

RECOIL CORRECTIONS TO THE HYDROGEN HYPERFINE
SPLITTING OF ORDER $(m_e/m_p)^6 \ln^{-1}$

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This is to certify that the
 thesis entitled
 RECOIL CORRECTIONS TO THE HYDROGEN HYPERFINE
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Robert S. Cole
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Wayne W. Repko
 Major professor

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ABSTRACT

RECOIL CORRECTIONS TO THE HYDROGEN HYPERFINE SPLITTING OF ORDER $(m_e/m_p)\alpha^6\ln\alpha^{-1}$

By

Robert S. Cole

A calculation of the $(m_e/m_p)\alpha^6\ln\alpha^{-1}$ recoil corrections to the hydrogen hyperfine splitting is presented. The result is

$$\begin{aligned}\Delta\nu_H &= \frac{8}{3}\left(\frac{m_e}{m_p}\right)^2\left(1+\frac{m_e}{m_p}\right)^{-5}\alpha^4cRy_\infty\ln\alpha^{-1}\left[3-K\left(13+7\frac{m_e}{m_p}\right)+K^2\left(7-14\frac{m_e}{m_p}\right)\right] \\ &= 2.899 \times 10^{-4} \text{ MHz} \quad (0.2 \text{ ppm})\end{aligned}$$

The calculation is performed using a perturbation theory based on the Bethe-Salpeter equation and on a wave function obtained by a single iteration from the nonrelativistic Pauli wave function. Form factors were included to account for proton structure, but it is shown that the $(m_e/m_p)\alpha^6\ln\alpha^{-1}$ is totally insensitive to the details of the proton form factors.

The recoil corrections calculated here, when combined with similar results for muonium, can be used to set an upper limit on the proton polarizability.

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Robert S. Cole

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I. INTRODUCTION

A calculation of the $(m_e/m_p)\alpha^2 \ln \bar{\alpha}^1$ recoil contributions to the triplet-singlet splitting of the hydrogen ground state (the hydrogen hyperfine splitting) is presented. The result to this order is

$$\delta_p(\alpha^2 \ln \alpha^{-1}) = \frac{1}{2} \frac{m_e}{m_p} \frac{1}{1+\kappa} \alpha^2 \ln \alpha^{-1} \left(1 + \frac{m_e}{m_p}\right)^{-2} \left\{ 9 - \kappa(13 + 7 \frac{m_e}{m_p}) + \kappa^2(7 - 14 \frac{m_e}{m_p}) \right\} \quad (1.1)$$

where κ is the proton anomalous magnetic moment ($\kappa \approx 1.79$). The calculation was done using the perturbation theory treatment (developed by Karplus and Klein⁽¹⁾ and Fulton and Martin⁽²⁾) of the Bethe-Salpeter (BS) equation⁽³⁾. The wave function used was a perturbation approximation to the Coulomb wave function obtained by a single iteration of the BS equation containing only the Coulomb interaction. In addition, the calculation included proton form factors to account for proton structure.

The theoretical expression for the hydrogen hyperfine interval, ν_H , can be written as

$$\nu_H = \frac{16}{3} R_y \alpha^2 c \frac{\mu_p}{\mu_B} \left(1 + \frac{m_e}{m_p}\right)^{-3} \left\{ 1 + \rho + \delta_p + \delta_p' \right\} \quad (1.2)$$

where μ_p and μ_B are the proton magnetic moment and Bohr magneton respectively, and where ρ represents the radiative corrections to the electron magnetic moment, δ_p the nuclear size

and recoil terms, and δ'_p the proton polarizability (proton structure) terms.

Equation (1.2) takes the form of the Fermi frequency (namely the first term in the expression) due to a point magnetic dipole in the proton interacting at zero range with the electron moment in a spherical S state, modified by radiative corrections, nuclear size and recoil corrections, and polarizability corrections. The leading term of the radiative corrections to the electron magnetic moment is of order α ($\approx 1/137$.) relative to the Fermi term. These radiative corrections have been completely calculated to order $\alpha(Z\alpha)^2 \log^2(Z\alpha)^{-2}$ relative to the Fermi frequency, and the dominant α^3 terms have been computed. The leading recoil correction from the static magnetic moment and charge distribution is of order $(m_e/m_p)\alpha$ relative to the Fermi term, and hence the $(m_e/m_p)\alpha^2 \ln \alpha^{-1}$ correction is of comparable magnitude to the α^3 terms in ρ .

The nuclear polarization (having to do with the "non-rigidity" of an extended proton) is an extremely important yet difficult term to deal with rigorously. It cannot be calculated since it involves, in principle, most of the uncertainties of proton dynamics. Attempts have been made to calculate δ'_p using inelastic electron - proton scattering data⁽⁴⁾, but at present such calculations are highly model dependent. Most of these model dependent calculations find the proton polarizability to be of the order of $|1|$ to $|2|$ ppm

(a recent calculation by Jensen, Kovesi-Domokos, and Schonberg⁽⁵⁾ places δ'_p at 1 ppm).

While the direct calculation of the proton polarizability appears to be far in the future, new experimental measurements of the hyperfine interval of muonium, ($\mu^+ e^-$ bound system), when coupled with the calculation of the recoil terms, provides an exceedingly accurate measure of δ'_p . Specifically the theoretical expression for the muonium hyperfine interval is

$$\mathcal{V}_\mu = \frac{16}{3} R_y \alpha^2 \cancel{\mathcal{L}} \frac{\mu_\mu}{\mu_B} \left(1 + \frac{m_e}{m_\mu}\right)^{-3} \left\{ 1 + \rho + \delta_\mu \right\} \quad (1.3)$$

where, again, ρ represents the radiative corrections to the electron moment, δ_μ the recoil terms, and both are, in principle, calculable. Taking the ratio of (1.3) to (1.2), we obtain:

$$\frac{\mathcal{V}'_\mu}{\mathcal{V}_H} = \frac{\mu_\mu}{\mu_p} \frac{\left(1 + \frac{m_e}{m_p}\right)^3}{\left(1 + \frac{m_e}{m_\mu}\right)^3} \left\{ 1 + \delta_\mu - \delta_p - \delta_p' \right\} \quad (1.4)$$

The important point to note about (1.4) is that the radiative corrections to the electron moment have cancelled out, as has the leading α dependent coefficient. Thus, the ratio is free of uncertainties in α and affords an opportunity to directly measure the (calculable) recoil terms and the (non-calculable) polarizability terms. Hence, although the absolute size of the $(m_e/m_p)\alpha^2 \ln \alpha^{-1}$ recoil correction presented here is small, it is nonetheless significant because

its effect can be compared directly with experimental data. Moreover a precise calculation of the recoil terms provides an accurate measurement of the proton polarizability.

Another noteworthy feature of the $(m_e/m_p)\alpha^2\ln\alpha^{-1}$ recoil correction is that, as will be shown, it is totally insensitive to the details of the proton form factors. This is due to the fact that the logarithmic dependence arises from the low momentum limit of the exchanged quanta. Hence, no further ambiguity or uncertainty from the proton form factors is introduced into the theoretical result for ν_H .

Finally, although the calculation is presented here in the Lorentz gauge, it was also done in the Coulomb gauge as a check on gauge invariance of the perturbation formalism.⁽⁶⁾

In the following calculations we denote the space-time coordinates by $x_\mu = (x, ix_0)$ and take γ_μ to be Hermitian matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The symbol m_1 will refer to the electron mass, and m_2 will refer to the proton mass. M will denote the total mass of the system, $(m_1 + m_2)$, and μ the reduced mass of the system, $\mu = m_1 m_2 / M$.

II. BACKGROUND

The starting point for the calculation of the hyperfine splitting to our order is the Bethe-Salpeter⁽³⁾ equation. We use the perturbation theory treatment of the Bethe-Salpeter equation as developed by Karplus and Klein⁽¹⁾. The essential idea is to separate the instantaneous Coulomb interaction, which accounts for the major part of the binding, from the total interaction. Using an approximate solution to the Bethe-Salpeter equation containing only the Coulomb interaction, the remaining time dependent interaction terms are treated by perturbation theory.

We start with a two body Green function defined by:

$$G(x_1, x_2, x_3, x_4) = \langle 0 | T \{ \psi_H''(x_1) \psi_H''(x_2) \bar{\psi}_H''(x_3) \bar{\psi}_H''(x_4) \} | 0 \rangle \quad (2.1)$$

Note that the spinor indices are suppressed, that the subscript H denotes the Heisenberg operator, and the particles are labeled (1) and (2) respectively. Using well known techniques, (2.1) can be transformed to the interaction picture as:

$$G(x_1, x_2, x_3, x_4) = \frac{\langle 0 | T \{ \psi''(x_1) \psi''(x_2) \bar{\psi}''(x_3) \bar{\psi}''(x_4) S(\infty, -\infty) \} | 0 \rangle}{\langle 0 | S(\infty, -\infty) | 0 \rangle} \quad (2.2)$$

where the ψ are the interaction picture field operators, $|0\rangle$ is the unperturbed vacuum state, and

$$S(t, t') = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{t'}^t dt_1 \cdots \int_{t'}^t dt_n T \{ H_I(t_1) \cdots H_I(t_n) \} \quad (2.2a)$$

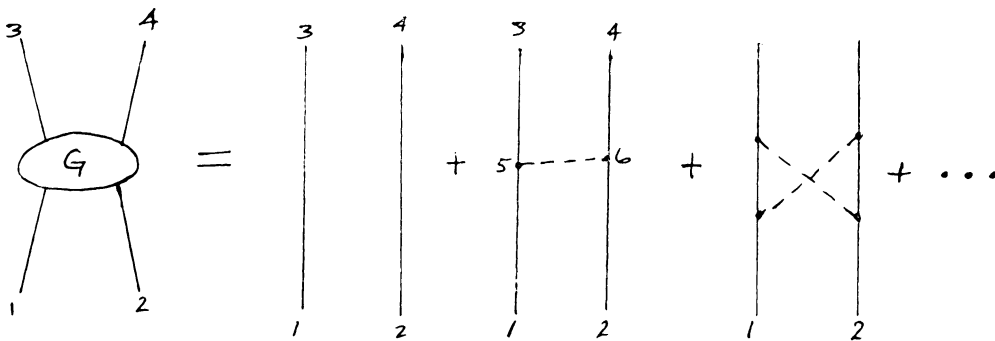
where H_I is the interaction Hamiltonian given by:⁽⁷⁾

$$-ie \bar{\psi}^{(1)}(x) \gamma_{\mu} A_{\mu}(x) \psi^{(1)}(x) - ie \bar{\psi}^{(2)}(x) \gamma_{\nu} A_{\nu}(x) \psi^{(2)}(x) \quad (2.2b)$$

Using the Wick theorem⁽⁸⁾ on (2.2) one derives the perturbation expansion:

$$\begin{aligned} G(x_1, x_2, x_3, x_4) = & S_F^{(1)}(x_1 - x_3) S_F^{(2)}(x_2 - x_4) - e^2 \int d^4x_5 d^4x_6 S_F^{(1)}(x_1 - x_5) S_F^{(2)}(x_2 - x_6) \times \\ & \times \gamma_{\mu}^{(1)} \gamma_{\nu}^{(2)} D_{\mu\nu, F}(x_5 - x_6) S_F^{(1)}(x_5 - x_3) S_F^{(2)}(x_6 - x_4) + \\ & + \dots \end{aligned} \quad (2.3)$$

which can be exhibited diagrammatically as:



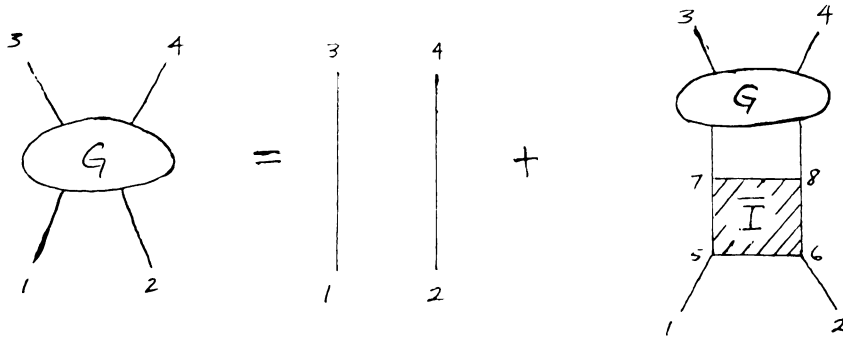
Formally we write the infinite series in (2.3) as:

$$\begin{aligned}
G(x_1, x_2, x_3, x_4) = & S_F^{(1)}(x_1 - x_3) S_F^{(2)}(x_2 - x_4) + \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 S_F^{(1)}(x_1 - x_5) \\
& \times S_F^{(2)}(x_2 - x_6) \bar{I}(x_5, x_6, x_7, x_8) S_F^{(1)}(x_7 - x_3) S_F^{(2)}(x_8 - x_4)
\end{aligned}
\tag{2.4}$$

where I is the sum of all graphs contributing to the two particle interaction. One casts this in the form of an integral equation by denoting $\bar{I}$ as that part of I which consists of the sum of all irreducible graphs⁽³⁾. Then (2.4) can be written:

$$\begin{aligned}
G(x_1, x_2, x_3, x_4) = & S_F^{(1)}(x_1 - x_3) S_F^{(2)}(x_2 - x_4) + \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 S_F^{(1)}(x_1 - x_5) \\
& \times S_F^{(2)}(x_2 - x_6) \bar{I}(x_5, x_6, x_7, x_8) G(x_7, x_8, x_3, x_4)
\end{aligned}
\tag{2.5}$$

which is schematically shown as



Equation (2.5) is the Bethe-Salpeter equation⁽³⁾. From equation (2.5) one can rigorously derive a form for the two particle wave function of a bound state⁽⁹⁾:

$$\begin{aligned} \Psi(x_1, x_2) = & \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 S_F^{(1)}(x_1 - x_5) S_F^{(2)}(x_2 - x_6) \\ & \times \bar{I}(x_5, x_6, x_7, x_8) \Psi(x_7, x_8) \end{aligned} \quad (2.6)$$

For convenience we will henceforth use the convention that all repeated indices are to be summed (integrated) over. Equation (2.6) is the basic equation used in calculating the hyperfine splitting. Introducing center of mass and relative coordinates X and x , where:

$$\begin{aligned} X &\equiv \eta_1 x_1 + \eta_2 x_2 & x_1 &= X + \eta_2 x \\ x &\equiv x_1 - x_2 & x_2 &= X - \eta_1 x \\ \eta_1 &= \frac{m_1}{m_1 + m_2} & ; \quad \eta_2 &= \frac{m_2}{m_1 + m_2} \end{aligned} \quad (2.7)$$

we can write (2.6) as:

$$\varphi_K(x) = G_K(x, x') I_K(x', x'') \varphi_K(x'') \quad (2.8)$$

where
$$\varphi_K(x) = e^{-iK \cdot X} (2\pi)^{-2} \gamma(x_1, x_2), \quad (2.8a)$$

$$G_K(x, x') = \int d^4 X' e^{-iK(X-X')} S_F^{(1)}[X-X' + \eta_2(x-x')] \times S_F^{(2)}[X-X' + \eta_1(x-x')] \quad (2.8b)$$

and
$$I_K(x, x') = \int d^4 X' e^{iK(X-X')} \bar{I}(x, x'; X-X') \quad (2.8c)$$

We have used momentum K as conjugate to center of mass position X . For convenience we will work in the system where the center of mass is at rest so that $K_\mu = (0, iK_0)$. We will also take p to be the relative momentum, conjugate to x . Thus we can rewrite (2.8b) as

$$G_K(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-x')} S_F^{(1)}(p + \eta_1 K) S_F^{(2)}(-p + \eta_2 K) \quad (2.8d)$$

We now begin the perturbation theory by separating the instantaneous Coulomb interaction from the interaction kernel:

$$I_K = I^c + I'_K \quad (2.9)$$

where
$$I^c = -i\alpha \delta(x-x') \delta(x_0) \gamma_4^{(1)} \gamma_4^{(2)} / r \quad (2.9a)$$

and I'_K the remaining part of the interaction kernel.

Hence (2.8) can be written:

$$\begin{aligned} \varphi_K(x) = & -i\alpha G_K(x, x') \delta(x'-x'') \delta(x_0) \frac{\gamma_4^{(1)} \gamma_4^{(2)}}{r} \varphi_K(x'') \\ & + G_K(x, x') I'_K(x', x'') \varphi_K(x'') \end{aligned} \quad (2.10)$$

For the unperturbed (Coulomb interaction) equation, having total energy K_0^C , we take:

$$\varphi_{K^C}(x) = -i\alpha G_{K^C}(x, x') \delta(x' - x'') \delta(x_0) \frac{y_4^{(1)} y_4^{(2)}}{\mathcal{L}} \varphi_{K^C}(x'') \quad (2.11)$$

We then want to evaluate the energy difference

$\Delta E \equiv K_0 - K_0^C$. In order to get at this energy difference, it is useful to factor $G_K(x, x')$ into the following form:

$$\begin{aligned} G_K(x, x') &= g_K(x, y) \Lambda_K(y, x') y_4^{(1)} y_4^{(2)} \\ &= \Lambda_K(x, y) g_K(y, x') y_4^{(1)} y_4^{(2)} \end{aligned} \quad (2.12)$$

where

$$g_K(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{i p(x-y)} \left[H^{(1)}(p) + H^{(2)}(-p) - K_0 \right]^{-1} \quad (2.12a)$$

and

$$\Lambda_K(y, x') = \int \frac{d^4 p}{(2\pi)^4} e^{i p(y-x')} \left\{ \left[H^{(1)}(p) - (p_0 + \mathcal{L}_2 K_0) \right]^{-1} + \left[H^{(2)}(-p) - (-p_0 + \mathcal{L}_2 K_0) \right]^{-1} \right\} \quad (2.12b)$$

where $H^{(i)}(p) = \alpha^{(i)} \cdot p + \beta^{(i)} m_i$ is the Dirac Hamiltonian.

A similar factorization holds for $G_K^C(x, x')$. Then we note that:

$$g_K(x, y) - g_{K^C}(x, y) = \Delta E \int \frac{d^4 p}{(2\pi)^4} e^{i p(x-y)} \left[H^{(1)}(p) + H^{(2)}(-p) - K_0 \right]^{-1} \left[H^{(1)}(p) + H^{(2)}(-p) - K_0^C \right]^{-1} \quad (2.13)$$

Multiplying (2.13) by $\Lambda_K(y, x')$ we obtain:

$$\begin{aligned}
g_K(x, y) \Lambda_K(y, x') - g_{K^c}(x, y) \Lambda_K(y, x') &= \Delta E G_K(x, z) \Lambda_K(z, y) g_{K^c}(y, x') \\
&= G_K(x, x') \delta_4^{(1)} \delta_4^{(2)} - g_{K^c}(x, y) \Lambda_K(y, x') = \Delta E G_K(x, y) \delta_4^{(1)} \delta_4^{(2)} g_{K^c}(y, x')
\end{aligned} \tag{2.14}$$

In order to utilize this equation we define a hybrid wave function, (which differs from $\phi_K^c(x)$ by terms of order ΔE):

$$\varphi_K'(x) \equiv g_{K^c}(x, y) \Lambda_K(y, x') \left(-\frac{i\alpha}{r} d(x_0') \varphi_{K^c}(x') \right) \tag{2.15}$$

$$= -i \Lambda_K(x, y) d(y_0) \mathcal{X}(y) \tag{2.15a}$$

$$\text{where } d(y_0) \mathcal{X}(y) \equiv \alpha g_{K^c}(y, x') d(x_0') \varphi_{K^c}(x') / r \tag{2.15b}$$

Then multiplying (2.14) by $-i\alpha d(x_0') \phi_K^c(x') / r$ we have:

$$\begin{aligned}
G_K(x, x') I_K^c(x', x'') \varphi_K'(x'') - \varphi_K'(x) &= \Delta E G_K(x, y) \delta_4^{(1)} \delta_4^{(2)} g_{K^c}(y, x') \\
&\quad \times \left(-i\alpha \frac{d(x_0')}{r'} \varphi_{K^c}(x') \right)
\end{aligned} \tag{2.16}$$

where we've used $d(x_0) \varphi_K'(x) / r = d(x_0) \varphi_{K^c}(x) / r$ which follows

because $\delta(x_0) \Lambda_K(x, y) \delta(y_0)$ is independent of K . Equation (2.16) then becomes:

$$\varphi_K'(x) - G_K(x, x') I_K^c(x', x'') \varphi_K'(x'') = i \Delta E G_K(x, y) \delta_4^{(1)} \delta_4^{(2)} d(y_0) \mathcal{X}(y) \tag{2.17}$$

We then write (2.17) as:

$$\left[G_K^{-1}(x, x') - I_K^c(x, x') \right] \varphi_K'(x') = i \Delta E \delta_4^{(1)} \delta_4^{(2)} d(x_0) \mathcal{X}(x) \tag{2.17a}$$

whose adjoint is

$$\overline{\varphi'_k(x')} \left[G_k^{-1}(x', x) - I_k^c(x', x) \right] = i \Delta E \delta(x_0) \chi^*(x) \quad (2.18)$$

We then multiply by $\phi_K(x)$:

$$\varphi'_k(x') \left[G_k^{-1}(x', x) - I_k^c(x', x) \right] \varphi_k(x) = i \Delta E$$

where we've taken the normalization $\chi^*(x) \delta(x_0) \varphi_k(x) = 1$

Adding and subtracting $I'_k(x, x')$:

$$\begin{aligned} \varphi'_k(x') \left[G_k^{-1}(x', x) - I_k^c(x', x) + I'_k(x', x) \right] \varphi_k(x) &= i \Delta E \\ &= \varphi'_k(x') I'_k(x', x) \varphi_k(x) = i \Delta E \end{aligned} \quad (2.19)$$

where we've used (2.8) and (2.9). We have already seen

that ϕ'_K differs from ϕ_K^c by terms of order ΔE , which is negligible to our order of accuracy. Hence we wish to express

(2.19) in terms of ϕ'_K which we will then replace with ϕ_K^c .

To find the relationship between ϕ_K and ϕ'_K we write (2.17a)

as:

$$\begin{aligned} \left[G_k^{-1}(x, x') - I_k^c(x, x') - I'_k(x, x') + I'_k(x, x') \right] \varphi'_k(x') &= i \Delta E \delta_+^{(1)} \delta_+^{(2)} \delta(x_0) \chi(x) \\ \cong G_k^{-1}(x, x') \varphi'_k(x') - G_k^{-1}(x, x') \varphi_k(x') + I'_k(x, x') \varphi'_k(x') &= i \Delta E \delta_+^{(1)} \delta_+^{(2)} \delta(x_0) \chi(x) \end{aligned}$$

so that

$$\varphi'_k(x) - \varphi_k(x) + G_k(x, x') I'_k(x', x'') \varphi'_k(x'') = i \Delta E G_k(x, y) \delta_+^{(1)} \delta_+^{(2)} \delta(y_0) \chi(y) \quad (2.20)$$

Expanding the exact Green function in terms of the Coulomb Green function:

$$G(x, x') = G_{\kappa^c}(x, x') + G_{\kappa}^c(x, y) I_{\kappa}'(y, y') G_{\kappa}^c(y', x') + \dots$$

and neglecting the term in ΔE in (2.20) we write (2.19) as:

$$\Delta E = -i \overline{\varphi_{\kappa^c}(x)} \left[I_{\kappa}'(x, x') + I_{\kappa}'(x, y) G_{\kappa}^c(y, y') I_{\kappa}'(y', x') \right] \varphi_{\kappa^c}(x') \quad (2.21)$$

This expression is the one that will be used in the following calculations. But first we must derive an approximation to the Coulomb wave function.

III. FIRST ITERATED WAVE FUNCTION

In order to make use of the expression for energy shift, eq. (2.21), we need an approximation to the Coulomb wave function $\phi_K^C(x)$. This may be obtained by using (2.11) as the basis of an iteration procedure⁽¹⁰⁾ beginning with the non-relativistic Pauli wave function $\phi_p(x)$. Specifically the first iterated wave function is written as

$$\varphi_K^{(1)}(x) = \int d^4x' d^4x'' G_K(x, x') I_{K^c}(x', x'') \varphi_p(x'') \quad (3.1)$$

$$\text{where } G_K(x, x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} S_F^{(1)}(p+\eta, K) S_F^{(2)}(-p+\eta, K)$$

$$\text{and } I^c(x, x'') = -i 4\pi\alpha \gamma_4^{(1)} \gamma_4^{(2)} d(x'-x'') d(t'') \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot x''}}{|p|^2}$$

$$\text{and } \varphi_p(x'') = \int \frac{d^3p'}{(2\pi)^{3/2}} e^{ip' \cdot x''} \left(\frac{8\pi\gamma \varphi_p(0)}{(2\pi)^{3/2} (p'^2 + \gamma^2)^{1/2}} \right)$$

where $\gamma \equiv \mu\alpha$, and $\phi_p(0) = \gamma^{3/2} \pi^{-1/2} (\text{spin ftn}) \equiv \phi(0) (\text{spin ftn})$

The x'' integration is trivial using $\delta(x'-x'')$. The x' integration yields $(2\pi)^3 \delta(-p^1 - q)$. Hence the now trivial q integration yields

$$\varphi_K^{(1)}(x) = \frac{-i 4\pi\alpha \gamma_4^{(1)} \gamma_4^{(2)}}{\pi (2\pi)^4} \int \frac{d^4p d^3p'}{(p'^2 + \gamma^2)^{1/2} |p-p'|^2} e^{ip \cdot x} e^{-ip_0 t} S_F^{(1)}(p+\eta, K) S_F^{(2)}(p+\eta, K) \quad (3.2)$$

but $\int \frac{d^3 p'}{(p'^2 + y^2)^2 |x - p'|^2} = \frac{\pi^2}{y^4 (p'^2 + y^2)}$, so

$$\varphi_{K^c}^{(1)}(x) = -i \frac{4\pi x y_4^{(1)} y_4^{(2)}}{(2\pi)^4} \int \frac{d^4 p e^{i p \cdot x} e^{-i p_0 t}}{(p^2 + y^2)^2} S_F^{(1)}(p + \eta, K) S_F^{(2)}(-p + \eta_2, K) \varphi_p^{(0)} \quad (3.3)$$

In order to do the p_0 integration, we expand the propagators:

$$\begin{aligned} S_F^{(1)}(p + \eta, K) S_F^{(2)}(-p + \eta_2, K) = & \left\{ \frac{\Lambda_+^{(1)}(p) \Lambda_+^{(2)}(p)}{(p_0 + \eta_1 K_0 - E_1)(-p_0 + \eta_2 K_0 - E_2)} + \frac{\Lambda_+^{(1)} \Lambda_-^{(2)}}{(p_0 + \eta_1 K_0 - E_1)(-p_0 + \eta_2 K_0 + E_2)} \right. \\ & \left. + \frac{\Lambda_-^{(1)}(p) \Lambda_+^{(2)}(-p)}{(p_0 + \eta_1 K_0 + E_1)(-p_0 + \eta_2 K_0 - E_2)} + \frac{\Lambda_-^{(1)}(p) \Lambda_-^{(2)}(-p)}{(p_0 + \eta_1 K_0 + E_1)(-p_0 + \eta_2 K_0 + E_2)} \right\} y_4^{(1)} y_4^{(2)} \end{aligned} \quad (3.4)$$

The p_0 integration is then accomplished by closing the integration contour in the lower half of the complex plane if $t > 0$ and in the upper half of the complex plane if $t < 0$. For example: (++) term):

$$\begin{aligned} & (-1) \int_{-\infty}^{\infty} d p_0 e^{-i p_0 t} \left[p_0 - (-\eta_1 K_0 + E_1(p) - i\epsilon) \right]^{-1} \left[p_0 - (\eta_2 K_0 - E_2(p) + i\epsilon) \right]^{-1} \\ & = 2\pi i \left\{ \frac{\theta(t) e^{-i(E_1 - \eta_1 K_0)|t|}}{E_1 + E_2 - K_0} + \frac{\theta(-t) e^{-i(E_2 - \eta_2 K_0)|t|}}{E_1 + E_2 - K_0} \right\} \end{aligned}$$

where $\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

and using $\eta_1 + \eta_2 = 1$

(3.5a)

Similarly the (+-) term yields

$$-(2\pi i) \left\{ \frac{e^{-i(E_1 - \eta_1 K_0)|t|}}{E_2 - E_1 + K_0} - \frac{e^{-i(E_2 + \eta_2 K_0)|t|}}{E_2 - E_1 + K_0} \right\} \theta(t) \quad (3.5b)$$

and the $(-+)$ term

$$-(2\pi i) \left\{ \frac{e^{-i(E_1 + \eta_1 K_0)|t|}}{E_2 - E_1 + K_0} - \frac{e^{-i(E_2 - \eta_2 K_0)|t|}}{E_2 - E_1 + K_0} \right\} \theta(-t) \quad (3.5c)$$

and the $(--)$ term

$$(2\pi i) \left\{ \theta(-t) \frac{e^{-i(E_1 + \eta_1 K_0)|t|}}{E_1 + E_2 + K_0} + \theta(t) \frac{e^{-i(E_2 + \eta_2 K_0)|t|}}{E_1 + E_2 + K_0} \right\} \quad (3.5d)$$

