter.

Referring to Figure 28 find the symmetrical components of the generated voltages in terms of the symmetrical components of the currents.

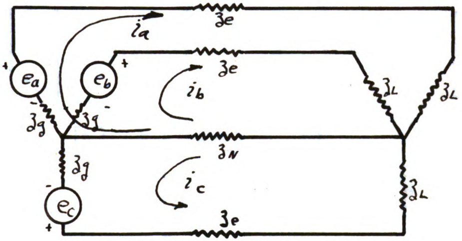


Figure 28

Solution:

$$e_a = (z_g + z_e + z_L + z_N) i_a + z_N i_b + z_N i_c$$

$$e_b = z_N i_a + (z_g + z_e + z_L + z_N) i_b + z_N i_c$$

$$e_c = z_N i_a + z_N i_b + (z_g + z_e + z_L + z_N) i_c$$

Let $z = z_g + z_e + z_L$

And
$$\begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \begin{bmatrix} (z+z_N) & z_N & z_N \\ z_N & (z+z_N) & z_N \\ z_N & z_N & (z+z_N) \end{bmatrix} \mathrm{x} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

But $[E_a]_s = [T] \mathrm{~x~} [E]$ and $[I] = [T]^{-1} \mathrm{~x~} [I_a]_s$

Therefore $[E_a]_s = [T] \mathrm{~x~} [Z] \mathrm{~x~} [T]^{-1} \mathrm{~x~} [I_a]_s$

And if the indicated operation is carried out the result is:

$$\begin{bmatrix} e_{a_0} \\ e_{a_1} \\ e_{a_2} \end{bmatrix} = \begin{bmatrix} (z+3z_N) & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} \times \begin{bmatrix} i_{a_0} \\ i_{a_1} \\ i_{a_2} \end{bmatrix}$$

Also
$$\begin{bmatrix} e_{b_0} \\ e_{b_1} \\ e_{b_2} \end{bmatrix} = [T]_r \times \begin{bmatrix} (z+3z_N) & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} \times [T]_r^{-1} \times \begin{bmatrix} i_{b_0} \\ i_{b_1} \\ i_{b_2} \end{bmatrix}$$

Or
$$[E_b]_s = \begin{bmatrix} (z+3z_N) & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} \times [I_b]_s$$

And
$$[E_c]_s = \begin{bmatrix} (z+3z_N) & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} \times [I_c]_s$$

It will be noticed in Figure 28 that each phase has the same impedance and, therefore, there is no coupling between the sequences; that is, e_{a_0} is a function of i_{a_0} only, etc., also that the zero sequence impedance equals the phase impedance plus three times the return impedance, when the load is balanced.

Referring to the possible technique for solution of circuit problems mentioned above, it will also be noticed from Figure 28 and its [Z] matrix that a certain symmetry is present. All the impedances on the diagonal are equal and all the impedances off the diagonal are equal, and it was found that premultiplying this matrix by $[T]$ and postmultiplying it by $[T]^{-1}$ resulted in a diagonalized matrix. The result is that three simultaneous equations in three unknowns is reduced to three simple ratios. This subject will be covered more thoroughly in the next chapter.

One more example will now be given illustrating the use of matrices to symmetrical component problems. Symmetrical components are used primarily for the determination of fault currents in power systems. The following example, therefore, will be a simple problem on a single line to ground fault. This and other problems are given by Reed.[13] Referring to Figure 29 the problem

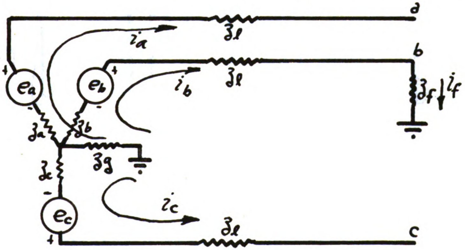


Figure 29

[13] Ibid., pp.498-511.

is to find the fault current i_f. The equations for Figure 29 are:

$$e_a = (z_a + z_1 + z_g) i_a + z_g i_b + z_g i_c + V_{af}$$
$$e_b = z_g i_a + (z_b + z_1 + z_g) i_b + z_g i_c + z_f i_f$$
$$e_c = z_g i_a + z_g i_b + (z_c + z_1 + z_g) i_c + V_{cf}$$

And therefore

$$\begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \left\{ \begin{bmatrix} z_a & 0 & 0 \\ 0 & z_b & 0 \\ 0 & 0 & z_c \end{bmatrix} + \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_1 \end{bmatrix} + z_g \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} x \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \begin{bmatrix} V_{af} \\ z_f i_f \\ V_{cf} \end{bmatrix}$$

But $[E_a]_s = [T] \; x \; [E] = \begin{bmatrix} e_{a0} \\ e_{a1} \\ e_{a2} \end{bmatrix}$ and the assumption may be made that the generated voltages are balanced and of positive sequence; therefore, $e_{a0} = e_{a2} = 0$.

Also $[I] = [T]^{-1} \; x \; [I_a]_s$ and therefore

$$\begin{bmatrix} 0 \\ e_{a1} \\ 0 \end{bmatrix} = [T] \, x \left\{ \begin{bmatrix} z_a & 0 & 0 \\ 0 & z_b & 0 \\ 0 & 0 & z_c \end{bmatrix} + \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_1 \end{bmatrix} + z_g \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} x \, [T]^{-1} \, x \begin{bmatrix} i_{a0} \\ i_{a1} \\ i_{a2} \end{bmatrix} + [T] \, x \begin{bmatrix} V_{af} \\ z_f i_f \\ V_{cf} \end{bmatrix}$$

z_a, z_b, z_c are the phase impedances of the generator.

