ESTIMATES ON SINGULAR VALUES OF FUNCTIONS OF PERTURBED OPERATORS

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ABSTRACT

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In this thesis we study the behavior of functions of operators under perturbations. We prove that if function f belongs to the class $\Lambda_\omega \stackrel{\text{def}}{=} \{f : \omega_f(\delta) \leq \text{const}\, \omega(\delta)\}$ for an arbitrary modulus of continuity ω, then $s_j(f(A) - f(B)) \leq c \cdot \omega_*\big((1+j)^{-\frac{1}{p}} \|A - B\|_{S_p^l}\big) \cdot \|f\|_{\Lambda_\omega}$ for arbitrary self-adjoint operators A, B and all $1 \leq j \leq l$, where $\omega_*(x) \stackrel{\text{def}}{=} x \int_x^\infty \frac{\omega(t)}{t^2} dt$ $(x > 0)$. The result is then generalized to contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators.

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PREFACE

It is well known that a Lipschitz function on the real line does not have to be operator Lipschitz. The situation changes dramatically if we consider the Hölder class of functions. In [1] and [3], it was proved that if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$, then $\|f(A) - f(B)\| \le \text{const}\,\|f\|_{\Lambda_\alpha}\|A - B\|^\alpha$ for all pairs of self-adjoint or unitary operators A and B. The authors also generalized their results to the class Λ_ω, and obtained estimate $\|f(A) - f(B)\| \le \text{const}\,\|f\|_{\Lambda_\omega}\omega_*\|A - B\|$.

In [2], it was shown that for functions f in the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$ and if $1 < p < \infty$, the operator $f(A) - f(B)$ belongs to $\boldsymbol{S}_{p/\alpha}$, whenever A and B are arbitrary self-adjoint operators such that $A - B \in \boldsymbol{S}_p$. In particular, it was proved that if $0 < \alpha < 1$, then there exists a constant $c > 0$ such that for every $l \ge 0$, $p \in [1, \infty)$, $f \in \Lambda_\alpha(\mathbb{R})$, and for arbitrary self-adjoint operators A and B on Hilbert space with bounded $A - B$, the following inequality holds for every $j \le l$:

$$s_j(f(A) - f(B)) \le c\,\|f\|_{\Lambda_\alpha(\mathbb{R})}(1 + j)^{-\frac{\alpha}{p}}\|A - B\|^\alpha_{\boldsymbol{S}_p^l} \text{(see (3.1.1) for definition).}$$

In section §3.2, we generalize this estimate to the class Λ_ω. We prove that if function f belongs to the class Λ_ω for an arbitrary modulus of continuity ω, then $s_j(f(A) - f(B)) \le c\,\omega_*\big((1 + j)^{-\frac{1}{p}}\|A - B\|_{\boldsymbol{S}_p^l}\big)\|f\|_{\Lambda_\omega}$ for arbitrary self-adjoint operators A, B and all $1 \le j \le l$. The result is then generalized to contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators. We also obtain some lower-bound estimates for rank one perturbations which also extend the results in [2]. In section §3.3, similar estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators.

In chapter 1, we give a brief introduction to the theory of double operator integrals and their applications to the perturbation theory. We refer the reader to [21] for more details.

Necessary information on function spaces $B^s_{p,q}$ and Λ_ω are given in section §2.2. We refer the reader to [1] for more detailed information.

The results obtained in section §3.2 and §3.3 were proved in [14], submitted to the Indiana University Mathematics Journal in April, 2016.

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Chapter 1

A brief note on double operator integrals

1.1 Introduction

1.1.1 Formal definition of double operator integrals

Formally, Double Operator Integrals (DOI) are objects of the form

$$T = \int_X \int_Y \Phi(x, y) dE_1(x) Q dE_2(y). \tag{1.1.1}$$

In (1.1.1) $(X, E_1(\cdot))$ and $(Y, E_2(\cdot))$ are two spaces with spectral measure. The values of the measure $E_1(\cdot)$ are orthogonal projections in a separable Hilbert space H_1, and similar for the measure $E_2(\cdot)$ in the Hilbert space H_2. The scalar-valued function $\Phi(x, y)$ (*the symbol* of the DOI) is defined on $X \times Y$. Finally, Q is a linear bounded operator acting from H_2 to H_1, or $Q \in \mathcal{B}(H_2, H_1)$. Under reasonable definition the result T is also an operator acting from H_2 to H_1. Hence, the integral (1.1.1) defines a linear mapping (*transformer*)

$$\mathscr{T}_\Phi^{E_1, E_2} : Q \mapsto T. \tag{1.1.2}$$

$\mathscr{T}_\Phi^{E_1,E_2}$ is often written as $\mathscr{T}_\Phi$ for short, especially when the spectral measures E_1, E_2 are fixed. Sometimes we write

$$T_\Phi := \mathscr{T}_\Phi^{E_1,E_2} Q \tag{1.1.3}$$

If E_1, E_2 are the spectral measures of self-adjoint operators A, B ($E_1 = E_1^A, E_2 = E_2^B$), then instead of (1.1.3) we write

$$T_\Phi := \mathscr{T}_\Phi^{A,B} Q \tag{1.1.4}$$

Rigorous definition of the integral (1.1.1) depends on the space of operators we wish to deal with and the class of admissible symbols is also determined by the choice of this space. In the case of the space $\boldsymbol{S}_2$ of Hilbert–Schmidt operators, the integral (1.1.1) can be well defined for an arbitrary bounded and measurable symbol with respect to an appropriate measure μ on $X \times Y$. The measure μ is determined by the given spectral measures E_1 and E_2; the operator T_Φ is also Hilbert–Schmidt and moreover,

$$\|T_\Phi\|_{\boldsymbol{S}_2} \leq (\mu)\text{-sup}|\Phi| \|Q\|_{\boldsymbol{S}_2}. \tag{1.1.5}$$

All this, including the construction of the measure σ, will be explained in section §1.2. For other spaces of operators the situation is more complex. One of the most important cases is when the integral (1.1.1) can be well defined for any bounded operator Q and the resulting operator T_Φ is also bounded. Then the transformer $\mathscr{T}_\Phi^{E_1,E_2}$ acts in the space $\mathcal{B}(H_2, H_1)$ and is bounded by Closed Graph Theorem. Theorem 1.3.1 gives a full description of the class $\mathfrak{M}$ of all admissible symbols of this type. If $\Phi \in \mathfrak{M}$, then the transformer $\mathscr{T}_\Phi^{E_1,E_2}$ is also bounded in the space $\boldsymbol{S}_1$ of all trace class operators and in the space $\boldsymbol{S}_\infty$ of all compact operators. It is possible to consider the action of the integral (1.1.1) between other spaces of operators, and the spaces for Q and T may differ from each other and the exhaustive description of the class of admissible symbols for the most of cases is not know. However, there are many sufficient conditions which allow one to apply the general results of the theory of DOI.

1.1.2 Functions of non-commuting operators

Suppose that $H_2 = H_1$ and in (1.1.1) $X = Y = \mathbb{R}$, $E_1 = E_1^A$, $E_2 = E_2^B$ where A, B are self-adjoint operators. Then it is natural to regard T_Φ as the function Φ of the pair (A, B), separated by the operator Q. The operators A and B are not assumed commuting, since the presence of the operator Q prevents any possible gains which might come from the commutation of A and B. For the simple case when $\Phi(x, y) = \phi(x)\psi(y)$ where ϕ and ψ are bounded functions, we have by Spectral Theorem

$$\phi(A)Q\psi(B) = \int \phi(x)dE_1(x)Q \int \psi(y)dE_2(y).$$

Formally, this can be re-written as

$$T_\Phi = \phi(A)Q\psi(B) = \int_{\sigma(A)} \int_{\sigma(B)} \phi(x)\psi(y)dE_1(x)QdE_2(y). \tag{1.1.6}$$

Moreover, we have

$$\|T_\Phi\| \leq \|\phi\|_{L^\infty(A;E_1)} \|\psi\|_{L^\infty(B;E_2)} \|Q\|. \tag{1.1.7}$$

The equality (1.1.6) can serve as the definition of the integral (1.1.1) for the function $\Phi(x, y) = \phi(x)\psi(y)$. This definition extends naturally to the finite sums

$$\Phi(x, y) = \sum_{1 \le k \le N} \phi_k(x)\psi_k(y),$$

in particular to the case when Φ is a polynomial in x, y and the operators A, B are bounded. However, the estimate similar to (1.1.7), i.e.

$$\|T_\Phi\| \le \|\Phi\|_{L^\infty} \|Q\|$$

is no longer valid. Theorem 1.3.1 will give an estimate of the operator norm in a more general situation. If one is only interested in the Hilbert–Schmidt norm, the estimate (1.1.5) gives the desired result.

1.2 DOI on $\boldsymbol{S}_2$

Let (X, E_1) and (Y, E_2) be two spectral measures in the space H_1 and H_2 respectively. The Hilbert–Schmidt class $\boldsymbol{S}_2 = \boldsymbol{S}_2(H_2, H_1)$ is a Hilbert space, with respect to the scalar product

$$\langle Q, R \rangle = \operatorname{tr}(QR^*) = \operatorname{tr}(R^*Q). \tag{1.2.1}$$

We will construct a certain spectral measure on $\boldsymbol{S}_2$, the *tensor product* of measures (X, E_1) and (Y, E_2), and define the DOI $\mathscr{T}_\Phi$ as integral with respect to this spectral measure.

Consider the mappings

$$\begin{cases} \mathcal{E}_1(\delta) : Q \mapsto E_1(\delta)Q, & \text{for } \delta \subset X,\ Q \in \boldsymbol{S}_2; \\ \mathcal{E}_2(\partial) : Q \mapsto QE_2(\partial), & \text{for } \partial \subset Y,\ Q \in \boldsymbol{S}_2. \end{cases} \tag{1.2.2}$$

Each operator $\mathcal{E}_1(\delta)$ is an orthogonal projection in $\boldsymbol{S}_2$, the mapping $\delta \mapsto \mathcal{E}_1(\delta)$ is σ-additive, and $\mathcal{E}_1(X) = \mathcal{I}$ (the identity transformer on $\boldsymbol{S}_2$). So we see that $\mathcal{E}_1$ is a spectral measure in $\boldsymbol{S}_2$, and the same for $\mathcal{E}_2$. The types of $\mathcal{E}_1$ and $\mathcal{E}_2$ coincide with that of E_1 and E_2 respectively. Thus for any bounded measurable functions $\phi(x)$, $\psi(y)$ we have

$$\int_X \phi(x) d(\mathcal{E}_1(x)Q) = \int_X \phi(x) dE_1(x) \cdot Q$$

and

$$\int_Y \psi(y) d(\mathcal{E}_2(y)Q) = Q \cdot \int_Y \psi(y) dE_2(y).$$

The measures $\mathcal{E}_1$ and $\mathcal{E}_2$ commute, since one corresponds to the multiplication from the left and the other from the right.

The mapping

$$\mathcal{E}(\delta \times \partial) = \mathcal{E}_1(\delta)\mathcal{E}_2(\partial) : Q \mapsto E_1(\delta)QE_2(\partial) \tag{1.2.3}$$

is an additive projection-valued function on the set of all "measurable rectangles" $\delta \times \partial \subset X \times Y$ (orthogonal projections on $\boldsymbol{S}_2$). It turns out (see [20]) that this function is σ-additive. The σ-additive projection-valued function $\mathcal{E}(\Delta)$ extends, in a unique way, from the set of measurable rectangles $\Delta = \delta \times \partial$ to the minimal σ-algebra $\mathcal{A}_0$ of subsets in $X \times Y$, generated by such rectangles, and the extension is σ-algebra, so it is a spectral measure in $\boldsymbol{S}_2$. We denote it by the same notation $\mathcal{E}$. It is convenient to add to $\mathcal{A}_0$ all the subsets $\epsilon' \subset \epsilon$ of sets

$\epsilon \in \mathcal{A}_0$ of $\mathcal{E}$-measure zero, putting $\mathcal{E}(\epsilon') = 0$. The resulting family $\mathcal{A}$ is also a σ-algebra, and the spectral measure $\mathcal{E}$ on $\mathcal{A}$ is *N–full* (see Birman section I.3.7). A scalar measure of type $\mathcal{E}$ can be chosen as the measure μ in (1.1.5).

Now we take by definition

$$\mathscr{T}_\Phi = \int_{X\times Y} \Phi(x,y) d\mathcal{E}(x,y), \tag{1.2.4}$$

or

$$\mathscr{T}_\Phi Q = \int_{X\times Y} \Phi(x,y) d(\mathcal{E}(x,y)Q). \tag{1.2.5}$$

So, for bounded Φ this is a bounded transformer in $\boldsymbol{S}_2$. Then we have

$$\mathscr{T}_{\Phi_1+\Phi_2} = \mathscr{T}_{\Phi_1} + \mathscr{T}_{\Phi_2},\ \mathscr{T}_{\Phi_1\Phi_2} = \mathscr{T}_{\Phi_1}\mathscr{T}_{\Phi_2}; \tag{1.2.6}$$

$$\mathscr{T}_{\bar{\Phi}} = \mathscr{T}^*_\Phi; \tag{1.2.7}$$

$$\|\mathscr{T}_\Phi\| = \|\Phi\|_{L^\infty(X\times Y)}. \tag{1.2.8}$$

If $\Phi(x,y) = \phi(x)$, then $\mathscr{T}_\Phi = \int_X \phi(x) d\mathcal{E}_1(x)$, or $\mathscr{T}_\Phi Q = \int_X \phi(x) dE_1(x) \cdot Q$. The similar formula is valid for $\Phi(x,y) = \psi(y)$. From this observation and (1.2.6), we see that

$$\int_{X\times Y} \phi(x)\psi(y) d(\mathcal{E}(x,y)Q) = \int_X \phi(x) dE_1(x) \cdot Q \cdot \int_Y \psi(y) dE_2(y).$$

1.3 DOI on $\boldsymbol{S}_1$ and $\mathcal{B}$

1.3.1 Class $\mathfrak{M}$

Now we extend the definition of $\mathscr{T}_\Phi$ to the space $\mathcal{B} = \mathcal{B}(H_2, H_1)$ of all bounded operators. To do this we need some additional assumptions on the symbol Φ since it is not always possible.