Substituting (3.5a - 3.5d) into (3.4) and rearranging we have:

$$\begin{aligned} & \frac{2\pi i \theta(t)}{4E_1(p)E_2(p)} \left\{ \frac{[E_1 + H^{(1)}(p)][E_2 + H^{(2)}(p)]}{E_1 + E_2 - K_0} e^{-i(E_1 - \eta_1 K_0)|t|} \right. \\ & \quad - \frac{[E_1 + H^{(1)}(p)][E_2 - H^{(2)}(p)]}{E_2 - E_1 + K_0} \left(e^{-i(E_1 - \eta_1 K_0)|t|} - e^{-i(E_2 + \eta_2 K_0)|t|} \right) \\ & \quad \left. + \frac{[E_1 - H^{(1)}(p)][E_2 - H^{(2)}(p)]}{E_1 + E_2 + K_0} e^{-i(E_2 + \eta_2 K_0)|t|} \right\} \\ & + \frac{2\pi i \theta(-t)}{4E_1(p)E_2(p)} \left\{ \frac{[E_1 + H^{(1)}(p)][E_2 + H^{(2)}(p)]}{E_1 + E_2 - K_0} e^{-i(E_2 - \eta_2 K_0)|t|} \right. \\ & \quad - \frac{[E_1 - H^{(1)}(p)][E_2 + H^{(2)}(p)]}{E_2 - E_1 + K_0} \left(e^{-i(E_1 + \eta_1 K_0)|t|} - e^{-i(E_2 - \eta_2 K_0)|t|} \right) \\ & \quad \left. + \frac{[E_1 - H^{(1)}(p)][E_2 - H^{(2)}(p)]}{E_1 + E_2 + K_0} e^{-i(E_1 + \eta_1 K_0)|t|} \right\} \quad (3.6) \end{aligned}$$

This equation can be rearranged by using $\theta(t) = \frac{1}{2}(1 + \epsilon(t))$

$$\text{and } \theta(-t) = \frac{1}{2}(1 - \epsilon(t)) \quad \text{where } \epsilon(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

Then we use $H^{(i)}(p) = \alpha^{(i)} \cdot p + \beta^{(i)} m_i$, ($i=1,2$) and the fact that $\beta^{(i)} \phi_p(0) = \phi_p(0)$ so that we can effectively set $\beta^{(i)}=1$ ($i=1,2$). We combine the exponentials according to the formula:

$$f_{\pm}^{(j)} = \frac{1}{2} \left[\exp\{-i(E_j - \pi_j \cdot k_0)/t\} \pm \exp\{-i(E_j + \pi_j \cdot k_0)/t\} \right] \quad (3.7)$$

Terms of order α^2 in the numerator are neglected (i.e. $K_0 \approx M - \mu\alpha^2/2 \approx M$) while the denominators are expanded in exact powers of momentum:

$$\begin{aligned} \frac{1}{(E_1 + E_2)^2 - K_0^2} &= \frac{(E_2 - E_1)^2 - K_0^2}{[(E_1 + E_2)^2 - K_0^2][(E_2 - E_1)^2 - K_0^2]} \\ &\approx \frac{(E_2 - E_1)^2 - M^2 + \mathcal{O}(K^2)}{-4M^2(p^2 + k^2) + \mathcal{O}(K^4)} \end{aligned} \quad (3.8)$$

and similarly

$$\frac{1}{(E_2 - E_1)^2 - K_0^2} = \frac{(E_2 + E_1)^2 - M^2 + \mathcal{O}(K^2)}{-4M^2(p^2 + k^2) + \mathcal{O}(K^4)} \quad (3.9)$$

After a bit of manipulation we obtain

$$\varphi''(p, t) \equiv \varphi_c''' + \delta\varphi + \delta\varphi' \quad (3.10)$$



where

$$\varphi_c''' = 4\mu M \left(1 + \frac{\alpha^{(1)} \cdot p}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p}{2m_2}\right) [F(t) + \epsilon(t)G(t)] \frac{\pi \alpha \varphi_p(0)}{(2\pi)^{3/2} M (p^2 + y^2)^2} \quad (3.10a)$$

and

$$\delta\varphi = p^2 \left[\delta F(t) + \epsilon(t) \delta G(t) \right] \frac{\pi \alpha \varphi_p(0)}{(2\pi)^{3/2} M (p^2 + y^2)^2} \quad (3.10b)$$

and

$$\delta\varphi' = -p^2 (\alpha^{(1)} \cdot p + \alpha^{(2)} \cdot p) \left[\delta F'(t) + \epsilon(t) \delta G'(t) \right] \frac{\pi \alpha \varphi_p(0)}{(2\pi)^{3/2} M (p^2 + y^2)^2} \quad (3.10c)$$

with

$$F(t) = \left[\left(\frac{m_1}{E_1} f_+'' + f_-'' \right) + (1 \leftrightarrow 2) \right]$$

$$G(t) = \left[\left(\frac{m_1}{E_1} f_-'' + f_+'' \right) - (1 \leftrightarrow 2) \right]$$

$$\delta F(t) = \left[\left(\frac{2m_2 - m_1}{E_1} f_+'' - f_-'' \right) + (1 \leftrightarrow 2) \right]$$

$$\delta G(t) = \left[\left(\frac{2m_2 - m_1}{E_1} f_-'' - f_+'' \right) - (1 \leftrightarrow 2) \right] \quad (3.10d)$$

$$\delta F'(t) = \left[\frac{f_+'''}{E_1} - (1 \leftrightarrow 2) \right]$$

$$\delta G'(t) = \left[\frac{f_-'''}{E_1} - (1 \leftrightarrow 2) \right]$$

It is also useful to note at this point that the static wave function (i.e. at $t = 0$) is given by

$$\psi'''(\mathbf{p}, 0) = \left\{ D(\mathbf{p}) + F(\mathbf{p}) \alpha^{(1)} \cdot \mathbf{p} - \bar{F}(\mathbf{p}) \alpha^{(2)} \cdot \mathbf{p} - G(\mathbf{p}) \alpha^{(1)} \cdot \mathbf{p} \alpha^{(2)} \cdot \mathbf{p} \right\} \frac{\pi \alpha \varphi_p(0)}{(2\pi)^{3/2} M(p^2 + g^2)^2} \quad (3.11)$$

where

$$\begin{aligned} D(\mathbf{p}) &= 4\mu M \left(\frac{m_1}{E_1} + \frac{m_2}{E_2} \right) + \frac{p^2}{2} \left[M \left(\frac{1}{E_1} + \frac{1}{E_2} \right) + 3(m_2 - m_1) \left(\frac{1}{E_1} - \frac{1}{E_2} \right) \right] \\ F(\mathbf{p}) &= 2m_2 \left(\frac{m_1}{E_1} + \frac{m_2}{E_2} \right) - p^2 \left(\frac{1}{E_1} - \frac{1}{E_2} \right) \\ \bar{F}(\mathbf{p}) &= 2m_1 \left(\frac{m_1}{E_1} + \frac{m_2}{E_2} \right) + p^2 \left(\frac{1}{E_1} - \frac{1}{E_2} \right) \\ G(\mathbf{p}) &= \left(\frac{m_1}{E_1} + \frac{m_2}{E_2} \right) \end{aligned} \quad (3.12)$$

Both $\delta\phi$ and $\delta\phi'$ are proportional to p^2 and hence represent those terms of the first approximation to the true Coulomb wave function which may be in error. Only a second iteration process would resolve the uncertainty in these terms, but a second iteration calculation appears to involve a prohibitive amount of unnecessary work at this stage.

The only other uncertainties or approximations made in the calculation of $\phi_K^{c(1)}$ are that terms of order α^2 were neglected in the numerator and terms of order α^4 were neglected in the denominator. In each case such terms would contribute only to an energy splitting of order α^7 or higher, hence we are justified in dropping them.

IV. CALCULATIONS

Chapter IV consists of the details of the calculations involved in the computation of the $\alpha^6 \ln \alpha^{-1}$ hyperfine splitting. For convenience this chapter will be broken up into five sections: A. The Coulomb recoil, B. The one photon exchange, C. The two photon ladder exchange, D. The two photon crossed exchange, and E. Non-contributing diagrams.

A. Coulomb Recoil

The first source of potential $\alpha^6 \ln \alpha^{-1}$ contribution comes from the Coulomb recoil. In order to determine ΔE (eq. (2.21)) to order $\alpha^6 \ln \alpha^{-1}$ we must first determine the "unperturbed" energy, K_0^C , to the same order. The $\alpha^6 \ln \alpha^{-1}$ component of K_0^C is called the Coulomb recoil.

We begin by writing (2.11) as⁽²⁾:

$$\begin{aligned} & \left[K_0^C - H^{(1)}(\varphi) - H^{(2)}(\varphi) \right] \varphi_K^C(\underline{r}, 0) \\ & + \int \frac{d^3 r' d^3 k}{(2\pi)^3} e^{i \underline{k} \cdot (\underline{r} - \underline{r}')} \left\{ \Lambda_+^{(1)}(\underline{k}) \Lambda_+^{(2)}(-\underline{k}) - \Lambda_-^{(1)}(\underline{k}) \Lambda_-^{(2)}(-\underline{k}) \right\} \frac{\alpha}{r'} \varphi_K^C(\underline{r}', 0) = 0 \end{aligned} \quad (4A.1)$$

Adding and subtracting $(\alpha/r) \varphi_K^C(r, 0)$ and using the property of projection operators that

$$\Lambda_+^{(1)} \Lambda_+^{(2)} + \Lambda_+^{(1)} \Lambda_-^{(2)} + \Lambda_-^{(1)} \Lambda_+^{(2)} + \Lambda_-^{(1)} \Lambda_-^{(2)} = 1$$

we can write (4A.1) as:

$$\left[K_0^c - H^{(1)}(\mathbf{p}) - H^{(2)}(\mathbf{p}) + \frac{\alpha}{r} \right] \varphi_{K^c}(\mathbf{r}, 0) \quad (4A.2)$$

$$- \int d^3 r' \Omega(\mathbf{r}, \mathbf{r}') \frac{\alpha}{r'} \varphi_{K^c}(\mathbf{r}', 0) = 0$$

where

$$\begin{aligned} \Omega(\mathbf{r}, \mathbf{r}') = \int \frac{d^3 k}{(2\pi)^3} \{ & \Lambda_+^{(1)}(\mathbf{k}) \Lambda_-^{(2)}(-\mathbf{k}) + \Lambda_-^{(1)}(\mathbf{k}) \Lambda_+^{(2)}(-\mathbf{k}) \\ & + 2 \Lambda_-^{(1)}(\mathbf{k}) \Lambda_-^{(2)}(-\mathbf{k}) \} e^{i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \end{aligned} \quad (4A.3)$$

Equation (4A.2) is very similar to the Breit equation:

$$\left[K_0^{c'} - H^{(1)}(\mathbf{p}) - H^{(2)}(\mathbf{p}) + \frac{\alpha}{r} \right] \varphi_B(\mathbf{r}, 0) = 0 \quad (4A.4)$$

The energy levels of the Breit equation can be expressed as a power series in α^2 (11) with no $\alpha^6 \ln \alpha^{-1}$ term in the expansion.

We then take the $\Omega(\alpha/r)$ term of (4A.2) as a small perturbation of (4A.4) and compute the difference of K_0^c and $K_0^{c'}$ to order $\alpha^6 \ln \alpha^{-1}$. Since $K_0^{c'}$ contains no $\alpha^6 \ln \alpha^{-1}$ term, the difference mentioned above will be the Coulomb recoil to order $\alpha^6 \ln \alpha^{-1}$.

Hence we write:

$$\Delta E_{\text{Coul}} = K_0^c - K_0^{c'} = \frac{4\pi\alpha}{(2\pi)^3} \int d^3 p d^3 p' \frac{\varphi_B^*(\mathbf{p}') \Omega(\mathbf{p}')}{|\mathbf{p} - \mathbf{p}'|^2} \varphi_B(\mathbf{p}) \quad (4A.5)$$

where $\Omega(p')$ is the Fourier transform of (4A.3) and where

$$\varphi_B(p) = \left[H^{(1)}(p) + H^{(2)}(-p) - K_0^{c'} \right]^{-1} \int \frac{d\Omega'}{(2\pi)^{3/2}} e^{-i p \cdot \Omega'} \left\{ \frac{\alpha}{\Omega'} \varphi_B(\Omega') \right\} \quad (4A.6)$$

which is the Fourier transform of (4A.4). Taking the first iteration of (4A.6) by inserting

$$\varphi_p(\Omega') = e^{-\gamma |\Omega'|} \varphi_p(0)$$

on the right hand side, we obtain:

$$\varphi_B(p) = \left[H^{(1)}(p) + H^{(2)}(-p) - K_0^{c'} \right]^{-1} \frac{4\pi\alpha \varphi_p(0)}{(2\pi)^{3/2} (p^2 + \gamma^2)} \quad (4A.7)$$

as the first iteration. The operator $[H^{(1)}(p) + H^{(2)}(-p) - K_0^{c'}]^{-1}$ can be explicitly written as:

$$\begin{aligned} & \left[H^{(1)}(p) + H^{(2)}(-p) - K_0 \right]^{-1} = \\ & = \left\{ \frac{\Lambda_+^{(1)} \Lambda_+^{(2)}}{E_1 + E_2 - K_0} + \frac{\Lambda_+^{(1)} \Lambda_-^{(2)}}{E_1 - E_2 - K_0} + \frac{\Lambda_-^{(1)} \Lambda_+^{(2)}}{E_2 - E_1 - K_0} - \frac{\Lambda_-^{(1)} \Lambda_-^{(2)}}{E_1 + E_2 + K_0} \right\} \end{aligned} \quad (4A.7a)$$

Thus from (4A.3) and (4A.7a) we have:

$$\varphi_B^*(p') \Omega(p') = \frac{4\pi\alpha \varphi_p^\dagger(0)}{(2\pi)^{3/2} (p'^2 + \gamma^2)} \left\{ - \frac{\Lambda_+^{(1)} \Lambda_-^{(2)}}{E_2' - E_1' + K_0} + \frac{\Lambda_-^{(1)} \Lambda_+^{(2)}}{E_2' - E_1' - K_0} - \frac{\Lambda_-^{(1)} \Lambda_-^{(2)}}{E_1' + E_2' + K_0} \right\} \quad (4A.8)$$

Setting $\beta^{(i)} = 1$ ($i=1,2$) the projection operators are effectively:

$$\begin{aligned}
\Lambda_+^{(1)} \Lambda_+^{(2)} &= \frac{1}{4E_1 E_2} \left\{ (E_1 + m_1)(E_2 + m_2) + (E_2 + m_2) \underline{\alpha}^{(1)} \cdot \underline{p} - (E_1 + m_1) \underline{\alpha}^{(2)} \cdot \underline{p} - \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \right\} \\
\Lambda_+^{(1)} \Lambda_-^{(2)} &= \frac{1}{4E_1 E_2} \left\{ (E_1 + m_1)(E_2 - m_2) + (E_2 - m_2) \underline{\alpha}^{(1)} \cdot \underline{p} + (E_1 + m_1) \underline{\alpha}^{(2)} \cdot \underline{p} + \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \right\} \\
\Lambda_-^{(1)} \Lambda_+^{(2)} &= \frac{1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2 + m_2) - (E_2 + m_2) \underline{\alpha}^{(1)} \cdot \underline{p} - (E_1 - m_1) \underline{\alpha}^{(2)} \cdot \underline{p} + \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \right\} \\
\Lambda_-^{(1)} \Lambda_-^{(2)} &= \frac{1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2 - m_2) - (E_2 - m_2) \underline{\alpha}^{(1)} \cdot \underline{p} + (E_1 - m_1) \underline{\alpha}^{(2)} \cdot \underline{p} - \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \right\}
\end{aligned} \tag{4A.9}$$

Since we will need only the $\underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}'$ term in computing the spin-spin term (see Section 4C and Appendix B) we can write (4A.8) as:

$$\begin{aligned}
\varphi_B^*(\underline{p}') \varphi_B(\underline{p}') &= \frac{4\pi\alpha \varphi_p^{\dagger(0)}}{(2\pi)^{3/2} (p'^2 + \gamma^2)} \left\{ - \frac{\underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}'}{4E_1' E_2' (E_2' - E_1' + K_0)} + \frac{\underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}'}{4E_1' E_2' (E_2' - E_1' - K_0)} \right. \\
&\quad \left. - \frac{\underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}'}{4E_1' E_2' (E_2' + E_1' + K_0)} \right\}
\end{aligned} \tag{4A.10}$$

Similarly the only part of $\phi_B(\underline{p})$ that can possibly contribute to the spin-spin interaction is:

$$\begin{aligned}
\varphi_B(\underline{p}) &= \frac{4\pi\alpha \varphi_p^{(0)}}{(2\pi)^{3/2} (p^2 + \gamma^2)} \left\{ \frac{-\underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p}}{4E_1 E_2 (E_1 + E_2 - K_0)} - \frac{\underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p}}{4E_1 E_2 (E_2 - E_1 + K_0)} \right. \\
&\quad \left. + \frac{\underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p}}{4E_1 E_2 (E_2 - E_1 - K_0)} - \frac{\underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p}}{4E_1 E_2 (E_1 + E_2 + K_0)} \right\}
\end{aligned} \tag{4A.11}$$

Using the fact that $\langle \alpha^{(1)} \cdot p' \alpha^{(2)} \cdot p' \alpha^{(1)} \cdot p \alpha^{(2)} \cdot p \rangle \rightarrow -\frac{1}{3} |p' \times p|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle$
 (4A.5) becomes:

$$\begin{aligned} \Delta E_{\text{coul}} = & -\frac{4\pi\alpha}{(2\pi)^3} \frac{\pi^2\alpha^2}{(2\pi)^3} |\psi(0)|^2 \frac{\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle}{3} \int \frac{d^3p d^3p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)} \frac{|p' \times p|^2}{|p - p'|^2} \\ & \times \left\{ \frac{-1}{E_1' E_2' (E_2' - E_1' + K_0)} + \frac{1}{E_1' E_2' (E_2' - E_1' - K_0)} - \frac{1}{E_1' E_2' (E_2' + E_1' + K_0)} \right\} \\ & \times \left[\frac{-1}{E_1 E_2 (E_1 + E_2 - K_0)} - \frac{1}{E_1 E_2 (E_2 - E_1 + K_0)} + \frac{1}{E_1 E_2 (E_2 - E_1 - K_0)} - \frac{1}{E_1 E_2 (E_1 + E_2 + K_0)} \right] \end{aligned}$$

(4A.12)

Now

$$\frac{1}{E_2' - E_1' + K_0} \simeq \frac{1}{(E_2' + m_2) - (E_1' - m_1) - \frac{1}{2}\mu\alpha^2} \sim \frac{1}{2m_2 + \mathcal{O}(p'^2)}$$

$$\frac{1}{E_2' - E_1' - K_0} \sim \frac{-1}{2m_1 + \mathcal{O}(p'^2)}$$

$$\frac{1}{E_1 + E_2 - K_0} \simeq \frac{1}{(E_1 - m_1) + (E_2 - m_2) + \frac{1}{2}\mu\alpha^2} \sim \frac{2\mu}{p^2 + \gamma^2}$$

The most singular term in (4A.12) (the term involving $(E_1 + E_2 - K_0)^{-1}$) has dimension [momentum]⁰ at small momentum and hence⁽¹²⁾ will not contribute to order $\alpha^6 \ln \alpha^{-1}$.

Thus, to this order, the Coulomb recoil is zero.

B. One Photon Exchange

The simplest interaction to yield an $\alpha^6 \ln \alpha^{-1}$ contribution is the exchange of a single photon. We separate the instantaneous Coulomb interaction (which accounts for the major part of the binding) from the covariant single photon exchange, and then utilize (2.21) from perturbation theory to approximate the remaining time dependent contribution.

The single photon exchange kernel is (see Appendix A):

$$I(x, x') = -i 4\pi\alpha \delta(x-x') \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ikx}}{k^2} \left\{ F_1(k^2) \gamma_\mu^{(1)} \gamma_\mu^{(2)} + \frac{F_2(k^2)}{2m_2} \gamma_\mu^{(1)} \tilde{\sigma}_{\mu\nu}^{(2)} k_\nu \right\} \quad (4B.1)$$

where k_μ is the four momentum of the exchanged photon. Subtracting out the instantaneous Coulomb interaction, $i(4\pi\alpha) \gamma_4^{(1)} \gamma_4^{(2)} / |k|^2$, we obtain:

$$I^{(1)}(x, x') = -i 4\pi\alpha \delta(x-x') \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ikx}}{k^2} \left\{ F_1(k^2) \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \tilde{\alpha}^{(1)} \cdot \tilde{\alpha}^{(2)} \right) + \frac{F_2(k^2)}{2m_2} \gamma_\mu^{(1)} \tilde{\sigma}_{\mu\nu}^{(2)} k_\nu \right\} \quad (4B.2)$$

Thus we substitute this kernel into

$$\Delta E_1 = -i \int d^4 x d^4 x' \overline{\varphi_{K^c}^{(1)}(x)} I^{(1)}(x, x') \varphi_{K^c}^{(1)}(x) \quad (4B.3)$$

where $\varphi_{K^c}^{(1)}(x)$ is the first iterated Coulomb wave function.

The kernel (4B.2) very naturally separates itself into a "charge" part $I^c(x, x')$ and a "dipole" part $I^d(x, x')$. We

compute each part separately for computational ease.

1. Charge Component

For the charge component it is sufficient to set $F_1(k^2) = F_1(0) = 1^{(13)}$ since the $\alpha^6 \ln \alpha^{-1}$ contribution arises from low momentum recoil terms. Indeed for any term to give a logarithmic contribution, we can take $k \ll m_1^{(14)}$. Equation (4B.3) in momentum space is

$$\Delta E_1^c = -i \int \frac{d^4 p d^4 p'}{(2\pi)^4} \varphi^{(1)}(p) I^{(1)c}(p, p') \varphi^{(1)}(p') \quad (4B.4)$$

where $\varphi^{(1)}(p) = \int \frac{dt}{(2\pi)^{1/2}} e^{i p_0 t} \varphi^{(1)}(p, t)$; $\varphi^{(1)}(p, t)$ given by (3.10)

and

$$I^{(1)c} = -i (4\pi\alpha) \frac{\delta_4^{(1)} \delta_4^{(2)}}{k^2} \left(\frac{k_0^2}{|k|^2} - \alpha^{(1)} \cdot \alpha^{(2)} \right), \quad k_\mu \equiv p_\mu - p'_\mu$$

Thus we can write

$$\begin{aligned} \Delta E_1^c = & -4\pi\alpha \int \frac{d^4 p d^4 p' dt dt'}{(2\pi)^5} e^{-i p_0 t} e^{+i p'_0 t'} \varphi^{(1)}(p, t) \frac{\delta_4^{(1)} \delta_4^{(2)}}{k^2} \times \\ & \times \left(\frac{k_0^2}{|k|^2} - \alpha^{(1)} \cdot \alpha^{(2)} \right) \varphi^{(1)}(p', t) \end{aligned}$$

To do the p_0 and p'_0 integrations we invoke the following formulas (based on the outgoing wave prescription):

$$\int_{-\infty}^{\infty} dp_0 \frac{e^{-ip_0 t}}{p_0^2 - |\underline{k}|^2} = -\pi i \frac{e^{-i|\underline{k}||t|}}{|\underline{k}|} \quad (4B.5a)$$

$$\int_{-\infty}^{\infty} dp_0 \frac{p_0^2 e^{-ip_0 t}}{p_0^2 - |\underline{k}|^2} = \left\{ 2\pi \delta(t) - \pi i |\underline{k}| e^{-i|\underline{k}||t|} \right\} \quad (4B.5b)$$

and quickly obtain

$$\begin{aligned} \Delta E_i^c = 4\pi^2 i \alpha \int \frac{d^3 p d^3 p' dt}{(2\pi)^4} \overline{\varphi''(\underline{p}, t)} \frac{\gamma_4^{(1)} \gamma_4^{(2)}}{|\underline{k}|} e^{-i|\underline{k}||t|} \\ \times \left\{ \tilde{\alpha}^{(1)} \cdot \tilde{\alpha}^{(2)} + \frac{2 \delta(t) e^{+i|\underline{k}||t|}}{i|\underline{k}|} - 1 \right\} \varphi''(\underline{p}', t) \end{aligned} \quad (4B.6)$$

It is worth noting that $\overline{\varphi(\underline{p}, t)} = \varphi^\dagger(\underline{p}, -t) \gamma_4^{(1)} \gamma_4^{(2)}$ (4B.7)

which can be seen by examination of the steps leading up to (3.5). We also have $\phi^1(\underline{p}, t) = \phi_C^1 + \delta\phi + \delta\phi'$ where both $\delta\phi$ and $\delta\phi'$ are of order p^2 compared to ϕ_C^1 . Since they are of higher order in momentum they won't make a logarithmic contribution, and hence we need examine only $\phi_C^1 K \phi_C^1$.

We first examine the part of (4B.6) that contains

$$\tilde{\alpha}^1 \cdot \tilde{\alpha}^1:$$

$$\begin{aligned}
(\Delta E_i) &= \frac{4\pi^2 i \alpha}{(2\pi)^4} \int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} \frac{\pi^2 \alpha^2 |\psi(0)|^2}{M^2 (2\pi)^3} (4\mu M)^2 \\
&\quad \left\langle \left(1 + \frac{\alpha^{(1)} \cdot p}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p}{2m_2}\right) \alpha^{(1)} \cdot \alpha^{(2)} \left(1 + \frac{\alpha^{(1)} \cdot p'}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p'}{2m_2}\right) \right\rangle \\
&\quad \times \int_{-\infty}^{\infty} dt \left\{ \left[\left(\frac{m_1}{E_i} f_+^{(1)} + f_-^{(1)} \right) + (1 \leftrightarrow 2) \right] \left[\left(\frac{m_1}{E_i'} f_+^{(1)} + f_-^{(1)} \right) + (1 \leftrightarrow 2) \right] \right. \\
&\quad \left. - \epsilon^2(t) \left[\left(\frac{m_1}{E_i} f_-^{(1)} + f_+^{(1)} \right) - (1 \leftrightarrow 2) \right] \left[\left(\frac{m_1}{E_i'} f_-^{(1)} + f_+^{(1)} \right) - (1 \leftrightarrow 2) \right] \right\} \\
&\quad \times \frac{e^{-i|k||t|}}{|k|} \tag{4B.8}
\end{aligned}$$

We have eliminated all terms linear in $\epsilon(t)$ as they would lead to an integrand that is an odd function of time. The time integration can then be put in the following form (using the fact that $\epsilon^2(t) = 1$ except for a point discontinuity at $t = 0$):

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt \left\{ \frac{m_1^2 - E_i E_i'}{E_i E_i'} (f_+^{(1)} f_+^{(1)} - f_-^{(1)} f_-^{(1)}) + \left(\frac{m_1}{E_i} - \frac{m_1}{E_i'} \right) (f_+^{(1)} f_-^{(1)} - f_-^{(1)} f_+^{(1)}) \right. \\
&\quad + \frac{m_1 m_2 + E_i E_i'}{E_i E_i'} (f_+^{(1)} f_+^{(2)} + f_-^{(1)} f_-^{(2)}) + \left(\frac{m_1}{E_i} + \frac{m_2}{E_i'} \right) (f_+^{(1)} f_-^{(2)} + f_-^{(1)} f_+^{(2)}) \\
&\quad \left. + (1 \leftrightarrow 2) \right\} \frac{e^{-i|k||t|}}{|k|} \tag{4B.9}
\end{aligned}$$

We note that we encounter integrals of the form

$\int_{-\infty}^{\infty} dt e^{-ia|t|} e^{-ik|t|} e^{-ib|t|}$ which, since the integrand is an even function of time, can be written as

$$2 \int_0^{\infty} dt e^{-ia|t|} e^{-ik|t|} e^{-ib|t|} = -2i(a+k+b)^{-1}.$$

Hence we can draw up Table 1.