Symmetrical components are applied to problems involving rotating machinery because of their ability to represent a non-linear problem as a linear one with a great deal of accuracy. It has been found through experience that the best results are obtained for generators

if the positive, negative, and zero sequence generator impedances have certain values. These values are found empirically and will be represented in this problem as z_{G_0}, z_{G_1}, and z_{G_2} for the zero, positive, and negative sequence impedances respectively.

Therefore $[T] \times \begin{bmatrix} z_a & 0 & 0 \\ 0 & z_b & 0 \\ 0 & 0 & z_c \end{bmatrix} \times [T]^{-1} = \begin{bmatrix} z_{G_0} & 0 & 0 \\ 0 & z_{G_1} & 0 \\ 0 & 0 & z_{G_2} \end{bmatrix}$

Also $[T] \times \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_1 \end{bmatrix} \times [T]^{-1} = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_1 \end{bmatrix}$

And $z_g[T] \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times [T]^{-1} = z_g \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Which gives

$$\begin{bmatrix} 0 \\ e_{a_1} \\ 0 \end{bmatrix} = \begin{bmatrix} (z_{G_0} + z_1 + 3z_g) & 0 & 0 \\ 0 & (z_{G_1} + z_1) & 0 \\ 0 & 0 & (z_{G_2} + z_1) \end{bmatrix} \times \begin{bmatrix} i_{a_0} \\ i_{a_1} \\ i_{a_2} \end{bmatrix} + [T] \times \begin{bmatrix} V_{af} \\ z_f i_f \\ V_{cf} \end{bmatrix}$$

But $[I_a]_s = [T] \times [I] = [T] \times \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$ Where $i_a = i_c = 0$ and $i_b = i_f$

Therefore $[I_a]_s = 1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \times \begin{bmatrix} 0 \\ i_f \\ 0 \end{bmatrix} = \frac{i_f}{3} \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$

And $$\begin{bmatrix} 0 \\ e_{a1} \\ 0 \end{bmatrix} = \begin{bmatrix} (z_{G_0}+z_1+3z_g) & 0 & 0 \\ 0 & (z_{G_1}+z_1) & 0 \\ 0 & 0 & (z_{G_2}+z_1) \end{bmatrix} \times \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} \times \frac{i_f}{3} + \frac{1}{3} \begin{bmatrix} V_{af} + z_f i_f + V_{cf} \\ V_{af} + a z_f i_f + a^2 V_{cf} \\ V_{af} + a^2 z_f i_f + a V_{cf} \end{bmatrix}$$

The above equation, however, represents the fault in phase b in terms of the components of phase a. If the equation is made to be a function of the components of phase b an important symmetry presents itself.

Since $[E_b]_s = [T]_r \times [E_a]_s$ and $[I_a]_s = [T]_r^{-1} \times [I_b]_s$

$$[T]_r^{-} \times [E_a]_s = [T]_r \times [Z] \times [T]_r^{-1} \times [I_b]_s + \frac{[T]_r}{3} \times \begin{bmatrix} (V_{af}+z_f i_f+V_{cf}) \\ (V_{af}+ a z_f i_f+a^2 V_{cf}) \\ (V_{af}+ a^2 z_f i_f+a V_{cf}) \end{bmatrix}$$

And

$$\begin{bmatrix} 0 \\ e_{b1} \\ 0 \end{bmatrix} = \begin{bmatrix} (z_{G_0}+z_1+3z_g) & 0 & 0 \\ 0 & (z_{G_1}+z_1) & 0 \\ 0 & 0 & (z_{G_2}+z_1) \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \frac{i_f}{3} + \frac{1}{3} \begin{bmatrix} V_{af}+z_f i_f+V_{cf} \\ a^2 V_{af}+z_f i_f+a V_{cf} \\ a V_{af}+z_f i_f+a^2 V_{cf} \end{bmatrix}$$

Multiplying the above equation by the row matrix, $[1,1,1]$, gives: $e_{b_1} = \left\{(z_{G_0}+z_1+3z_g)+(z_{G_1}+z_1)+(z_{G_2}+z_1)\right\} \frac{i_f}{3} + z_f i_f$

Therefore $\frac{i_f}{3} = I_{b_0} = I_{b_1} = I_{b_2} = \dfrac{e_{b_1}}{(z_{G_0}+z_1+3z_g)+(z_{G_1}+z_1)+(z_{G_2}+z_1)+3z_f}$

And $V_{bf_0} = \dfrac{V_{af}+z_f i_f+V_{cf}}{3} = -(z_{G_0}+z_1+3z_g) \times i_{b_0}$

$$V_{bf_1} = \frac{a^2V_{af}+z_f i_f+aV_{cf}}{3} = e_{b_1} -(z_{G_1}+z_1)i_{b_1}$$

$$V_{bf_2} = \frac{aV_{af}+z_f i_f+a^2V_{cf}}{3} = -(z_{G_2} + z_1)\, i_{b_2}$$

The above equations indicate that the fault may be represented by a series circuit as shown in Figure 30 where the total fault current i_f is equal to $i_b = i_{b_0}+i_{b_1}+i_{b_2}$.