Let $\boldsymbol{S}_1$ be the trace class of operators, then

$$\boldsymbol{S}_1 \subset \boldsymbol{S}_2 \subset \mathcal{B}. \tag{1.3.1}$$

Moreover, the space $\mathcal{B}$ is adjoint to $\boldsymbol{S}_1$, with repect to the duality given by (1.2.1):

$$\langle Q, R \rangle = \operatorname{tr}(QR^*), \quad Q \in \boldsymbol{S}_1, R \in \mathcal{B}. \tag{1.3.2}$$

Clearly, any transformer $\mathscr{T}_\Phi$ with a L^∞–symbol maps $\boldsymbol{S}_1$ into $\boldsymbol{S}_2$. Suppose that $\mathscr{T}_\Phi$ is a bounded transformer from $\boldsymbol{S}_1$ into $\boldsymbol{S}_1$ itself for a given function Φ. Then the transformer $\mathscr{T}_{\bar\Phi}$ is also bounded in $\boldsymbol{S}_1$ and has the same norm. The adjoint transformer $\mathscr{T}^*_{\bar\Phi}$ acts in the space $\mathcal{B}$. The equality (1.2.7) shows that it is natural to define

$$\mathscr{T}_\Phi Q = (\mathscr{T}_{\bar\Phi} | \boldsymbol{S}_1)^* Q, \quad \forall Q \in \mathcal{B}. \tag{1.3.3}$$

The properties (1.2.6) of the transformers $\mathscr{T}_\Phi$ extend to the whole of $\mathcal{B}$.

Let $\mathscr{T}_\Phi$ be a bounded transformer with a L^∞–symbol that maps from $\boldsymbol{S}_1$ into $\boldsymbol{S}_1$. If $Q \in \boldsymbol{S}_\infty$ (the space of all compact operators), then $\mathscr{T}_\Phi Q \in \boldsymbol{S}_\infty$. Indeed, it is sufficient to show this for the dense in $\boldsymbol{S}_\infty$ subset $\mathcal{K}$ of finite rank operators. But if $Q \in \mathcal{K}$, then

$\mathscr{T}_\Phi Q \in \boldsymbol{S}_1 \subset \boldsymbol{S}_\infty$. So $\mathscr{T}_\Phi$ acts from $\boldsymbol{S}_\infty$ into $\boldsymbol{S}_\infty$ and

$$\|\mathscr{T}_\Phi\|_{\mathcal{B}\to\mathcal{B}} = \|\mathscr{T}_\Phi\|_{\boldsymbol{S}_1\to\boldsymbol{S}_1} = \|\mathscr{T}_\Phi\|_{\boldsymbol{S}_\infty\to\boldsymbol{S}_\infty}. \tag{1.3.4}$$

By interpolation, we get

$$\|\mathscr{T}_\Phi\|_{\mathcal{B}\to\mathcal{B}} \geq \|\mathscr{T}_\Phi\|_{\boldsymbol{S}_2\to\boldsymbol{S}_2} = \|\Phi\|_{L^\infty}. \tag{1.3.5}$$

Denote by $\mathfrak{M}_\mathcal{B}$ the set of all functions Φ on $X \times Y$, such that the transformer $\mathscr{T}_\Phi$ is bounded on $\mathcal{B}$. This is a normed algebra of function, with respect to the norm

$$\|\Phi\|_{\mathfrak{M}_\mathcal{B}} = \|\mathscr{T}_\Phi\|_{\mathcal{B}\to\mathcal{B}}.$$

The mapping $\Phi \mapsto \bar{\Phi}$ is an involution in $\mathfrak{M}_\mathcal{B}$. It then follows from (1.3.5) that the algebra $\mathfrak{M}_\mathcal{B}$ is complete and hence, is a Banach C^*–algebra. The Banach algebras $\mathfrak{M}_{\boldsymbol{S}_1}$ and $\mathfrak{M}_{\boldsymbol{S}_\infty}$ are introduced in the same way. It follows from duality that

$$\mathfrak{M} := \mathfrak{M}_\mathcal{B} = \mathfrak{M}_{\boldsymbol{S}_1} = \mathfrak{M}_{\boldsymbol{S}_\infty},$$

including equality of the corresponding norms.

The class $\mathfrak{M}$ depends on the choice of the spectral measures E_1 and E_2. We shall use $\mathfrak{M}(E_1, E_2)$ when it is useful to reflect this dependence explicitly.

1.3.2 Criterion of $\Phi \in \mathfrak{M}$

Let (X, E_1) and (Y, E_2) be two spectral measures in the space H_1 and H_2 respectively. For each $h_1 \in H_1$, the function $\rho_{h_1}(\cdot) = (E_1(\cdot)h_1, h_1)$ is a finite scalar measure. Similarly , the function $\tau_{h_2}(\cdot) = (E_2(\cdot)h_2, h_2)$ is defined for each $h_2 \in H_2$. The class $\mathfrak{M}(E_1, E_2)$ admits the following description.

Theorem 1.3.1. [17, 18, 23] *Let $\Phi \in L^\infty(E_1, E_2)$. Then the following statements are equivalent:*

(i) $\Phi \in \mathfrak{M} = \mathfrak{M}(E_1, E_2)$.

(ii) *For any $h_2 \in H_2$, $h_1 \in H_1$ the integral operator*

$\mathbf{K}_{h_2,h_1} : L^2(Y; \tau_{h_2}) \to L^2(X; \rho_{h_1}), \quad (\mathbf{K}_{h_2,h_1}u)(x) = \int_Y \Phi(x, y)u(y)d\tau_{h_2}(y)$

belongs to $\boldsymbol{S}_1$, and

$$\sup_{\|h_1\|=\|h_2\|=1} \|\mathbf{K}_{h_2,h_1}\|_{\boldsymbol{S}_1} =: C < \infty.$$

Moreover,

$$\|\Phi\|_{\mathfrak{M}} = C.$$

(iii) *There exist a measure space (Z, η) and measurable functions α on $X \times Z$, β on $Y \times Z$ such that*

$$\Phi(x, y) = \int_Z \alpha(x, z)\beta(y, z)d\eta(z) \tag{1.3.6}$$

and

$$\begin{cases} A^2 := (E_1)\text{-}\sup\limits_x \int_Z |\alpha(x, z)|^2 d\eta(z) < \infty; \\ B^2 := (E_2)\text{-}\sup\limits_y \int_Z |\beta(y, z)|^2 d\eta(z) < \infty. \end{cases} \tag{1.3.7}$$

For any such factorization

$$\|\Phi\|_{\mathfrak{M}} \le AB, \tag{1.3.8}$$

and there exists a factorization such that

$$cAB \leq \|\Phi\|_{\mathfrak{M}}, \quad c > 0. \tag{1.3.9}$$

The constant c does not depend on the spectral measures E_1, E_2.

For the proof, see [17], [18] and [23]. The set of functions that admit the representation in (1.3.6) and (1.3.7) is called *the integral projective tensor product* of spaces $L^\infty(E_1)$ and $L^\infty(E_2)$.

1.4 Transformers on other classes

Let $\mathcal{B} = \mathcal{B} = (H_2, H_1)$, where H_1 and H_2 be two given separable Hilbert spaces. For each $Q \in \mathcal{B}$, the singular values s_n is defined by $s_n(Q) := \lambda_n(\sqrt{Q^*Q})$, $n \geq 0$. The Schatten ideals $\boldsymbol{S}_p$, weak $\boldsymbol{S}_p$–ideals $\boldsymbol{S}_{p,w}$,ideals $\boldsymbol{S}^\circ_{p,w}$ and spaces $\boldsymbol{S}_{p,1}$ are defined by

$$\boldsymbol{S}_p = \{Q \in \boldsymbol{S}_\infty : \{s_n(Q)\} \in l_p\}, \quad 0 < p < \infty. \tag{1.4.1}$$

$$\boldsymbol{S}_{p,w} = \{Q \in \boldsymbol{S}_\infty : s_n(Q) = O(n^{-1/p})\}, \quad 0 < p < \infty. \tag{1.4.2}$$

$$\boldsymbol{S}^\circ_{p,w} = \{Q \in \boldsymbol{S}_\infty : s_n(Q) = o(n^{-1/p})\}, \quad 0 < p < \infty. \tag{1.4.3}$$

$$\boldsymbol{S}_{p,1} = \{Q \in \boldsymbol{S}_\infty : \sum_n (n+1)^{p^{-1}-1} s_n(Q) < \infty\}, \quad 0 < p < \infty. \tag{1.4.4}$$

For $1 < p < \infty$, certain norms can be introduced such that these spaces become Banach algebras. They will be called the nice symmetrically-normed ideals (See [12] for more details on symmetrically-normed ideals (SNI)) and we have the following duality relations given by

(1.2.1)

$$\boldsymbol{S}_p^* = \boldsymbol{S}_{p'};\ (\boldsymbol{S}_{p,w}^\circ)^* = \boldsymbol{S}_{p',1};\ \boldsymbol{S}_{p,1}^* = \boldsymbol{S}_{p',w}, \quad 1/p' = 1 - 1/p. \tag{1.4.5}$$

Given a SNI $\boldsymbol{S}$, the set of symbols Φ, such that the transformer $\mathscr{T}_\Phi$ is bounded on $\boldsymbol{S}$, form a commutative Banach algebra of functions on $X \times Y$, with complex conjugation as the involution. We denote this algebra as $\mathfrak{M}_{\boldsymbol{S}}$. It follows from the duality arguments and interpolation that for $1 < p < \infty$

$$\mathfrak{M}_{\boldsymbol{S}_p} = \mathfrak{M}_{\boldsymbol{S}_{p'}};\ \ \mathfrak{M}_{\boldsymbol{S}_{p,w}^\circ} = \mathfrak{M}_{\boldsymbol{S}_{p',1}} = \mathfrak{M}_{\boldsymbol{S}_{p,w}} \tag{1.4.6}$$

and for any nice SNI $\boldsymbol{S}$, the following topological imbeddings hold:

$$\mathfrak{M} \subset \mathfrak{M}_{\boldsymbol{S}} \subset \mathfrak{M}_{\boldsymbol{S}_2} = L^\infty(X \times Y). \tag{1.4.7}$$

Furthermore, if $\Phi \in \mathfrak{M}$, then $\|\Phi\|_{\mathfrak{M}_{\boldsymbol{S}}} \le \|\Phi\|_{\mathfrak{M}}$ for any nice SNI $\boldsymbol{S}$.

1.5 Applications of DOI to the perturbation theory

1.5.1 Transformers Z_ϕ

Let $(\mathbb{R}, E_1)$ and $(\mathbb{R}, E_2)$ be two spectral measures in the separable Hilbert space H_1 and H_2 respectively. The spectral measure $\mathcal{E}$ is defined as in §1.2. Denote the diagonal of $\mathbb{R}^2$ by **diag**, i.e. $\mathbf{diag} \stackrel{\text{def}}{=} \{(x,x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Then it is shown in [21] that $\mathcal{E}|\mathbf{diag}$ is an atomic measure.

Let A and B be self-adjoint operators in the Hilbert spaces H_1 and H_2 respectively. If ϕ

is a uniformly Lipschitz function on $\mathbb{R}$, then the function

$$\check{\phi}(x,y) = \frac{\phi(x) - \phi(y)}{x-y}$$

is well defined and continuous outside the diagonal and bounded. Suppose that it is somehow extended to **diag** and the extended function is bounded on $\mathbb{R}^2$. Note that this function is always $\mathcal{E}$–measure since $\mathcal{E}$ is N–full. If at some point $x \in \mathbb{R}$ the function ϕ is differentiable, the natural choice of extension is $\check{\phi}(x,x) \overset{\text{def}}{=} \phi'(x)$. Otherwise, the value of $\check{\phi}(x,x)$ can be chosen arbitrary.

Below we suppose that some extension of $\check{\phi}$ to the whole of $\mathbb{R}^2$ is chosen and fixed. Then the transformer

$$Z_{\phi}^{A,B} \overset{\text{def}}{=} \mathscr{T}_{\check{\phi}}^{A,B} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{x-y} dE_1^A(x)(\cdot) dE_2^B(y) \tag{1.5.1}$$

is well defined, at least on the class $\boldsymbol{S}_2$. We do not reflect the choice of extension in the notations, since the formulas presented in Theorem 1.5.1, 1.5.2 hold true independently of it. Moreover, for any SNI $\boldsymbol{S}$ the membership $\check{\phi} \in \mathfrak{M}_{\boldsymbol{S}}$ does not depend on this choice. This follows from section 7.1 of [21].