Thus (4B.9) will yield

$$\begin{aligned} & \frac{-2i}{|k|} \left\{ \frac{1}{E_1 E_1'} \left(\frac{m_1^2 - E_1 E_1'}{k + E_1 + E_1'} \right) + \frac{1}{2E_1 E_1'} \left(\frac{(E_1 + m_1)(E_2' + m_2)}{k + E_1 + E_2' - M} \right) \right. \\ & \quad \left. + \frac{1}{2E_1 E_2'} \left(\frac{(E_1 - m_1)(E_2' - m_2)}{k + E_1 + E_2' + M} \right) + (1 \leftrightarrow 2) \right\} \end{aligned} \quad (4B.10)$$

We then expand the denominators according to

$$\frac{1}{k + E_1 + E_1'} = \frac{1}{k} \left(1 - \frac{E_1 + E_1'}{k + E_1 + E_1'} \right), \quad (4B.11)$$

so that equation (4B.11) can be written as

$$\begin{aligned} & \frac{-2i}{k^2} \left\{ \frac{m_1 m_2}{E_1 E_2'} + \frac{m_1^2}{E_1 E_1'} + \frac{p^2 E_1' + p'^2 E_1}{E_1 E_1' (k + E_1 + E_1')} - \frac{p^2 (E_2' + m_2) + p'^2 (E_1 + m_1)}{2E_1 E_2' (k + E_1 + E_2' - M)} \right. \\ & \quad \left. - \frac{p^2 (E_2' - m_2) + p'^2 (E_1 - m_1)}{2E_1 E_2' (k + E_1 + E_2' + M)} + (1 \leftrightarrow 2) \right\} \end{aligned} \quad (4B.12)$$

Referring to Table 2 we see that the spin averages (see Section 4C and Appendix B for the details of their computation) are

$$(4\mu M)^{-1} (\Phi + \Omega + 2\Psi) \quad (4B.13)$$

Table I. Time Integrations for One Photon Exchange.

$$\int_{-\infty}^{\infty} dt (f_+^{(1)} f_+^{(1)} - f_-^{(1)} f_-^{(1)}) \frac{e^{-i k |t|}}{k} = - \frac{2i}{k} \left(\frac{1}{k + E_1 + E_1'} \right)$$

$$\int_{-\infty}^{\infty} dt (f_+^{(1)} f_-^{(1)} - f_-^{(1)} f_+^{(1)}) \frac{e^{-i k |t|}}{k} = 0$$

$$\int_{-\infty}^{\infty} dt (f_+^{(1)} f_+^{(2)} + f_-^{(1)} f_-^{(2)}) \frac{e^{-i k |t|}}{k} = \frac{-i}{k} \left(\frac{1}{k + E_1 + E_2' - M} + \frac{1}{k + E_1 + E_2' + M} \right)$$

$$\int_{-\infty}^{\infty} dt (f_+^{(1)} f_-^{(2)} + f_-^{(1)} f_+^{(2)}) \frac{e^{-i k |t|}}{k} = \frac{-i}{k} \left(\frac{1}{k + E_1 + E_2' - M} - \frac{1}{k + E_1 + E_2' + M} \right)$$

Thus we can write (4B.8) as

$$(\Delta E_1^c) = - \frac{\pi \chi^3 |\varphi(0)|^2}{M^3 (2\pi)^4} \int \frac{d^3 p d^3 p'}{(p^2 + p'^2)^2 (p'^2 + p'^2)^2} \frac{4\mu M (\Phi + \Omega + 2\Psi)}{|p - p'|^4} \quad (4B.14)$$

$$\times \left\{ (4B.12) \text{ IN BRACES} \right\}$$

The only terms that will yield a logarithmic dependence are those terms with an integrand of dimension [momentum]⁻² at small momentum and with a coupling between the p and p' integrations⁽¹²⁾. Inspection will show that the

$\frac{p'^2(E_1+m_1) + p^2(E_1'+m_1)}{2E_1 E_2' (k+E_1+E_2'-M)}$ term will yield a potential logarithmic contribution. Since the integrand of this part has dimension [momentum]⁻³ at small momentum, we expand in powers of momentum to extract the [momentum]⁻² behavior.

$$\frac{p^2}{k+E_1+E_2'-M} = \frac{p^2}{k+E_1-E_1'} \left(1 - \frac{E_2'+E_1'-M}{k+E_1+E_2'-M} \right)$$

$$\simeq \frac{p^2}{k} \left(1 - \frac{p'^2}{2\mu k} \right) \quad (4B.15)$$

so that

$$- \frac{p'^2(E_1+m_1) + p^2(E_2'+m_2)}{2E_1 E_2' (k+E_1+E_2'-M)} \simeq - \frac{p'^2}{E_2'} \left(\frac{1}{k} - \frac{p^2}{2\mu k^2} \right) - \frac{p^2}{E_1} \left(\frac{1}{k} - \frac{p'^2}{2\mu k^2} \right) \quad (4B.16)$$

with the potential logarithmic contribution underlined. We are thus left with

Table II. Spin averages for one photon exchange.

Spin average	Coefficient of $\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle$
$\Phi \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{p} \cdot \underline{p}' \rangle$	$\longrightarrow -\frac{2}{3} p'^2$
$\Psi \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(2)} \cdot \underline{p}' \rangle$ $= \langle \underline{\alpha}^{(2)} \cdot \underline{p} \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1)} \cdot \underline{p}' \rangle$	$\longrightarrow \frac{2}{3} \underline{p} \cdot \underline{p}'$
$\Omega \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \alpha^{(1)} \cdot \alpha^{(2)} \rangle$	$\longrightarrow -\frac{2}{3} p^2$
$\mathcal{R} \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}' \rangle$	$\longrightarrow -\frac{1}{3} \underline{p} \times \underline{p}' ^2$
$\mathcal{S} \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{k} \underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}' \rangle$	$\longrightarrow -\frac{1}{3} \underline{p} \times \underline{p}' ^2$
$\mathcal{I} \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{k} \underline{\alpha}^{(1)} \cdot \underline{p}' \rangle$	$\longrightarrow \frac{1}{3} \underline{p} \times \underline{p}' ^2$
$\mathcal{U} \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{p}' \rangle$	$\longrightarrow \frac{1}{3} \underline{p} \times \underline{p}' ^2$
$\mathcal{V} \equiv \langle \underline{\alpha}^{(2)} \cdot \underline{p} \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(1)} \cdot \underline{p}' \underline{\alpha}^{(2)} \cdot \underline{p}' \rangle$	$\longrightarrow -\frac{1}{3} \underline{p} \times \underline{p}' ^2$
$\mathcal{W} \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} \underline{\alpha}^{(1)} \cdot \underline{p}' \rangle$	$\longrightarrow -\frac{2}{3} (\underline{p} \cdot \underline{p}' - p'^2)$
$\mathcal{X} \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} \rangle$	$\longrightarrow \frac{2}{3} (p^2 - \underline{p} \cdot \underline{p}')$
$\mathcal{Y} \equiv \langle \underline{\alpha}^{(2)} \cdot \underline{p} \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p}' \rangle$	$\longrightarrow \frac{2}{3} [(p \cdot p')^2 - p'^2 (\underline{p} \cdot \underline{p}')]]$
$\mathcal{Z} \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} \underline{\alpha}^{(1)} \cdot \underline{p}' \rangle$	$\longrightarrow \frac{2}{3} [p^2 p'^2 - p' (\underline{p} \cdot \underline{p}')]]$

$$\begin{aligned}
& \frac{-\pi \alpha^3 |\varphi(0)|^2}{M^2 (2\pi)^4} \left(-\frac{8}{3} \mu M \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \right) \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \frac{|p - p'|^2}{|p - p'|^2} \times \\
& \times \left\{ \frac{p'^2 p^2}{2\mu E_2' |p - p'|^2} + \frac{p^2 p'^2}{2\mu E_1 |p - p'|^2} + (1 \leftrightarrow 2) \right\}
\end{aligned} \tag{4B.17}$$

Using Table III (see Appendix C for a derivation of the integrals) we obtain:

$$\frac{2}{3} \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{\mu M} \tag{4B.18}$$

We now return to that part of (4B.6) that has no α matrices in the kernel. The only possible spin-spin contribution comes from $\mathcal{R}$ so that we have

$$\begin{aligned}
(\Delta E_c)_2 &= \frac{4\pi^2 i \alpha}{(2\pi)^4} \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \frac{\pi^2 \alpha^2 |\varphi(0)|^2}{M^2 (2\pi)^3} \mathcal{R} \\
& \times \int_{-\infty}^{\infty} dt [F(t) + \epsilon(t) G(t)] [F'(t) + \epsilon(t) G'(t)] \left\{ \frac{2\delta(t)}{i|k|^2} - \frac{e^{-i|k||t|}}{|k|} \right\}
\end{aligned} \tag{4B.19}$$

The integral over the δ function is trivial, while the integral over the other term is identical to that done above.

Inspection of (4B.12) shows that only the

$$\left(\frac{m_1 m_2}{E_1 E_2'} + \frac{m_1'^2}{E_1 E_1'} \right) \text{ portion will be singular enough to}$$

yield a logarithmic contribution, so that we have:



Table III. $\alpha^6 \ln \alpha^{-1}$ momentum integrals for one photon exchange.

$$\int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} \frac{p^2 p'^2}{|k - p'|^2 E_i'} = \frac{4\pi^4}{m_i} \ln \alpha^{-1}$$

$$\int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} \frac{|k \times p'|^2}{|k - p'|^2 E_i'} = \frac{2\pi^4}{m_i} \ln \alpha^{-1}$$

$$\int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} \frac{p^2 (k' \cdot p)}{|k - p'|^2 E_i'} = \frac{2\pi^4}{m_i} \ln \alpha^{-1}$$

$$\int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} \frac{(k' \cdot p)^2}{|k - p'|^2 E_i'} = \frac{2\pi^4}{m_i} \ln \alpha^{-1}$$

$$\int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} \frac{p'^2 |k \times p'|^2}{|k - p'|^4 E_i'} = \frac{2\pi^4}{m_i} \ln \alpha^{-1}$$

$$\begin{aligned}
 (\Delta E_c)_2 = & \frac{4\pi^2 i \alpha}{(2\pi)^4} \int \frac{d^3 p d^3 p'}{(p^2 + y^2)^2 (p'^2 + y^2)^2} \frac{\pi^2 \alpha^2 |\varphi(0)|^2 \mathcal{R}}{M^2 (2\pi)^3} \\
 & \times \left(\frac{m_1 m_2}{E_1 E_2'} + \frac{m_1^2}{E_1 E_1'} + (1 \leftrightarrow 2) \right) \left[\frac{2}{i|k|^2} - \frac{2}{i|k'|^2} \right]
 \end{aligned}
 \tag{4B.20}$$

which vanishes identically.

Hence the total contribution of the charge component of the one photon exchange is

$$\Delta E_c = \frac{2}{3} \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{\mu M}
 \tag{4B.21}$$

2. Linear Dipole Component

We now turn to the linear dipole component of (4B.2).

We have

$$\Delta E_c^d = -i \int d^4 x d^4 x' \overline{\varphi_{K^c}^{(11)}(x)} I^{(1)d}(x, x') \varphi_{K^c}^{(11)}(x')
 \tag{4B.22}$$

$$\text{where } I^{(1)d}(x, x') = -i 4\pi \alpha d(x-x') \int \frac{d^4 k e^{ikx}}{(2\pi)^4 k^2} \left(+ \frac{F_2(k^2)}{2m_1} \gamma_\mu^{(1)} T_{\mu\nu}^{(2)} k_\nu \right)
 \tag{4B.23}$$

Equation (4B.22) in momentum space is

$$\Delta E_c^d = -i \int \frac{d^4 p d^4 p'}{(2\pi)^4} \overline{\varphi^{(1)}(p)} I^{(1)d}(p, p') \varphi^{(1)}(p')
 \tag{4B.24}$$

$$\text{where } \varphi^{(1)}(p) = \int \frac{dt}{(2\pi)^4} e^{ip_0 t} \varphi^{(1)}(\underline{p}, t) ; \quad \varphi^{(1)}(\underline{p}, t) \text{ given by (3.10)}$$

and $k_\mu \equiv (p_\mu - p'_\mu)$

Thus

$$\Delta E_i^d = -4\pi\alpha \int \frac{d^4 p d^4 p'}{(2\pi)^4} \frac{d^4 t d^4 t'}{(2\pi)^4} \varphi^{(1)}(\underline{p}, t) e^{-ip_0 t} \left(+ \frac{F_2(k^2)}{2m_2} \right) \frac{\gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu}{k^2} \varphi^{(1)}(\underline{p}', t') e^{ip'_0 t'} \quad (4B.25)$$

We can do the p_0 and p'_0 integrations by first expanding

$$\gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu = \gamma_\mu^{(1)} \left[\alpha^{(1)} \underline{k} - \alpha^{(2)} \underline{k} + \alpha^{(1)} \alpha^{(2)} (\underline{k}_0 - \alpha^{(2)} \underline{k}) \right] \quad (4B.26)$$

and then invoking the following formulas (based on the outgoing wave prescription):

$$\int_{-\infty}^{\infty} dp_0 \frac{\kappa \Lambda^2 e^{-ip_0 t}}{(p_0^2 - |\underline{k}|^2)(p_0^2 - (|\underline{k}|^2 + \Lambda^2))} = \kappa\pi i \left(e^{-i|\underline{k}|t} - e^{-i(k^2 + \Lambda^2)^{1/2}|t|} \right) \epsilon(t) \quad (4B.27)$$

$$\int_{-\infty}^{\infty} dp_0 \frac{\kappa \Lambda^2 e^{-ip_0 t}}{(p_0^2 - |\underline{k}|^2)(p_0^2 - (|\underline{k}|^2 + \Lambda^2))} = +\kappa\pi i \left(\frac{e^{-i|\underline{k}|t}}{k} - \frac{e^{-i(k^2 + \Lambda^2)^{1/2}|t|}}{(k^2 + \Lambda^2)^{1/2}} \right) \quad (4B.28)$$

For $F_2(k^2)$ we have written $\kappa\Lambda^2/(\Lambda^2 + k^2)$ where κ is the proton anomalous magnetic moment⁽¹³⁾. This particular form factor is quite suitable for low momentum processes, which is what the $\alpha^6 \ln \alpha^{-1}$ contribution is. It will be seen that the exact functional form of $F_2(k^2)$ is unimportant, and indeed, the results to our order would be unchanged if we had used a point proton (i.e. let $\Lambda^2 \rightarrow \infty$). This is not surprising since again all $\alpha^6 \ln \alpha^{-1}$ contributions come from low momentum recoil effects.

Thus we can write (4B.36) as

$$\begin{aligned}
(\Delta E_1^d) = & 4\pi\alpha \int \frac{d^3p d^3p'}{(2\pi)^4} \frac{dt dt'}{2m_2} \delta(t-t') \overline{\varphi^{(1)}(\mathbf{p}, t)} \left\{ -\kappa\pi i \gamma_4^{(1)} \left[\underline{\alpha}^{(1)} \cdot \underline{k} - \underline{\alpha}^{(2)} \cdot \underline{k} - \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} \right] \right. \\
& \times \left(\frac{e^{-ik|t|}}{k} - \frac{e^{-i(k^2 + \Lambda^2)^{1/2}|t|}}{(k^2 + \Lambda^2)^{1/2}} \right) \\
& \left. - \kappa\pi i \gamma_4^{(1)} \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \left(e^{-ik|t|} - e^{-i(k^2 + \Lambda^2)^{1/2}|t|} \right) \epsilon(t) \right\} \varphi^{(1)}(\mathbf{p}', t)
\end{aligned}$$

(4B.29)

We now do the trivial t' integration and insert the wave function (3.10) recalling that $\overline{\varphi^{(1)}(\mathbf{p}, t)} = \varphi^{(1)\dagger}(\mathbf{p}, t) \gamma_4^{(1)} \gamma_4^{(2)}$

and $\epsilon^*(t) = -\epsilon(t)$. Again we keep only even powers of $\epsilon(t)$, as $\epsilon(t)$ is itself an odd function of time, and the rest of the time integrand is an even function of time. As was the case with the charge component we separate out the part from the remainder terms (involving $\delta\phi$ or $\delta\phi'$). First the $\overline{\varphi_c^{(1)}} K \varphi_c^{(1)}$ part:

$$\begin{aligned}
(\Delta E_i^d) = & \frac{2\pi\alpha}{m_2} \frac{-\kappa\pi i}{(2\pi)^4} \frac{\pi^2\alpha^2 |\varphi(0)|^2}{M^2 (2\pi)^3} \int \frac{d^3p d^3p'}{(p^2+\lambda^2)^2 (p'^2+\lambda^2)^2} (4\mu M)^2 \\
& \times \left\{ \left\langle \left(1 + \frac{\alpha^{(1)} \cdot p}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p}{2m_2}\right) \gamma_4^{(1)} \gamma_4^{(2)} \gamma_4^{(1)} \left[\underline{\alpha}^{(1)} \cdot \underline{k} - \underline{\alpha}^{(2)} \cdot \underline{k} - \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} \right] \right\rangle \right. \\
& \quad \times \left. \left(1 + \frac{\alpha^{(1)} \cdot p'}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p'}{2m_2}\right) \right\rangle \times \\
& \times \int_{-\infty}^{\infty} dt \left\{ \left[\left(\frac{m_1}{E_i} f_+^{(1)} + f_-^{(1)} \right) + (1 \leftrightarrow 2) \right] \left[\left(\frac{m_1}{E_i} f_+^{(1)} + f_-^{(1)} \right) + (1 \leftrightarrow 2) \right] \right. \\
& \quad \left. - \epsilon^2(t) \left[\left(\frac{m_1}{E_i} f_-^{(1)} + f_+^{(1)} \right) - (1 \leftrightarrow 2) \right] \left[\left(\frac{m_1}{E_i} f_-^{(1)} + f_+^{(1)} \right) - (1 \leftrightarrow 2) \right] \right\} \\
& \times \left[\frac{e^{-ik|t|}}{k} - \frac{e^{-i(k^2+\lambda^2)^{1/2}|t|}}{(k^2+\lambda^2)^{1/2}} \right] \quad (4B.30)
\end{aligned}$$

$$\begin{aligned}
& + \left\langle \left(1 + \frac{\alpha^{(1)} \cdot p}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p}{2m_2}\right) \gamma_4^{(1)} \gamma_4^{(2)} \gamma_4^{(1)} \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \left(1 + \frac{\alpha^{(1)} \cdot p'}{2m_1}\right) \left(1 - \frac{\alpha^{(2)} \cdot p'}{2m_2}\right) \right\rangle \\
& \times \int_{-\infty}^{\infty} dt \left\{ \left[\left(\frac{m_1}{E_i} f_+^{(1)} + f_-^{(1)} \right) + (1 \leftrightarrow 2) \right] \left[\left(\frac{m_1}{E_i} f_-^{(1)} + f_+^{(1)} \right) - (1 \leftrightarrow 2) \right] \epsilon^2(t) \right. \\
& \quad \left. - \left[\left(\frac{m_1}{E_i} f_-^{(1)} + f_+^{(1)} \right) - (1 \leftrightarrow 2) \right] \left[\left(\frac{m_1}{E_i} f_+^{(1)} + f_-^{(1)} \right) + (1 \leftrightarrow 2) \right] \epsilon^2(t) \right\} \\
& \times \left[e^{-ik|t|} - e^{-i(k^2+\lambda^2)^{1/2}|t|} \right] \Bigg\}
\end{aligned}$$

(4B.30)

The spin average for the first term can be written (see Table II)

$$(8m_1\mu M)^{-1}(\mathcal{J}-\mathcal{I}) - (8m_2\mu M)^{-1}(\mathcal{U}+\mathcal{V}) - (2m_1)^{-1}(\chi+\mathcal{W}) - 8(m_2\mu M)^{-1}(\mathcal{Y}+\mathcal{Z}) \quad (4B.31)$$

while the spin averages for the second term can be written

$$(4\mu M)^{-1}(\Omega - \Phi) \quad (4B.32)$$

The momentum integrations in (4B.30) thus become

$$\int \frac{d^3p d^3p'}{(p^2+y^2)^2 (p'^2+y^2)^2} \left[2m_2(\mathcal{J}-\mathcal{I}) + 2m_1(\mathcal{Y}+\mathcal{Z}) - 8m_2\mu M(\chi+\mathcal{W}) \right] \quad (4B.33)$$

$$\text{and } \int \frac{d^3p d^3p'}{(p^2+y^2)^2 (p'^2+y^2)^2} \left[4\mu M(\Omega - \Phi) \right] \quad (4B.34)$$

Next we turn to the time integrations. The first time integration in (4B.30) has been done previously (see (4B.12)) and can be written as

$$\begin{aligned} -2i \left\{ \left\{ \frac{m_1 m_2}{E_1 E_2' k^2} + \frac{m_1^2}{E_1 E_1' k^2} + \frac{p^2 E_1' + p'^2 E_1}{E_1 E_1' k^2 (k + E_1 + E_1')} - \frac{p^2 (E_2' + m_2) + p'^2 (E_1 + m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' - M)} \right. \right. \\ \left. \left. - \frac{p^2 (E_2' - m_2) + p'^2 (E_1 - m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' + M)} + (1 \leftrightarrow 2) \right\} - (k \leftrightarrow (k^2 + \Lambda^2)^{1/2}) \right\} \end{aligned} \quad (4B.35)$$

The second time integration in (4B.30) is

$$\begin{aligned} -2i \left\{ \left\{ \frac{m_1}{k + E_1 + E_1'} \left(\frac{1}{E_1} - \frac{1}{E_1'} \right) - \frac{(E_1 + m_1)(E_2' + m_2)}{2 E_1 E_2' (k + E_1 + E_2' - M)} + \frac{(E_1 - m_1)(E_2' - m_2)}{2 E_1 E_2' (k + E_1 + E_2' + M)} \right. \right. \\ \left. \left. - (1 \leftrightarrow 2) \right\} - (k \leftrightarrow (k^2 + \Lambda^2)^{1/2}) \right\} \end{aligned} \quad (4B.36)$$

Combining (4B.33) and (4B.35) and taking the spin average terms one at a time we have for the (S-T) term,

$$\begin{aligned}
 & \frac{8m_2 i}{3} \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} |\mathbf{p} \times \mathbf{p}'|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \left\{ \left\{ \frac{m_1 m_2}{\underline{k^2 E_1 E_2'}} + \frac{m_1^2}{\underline{k^2 E_1 E_1'}} \right. \right. \\
 & + \frac{p^2 E_1' + p'^2 E_1}{k^2 E_1 E_1' (k + E_1 + E_1')} - \frac{p^2 (E_2' + m_2) + p'^2 (E_1 + m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' - M)} \\
 & + \left. \frac{p^2 (E_2' - m_2) + p'^2 (E_1 - m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' + M)} + (1 \leftrightarrow 2) \right\} - (k \leftrightarrow (k^2 + \Lambda^2)^{1/2}) \Big\} \\
 & \hspace{15em} (4B.37)
 \end{aligned}$$

Only the underlined terms are of the proper dimension to give a logarithmic contribution. Hence the relevant integral in (4B.37) is:

$$\begin{aligned}
 & \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \left\{ \frac{m_1 m_2}{E_1 E_2' |\mathbf{p} - \mathbf{p}'|^2} + \frac{m_1^2}{E_1 E_1' |\mathbf{p} - \mathbf{p}'|^2} + (1 \leftrightarrow 2) \right. \\
 & \left. - \frac{m_1 m_2}{E_1 E_2' (|\mathbf{p} - \mathbf{p}'|^2 + \Lambda^2)} - \frac{m_1^2}{E_1 E_1' (|\mathbf{p} - \mathbf{p}'|^2 + \Lambda^2)} - (1 \leftrightarrow 2) \right\} \\
 & \hspace{15em} (4B.38)
 \end{aligned}$$

Since Λ is of the order of m_2 , the Λ dependent terms will not contribute to order $\alpha^6 \ln \alpha^{-1}$ since they are not singular enough to give a logarithmic contribution. The remaining terms can be integrated with the help of Table III and we obtain for the $\mathcal{J}$ - $\mathcal{J}$ term:

$$\frac{4\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{3 M^2} \hspace{10em} (4B.39)$$

The next term of (4B.33) and (4B.35) is the $\gamma + z$ term:

$$\begin{aligned}
 & - \frac{8m_1 i}{3} \int \frac{d^3 p \, d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} (p^2 p'^2 + (p \cdot p')^2 - (p^2 + p'^2)(p' \cdot p)) \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \\
 & \times \left\{ \left\{ \frac{m_1 m_2}{E_1 E_2' k^2} + \frac{m_1^2}{E_1 E_1' k^2} + \frac{p^2 E_1' + p'^2 E_1}{E_1 E_1' k^2 (k + E_1 + E_1')} - \frac{p^2 (E_2' + m_2) + p'^2 (E_1 + m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' - M)} \right. \right. \\
 & \quad \left. \left. - \frac{p^2 (E_2' - m_2) + p'^2 (E_1 - m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' + M)} + (1 \leftrightarrow 2) \right\} - (k \leftrightarrow (k^2 + \Lambda^2)^{1/2}) \right\} \\
 & \hspace{15em} (4B.40)
 \end{aligned}$$

Again the terms singular enough to contribute to order $\alpha^6 \ln \alpha^{-1}$ are underlined, and as before the Λ dependent terms will not contribute as they are not singular enough. The relevant integral is

$$\int \frac{d^3 p \, d^3 p' [p^2 p'^2 + (p \cdot p')^2 - (p'^2 + p^2)(p' \cdot p)]}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2 |p - p'|^2} \left(\frac{m_1 m_2}{E_1 E_2'} + \frac{m_1^2}{E_1 E_1'} + (1 \leftrightarrow 2) \right) \quad (4B.41)$$

Using Table III we obtain for the $\gamma + z$ term:

$$-K \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{3 M^2} \left(\frac{m_1}{m_2} \right) \quad (4B.42)$$

Finally the $\chi + w$ term in (4B.33) and (4B.35):

$$\begin{aligned}
& \frac{32 m_2 \mu M i}{3} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} (p - p')^2 \left\{ \frac{m_1 m_2}{E_1 E_2' k^2} + \frac{m_1^2}{E_1 E_1' k^2} \right. \\
& + \frac{p^2 E_1' + p'^2 E_1}{E_1 E_1' k^2 (k + E_1 + E_1')} - \frac{p^2 (E_2' + m_2) + p'^2 (E_1 + m_1)}{\underline{2 E_1 E_2' k^2 (k + E_1 + E_2' - M)}} \quad (4B.43) \\
& \left. - \frac{p^2 (E_2' - m_2) + p'^2 (E_1 - m_1)}{2 E_1 E_2' k^2 (k + E_1 + E_2' + M)} + (1 \leftrightarrow 2) \right\} - (k \leftrightarrow (k^2 + \Lambda^2)^{1/2}) \Big\}
\end{aligned}$$

(4B.43)

The only term singular enough to contribute to our order is the underlined term in (4B.43). Again the Λ dependent terms do not contribute to order $\alpha^6 \ln \alpha^{-1}$. The remaining integral is

$$\int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \left\{ \frac{p^2 (E_2' + m_2) + p'^2 (E_1 + m_1)}{2 E_1 E_2' (k + E_1 + E_2' - M)} + (1 \leftrightarrow 2) \right\} \quad (4B.44)$$

which is identical with the one leading up to (4B.16) so that we obtain for the $\chi + \omega$ term:

$$\frac{2}{3} \frac{\kappa \pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{\mu M} \quad (4B.45)$$

In order to evaluate the remaining linear dipole contribution, namely the second term in (4B.30) we combine (4B.34) with (4B.36) to obtain:

$$\begin{aligned}
& - \frac{16 i \mu M \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle}{3} \int \frac{d^3 p d^3 p'}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2} (p'^2 - p^2) \left\{ \left\{ \frac{m_1}{k + E_1 + E_1'} \left(\frac{1}{E_1} - \frac{1}{E_1'} \right) \right. \right. \\
& \left. \left. - \frac{(E_1 + m_1)(E_2' + m_2)}{2 E_1 E_2' (k + E_1 + E_2' - M)} + \frac{(E_1 - m_1)(E_2' - m_2)}{2 E_1 E_2' (k + E_1 + E_2' + M)} - (1 \leftrightarrow 2) \right\} - \left(k \leftrightarrow (k^2 + \Lambda^2)^{1/2} \right) \right\} \\
& \hspace{15em} (4B.46)
\end{aligned}$$

Only the underlined terms in (4B.46) are singular enough to contribute to our order. The first term can be written:

$$\begin{aligned}
& m_1 \left(\frac{E_1' - E_1}{E_1' E_1} \right) \frac{1}{k + E_1 + E_1'} = m_1 \left(\frac{E_1' - E_1}{E_1' E_1} \right) \frac{1}{k} \left(1 - \frac{E_1' + E_1}{k + E_1 + E_1'} \right) \\
& = m_1 \frac{E_1' - E_1}{k E_1' E_1} - m_1 \frac{p'^2 - p^2}{k (k + E_1 + E_1')} \hspace{10em} (4B.47) \\
& \simeq \frac{p'^2 - p^2}{2 k E_1 E_1'} - \frac{m_1 (p'^2 - p^2)}{k (E_1 + E_1')}
\end{aligned}$$

Both terms are of dimension [momentum]⁺¹ so they will not give a singular enough integrand in (4B.46) to yield a logarithmic contribution. The Λ dependent term in the form of (4B.47) is even less singular. The second term in (4B.46) can be written:

$$- \frac{(E_2' + m_2)(E_1 + m_1)}{2E_1 E_2' (k + E_1 + E_2' - M)} = - \frac{(E_2' + m_2)(E_1 + m_1)}{2k E_1 E_2'} + \frac{p'^2(E_1 + m_1) + p^2(E_2' + m_2)}{2k E_1 E_2' (k + E_1 + E_2' - M)} \quad (4B.48)$$

$$\approx - \frac{(E_1 + m_1)(E_2' + m_2)}{2k E_1 E_2'} + \frac{\underline{p'^2 2m_1}}{2k^2 E_1 E_2'} + \frac{\underline{p^2 2m_2}}{2k^2 E_1 E_2'}$$

In (4B.48) the underlined terms represent the potential logarithmic contribution from the Λ independent terms, while the only Λ dependent term that could possibly contribute is

$$\int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \left\{ - \frac{(E_1 + m_1)(E_2' + m_2)}{2E_1 E_2' (k^2 + \Lambda^2)^{1/2}} - (1 \leftrightarrow 2) \right\} \quad (4B.49)$$

Again using the fact that $\Lambda \geq m_2$, and keeping only the leading terms in the numerator, we could write (4B.49) as:

$$\int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2 \Lambda} \left\{ - \frac{p'^2}{E_1 E_2'} + \frac{p'^2}{E_1' E_2} + \frac{p^2}{E_1 E_2'} - \frac{p^2}{E_1' E_2} \right\} \quad (4B.50)$$

Letting $p \leftrightarrow p'$ in the last two terms we obtain:

$$4m_1 m_2 \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2 \Lambda} \left\{ \frac{p'^2}{E_1' E_2} - \frac{p^2}{E_1 E_2'} \right\} \quad (4B.51)$$

the leading term of which is

$$4m_1 m_2 \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2 \Lambda} \left\{ p'^2 \left(\frac{(m_2 - m_1)(p^2 - p'^2)}{2E_1 E_1' E_2 E_2'} \right) \right\} \quad (4B.52)$$

which has an integrand of dimension [momentum]⁰ at low momentum and hence won't yield a logarithmic contribution.