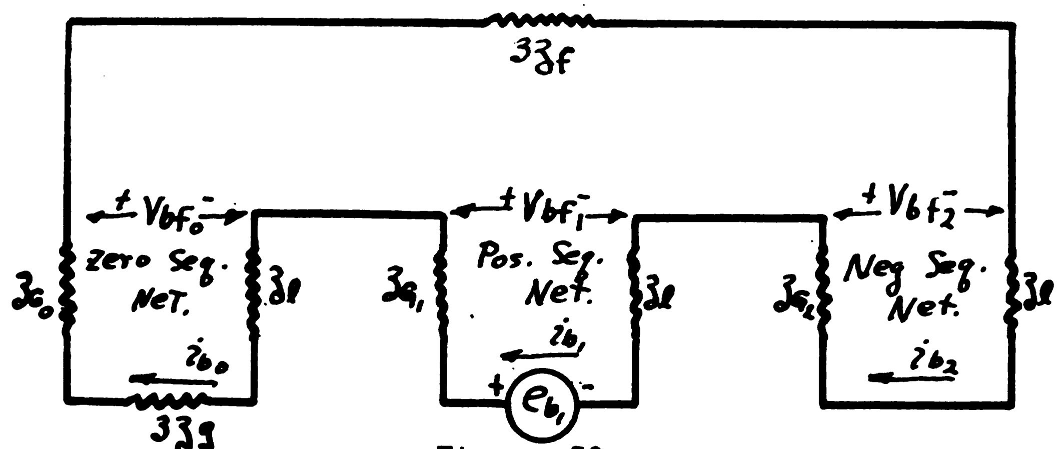


Figure 30

This chapter has shown how matrices may be applied with advantage to problems involving symmetrical components. The usual process of solving equations to get the relations between the voltages, currents, and their symmetrical components has been reduced to the simple procedure of multiplying by $[T]$, $[T]^{-1}$, $[T]_r$, and $[T]_r^{-1}$. The result is that the thread of the problem is never lost.

CHAPTER V

DIAGONALIZATION

For a given circuit problem, the matrix equation $[E] = [Z] \times [I]$ is obtained. When voltage equations are written, the voltage and impedances are known and the currents are desired. Therefore, to find the currents the above equation must be written as $[I] = [Z]^{-1} \times [E]$. Similarly, if node equations are written, $[I] = [Y] \times [E]$, the currents and admittances are known and the voltages are desired. This means that the voltages must be found as $[E] = [Y]^{-1} \times [I]$. In either case, an inverse matrix must be computed and for a large number of meshes or nodes this can be a considerable task.

The determination of the inverse is nearly as difficult as, or equally as difficult as the method of determinants (Chapter I). If, however, the matrices $[P]$ and $[Q]$ could be found such that $[P] \times [Z] \times [Q]$ gave a matrix with elements on the principle diagonal only, then the new equation $[\overset{N}{E}] = [\overset{N}{Z}] \times [\overset{N}{I}]$, where $[\overset{N}{Z}]$ is the diagonalized matrix, would result in n simple equations of the type, $\dfrac{\overset{N}{e}_1}{\overset{N}{z}_1} = \overset{N}{i}_1$, where n is the order of the

$[Z]$ matrix. Once the new currents are found, the desired currents (old) could be obtained by a simple multiplication between the $[Q]$ matrix and the $\overset{N}{[I]}$ matrix.

This method will now be illustrated with the following example. Find the currents of Figure 31 by diagonalization and by the method of computing the inverse.

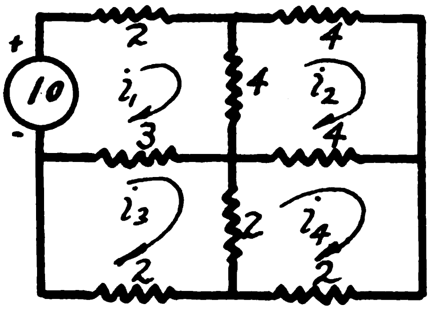


Figure 31

Mesh equations:

$$10 = (2+4+3)i_1 - 4i_2 - 3i_3 - 0i_4$$

$$0 = -4i_1 + (4+4+4)i_2 - 0i_3 - 4i_4$$

$$0 = -3i_1 - 0i_2 + (2+2+3)i_3 - 2i_4$$

$$0 = 0i_1 - 4i_2 - 2i_3 + (2+2+4)i_4$$

Matrix equation:

$$\begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (9) & (-4) & (-3) & (0) \\ (-4) & (12) & (0) & (-4) \\ (-3) & (0) & (7) & (-2) \\ (0) & (-4) & (-2) & (8) \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

That is $[E] = [Z] \times [I]$

By the method of diagonalization, the following procedure is used:

Premultiply the matrix equation by $[P]$ giving $[P] \times [E] = [P] \times [Z] \times [1] \times [I]$. The $[1]$ matrix may be inserted between $[Z]$ and $[I]$ because it does not affect the equality of the equation. But $[Q] \times [Q]^{-1} = [1]$, therefore, insert this in the place of $[1]$ giving, $[P] \times [E] = [P] \times [Z] \times [Q] \times [Q]^{-1} \times [I]$. The equation $\overset{N}{[E]} = \overset{N}{[Z]} \times \overset{N}{[I]}$ may now be written, where: $\overset{N}{[E]} = [P] \times [E]$, $\overset{N}{[Z]} = [P] \times [Z] \times [Q]$ (the diagonalized matrix) and $\overset{N}{[I]} = [Q]^{-1} \times [I]$ are unknown currents.