Theorem 1.5.1. [18] *Let $H_1 = H_2$ be a Hilbert space and A, B be self-adjoint operators with the same domain in H_1, and suppose that $B - A \in \boldsymbol{S}$ where $\boldsymbol{S}$ is a nice SNI. Suppose also that the function ϕ is such that $\check{\phi} \in \mathfrak{M}_{\boldsymbol{S}}$. Then,*

$$\phi(B) - \phi(A) = Z_{\phi}^{A,B}(B-A). \tag{1.5.2}$$

It allows the operators A, B to be unbounded.

Formula (1.5.2) is called *the Birman–Solomyak formula*. A similar formula also holds for unitary operators, in which case we have to integrate $\check{\phi}$ of a function ϕ on the unit circle with with respect to the spectral measures of the corresponding operator integrals.

Theorem 1.5.1 extends to the quasi–commutators $JB - AJ$. Here J is a linear bounded operator acting from H_2 to H_1. The operators A, B are not supposed bounded, and $JB - AJ$ is understood as the operator generated by the sesqui-linear form $(JBh_2, h_1) - (Jh_2, Ah_1)$ where $h_1 \in \mathrm{Dom}\ A$, $h_2 \in \mathrm{Dom}\ B$.

Theorem 1.5.2. [18] *Let A and B be self-adjoint operators in Hilbert space H_1 and H_2 respectively and let $J \in \mathcal{B}(H_2, H_1)$. Suppose that $JB - BA \in \boldsymbol{S}$ where $\boldsymbol{S}$ is a nice SNI, and that $\check{\phi} \in \mathfrak{M}_{\boldsymbol{S}}$. Then, independently on the way $\check{\phi}$ is defined on the diagonal,*

$$J\phi(B) - \phi(A)J = Z_{\phi}^{A,B}(JB - AJ). \tag{1.5.3}$$

Theorem 1.5.2 turns into Theorem 1.5.1 if we take $H_2 = H_1$ and $J = I$. Both theorems were proved in [18].

1.5.2 Tests for $\check{\phi} \in \mathfrak{M}_{\boldsymbol{S}}$

For practical usage of Theorem 1.5.1, 1.5.2 one needs tools for checking the inclusion $\check{\phi} \in \mathfrak{M}_{\boldsymbol{S}}$ for a given SNI $\boldsymbol{S}$. A particular case of Theorem 1.5.1 says that for self-adjoint operators A and B

$$|\phi(x) - \phi(y)| \le L|x - y| \Rightarrow \|\phi(A) - \phi(B)\|_{\boldsymbol{S}_2} \le L\|A - B\|_{\boldsymbol{S}_2}.$$

It is well known that a Lipschitz function on the real line is not necessarily *operator Lipschitz*, i.e., the condition

$$|\phi(x) - \phi(y)| \le \text{const}|x - y|$$

does not imply that for self-adjoint operators A and B

$$\|\phi(A) - \phi(B)\| \le \text{const}\|A - B\|.$$

Denote $\mathfrak{M}(\mathbb{R}, \mathbb{R})$ and $\mathfrak{M}(\mathbb{T}, \mathbb{T})$ by $\mathfrak{M}(\mathbb{R})$ and $\mathfrak{M}(\mathbb{T})$ respectively. It was shown in [23] that if ϕ is a trigonometric polynomial of degree d, then $\check{\phi} \in \mathfrak{M}(\mathbb{T})$ and

$$\|\check{\phi}\|_{\mathfrak{M}(\mathbb{T})} \le \text{const}\, d\, \|\phi\|_{L^\infty}. \tag{1.5.4}$$

On the other hand, it was shown in [25] that if ϕ is a bounded function on $\mathbb{R}$ whose Fourier transform is supported on $[-\sigma, \sigma]$ (in other words, ϕ is an entire function of exponential type at most σ that is bounded on $\mathbb{R}$), then $\check{\phi} \in \mathfrak{M}(\mathbb{R})$ and

$$\|\check{\phi}\|_{\mathfrak{M}(\mathbb{R})} \le \text{const}\, \sigma\, \|\phi\|_{L^\infty}. \tag{1.5.5}$$

Inequalities (1.5.4) and (1.5.5) led in [23] and [25] to the fact that functions in functions in the Besov spaces $B^1_{\infty,1}(\mathbb{T})$ and $B^1_{\infty,1}(\mathbb{R})$ are operator Lipschitz. The gegeral Besov space $B^s_{p,q}$ will be defined in §2.1 for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. Here we only give an equivalent description for $B^1_{\infty,1}(\mathbb{R})$. A function ϕ on $\mathbb{R}$ belongs to for $B^1_{\infty,1}(\mathbb{R})$ if

$$r_0(\phi) \stackrel{\text{def}}{=} \int_0^\infty (\sup_{x\in\mathbb{R}} |\phi(x+t) - 2\phi(x) + \phi(x-t)|) \frac{dt}{t^2} < \infty. \tag{1.5.6}$$

Any function $\phi \in B^1_{\infty,1}(\mathbb{R})$ has uniformly bounded continuous derivative, and we denote

$$r(\phi) \stackrel{\text{def}}{=} r_0(\phi) + \sup_{x\in\mathbb{R}} |\phi'(x)|.$$

The space $B^1_{\infty,1}(\mathbb{T})$ can be defined similarly.

Theorem 1.5.3. [25] *Let $\phi \in B^1_{\infty,1}(\mathbb{R})$. Then $\check{\phi} \in \mathfrak{M}(\mathbb{R})$ for any spectral measures E_1, E_2, and*

$$\|\check{\phi}\|_{\mathfrak{M}(\mathbb{R})} \leq Cr(\phi),$$

where the constant C is independent of the spectral measures E_1, E_2. In particular, for any nice SNI $\boldsymbol{S}$ we have

$$\|J\phi(B) - \phi(A)J\|_{\boldsymbol{S}} \leq Cr(\phi)\|JB - BA\| \tag{1.5.7}$$

for self–adjoint operators A, B and bounded operator J as in Theorem 1.5.2.

A similar result also holds when $\phi \in B^1_{\infty,1}(\mathbb{T})$.

In chapter 2 we will explain the result found in [1] that if ϕ belongs to the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$, then $\|\phi(A) - \phi(B)\| \leq \text{const}\|A - B\|^\alpha$ for arbitrary self-adjoint operators A and B.

1.6 DOI with respect to semi-spectral measures

Let H be a Hilbert space and let $(X, \mathcal{A})$ be a measurable space. A map $\mathcal{F}$ from $\mathcal{A}$ to the algebra $\mathcal{B}(H)$ of all bounded operators on H is called a *semi–spectral* measure if

$$\mathcal{F}(\Delta) \geq \mathbf{0}, \quad \Delta \in \mathcal{A},$$

$$\mathcal{F}(\emptyset) = \mathbf{0} \ \text{ and } \ \mathcal{F}(X) = I,$$

and for a sequence $\{\Delta_j\}_{j\geq 1}$ of disjoint sets in $\mathcal{A}$,

$$\mathcal{F}(\mathop{\cup}_{j=1}^{\infty} \Delta_j) = \lim_{N\to\infty} \sum_{j=1}^{N} \mathcal{F}(\Delta_j) \quad \text{in the weak operator topology.}$$

If K is a Hilbert space, $(X, \mathcal{A})$ is a measurable space, $F : \mathcal{A} \mapsto \mathcal{B}(K)$ is a spectral measure, and H is a subspace of K, then it is easy to see that the map $\mathcal{F} : \mathcal{A} \mapsto \mathcal{B}(H)$ defined by

$$\mathcal{F}(\Delta) = P_H F(\Delta)|H, \quad \Delta \in \mathcal{A} \tag{1.6.1}$$

is a semi-spectral measure. Here P_H stands for the orthogonal projection onto H.

Naimark proved in [15] that all semi–spectral measures can be obtained in this way, i.e., a semi–spectral measure is always a *compression* of a spectral measure. A spectral measure F satisfying (1.6.1) is called a *spectral dilation of the semi–spectral measure* $\mathcal{F}$.

A spectral dilation F of a semi-spectral measure $\mathcal{F}$ is called *minimal* if

$$K = \text{clos span}\{F(\Delta)H : \quad \Delta \in \mathcal{A}\}.$$

It was shown in [16] that if F is a minimal spectral dilation of a semi–spectral measure $\mathcal{F}$, then F and $\mathcal{F}$ are mutually absolutely continuous and all minimal spectral dilations of a semi–spectral measure are isomorphic in the natural sense.

If ϕ is a bounded measurable function X and $\mathcal{F} : \mathcal{A} \mapsto \mathcal{B}(H)$ is a semi–spectral measure, then the integral

$$\int_X \phi(x) d\mathcal{F}(x) \tag{1.6.2}$$

can be defined as

$$\int_X \phi(x)d\mathcal{F}(x) = P_H(\int_X \phi(x)dF(x))\Big| H, \tag{1.6.3}$$

where F is a spectral dilation of $\mathcal{F}$. It is easy to see that the right–hand side of (1.6.3) does not depend on the choice of a spectral dilation. The integral (1.6.2) can also be computed as the limit of sums

$$\sum \phi(x_\alpha)\mathcal{F}(\Delta_\alpha), \quad x_\alpha \in \Delta_\alpha,$$

over all finite measurable partitions $\{\Delta_\alpha\}_\alpha$ of X.

If T is a contraction on a Hilbert space H, then by the Sz.–Nagy dilation theorem (see [8]), T has a unitary dilation, i.e., there exist a Hilbert space K such that $H \subset K$ and a unitary operator U on K such that

$$T^n = P_H U^n | H, \quad n \geq 0, \tag{1.6.4}$$

where P_H is the orthogonal projection onto H. Let F_U be the spectral measure of U. Consider the operator set funtion $\mathcal{F}$ defined on the Borel subsets of the unit circle $\mathbb{T}$ by

$$\mathcal{F}(\Delta) = P_H F_U(\Delta) | H, \quad \Delta \subset \mathbb{T}.$$

Then $\mathcal{F}$ is a semi–spectral measure. It follows from (1.6.4) that

$$T^n = \int_{\mathbb{T}} \zeta^n d\mathcal{F}(\zeta) = P_H \int_{\mathbb{T}} \zeta^n dF_U(\zeta)\Big| H, \quad n \geq 0. \tag{1.6.5}$$

Such a semi–spectral measure $\mathcal{F}$ is called a *semi–spectral measure* of $\mathbb{T}$. Note that it is not unique. To have uniqueness, we consider a minimal unitary dilation U of T, which is unique up to an isomorphism (see [8]).

It follows easily from (1.6.5) that

$$\phi(T) = P_H \int_{\mathbb{T}} \phi(\zeta) dF_U(\zeta) \Big| H$$

for an arbitrary function ϕ in the disk–algebra C_A.

In [24] and [27] DOI with respect to semi–spectral measures were introduced.

Suppose that $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ are measurable spaces, and $\mathcal{F}_1 : \mathcal{A}_1 \mapsto \mathcal{B}(H_1)$ and $\mathcal{F}_2 : \mathcal{A}_2 \mapsto \mathcal{B}(H_2)$ are semi–spectral measures. Then double operator integral

$$\int_{X_1 \times X_2} \Phi(x_1, x_2) d\mathcal{F}_1(x_1) Q d\mathcal{F}_2(x_2)$$

were defined in [27] in the case when $Q \in \boldsymbol{S}_2$ and Φ is a bounded symbol. DOI were also defined in [27] in the case when Q is a bounded linear operator and Φ belongs to the integral projective tensor product of the spaces $L^\infty(\mathcal{F}_1)$ and $L^\infty(\mathcal{F}_2)$.

In particular, the following Birman–Solomyak formula holds:

$$\phi(R) - \phi(Q) = \int_{\mathbb{T} \times \mathbb{T}} \check{\phi}(\zeta, \tau) d\mathcal{F}_R(\zeta)(R - Q) d\mathcal{F}_Q(\tau). \tag{1.6.6}$$

Here R and Q are contractions on Hilbert space. $\mathcal{F}_R$ and $\mathcal{F}_Q$ are their semi–spectral measures, and ϕ is an analytic in $\mathbb{D}$ of class $(B^1_{\infty,1})_+$ (For definition, see section §2.2.1).

Chapter 2

Operator Hölder Functions and arbitrary moduli of continuity

2.1 Introduction

Let $0 < \alpha < 1$. In was proved in [1] that the functions in the Hölder class Λ_α are also operator Hölder continuous for arbitrary self–adjoint operators, unitary operators and contractions, and their sharp estimates are also obtained. Similar results for maximal dissipative operators, normal operators and n–tuples of self–adjoint oprators are also obtained in [1], [4], [5] and [10]. In those papers, the authors also extended the results to class $\Lambda\omega$. We will show the proof for self–adjoint operators and unitary operators and give a short discussion for other types of operators. An introduction to the function spaces Λ_α and $\Lambda\omega$ are given below in §2.2.1 and §2.2.2.

2.2 Function spaces

2.2.1 Besov classes

In this subsection we give a brief introduction to the Besov spaces that play an important role in the problems of perturbation theory. We start with Besov spaces on the unit circle.

Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. The Besov class $B^s_{p,q}$ of functions (or distributions) on $\mathbb{T}$ can be defined in the following way. Let w be an infinitely differentiable function on $\mathbb{R}$ such that

$$w \ge 0, \text{supp } w \subset [\frac{1}{2}, 2], \text{ and } w(x) = 1 - w(\frac{x}{2}) \text{ for } x \in [1, 2]. \tag{2.2.1}$$

Define a C^∞ function v on $\mathbb{R}$ by

$$v(x) = 1 \text{ for } x \in [-1, 1] \text{ and } v(x) = w(|x|) \text{ if } |x| \ge 1. \tag{2.2.2}$$

Define trigonometric polynomials W_n, $W_n^\sharp$ and V_n by

$$W_n(z) = \sum_{k \in \mathbb{Z}} w(\frac{k}{2^n}) z^k, n \ge 1, \; W_0(z) = \bar{z} + 1 + z, \text{ and } W_n^\sharp(z) = \overline{W_n(z)}, n \ge 0$$

and

$$V_n(z) = \sum_{k \in \mathbb{Z}} v(\frac{k}{2^n}) z^k, n \ge 1.$$

V_n is called de la Vallée Poussin type kernel.

If f is a distribution on $\mathbb{T}$, we define f_n, $n \ge 0$ by

$$f_n = f * W_n + f * W_n^\sharp, \; n \ge 1, \text{ and } f_0 = f * W_0,$$

Then $f = \sum_{n \ge 0} f_n$ and $f - f * V_n = \sum_{k=n+1}^\infty f_n$.

The Besov class $B^s_{p,q}$ consists of functions (in the case $s > 0$) or distributions f on $\mathbb{T}$ such that

$$\{\|2^{ns} f * W_n\|_{L^p}\}_{n \ge 1} \in \ell^q \text{ and } \{\|2^{ns} f * W_n^\sharp\|_{L^p}\}_{n \ge 1} \in \ell^q \tag{2.2.3}$$

Besov classes admit many other descriptions. In particular, for $s > 0$, the space $B^s_{p,q}$ admits the following characterization. A function $f \in L^p$ belongs to $B^s_{p,q}$, $s > 0$, if and only if

$$\begin{cases} \int_{\mathbb{T}} \frac{\|\Delta^n_\tau f\|^q_{L^p}}{|1-\tau|^{1+sq}} d\boldsymbol{m}(\tau), & \text{for} \quad q < \infty; \\ \sup\limits_{\tau \neq 1} \frac{\|\Delta^n_\tau f\|_{L^p}}{|1-\tau|^s} < \infty, & \text{for} \quad q = \infty. \end{cases} \tag{2.2.4}$$

Here $\boldsymbol{m}$ is the normalized Lebesgue measure on $\mathbb{T}$, n is an integer greater than s, and Δ_τ, $\tau \in \mathbb{T}$, is the difference operator:

$$(\Delta_\tau f)(\zeta) = f(\tau\zeta) - f(\zeta), \quad \zeta \in \mathbb{T}.$$

We use the notation B^s_p for $B^s_{p,p}$.

The spaces $\Lambda_\alpha \stackrel{\text{def}}{=} B^\alpha_\infty$ form the *Hölder–Zygmund class.* If $0 < \alpha < 1$, then $f \in \Lambda_\alpha$ if and only if

$$|f(\zeta) - f(\tau)| \le \text{const}|\zeta - \tau|^\alpha, \quad \zeta, \tau \in \mathbb{T}.$$

These spaces are called the *Hölder spaces.* A function $f \in \Lambda_1$ if and only if f is continuous and

$$|f(\zeta\tau) - 2f(\zeta) + f(\zeta\bar{\tau})| \le \text{const}|1 - \tau|, \quad \zeta, \tau \in \mathbb{T}.$$

By (2.2.4), for $\alpha > 0$, $f \in \Lambda_\alpha$ if and only if f is continuous and

$$|(\Delta^n_\tau f)(\zeta)| \le \text{const}|1 - \tau|^\alpha,$$

where n is a positive integer such that $n > \alpha$.

Note that the (semi)norm of a function f in Λ_α is equivalent to

$$\sup_{n\geq 1} 2^{n\alpha}(\|f * W_n\|_{L^\infty} + \|f * W_n^\sharp\|_{L^\infty}).$$

It is easy to see from the definition of Besov classes that the Riesz projection $\mathbb{P}_+$,

$$\mathbb{P}_+ f = \sum_{n\geq 0} \hat{f}(n) z^n,$$

is bounded on $B^s_{p,q}$. Functions in $(B^s_{p,q})_+ \stackrel{\text{def}}{=} \mathbb{P}_+ B^s_{p,q}$ admit a natural extension to analytic functions in the unit disk $\mathbb{D}$. It is well known that the functions in $(B^s_{p,q})_+$ admit the following description:

$$f \in (B^s_{p,q})_+ \Leftrightarrow \int_0^1 (1-r)^{q(n-s)-1} \|f_r^{(n)}\|_p^q dr < \infty, \quad q < \infty,$$

and

$$f \in (B^s_{p,\infty})_+ \Leftrightarrow \sup_{0<r<1} (1-r)^{(n-s)} \|f_r^{(n)}\|_p < \infty,$$

where $f_r(\zeta) \stackrel{\text{def}}{=} f(r\zeta)$ and n is a nonnegative integer greater than s.

Let us proceed now to Besov spaces on the real line. We consider homogeneous Besov spaces $B^s_{p,q}(\mathbb{R})$ of functions (distributions) on $\mathbb{R}$. We use the same functions w, v as in (2.2.1), (2.2.2) , and define functions W_n, $W_n^\sharp$ and V_n on $\mathbb{R}$ by

$$\mathscr{F} W_n(x) = w(\frac{x}{2^n}),\ \mathscr{F} W_n^\sharp(x) = \mathscr{F} W_n(-x),\ n \in \mathbb{Z}$$

and

$$\mathscr{F}V_n(x) = v(\frac{x}{2^n}),\ n \in \mathbb{Z},$$

where $\mathscr{F}$ is the *Fourier transform*:

$$(\mathscr{F}f)(t) = \int_{\mathbb{R}} f(x)e^{-ixt}dx,\ f \in L^1.$$

V_n is also called de la Vallée Poussin type kernel.

If f belongs to $\mathscr{S}'(\mathbb{R})$, the space of tempered distribution on $\mathbb{R}$, we define f_n by

$$f_n = f * W_n + f * W_n^\sharp,\ n \in \mathbb{Z}.$$

Initially we define the (homogeneous) Besov class $\dot{B}^s_{p,q}(\mathbb{R})$ as the set of all $f \in \mathscr{S}'(\mathbb{R})$ such that

$$\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}). \tag{2.2.5}$$

According to this definition, the space $\dot{B}^s_{p,q}(\mathbb{R})$ contains all polynomials. Moreover, the distribution f is defined by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ uniquely up to a polynomial. It is easy to see that the series $\sum_{n\geq 0} f_n$ converges in $\mathscr{S}'(\mathbb{R})$. However, the series $\sum_{n<0} f_n$ can diverge in general. It is easy to prove that the series $\sum_{n<0} f_n^{(r)}$ converges uniformly on $\mathbb{R}$ for each nonnegative integer $r > s - 1/p$. Note that in the case $q = 1$ the series $\sum_{n<0} f_n^{(r)}$ converges uniformly, whenever $r \geq s - 1/p$.

Now we define the modified (homogeneous) Besov class $B^s_{p,q}(\mathbb{R})$. We say that a distribution f belongs to $B^s_{p,q}(\mathbb{R})$ if $\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$ and $f^{(r)} = \sum_{n\in\mathbb{Z}} f_n^{(r)}$ in the space $\mathscr{S}'(\mathbb{R})$, where r is the minimal nonnegative integer such that $r > s - 1/p$ ($r \geq s - 1/p$

if $q = 1$). Now the function f is determined uniquely by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ up to a polynomial of degree less than r, and a polynomial ϕ belongs to $B^s_{p,q}(\mathbb{R})$ if and only if $\deg \phi < r$.

Besov spaces $B^s_{p,q}(\mathbb{R})$ admit equivalent definitions that are similar to those discussed above in case of Besov spaces of functions on $\mathbb{T}$. In particular, the Hölder–Zygmund classes $\Lambda_\alpha(\mathbb{R}) \stackrel{\mathrm{def}}{=} B^\alpha_\infty(\mathbb{R})$, $\alpha > 0$, can be described as the classes of continuous functions f on $\mathbb{R}$ such that

$$|(\Delta^m_t)(x)| \le \mathrm{const}|t|^\alpha, \quad t \in \mathbb{R},$$

where the difference operator Δ_t is defined by

$$(\Delta_t f)(x) = f(x+t) - f(x), \quad x \in \mathbb{R},$$

and m is an integer greater than α.

As in the case of functions on the unit circle, we can introduce the following equivalent (semi)norm on $\Lambda_\alpha(\mathbb{R})$:

$$\sup_{n\in\mathbb{Z}} 2^{n\alpha}(\|f * W_n\|_{L^\infty} + \|f * W^\sharp_n\|_{L^\infty}), \quad f \in \Lambda_\alpha(\mathbb{R}).$$

The following result will be used in §2.3.

Theorem 2.2.1. [1] *Let $\alpha > 0$. Then for each $\epsilon > 0$ and each function $f \in \Lambda_\alpha(\mathbb{R})$ there exists a function $g \in \Lambda_\alpha(\mathbb{R})$ with compact support such that $f(t) = g(t)$ for $t \in [0,1]$ and*

$$\|g\|_{\Lambda_\alpha} \le \|f\|_{\Lambda_\alpha} + \epsilon,$$

where the constant can depend only on α.

To prove Theorem 2.2.1, we use the well–known fact that if ϕ and f are functions in $\Lambda_\alpha(\mathbb{R})$ and ϕ has compact support, then $\phi f \in \Lambda_\alpha(\mathbb{R})$. We refer the reader to [11], Section 4.5.2 for the proof.

Denote by $\mathscr{S}'_+(\mathbb{R})$ the set of all $f \in \mathscr{S}'(\mathbb{R})$ such that $\operatorname{supp} \mathscr{F} f \subset [0,\infty)$. We define the analytic Besov space $\big(B^s_{p,q}(\mathbb{R})\big)_+$ as $B^s_{p,q}(\mathbb{R}) \cup \mathscr{S}'_+(\mathbb{R})$. Put $\big(\Lambda_\alpha(\mathbb{R})\big)_+ \stackrel{\text{def}}{=} \Lambda_\alpha(\mathbb{R}) \cup \mathscr{S}'_+(\mathbb{R})$. For $f \in \mathscr{S}'_+(\mathbb{R})$, we have $f * W_n^\sharp = 0$, $n \in \mathbb{Z}$.

We refer the reader to [1], [13] and [26] for more detailed information on Besov spaces.

2.2.2 Spaces Λ_ω

Let ω be a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0) = 0$, $\omega(x) > 0$ for $x > 0$, and

$$\omega(x+y) \le \omega(x) + \omega(y), x, y \in [0,\infty).$$

We denote by $\Lambda_\omega(\mathbb{R})$ the space of functions on $\mathbb{R}$ such that

$$\|f\|_{\Lambda_\omega(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x \ne y} \frac{|f(x) - f(y)|}{\omega(|x-y|)}.$$

The space $\Lambda_\omega(\mathbb{T})$ on the unit circle can be defined in a similar way.

Theorem 2.2.2. [1] *There exists a constant c such that for an arbitrary modulus of continuity ω and for an arbitrary function f in $\Lambda_\omega(\mathbb{R})$, the following inequalities hold for all $n \in \mathbb{Z}$:*

$$\|f - f * V_n\|_{L^\infty} \le c\, \omega(2^{-n}) \|f\|_{\Lambda_\omega(\mathbb{R})}. \tag{2.2.6}$$

Proof. We have

$$\begin{aligned}
|f(x)-(f*V_n)(x)| &= 2^n\left|\int_{\mathbb{R}}(f(x)-f(x-y))V(2^ny)dy\right| \\
&\le 2^n\|f\|_{\Lambda_\omega(\mathbb{R})}\int_{\mathbb{R}}\omega(|y|)|V(2^ny)|dy \\
&\le 2^n\|f\|_{\Lambda_\omega(\mathbb{R})}\int_{-2^{-n}}^{2^{-n}}\omega(|y|)|V(2^ny)|dy \\
&+ 2^{n+1}\|f\|_{\Lambda_\omega(\mathbb{R})}\int_{2^{-n}}^{\infty}\omega(y)|V(2^ny)|dy.
\end{aligned}$$

Clearly,

$$2^n\int_{-2^{-n}}^{2^{-n}}\omega(|y|)|V(2^ny)|dy \le \omega(2^{-n})\|V\|_{L^1}.$$

On the other hand, keeping in mind the obvious inequality $2^{-n}\omega(y)\le 2y\omega(2^{-n})$ for $y\ge 2^{-n}$, we obtain

$$\begin{aligned}
2^{n+1}\int_{2^{-n}}^{\infty}\omega(y)|V(2^ny)|dy &\le 4\cdot 2^{2n}\omega(2^{-n})\int_{2^{-n}}^{\infty}y|V(2^ny)|dy \\
&= 4\omega(2^{-n})\int_{1}^{\infty}y|V(y)|dy \le \text{const }\omega(2^{-n})
\end{aligned}$$

This proves (2.2.6). □

Remark 2.2.3. [1] A similar inequality holds for functions on $\mathbb{T}$ of class Λ_ω:

$$\|f-f*V_n\|_{L^\infty}\le c\,\omega(2^{-n})\|f\|_{\Lambda_\omega},\quad n>0.$$

To prove it, it suffices to extend f as a 2π–periodic function on $\mathbb{R}$ and apply Theorem 2.2.2.