Returning to the Λ independent terms of (4B.48) that are underlined we find the potential logarithmic contribution of (4B.46) to be:

$$\frac{-16i\mu M \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle}{3} \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \left\{ \left(-\frac{m_1 p'^2}{E_1 E_2' k^2} - \frac{m_2 p^2}{E_1 E_2' k^2} \right) - (1 \leftrightarrow 2) \right\} \quad (4B.53)$$

which can be written:

$$C_1 \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \left\{ m_2 \left(\frac{p'^2}{E_1' E_2} - \frac{p^2}{E_1 E_2'} \right) - m_1 \left(\frac{p'^2}{E_1 E_2'} - \frac{p^2}{E_1' E_2} \right) \right\} \quad (4B.54)$$

The leading behavior of the expression in braces is

$$\left\{ m_2 \left(\frac{m_1 m_2 [p'^2 - p^2 + \frac{p'^4 - p^4}{2m_1} + \dots]}{E_1 E_2 E_1' E_2'} \right) \right. \quad (4B.55)$$

$$\left. - m_1 \left(\frac{m_1 m_2 [p'^2 - p^2 + \frac{p'^4 - p^4}{2m_2} + \dots]}{E_1 E_2 E_1' E_2'} \right) \right\}$$

The only terms in (4B.55) yielding an integrand of proper dimension (at small momentum) are the lowest powers of momentum. Thus (4B.54) is to our order:

$$C_1 \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \left(\frac{(m_2 - m_1)(p'^2 - p^2)}{E_1 E_2 E_1' E_2'} \right) \quad (4B.56)$$

$$= C_1 m_1 m_2 (m_2 - m_1) \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \frac{(p' + p)^2}{E_1 E_2 E_1' E_2'}$$

Using the symmetry of p' and p the integral in (4B.56) can be written:

$$2 \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \frac{p'^2}{E_1 E_2 E_1' E_2'} \quad (4B.57)$$

and (4B.57) contains no $\ln \alpha^{-1}$ dependence whatsoever. This can be seen by writing:

$$\int \frac{d^3 p}{(p^2 + \gamma^2)^2 E_1 E_2} \left\{ \int \frac{d^3 p'}{(p'^2 + \gamma^2)^2 E_1 E_2'} \right\}$$

The p' integration yields a number independent of α , while the p integration can be approximated as $m_2^{-1} \int d^3 p (p^2 + \gamma^2)^{-2} E_1^{-1}$

$$= \frac{1}{m_2} \left\{ \frac{2\pi}{\gamma (m_1^2 - \gamma^2)^{1/2}} \tan^{-1} \left[\frac{(m_1^2 - \gamma^2)^{1/2}}{\gamma} \right] - \frac{2\pi}{m_1^2 - \gamma^2} + \frac{2\pi \gamma}{(m_1^2 - \gamma^2)^{3/2}} \tan^{-1} \left[\frac{(m_1^2 - \gamma^2)^{1/2}}{\gamma} \right] \right\}$$

which has no logarithmic dependence whatsoever.

Thus we can finally write (4B.30) as equal to (see (4B.39), (4B.42) and (4B.45)):

$$\Delta E_1^d = K \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left(\frac{2}{3} \frac{M}{\mu} + \frac{1}{3} - \frac{1}{3} \frac{m_1}{m_2} \right) \quad (4B.58)$$

Thus the total $\alpha^6 \ln \alpha^{-1}$ contribution from the exchange of a single photon is (see (4B.21 and (4B.58))

$$\Delta E_1 = \eta \left(\frac{2}{3} \frac{M}{\mu} \right) + K \eta \left(\frac{2}{3} \frac{M}{\mu} + \frac{1}{3} - \frac{1}{3} \frac{m_1}{m_2} \right) \quad (4B.59)$$

$$\text{where } \eta \equiv \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2}$$

C. The Ladder Diagram

The "ladder diagram" contribution to the energy shift comes from the second order perturbation formula of the one photon exchange. Specifically we have⁽¹²⁾:

$$\Delta E_L = -2i \int d^4x d^4x' \varphi_p(0) I_K(x, x') \varphi_{K^0}^{(u)}(x') \quad (4C.1)$$

where $I_K(x, x')$ is derived in Appendix A and is:

$$I_K(x, x') = -\frac{(4\pi\alpha)^4}{(2\pi)^8} \int \frac{d^4k d^4k'}{k^2 k'^2} e^{ikx} e^{ik'x'} \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha^{(1)} \cdot \alpha^{(2)} \right) - \frac{F_2(k^2) \gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu}{2m_2} \right\} \\ \times G_K(x-x') \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha^{(1)} \cdot \alpha^{(2)} \right) - \frac{F_2(k'^2) \gamma_\lambda^{(1)} \sigma_{\lambda\rho}^{(2)} k'_\rho}{2m_2} \right\} \quad (4C.2)$$

$$\text{where } G_K(x-x') = \int \frac{d^4p'}{(2\pi)^4} e^{ip'(x-x')} S_F^{(u)}(p' + \eta, K) S_F^{(u)}(-p' + \eta, K)$$

Inserting (4C.2) into (4C.1), the integration proceeds as

follows: do the x integration obtaining $(2\pi)^4 \delta(-k-p')$; do the p' integration using the δ function; do the x' integration obtaining $(2\pi)^4 \delta(-k'-k-p) \delta(k'_0 + k_0)$. Then change variables: $k' \rightarrow -k'$ and we obtain:

The charge component of ΔE_L :

$$\Delta E_L^c = \frac{2i(4\pi\alpha)^2}{(2\pi)^4} \int \frac{d^3p d^4k d^4k'}{k^2 k'^2} \frac{\delta(k' - k - p) \delta(k_0 - k_0')}{\varphi_p(0)} \\ \times \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha_i^{(1)} \alpha_i^{(2)} \right) S_F^{(1)}(-k + \eta_1, k) S_F^{(2)}(k + \eta_2, k) \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0'^2}{|k'|^2} - \alpha_j^{(1)} \alpha_j^{(2)} \right) \right\} \frac{\varphi_{k^c}^{(1)}(p)}{(2\pi)^{3/2}} \quad (4C.3)$$

the linear dipole component of ΔE_L :

$$\Delta E_L^d = \frac{2i(4\pi\alpha)^2}{(2\pi)^4} \int \frac{d^3p d^4k d^4k'}{k^2 k'^2} \frac{\delta(k' - k - p) \delta(k_0 - k_0')}{\varphi_p(0)} \\ \times \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha_i^{(1)} \alpha_i^{(2)} \right) S_F^{(1)}(-k + \eta_1, k) S_F^{(2)}(k + \eta_2, k) \frac{F_2(k^{(2)}) \gamma_\lambda^{(1)} \sigma_{\lambda\rho}^{(2)} k'_\rho}{2m_2} \frac{\varphi_{k^c}^{(1)}(p)}{(2\pi)^{3/2}} \right. \\ \left. - \frac{F_2(k^{(1)}) \gamma_{\mu\nu}^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu}{2m_2} S_F^{(1)}(k + \eta_1, k) S_F^{(2)}(k + \eta_2, k) \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0'^2}{|k'|^2} - \alpha_j^{(1)} \alpha_j^{(2)} \right) \frac{\varphi_{k^c}^{(1)}(p)}{(2\pi)^{3/2}} \right\} \quad (4C.4)$$

the quadratic dipole component of ΔE_L :

$$\Delta E_L^{dd} = \frac{2i(4\pi\alpha)^2}{(2\pi)^4} \int \frac{d^3p d^4k d^4k'}{k^2 k'^2} \frac{\delta(k' - k - p) \delta(k_0 - k'_0)}{\varphi_p^{(0)}} \times$$

$$\left\{ \frac{-F_2(k^2) \gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu}{2m_2} S_F^{(1)}(-k + \eta_1 k) S_F^{(2)}(k + \eta_2 k) \frac{F_2(k'^2) \gamma_\lambda^{(1)} \sigma_{\lambda\rho}^{(2)} k'_\rho}{2m_2} \right\} \frac{\varphi_{k'}^{(1)}(p)}{(2\pi)^{3/2}}$$

(4C.5)

1. Charge Component

We turn our attention first to the charge component given by (4C.3). The k'_0 integration is trivial using $\delta(k_0 - k'_0)$. Inserting the first iterated static wave function (3.11) we have:

$$\Delta E_L^c = \frac{4i\pi^3 \alpha^3 |\varphi^{(0)}|^2}{\pi^3 M (2\pi)^4} \int \frac{d^3p d^4k d^4k'}{k^2 k'^2 (p^2 + \gamma^2)^2} \times$$

$$\left\langle \left(\frac{k_0^2}{|k|^2} - \alpha_i^{(1)} \alpha_i^{(1)} \right) S_F^{(1)}(-k + \eta_1 k) S_F^{(2)}(k + \eta_2 k) \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha_j^{(1)} \alpha_j^{(1)} \right) \right.$$

$$\left. \times \left[D(p) + F(p) \alpha^{(1)} \cdot p - \bar{F}(p) \alpha^{(2)} \cdot p - G(p) \alpha^{(1)} \cdot p \alpha^{(2)} \cdot p \right] \right\rangle$$

(4C.6)

The product of propagators can be expanded as a series of projection operators times $\gamma_4^1 \gamma_4^2$

$S_F^{(1)} S_F^{(2)} = [] \gamma_4^1 \gamma_4^2$ (see (3.4)). Thus the expression in the angular brackets can be written as the sum of four terms:

$$\begin{aligned}
\left\langle \frac{k_0^4}{|k|^2 |k'|^2} [] \{w.f.\} - \frac{k_0^2}{|k|^2} [] \alpha_j^{(1)} \alpha_j^{(2)} \{w.f.\} - \frac{k_0^2}{|k'|^2} \alpha_i^{(1)} \alpha_i^{(2)} [] \{w.f.\} \right. \\
\left. + \alpha_i^{(1)} \alpha_i^{(2)} [] \alpha_j^{(1)} \alpha_j^{(2)} \{w.f.\} \right\rangle \quad (4C.7)
\end{aligned}$$

where $[]$ is the product of projection operators associated with $S_F^{(1)} S_F^{(2)}$ (see (3.4)) and $\{w.f.\}$ is

$$[D(p) + F(p) \underline{\alpha}^{(1)} \cdot p - \bar{F}(p) \underline{\alpha}^{(2)} \cdot p - G(p) \underline{\alpha}^{(1)} \cdot p \underline{\alpha}^{(2)} \cdot p]$$

When we insert the product of projection operators, (3.4) into (4C.7) and anti-commute the $\beta^{(i)}$ ($i=1,2$) toward the left we will get different results depending on the α matrices multiplying $[]$ on the left. Specifically we can write:

For $\alpha^{(1)} \alpha^{(2)}$ on the left:

$$\begin{aligned}
[] = \frac{1}{4E_1(k)E_2(k)} \left\{ \frac{(E_1 - m_1)(E_2 - m_2) - (E_2 - m_2) \underline{\alpha}^{(1)} \cdot k + (E_1 - m_1) \underline{\alpha}^{(2)} \cdot k - \underline{\alpha}^{(1)} \cdot k \underline{\alpha}^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 - E_1)(k_0 + \eta_2 k_0 - E_2)} \right. \\
+ \frac{(E_1 - m_1)(E_2 + m_2) - (E_2 + m_2) \underline{\alpha}^{(1)} \cdot k - (E_1 - m_1) \underline{\alpha}^{(2)} \cdot k + \underline{\alpha}^{(1)} \cdot k \underline{\alpha}^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 - E_1)(k_0 + \eta_2 k_0 + E_2)} \\
+ \frac{(E_1 + m_1)(E_2 - m_2) + (E_2 - m_2) \underline{\alpha}^{(1)} \cdot k + (E_1 - m_1) \underline{\alpha}^{(2)} \cdot k + \underline{\alpha}^{(1)} \cdot k \underline{\alpha}^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 + E_1)(k_0 + \eta_2 k_0 - E_2)} \\
\left. + \frac{(E_1 + m_1)(E_2 + m_2) + (E_2 + m_2) \underline{\alpha}^{(1)} \cdot k - (E_1 + m_1) \underline{\alpha}^{(2)} \cdot k - \underline{\alpha}^{(1)} \cdot k \underline{\alpha}^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 + E_1)(k_0 + \eta_2 k_0 - E_2)} \right\} \quad (4C.8)
\end{aligned}$$

and for no α matrices on the left:

$$\begin{aligned}
 [] = & \frac{1}{4E_1(k)E_2(k)} \left\{ \frac{(E_1+m_1)(E_2+m_2) - (E_2+m_2)\alpha^{(1)} \cdot k + (E_1+m_1)\alpha^{(2)} \cdot k - \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 - E_1)(k_0 + \eta_2 k_0 - E_2)} \right. \\
 & + \frac{(E_1+m_1)(E_2-m_2) - (E_2-m_2)\alpha^{(1)} \cdot k - (E_1+m_1)\alpha^{(2)} \cdot k + \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 - E_1)(k_0 + \eta_2 k_0 + E_2)} \\
 & + \frac{(E_1-m_1)(E_2+m_2) + (E_2+m_2)\alpha^{(1)} \cdot k + (E_1-m_1)\alpha^{(2)} \cdot k + \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 + E_1)(k_0 + \eta_2 k_0 - E_2)} \\
 & \left. + \frac{(E_1-m_1)(E_2-m_2) + (E_2-m_2)\alpha^{(1)} \cdot k - (E_1-m_1)\alpha^{(2)} \cdot k - \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k}{(-k_0 + \eta_1 k_0 + E_1)(k_0 + \eta_2 k_0 + E_2)} \right\}
 \end{aligned}
 \tag{4C.9}$$

The spin-spin interaction portion of ΔE_L^C is extracted in the following manner: Repeated use of the relation $\alpha_i \alpha_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k$ is made to reduce a product of α matrices to a σ matrix. Then use is made of the relation

$$\left\langle \sigma(1) \cdot A \sigma(2) \cdot B \right\rangle \rightarrow \frac{1}{3} A \cdot B \left\langle \sigma(1) \cdot \sigma(2) \right\rangle$$

(see Appendix B) to separate the momentum vectors from the spin variables. It is important to note that in order to yield a spin-spin contribution, and given product of α matrices must have an even number of α matrices for particle (1) and an even number of α matrices for particle (2). An odd number of α matrices will not result in a $\left\langle \sigma(1) \cdot \sigma(2) \right\rangle$ term. An example of this method follows:

Table IV. Spin averages for two photon ladder diagram.

Spin average	Coefficient of $\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle$
$A \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(1')} \cdot \underline{\alpha}^{(2')} \rangle$	-2
$B \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2')} \underline{\alpha}^{(1')} \cdot p \rangle$	$\frac{2}{3} \underline{k} \cdot p$
$C \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \underline{\alpha}^{(1')} \cdot \alpha^{(2')} \underline{\alpha}^{(1')} \cdot p \underline{\alpha}^{(2')} \cdot p \rangle$	$-\frac{6}{3} k^2 p^2 - \frac{1}{3} \underline{k} \times p ^2$
$D \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot \underline{k}' \rangle$	$+\frac{2}{3} \underline{k}' \cdot \underline{k}$
$F \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \alpha^{(1')} \cdot \alpha^{(2')} \underline{\alpha}^{(2')} \cdot \underline{k}' \underline{\alpha}^{(2')} \cdot p \rangle$	$O(k^2)$
$G \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2')} \rangle$	$-\frac{6}{3} k^2$
$L \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot \underline{k}' \underline{\alpha}^{(2')} \cdot p \underline{\alpha}^{(2')} \cdot p \rangle$	$O(k^4)$
$M \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot \underline{k}' \underline{\alpha}^{(2')} \cdot p \rangle$	$-\frac{6}{3} k^2 (\underline{k}' \cdot p) - \frac{1}{3} \underline{k} \times p ^2$
$N \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot p \rangle$	$\frac{2}{3} k^2 (\underline{k} \cdot p)$
$O \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot \underline{k}' \underline{\alpha}^{(2')} \cdot p \rangle$	$O(k^4)$
$P \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot \underline{k}' \underline{\alpha}^{(2')} \cdot p \underline{\alpha}^{(2')} \cdot p \rangle$	$O(k^6)$
$Q \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot p \underline{\alpha}^{(2')} \cdot p \rangle$	$-\frac{2}{3} p^2$
$B \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \underline{\alpha}^{(2')} \cdot p \rangle = (1 \leftrightarrow 2)$	$\frac{2}{3} \underline{k} \cdot p$
$P \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \rangle$	$-\frac{2}{3} k^2$
$D \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot p \rangle = (1 \leftrightarrow 2)$	$-\frac{2}{3} \underline{k} \cdot p$
$D' \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k}' \underline{\alpha}^{(2')} \cdot p \rangle = (1 \leftrightarrow 2)$	$-\frac{2}{3} \underline{k}' \cdot p$
$E \equiv \langle \underline{\alpha}^{(1)} \cdot \underline{k} \alpha^{(1')} \cdot \alpha^{(2)} \alpha^{(2')} \cdot \underline{k}' \rangle$	$+\frac{2}{3} \underline{k}' \cdot \underline{k}$
$F \equiv \langle \alpha^{(1)} \cdot \alpha^{(2)} \underline{\alpha}^{(1')} \cdot \underline{k} \underline{\alpha}^{(2')} \cdot \underline{k} \rangle$	$-\frac{2}{3} k^2$



Table IV (continued)

Spin average	Coefficient of $\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle$
$\mathcal{F}' \equiv \langle \alpha^{(1)} \alpha^{(2)} \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k}' \rangle = (1 \leftrightarrow 2)$	$-\frac{2}{3} \underline{k}' \cdot \underline{k}$
$\mathcal{G} \equiv \langle \alpha^{(1)} \alpha^{(2)} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{p} \alpha^{(2)} \underline{p} \rangle$	$-\frac{2}{3} k^2 p^2$
$\mathcal{G}' \equiv \langle \alpha^{(1)} \alpha^{(2)} \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{k}' \alpha^{(1)} \underline{p} \alpha^{(2)} \underline{p} \rangle = (1 \leftrightarrow 2)$	$-\frac{2}{3} p^2 (\underline{k}' \cdot \underline{k})$
$\mathcal{H} \equiv \langle \alpha^{(1)} \alpha^{(2)} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{p} \rangle$	$-\frac{2}{3} k^2 (\underline{k} \cdot \underline{p})$
$\mathcal{H}' \equiv \langle \alpha^{(1)} \alpha^{(2)} \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{k}' \alpha^{(1)} \underline{p} \rangle = (1 \leftrightarrow 2)$	$-\frac{2}{3} k^2 (\underline{k}' \cdot \underline{p})$
$\mathcal{J} \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(1)} \alpha^{(2)} \alpha^{(1)} \underline{p} \rangle = (1 \leftrightarrow 2)$	$+\frac{2}{3} k^2 (\underline{k} \cdot \underline{p})$
$\mathcal{O} \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(1)} \alpha^{(2)} \alpha^{(1)} \underline{p} \alpha^{(2)} \underline{p} \rangle = (1 \leftrightarrow 2)$	$-\frac{2}{3} k^2 p^2$
$\mathcal{P} \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{p} \alpha^{(2)} \underline{p} \rangle$	$-\frac{1}{3} \underline{k} \times \underline{p} ^2$
$\mathcal{P}' \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k}' \alpha^{(1)} \underline{p} \alpha^{(2)} \underline{p} \rangle = (1 \leftrightarrow 2)$	$-\frac{1}{3} \underline{k} \times \underline{p} ^2$
$\mathcal{Q} \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{p} \rangle = (1 \leftrightarrow 2)$	0
$\mathcal{Q}' \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{k}' \alpha^{(2)} \underline{p} \rangle = (1 \leftrightarrow 2)$	$-\frac{1}{3} \underline{k} \times \underline{p} ^2$
$\mathcal{W} \equiv \langle \alpha^{(2)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{k}' \alpha^{(1)} \underline{p} \rangle$	0
$\mathcal{V} \equiv \langle \alpha^{(1)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(2)} \underline{k} \alpha^{(1)} \underline{k}' \rangle$	0



$$\begin{aligned}
\langle \underline{\alpha}^{(1)} \cdot \underline{A} \quad \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \quad \underline{\alpha}^{(2)} \cdot \underline{B} \rangle &= \langle A_\ell B_m \alpha_\ell^{(1)} \alpha_i^{(1)} \alpha_i^{(2)} \alpha_m^{(2)} \rangle \\
&= \langle A_\ell B_m (\delta_{\ell i} + i \epsilon_{\ell i n} \sigma_n^{(1)}) (\delta_{i m} + i \epsilon_{i m s} \sigma_s^{(2)}) \rangle \\
&\rightarrow i \langle A_\ell B_m \epsilon_{\ell i n} \epsilon_{i m s} \sigma_n^{(1)} \sigma_s^{(2)} \rangle = \langle A_\ell B_m (\delta_{\ell m} \delta_{n s} - \delta_{\ell s} \delta_{n m}) \sigma_n^{(1)} \sigma_s^{(2)} \rangle \\
&= \underline{A} \cdot \underline{B} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle - \frac{1}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle = \frac{2}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle
\end{aligned}$$

Tabulation of all spin averages for the ladder diagram occurs in Table IV.

We now extract the k_0 integration from (4C.6) and apply it to (4C.7):

$$\begin{aligned}
\int_{-\infty}^{\infty} dk_0 \left\langle \frac{k_0^4}{|\underline{k}|^2 |\underline{k}'|^2} [] \{w.f.\} - \frac{k_0^2}{|\underline{k}|^2} [] \alpha_j^{(1)} \alpha_j^{(2)} \{w.f.\} - \frac{k_0^2}{|\underline{k}'|^2} \alpha_i^{(1)} \alpha_i^{(2)} [] \{w.f.\} \right. \\
\left. + \alpha_i^{(1)} \alpha_i^{(2)} [] \alpha_j^{(1)} \alpha_j^{(2)} \{w.f.\} \right\rangle \quad (4C.7a)
\end{aligned}$$

Since both $[]$ and $\{w.f.\}$ contain terms independent of α , dependent on $\alpha^{(1)}$, or on $\alpha^{(2)}$, or on $\alpha^{(1)} \alpha^{(2)}$, one can extract the spin-spin contribution (remembering to keep an even number of α matrices for each particle) by inspection.

$k_0^4 |\underline{k}|^{-2} |\underline{k}'|^{-2} [] \{w.f.\}$ term:

Clearly the only spin average to contribute to the spin-spin interaction is $\langle \underline{\alpha}^{(1)} \cdot \underline{k} \quad \underline{\alpha}^{(2)} \cdot \underline{k} \quad \underline{\alpha}^{(1)} \cdot \underline{p} \quad \underline{\alpha}^{(2)} \cdot \underline{p} \rangle \equiv \mathcal{V}$



Thus the above term contributes

$$\frac{-1}{4E_1(k)E_2(k)} \left\{ I_{++}^{(IV)} + I_{+-}^{(IV)} + I_{-+}^{(IV)} + I_{--}^{(IV)} \right\} \frac{\gamma}{|k|^2 |k'|^2} \quad (4C.10)$$

$$\text{where } I_{\pm\pm}^{(IV)} = \int_0^\infty dk_0 \frac{k_0^4}{(k_0^2 - |k|^2)(k_0^2 - |k'|^2)} \frac{-1}{[k_0 - (\eta_1 k_0 \mp E_1 \pm i\epsilon)][k_0 - (\eta_2 k_0 \pm E_2 \pm i\epsilon)]} \quad (4C.11)$$

All such k_0 integrations for the ladder diagram are tabulated in Table V.