The problem, now, is to find $[P]$ and $[Q]$ such that $[P] \times [Z] \times [Q] = \overset{N}{[Z]}$ will result in a diagonal matrix. This is done as follows:

$$[Z] = \begin{bmatrix} 9 & -4 & -3 & 0 \\ -4 & 12 & 0 & -4 \\ -3 & 0 & 7 & -2 \\ 0 & -4 & -2 & 8 \end{bmatrix}$$

Add 4/9 x 1st row to 2nd row giving:

$$[Z]' = \begin{bmatrix} 9 & -4 & -3 & 0 \\ 0 & \frac{92}{9} & \frac{-4}{3} & -4 \\ -3 & 0 & 7 & -2 \\ 0 & -4 & -2 & 8 \end{bmatrix}$$

Add 1/3 x 1st row to 3rd row giving:

$$[Z]'' = \begin{bmatrix} 9 & -4 & -3 & 0 \\ 0 & \frac{92}{9} & \frac{-4}{3} & -4 \\ 0 & \frac{-4}{3} & 6 & -2 \\ 0 & -4 & -2 & 8 \end{bmatrix}$$

Add 3/23 x 2nd row to 3rd row giving:

$$[Z]''' = \begin{bmatrix} 9 & -4 & -3 & 0 \\ 0 & \frac{92}{9} & \frac{-4}{3} & -4 \\ 0 & 0 & \frac{134}{23} & \frac{-58}{23} \\ 0 & -4 & -2 & 8 \end{bmatrix}$$

Add 9/23 x 2nd row to 4th row giving:

$$[Z]'''' = \begin{bmatrix} 9 & -4 & -3 & 0 \\ 0 & \frac{92}{9} & \frac{-4}{3} & -4 \\ 0 & 0 & \frac{134}{23} & \frac{-58}{23} \\ 0 & 0 & \frac{-58}{23} & \frac{148}{23} \end{bmatrix}$$

Add 29/67 x 3rd row to 4th row giving:

$$[Z]''''' = \begin{bmatrix} 9 & -4 & -3 & 0 \\ 0 & \frac{92}{9} & \frac{-4}{3} & -4 \\ 0 & 0 & \frac{134}{23} & \frac{-58}{23} \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix}$$

Referring to Linear Transformations, Chapter I, the first operation could have been performed by a premultiplier. This premultiplier is, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{4}{9} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, where 4/9 of 1st row is added to 2nd row.

Therefore $$[Z]' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{4}{9} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times [Z]$$

By the same reasoning $[Z]'''''$ is:

$$[Z]''''' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{23}{67} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{9}{23} & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{3}{23} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{4}{9} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times [Z]$$

But $[Z]''''' = [P] \times [Z]$ and therefore,

$$[P] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{4}{9} & 1 & 0 & 0 \\ \frac{27}{69} & \frac{3}{23} & 1 & 0 \\ \frac{1587}{4623} & \frac{690}{1541} & \frac{26}{27} & 1 \end{bmatrix}$$

Since all we have done is add multiples of one row to another, the determinant of $[Z]'''''$ must equal the determinant of $[Z]$ and $|P|$ must equal 1, and this is the case.

Next operation:

$$[Z]''''' = \begin{bmatrix} 9 & -4 & -3 & 0 \\ 0 & \frac{83}{9} & \frac{-4}{3} & -4 \\ 0 & 0 & \frac{134}{23} & \frac{-58}{23} \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix}$$

Add 4/9 x 1st column to 2nd giving:

$$
{}^{\prime}[Z]^{\prime\prime\prime\prime\prime} = \begin{bmatrix} 9 & 0 & -3 & 0 \\ 0 & \frac{23}{9} & -\frac{4}{3} & -4 \\ 0 & 0 & \frac{134}{23} & -\frac{58}{23} \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix}
$$

Add 1/3 x 1st column to 3rd column giving:

$$
{}^{\prime\prime}[Z]^{\prime\prime\prime\prime\prime} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & \frac{23}{9} & -\frac{4}{3} & -4 \\ 0 & 0 & \frac{134}{23} & -\frac{58}{23} \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix}
$$

Add 3/23 x 2nd column to 3rd column giving:

$$
{}^{\prime\prime\prime}[Z]^{\prime\prime\prime\prime\prime} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & \frac{23}{9} & 0 & -4 \\ 0 & 0 & \frac{134}{23} & -\frac{58}{23} \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix}
$$

Add 9/23 x 2nd column to 4th column giving:

$$
{}^{\prime\prime\prime\prime}[Z]^{\prime\prime\prime\prime\prime} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & \frac{23}{9} & 0 & 0 \\ 0 & 0 & \frac{134}{23} & -\frac{58}{23} \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix}
$$

Add 29/67 x 3rd column to 4th column giving:

$$\overset{N}{[Z]} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & \frac{92}{9} & 0 & 0 \\ 0 & 0 & \frac{134}{23} & 0 \\ 0 & 0 & 0 & \frac{8334}{1541} \end{bmatrix}$$

Referring again to Linear Transformations, Chapter 1, these operations could have been performed by a post-multiplier. The first postmultiplier is, $\begin{bmatrix} 1 & \frac{4}{9} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, where