Corollary 2.2.4. [1] *There exists a constant c such that for an arbitrary modulus of continuity ω and for an arbitrary function f in Λ_ω, the following inequalities hold for all $n \in \mathbb{Z}$, in $\mathbb{R}$ case, or for all $n \geq 0$, in $\mathbb{T}$ case:*

$$\|f * W_n\|_{L^\infty} \leq c\,\omega(2^{-n})\|f\|_{\Lambda_\omega}, \ \|f * W_n^\sharp\|_{L^\infty} \leq c\,\omega(2^{-n})\|f\|_{\Lambda_\omega}. \tag{2.2.7}$$

Put $\big(\Lambda_\omega(\mathbb{R})\big)_+ \stackrel{\text{def}}{=} \Lambda_\omega(\mathbb{R}) \cap \mathscr{S}'_+(\mathbb{R})$ and $\mathbb{C}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Then a function in $\Lambda_\omega(\mathbb{R})$ belongs to the space $\big(\Lambda_\omega(\mathbb{R})\big)_+$ if and only if it has a (unique) continuous extension to the closed upper half-plance $\operatorname{clos}\mathbb{C}_+$ that is analytic in the open upper half-plane $\mathbb{C}_+$ with at most a polynomial growth rate at infinity.

2.3 Hölder estimates for self–adjoint operators

In this section we show that Hölder functions on $\mathbb{R}$ of order α, $0 < \alpha < 1$, must also be operator Hölder of order α. Note that if A and B are self–adjoint operators, we say that operator $A - B$ is bounded if $B = A + K$ for some bounded self–adjoint operator K. In particular, this implies that $\operatorname{Dom} A = \operatorname{Dom} B$. We say that $\|A - B\| = \infty$ if there is no such a bounded operator K that $B = A + K$.

Lemma 2.3.1. [3] *Let A and B be self–adjoint operators and let R be an operator of norm 1. Then there exist a sequence of operators $\{R_n\}_{n\geq 1}$ and sequences of bounded self–adjoint operators $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ such that*

(i) *the sequence $\{\|R_n\|\}_{n\geq 1}$ is nondecreasing and $\lim_{n\to\infty} \|R_n\| = 1$;*

(ii) *$\lim_{n\to\infty} R_n = R$ in the strong operator topology;*

(iii) *for every continuous function f on $\mathbb{R}$, the sequence*

$$\{\|f(A_n)R_n - R_n f(B_n)\|\}_{n\geq 1}$$

is nondecreasing and

$$\lim_{n\to\infty} \|f(A_n)R_n - R_n f(B_n)\| = \|f(A)R - Rf(B)\|;$$

(iv) *if f is a continuous function on $\mathbb{R}$ such that $\|f(A)R - Rf(B)\| < \infty$, then*

$$\lim_{n\to\infty} f(A_n)R_n - R_n f(B_n) = f(A)R - Rf(B)$$

in the strong operator topology;

(v) *if f is a continuous function on $\mathbb{R}$ such that $\|f(A)R - Rf(B)\| < \infty$, then the sequence*

$$\{s_j(f(A_n)R_n - R_n f(B_n))\}_{n\geq 1}$$

is nondecreasing for every $j \geq 0$ and

$$\lim_{n\to\infty} s_j(f(A_n)R_n - R_n f(B_n)) = s_j(f(A)R - Rf(B)).$$

Proof. Put $P_n \stackrel{\text{def}}{=} E_A([-n,n])$ and $Q_n \stackrel{\text{def}}{=} E_B([-n,n])$ where E_A and E_B are the spectral measures of A and B. Put $A_n \stackrel{\text{def}}{=} P_nA = AP_n$ and $B_n \stackrel{\text{def}}{=} Q_nB = BQ_n$. Clearly,

$$P_n(f(A)R - Rf(B))Q_n = f(A_n)P_nRQ_n - P_nRQ_nf(B_n), \quad n \geq 1. \tag{2.3.1}$$

It remains to put $R_n \stackrel{\text{def}}{=} P_n R Q_n$. $\square$

Theorem 2.3.2. [1] *Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R})$ and for arbitrary self–adjoint operators A and B on Hilbert space the following inequality holds:*

$$\|f(A) - f(B)\| \le c\,\|f\|_{\Lambda_\alpha(\mathbb{R})} \cdot \|A - B\|^\alpha. \tag{2.3.2}$$

Proof. Due to Lemma 2.3.1, we can assume that A and B are bounded operators. It then follows from Theorem 2.2.1 that we may assume that $f \in L^\infty(\mathbb{R})$ and we have to obtain an estimate for $\|f(A) - f(B)\|$ that does not depend on $\|f\|_{L^\infty}$. Put

$$f_n = f * W_n + f * W_n^\sharp.$$

Let us show that

$$f(A) - f(B) = \sum_{n=-\infty}^{\infty} (f_n(A) - f_n(B)) \tag{2.3.3}$$

and the series on the right converges absolutely in the operator norm.

For $N \in \mathbb{Z}$, we put $g_N \stackrel{\text{def}}{=} f * V_N$. Clearly,

$$f = f * V_N + \sum_{n>N} f_n$$

and the series on the right converges absolutely in the L^∞ norm. Thus

$$f(A) = (f * V_N)(A) + \sum_{n>N} f_n(A) \quad \text{and} \quad f(B) = (f * V_N)(B) + \sum_{n>N} f_n(B)$$

and the series converge absolutely in the operator norm. We have

$$f(A)-f(B)-\sum_{n>N}(f_n(A)-f_n(B))=\Big(f(A)-\sum_{n>N}f_n(A)\Big)-\Big(f(B)-\sum_{n>N}f_n(B)\Big)$$
$$g_N(A)-g_N(B).$$

Since $g_N\in L^\infty(\mathbb{R})$ and g_N is an entire function of exponential type at most 2^{N+1}, it follows from (1.5.2) and (1.5.5) that

$$\|g_N(A)-g_N(B)\|\le\operatorname{const}2^N\|f*V_N\|_{L^\infty}\|A-B\|\le\operatorname{const}2^N\|f\|_{L^\infty}\|A-B\|\to 0$$

as $N\to-\infty$. This proves (2.3.3).

Let N be the integer such that

$$2^{-N}<\|A-B\|\le 2^{-N+1}. \tag{2.3.4}$$

We have

$$f(A)-f(B)=\sum_{n\le N}(f_n(A)-f_n(B))+\sum_{n>N}(f_n(A)-f_n(B)).$$

It follows from (2.2.5) and (2.3.4) that

$$\begin{aligned}\Big\|\sum_{n\le N}(f_n(A)-f_n(B))\Big\| &\le \sum_{n\le N}\|(f_n(A)-f_n(B))\|\\ &\le \mathrm{const}\sum_{n\le N}2^n\|f_n\|_{L^\infty}\|A-B\|\\ &\le \mathrm{const}\sum_{n\le N}2^n2^{-n\alpha}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|\\ &\le \mathrm{const}\,2^{N(1-\alpha)}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|\\ &\le \mathrm{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|^\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big\|\sum_{n>N}(f_n(A)-f_n(B))\Big\| &\le \sum_{n>N}(\|f_n(A)\|+\|f_n(B)\|)\\ &\le 2\sum_{n>N}\|f_n\|_{L^\infty}\le \mathrm{const}\sum_{n>N}2^{-N\alpha}\|f\|_{\Lambda_\alpha(\mathbb{R})}\\ &\le \mathrm{const}\,2^{-N\alpha}\|f\|_{\Lambda_\alpha(\mathbb{R})}\le \mathrm{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|^\alpha\end{aligned}$$

by (2.3.4). This completes the proof. □

2.4 Hölder estimates for other classes of operators

In this section we obtain analogs of the result of the previous section for functions of unitary operators, contractions, maximal dissipative operators, normal operators and n–Tuples of commuting self–adjoint operators.

2.4.1 The case of unitary operators

Theorem 2.4.1. [1] *Let $0 < \alpha < 1$. Then there exists a constant $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{T})$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:*

$$\|f(U) - f(V)\| \le c\,\|f\|_{\Lambda_\alpha(\mathbb{T})} \cdot \|U - V\|^\alpha. \tag{2.4.1}$$

Proof. Let $f \in \Lambda_\alpha(\mathbb{T})$. We have

$$f = \mathbb{P}_+ f + \mathbb{P}_- f = f_+ + f_-.$$

We estimate $\|f_+(U) - f_+(V)\|$. The norm of $\|f_-(U) - f_-(V)\|$ can be obtained in the same way. Thus we assume that $f = f_+$. Let

$$f_n \stackrel{\text{def}}{=} f * W_n.$$

Then

$$f = \sum_{n \ge 0} f_n. \tag{2.4.2}$$

Clearly, we may assume $U \ne V$. Let N be the nonnegative integer such that

$$2^{-N} < \|U - V\| \le 2^{-N+1}. \tag{2.4.3}$$

We have

$$f(U) - f(V) = \sum_{n \le N} (f_n(U) - f_n(V)) + \sum_{n > N} (f_n(U) - f_n(V)).$$

It follows from the Birman–Solomyak formula for unitary operators and (1.5.4) that

$$\begin{aligned}\left\|\sum_{n\le N}(f_n(U)-f_n(V))\right\| &\le \sum_{n\le N}\|(f_n(U)-f_n(V))\|\\ &\le \operatorname{const}\sum_{n\le N}2^n\|f_n\|_{L^\infty}\|U-V\|\\ &\le \operatorname{const}\sum_{n\le N}2^n2^{-n\alpha}\|f\|_{\Lambda_\alpha(\mathbb{T})}\|U-V\|\\ &\le \operatorname{const}2^{N(1-\alpha)}\|f\|_{\Lambda_\alpha(\mathbb{T})}\|U-V\|\\ &\le \operatorname{const}\|f\|_{\Lambda_\alpha(\mathbb{T})}\|U-V\|^\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\sum_{n>N}(f_n(U)-f_n(V))\right\| &\le \sum_{n>N}(\|f_n(U)\|+\|f_n(V)\|)\\ &\le 2\sum_{n>N}\|f_n\|_{L^\infty}\le \operatorname{const}\sum_{n>N}2^{-N\alpha}\|f\|_{\Lambda_\alpha(\mathbb{T})}\\ &\le \operatorname{const}2^{-N\alpha}\|f\|_{\Lambda_\alpha(\mathbb{T})}\le \operatorname{const}\|f\|_{\Lambda_\alpha(\mathbb{T})}\|U-V\|^\alpha\end{aligned}$$

by (2.4.3). This completes the proof. □

2.4.2 The case of contractions

Recall that if T is a contraction on Hilbert space, it follows from von Neumann's inequality that the polynomial functional calculus $f \mapsto f(T)$ extends to the disk–algebra C_A and $\|f(T)\| \le \|f\|_{C_A}$, $f \in C_A$.

Theorem 2.4.2. [1] *Let $0 < \alpha < 1$. Then there exists a constant $c > 0$ such that for every $f \in (\Lambda_\alpha)_+$ and for arbitrary contractions T and R on Hilbert space the following inequality* *holds:*

$$\|f(T) - f(R)\| \le c\,\|f\|_{\Lambda_\alpha} \cdot \|T - R\|^\alpha. \tag{2.4.4}$$

Proof. The proof of Theorem 2.4.2 is almost the same as the proof of Theorem 2.4.1. For $f \in (\Lambda_\alpha)_+$, we use expansion (2.4.2) and choose N such that

$$2^{-N} < \|T - R\| \le 2^{-N+1}.$$

Thus as in the proof of Theorem 2.4.2, for $n \le N$, we estimate $\|f_n(T) - f_n(R)\|$ in terms of $\mathrm{const}\,2^{-n}\|T-R\|$ (see (1.6.6) and (1.5.4)), while for $n > N$ we use von Neumann's inequality to estimate $\|f_n(T) - f_n(R)\|$ in terms of $2\|f_n\|_{L^\infty}$. The rest of the proof is the same. $\square$

Corollary 2.4.3. [1] *Let f be a function in the disk–algebra and $0 < \alpha < 1$. Then the following two statements are equivalent:*

(i) $\|f(T) - f(R)\| \le \mathrm{const}\,\|T - R\|_\alpha$ *for all contractions T and R,*

(ii) $\|f(U) - f(V)\| \le \mathrm{const}\,\|U - V\|_\alpha$ *for all unitary operators U and V*

Remark 2.4.4. [1, 9] This corollary is also true for $\alpha = 1$. This was proved by Kissin and Shulman (see [9]).

2.4.3 The case of maximal dissipative operators

2.4.3.1 Dissipative operators

In this section we give necessary information of dissipative operators in order to interpret the construction of the semi–spectral measure of a maximal dissipative operator. We refer the reader to [4], [8] and [7] for more information.

Definition 2.4.5. Let $\mathscr{H}$ be a Hilbert space. An operator L (not necessarily bounded) with dense domain $\mathscr{D}_L$ in $\mathscr{H}$ is called *dissipative* if

$$Im(Lu,u) \geq 0 \quad , u \in \mathscr{D}_L.$$

A dissipative operator is called *maximal dissipative* if it has no proper dissipative extension.

Note that if L is a symmetric operator (i.e., $(Lu,u) \in \mathbb{R}$ for every $u \in \mathscr{D}_L$), then L is dissipative. However, it can happen that L is maximal symmetric, but not maximal dissipative.