In like manner the $-k_0^2 |k|^{-2} [\alpha_i''' \alpha_j^{(2)} \{ \omega, f. \}]$ term will contribute

$$\begin{aligned} \frac{-|k|^{-2}}{4E_1 E_2} \left\{ \begin{aligned} & \underbrace{-(E_1 + m_1)(E_2 + m_2)G(p)A}_{\text{wavy}} + \underbrace{(E_2 + m_2)\bar{F}(p)B}_{\text{wavy}} + \underbrace{(E_1 + m_1)F(p)B}_{\text{wavy}} - \underbrace{D(p)C}_{\text{wavy}} \end{aligned} \right\} I_{++}'' \\ + \left\{ \begin{aligned} & -(E_1 + m_1)(E_2 - m_2)G(p)A + (E_2 - m_2)\bar{F}(p)B - (E_1 + m_1)F(p)B + D(p)C \end{aligned} \right\} I_{+-}'' \\ + \left\{ \begin{aligned} & -(E_1 - m_1)(E_2 + m_2)G(p)A - (E_2 + m_2)\bar{F}(p)B + (E_1 - m_1)F(p)B + D(p)C \end{aligned} \right\} I_{-+}'' \\ + \left\{ \begin{aligned} & -(E_1 - m_1)(E_2 - m_2)G(p)A - (E_2 - m_2)\bar{F}(p)B - (E_1 - m_1)F(p)B - D(p)C \end{aligned} \right\} I_{--}'' \end{aligned} \quad (4C.12)$$

Similarly the $-k_0^2 |k'|^{-2} \alpha_i''' \alpha_j^{(2)} [\omega, f.]$ term will contribute:

$$\begin{aligned} \frac{-|k'|^{-2}}{4E_1 E_2} \left\{ \begin{aligned} & -(E_1 - m_1)(E_2 - m_2)G(p)A + (E_2 - m_2)\bar{F}(p)D + (E_1 - m_1)F(p)D - \underbrace{D(p)C}_{\text{wavy}} \end{aligned} \right\} I_{++}'' \\ + \left\{ \begin{aligned} & -(E_1 - m_1)(E_2 + m_2)G(p)A + (E_2 + m_2)\bar{F}(p)D - (E_1 - m_1)F(p)D + D(p)C \end{aligned} \right\} I_{+-}'' \\ + \left\{ \begin{aligned} & -(E_1 + m_1)(E_2 - m_2)G(p)A - (E_2 - m_2)\bar{F}(p)D + (E_1 + m_1)F(p)D + D(p)C \end{aligned} \right\} I_{-+}'' \\ + \left\{ \begin{aligned} & -(E_1 + m_1)(E_2 + m_2)G(p)A - (E_2 + m_2)\bar{F}(p)D - (E_1 + m_1)F(p)D - D(p)C \end{aligned} \right\} I_{--}'' \end{aligned} \quad (4C.13)$$

And the $\alpha_i^{(1)} \alpha_i^{(2)} [] \alpha_j^{(1)} \alpha_j^{(2)} \{ \omega, f. \}$ term contributes:

$$\begin{aligned}
 \frac{1}{4E_1 E_2} \left\{ \right. & \left\{ \underbrace{(E_1 - m_1)(E_2 - m_2) D(p) A}_{\text{underlined}} - \underbrace{(E_2 - m_2) F(p) B}_{\text{underlined}} - \underbrace{(E_1 - m_1) \bar{F}(p) B}_{\text{underlined}} + \underbrace{G(p) C}_{\text{underlined}} \right\} I_{++} \\
 & + \left\{ \underbrace{(E_1 - m_1)(E_2 + m_2) D(p) A}_{\text{underlined}} - \underbrace{(E_2 + m_2) F(p) B}_{\text{underlined}} + \underbrace{(E_1 - m_1) \bar{F}(p) B}_{\text{underlined}} - \underbrace{G(p) C}_{\text{underlined}} \right\} I_{+-} \\
 & + \left\{ \underbrace{(E_1 + m_1)(E_2 - m_2) D(p) A}_{\text{underlined}} + \underbrace{(E_2 - m_2) F(p) B}_{\text{underlined}} - \underbrace{(E_1 + m_1) \bar{F}(p) B}_{\text{underlined}} - \underbrace{G(p) C}_{\text{underlined}} \right\} I_{-+} \\
 & + \left\{ \underbrace{(E_1 + m_1)(E_2 + m_2) D(p) A}_{\text{underlined}} + \underbrace{(E_2 + m_2) F(p) B}_{\text{underlined}} + \underbrace{(E_1 + m_1) \bar{F}(p) B}_{\text{underlined}} + \underbrace{G(p) C}_{\text{underlined}} \right\} I_{--} \left. \right\}
 \end{aligned}
 \tag{4C.14}$$

Previous results⁽¹²⁾ have shown that the only terms which can give an $\alpha^6 \ln \alpha^{-1}$ contribution are those with an integrand of dimension [momentum]⁻² in the limit of low momentum. Such terms are underlined in equations (4C.10), (4C.12), (4C.13), and (4C.14). The term underlined twice in (4C.14) is of dimension [momentum]⁻³ in the low momentum limit. The remaining, non-underlined, terms are not singular enough to give a logarithmic contribution and hence will contribute to order α^6 or higher. Thus the entire $\alpha^6 \ln \alpha^{-1}$ contribution from the charge component of the ladder diagram is:

Table V. k_0 integrations for ladder diagram.

$$I_{\pm\pm}^{(m)} = \int_{-\infty}^{\infty} dk_0 \frac{k_0^{(m)}}{k_0^2 - |k|^2} \frac{1}{k_0^2 - |k'|^2} \frac{-1}{[k_0 - (\eta_1 \mu_0 \mp E_1 \pm i\epsilon)]} \frac{1}{[k_0 - (-\eta_2 \mu_0 \pm E_2 \mp i\epsilon)]}$$

$$I_{++} \rightarrow 2\pi i \left\{ \frac{2\mu}{|k'|^2 |k|^2 (|k|^2 + \gamma^2)} - \frac{|k'|^2 + |k||k'| + |k|^2}{2|k'|^3 |k|^3 (|k'| + |k|)} \right\}$$

$$I_{+-} \rightarrow 2\pi i \left\{ \frac{-1}{4m_2 |k|^2 |k'|^2} \right\}$$

$$I_{-+} \rightarrow 2\pi i \left\{ \frac{-1}{4m_1 |k|^2 |k'|^2} \right\}$$

$$I_{--} \rightarrow 2\pi i \left\{ \frac{1}{8m_1 m_2 (|k'| + |k|)} \right\}$$

$$I'_{++} \rightarrow 2\pi i \left\{ \frac{-\mu/m_1}{|k|^2 (|k|^2 + \gamma^2)} + \frac{1}{2|k'|^2 |k|^2} \right\}$$

$$I'_{+-} \rightarrow 2\pi i \left\{ \frac{-1}{4m_2 |k'| |k| (|k'| + |k|)} \right\}$$

$$I'_{-+} \rightarrow 2\pi i \left\{ \frac{-1}{4m_1 |k'| |k| (|k'| + |k|)} \right\}$$

$$I'_{--} \rightarrow 2\pi i \left\{ \frac{-1}{16m_1^2 m_2 (|k'| + |k|)} + \frac{-1}{16m_1 m_2^2 (|k'| + |k|)} \right\}$$

$$I''_{++} \rightarrow 2\pi i \left\{ \frac{1}{2|k'| |k| (|k'| + |k|)} + \frac{(\mu/2m_1^2) |k|^2}{|k'|^2 (|k|^2 + \gamma^2)} \right\}$$

$$I''_{+-} \rightarrow 2\pi i \left\{ \frac{-1}{8m_2^2 (|k'| + |k|)} \right\}$$

$$I''_{-+} \rightarrow 2\pi i \left\{ \frac{-1}{8m_1^2 (|k'| + |k|)} \right\}$$

$$I''_{--} \rightarrow 2\pi i \left\{ O(k^0) \right\}$$

Table V (continued)

$$I_{++}''' \rightarrow 2\pi i \left\{ \frac{(\mu/4m_1^3) |k|^4}{|k'|^2 (|k|^2 + \gamma^2)} \right\}$$

$$I_{+-}''' \rightarrow 2\pi i \left\{ \sigma(k^0) \right\}$$

$$I_{-+}''' \rightarrow 2\pi i \left\{ \sigma(k^0) \right\}$$

$$I_{--}''' \rightarrow 2\pi i \left\{ \sigma(k^1) \right\}$$

$$I_{++}^{(\text{IX})} \rightarrow 2\pi i \left\{ \frac{1}{2(|k'| + |k|)} \right\}$$

$$I_{+-}^{(\text{IX})} \rightarrow 2\pi i \left\{ \sigma(k^1) \right\}$$

$$I_{-+}^{(\text{IX})} \rightarrow 2\pi i \left\{ \sigma(k^1) \right\}$$

$$I_{--}^{(\text{IX})} \rightarrow 2\pi i \left\{ \sigma(k^2) \right\}$$



$$\begin{aligned}
(\Delta E_L^c) = & \frac{i \alpha^3 |\psi(0)|^2}{M (2\pi)^4} \int \frac{d^3k d^3p d^3k' \delta(k' - k - p)}{(p^2 + \mu^2)^2 E_1(k) E_2(k)} \\
& \times \left\{ \left[-(E_1 + m_1)(E_2 + m_2) G(p) A + (E_2 + m_2) \bar{F}(p) B + (E_1 + m_1) F(p) B - D(p) C \right] \frac{I_{++}''}{|k|^2} \right. \\
& + \left[(E_1 - m_1)(E_2 - m_2) D(p) A - (E_2 - m_2) F(p) B - (E_1 - m_1) \bar{F}(p) B + G(p) C \right] I_{++} \\
& - D(p) \frac{I_{++}''}{|k|^2} + \left[(E_1 - m_1)(E_2 + m_2) D(p) A - (E_2 + m_2) F(p) B \right] I_{+-} \\
& + \left[(E_1 + m_1)(E_2 - m_2) D(p) A - (E_1 + m_1) \bar{F}(p) B \right] I_{-+} \\
& \left. + \left[(E_1 + m_1)(E_2 + m_2) D(p) A \right] I_{--} \right\}
\end{aligned}
\tag{4C.15}$$

Since it is the low momentum region that will make the logarithmic contribution, we can make certain approximations in the above expression⁽¹⁴⁾. Namely we can expand the E_i in the numerators, and keep only the leading terms as they occur. Thus $(E_i + m_i) \approx 2m_i$ ($i=1,2$) and $(E_i - m_i) \approx k^2/2m_i$. We must be a bit more careful in the denominators. Here we have the product $E_1(k)E_2(k)E_1(p)E_2(p)$, where $E_1(k)E_2(k)$ comes from the projection operator expansion of the particle propagators, and $E_1(p)E_2(p)$ comes from the wave function coefficients of the matrices. It can be shown (Appendix E) that it is sufficient to set $E_1(p)E_2(p) = m_1m_2$ in the denominator so that the only explicit energy dependence left is $E_1(k)E_2(k)$ in the denominator. Thus for purposes of

calculation to order $\alpha^6 \ln \alpha^{-1}$ we can effectively set:

$$\begin{aligned}
 D(\underline{p}) &\simeq 8 \\
 F(\underline{p}) &\simeq 4m_2 \\
 \bar{F}(\underline{p}) &\simeq 4m_1 \\
 G(\underline{p}) &\simeq 2
 \end{aligned}
 \tag{4C.16}$$

Hence we can reduce (4C.15) to the point where we can then use the table of integrals, Table VI.

For example, the first I''_{++} term inside the braces of (4C.15) can be written:

$$\frac{16}{3} \mu M \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \frac{1}{|\underline{k}|^2} \left(\frac{2\pi i \mu}{m_1^2 (|\underline{k} + \underline{p}|^2)} \right) [p^2 + 2\underline{k} \cdot \underline{p} + k^2]
 \tag{4C.17}$$

where we have used only the term with dimension [momentum] $^{-2}$ from I''_{++} , since the other terms would not give a logarithmic contribution. If we now integrate over (4C.17) using Table VI we obtain

$$\int \frac{d^3 p d^3 k [p^2 + 2\underline{k} \cdot \underline{p} + k^2]}{(p^2 + \gamma^2)^2 E_1(k) E_2(k) |\underline{k}|^2 |\underline{k} + \underline{p}|^2} \rightarrow 0$$

to order $\alpha^6 \ln \alpha^{-1}$. Similar methods are applied to the other terms in (4C.15). Only the I_{--} term deserves special mention. I_{--} contains terms of dimension [momentum] $^{-3}$ and [momentum] $^{-1}$ but none of dimension [momentum] $^{-2}$. The term

Table VI. $\alpha^6 \ln \alpha^{-1}$ momentum integrals for two photon exchange.

$$\int \frac{d^3 k d^3 p}{(p^2 + \gamma^2)^2 E_1(k) E_2(k)} \frac{p^2}{k^2 |\underline{k} + \underline{p}|^2} = \frac{4\pi^4}{m_1 m_2} \ln \alpha^{-1}$$

$$\int \frac{d^3 k d^3 p}{(p^2 + \gamma^2)^2 E_1(k) E_2(k)} \frac{(k \cdot p)}{k^2 |\underline{k} + \underline{p}|^2} = - \frac{2\pi^4}{m_1 m_2} \ln \alpha^{-1}$$

$$\int \frac{d^3 k d^3 p}{(p^2 + \gamma^2)^2 E_1(k) E_2(k)} \frac{k^2}{k^2 |\underline{k} + \underline{p}|^2} = 0$$

$$\int \frac{d^3 k d^3 p}{(p^2 + \gamma^2)^2 E_1(k) E_2(k)} \frac{|\underline{k} \times \underline{p}|^2}{k^4 |\underline{k} + \underline{p}|^2} = \frac{2\pi^4}{m_1 m_2} \ln \alpha^{-1}$$

$$\int \frac{d^3 k d^3 p}{(p^2 + \gamma^2)^2 E_1(k) E_2(k)} \frac{|\underline{k} \times \underline{p}|^2}{k^2 |\underline{k} + \underline{p}|^4} = \frac{2\pi^4}{m_1 m_2} \ln \alpha^{-1}$$



of dimension [momentum]⁻³ contributes to order α⁵ and the term of dimension [momentum]⁻¹ contributes to order α⁶, but there is no α⁶lnα⁻¹ dependence to this term. Hence summing up all of the terms in (4C.15) we obtain

$$(\Delta E_L^c) = \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left[3 + \frac{1}{3} \frac{M}{\mu} \right] \quad (4C.18)$$

2. Linear Dipole Component

The linear dipole component of ΔE_L, (4C.4), is computed in the same way as the charge component, although the situation is slightly more messy. For computational convenience we break (4C.4) up into two terms:

$$(\Delta E_L^d)_0 = \frac{2i(4\pi\alpha)^2}{(2\pi)^4} \int \frac{d^3p \, d^4k \, d^4k' \, \delta(k' - k - p) \, \delta(k_0 - k_0')}{k^2 \, k'^2} \overline{\varphi_p(0)} \\ \times \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha^{(1)} \alpha^{(2)} \right) S_F^{(1)}(-k + \eta, \kappa) S_F^{(2)}(k + \eta_2, \kappa) \frac{F_2(k'^2) \gamma_\lambda^{(1)} \sigma_{\lambda p}^{(2)} k'_p}{2m_2} \frac{\varphi_{k^c}^{(1)}(p)}{(2\pi)^{3/2}} \right\} \quad (4C.4a)$$

$$(\Delta E_L^d)_0 = \frac{2i(4\pi\alpha)^2}{(2\pi)^4} \int \frac{d^3p \, d^4k \, d^4k' \, \delta(k' - k - p) \, \delta(k_0 - k_0')}{k^2 \, k'^2} \overline{\varphi_p(0)} \\ \times \left\{ - \frac{F_2(k'^2) \gamma_\mu^{(1)} \tilde{T}_{\mu\nu}^{(2)} k_\nu}{2m_2} S_F^{(1)}(-k + \eta, \kappa) S_F^{(2)}(k + \eta_2, \kappa) \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \alpha^{(1)} \alpha^{(2)} \right) \frac{\varphi_{k^c}^{(1)}(p)}{(2\pi)^{3/2}} \right\} \quad (4C.4b)$$

We deal first with (ΔE_L^d)₁. The matrix product

is reduced using σ_{μν} = -iγ_μγ_ν + iδ_{μν}, and γ_μk_ν = γ₄ik₀ - iγ₄α_i·k_i

We readily obtain:

$$\gamma_{\lambda}^{(1)} \sigma_{\lambda\rho}^{(2)} k_{\rho}' = \gamma_4^{(1)} [\underline{\alpha}^{(1)'} \cdot \underline{k}' - \underline{\alpha}^{(2)'} \cdot \underline{k}' + \underline{\alpha}^{(1)'} \cdot \underline{\alpha}^{(2)'} (\underline{k}_0' - \underline{\alpha}^{(2)'} \cdot \underline{k}')] \quad (4C.19)$$

We again expand the propagators in terms of projection operators (see (3.4)): $S_F^{(1)} S_F^{(2)} = [] \gamma_4^1 \alpha_4^2$. Incorporating the matrix portions of the wave functions, we can write the matrix portion of (4C.4a) as:

$$\begin{aligned} & \left\langle \gamma_4^{(1)} \gamma_4^{(2)} \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|\underline{k}|^2} - \underline{\alpha}^{(1)'} \cdot \underline{\alpha}^{(2)'} \right) [] \gamma_4^{(1)} \gamma_4^{(2)} \left(\gamma_4^{(1)'} [\underline{\alpha}^{(1)'} \cdot \underline{k}' - \underline{\alpha}^{(2)'} \cdot \underline{k}' + \underline{\alpha}^{(1)'} \cdot \underline{\alpha}^{(2)'} (\underline{k}_0' - \underline{\alpha}^{(2)'} \cdot \underline{k}')] \right) \right. \\ & \quad \left. \times \{ D(p) + F(p) \underline{\alpha}^{(1)'} \cdot p - \overline{F}(p) \underline{\alpha}^{(2)'} \cdot p - G(p) \underline{\alpha}^{(1)'} \cdot p \underline{\alpha}^{(2)'} \cdot p \} \right\rangle \\ &= \left\langle \left(\frac{k_0^2}{|\underline{k}|^2} - \underline{\alpha}^{(1)'} \cdot \underline{\alpha}^{(2)'} \right) [] [\underline{\alpha}^{(1)'} \cdot \underline{k}' + \underline{\alpha}^{(2)'} \cdot \underline{k}' - \underline{\alpha}^{(1)'} \cdot \underline{\alpha}^{(2)'} (\underline{k}_0' + \underline{\alpha}^{(2)'} \cdot \underline{k}')] \right. \\ & \quad \left. \times \{ D(p) + F(p) \underline{\alpha}^{(1)'} \cdot p + \overline{F}(p) \underline{\alpha}^{(2)'} \cdot p + G(p) \underline{\alpha}^{(1)'} \cdot p \underline{\alpha}^{(2)'} \cdot p \} \right\rangle \quad (4C.20) \end{aligned}$$

where we have anti-commuted a γ_4^2 to the right through the wave function.

We then integrate over k_0' using $(k_0 - k_0')$ and explicitly indicate the k_0 integration so that we can write (4C.20) as:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dk_0 \left\langle \frac{k_0^2}{|k|^2} [] (\underline{\alpha}^{(1)} \cdot \underline{k}' + \underline{\alpha}^{(2)} \cdot \underline{k}') \{ \omega.f. \} - \frac{k_0^3}{|k|^2} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \{ \omega.f. \} \right. \\
& \quad - \frac{k_0^2}{|k|^2} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k}' \{ \omega.f. \} - \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] (\underline{\alpha}^{(1)} \cdot \underline{k}' + \underline{\alpha}^{(2)} \cdot \underline{k}') \{ \omega.f. \} \\
& \quad \left. + k_0 \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \{ \omega.f. \} + \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k}' \{ \omega.f. \} \right\rangle
\end{aligned}
\tag{4C.20a}$$

When we insert the product of projection operators, (3.4) into (4C.20a) and anticommute the $\beta^{(i)}$, ($i=1,2$) toward the left we will get different results depending on the α matrices multiplying $[]$ on the left. For the calculation at hand we can use (4C.8) and (4C.9). As before we extract the spin-spin contribution by retaining only those terms with an even number of α matrices for each particle. We thus obtain

$$\begin{aligned}
& k_0^2 [] (\underline{\alpha}^{(1)} \cdot \underline{k}' + \underline{\alpha}^{(2)} \cdot \underline{k}') \{ \omega.f. \} \quad \text{term:} \\
& \frac{|k|^{-2}}{4 E_1 E_2} \left\{ \left\{ -(E_2 + m_2) G_1(p) \gamma' + (E_1 + m_1) G_1(p) \gamma' - \bar{F}(p) \zeta' - F(p) \zeta' \right\} I_{++}'' \right. \\
& \quad + \left\{ -(E_2 - m_2) G_1(p) \gamma' - (E_1 + m_1) G_1(p) \gamma' + \bar{F}(p) \zeta' + F(p) \zeta' \right\} I_{+-}'' \\
& \quad + \left\{ (E_2 + m_2) G_1(p) \gamma' + (E_1 - m_1) G_1(p) \gamma' + \bar{F}(p) \zeta' + F(p) \zeta' \right\} I_{-+}'' \\
& \quad \left. + \left\{ (E_2 - m_2) G_1(p) \gamma' - (E_1 - m_1) G_1(p) \gamma' - \bar{F}(p) \zeta' - F(p) \zeta' \right\} I_{--}'' \right\}
\end{aligned}$$

Similarly the $-\frac{k_0^3}{|k|^2} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \{ \omega.f. \}$ term is

$$\begin{aligned}
& \frac{-|\underline{k}|^{-2}}{4 E_1 E_2} \left\{ \left\{ (E_1 + m_1)(E_2 + m_2) G(p) \mathcal{A} - (E_2 + m_2) \bar{F}(p) \mathcal{B} + (E_1 + m_1) F(p) \mathcal{B} - D(p) \mathcal{C} \right\} I_{++}''' \right. \\
& + \left\{ (E_1 + m_1)(E_2 - m_2) G(p) \mathcal{A} - (E_2 - m_2) \bar{F}(p) \mathcal{B} - (E_1 + m_1) F(p) \mathcal{B} + D(p) \mathcal{C} \right\} I_{+-}''' \\
& + \left\{ (E_1 - m_1)(E_2 + m_2) G(p) \mathcal{A} + (E_2 + m_2) \bar{F}(p) \mathcal{B} + (E_1 - m_1) F(p) \mathcal{B} + D(p) \mathcal{C} \right\} I_{-+}''' \\
& \left. + \left\{ (E_1 - m_1)(E_2 - m_2) G(p) \mathcal{A} + (E_2 - m_2) \bar{F}(p) \mathcal{B} - (E_1 - m_1) F(p) \mathcal{B} - D(p) \mathcal{C} \right\} I_{--}''' \right\}
\end{aligned}$$

and the $-k_0^2 |\underline{k}|^{-2} [] \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \underline{k}' \{ \omega. f. \}$ term

$$\begin{aligned}
& \frac{-|\underline{k}|^{-2}}{4 E_1 E_2} \left\{ \left\{ (E_1 + m_1)(E_2 + m_2) F(p) \mathcal{D}' - (E_2 + m_2) D(p) \mathcal{E} + (E_1 + m_1) G(p) \mathcal{H} - \bar{F}(p) \mathcal{L} \right\} I_{++}'' \right. \\
& + \left\{ (E_1 + m_1)(E_2 - m_2) F(p) \mathcal{D}' - (E_2 - m_2) D(p) \mathcal{E} - (E_1 + m_1) G(p) \mathcal{H} + \bar{F}(p) \mathcal{L} \right\} I_{+-}'' \\
& + \left\{ (E_1 - m_1)(E_2 + m_2) F(p) \mathcal{D}' + (E_2 + m_2) D(p) \mathcal{E} + (E_1 - m_1) G(p) \mathcal{H} + \bar{F}(p) \mathcal{L} \right\} I_{-+}'' \\
& \left. + \left\{ (E_1 - m_1)(E_2 - m_2) F(p) \mathcal{D}' + (E_2 - m_2) D(p) \mathcal{E} - (E_1 - m_1) G(p) \mathcal{H} - \bar{F}(p) \mathcal{L} \right\} I_{--}'' \right\}
\end{aligned}$$

and the $-\underline{\alpha}^{(1)} \underline{\alpha}^{(2)} [] \underline{\alpha}^{(1)} \underline{k}' \{ \omega. f. \}$ term

$$\begin{aligned}
& \frac{-1}{4 E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) \bar{F}(p) \mathcal{D}' - (E_2 - m_2) G(p) \mathcal{Y}' + (E_1 - m_1) D(p) \mathcal{Z}' - F(p) \mathcal{H}' \right\} I_{++} \right. \\
& + \left\{ (E_1 - m_1)(E_2 + m_2) \bar{F}(p) \mathcal{D}' - (E_2 + m_2) G(p) \mathcal{Y}' - (E_1 - m_1) D(p) \mathcal{Z}' + F(p) \mathcal{H}' \right\} I_{+-} \\
& + \left\{ (E_1 + m_1)(E_2 - m_2) \bar{F}(p) \mathcal{D}' + (E_2 - m_2) G(p) \mathcal{Y}' + (E_1 + m_1) D(p) \mathcal{Z}' + F(p) \mathcal{H}' \right\} I_{-+} \\
& \left. + \left\{ (E_1 + m_1)(E_2 + m_2) \bar{F}(p) \mathcal{D}' + (E_2 + m_2) G(p) \mathcal{Y}' - (E_1 + m_1) D(p) \mathcal{Z}' - F(p) \mathcal{H}' \right\} I_{--} \right\}
\end{aligned}$$

and the $-\underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] \underline{\alpha}^{(2)} \cdot \underline{k}' \{w.f.\}$ term

$$\begin{aligned} \frac{-1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) F(p) \underline{D}' - \underline{(E_2 - m_2) D(p) \underline{D}'} + (E_1 - m_1) G(p) \underline{Y}' - F(p) \underline{H}' \right\} I_{++} \right. \\ \left. + \left\{ (E_1 - m_1)(E_2 + m_2) F(p) \underline{D}' - \underline{(E_2 + m_2) D(p) \underline{D}'} - (E_1 - m_1) G(p) \underline{Y}' + F(p) \underline{H}' \right\} I_{+-} \right. \\ \left. + \left\{ (E_1 + m_1)(E_2 - m_2) F(p) \underline{D}' + (E_2 - m_2) D(p) \underline{D}' + (E_1 + m_1) G(p) \underline{Y}' + F(p) \underline{H}' \right\} I_{-+} \right. \\ \left. + \left\{ (E_1 + m_1)(E_2 + m_2) F(p) \underline{D}' + (E_2 + m_2) D(p) \underline{D}' - (E_1 + m_1) G(p) \underline{Y}' - F(p) \underline{H}' \right\} I_{--} \right\} \end{aligned}$$

and the $k_0 \underline{\alpha}^1 \cdot \underline{\alpha}^2 [] \underline{\alpha}^1 \cdot \underline{\alpha}^2 \{w.f.\}$ term

$$\begin{aligned} \frac{1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) D(p) A - (E_2 - m_2) F(p) B + (E_1 - m_1) \bar{F}(p) B - G(p) C \right\} I_{++} \right. \\ \left. + \left\{ (E_1 - m_1)(E_2 + m_2) D(p) A - (E_2 + m_2) F(p) B - (E_1 - m_1) \bar{F}(p) B + G(p) C \right\} I_{+-} \right. \\ \left. + \left\{ (E_1 + m_1)(E_2 - m_2) D(p) A + (E_2 - m_2) F(p) B + (E_1 + m_1) \bar{F}(p) B + G(p) C \right\} I_{-+} \right. \\ \left. + \left\{ (E_1 + m_1)(E_2 + m_2) D(p) A + (E_2 + m_2) F(p) B - (E_1 + m_1) \bar{F}(p) B - G(p) C \right\} I_{--} \right\} \end{aligned}$$

and finally the $\alpha^1 \cdot \alpha^2 [] \alpha^1 \cdot \alpha^2 \alpha^2 \cdot k' \{w.f.\}$ term:

$$\begin{aligned} \frac{1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) \bar{F}(p) F - (E_2 - m_2) G(p) L + \underline{(E_1 - m_1) D(p) D} - \underline{F(p) M} \right\} I_{++} \right. \\ \left. + \left\{ (E_1 - m_1)(E_2 + m_2) \bar{F}(p) F - (E_2 + m_2) G(p) L - (E_1 - m_1) D(p) D + F(p) M \right\} I_{+-} \right. \\ \left. + \left\{ (E_1 + m_1)(E_2 - m_2) \bar{F}(p) F + (E_2 - m_2) G(p) L + \underline{(E_1 + m_1) D(p) D} + \underline{F(p) M} \right\} I_{-+} \right. \\ \left. + \left\{ (E_1 + m_1)(E_2 + m_2) \bar{F}(p) F + (E_2 + m_2) G(p) L - (E_1 + m_1) D(p) D - F(p) M \right\} I_{--} \right\} \end{aligned}$$

As before, all terms underlined once are of dimension [momentum]⁻² while those underlined twice are of dimension [momentum]⁻³. All other terms are less singular in momentum and hence won't contribute to order $\alpha^6 \ln \alpha^{-1}$. Collecting all terms with a potential logarithmic contribution we have

$$\begin{aligned}
 (\Delta E_L^d)_0 &= \frac{i X^3 |\varphi(0)|^2}{2m_2 M (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k'}{(p^2 + \delta^2)^2 E_1(k) E_2(k)} \delta(k' - k - p) \\
 &\times \left\{ \begin{aligned}
 &[-(E_1 + m_1)(E_2 + m_2) F(p) \mathcal{D}' - (E_2 + m_2) \mathcal{D}(p) \mathcal{E}] I_{++}'' |k|^{-2} \\
 &- (E_1 - m_1) \mathcal{D}(p) \mathcal{D} I_{++} - (E_1 + m_1) \mathcal{D}(p) \mathcal{D}' I_{-+} + (E_2 - m_2) \mathcal{D}(p) \mathcal{D}' I_{++} \\
 &+ (E_2 + m_2) \mathcal{D}(p) \mathcal{D}' I_{+-} - F(p) M I_{++} + (E_1 - m_1) \mathcal{D}(p) \mathcal{D} I_{++} \\
 &+ (E_1 + m_1) \mathcal{D}(p) \mathcal{D} I_{-+} \end{aligned} \right\} \quad (4C.21)
 \end{aligned}$$

Only the I_{++}'' term merits special attention as it is of dimension [momentum]⁻³. A careful calculation of I_{++}'' shows that it contains a term of dimension [momentum]⁻³ plus a term of dimension [momentum]⁻². Explicit calculation of the integral involving the integrand of dimension [momentum]⁻³ shows no logarithmic dependence, while explicit calculation of the integral with the integrand of dimension [momentum]⁻²

reveals the integral $\int \frac{d^3 k d^3 p [p^2 + 2k \cdot p + k^2]}{(p^2 + \delta^2)^2 E_1(k) E_2(k) |k|^2 |k + p|^2}$ which,

from Table VI, is zero to order $\alpha^6 \ln \alpha^{-1}$.