4/9 of the 1st column is added to 2nd column.

$$\text{Therefore, } \overset{N}{[Z]} = [P] \times [Z] \times \begin{bmatrix} 1 & \frac{4}{9} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{9}{23} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{29}{67} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{But } \overset{N}{[Z]} = [P] \times [Z] \times [Q]$$

$$\text{Therefore } [Q] = \begin{bmatrix} 1 & \frac{4}{9} & \frac{9}{23} & \frac{523}{1541} \\ 0 & 1 & \frac{3}{23} & \frac{690}{1541} \\ 0 & 0 & 1 & \frac{29}{67} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{N}{[E]} = [P] \times [E] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{4}{9} & 1 & 0 & 0 \\ \frac{27}{69} & \frac{3}{23} & 1 & 0 \\ \frac{1587}{4623} & \frac{690}{1541} & \frac{23}{27} & 1 \end{bmatrix} \times \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{40}{9} \\ \frac{270}{69} \\ \frac{15870}{4623} \end{bmatrix}$$

$$\overset{N}{[E]} = \overset{N}{[Z]} \times \overset{N}{[I]} = \begin{bmatrix} 10 \\ \frac{40}{9} \\ \frac{270}{69} \\ \frac{15870}{4623} \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & \frac{92}{9} & 0 & 0 \\ 0 & 0 & \frac{184}{23} & 0 \\ 0 & 0 & 0 & \frac{8234}{1541} \end{bmatrix} \times \begin{bmatrix} \overset{N}{i_1} \\ \overset{N}{i_2} \\ \overset{N}{i_3} \\ \overset{N}{i_4} \end{bmatrix}$$

Which gives, on multiplying:

$$\overset{N}{i_1} = \frac{10}{9}, \quad \overset{N}{i_2} = \frac{40}{92}, \quad \overset{N}{i_3} = \frac{90}{134}, \quad \overset{N}{i_4} = \frac{15,870}{24,702}$$

But $\overset{N}{[I]} = [Q]^{-1} \times [I]$ and therefore $[I] = [Q] \times \overset{N}{[I]}$

Therefore:

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{9} & \frac{9}{23} & \frac{529}{1541} \\ 0 & 1 & \frac{3}{23} & \frac{690}{1541} \\ 0 & 0 & 1 & \frac{29}{67} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{10}{9} \\ \frac{40}{92} \\ \frac{90}{134} \\ \frac{15,870}{24,702} \end{bmatrix}$$

$$i_1 = \frac{10}{9} + \frac{160}{828} + \frac{810}{3082} + \frac{8,395,330}{38,065,782}$$

$$= 1.1111 + 0.1932 + 0.2628 + .2205$$

$i_1 = 1.7876$ amps $\qquad$ $i_2 = 0.8101$ amps

$i_3 = 0.950$ amps $\qquad$ $i_4 = 0.6425$ amps

If the usual method of computing the inverse is used to determine the currents, the result will be as follows:

$$[I] = [Z]^{-1} \times [E], \quad \text{That is,} \quad \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \frac{1}{|Z|} \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix} \times \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Where Z_{ij} is the cofactor of z_{ij}.

Therefore $i_1 = \dfrac{Z_{11}}{|Z|} \times 10, \quad i_2 = \dfrac{Z_{21}}{|Z|} \times 10$

$$i_3 = \frac{Z_{31}}{|Z|} \times 10, \quad i_4 = \frac{Z_{41}}{|Z|} \times 10$$

$$|Z| = 9 \times Z_{11} - 4 \times Z_{21} - 3 \times Z_{31} + 0 \times Z_{41}$$

And $Z_{11} = \begin{vmatrix} 12 & 0 & -4 \\ 0 & 7 & -2 \\ -4 & -2 & 8 \end{vmatrix} = 512, \quad Z_{21} = -1 \times \begin{vmatrix} -4 & -3 & 0 \\ 0 & 7 & -2 \\ -4 & -2 & 8 \end{vmatrix} = 232$

$$Z_{31} = \begin{vmatrix} -4 & -3 & 0 \\ 12 & 0 & -4 \\ -4 & -2 & 8 \end{vmatrix} = 272, \quad Z_{41} = -1 \times \begin{vmatrix} -4 & -3 & 0 \\ 12 & 0 & -4 \\ 0 & 7 & -2 \end{vmatrix} = 184$$

Therefore, $|Z| = 9 \times 512 - 4 \times 232 - 3 \times 272 + 0 \times 184 = 2864$

And $i_1 = \dfrac{5120}{2864} = 1.7877$ amps $\quad i_3 = \dfrac{2720}{2864} = 0.950$ amps

$$i_2 = \frac{2320}{2864} = 0.810 \text{ amps} \quad i_4 = \frac{1840}{2864} = 0.6425 \text{ amps}$$

If an easy method for determining the currents is what we are after, then diagonalization by the [P] and

[Q] matrices should not be used. This past example illustrates the difficulty and the time consumed in determining [P] and [Q] before the problem can even be worked. And if the impedances are complex, it is next to impossible to determine the proper [P] and [Q] . It is also apparent from the past illustration that a different [P] and [Q] is required for every [Z] matrix.