The *Cayley transform* of a dissipative operator L is defined by

$$T \stackrel{\text{def}}{=} (L-iI)(L+iI)^{-1}$$

with domain $\mathscr{D}_T = (L+iI)\mathscr{D}_L$ and range $\operatorname{Range} T = (L-iI)\mathscr{D}_L$ (the operator T is not densely defined in general). T is a contraction, i.e., $\|T(u)\| \leq \|u\|$, $u \in \mathscr{D}_T$, 1 is not an eigenvalue of T, and $Range\,(I-T) \stackrel{\text{def}}{=} \{u - Tu : \ u \in \mathscr{D}_T\}$ is dense.

Conversely, if T is a contraction defined on its domain $\mathscr{D}_T$, 1 is not an eigenvalue of T, and $Range\,(I-T)$ is dense, then it is the Cayley transform of a dissipative operator L and L is the inverse Cayley transform of T:

$$L = i(I+T)(I-T)^{-1}, \quad \mathscr{D}_L = Range\,(I-T).$$

A dissipative operator is maximal if and only if the domain of its Cayley transform is the whole Hilbert space.

Every dissipative operator has a maximal dissipative extension. Every maximal dissipative operator is necessarily closed.

If L is a maximal dissipative operator, then $-L^*$ is also maximal dissipative.

If L is a maximal dissipative operator, then its spectrum $\sigma(L)$ is contained in the closed upper half–plane $\operatorname{clos}\mathbb{C}_+$ and

$$\|(L-\lambda I)^{-1}\| \le \frac{1}{|\operatorname{Im}\lambda|}, \quad \operatorname{Im}\lambda < 0. \tag{2.4.5}$$

If L and M are maximal dissipative operators , we say that the difference operator $L-M$ is *bounded* if there exists a bounded operator K such that $L = M + K$. An elementary fact is (see [4] for the proof) that if L is a maximal dissipative operator and M is a dissipative operator such that $L-M$ is bounded, then M is also maximal dissipative.

The construction of the functional calculus for dissipative operators was given in [4].

Let L be a maximal dissipative operator and let T be its Cayley transform. Consider its minimal unitary dilation U, i.e., U is a unitary operator defined on a Hilbert space $\mathscr{K}$ that contains $\mathscr{H}$ such that

$$T^n = P_{\mathscr{H}} U^n | \mathscr{H}, \quad n \ge 0,$$

and $\mathscr{K} = \operatorname{clos}\operatorname{span}\{U^n h : \ h \in \mathscr{H}\}$. Since 1 is not an eigenvalue of T, it follows that 1 is not an eigenvalue of U (see [8], Ch. II, §6).

The Sz–Nagy–Foiaş functional calculus allows us define a functional calculus for T on the Banach algebra

$$C_{A,1} \stackrel{\mathrm{def}}{=} \big\{ g \in \mathscr{H}^\infty : \ \ g \ \text{ is continuous on } \ \mathbb{T}\setminus\{1\} \big\}.$$

If $g \in C_{A,1}$, we put

$$g(T) \stackrel{\text{def}}{=} P_{\mathscr{H}} g(U) | \mathscr{H}.$$

This functional calculus is linear and multiplicative and

$$\|g(T)\| \le \|g\|_{H^\infty}, \quad g \in C_{A,1}.$$

A functional calculus for the dissipative operator on the Banach algebra

$$C_{A,\infty} \stackrel{\text{def}}{=} \{f \in H^\infty(\mathbb{C}_+) : \quad f \text{ is continuous on } \mathbb{R}\}$$

by

$$f(L) \stackrel{\text{def}}{=} (f \circ \omega)(T), \quad f \in C_{A,\infty},$$

where ω is the conformal map of $\mathbb{D}$ onto $\mathbb{C}_+$ defined by $\omega(\zeta) \stackrel{\text{def}}{=} i(1+\zeta)(1-\zeta)^{-1}$, $\zeta \in \mathbb{D}$.

Let L be a maximal dissipative operator, T be its Cayley transform and let $\mathcal{E}_T$ be the semi–spectral measure of T on the unit circle. Then

$$g(T) = \int_{\mathbb{T}} g(\zeta) d\mathcal{E}_T(\zeta), \quad g \in C_{A,1}. \tag{2.4.6}$$

The semi–spectral measure $\mathcal{E}_L$ of L can be defined by

$$\mathcal{E}_L(\Delta) \stackrel{\text{def}}{=} \mathcal{E}_T(\omega^{-1}(\Delta)), \quad \Delta \text{ is a Borel subset of } \mathbb{R}.$$

It folllows from (2.4.6) that

$$f(L) = \int_{\mathbb{R}} f(x) d\mathcal{E}_L(x), \quad f \in C_{A,\infty}. \tag{2.4.7}$$

2.4.3.2 Hölder Estimates

It was shown in [4] that if f is a bounded function on $\mathbb{R}$ whose Fourier transform has compact support in $(0, \infty)$, and if L and M are maximal dissipative operators such that $L - M$ is bounded, then the Birman–Solomyak formula holds for L and M with respect to their semi–spectral measures and

$$\|f(L) - f(M)\| \leq 8\sigma \|f\|_{L^\infty(\mathbb{R})} \|L - M\|. \tag{2.4.8}$$

It then follows that if $f \in \big(B^1_{\infty,1}(\mathbb{R})\big)_+$, we can associate with f the sequence $\{f_n\}_{n\in\mathbb{Z}}$ defined by $f_n \stackrel{\text{def}}{=} f * W_n$, which gives

$$\check{f} = \sum_{n=-\infty}^{\infty} \check{f}_n.$$

The series converges uniformly. Then the Birman–Solomyak formula also holds for f. Note that f is not necessarily bounded, and when it is not bounded, the difference operator $f(L) - f(M)$ is defined by

$$f(L) - f(M) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \big((f_n(L) - f_n(M))\big). \tag{2.4.9}$$

As in the case of self–adjoint operators, the series on the right converges absolutely and the definition does not depend on the choice of the functions W_n. Furthermore, the functions in

$\big(B^1_{\infty,1}(\mathbb{R})\big)_+$ are operator Lipschitz on the class of maximal dissipative operators.

Theorem 2.4.6. [4] *There is a constant $c > 0$ such that for every $\alpha \in (0,1)$, for arbitrary $f \in \big(\Lambda_\alpha(\mathbb{R})\big)_+$, and for arbitrary maximal dissipative operators L and M with bounded $L - M$, the following inequality holds:*

$$\|f(L) - f(M)\| \le c\,(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|L - M\|^\alpha, \tag{2.4.10}$$

where $f(L) - f(M)$ is defined by (2.4.9).

Proof. Using the same arguments as in the proof of Theorem 2.3.2 , we get

$$\|f(L) - f(M)\| \le \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|L - M\|^\alpha.$$

The fact that the constant in this inequality can be estimated in terms of $c\,(1-\alpha)^{-1}$ follows immediately form Theorem 2.5.5 below. $\square$

2.4.4 The case of normal operators

In [5] it was shown that the Birman–Solomyak formula holds for arbitrary normal operators only for linear functions and a new formula for the difference $f(A) - f(B)$ was established for functions in the Besov space $B^1_{\infty,1}(\mathbb{R}^2)$ and normal operators N_1, N_2 in terms of DOI. Readers are referred to [5] for the definition of $B^1_{\infty,1}(\mathbb{R}^2)$ and the construction of the theory of DOI for normal operators.

Also denote by $\mathscr{F}$ the *Fourier transform* on $L_1(\mathbb{R}^n)$, $n \ge 1$ by:

$$(\mathscr{F}f)(t) = \int_{\mathbb{R}^{\ltimes}} f(x)e^{-i(x,t)}dx, \quad \text{where}$$

$$x = (x_1, ..., x_n),\ t = (t_1, ..., t_n),\ (x, t) \stackrel{\text{def}}{=} x_1 t_1 + ... + x_n t_n.$$

The following important result was proved in [5]:

Let f be a bounded continuous function on $\mathbb{R}^2$ such that

$$\operatorname{supp} \mathscr{F} f \subset \{\zeta \in \mathbb{C} : |\zeta| \le \sigma\}, \quad \sigma > 0.$$

There exists a constant $c > 0$ such that for arbitrary normal operators N_1 and N_2,

$$\|(f(N_1) - f(N_2)\| \le c\,\sigma \|f\|_{L^\infty} \|N_1 - N_2\|. \tag{2.4.11}$$

The class $\Lambda_\alpha(\mathbb{R}^2)$ of Hölder functions of order α, $0 < \alpha < 1$,is defined by:

$$\Lambda_\alpha(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{ f : \ \ \|f\|_{\Lambda_\alpha(\mathbb{R}^2)} = \sup_{z_1 \ne z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} < \infty. \right\}$$

The class $\Lambda_\alpha(\mathbb{R}^n)$, $n > 2$ is defined in the same way.

Using (2.4.11), it was proved in [5] that the functions in $B^1_{\infty,1}(\mathbb{R}^2)$ are operator Lipschitz for normal operators and there exists a constant $c > 0$ such that $\|f(N_1) - f(N_2)\| \le$ c $\|f\|_{\Lambda_\alpha} \|N_1 - N_2\|^\alpha$ for every function in $\Lambda_\alpha(\mathbb{R}^2)$ and arbitrary normal operators N_1 and N_2.

2.4.5 The case of n–Tuples of commuting self–adjoint operators

In [10], another formula for the difference $f(A_1, ..., A_n) - f(B_1, ..., B_n)$ was established for functions in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$ and n–tuples of commuting self–adjoint operators $(A_1, ..., A_n)$, $(B_1, ..., B_n)$ in terms of DOI. Readers are referred to [10] for the definition of

$B^1_{\infty,1}(\mathbb{R}^n)$ and the construction of the theory of DOI for n–Tuples of commuting self–adjoint operators.

The following important result was proved in [10]:

Let f be a bounded continuous function on $\mathbb{R}^n$ such that

$$\operatorname{supp} \mathscr{F} f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \sigma\}, \quad \sigma > 0.$$

There exists a constant $c_n > 0$ such that for arbitrary n–tuples of commuting self-adjoint operators $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$,

$$\|f(A_1, ..., A_n) - f(B_1, ..., B_n)\| \leq c_n\, \sigma \|f\|_{L^\infty} \max_{1 \leq j \leq n} \|A_j - B_j\|. \tag{2.4.12}$$

Using (2.4.12), it was proved in [10] that the functions in $B^1_{\infty,1}(\mathbb{R}^n)$ are operator Lipschitz for normal operators and there exists a constant $c_n > 0$ such that $\|f(A_1, ..., A_n) - f(B_1, ..., B_n)\| \leq c_n(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha} \max_{1\leq j\leq n} \|A_j - B_j\|^\alpha$ for every function in $\Lambda_\alpha(\mathbb{R}^n)$ and n–tuples of commuting self-adjoint operators $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$.

2.5 Arbitrary moduli of continuity

In this section we consider the problem of estimating $\|f(A)-f(B)\|$ for self–adjoint operators A and B and functions in the space Λ_ω, where ω is an arbitrary modulus of continuity. We also show similar results for unitary operators, contractions, maximal dissipative operators, and n–tuples of commuting self–adjoint operators.

Given a modulus of continuity ω, we define the function ω_* and $\omega_\sharp$ by

$$\omega_*(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt \ , \ x > 0$$

and

$$\omega_\sharp(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt \ + \int_0^x \frac{\omega(t)}{t} dt \ , \ x > 0.$$

In this paper, we assume that $\omega_\sharp$ is finite valued whenever it is used.

For example, if we define ω by

$$\omega(x) = x^\alpha, \ x > 0, \ 0 < \alpha < 1,$$

then $\omega_\sharp(x) \leq \text{const } \omega(x)$.

It is well known(see [6], Ch.3, Theorem 13.30) that if ω is a modulus of continuity, then the Hilbert transform maps Λ_ω into itself if and only if $\omega_\sharp(x) \leq \text{const } \omega(x)$.

Theorem 2.5.1. [1] *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B, the following inequality holds:*

$$\|f(A) - f(B)\| \leq c \, \|f\|_{\Lambda_\omega(\mathbb{R})} \omega_*(\|A - B\|). \tag{2.5.1}$$

Proof. Due to Lemma 2.3.1, we can assume that A and B are bounded operators and their spectra are contained in $[a, b]$. We replace the function $f \in \Lambda_\omega(\mathbb{R})$ with the bounded function

$f_\flat$ defined by

$$f_\flat(x) = \begin{cases} f(b), & x > b; \\ f(x), & x \in [a,b]; \\ f(a), & x < a. \end{cases} \tag{2.5.2}$$

Clearly, $\|f_\flat\|_{\Lambda_\omega(\mathbb{R})} \le \|f\|_{\Lambda_\omega(\mathbb{R})}$. Thus we may assume that f is bounded.