The remaining terms in (4C.21) are quickly calculated using the low momentum approximations mentioned before equation (4C.17) and Table VI. We obtain

$$(\Delta E_L^d)_0 = -\kappa \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left[1 + \frac{2}{3} \frac{m_1}{m_2} + \frac{1}{3} \frac{M}{m_2} \right] \quad (4C.22)$$

We turn now to $(\Delta E_L^d)_2$ as given in (4C.4b). As before we write $\gamma_\mu^1 \sigma_{\mu\nu}^2 k_\nu = \gamma_4^1 [\alpha^1 \cdot k - \alpha^2 \cdot k + \alpha^1 \cdot \alpha^2 (k_0 - \alpha^2 \cdot k)]$ and expand the propagators as in (3.4) so that the matrix portion of (4C.4b) can be written as

$$\begin{aligned} & \left\langle \left[\underline{\alpha}^{(1)} \cdot \underline{k} - \underline{\alpha}^{(2)} \cdot \underline{k} + \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} (\underline{k}_0 - \underline{\alpha}^{(2)} \cdot \underline{k}) \right] [] \left(\frac{\underline{k}_0'^2}{|\underline{k}'|^2} - \alpha_j^{(1)} \alpha_j^{(2)} \right) \right. \\ & \quad \times \left. \left\{ D(p) + F(p) \underline{\alpha}^{(1)} \cdot \underline{p} - \bar{F}(p) \underline{\alpha}^{(2)} \cdot \underline{p} - G(p) \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \right\} \right. \end{aligned} \quad (4C.23)$$

When we insert the product of projection operators (3.4) into (4C.23) and anti-commute the $\beta^{(i)}$, ($i=1,2$) toward the left we will get different results depending on the α matrices multiplying $[]$ on the left. For the present calculation we need (4C.8), (4C.9) as well as, for $\alpha^{(1)}_{(1)}$ (or $\alpha^{(1)}_{(1)} \alpha^{(2)}_{(2)}$) on the left:

$$\begin{aligned}
[] = \frac{1}{4E_1(k)E_2(k)} & \left\{ \frac{(E_1 - m_1)(E_2 + m_2) - (E_2 + m_2)\underline{\alpha}^{(1)}\underline{k} + (E_1 - m_1)\underline{\alpha}^{(2)}\underline{k} - \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 - E_1)(k_0 + \eta_2 K_0 - E_2)} \right. \\
& + \frac{(E_1 - m_1)(E_2 - m_2) - (E_2 - m_2)\underline{\alpha}^{(1)}\underline{k} - (E_1 - m_1)\underline{\alpha}^{(2)}\underline{k} + \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 - E_1)(k_0 + \eta_2 K_0 + E_2)} \\
& + \frac{(E_1 + m_1)(E_2 + m_2) + (E_2 + m_2)\underline{\alpha}^{(1)}\underline{k} + (E_1 + m_1)\underline{\alpha}^{(2)}\underline{k} + \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 + E_1)(k_0 + \eta_2 K_0 - E_2)} \\
& \left. + \frac{(E_1 + m_1)(E_2 - m_2) + (E_2 - m_2)\underline{\alpha}^{(1)}\underline{k} - (E_1 + m_1)\underline{\alpha}^{(2)}\underline{k} - \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 + E_1)(k_0 + \eta_2 K_0 + E_2)} \right\}
\end{aligned}
\tag{4C.24}$$

and for $\alpha^{(2)}$ on the left

$$\begin{aligned}
[] = \frac{1}{4E_1(k)E_2(k)} & \left\{ \frac{(E_1 + m_1)(E_2 - m_2) - (E_2 - m_2)\underline{\alpha}^{(1)}\underline{k} + (E_1 + m_1)\underline{\alpha}^{(2)}\underline{k} - \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 - E_1)(k_0 + \eta_2 K_0 - E_2)} \right. \\
& + \frac{(E_1 + m_1)(E_2 + m_2) - (E_2 + m_2)\underline{\alpha}^{(1)}\underline{k} - (E_1 + m_1)\underline{\alpha}^{(2)}\underline{k} + \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 - E_1)(k_0 + \eta_2 K_0 + E_2)} \\
& + \frac{(E_1 - m_1)(E_2 - m_2) + (E_2 - m_2)\underline{\alpha}^{(1)}\underline{k} + (E_1 - m_1)\underline{\alpha}^{(2)}\underline{k} + \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 + E_1)(k_0 + \eta_2 K_0 - E_2)} \\
& \left. + \frac{(E_1 - m_1)(E_2 + m_2) + (E_2 + m_2)\underline{\alpha}^{(1)}\underline{k} - (E_1 - m_1)\underline{\alpha}^{(2)}\underline{k} - \underline{\alpha}^{(1)}\underline{k}\underline{\alpha}^{(2)}\underline{k}}{(-k_0 + \eta_1 K_0 + E_1)(k_0 + \eta_2 K_0 + E_2)} \right\}
\end{aligned}
\tag{4C.25}$$

Again we extract the spin-spin content by retaining only those terms with an even number of α matrices for each particle. Doing the trivial k'_0 integration, we obtain:

$$k_0^2 |k'|^{-2} \underline{\alpha}^{(1)} \cdot \underline{k} [] \{ \omega. f. \} \quad \text{term}$$

$$\frac{|k'|^{-2}}{4E_1 E_2} \left\{ \left\{ -(E_1 - m_1) G(p) \gamma + F(p) \gamma \right\} I_{++}'' + \left\{ (E_1 - m_1) G(p) \gamma - \bar{F}(p) \gamma \right\} I_{+-}'' \right. \\ \left. + \left\{ -(E_1 + m_1) G(p) \gamma - \bar{F}(p) \gamma \right\} I_{-+}'' + \left\{ (E_1 + m_1) G(p) \gamma + \bar{F}(p) \gamma \right\} I_{--}'' \right\}$$

Similarly the $-k_0^2 |k'|^{-2} \underline{\alpha}^{(1)} \cdot \underline{k} [] \{ \omega. f. \}$ term is

$$\frac{-|k'|^{-2}}{4E_1 E_2} \left\{ \left\{ (E_2 - m_2) G(p) \gamma - F(p) \gamma \right\} I_{++}'' + \left\{ (E_2 + m_2) G(p) \gamma + F(p) \gamma \right\} I_{+-}'' \right. \\ \left. + \left\{ -(E_2 - m_2) G(p) \gamma + F(p) \gamma \right\} I_{-+}'' + \left\{ -(E_2 + m_2) G(p) \gamma - F(p) \gamma \right\} I_{--}'' \right\}$$

and the $k_0^3 |k'|^{-2} \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] \{ \omega. f. \}$ term

$$\frac{|k'|^{-2}}{4E_1 E_2} \left\{ \left\{ -(E_1 - m_1)(E_2 - m_2) G(p) \alpha + (E_2 - m_2) \bar{F}(p) \alpha + (E_1 - m_1) F(p) \alpha - D(p) \gamma \right\} I_{++}''' \right. \\ \left. + \left\{ -(E_1 - m_1)(E_2 + m_2) G(p) \alpha + (E_2 + m_2) \bar{F}(p) \alpha - (E_1 - m_1) F(p) \alpha + D(p) \gamma \right\} I_{+-}''' \right. \\ \left. + \left\{ -(E_1 + m_1)(E_2 - m_2) G(p) \alpha - (E_2 - m_2) \bar{F}(p) \alpha + (E_1 - m_1) F(p) \alpha + D(p) \gamma \right\} I_{-+}''' \right. \\ \left. + \left\{ -(E_1 + m_1)(E_2 + m_2) G(p) \alpha - (E_2 + m_2) \bar{F}(p) \alpha - (E_1 + m_1) F(p) \alpha - D(p) \gamma \right\} I_{--}''' \right\}$$

and the $-k_0^3 |k'|^{-2} \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \{ \omega. f. \}$ term

$$\begin{aligned}
& \frac{-|k'|^{-2}}{4 E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 + m_2) F(p) \mathcal{D} - \underbrace{(E_2 + m_2) D(p) \mathcal{F}}_{\text{wavy}} - (E_1 - m_1) G(p) \mathcal{U} + \bar{F}(p) \mathcal{V} \right\} I_{++}'' \right. \\
& + \left\{ (E_1 - m_1)(E_2 - m_2) F(p) \mathcal{D} - (E_2 - m_2) D(p) \mathcal{F} + (E_1 - m_1) G(p) \mathcal{U} - \bar{F}(p) \mathcal{V} \right\} I_{+-}'' \\
& + \left\{ (E_1 + m_1)(E_2 + m_2) F(p) \mathcal{D} + (E_2 + m_2) D(p) \mathcal{F} - (E_1 + m_1) G(p) \mathcal{U} - \bar{F}(p) \mathcal{V} \right\} I_{-+}'' \\
& \left. + \left\{ (E_1 + m_1)(E_2 - m_2) F(p) \mathcal{D} + (E_2 - m_2) D(p) \mathcal{F} + (E_1 + m_1) G(p) \mathcal{U} + \bar{F}(p) \mathcal{V} \right\} I_{--}'' \right\}
\end{aligned}$$

and the $\alpha^{(1)} \cdot \underline{k} [] \alpha^{(2)} \cdot \alpha^{(2)} \{ \omega, f. \}$ term

$$\begin{aligned}
& \frac{-1}{4 E_1 E_2} \left\{ \left\{ -(E_1 - m_1)(E_2 + m_2) \bar{F}(p) \mathcal{B} + \underbrace{(E_2 + m_2) G(p) \mathcal{J}}_{\text{wavy}} + (E_1 - m_1) D(p) \mathcal{Q} - \underbrace{\bar{F}(p) \mathcal{V}}_{\text{wavy}} \right\} I_{++} \right. \\
& + \left\{ -(E_1 - m_1)(E_2 - m_2) \bar{F}(p) \mathcal{B} + (E_2 - m_2) G(p) \mathcal{J} - (E_1 - m_1) D(p) \mathcal{Q} + \bar{F}(p) \mathcal{V} \right\} I_{+-} \\
& + \left\{ -(E_1 + m_1)(E_2 + m_2) \bar{F}(p) \mathcal{B} - \underbrace{(E_2 + m_2) G(p) \mathcal{J}}_{\text{wavy}} + (E_1 + m_1) D(p) \mathcal{Q} + \bar{F}(p) \mathcal{V} \right\} I_{-+} \\
& \left. + \left\{ -(E_1 + m_1)(E_2 - m_2) \bar{F}(p) \mathcal{B} - (E_2 - m_2) G(p) \mathcal{J} - (E_1 + m_1) D(p) \mathcal{Q} - \bar{F}(p) \mathcal{V} \right\} I_{--} \right\}
\end{aligned}$$

and the $\alpha^{(2)} \cdot \underline{k} [] \alpha^{(1)} \cdot \alpha^{(2)} \{ \omega, f. \}$ term

$$\begin{aligned}
& \frac{1}{4 E_1 E_2} \left\{ \left\{ \underbrace{(E_1 + m_1)(E_2 - m_2) F(p) \mathcal{B}}_{\text{wavy}} - \underbrace{(E_2 - m_2) D(p) \mathcal{Q}}_{\text{wavy}} - \underbrace{(E_1 + m_1) G(p) \mathcal{J}}_{\text{wavy}} + \underbrace{\bar{F}(p) \mathcal{V}}_{\text{wavy}} \right\} I_{++} \right. \\
& + \left\{ \underbrace{(E_1 + m_1)(E_2 + m_2) F(p) \mathcal{B}}_{\text{wavy}} - \underbrace{(E_2 + m_2) D(p) \mathcal{Q}}_{\text{wavy}} + (E_1 + m_1) G(p) \mathcal{J} - \bar{F}(p) \mathcal{V} \right\} I_{+-} \\
& + \left\{ (E_1 - m_1)(E_2 - m_2) F(p) \mathcal{B} + (E_2 - m_2) D(p) \mathcal{Q} - (E_1 - m_1) G(p) \mathcal{J} - \bar{F}(p) \mathcal{V} \right\} I_{-+} \\
& \left. + \left\{ (E_1 - m_1)(E_2 + m_2) F(p) \mathcal{B} + (E_2 + m_2) D(p) \mathcal{Q} + (E_1 - m_1) G(p) \mathcal{J} + \bar{F}(p) \mathcal{V} \right\} I_{--} \right\}
\end{aligned}$$

and the $-k_0 \alpha^{(1)} \alpha^{(2)} [] \alpha^{(1)} \alpha^{(2)} \{ \omega, f. \}$ term

$$\begin{aligned} & \frac{-1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) D(p) A - (E_2 - m_2) F(p) B - (E_1 - m_1) \overline{F}(p) B + G(p) C \right\} I_{++}' \right. \\ & \quad + \left\{ (E_1 - m_1)(E_2 + m_2) D(p) A - (E_2 + m_2) F(p) B + (E_1 - m_1) \overline{F}(p) B - G(p) C \right\} I_{+-}' \\ & \quad + \left\{ (E_1 + m_1)(E_2 - m_2) D(p) A + (E_2 - m_2) F(p) B - (E_1 + m_1) \overline{F}(p) B - G(p) C \right\} I_{-+}' \\ & \quad \left. + \left\{ (E_1 + m_1)(E_2 + m_2) D(p) A - (E_2 + m_2) F(p) B + (E_1 + m_1) \overline{F}(p) B + G(p) C \right\} I_{--}' \right\} \end{aligned}$$

and finally the $\alpha^{(1)} \alpha^{(2)} \underline{k} [] \alpha^{(1)} \alpha^{(2)} \{ \omega, f. \}$ term

$$\begin{aligned} & \frac{1}{4E_1 E_2} \left\{ \left\{ \underbrace{-(E_1 - m_1)(E_2 + m_2) \overline{F}(p) B}_{\text{once}} + \underbrace{(E_2 + m_2) G(p) C}_{\text{once}} + \underbrace{(E_1 - m_1) D(p) G}_{\text{once}} - \underbrace{F(p) N}_{\text{once}} \right\} I_{++} \right. \\ & \quad + \left\{ \underbrace{-(E_1 - m_1)(E_2 - m_2) \overline{F}(p) B}_{\text{once}} + \underbrace{(E_2 - m_2) G(p) C}_{\text{once}} - \underbrace{(E_1 - m_1) D(p) G}_{\text{once}} + \underbrace{F(p) N}_{\text{once}} \right\} I_{+-} \\ & \quad + \left\{ \underbrace{-(E_1 + m_1)(E_2 + m_2) \overline{F}(p) B}_{\text{once}} - \underbrace{(E_2 + m_2) G(p) C}_{\text{once}} + \underbrace{(E_1 + m_1) D(p) G}_{\text{once}} + \underbrace{F(p) N}_{\text{once}} \right\} I_{-+} \\ & \quad \left. + \left\{ \underbrace{-(E_1 + m_1)(E_2 + m_2) \overline{F}(p) B}_{\text{once}} - \underbrace{(E_2 - m_2) G(p) C}_{\text{once}} - \underbrace{(E_1 + m_1) D(p) G}_{\text{once}} - \underbrace{F(p) N}_{\text{once}} \right\} I_{--} \right\} \end{aligned}$$

As before, all terms underlined once are of dimension [momentum]⁻² while those underlined twice are of dimension [momentum]⁻³. All other terms won't contribute to our order. Thus we can write:

$$\begin{aligned}
(\Delta E_L^d)_{(2)} = & \frac{-i\alpha^3 |\varphi(0)|^2}{2m_2 M (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k'}{(p^2 + \mu^2)^2 E_1(k) E_2(k)} \delta(k' - k - p) \left\{ \frac{(E_2 + m_2) D(p)}{|k'|^2} I_{++}'' \right. \\
& - [(E_2 + m_2) G(p) A - F(p) V] I_{++} + [(E_1 + m_1)(E_2 + m_2) F(p) B - (E_1 + m_1) D(p) C] I_{-+} \\
& + [(E_1 + m_1)(E_2 - m_2) F(p) B - (E_2 - m_2) D(p) C - (E_1 + m_1) G(p) A + F(p) V] I_{++} \\
& + [-(E_1 - m_1)(E_2 + m_2) \bar{F}(p) B + (E_2 + m_2) G(p) C + (E_1 - m_1) D(p) A - F(p) V] I_{++} \\
& + [(E_1 + m_1)(E_2 + m_2) F(p) B - (E_2 + m_2) D(p) C] I_{+-} \\
& \left. + [-(E_1 + m_1)(E_2 + m_2) \bar{F}(p) B + (E_1 + m_1) D(p) C] I_{-+} \right\} \quad (4C.26)
\end{aligned}$$

As in the $(\Delta E_L^d)_1$ case the term with integrand of dimension [momentum] $^{-3}$ (the I_{++}'' term above) will contribute zero for exactly the same reasons as in the $(\Delta E_L^d)_1$ case. The remaining terms in (4C.26) are quickly calculated using the low momentum approximations mentioned before (4C.17) and using Table VI. We obtain:

$$(\Delta E_L^d)_{(2)} = -K \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left[3 + \frac{2}{3} \frac{m_1}{m_2} - \frac{M}{m_2} \right] \quad (4C.27)$$

Adding (4C.22) and (4C.27):

$$\Delta E_L^d = -K \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left[4 + \frac{4}{3} \frac{m_1}{m_2} - \frac{2}{3} \frac{M}{m_2} \right] \quad (4C.28)$$

3. Quadratic Dipole Component

The quadratic dipole component of ΔE_L is computed using identical methods as before.

$$\Delta E_L^{dd} = \frac{2i(4\pi\alpha)^2}{(2\pi)^9} \int \frac{d^3p d^4k d^4k'}{k^2 k'^2} \frac{\delta(k' - k - p) \delta(k_0 - k_0')}{\varphi_p(0)} \times \left\{ -\frac{\kappa \gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu}{2m_2} S_F^{(1)}(k + \eta_1 k) S_F^{(2)}(k + \eta_2 k) \frac{\kappa \gamma_\lambda^{(1)} \sigma_{\lambda\rho}^{(2)} k'_\rho}{2m_2} \right\} \frac{\varphi_{k'}^{(1)}(p)}{(2\pi)^{3/2}} \quad (4C.5)$$

Using (4C.19) and (3.4) we can write the matrix part of (4C.5) as:

$$\begin{aligned} & \left\langle \underline{\alpha}^{(1)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{k}' - \underline{\alpha}^{(2)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{k}' + \underline{\alpha}^{(1)} \cdot \underline{k} [] \underline{\alpha}^{(2)} \cdot \underline{k}' - \underline{\alpha}^{(2)} \cdot \underline{k} [] \underline{\alpha}^{(2)} \cdot \underline{k}' \right. \\ & - \underline{\alpha}^{(1)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} (\underline{k}_0 + \underline{\alpha}^{(2)} \cdot \underline{k}') + \underline{\alpha}^{(2)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} (\underline{k}_0 + \underline{\alpha}^{(2)} \cdot \underline{k}') \\ & + \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} (\underline{k}_0 - \underline{\alpha}^{(2)} \cdot \underline{k}) [] \underline{\alpha}^{(1)} \cdot \underline{k}' + \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} (\underline{k}_0 - \underline{\alpha}^{(2)} \cdot \underline{k}) [] \underline{\alpha}^{(2)} \cdot \underline{k}' \\ & - \underline{k}_0^2 \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} - \underline{k}_0 \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k}' \\ & \left. + \underline{k}_0 \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} + \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k}' \right\rangle \{\omega, f.\} \end{aligned}$$

where $\{\omega, f.\} = \{ D(p) + F(p) \underline{\alpha}^{(1)} \cdot p + \bar{F}(p) \underline{\alpha}^{(2)} \cdot p + G(p) \underline{\alpha}^{(1)} \cdot p \underline{\alpha}^{(2)} \cdot p \}$

Again using the same techniques as before we can extract the spin-spin content.

The $\underline{\alpha}^{(1)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{k}' \{\omega, f.\}$ term

$$\frac{1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1) \bar{F}(p) \zeta' - G(p) \theta \right\} I_{++} + \left\{ -(E_1 - m_1) \bar{F}(p) \zeta' + G(p) \theta \right\} I_{+-} \right. \\ \left. + \left\{ (E_1 + m_1) \bar{F}(p) \zeta' + G(p) \theta \right\} I_{-+} + \left\{ -(E_1 + m_1) \bar{F}(p) \zeta' - G(p) \theta \right\} I_{--} \right\}$$

while the $-\alpha^{(2)} \cdot \underline{k} [] \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term yields

$$\frac{-1}{4E_1 E_2} \left\{ \left\{ -(E_2 - m_2) \bar{F}(p) \zeta' - G(p) \theta \right\} I_{++} + \left\{ -(E_2 + m_2) \bar{F}(p) \zeta' + G(p) \theta \right\} I_{+-} \right. \\ \left. + \left\{ (E_2 - m_2) \bar{F}(p) \zeta' + G(p) \theta \right\} I_{-+} + \left\{ (E_2 + m_2) \bar{F}(p) \zeta' - G(p) \theta \right\} I_{--} \right\}$$

and the $-\alpha^{(2)} \cdot \underline{k} [] \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term; since the spin averages are of dimension [momentum]⁴ we need examine only I_{++} ,

$$\frac{-1}{4E_1 E_2} \left\{ (E_1 + m_1)(E_2 - m_2) G(p) \gamma' - (E_2 - m_2) \bar{F}(p) \zeta' + \underbrace{(E_1 + m_1) F(p) \omega}_{\text{minimum}} - \underbrace{D(p) \nu}_{\text{minimum}} \right\} I_{++}$$

and the $\alpha^{(2)} \cdot \underline{k} [] \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term: since the spin averages are of dimension [momentum]⁴ we need examine only I_{++} ,

$$\frac{1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2 + m_2) G(p) \gamma' - \underbrace{(E_2 + m_2) \bar{F}(p) \omega}_{\text{minimum}} + (E_1 - m_1) F(p) \zeta' - \underbrace{D(p) \nu}_{\text{minimum}} \right\} I_{++}$$

and the $-k_0 \alpha^{(1)} \cdot \underline{k} [] \alpha^{(1)} \cdot \underline{k}' \{ \omega, f. \}$ term

$$\frac{-1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2 + m_2) \bar{F}(p) \beta - (E_2 + m_2) G(p) \alpha + (E_1 - m_1) D(p) \alpha - F(p) \gamma \right\} I_{++}'$$

All terms in I_{++}' are of dimension [momentum]⁻¹ so all terms in I_{+-}' , I_{-+}' , or I_{--}' will be of the same dimension or even less singular. Hence no $\alpha^6 \ln \alpha^{-1}$ contribution.

The $k_0 \alpha^{(2)} \cdot \underline{k} [] \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term,

$$\frac{1}{4E_1E_2} \left\{ (E_1+m_1)(E_2-m_2)F(p)B - (E_2-m_2)D(p)C + (E_1+m_1)G(p)J - F(p)A \right\} I_{++}'$$

There is no contribution to our order here for the same reason as in the previous paragraph.

The $-\alpha^{(1)} \cdot \underline{k} [] \alpha^{(1)} \alpha^{(2)} \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term,

$$\begin{aligned} & \frac{-1}{4E_1E_2} \left\{ \left\{ \underbrace{(E_1-m_1)(E_2+m_2)D(p)E - (E_2-m_2)F(p)K + (E_1-m_1)\bar{F}(p)L - G(p)M}_{\text{}} \right\} I_{++} \right. \\ & \quad + \left\{ (E_1-m_1)(E_2-m_2)D(p)E - (E_2-m_2)F(p)K - (E_1-m_1)\bar{F}(p)L + G(p)M \right\} I_{+-} \\ & \quad + \left\{ \underbrace{(E_1+m_1)(E_2+m_2)D(p)E + (E_2+m_2)F(p)K + (E_1+m_1)\bar{F}(p)L + G(p)M}_{\text{}} \right\} I_{-+} \\ & \quad \left. + \left\{ (E_1+m_1)(E_2+m_2)D(p)E + (E_2-m_2)F(p)K - (E_1+m_1)\bar{F}(p)L - G(p)M \right\} I_{--} \right\} \end{aligned}$$

and the $\alpha^{(2)} \cdot \underline{k} [] \alpha^{(1)} \alpha^{(2)} \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term,

$$\begin{aligned} & \frac{1}{4E_1E_2} \left\{ \left\{ \underbrace{(E_1+m_1)(E_2-m_2)G(p)N - (E_2-m_2)\bar{F}(p)O + (E_1+m_1)F(p)P - D(p)Q}_{\text{}} \right\} I_{++} \right. \\ & \quad + \left\{ (E_1+m_1)(E_2+m_2)G(p)N - (E_2+m_2)\bar{F}(p)O - (E_1+m_1)F(p)P + D(p)Q \right\} I_{+-} \\ & \quad + \left\{ (E_1-m_1)(E_2-m_2)G(p)N + (E_2-m_2)\bar{F}(p)O + (E_1-m_1)\bar{F}(p)P + D(p)Q \right\} I_{-+} \\ & \quad \left. + \left\{ (E_1-m_1)(E_2+m_2)G(p)N + (E_2+m_2)\bar{F}(p)O - (E_1-m_1)\bar{F}(p)P - D(p)Q \right\} I_{--} \right\} \end{aligned}$$

and the $\alpha^{(2)} \cdot \underline{k} [] \alpha^{(1)} \alpha^{(2)} \alpha^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term,

$$\frac{1}{4E_1E_2} \left\{ (E_1-m_1)(E_2-m_2)\bar{F}(p)S' - (E_2-m_2)G(p)U' + (E_1-m_1)D(p)T' - F(p)V' \right\} I_{++}$$

the most singular term here is of dimension [momentum]⁻¹
 so the terms in I'_{+-} , I'_{++} , or I'_{--} will be of the same dimension or even less singular.

Identical arguments apply to the $k_0 \alpha^{(1)} \alpha^{(2)} [\alpha^{(2)} k' \{ \omega.f. \}]$ term whose most singular term is:

$$\frac{1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2 - m_2) \bar{F}(p) \not{D}' - (E_2 - m_2) \not{D}(p) \not{F}' + (E_1 - m_1) G(p) \not{U}' - \bar{F}(p) \not{H}' \right\} I'_{++}$$

The $-\alpha^{(1)} \alpha^{(2)} \alpha^{(2)} k [\alpha^{(2)} k' \{ \omega.f. \}]$ term contributes

$$\begin{aligned} & \frac{-1}{4E_1 E_2} \left\{ \left\{ \underbrace{(E_1 - m_1)(E_2 + m_2) \not{D}(p) \not{F}'}_{\text{}} - \underbrace{(E_2 + m_2) F(p) \not{H}'}_{\text{}} + (E_1 - m_1) \bar{F}(p) R - G(p) \not{J} \right\} I_{++} \right. \\ & + \left\{ (E_1 - m_1)(E_2 - m_2) \not{D}(p) \not{F}' - (E_2 - m_2) F(p) \not{H}' - (E_1 - m_1) \bar{F}(p) R + G(p) \not{J} \right\} I_{+-} \\ & + \left\{ (E_1 + m_1)(E_2 + m_2) \not{D}(p) \not{F}' + (E_2 + m_2) F(p) \not{H}' + (E_1 + m_1) \bar{F}(p) R + G(p) \not{J} \right\} I_{-+} \\ & \left. + \left\{ (E_1 + m_1)(E_2 + m_2) \not{D}(p) \not{F}' + (E_2 - m_2) F(p) \not{H}' - (E_1 + m_1) \bar{F}(p) R - G(p) \not{J} \right\} I_{--} \right\} \end{aligned}$$

and the $-\alpha^{(1)} \alpha^{(2)} \alpha^{(2)} k [\alpha^{(2)} k' \{ \omega.f. \}]$ term,

$$\begin{aligned} & \frac{-1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 + m_2) G(p) \not{U}' - \underbrace{(E_2 + m_2) \bar{F}(p) \not{H}'}_{\text{}} + (E_1 - m_1) F(p) R - \underbrace{D(p) K}_{\text{}} \right\} I_{++} \right. \\ & + \left\{ (E_1 - m_1)(E_2 - m_2) G(p) \not{U}' - (E_2 - m_2) \bar{F}(p) \not{H}' - (E_1 - m_1) F(p) R + D(p) K \right\} I_{+-} \\ & + \left\{ (E_1 + m_1)(E_2 + m_2) G(p) \not{U}' + (E_2 + m_2) \bar{F}(p) \not{H}' + (E_1 + m_1) F(p) R + D(p) K \right\} I_{-+} \\ & \left. + \left\{ (E_1 + m_1)(E_2 - m_2) G(p) \not{U}' + (E_2 - m_2) \bar{F}(p) \not{H}' - (E_1 + m_1) F(p) R - D(p) K \right\} I_{--} \right\} \end{aligned}$$

and the $-k_0^2 \alpha^{(1)} \alpha^{(2)} [\alpha^{(1)} \alpha^{(2)} \{ \omega.f. \}]$ term

$$\begin{aligned}
& \frac{-1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) D(p) A - (E_2 - m_2) F(p) B + (E_1 - m_1) \bar{F}(p) B - G(p) C \right\} I_{++}'' \right. \\
& \quad + \left\{ (E_1 - m_1)(E_2 + m_2) D(p) A - (E_2 + m_2) F(p) B - (E_1 - m_1) \bar{F}(p) B + G(p) C \right\} I_{+-}'' \\
& \quad + \left\{ (E_1 + m_1)(E_2 - m_2) D(p) A + (E_2 - m_2) F(p) B + (E_1 + m_1) \bar{F}(p) B + G(p) C \right\} I_{-+}'' \\
& \quad \left. + \left\{ (E_1 + m_1)(E_2 + m_2) D(p) A + (E_2 + m_2) F(p) B - (E_1 + m_1) \bar{F}(p) B - G(p) C \right\} I_{--}'' \right\}
\end{aligned}$$

and the $-k_0 \alpha^{(1)} \cdot \alpha^{(2)} [\] \alpha^{(1)} \alpha^{(2)} \alpha^{(2)} \cdot k [\] \omega, f. \}$ term

$$\begin{aligned}
& \frac{-1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 - m_2) \bar{F}(p) F - (E_2 - m_2) G(p) L + (E_1 - m_1) D(p) D - F(p) M \right\} I_{++}' \right. \\
& \quad + \left\{ (E_1 - m_1)(E_2 + m_2) \bar{F}(p) F - (E_2 + m_2) G(p) L - (E_1 - m_1) D(p) D + F(p) M \right\} I_{+-}' \\
& \quad + \left\{ (E_1 + m_1)(E_2 - m_2) \bar{F}(p) F + (E_2 - m_2) G(p) L + (E_1 + m_1) D(p) D + F(p) M \right\} I_{-+}' \\
& \quad \left. + \left\{ (E_1 + m_1)(E_2 + m_2) \bar{F}(p) F + (E_2 + m_2) G(p) L - (E_1 + m_1) D(p) D - F(p) M \right\} I_{--}' \right\}
\end{aligned}$$

and the $k_0 \alpha^{(1)} \cdot \alpha^{(2)} \alpha^{(2)} \cdot k [\] \alpha^{(1)} \alpha^{(2)} \omega, f. \}$ term

$$\begin{aligned}
& \frac{1}{4E_1 E_2} \left\{ \left\{ (E_1 - m_1)(E_2 + m_2) \bar{F}(p) B - (E_2 + m_2) G(p) C + (E_1 - m_1) D(p) G - F(p) N \right\} I_{++}' \right. \\
& \quad + \left\{ (E_1 - m_1)(E_2 - m_2) \bar{F}(p) B - (E_2 - m_2) G(p) C - (E_1 - m_1) D(p) G + F(p) N \right\} I_{+-}' \\
& \quad + \left\{ (E_1 + m_1)(E_2 + m_2) \bar{F}(p) B + (E_2 + m_2) G(p) C + (E_1 + m_1) D(p) G + F(p) N \right\} I_{-+}' \\
& \quad \left. + \left\{ (E_1 + m_1)(E_2 - m_2) \bar{F}(p) B + (E_2 - m_2) G(p) C - (E_1 + m_1) D(p) G - F(p) N \right\} I_{--}' \right\}
\end{aligned}$$

and finally the $\underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \underline{\alpha}^{(1)} \cdot \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k}' \{ \omega, f. \}$ term

$$\begin{aligned} \frac{1}{4E_1 E_2} & \left\{ \left\{ \underbrace{(E_1 - m_1)(E_2 + m_2) D(p) D}_{\text{m}} - \underbrace{(E_2 + m_2) F(p) C}_{\text{m}} + (E_1 - m_1) \bar{F}(p) O - G(p) P \right\} I_{++} \right. \\ & + \left\{ (E_1 - m_1)(E_2 - m_2) D(p) D - (E_2 - m_2) F(p) C - (E_1 - m_1) \bar{F}(p) O + G(p) P \right\} I_{+-} \\ & + \left\{ \underbrace{(E_1 + m_1)(E_2 + m_2) D(p) D}_{\text{m}} + (E_2 + m_2) F(p) C + (E_1 + m_1) \bar{F}(p) O + G(p) P \right\} I_{-+} \\ & \left. + \left\{ (E_1 + m_1)(E_2 - m_2) D(p) D + (E_2 - m_2) F(p) C - (E_1 + m_1) \bar{F}(p) O - G(p) P \right\} I_{--} \right\} \end{aligned}$$