The problem of diagonalization is not hopeless, however. It was found in the last chapter that for a third order [Z] matrix, with all elements on the diagonal equal to one value and all other elements equal to a different value, the matrix [T] could be used to diagonalize it; that is, $\overset{N}{[Z]} = [T] \times [Z] \times [T]^{-1}$. This type of symmetry is referred to, by Pipes,[14] as E symmetry. He then states that a [Z] matrix of n order with E symmetry may be diagonalized by the use of an [S] matrix; that is, $\overset{N}{[Z]} = [S]^{-1} \times [Z] \times [S]$ where $[S] = [S_{rs}]$ and $S_{rs} = a^{-(r-1)(s-1)}$ with $r = 1, 2, 3 \ldots n$, $s = 1, 2, 3 \ldots n$, and $a = \epsilon^{\frac{j2\pi}{n}}$. The use of the [S] matrix is an extension of the method of symmetrical components.

It might be worth noticing, at this point, that to obtain $\overset{N}{[Z]}$, in the last chapter, [Z] was premultiplied

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[14] L. A. Pipes, "Transient Analysis of Symmetrical Networks by the Method of Symmetrical Components," <u>AIEE Transactions</u>, (1940), p. 457.

by $[T]$ and postmultiplied by $[T]^{-1}$ while Pipes (above) premultiplies by $[S]^{-1}$ and postmultiplies by $[S]$. If the above definition of $[S]$ is used and $[S]$ is obtained, it will be:

$$[S] = \begin{bmatrix} 1 & 1 & 1 & . & . & . & . & . & 1 \\ 1 & a^{-1} & a^{-2} & . & . & . & . & . & a^{-(n-1)} \\ 1 & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . \\ 1 & a^{-(n-1)} & & . & . & . & . & & a^{-(n-1)(n-1)} \end{bmatrix}$$

Therefore $[S] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^{-1} & a^{-2} \\ 1 & a^{-2} & a^{-4} \end{bmatrix}$ and $a = \epsilon^{j\frac{2\pi}{3}} = \epsilon^{j120^\circ}$
$(n=3)$

therefore $a^3 = 1$

Multiplying each element of $[S]$ by a^3 gives:

$$[S] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} = [T]^{-1}$$

And therefore $[S]^{-1} = [T]$

Fundamentally, then, Pipes has defined the same operator as is used in symmetrical components in the previous chapter. However, in so doing he has extended it to cover problems other than three phase and has introduced the negative exponent. For a given problem the negative exponents of a can always be made positive by multiplying each element of [S] by $\epsilon^{(j\frac{2\pi}{n})m} = 1$.

In the discussion of Pipes' article in the A. I. E. E. Transactions,[15] the point is brought out that this same [S] matrix will also diagonalize a [Z] matrix that has "ring" symmetry. "Ring" symmetry is not defined, but the two examples given [16] are shown in Figure 32.

The red lines on Figure 32 indicate a symmetry that will allow diagonalization with the use of the [S] matrix.

$$\begin{bmatrix} z & z_{12} & z_{13} & z_{14} & z_{13} & z_{12} \\ z_{12} & z & z_{12} & z_{13} & z_{14} & z_{13} \\ z_{13} & z_{12} & z & z_{12} & z_{13} & z_{14} \\ z_{14} & z_{13} & z_{12} & z & z_{12} & z_{13} \\ z_{13} & z_{14} & z_{13} & z_{12} & z & z_{12} \\ z_{12} & z_{13} & z_{14} & z_{13} & z_{12} & z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z & z_{12} & z_{13} & z_{12} \\ z_{12} & z & z_{12} & z_{13} \\ z_{13} & z_{12} & z & z_{12} \\ z_{12} & z_{13} & z_{12} & z \end{bmatrix}$$

(a) (b)

Figure 32

Matrix (a) of Figure 32 gives the following diagonalized form:

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[15] Ibid., p.1109.

[16] Ibid., pp.1109-1110.

$$\begin{bmatrix} z_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_5 \end{bmatrix}$$

Where $z_0 = (z+3z_{12}+2z_{13}+z_{14})$

$z_1 = z_5 = (z+z_{12} - z_{13} - z_{14})$

$z_2 = z_4 = (z - z_{12} - z_{13} + z_{14})$

$z_3 = (z-2z_{12}+2z_{13}-z_{14})$

Matrix (b) of Figure 32 gives the following diagonalized form:

$$\begin{bmatrix} z_0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_3 \end{bmatrix}$$

Where $z_0 = (z+2z_{12}+z_{13})$

$z_1 = z_3 = (z - z_{13})$

$z_2 = (z + z_{13} - 2z_{12})$

The matrices of Figure 33(a) and (b) have the symmetry indicated, in red, in Figure 32. These matrices did diagonalize and their results are given in Figure 33(c) and (d).

$$\begin{bmatrix} z_a & z_b & z_c & z_d \\ z_d & z_a & z_b & z_c \\ z_c & z_d & z_a & z_b \\ z_b & z_c & z_d & z_a \end{bmatrix}$$

(a)

$$\begin{bmatrix} z_0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_3 \end{bmatrix}$$

(c)

Where

$z_0=(z_a+z_b+z_c+z_d)$

$z_1=(z_a-jz_b-z_c+jz_d)$

$z_2=(z_a-z_b+z_c-z_d)$

$z_3=(z_a+jz_b-z_c-jz_d)$

$$\begin{bmatrix} z_a & z_b & z_c \\ z_c & z_a & z_b \\ z_b & z_c & z_a \end{bmatrix}$$

(b)

$$\begin{bmatrix} z_0 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{bmatrix}$$

(d)

Where

$z_0 =(z_a+z_b+z_c)$

$z_1 =(z_a+a^2z_b+az_c)$

$z_2 =(z_a+az_b+a^2z_c)$

Figure 33

The matrices of Figure 33 are of little use in most circuit problems, because they lack the symmetry about the diagonal characterized by most [Z] matrices; that is, $z_{ij} \neq z_{ji}$ for the matrices of Figure 33(a) and (b). However, they have served a purpose in indicating a type of symmetry necessary for diagonalization by [S] . Also of importance is the apparent fact that if $z_{ij} = z_{ji}$ in the [Z] matrix, then the order of [Z] must be even if diagonalization is to be possible, unless, of course, E symmetry is present.