Let N be an integer, we claim that

$$f(A) - f(B) = \sum_{n=-\infty}^{N} \big(f_n(A) - f_n(B)\big) + \big((f - f * V_n)(A) - (f - f * V_n)(B)\big), \tag{2.5.3}$$

and the series converges absolutely in the operator norm. Here $f_n = f * W_n + f * W_n^\sharp$ and the de la Vallée Poussin type kernel V_n is defined as in §2.2.1. Suppose that $M < N$, it is easy to see that

$$\begin{aligned} f(A) - f(B) - &\left(\sum_{n=M+1}^{N} \big(f_n(A) - f_n(B)\big) + \big((f - f * V_N)(A) - (f - f * V_N)(B)\right) \\ &= \big((f - f * V_M)(A) - (f - f * V_M)(B)\big). \end{aligned}$$

Clearly, $f - f * V_M$ is an entire function of exponential type at most 2^{M+1}. Thus it follows from (1.5.5) that

$$\|(f - f * V_M)(A) - (f - f * V_M)(B)\| \le \text{const}\, 2^M \|f\|_{L^\infty} \|A - B\| \ \to 0 \ \text{ as } \ M \to -\infty.$$

Suppose now that N is the integer satisfying (2.3.4). It follows from Theorem 2.2.2 that

$$\begin{aligned}\|(f - f * V_N)(A) - (f - f * V_N)(B)\| &\le 2\|f - f * V_n\|_{L^\infty} \\ &\le \text{const}\,\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(2^{-N}) \le \text{const}\,\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(\|A - B\|).\end{aligned}$$

On the other hand, it follows from Corollary 2.2.4 and from (1.5.5) that

$$\begin{aligned}\sum_{n=-\infty}^{N}\|f_n(A) - f_n(B)\| &\le \text{const}\sum_{n=-\infty}^{N}2^n\|f_n\|_{L^\infty}\|A - B\| \\ &\le \text{const}\sum_{n=-\infty}^{N}2^n\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(2^{-N})\|A - B\| \\ &= \text{const}\sum_{k\ge 0}2^{N-k}\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(2^{-(N-k)})\|A - B\| \\ &\le \text{const}\left(\int_{2^{-N}}^{\infty}\frac{\omega(t)}{t^2}dt\right)\|f\|_{\Lambda_\omega(\mathbb{R})}\|A - B\| \\ &= \text{const}\,2^N\omega_*(2^{-N})\|f\|_{\Lambda_\omega(\mathbb{R})}\|A - B\| \\ &\le \text{const}\,\|f\|_{\Lambda_\omega(\mathbb{R})}\omega_*(\|A - B\|)\end{aligned}$$

The result now follows from the obvious inequality $\omega(x) \le \omega_*(x), \quad x > 0.$ $\square$

Corollary 2.5.2. [1] *Let ω be a modulus of continuity such that $\omega_x \le \text{const}\,\omega(x), \quad x > 0$. Then for an arbitrary function $f \in \Lambda_\omega(\mathbb{R})$ and for arbitrary self–adjoint operators A and B on Hilbert space the following inequality holds:*

$$\|f(A) - f(B)\| \le c\,\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(\|A - B\|). \tag{2.5.4}$$

Below we give similar results for other types of operators. Their proofs are similar to the proof of Theorem 2.5.1.

Theorem 2.5.3. [1] *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in Λ_ω and for arbitrary unitary operators U and V, the following inequality holds:*

$$\|f(U) - f(V)\| \le c\,\|f\|_{\Lambda_\omega}\omega_*(\|U - V\|). \tag{2.5.5}$$

Theorem 2.5.4. [1] *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $(\Lambda_\omega)_+$ and for arbitrary contractions T and R, the following inequality holds:*

$$\|f(T) - f(R)\| \le c\,\|f\|_{\Lambda_\omega}\omega_*(\|T - R\|). \tag{2.5.6}$$

Theorem 2.5.5. [4] *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $(\Lambda_\omega)_+$ and for maximal dissipative operators L and M with bounded difference, the following inequality holds:*

$$\|f(L) - f(M)\| \le c\,\|f\|_{\Lambda_\omega}\omega_*(\|L - M\|). \tag{2.5.7}$$

Let ω be a modulus of continuity, the class $\Lambda_\omega(\mathbb{R}^2)$ is defined by:

$$\Lambda_\omega(\mathbb{R}^2) \overset{\text{def}}{=} \left\{ f : \quad \|f\|_{\Lambda_\omega(\mathbb{R}^2)} = \sup_{z_1 \ne z_2} \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} < \infty. \right\}$$

The class $\Lambda_\omega(\mathbb{R}^n)$, $n > 2$ is defined in the same way.

Theorem 2.5.6. [5] *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R}^2)$ and for arbitrary normal operators N_1 and N_2, the following inequality* *holds:*

$$\|f(N_1) - f(N_2)\| \le c\,\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega_*(\|N_1 - N_2\|). \qquad (2.5.8)$$

Theorem 2.5.7. [10] *Let n be a positive integer. There exists a constant $c_n > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R}^n)$ and for arbitrary n–tuples of commuting self–adjoint operators $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$, the following inequality holds:*

$$\|f(A_1, ..., A_n) - f(B_1, ..., B_n)\| \le c_n\,\|f\|_{\Lambda_\omega} \max_{1\le j\le n} \omega_*(\|A_j - B_j\|). \qquad (2.5.9)$$

Chapter 3

Estimates on singular values

3.1 Results for perturbation of class $\boldsymbol{S}_p$

Let $l \geq 0$ be an integer and $p \geq 1$.Denote by $\boldsymbol{S}_p^l$ the normed ideal that consists of all bounded linear operators equipped with norm

$$\|T\|_{\boldsymbol{S}_p^l} \stackrel{\text{def}}{=} \Big(\sum_{j=0}^{l}(s_j(T))^p\Big)^{\frac{1}{p}}. \tag{3.1.1}$$

Classes $\boldsymbol{S}_p^l$ and $\boldsymbol{S}_p$ are both nice SNI. Thus Theorem 1.5.1 and (1.5.5) can be applied to them, i.e., if f is an exponential function of finite type at most σ that is bounded on R, then for arbitrary self–adjoint operators A and B, we have:

$$\|f(A) - f(B)\|_{\boldsymbol{S}_p^l} \leq \operatorname{const} \sigma \|f\|_{L^\infty} \|A - B\|_{\boldsymbol{S}_p^l}, \tag{3.1.2}$$

and

$$\|f(A) - f(B)\|_{\boldsymbol{S}_p} \leq \operatorname{const} \sigma \|f\|_{L^\infty} \|A - B\|_{\boldsymbol{S}_p}. \tag{3.1.3}$$

Similar results also hold for maximal dissipative operators, normal operators and n–tuples of self–adjoint operators.

We also have if f is a trigonometric polynomial of degree d, then for arbitrary unitary operators U and V,

$$\|f(U)-f(V)\|_{\boldsymbol{S}_p^l} \le \operatorname{const} d\|f\|_{L^\infty}\|U-V\|_{\boldsymbol{S}_p^l},$$

and

$$\|f(U)-f(V)\|_{\boldsymbol{S}_p} \le \operatorname{const} d\|f\|_{L^\infty}\|U-V\|_{\boldsymbol{S}_p}.$$

Similar results also hold for contractions.

Theorem 3.1.1. *Let $0<\alpha<1$. Then there exists a constant $c>0$ such that for every $l\ge 0$, $p\in[1,\infty)$, $f\in\Lambda_\alpha(\mathbb{R})$, and for arbitrary self–adjoint operators A and B on Hilbert space with bounded $A-B$, the following inequality holds for every every $j\le l$:*

$$s_j\big(f(A)-f(B)\big) \le c\,\|f\|_{\Lambda_\alpha(\mathbb{R})}(1+j)^{-\alpha/p}\|A-B\|^\alpha_{\boldsymbol{S}_p^l}. \tag{3.1.4}$$

Proof. Put $f_n \stackrel{\text{def}}{=} f*W_n + f*W_n^\sharp, n\in\mathbb{Z}$, and fix an integer N. We have

$$\begin{aligned}
\left\|\sum_{n=-\infty}^{N}\big(f_n(A)-f_n(B)\big)\right\|_{\boldsymbol{S}_p^l} &\le \sum_{n=-\infty}^{N}\|f_n(A)-f_n(B)\|_{\boldsymbol{S}_p^l}\\
&\le \operatorname{const}\sum_{n=-\infty}^{N}2^n\|f_n\|_{L^\infty}\|A-B\|_{\boldsymbol{S}_p^l}\\
&\le \operatorname{const}\|f\|_{\Lambda_\alpha(\mathbb{R})}\sum_{n=-\infty}^{N}2^{n(1-\alpha)}\|A-B\|_{\boldsymbol{S}_p^l}\\
&\le \operatorname{const}2^{N(1-\alpha)}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|_{\boldsymbol{S}_p^l}.
\end{aligned}$$

On the other hand,

$$\Big\|\sum_{n>N}\big(f_n(A)-f_n(B)\big)\Big\| \le 2\sum_{n>N}\|f_n\|_{L^\infty}$$
$$\le \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\sum_{n>N}2^{-n\alpha}$$
$$\le \text{const}\,2^{-N\alpha}\|f\|_{\Lambda_\alpha(\mathbb{R})}.$$

Put $R_N \stackrel{\text{def}}{=} \sum\limits_{n=-\infty}^{N}\big(f_n(A)-f_n(B)\big)$ and $Q_N \stackrel{\text{def}}{=} \sum\limits_{n>N}\big(f_n(A)-f_n(B)\big)$. Clearly, for $j\le l$,

$$s_j\big(f(A)-f(B)\big) \le s_j(R_N)+\|Q_N\|$$
$$\le (1+j)^{-1/p}\|R_N\|_{\boldsymbol{S}_p^l}+\|Q_N\|$$
$$\le \text{const}\,\Big((1+j)^{\frac{-1}{p}}2^{N(1-\alpha)}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|_{\boldsymbol{S}_p^l}+2^{-N\alpha}\|f\|_{\Lambda_\alpha(\mathbb{R})}\Big).$$

To obtain the desired estimate, it suffices to choose the number N such that

$$2^{-N} < (1+j)^{-1/p}\|A-B\|_{\boldsymbol{S}_p^l} \le 2^{-N+1}.$$

$\square$

Using the same type of arguments, we can get similar estimates for unitary operators, contractions, maximal dissipative operators, normal operators and n–tuples of self–adjoint operators.

3.2 Estimates on singular values of functions of perturbed self-adjoint and unitary operators

In this section, we generalize the estimate in §3.1 to the class Λ_ω and also obtain some lower–bound estimates for rank one perturbations which also extend the results in [2]. In section §3.3, similar estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n–tuples of commuting self–adjoint operators.

Theorem 3.2.1. *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B, the following inequality holds for all l and for all j, $1 \le j \le l$:*

$$s_j(f(A) - f(B)) \le c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|A - B\|_{\boldsymbol{S}_p^l}\big)\|f\|_{\Lambda_\omega}. \tag{3.2.1}$$

Proof. Due to Lemma 2.3.1, A and B can be taken as bounded operators, then we may further assume f is bounded. Let $R_N = \sum_{n=-\infty}^{N}(f_n(A) - f_n(B))$, $Q_N = (f - f * V_N)(A) - (f - f * V_N)(B)$. Here f_n and the de la Vallée Poussín type kernel V_N are defined as in §2.2.1. Then $f(A) - f(B) = R_N + Q_N$, with convergence in the uniform operator topology. Note that for any integer $m \in \mathbb{Z}$, functions f_m and $f - f * V_m$ are entire functions of exponential type at most 2^{m+1}. Thus it follows from (3.1.2), (2.2.6) and (2.2.7) that

$$\|Q_N\| \le c\,\omega(2^{-N})\|f\|_{\Lambda_\omega}$$

and

$$\begin{aligned}\|R_N\|_{\boldsymbol{S}_p^l} &\le \sum_{n=-\infty}^{N} \|f_n(A) - f_n(B)\|_{\boldsymbol{S}_p^l} \\ &\le c \sum_{n=-\infty}^{N} \big(2^n \|f_n\|_{L^\infty}\big) \|A - B\|_{\boldsymbol{S}_p^l} \\ &\le c\, 2^N \omega_*(2^{-N}) \|A - B\|_{\boldsymbol{S}_p^l} \|f\|_{\Lambda_\omega}.\end{aligned}$$

Then

$$\begin{aligned} s_j(f(A) - f(B)) &\le s_j(R_N) + \|Q_N\| \le (1+j)^{-1} \|R_N\|_{\boldsymbol{S}_p^l} + \|Q_N\| \\ &\le c\Big((1+j)^{-\frac{1}{p}}\, 2^N \omega_*(2^{-N}) \|A - B\|_{\boldsymbol{S}_p^l} + \omega(2^{-N})\Big) \|f\|_{\Lambda_\omega}.\end{aligned}$$

Take N such that $1 \le (1+j)^{-\frac{1}{p}}\, 2^N \|A - B\|_{\boldsymbol{S}_p^l} \le 2$ and use the fact that $\omega(t) \le \omega_*(t)$ for any $t > 0$, we get (3.2.1). □

Theorem 3.2.2. *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{T})$ and for arbitrary unitary operators U and V, the following inequality holds for all l and for all j, $1 \le j \le l$:*

$$s_j(f(U) - f(V)) \le c\, \omega_*\Big((1+j)^{-\frac{1}{p}} \|U - V\|_{\boldsymbol{S}_p^l}\Big) \|f\|_{\Lambda_\omega}. \tag{3.2.2}$$

Proof. If $(1+j)^{-\frac{1}{p}} \|U - V\|_{\boldsymbol{S}_p^l} \le 2$, the proof is similar to Theorem 3.2.1 with $R_N = \sum_{n=0}^{N} (f_n(U) - f_n(U))$; if $(1+j)^{-\frac{1}{p}} \|U - V\|_{\boldsymbol{S}_p^l} > 2$, then

$$s_j(f(U) - f(V)) \le \|f(U) - f(V)\| \le c\, \omega_*(\|U - V\|) \|f\|_{\Lambda_\omega} \le c\, \omega_*(2) \|f\|_{\Lambda_\omega}.$$

$\square$

Corollary 3.2.3. *Let ω be a modulus of continuity such that*

$$\omega_*(x) \le \text{const } \omega(x), \quad x \ge 0.$$

Then for an arbitrary function $f \in \Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B, the following inequality holds for all l and for all j, $1 \le j \le l$:

$$s_j(f(A) - f(B)) \le \text{const } \omega\big((1+j)^{-\frac{1}{p}} \|A - B\|_{\boldsymbol{S}_p^l}\big) \|f\|_{\Lambda_\omega}.$$

Let H, $\mathcal{H}$ be the Hankel operators defined in [2].