Collecting those terms that will contribute to our order, we have:

$$\begin{aligned} \Delta E_L^{dd} = & - \frac{i \alpha^3 |\varphi(0)|^2}{64 \pi^4 M m_2^2} \int \frac{d^3 p d^3 k d^3 k' d(k' - k - p)}{(p^2 + \delta^2)^2 E_1(k) E_2(k)} \left\{ - [(E_1 - m_1)(E_2 + m_2) D(p) E] I_{++} \right. \\ & - [(E_1 + m_1) F(p) \omega + (E_2 + m_2) \bar{F}(p) \omega] I_{++} - (E_1 + m_1)(E_2 + m_2) D(p) \Sigma I_{+-} \\ & + [(E_1 + m_1) F(p) P - D(p) Q] I_{++} + [-(E_1 - m_1)(E_2 + m_2) D(p) \mathcal{D}' + (E_2 + m_2) F(p) \mathcal{D}'] I_{++} \\ & + (E_2 + m_2) \bar{F}(p) \mathcal{D}' - D(p) K + (E_1 - m_1)(E_2 + m_2) D(p) D - (E_2 + m_2) F(p) C] I_{++} \\ & \left. + [(E_1 + m_1)(E_2 + m_2) D(p) D] I_{-+} \right\} \end{aligned}$$

Using the same procedure as before, we quickly find that

$$\Delta E_L^{dd} = K^2 \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \quad \frac{11}{3} \quad (4C.29)$$

and finally the $\alpha^{(1)} \cdot \alpha^{(2)} \alpha^{(2)} \cdot k [] \alpha^{(1)} \cdot \alpha^{(2)} \alpha^{(2)} \cdot k' \{ \omega, f. \}$ term

$$\begin{aligned} \frac{1}{4 E_1 E_2} \left\{ \right. & \left\{ \underbrace{(E_1 - m_1)(E_2 + m_2) D(p) D}_{\text{}} - \underbrace{(E_2 + m_2) F(p) C}_{\text{}} + (E_1 - m_1) \bar{F}(p) O - G(p) P \right\} I_{++} \\ & + \left\{ (E_1 - m_1)(E_2 - m_2) D(p) D - (E_2 - m_2) F(p) C - (E_1 - m_1) \bar{F}(p) O + G(p) P \right\} I_{+-} \\ & + \left\{ \underbrace{(E_1 + m_1)(E_2 + m_2) D(p) D}_{\text{}} + (E_2 + m_2) F(p) C + (E_1 + m_1) \bar{F}(p) O + G(p) P \right\} I_{-+} \\ & + \left\{ (E_1 + m_1)(E_2 - m_2) D(p) D + (E_2 - m_2) F(p) C - (E_1 + m_1) \bar{F}(p) O - G(p) P \right\} I_{--} \left. \right\} \end{aligned}$$

Collecting those terms that will contribute to our order, we have:

$$\begin{aligned} \Delta E_L^{dd} = & - \frac{i \alpha^3 |\varphi(0)|^2}{64 \pi^4 M m_2^2} \int \frac{d^3 p d^3 k d^3 k' d(k' - k - p)}{(p^2 + \gamma^2)^2 E_1(k) E_2(k)} \left\{ - [(E_1 - m_1)(E_2 + m_2) D(p) E] I_{++} \right. \\ & - [(E_1 + m_1) F(p) \omega + (E_2 + m_2) \bar{F}(p) \omega] I_{++} - (E_1 + m_1)(E_2 + m_2) D(p) \Sigma I_{+-} \\ & + [(E_1 + m_1) F(p) P - D(p) Q] I_{++} + [-(E_1 - m_1)(E_2 + m_2) D(p) \bar{Q}' + (E_2 + m_2) F(p) \bar{Q}'] I_{++} \\ & + (E_2 + m_2) \bar{F}(p) \bar{Q}' - D(p) K + (E_1 - m_1)(E_2 + m_2) D(p) D - (E_2 + m_2) F(p) C] I_{++} \\ & \left. + [(E_1 + m_1)(E_2 + m_2) D(p) D] I_{-+} \right\} \end{aligned}$$

Using the same procedure as before, we quickly find that

$$\Delta E_L^{dd} = K^2 \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \quad \frac{11}{3} \quad (4C.29)$$

In summary the total ladder diagram contribution is:

$$\Delta E_L = \eta \left\{ \left(3 + \frac{1}{3} \frac{M}{\mu} \right) - \kappa \left(4 + \frac{4}{3} \frac{m_1}{m_2} - \frac{2}{3} \frac{M}{m_2} \right) + \kappa^2 \frac{11}{3} \right\} \quad (4C.30)$$

where $\eta = \frac{\pi \alpha^3 |\psi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2}$

D. Crossed Diagram

The crossed two photon contribution to the energy shift comes from the irreducible crossed two photon kernel, and is given by⁽¹²⁾:

$$\Delta E_c = -2i \int d^4x d^4x' \overline{\varphi_p^{(0)}} \widetilde{I}_{\widetilde{K}}(x, x') \varphi_{\kappa c}^{(1)}(x', 0) \quad (4D.1)$$

where $\widetilde{I}_{\widetilde{K}}(x, x')$ is derived in Appendix A and is

$$\begin{aligned} \widetilde{I}_{\widetilde{K}}(x, x') = & - \frac{(4\pi\alpha)^2}{(2\pi)^8} \int \frac{d^4k d^4k'}{k^2 k'^2} e^{i(\eta_2 k - \eta_1 k')x} e^{i(\eta_1 k - \eta_2 k')x'} \\ & \times \left\{ \gamma_{\mu}^{(1)} \gamma_{\nu}^{(2)} G_{\widetilde{K}}(x-x') \gamma_{\nu}^{(1)} \gamma_{\mu}^{(2)} + \frac{F_2(k^2)}{2m_2} \gamma_{\mu}^{(1)} \gamma_{\nu}^{(1)} G_{\widetilde{K}}(x-x') \gamma_{\nu}^{(1)} \gamma_{\mu\alpha}^{(2)} k_{\alpha} \right. \\ & - \frac{F_2(k^2)}{2m_2} \gamma_{\mu}^{(1)} \gamma_{\nu\beta}^{(2)} k'_{\beta} G_{\widetilde{K}}(x-x') \gamma_{\nu}^{(1)} \gamma_{\mu}^{(2)} \\ & \left. - \frac{F_2(k^2) F_2(k'^2)}{4m_2^2} \gamma_{\mu}^{(1)} \gamma_{\nu\beta}^{(2)} k'_{\beta} G_{\widetilde{K}}(x-x') \gamma_{\nu}^{(1)} \gamma_{\mu\alpha}^{(2)} k_{\alpha} \right\} \quad (4D.2) \end{aligned}$$

where $G_{\widetilde{K}}(x-x') = \int \frac{d^4p'}{(2\pi)^4} e^{ip'(x-x')} S_F^{(1)}(p' + \eta_1 \widetilde{K}) S_F^{(2)}(p' + \eta_2 \widetilde{K})$ (4D.2a)

Doing the x, x' and p' integrations when we substitute (4D.2) into (4D.1) we obtain:

$$\begin{aligned}
\Delta E_C = & \frac{4i\alpha}{\pi(2\pi)^4} \int \frac{d^3p d^4k d^4k'}{k^2 k'^2} \frac{\delta(k'-k-p) \delta(k_0-k'_0)}{\varphi_p(0)} \\
& \times \left\{ \gamma_\mu^{(1)} \gamma_\nu^{(2)} S_F^{(1)}(-k+\eta_1, k) S_F^{(2)}(k'+\eta_2, k) \gamma_\nu^{(1)} \gamma_\mu^{(2)} \right. \\
& + \frac{F_2(k^2)}{2m_2} \gamma_\mu^{(1)} \gamma_\nu^{(2)} S_F^{(1)}(-k+\eta_1, k) S_F^{(2)}(-k'+\eta_2, k) \gamma_\nu^{(1)} \sigma_{\mu\alpha}^{(2)} k_\alpha \\
& - \frac{F_2(k^2)}{2m_2} \gamma_\mu^{(1)} \sigma_{\nu\beta}^{(2)} k'_\beta S_F^{(1)}(-k+\eta_1, k) S_F^{(2)}(-k'+\eta_2, k) \gamma_\nu^{(1)} \gamma_\mu^{(2)} \\
& \left. - \frac{F_2(k^2) F_2(k'^2)}{4m_2^2} \gamma_\mu^{(1)} \sigma_{\nu\beta}^{(2)} k'_\beta S_F^{(1)}(-k+\eta_1, k) S_F^{(2)}(-k'+\eta_2, k) \gamma_\nu^{(1)} \sigma_{\mu\alpha}^{(2)} k_\alpha \right\} \times \\
& \times \varphi^{(1)}(p) (2\pi)^{3/2}
\end{aligned} \tag{4D.3}$$

The expression for ΔE_C rather logically breaks itself up into a charge component, a linear dipole component, and a quadratic dipole component. We consider first the charge component.

1. Charge Component

The matrix portion of the charge component of (4D.3) can be written as

$$\left\langle \gamma_4^{(1)} \gamma_4^{(2)} \gamma_\mu^{(1)} \gamma_\nu^{(2)} S_F^{(1)} S_F^{(2)} \gamma_\nu^{(1)} \gamma_\mu^{(2)} \left\{ D(p) + F(p) \underline{\alpha}^{(1)} \cdot \underline{p} - \bar{F}(p) \underline{\alpha}^{(2)} \cdot \underline{p} - G(p) \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p} \right\} \right\rangle \tag{4D.4}$$

We expand the product of propagators according to (3.4) so that (4D.4) can be rather quickly reduced to the following expression in brackets:

$$\left\langle [] \{w.f.\} - \alpha_j^{(2)} [] \alpha_j^{(1)} \{w.f.\} - \alpha_i^{(1)} [] \alpha_i^{(2)} \{w.f.\} + \alpha_i^{(1)} \alpha_j^{(2)} [] \alpha_j^{(1)} \alpha_i^{(2)} \{w.f.\} \right\rangle$$

(4D.5)

where $[]$ contains the projection operators and $\{w.f.\} = \{D(p) + F(p) \alpha^{(1)} \cdot p - \bar{F}(p) \alpha^{(2)} \cdot p - G(p) \alpha^{(1)} p \alpha^{(2)} \cdot p\}$. The k'_0 integration is trivial using $\delta(k_0 - k'_0)$, so that we can readily extract the spin-spin contribution using identical methods to those described earlier. All that is needed is the expression for the projection operators when various combinations of α matrices multiply it from the left. These are given by: for $\alpha^{(1)} \alpha^{(2)}$ on the left:

$$\begin{aligned} [] = \frac{1}{4E_1(k)E_2(k')} & \left\{ \frac{(E_1 - m_1)(E_2' - m_2) - (E_2' - m_2) \alpha^{(1)} \cdot k - (E_1 - m_1) \alpha^{(2)} \cdot k' + \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k'}{[k_0 - (\eta_1 k_0 - E_1)][k_0 - (\eta_2 k_0 - E_2')]} \right. \\ & + \frac{(E_1 - m_1)(E_2' + m_2) - (E_2' + m_2) \alpha^{(1)} \cdot k + (E_1 - m_1) \alpha^{(2)} \cdot k' - \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k'}{[k_0 - (\eta_1 k_0 - E_1)][k_0 - (\eta_2 k_0 + E_2')]} \\ & + \frac{(E_1 + m_1)(E_2' - m_2) + (E_2' - m_2) \alpha^{(1)} \cdot k - (E_1 + m_1) \alpha^{(2)} \cdot k' - \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k'}{[k_0 - (\eta_1 k_0 + E_1)][k_0 - (\eta_2 k_0 - E_2')]} \\ & \left. + \frac{(E_1 + m_1)(E_2' + m_2) + (E_2' + m_2) \alpha^{(1)} \cdot k + (E_1 + m_1) \alpha^{(2)} \cdot k' + \alpha^{(1)} \cdot k \alpha^{(2)} \cdot k'}{[k_0 - (\eta_1 k_0 + E_1)][k_0 - (\eta_2 k_0 + E_2')]} \right\} \end{aligned}$$

(4D.6)

for $\alpha^{(1)}$ on the left:

$$\begin{aligned}
[] = \frac{1}{4E_1 E_2'} \left\{ \frac{(E_1 - m_1)(E_2' + m_2) - (E_2' + m_2)\underline{\alpha}^{(1)} \cdot \underline{k} - (E_1 - m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' + \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 - E_1)][k_0 - (\eta_2 K_0 - E_2')]} \right. \\
+ \frac{(E_1 - m_1)(E_2' - m_2) - (E_2' - m_2)\underline{\alpha}^{(1)} \cdot \underline{k} + (E_1 - m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' - \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 - E_1)][k_0 - (\eta_2 K_0 + E_2')]} \\
+ \frac{(E_1 + m_1)(E_2' + m_2) + (E_2' + m_2)\underline{\alpha}^{(1)} \cdot \underline{k} - (E_1 + m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' - \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 + E_1)][k_0 - (\eta_2 K_0 - E_2')]} \\
\left. + \frac{(E_1 + m_1)(E_2' - m_2) + (E_2' - m_2)\underline{\alpha}^{(1)} \cdot \underline{k} + (E_1 + m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' + \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 + E_1)][k_0 - (\eta_2 K_0 + E_2')]} \right\}
\end{aligned}
\tag{4D.7}$$

for $\alpha^{(2)}$ on the left:

$$\begin{aligned}
[] = \frac{1}{4E_1 E_2'} \left\{ \frac{(E_1 + m_1)(E_2' - m_2) - (E_2' - m_2)\underline{\alpha}^{(1)} \cdot \underline{k} - (E_1 + m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' + \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 - E_1)][k_0 - (\eta_2 K_0 - E_2')]} \right. \\
+ \frac{(E_1 + m_1)(E_2' + m_2) - (E_2' + m_2)\underline{\alpha}^{(1)} \cdot \underline{k} + (E_1 + m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' - \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 - E_1)][k_0 - (\eta_2 K_0 + E_2')]} \\
+ \frac{(E_1 - m_1)(E_2' - m_2) + (E_2' - m_2)\underline{\alpha}^{(1)} \cdot \underline{k} - (E_1 - m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' - \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 + E_1)][k_0 - (\eta_2 K_0 - E_2')]} \\
\left. + \frac{(E_1 - m_1)(E_2' + m_2) + (E_2' + m_2)\underline{\alpha}^{(1)} \cdot \underline{k} + (E_1 - m_1)\underline{\alpha}^{(2)} \cdot \underline{k}' + \underline{\alpha}^{(1)} \cdot \underline{k} \underline{\alpha}^{(2)} \cdot \underline{k}'}{[k_0 - (\eta_1 K_0 + E_1)][k_0 - (\eta_2 K_0 + E_2')]} \right\}
\end{aligned}
\tag{4D.8}$$

for $\alpha^{(2)}\alpha^{(2)}$, or no α matrices on the left:

$$\begin{aligned}
 [] = \frac{1}{4E_1 E_2'} \left\{ \frac{(E_1 + m_1)(E_2' + m_2) - (E_2' + m_2)\alpha^{(1)}\underline{k} - (E_1 + m_1)\alpha^{(2)}\underline{k}' + \alpha^{(1)}\underline{k}\alpha^{(2)}\underline{k}'}{[k_0 - (\eta_1 k_0 - E_1)][k_0 - (\eta_2 k_0 - E_2')]} \right. \\
 + \frac{(E_1 + m_1)(E_2' - m_2) - (E_2' - m_2)\alpha^{(1)}\underline{k} + (E_1 + m_1)\alpha^{(2)}\underline{k}' - \alpha^{(1)}\underline{k}\alpha^{(2)}\underline{k}'}{[k_0 - (\eta_1 k_0 - E_1)][k_0 - (\eta_2 k_0 + E_2')]} \\
 + \frac{(E_1 - m_1)(E_2' + m_2) + (E_2' + m_2)\alpha^{(1)}\underline{k} - (E_1 - m_1)\alpha^{(2)}\underline{k}' - \alpha^{(1)}\underline{k}\alpha^{(2)}\underline{k}'}{[k_0 - (\eta_1 k_0 + E_1)][k_0 - (\eta_2 k_0 - E_2')]} \\
 \left. + \frac{(E_1 - m_1)(E_2' - m_2) + (E_2' - m_2)\alpha^{(1)}\underline{k} + (E_1 - m_1)\alpha^{(2)}\underline{k}' + \alpha^{(1)}\underline{k}\alpha^{(2)}\underline{k}'}{[k_0 - (\eta_1 k_0 + E_1)][k_0 - (\eta_2 k_0 + E_2')]} \right\}
 \end{aligned}
 \tag{4D.9}$$

From (4D.5) we now extract the spin-spin terms.

[] {w.f.} term

$$\frac{\gamma}{4E_1 E_2'} \left\{ \tilde{\mathcal{I}}_{++} - \tilde{\mathcal{I}}_{+-} - \tilde{\mathcal{I}}_{-+} + \tilde{\mathcal{I}}_{--} \right\} \quad \text{the most singular}$$

term here is of dimension [momentum]⁻¹ so there is no contribution to our order.

The $-\alpha_f^{(2)} [] \alpha_f^{(1)} \{w.f.\}$ term yields:

$$\begin{aligned}
& \frac{-1}{4E_1 E_2'} \left\{ \left\{ - (E_1 + m_1)(E_2' - m_2) G(p) \tilde{A} + (E_2' - m_2) \bar{F}(p) \tilde{B} - \underbrace{(E_1 + m_1) F(p) \tilde{C}}_{\text{wavy}} + \underbrace{D(p) \tilde{D}}_{\text{wavy}} \right\} \tilde{I}_{++} \right. \\
& + \left\{ - \underbrace{(E_1 + m_1)(E_2' + m_2) G(p) \tilde{A}}_{\text{wavy}} + \underbrace{(E_2' + m_2) \bar{F}(p) \tilde{B}}_{\text{wavy}} + \underbrace{(E_1 + m_1) F(p) \tilde{C}}_{\text{wavy}} - \underbrace{D(p) \tilde{D}}_{\text{wavy}} \right\} \tilde{I}_{+-} \\
& + \left\{ - (E_1 - m_1)(E_2' - m_2) G(p) \tilde{A} - (E_2' - m_2) \bar{F}(p) \tilde{B} - (E_1 - m_1) F(p) \tilde{C} - D(p) \tilde{D} \right\} \tilde{I}_{-+} \\
& \left. + \left\{ - (E_1 - m_1)(E_2' + m_2) G(p) \tilde{A} - (E_2' + m_2) \bar{F}(p) \tilde{B} + (E_1 - m_1) F(p) \tilde{C} + D(p) \tilde{D} \right\} \tilde{I}_{--} \right\}
\end{aligned}$$

and the $-\alpha_i^{(3)} [] \alpha_i^{(4)} \{ \omega.f. \}$ term:

$$\begin{aligned}
& \frac{-1}{4E_1 E_2'} \left\{ \left\{ - (E_1 - m_1)(E_2' + m_2) G(p) \tilde{A} + \underbrace{(E_2' + m_2) \bar{F}(p) \tilde{B}}_{\text{wavy}} - (E_1 - m_1) F(p) \tilde{C} + \underbrace{D(p) \tilde{D}}_{\text{wavy}} \right\} \tilde{I}_{++} \right. \\
& + \left\{ - (E_1 - m_1)(E_2' - m_2) G(p) \tilde{A} + (E_2' - m_2) \bar{F}(p) \tilde{B} + (E_1 - m_1) F(p) \tilde{C} - D(p) \tilde{D} \right\} \tilde{I}_{+-} \\
& + \left\{ - \underbrace{(E_1 + m_1)(E_2' + m_2) G(p) \tilde{A}}_{\text{wavy}} - \underbrace{(E_2' + m_2) \bar{F}(p) \tilde{B}}_{\text{wavy}} - \underbrace{(E_1 + m_1) F(p) \tilde{C}}_{\text{wavy}} - \underbrace{D(p) \tilde{D}}_{\text{wavy}} \right\} \tilde{I}_{-+} \\
& \left. + \left\{ - (E_1 + m_1)(E_2' - m_2) G(p) \tilde{A} - (E_2' - m_2) \bar{F}(p) \tilde{B} + (E_1 + m_1) F(p) \tilde{C} + D(p) \tilde{D} \right\} \tilde{I}_{--} \right\}
\end{aligned}$$

and the $\alpha_i^{(3)} \alpha_j^{(2)} [] \alpha_j^{(3)} \alpha_i^{(2)} \{ \omega.f. \}$ term:

$$\begin{aligned}
& \frac{1}{4E_1 E_2'} \left\{ \left\{ (E_1 - m_1)(E_2' - m_2) D(p) \tilde{A} - (E_2' - m_2) F(p) \tilde{B} + (E_1 - m_1) \bar{F}(p) \tilde{B}' - G(p) \tilde{C} \right\} \tilde{I}_{++} \right. \\
& + \left\{ \underbrace{(E_1 - m_1)(E_2' + m_2) D(p) \tilde{A}}_{\text{wavy}} - \underbrace{(E_2' + m_2) F(p) \tilde{B}}_{\text{wavy}} - (E_1 - m_1) \bar{F}(p) \tilde{B}' + G(p) \tilde{C} \right\} \tilde{I}_{+-} \\
& + \left\{ \underbrace{(E_1 + m_1)(E_2' - m_2) D(p) \tilde{A}}_{\text{wavy}} + \underbrace{(E_2' - m_2) F(p) \tilde{B}}_{\text{wavy}} + \underbrace{(E_1 + m_1) \bar{F}(p) \tilde{B}'}_{\text{wavy}} + G(p) \tilde{C} \right\} \tilde{I}_{-+} \\
& \left. + \left\{ \underbrace{(E_1 + m_1)(E_2' + m_2) D(p) \tilde{A}}_{\text{wavy}} + \underbrace{(E_2' + m_2) F(p) \tilde{B}}_{\text{wavy}} - (E_1 + m_1) \bar{F}(p) \tilde{B}' - G(p) \tilde{C} \right\} \tilde{I}_{--} \right\}
\end{aligned}$$

Thus we can write the entire charge component of the crossed diagram as:

$$\begin{aligned}
 (\Delta E_c^c) = & \frac{i \alpha^3 / |\varphi(0)|^2}{M (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k'}{(p^2 + \delta^2)^2} \frac{\delta(k' - k - p)}{E_1(k) E_2(k')} \left\{ \begin{aligned}
 & [(E_1 + m_1) F(p) \tilde{\mathcal{C}} - D(p) \tilde{\mathcal{D}}] \tilde{\mathcal{I}}_{++} \\
 & + [(E_1 + m_1) (E_2' + m_2) G(p) \tilde{\mathcal{A}} - (E_2' + m_2) \bar{F}(p) \tilde{\mathcal{B}} - (E_1 + m_1) F(p) \tilde{\mathcal{C}} + D(p) \tilde{\mathcal{D}}] \tilde{\mathcal{I}}_{+-} \\
 & + [(E_1 + m_1) (E_2' + m_2) G(p) \tilde{\mathcal{A}} + (E_2' + m_2) \bar{F}(p) \tilde{\mathcal{B}} + (E_1 + m_1) F(p) \tilde{\mathcal{C}} + D(p) \tilde{\mathcal{D}}] \tilde{\mathcal{I}}_{-+} \\
 & - [(E_2' + m_2) \bar{F}(p) \tilde{\mathcal{B}} + D(p) \tilde{\mathcal{D}}] \tilde{\mathcal{I}}_{++} + D(p) \tilde{\mathcal{D}} \tilde{\mathcal{I}}_{-+} + D(p) \tilde{\mathcal{D}} \tilde{\mathcal{I}}_{+-} \\
 & + [(E_1 - m_1) (E_2' + m_2) D(p) \tilde{\mathcal{A}} - (E_2' + m_2) \bar{F}(p) \tilde{\mathcal{B}}] \tilde{\mathcal{I}}_{+-} + (E_1 + m_1) \bar{F}(p) \tilde{\mathcal{B}} \tilde{\mathcal{I}}_{-+} \\
 & + [(E_1 + m_1) (E_2' + m_2) D(p) \tilde{\mathcal{A}}] \tilde{\mathcal{I}}_{--} \end{aligned} \right\} \quad (4D.10)
 \end{aligned}$$

The first two $\tilde{\mathcal{I}}_{++}$ terms, with integrands of dimension [momentum] $^{-3}$, will give zero contribution to our order. The remaining terms in (4D.10) are quickly calculated using Table VI, and we obtain:

$$\Delta E_c^c = - \frac{\pi \alpha^3 / |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \frac{M}{\mu} \quad (4D.11)$$

2. Linear Dipole Component

The linear dipole component is given by the second and third terms of (4D.3). For computational convenience we break the expression up into two terms, $(\Delta E_C^d)_1$ and $(\Delta E_C^d)_2$.

Table VII. Spin averages for two photon crossed diagram.