The following example will now be given to illustrate the advantage of diagonalizing when symmetry permits it.

Referring to Figure 34, the problem is to find the mesh currents as functions of time when a step voltage is

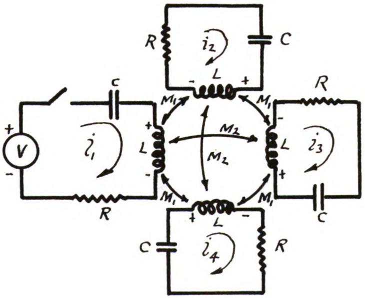

Figure 34

applied at $t = 0$; that is, v is a d. c. source and time is measured from the instant the switch is closed. It is assumed that energy storage in each mesh is zero before the switch is closed.

Solution:

The differential equations may be written as follows:

$$V = (Lp + R + \frac{1}{cp})i_1 + M_1 p i_2 + M_2 p i_3 + M_1 p i_4$$

$$0 = M_1 p i_1 + (Lp + R + \frac{1}{cp})i_2 + M_1 p i_3 + M_2 p i_4$$

$$0 = M_2 p i_1 + M_1 p i_2 + (Lp + R + \frac{1}{cp})i_3 + M_1 p i_4$$

$$0 = M_1 p i_1 + M_2 p i_2 + M_1 p i_3 + (Lp + R + \frac{1}{cp})i_4$$

Where $p = \frac{d}{dt}$ and $\frac{1}{p} = \quad dt$

If the laplace transform is taken on each side of the above equations, they may be written as:

$$e = z_a I_1 + z_b I_2 + z_c I_3 + z_b I_4$$

$$0 = z_b I_1 + z_a I_2 + z_b I_3 + z_c I_4$$

$$0 = z_c I_1 + z_b I_2 + z_a I_3 + z_b I_4$$

$$0 = z_b I_1 + z_c I_2 + z_b I_3 + z_a I_4$$

Where $e = \frac{V}{S}$, $z_a = (LS + R + \frac{1}{CS})$

$z_b = M_1 S$, $z_c = M_2 S$ $\quad I = \mathcal{L}(i)$

Because all initial conditions are zero.

Therefore
$$\begin{bmatrix} e \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_a & z_b & z_c & z_b \\ z_b & z_a & z_b & z_c \\ z_c & z_b & z_a & z_b \\ z_b & z_c & z_b & z_a \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$

And $[E] = [Z] \times [I]$

But it is noticed that the above $[Z]$ matrix has the same type of symmetry as Figure 32(b), and therefore, the $[S]$ matrix may be used.

$$[S] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a^{-1} & a^{-2} & a^{-3} \\ 1 & a^{-2} & a^{-4} & a^{-6} \\ 1 & a^{-3} & a^{-6} & a^{-9} \end{bmatrix} \quad \text{Where} \quad a = \epsilon^{j\frac{2\pi}{4}} = \epsilon^{j\frac{\pi}{2}} = j$$

$$a^4 = 1$$

$$[S] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad \text{and} \quad [S]^{-1} = \frac{1}{n}\left[\text{Conjugate } S\right]$$

(always)

$$\text{Therefore } [S]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$[E] = [Z] \times [I] \;\therefore\; \underbrace{[S]^{-1} \times [E]}_{\overset{N}{[E]}} = \underbrace{[S]^{-1} \times [Z] \times [S]}_{\overset{N}{[Z]}} \times \underbrace{[S]^{-1} \times [I]}_{\overset{N}{[I]}}$$

$$\overset{N}{[E]} = \overset{N}{[Z]} \times \overset{N}{[I]}$$

From Figure 32(b) $\overset{N}{[Z]}$ is found to be:

$$\overset{N}{[Z]} = \begin{bmatrix} z_0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_3 \end{bmatrix}$$

Where $z_0 = (z_a + 2z_b + z_c)$

$z_1 = z_3 = (z_a - z_c)$

$z_2 = (z_a + z_c - 2z_b)$

And $\overset{N}{[E]} = \frac{1}{4} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \times \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{e}{4} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

And therefore

$$\frac{e}{4} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} z_0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$

And $\overset{N}{I_1} = \frac{e}{4z_0}$, $\overset{N}{I_2} = \frac{e}{4z_1}$, $\overset{N}{I_3} = \frac{e}{4z_2}$, $\overset{N}{I_4} = \frac{e}{4z_1}$