Theorem 3.2.4. *Let ω be a modulus of continuity on $\mathbb{T}$. There exist unitary operators U, V and a real function h in $\Lambda_{\omega_\sharp}((T))$ such that*

$$\text{rank}(U - V) = 1 \qquad \textit{and} \qquad s_m(h(U) - h(V)) \ge \omega\big((1+m)^{-1}\big).$$

Proof. Consider the operators U and V on space $L_2(\mathbb{T})$ with respect to the normalized Lebesgue measure on $\mathbb{T}$ defined by (see [2])

$$Uf = \bar{z}f \text{ and } Vf = \bar{z}f - 2(f,\ 1)\bar{z},\ \ f \in L^2.$$

For $f \in C(\mathbb{T})$, we have

$$\big((f(U) - f(V))z^j, z^k\big) = -2 \begin{cases} \hat{f}(j-k), & \text{if } j \geq 0, k < 0; \\ \hat{f}(j-k), & \text{if } j < 0, k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Define function g by

$$g(\zeta) = \sum_{n=1}^{\infty} \omega(4^{-n})(\zeta^{4^n} + \bar{\zeta}^{4^n}), \qquad \zeta \in \mathbb{T}.$$

Then we have

$$\|g * W_n\|_{L^\infty} \leq \text{const}\, \omega(2^{-n}),\ \|g * W_n^\sharp\|_{L^\infty} \leq \text{const}\, \omega(2^{-n})\ ,\ n \geq 0.$$

Let ξ, η be two arbitrarily different fixed points on $\mathbb{T}$, choose $N \geq 0$ such that $\frac{1}{2} \leq \frac{2^{-N}}{|\xi-\eta|} \leq 1$, then

$$
\begin{aligned}
|g(\xi)-g(\eta)| &\le \sum_{n=0}^{N}|g_n(\xi)-g_n(\eta)|+|(g-g*V_N)(\xi)-(g-g*V_N)(\eta)| \\
&\le \sum_{n=0}^{N}|g_n(\xi)-g_n(\eta)|+2\sum_{n=N+1}^{\infty}\|g_n\|_{L^\infty} \\
&\le \text{const}\sum_{n=0}^{N}2^n\|g_n\|_{L^\infty}|\xi-\eta|+2\sum_{n=N+1}^{\infty}\|g_n\|_{L^\infty} \\
&\le \text{const}\sum_{n=0}^{N}2^n\omega(2^{-n})|\xi-\eta|+\text{const}\sum_{n=N+1}^{\infty}\omega(2^{-n}) \\
&\le \text{const}\ \omega_*(|\xi-\eta|)+\text{const}\int_0^{2^{-N}}\frac{\omega(t)}{t}dt \\
&\le \text{const}\ \omega_\sharp(|\xi-\eta|).
\end{aligned}
$$

Consider the matrix $\Gamma_g=\{\hat{g}(-j-k)\}_{j\ge 1,k\ge 0}=\{\hat{g}(j+k)\}_{j\ge 1,k\ge 0}$.

Let $n\ge 1$. Define matrix $T_n=\{\hat{g}(j+k+4^{n-1}+1)\}_{0\le j,k\le 3\cdot 4^{n-1}}$, then

$$
T_n=\begin{bmatrix}
 & & & \omega(4^{-n}) \\
 & & \omega(4^{-n}) & \\
 & \iddots & & \\
\omega(4^{-n}) & & &
\end{bmatrix}.
$$

If R is any matrix with the same size of T_n such that $\text{rank}(R)<3\cdot 4^{n-1}$, then $\|T_n-R\|\ge\omega(4^{-n})$. It follows that $s_j(T_n)\ge\omega(4^{-n})$ for $j<3\cdot 4^{n-1}$. For each T_n, there is some orthogonal projection P_n such that $T_n=P_n\Gamma_gP_n$, hence $s_j(\Gamma_g)\ge s_j(T_n)\ge\omega(4^{-n})$ for all

n and for all j, $j < 3 \cdot 4^{n-1}$. Thus for all $j \geq 0$, we have

$$s_j(\Gamma_g) \geq \omega\big(\frac{3}{16} \cdot (j+1)^{-1}\big) \geq \frac{3}{32} \cdot \omega\big((j+1)^{-1}\big).$$

To complete the proof, it suffices to take $h = \frac{32}{3} g$. $\square$

Corollary 3.2.5. *Let ω be a modulus of continuity such that*

$$\omega_{\sharp}(x) \leq \text{const}\ \omega(x), \quad 0 \leq x \leq 2.$$

There exist unitary operators U, V and a real function h in $\Lambda_\omega(T)$ such that

$$\operatorname{rank}(U-V) = 1 \qquad \text{and} \qquad s_m(h(U) - h(V)) \geq \omega\big((1+m)^{-1}\big).$$

Theorem 3.2.6. *Let ω be a modulus of continuity on $\mathbb{T}$ and f be a continuous function on $\mathbb{T}$. If for all unitary operators U and V, we have*

$$s_n(f(U) - f(V)) \leq \text{const}\ \omega\big((1+n)^{-\frac{1}{p}} \|U - V\|_{\boldsymbol{S}_p}\big), \ \textit{for all } n \geq 0\,,$$

then $f \in \Lambda_\omega(\mathbb{T})$.

Proof. Let $\zeta, \eta \in \mathbb{T}$, we can select commuting unitary operators U and V such that $s_0(U - V) = s_1(U-V) = \ldots = s_n(U-V) = |\zeta - \eta|$ and $s_k(U-V) = 0$, $k \geq n+1$. Then $s_n(f(U) - f(V)) = |f(\zeta) - f(\eta)|$, $\|U - V\|_{\boldsymbol{S}_p} = (1+n)^{\frac{1}{p}} \cdot |\zeta - \eta|$. $\square$

Theorem 3.2.7. *Let ω be a modulus of continuity on $\mathbb{R}$ and f be a continuous function on*

$\mathbb{R}$. *If for all self-adjoint operators A and B, we have*

$$s_n(f(A)-f(B)) \le \text{const}\ \omega\big((1+n)^{-\frac{1}{p}}\|A-B\|_{\boldsymbol{S}_p}\big),\ \textit{for all } n \ge 0\,,$$

then $f \in \Lambda_\omega(\mathbb{R})$.

Proof. Similar to Theorem 3.2.6. $\square$

Theorem 3.2.8. *Let ω be a modulus of continuity over $\mathbb{R}$. There exist self-adjoint operators A, B, and a real function f in $\Lambda_{\omega_\sharp}(\mathbb{R})$ such that*

$$\text{rank}(A-B)=1 \ \textit{and}\ s_m(f(A)-f(B)) \ge \omega\big((1+m)^{-1}\big),\ \textit{for all } m \ge 0.$$

Proof. WLOG, we assume $\omega(t)=\omega(2)$, for all $t \ge 2$, that is, ω can be regarded as a modulus of continuity on $\mathbb{T}$.

We then choose a function(see [2], Lemma 9.6) $\rho \in C^\infty(\mathbb{T})$ such that $\rho(\zeta)+\rho(i\zeta)=1$, $\rho(\zeta)=\rho(\bar\zeta)$ for all $\zeta \in \mathbb{T}$, and ρ vanishes in a neighborhood of the set $\{-1,1\}$. Note that $\rho \in \Lambda_\omega(\mathbb{T})$, since $\omega(st) \ge \frac{s}{2}\omega(t)$, for all $t \ge 0$ and s, $0<s<1$.

Define function g_1 by

$$g_1(\zeta)=\sum_{n=1}^{\infty}\omega(4^{-n})(\zeta^{4^n}+\bar\zeta^{4^n}), \qquad \zeta \in \mathbb{T}.$$

Then $g_1 \in \Lambda_{\omega_\sharp}(\mathbb{T})$. If $g_0 \stackrel{\text{def}}{=} C\rho g_1$ for a sufficient large number C, then $g_0 \in \Lambda_{\omega_\sharp}(\mathbb{T})$, vanishes in a neighborhood of the set $\{-1,1\}$ and $g_0(\zeta)=g_0(\bar\zeta)$ for all $\zeta \in \mathbb{T}$, and $s_m(H_{g_0}) \ge \omega\big((1+m)^{-1}\big)$ for all $m \ge 0$.

Define $\varphi(x)=(x^2+1)^{-1}$(as in [2], Theorem 9.9), then there exists a compactly supported real bounded function f such that $f(\varphi(x)) = g_0(\frac{x-i}{x+i})$ and a simple calculation shows that f belongs to $\Lambda_{\omega_\sharp}(\mathbb{R})$. Denote $L^2_e(\mathbb{R})$ the subspace of even functions in $L^2(\mathbb{R})$. Consider operators A and B on $L^2_e(\mathbb{R})$ defined by $A(g) = \mathbf{H}^{-1}M_\varphi\mathbf{H}(g)$ and $B(g) = \varphi g$, here $\mathbf{H}$ is the Hilbert transform defined on $L^2(\mathbb{R})$ (see [2]) and M_φ is the multiplication by φ. Then $\operatorname{rank}(A - B) = 1$, and we have

$$s_m(f(B) - f(A)) \geq \sqrt{2}s_m(\mathcal{H}_{f\circ\varphi}) = \sqrt{2}s_m(H_{g_0}) \geq \sqrt{2}\omega\big((1+m)^{-1}\big). \qquad \square$$

3.3 Estimates for other types of operators

The following estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators.

Theorem 3.3.1. *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\big(\Lambda_\omega(\mathbb{R})\big)_+$ and for arbitrary contractions T and R on Hilbert space, the following inequality holds for all l and for all j, $1 \leq j \leq l$:*

$$s_j(f(T) - f(R)) \leq c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|T - R\|_{\boldsymbol{S}_p^l}\big)\|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important(see [1], [2] and [24]):

There exists a constant c such that for arbitrary trigonometric polynomial f of degree n and for arbitrary contractions T and R on Hilbert space,

$$\|(f(T) - f(R)\|_{\boldsymbol{S}_p} \leq c\,n\|f\|_{L^\infty}\|T - R\|_{\boldsymbol{S}_p}.$$

Theorem 3.3.2. *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\big(\Lambda_\omega(\mathbb{R})\big)_+$ and for arbitrary maximal dissipative operators L and M with bounded difference, the following inequality holds for all l and for all j, $1 \le j \le l$:*

$$s_j(f(L) - f(M)) \le c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|L - M\|_{\boldsymbol{S}_p^l}\big)\|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important(see [4]):

There exists a constant $c > 0$ such that for every function f in $H^\infty(\mathbb{C}_+)$ with

$$\operatorname{supp} \mathscr{F} f \subset [0, \sigma], \quad \sigma > 0,$$

and for arbitrary maximal dissipative operators L and M with bounded difference,

$$\|(f(L) - f(M)\|_{\boldsymbol{S}_p} \le c\,\sigma\|f\|_{L^\infty}\|L - M\|_{\boldsymbol{S}_p}.$$

Theorem 3.3.3. *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R}^2)$ and for arbitrary normal operators N_1 and N_2, the following inequality holds for all l and for all j, $1 \le j \le l$:*

$$s_j(f(N_1) - f(N_2)) \le c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|N_1 - N_2\|_{\boldsymbol{S}_p^l}\big)\|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important(see [5]):

There exists a constant $c > 0$ such that for every bounded continuous function f on $\mathbb{R}^2$

with

$$\operatorname{supp} \mathscr{F} f \subset \{\zeta \in \mathbb{C} : |\zeta| \le \sigma\}, \quad \sigma > 0,$$

and for arbitrary normal operators N_1 and N_2,

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S}_p} \le c\,\sigma \|f\|_{L^\infty} \|N_1 - N_2\|_{\boldsymbol{S}_p}.$$

Theorem 3.3.4. *Let n be a positive integer and $p \ge 1$. There exists a positive number c_n such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R}^n)$ and for arbitrary n-tuples of commuting self-adjoint operators $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$, the following inequality holds for all l and for all j, $1 \le j \le l$:*

$$s_j(f(A_1, ..., A_n) - f(B_1, ..., B_n)) \le c_n \max_{1 \le j \le n} \omega_*\big((1+j)^{-\frac{1}{p}} \|A_j - B_j\|_{\boldsymbol{S}_p^l}\big) \|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important(see [10]):

There exists a constant $c_n > 0$ such that for every bounded continuous function f on $\mathbb{R}^n$ with

$$\operatorname{supp} \mathscr{F} f \subset \{\xi \in \mathbb{R}^n : |\xi| \le \sigma\}, \quad \sigma > 0,$$

and for arbitrary n-tuples of commuting self-adjoint operators $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$,

$$\|f(A_1, ..., A_n) - f(B_1, ..., B_n)\|_{\boldsymbol{S}_p} \le c_n\, \sigma \|f\|_{L^\infty} \max_{1 \le j \le n} \|A_j - B_j\|_{\boldsymbol{S}_p}.$$

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