But $\overset{N}{[I]} = [S]^{-1} \times [I]$ and therefore $[I] = [S] \times \overset{N}{[I]}$

$$[I] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \frac{e}{4} \times \begin{bmatrix} \frac{1}{z_0} \\ \frac{1}{z_1} \\ \frac{1}{z_3} \\ \frac{1}{z_1} \end{bmatrix} = \frac{e}{4} \times \begin{bmatrix} \left(\frac{1}{z_0} + \frac{1}{z_2} + \frac{2}{z_1}\right) \\ \left(\frac{1}{z_0} - \frac{1}{z_2}\right) \\ \left(\frac{1}{z_0} + \frac{1}{z_2} - \frac{2}{z1}\right) \\ \left(\frac{1}{z_0} - \frac{1}{z_2}\right) \end{bmatrix}$$

And $I_1 = \frac{e}{4}\left(\frac{1}{z_0} + \frac{1}{z_2} + \frac{2}{z_1}\right)$, $I_2 = I_4 = \frac{e}{4}\left(\frac{1}{z_0} - \frac{1}{z_2}\right)$

$I_3 = \frac{e}{4}\left(\frac{1}{z_0} + \frac{1}{z_2} - \frac{2}{z_1}\right)$

But $z_0 = z_a + 2z_b + z_c = R + LS + \frac{1}{CS} + 2M_1S + M_2S$

And $z_0 = R + S(L + 2M_1 + M_2) + \frac{1}{CS}$

$$z_1 = z_a - z_c = R + LS + \frac{1}{CS} - M_2 S$$

And $z_1 = R + S(L - M_2) + \frac{1}{CS}$

$$z_2 = z_a + z_c - 2z_b = R + LS + \frac{1}{CS} + M_2 S - 2M_1 S$$

And $z_2 = R + S(L - 2M_1 + M_2) + \frac{1}{CS}$

Therefore $I_1 = \mathcal{L}^{-1}\left\{\frac{V}{4}\left(\frac{1}{S^2(L+2M_1+M_2) + RS + \frac{1}{C}} + \frac{1}{S^2(L-2M_1+M_2)+RS+\frac{1}{C}} + \frac{2}{S^2(L-M_2)+RS+\frac{1}{C}}\right)\right\}$

$$I_2 = I_4 = \mathcal{L}^{-1}\left\{\frac{V}{4}\left(\frac{1}{S^2(L+2M_1+M_2)+RS+\frac{1}{C}} - \frac{1}{S^2(L-2M_1+M_2)+RS+\frac{1}{C}}\right)\right\}$$

$$I_3 = \mathcal{L}^{-1}\left\{\frac{V}{4}\left(\frac{1}{S^2(L+2M_1+M_2)+RS+\frac{1}{C}} + \frac{1}{S^2(L-2M_1+M_2)+RS+\frac{1}{C}} - \frac{2}{S^2(L-M_2)+RS+\frac{1}{C}}\right)\right\}$$

If $\alpha_1 = \frac{R}{2(L+2M_1+M_2)}$, $\beta_1 = \sqrt{\frac{1}{C(L+2M_1+M_2)} - \left(\frac{R}{2(L+2M_1+M_2)}\right)^2}$

$\alpha_2 = \frac{R}{2(L-2M_1+M_2)}$, $\beta_2 = \sqrt{\frac{1}{C(L-2M_1+M_2)} - \left(\frac{R}{2(L-2M_1+M_2)}\right)^2}$

$\alpha_3 = \frac{R}{2(L-M_2)}$, $\beta_3 = \sqrt{\frac{1}{C(L-M_2)} - \left(\frac{R}{2(L-M_2)}\right)^2}$

Then $i_1 = \frac{V}{4}\left\{\frac{\epsilon^{-\alpha_1 t}\sin\beta_1 t}{(L+2M_1+M_2)\beta_1} + \frac{\epsilon^{-\alpha_2 t}\sin\beta_2 t}{(L-2M_1+M_2)\beta_2} + \frac{2\,\epsilon^{-\alpha_3 t}\sin\beta_3 t}{(L-M_2)\beta_3}\right\}$

$$i_2 = i_4 = \frac{V}{4}\left\{\frac{\epsilon^{-\alpha_1 t}\sin\beta_1 t}{(L+2M_1+M_2)\beta_1} - \frac{\epsilon^{-\alpha_2 t}\sin\beta_2 t}{(L-2M_1+M_2)\beta_2}\right\}$$

And $i_3 = \frac{V}{4}\left\{\frac{\epsilon^{-\alpha_1 t}\sin\beta_1 t}{(L+2M_1+M_2)\beta_1} + \frac{\epsilon^{-\alpha_2 t}\sin\beta_2 t}{(L-2M_1+M_2)\beta_2} - \frac{2\,\epsilon^{-\alpha_3 t}\sin\beta_3 t}{(L-M_2)\beta_3}\right\}$

This example has illustrated, decisively, the advantage to be had in diagonalizing the [Z] matrix when E or ring symmetry is present. It is only hoped that further research along these lines will result in transform matrices that will diagonalize [Z] matrices of a more general type.

CONCLUSION

This thesis has presented some of the fundamentals of matrix algebra and showed how they may be applied to various types of circuit problems. It has not, by any means, covered all the possible applications of matrices.

Matrices are a field of mathematics in themselves and considerable work has been done with them. It has only been in the last few years, however, that an attempt has been made to apply them to circuit problems. Because of their compactness and ability to maintain the continuity of the problem their use in involved circuit problems is unlimited. It is hoped that in the future matrices will experience an even greater useage in the engineering problems of all the fields.

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