Spin average	Coefficient of $\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle$
$\tilde{A} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_j^{(1)} \alpha_i^{(2)} \rangle$	+ 2
$\tilde{B} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(1)} \underline{k} \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(1)} \underline{p} \rangle$	$-\frac{2}{3} \underline{k} \cdot \underline{p}$
$\tilde{B}' \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{p} \rangle$	$-\frac{2}{3} \underline{k}' \cdot \underline{p}$
$\tilde{C} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(1)} \underline{k} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(1)} \underline{p} \alpha_i^{(2)} \underline{p} \rangle$	$O(k^4)$
$\tilde{D} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{k} \rangle$	$-\frac{2}{3} \underline{k}' \cdot \underline{k}$
$\tilde{F} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{k} \alpha_i^{(2)} \underline{p} \rangle$	$O(k^2)$
$\tilde{G} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \rangle$	$\frac{6}{3} k'^2$
$\tilde{I} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(1)} \underline{k} \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{k} \alpha_i^{(1)} \underline{p} \alpha_i^{(2)} \underline{p} \rangle$	$O(k^4)$
$\tilde{M} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(1)} \underline{k} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{k} \alpha_i^{(1)} \underline{p} \rangle$	$O(k^4)$
$\tilde{N} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_i^{(1)} \underline{k} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(1)} \underline{p} \rangle$	$O(k^4)$
$\tilde{O} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_i^{(1)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{k} \alpha_i^{(2)} \underline{p} \rangle$	$O(k^4)$
$\tilde{P} \equiv \langle \alpha_i^{(1)} \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_i^{(1)} \underline{k} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(2)} \alpha_i^{(2)} \underline{k} \alpha_i^{(1)} \underline{p} \alpha_i^{(2)} \underline{p} \rangle$	$O(k^6)$
$\tilde{a} \equiv \langle \alpha_j^{(2)} \alpha_j^{(1)} \alpha_i^{(1)} \underline{p} \alpha_i^{(2)} \underline{p} \rangle$	$-\frac{2}{3} p^2$
$\tilde{B} \equiv \langle \alpha_j^{(2)} \alpha_i^{(1)} \underline{k} \alpha_j^{(1)} \alpha_i^{(2)} \underline{p} \rangle$	$+\frac{2}{3} \underline{k} \cdot \underline{p}$
$\tilde{B} \equiv \langle \alpha_i^{(1)} \alpha_i^{(1)} \underline{k} \alpha_i^{(2)} \alpha_i^{(2)} \underline{p} \rangle$	$-\frac{2}{3} \underline{k} \cdot \underline{p}$
$\tilde{D} \equiv \langle \alpha_j^{(2)} \alpha_i^{(2)} \underline{k}' \alpha_j^{(1)} \alpha_i^{(1)} \underline{p} \rangle$	$-\frac{2}{3} \underline{k}' \cdot \underline{p}$
$\tilde{D} \equiv \langle \alpha_i^{(1)} \alpha_i^{(2)} \underline{k}' \alpha_i^{(2)} \alpha_i^{(1)} \underline{p} \rangle$	$+\frac{2}{3} \underline{k}' \cdot \underline{p}$
$\tilde{D}' \equiv \langle \alpha_i^{(1)} \alpha_i^{(1)} \underline{k}' \alpha_i^{(2)} \alpha_i^{(1)} \underline{p} \rangle$	$-\frac{2}{3} \underline{k}' \cdot \underline{p}$

Table VIII. k_0 integrations for crossed diagram

$$\tilde{I}_{\pm\pm}^{(n)} = \int_{-\infty}^{\infty} dk_0 \frac{k_0^{(n)}}{k_0^2 - |k|^2} \frac{1}{k_0^2 - |k'|^2} \frac{1}{[k_0 - (\eta_1 k_0 \mp E_1 \pm i\epsilon)]} \frac{1}{[k_0 - (\eta_2 k_0 \mp E_2 \pm i\epsilon)]}$$

$$\tilde{I}_{++} \rightarrow 2\pi i \left\{ \frac{|k'|^2 + |k'| |k| + |k|^2}{2|k'|^3 |k|^3 (|k'| + |k|)} - \frac{1}{4\mu |k'|^2 |k|^2} \right\}$$

$$\tilde{I}_{+-} \rightarrow 2\pi i \left\{ \frac{-1}{4m_2 |k'|^2 |k|^2} \right\}$$

$$\tilde{I}_{-+} \rightarrow 2\pi i \left\{ \frac{-1}{4m_1 |k'|^2 |k|^2} \right\}$$

$$\tilde{I}_{--} \rightarrow 2\pi i \left\{ \frac{1}{8m_1 m_2 |k'| |k| (|k'| + |k|)} \right\}$$

$$\tilde{I}'_{++} \rightarrow 2\pi i \left\{ \frac{1}{2 |k'|^2 |k|^2} \right\}$$

$$\tilde{I}'_{+-} \rightarrow 2\pi i \left\{ \frac{-1}{4m_2 |k'| |k| (|k'| + |k|)} \right\}$$

$$\tilde{I}'_{-+} \rightarrow 2\pi i \left\{ \frac{-1}{4m_1 |k'| |k| (|k'| + |k|)} \right\}$$

$$\tilde{I}'_{--} \rightarrow 2\pi i \left\{ \frac{-1}{16m_1^2 m_2 (|k'| + |k|)} + \frac{-1}{16m_1 m_2^2 (|k'| + |k|)} \right\}$$

$$\tilde{I}''_{++} \rightarrow 2\pi i \left\{ \frac{1}{2 |k'| |k| (|k'| + |k|)} - \frac{1}{4} \left(\frac{1}{m_1 k'^2} + \frac{1}{m_2 k^2} \right) \right\}$$

$$\tilde{I}''_{+-} \rightarrow 2\pi i \left\{ \frac{-1}{8m_2^2 (|k'| + |k|)} \right\}$$

$$\tilde{I}''_{-+} \rightarrow 2\pi i \left\{ \frac{-1}{8m_1^2 (|k'| + |k|)} \right\}$$

$$\tilde{I}''_{--} \rightarrow 2\pi i \left\{ \mathcal{O}(k^0) \right\}$$

$$\begin{aligned}
\langle E_c^d \rangle_0 &= \frac{2i\alpha^2}{\pi m_2 (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k' \delta(k'-k-p) \delta(k_0-k'_0)}{k^2 k'^2} \overline{\varphi_p^{(0)}} F_2(k') \\
&\quad \times \left\{ \gamma_\mu^{(1)} \gamma_\nu^{(2)} S_F^{(1)}(-k+q, K) S_F^{(2)}(-k'+q, K) \gamma_\nu^{(1)} \gamma_\mu^{(2)} k_\alpha \right\} \varphi_{K^c}^{(1)}(p)(2\pi)^{3/2}
\end{aligned}
\tag{4D.12}$$

Using (3.4) and (4C.19), we can write the matrix portion of (4D.12) as

$$\begin{aligned}
&\left\langle [] \alpha^{(1)} k \{w.f.\} - \alpha_j^{(1)} [] \alpha_j^{(1)} \alpha^{(2)} k \{w.f.\} - \alpha_i^{(1)} [] \alpha_i^{(1)} (k_0 + \alpha^{(2)} k) \{w.f.\} \right. \\
&\quad \left. + \alpha^{(1)} k [] \{w.f.\} + \alpha_i^{(1)} \alpha_j^{(1)} [] \alpha_j^{(1)} \alpha_i^{(1)} (k_0 + \alpha^{(2)} k) \{w.f.\} - \alpha^{(1)} k \alpha_j^{(1)} [] \alpha_j^{(1)} \{w.f.\} \right\rangle
\end{aligned}
\tag{4D.13}$$

where $\{w.f.\} = \{D(p) + F(p) \alpha^{(1)} p + \overline{F}(p) \alpha^{(2)} p + G(p) \alpha^{(1)} p \alpha^{(2)} p\}$

We use (4D.6) through (4D.9) where appropriate and do the trivial k'_0 integration as well as the k_0 integration (see Table VIII). We then extract the spin-spin contribution. $[] \alpha^{(2)} k \{w.f.\}$ term: the spin average are of dimension [momentum]⁴ while the most singular k_0 integration, I_{++} , is of dimension [momentum]⁻⁵, so the integrand will not be singular enough for a contribution to our order.

The $-\alpha_j^{(1)} [] \alpha_j^{(1)} \alpha^{(2)} k \{w.f.\}$ term:

$$\begin{aligned}
& \frac{-1}{E_1 E_2'} \left\{ \left\{ (E_1 + m_1)(E_2' - m_2) F(p) \tilde{E} - (E_2' - m_2) D(p) \tilde{F} - (E_1 + m_1) G(p) \tilde{Y} + \bar{F}(p) \tilde{H} \right\} \tilde{I}_{++} \right. \\
& + \left\{ (E_1 + m_1)(E_2' + m_2) F(p) \tilde{E} - (E_2' + m_2) D(p) \tilde{F} + (E_1 + m_1) G(p) \tilde{Y} - \bar{F}(p) \tilde{H} \right\} \tilde{I}_{+-} \\
& + \left\{ (E_1 - m_1)(E_2' - m_2) F(p) \tilde{E} + (E_2' - m_2) D(p) \tilde{F} - (E_1 - m_1) G(p) \tilde{Y} - \bar{F}(p) \tilde{H} \right\} \tilde{I}_{-+} \\
& \left. + \left\{ (E_1 - m_1)(E_2' + m_2) F(p) \tilde{E} + (E_2' + m_2) D(p) \tilde{F} + (E_1 - m_1) G(p) \tilde{Y} + \bar{F}(p) \tilde{H} \right\} \tilde{I}_{--} \right\}
\end{aligned}$$

and the $-\alpha_i^{(1)} [\] \alpha_i^{(2)} (k_0 + \underline{q} \cdot \underline{k}) \{w.f.\}$ term:

$$\begin{aligned}
& \frac{-1}{4E_1 E_2'} \left\{ \left\{ (E_1 - m_1)(E_2' + m_2) G(p) \tilde{A} - (E_2' + m_2) \bar{F}(p) \tilde{B} - (E_1 - m_1) F(p) \tilde{C} + D(p) \tilde{D} \right\} \tilde{I}_{++}' \right. \\
& + \left\{ (E_1 - m_1)(E_2' - m_2) G(p) \tilde{A} - (E_2' - m_2) \bar{F}(p) \tilde{B} + (E_1 - m_1) F(p) \tilde{C} - D(p) \tilde{D} \right\} \tilde{I}_{+-}' \\
& + \left\{ (E_1 + m_1)(E_2' + m_2) G(p) \tilde{A} + (E_2' + m_2) \bar{F}(p) \tilde{B} - (E_1 + m_1) F(p) \tilde{C} - D(p) \tilde{D} \right\} \tilde{I}_{-+}' \\
& + \left\{ (E_1 + m_1)(E_2' - m_2) G(p) \tilde{A} + (E_2' - m_2) \bar{F}(p) \tilde{B} + (E_1 + m_1) F(p) \tilde{C} + D(p) \tilde{D} \right\} \tilde{I}_{--}' \\
& + \left\{ (E_1 - m_1)(E_2' + m_2) F(p) \tilde{E} - (E_2' + m_2) D(p) \tilde{F} - (E_1 - m_1) G(p) \tilde{Y} + \bar{F}(p) \tilde{H} \right\} \tilde{I}_{++}' \\
& + \left\{ (E_1 - m_1)(E_2' - m_2) F(p) \tilde{E} - (E_2' - m_2) D(p) \tilde{F} + (E_1 - m_1) G(p) \tilde{Y} - \bar{F}(p) \tilde{H} \right\} \tilde{I}_{+-}' \\
& + \left\{ (E_1 + m_1)(E_2' + m_2) F(p) \tilde{E} + (E_2' + m_2) D(p) \tilde{F} - (E_1 + m_1) G(p) \tilde{Y} - \bar{F}(p) \tilde{H} \right\} \tilde{I}_{-+}' \\
& \left. + \left\{ (E_1 + m_1)(E_2' - m_2) F(p) \tilde{E} + (E_2' - m_2) D(p) \tilde{F} + (E_1 + m_1) G(p) \tilde{Y} + \bar{F}(p) \tilde{H} \right\} \tilde{I}_{--}' \right\}
\end{aligned}$$

and the $\alpha^{(1)} \cdot k [\] \{w.f.\}$ term: Again the spin averages are of dimension [momentum]⁴ and the most singular of the k_0

integrations is of dimension [momentum]⁻⁵, so there will be no contribution to our order.

The $\alpha_i^{(2)} \alpha_j^{(2)} [] \alpha_j^{(1)} \{ \omega.f. \}$ term yields:

$$\begin{aligned} & \frac{-1}{4E_1 E_2'} \left\{ \left\{ (E_1 - m_1)(E_2' - m_2) \bar{F}(p) \tilde{B} - (E_2' - m_2) G(p) \tilde{M} - (E_1 - m_1) D(p) \tilde{D} + F(p) \tilde{N} \right\} \tilde{I}_{++} \right. \\ & + \left\{ (E_1 - m_1)(E_2' + m_2) \bar{F}(p) \tilde{B} - (E_2' + m_2) G(p) \tilde{M} + (E_1 - m_1) D(p) \tilde{D} - F(p) \tilde{N} \right\} \tilde{I}_{+-} \\ & + \left\{ (E_1 + m_1)(E_2' - m_2) \bar{F}(p) \tilde{B} + (E_2' - m_2) G(p) \tilde{M} - (E_1 + m_1) D(p) \tilde{D} - F(p) \tilde{N} \right\} \tilde{I}_{-+} \\ & \left. + \left\{ (E_1 + m_1)(E_2' + m_2) \bar{F}(p) \tilde{B} + (E_2' + m_2) G(p) \tilde{M} + (E_1 + m_1) D(p) \tilde{D} + F(p) \tilde{N} \right\} \tilde{I}_{--} \right\} \end{aligned}$$

and finally the $\alpha_i^{(2)} \alpha_j^{(2)} [] \alpha_j^{(1)} \alpha_i^{(1)} (k_0 + \alpha_i^{(1)} k) \{ \omega.f. \}$ term

$$\begin{aligned} & \frac{1}{4E_1 E_2'} \left\{ \left\{ (E_1 - m_1)(E_2' - m_2) D(p) \tilde{A} - (E_2' - m_2) F(p) \tilde{B} - (E_1 - m_1) \bar{F}(p) \tilde{B}' + G(p) \tilde{C} \right\} \tilde{I}_{++}' \right. \\ & + \left\{ (E_1 - m_1)(E_2' + m_2) D(p) \tilde{A} - (E_2' + m_2) F(p) \tilde{B} + (E_1 - m_1) \bar{F}(p) \tilde{B}' - G(p) \tilde{C} \right\} \tilde{I}_{+-}' \\ & + \left\{ (E_1 + m_1)(E_2' - m_2) D(p) \tilde{A} + (E_2' - m_2) F(p) \tilde{B} - (E_1 + m_1) \bar{F}(p) \tilde{B}' - G(p) \tilde{C} \right\} \tilde{I}_{-+}' \\ & + \left\{ (E_1 + m_1)(E_2' + m_2) D(p) \tilde{A} + (E_2' + m_2) F(p) \tilde{B} + (E_1 + m_1) \bar{F}(p) \tilde{B}' + G(p) \tilde{C} \right\} \tilde{I}_{--}' \\ & + \left\{ (E_1 - m_1)(E_2' - m_2) \bar{F}(p) \tilde{F} - (E_2' - m_2) G(p) \tilde{L} - (E_1 - m_1) D(p) \tilde{D} + F(p) \tilde{M} \right\} \tilde{I}_{++} \\ & + \left\{ (E_1 - m_1)(E_2' + m_2) \bar{F}(p) \tilde{F} - (E_2' + m_2) G(p) \tilde{L} + (E_1 - m_1) D(p) \tilde{D} - F(p) \tilde{M} \right\} \tilde{I}_{+-} \\ & + \left\{ (E_1 + m_1)(E_2' - m_2) \bar{F}(p) \tilde{F} + (E_2' - m_2) G(p) \tilde{L} - (E_1 + m_1) D(p) \tilde{D} - F(p) \tilde{M} \right\} \tilde{I}_{-+} \\ & \left. + \left\{ (E_1 + m_1)(E_2' + m_2) \bar{F}(p) \tilde{F} + (E_2' + m_2) G(p) \tilde{L} + (E_1 + m_1) D(p) \tilde{D} + F(p) \tilde{M} \right\} \tilde{I}_{--} \right\} \end{aligned}$$

Collecting the terms with a potential contribution to order $\alpha^6 \ln \alpha^{-1}$, we have

$$\begin{aligned}
 (\Delta E_c^d)_0 = & \frac{i \alpha^3 |\varphi(0)|^2}{2m_2 M (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k'}{(\rho^2 + \gamma^2)^2} \frac{\delta(k' - k - p)}{E_1(k) E_2(k')} \left\{ (E_2' + m_2) D(p) \tilde{F} \tilde{I}_{++} \right. \\
 & - [(E_1 + m_1)(E_2' + m_2) F(p) \tilde{E} - (E_2' + m_2) D(p) \tilde{F}] \tilde{I}_{+-} \\
 & - [(E_1 + m_1)(E_2' + m_2) F(p) \tilde{E} + (E_2' + m_2) D(p) \tilde{F} - (E_1 + m_1) D(p) \tilde{D} - (E_1 + m_1) D(p) \tilde{D}] \tilde{I}_{-+} \\
 & \left. - [D(p) \tilde{D} - (E_2' + m_2) F(p) \tilde{B}] \tilde{I}_{++}' \right\}
 \end{aligned} \tag{4D.14}$$

The integrand of dimension [momentum] $^{-3}$ (the last $\tilde{I}_{++}$ term in (4C.14)) yields zero. The remaining terms in (4C.14) are readily evaluated using Table VI:

$$(\Delta E_c^d)_0 = -K \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left(\frac{5}{3} \frac{M}{m_2} + \frac{1}{3} \frac{M}{m_1} \right) \tag{4D.15}$$

We now turn to $(\Delta E_c^d)_2$ which is given by

$$\begin{aligned}
 (\Delta E_c^d)_2 = & -\frac{2i \alpha^2}{\pi m_2 (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k'}{k^2 k'^2} \frac{\delta(k' - k - p) \delta(k_0 - k_0')}{\varphi_p^{(0)} F_2(k'^2)} \\
 & \times \left\{ \gamma_\mu^{(1)} \sigma_{\gamma\beta}^{(2)} k'_\beta S_F^{(1)}(-k + \eta, k) S_F^{(2)}(-k' + \eta_2, k) \delta_\nu^{(1)} \gamma_\mu^{(2)} \right\} \varphi_p^{(1)} (2\pi)^{3/2}
 \end{aligned} \tag{4D.16}$$

Using the same method outlined before (4D.13) we can write the matrix portion of (4D.16) as

$$\begin{aligned}
& \frac{-1}{4E_1 E_2'} \left\{ \left\{ \underbrace{(E_1 + m_1)(E_2' + m_2) F(p) \tilde{\mathcal{E}}}_{\text{~~~~~}} - \underbrace{(E_2' + m_2) D(p) \tilde{\mathcal{D}}}_{\text{~~~~~}} + (E_1 + m_1) G(p) \tilde{\mathcal{K}} - \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{++} \right. \\
& \quad + \left\{ (E_1 + m_1)(E_2' - m_2) F(p) \tilde{\mathcal{E}} - (E_2' - m_2) D(p) \tilde{\mathcal{D}} - (E_1 + m_1) G(p) \tilde{\mathcal{K}} + \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{+-} \\
& \quad + \left\{ (E_1 - m_1)(E_2' + m_2) F(p) \tilde{\mathcal{E}} + \underbrace{(E_2' + m_2) D(p) \tilde{\mathcal{D}}}_{\text{~~~~~}} + (E_1 - m_1) G(p) \tilde{\mathcal{K}} + \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{-+} \\
& \quad \left. + \left\{ (E_1 - m_1)(E_2' - m_2) F(p) \tilde{\mathcal{E}} + (E_2' - m_2) D(p) \tilde{\mathcal{D}} - (E_1 - m_1) G(p) \tilde{\mathcal{K}} - \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{--} \right\}
\end{aligned}$$

and the $\alpha_i^{(1)} \alpha_i^{(2)} \underline{k}' [] \alpha_i^{(2)} \{ \omega. f. \}$ term

$$\begin{aligned}
& \frac{1}{4E_1 E_2'} \left\{ \left\{ (E_1 - m_1)(E_2' - m_2) F(p) \tilde{\mathcal{E}} - (E_2' - m_2) D(p) \tilde{\mathcal{D}} + (E_1 - m_1) G(p) \tilde{\mathcal{K}} - \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{++} \right. \\
& \quad + \left\{ (E_1 - m_1)(E_2' + m_2) F(p) \tilde{\mathcal{E}} - (E_2' + m_2) D(p) \tilde{\mathcal{D}} - (E_1 - m_1) G(p) \tilde{\mathcal{K}} + \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{+-} \\
& \quad + \left\{ (E_1 + m_1)(E_2' - m_2) F(p) \tilde{\mathcal{E}} + (E_2' - m_2) D(p) \tilde{\mathcal{D}} + (E_1 + m_1) G(p) \tilde{\mathcal{K}} + \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{-+} \\
& \quad \left. + \left\{ (E_1 + m_1)(E_2' + m_2) F(p) \tilde{\mathcal{E}} + (E_2' + m_2) D(p) \tilde{\mathcal{D}} - (E_1 + m_1) G(p) \tilde{\mathcal{K}} - \bar{F}(p) \tilde{\mathcal{L}} \right\} \tilde{\mathcal{I}}_{--} \right\}
\end{aligned}$$

and the $-\alpha_i^{(1)} [] \alpha_i^{(1)} \underline{k}' \alpha_i^{(2)} \{ \omega. f. \}$ term

$$\begin{aligned}
& \frac{-1}{4E_1 E_2'} \left\{ \left\{ - (E_1 - m_1)(E_2' + m_2) \bar{F}(p) \tilde{\mathcal{E}}' + (E_2' + m_2) G(p) \tilde{\mathcal{H}}' - (E_1 - m_1) D(p) \tilde{\mathcal{D}}' + F(p) \tilde{\mathcal{L}}' \right\} \tilde{\mathcal{I}}_{++} \right. \\
& \quad + \left\{ - (E_1 - m_1)(E_2' - m_2) \bar{F}(p) \tilde{\mathcal{E}}' + (E_2' - m_2) G(p) \tilde{\mathcal{H}}' + (E_1 - m_1) D(p) \tilde{\mathcal{D}}' - F(p) \tilde{\mathcal{L}}' \right\} \tilde{\mathcal{I}}_{+-} \\
& \quad + \left\{ - \underbrace{(E_1 + m_1)(E_2' + m_2) \bar{F}(p) \tilde{\mathcal{E}}'}_{\text{~~~~~}} - (E_2' + m_2) G(p) \tilde{\mathcal{H}}' - \underbrace{(E_1 + m_1) D(p) \tilde{\mathcal{D}}'}_{\text{~~~~~}} - F(p) \tilde{\mathcal{L}}' \right\} \tilde{\mathcal{I}}_{-+} \\
& \quad \left. + \left\{ - (E_1 + m_1)(E_2' - m_2) \bar{F}(p) \tilde{\mathcal{E}}' - (E_2' - m_2) G(p) \tilde{\mathcal{H}}' + (E_1 + m_1) D(p) \tilde{\mathcal{D}}' + F(p) \tilde{\mathcal{L}}' \right\} \tilde{\mathcal{I}}_{--} \right\}
\end{aligned}$$

and finally the $-\alpha_i'' d_j^{(u)}(k_0 - \alpha_j^{(u)} k') [] d_j^{(u)} d_i^{(u)} \{ \omega. f. \}$ term:

$$\begin{aligned}
& \frac{1}{4E_1 E_2'} \left\{ - \left\{ (E_1 - m_1)(E_2' - m_2) D(p) \tilde{A} - (E_2' - m_2) F(p) \tilde{B} + (E_1 - m_1) \bar{F}(p) \tilde{B}' - G(p) \tilde{C} \right\} \tilde{I}_{++}' \right. \\
& \quad - \left\{ (E_1 - m_1)(E_2' + m_2) D(p) \tilde{A} - (E_2' + m_2) F(p) \tilde{B} - (E_1 - m_1) \bar{F}(p) \tilde{B}' + G(p) \tilde{C} \right\} \tilde{I}_{+-}' \\
& \quad - \left\{ (E_1 + m_1)(E_2' - m_2) D(p) \tilde{A} + (E_2' - m_2) F(p) \tilde{B} + (E_1 + m_1) \bar{F}(p) \tilde{B}' + G(p) \tilde{C} \right\} \tilde{I}_{-+}' \\
& \quad - \left\{ (E_1 + m_1)(E_2' + m_2) D(p) \tilde{A} + (E_2' + m_2) F(p) \tilde{B} - (E_1 + m_1) \bar{F}(p) \tilde{B}' - G(p) \tilde{C} \right\} \tilde{I}_{--}' \\
& \quad + \left\{ - (E_1 - m_1)(E_2' + m_2) \bar{F}(p) \tilde{B}' + (E_2' + m_2) G(p) \tilde{C} - (E_1 - m_1) D(p) \tilde{G} + F(p) \tilde{N} \right\} \tilde{I}_{++}' \\
& \quad + \left\{ - (E_1 - m_1)(E_2' - m_2) \bar{F}(p) \tilde{B}' + (E_2' - m_2) G(p) \tilde{C} + (E_1 - m_1) D(p) \tilde{G} - F(p) \tilde{N} \right\} \tilde{I}_{+-}' \\
& \quad + \left\{ - (E_1 + m_1)(E_2' + m_2) \bar{F}(p) \tilde{B}' - (E_2' + m_2) G(p) \tilde{C} - (E_1 + m_1) D(p) \tilde{G} - F(p) \tilde{N} \right\} \tilde{I}_{-+}' \\
& \quad + \left\{ - (E_1 + m_1)(E_2' - m_2) \bar{F}(p) \tilde{B}' - (E_2' - m_2) G(p) \tilde{C} + (E_1 + m_1) D(p) \tilde{G} + F(p) \tilde{N} \right\} \tilde{I}_{--}' \Big\}
\end{aligned}$$

Collecting all terms of potential contribution to our order,

we have:

$$\begin{aligned}
(\Delta E_c^d)_{(2)} &= \frac{-i \alpha^2 |\varphi(0)|^2}{2m_2 M (2\pi)^4} \int \frac{d^3 p d^3 k d^3 k'}{(p^2 + \gamma^2)^2} \frac{d^4(k' - k - p)}{E_1(k) E_2(k')} \left\{ [D(p) \tilde{\mathcal{D}} - (E_1 + m_1) F(p) \tilde{\mathcal{D}}] \tilde{I}_{++}' \right. \\
& \quad - (E_2' + m_2) D(p) \tilde{\mathcal{D}} \tilde{I}_{-+}' - (E_2' + m_2) D(p) \tilde{\mathcal{D}} \tilde{I}_{+-}' \\
& \quad + [(E_1 + m_1)(E_2' + m_2) \bar{F}(p) \tilde{\mathcal{D}}' + (E_1 + m_1) D(p) \tilde{\mathcal{D}}] \tilde{I}_{-+}' + (E_2' + m_2) D(p) \tilde{\mathcal{D}} \tilde{I}_{++}' \\
& \quad \left. - [(E_1 + m_1)(E_2' + m_2) \bar{F}(p) \tilde{B}' + (E_1 + m_1) D(p) \tilde{G}] \tilde{I}_{-+}' - (E_1 + m_1)(E_2' - m_2) F(p) \tilde{\mathcal{D}} \tilde{I}_{++}' \right\}
\end{aligned}$$

(4D.18)

The integrand of dimension [momentum]⁻³ (the $\tilde{I}_{++}'$ term above)

makes no contribution to order $\alpha^6 \ln \alpha^{-1}$, and the remaining terms are quickly calculated to be:

$$(\Delta E_c^d)_{(2)} = -\kappa \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left(\frac{1}{3} \frac{M}{m_2} + \frac{1}{3} \frac{M}{m_1} \right) \quad (4D.19)$$

Thus the total crossed diagram contribution to the linear dipole component is (see (4D.15) and (4D.19)):

$$\Delta E_c^d = -\kappa \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left(\frac{6}{3} \frac{M}{m_2} + \frac{2}{3} \frac{M}{m_1} \right) \quad (4D.20)$$

3. Quadratic Dipole Component

The quadratic dipole component of the crossed two photon diagram is given by the last term in (4D.3):

$$\Delta E_c^{dd} = -\frac{i \alpha^2}{\pi m_e^2 (2\pi)^4} \int \frac{d^3 p d^4 k d^4 k'}{k^2 k'^2} \frac{d(k' - k - p) d(k_0' - k_0)}{\varphi_p^{(0)} F_2(k') F_2(k')} \\ \times \left\{ \gamma_\mu^{(1)} \gamma_\nu^{(2)} k'_\beta S_F^{(1)}(k + \eta, \kappa) S_F^{(2)}(-k' + \eta, \kappa) \gamma_\nu^{(1)} \gamma_\mu^{(2)} k_\alpha \right\} \varphi^{(1)}(p) (2\pi)^{3/2} \quad (4D.21)$$

Using now familiar techniques we write the matrix portion of (4D.21) as:

$$\begin{aligned}
& \left\langle \left[-\underline{\alpha}^{(1)} \cdot \underline{k}' [] \underline{\alpha}^{(2)} \cdot \underline{k} - \underline{\alpha}^{(1)} \cdot \underline{k} [] \underline{\alpha}^{(2)} \cdot \underline{k}' - \alpha_f^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \alpha_f^{(1)} \alpha^{(2)} \cdot \underline{k}' \right. \right. \\
& + \alpha_i^{(1)} \underline{\alpha}^{(2)} \cdot \underline{k}' [] \alpha_i^{(2)} \alpha^{(2)} \cdot \underline{k} - \underline{\alpha}^{(1)} \cdot \underline{k} \alpha_f^{(2)} \alpha^{(2)} \cdot \underline{k}' [] \alpha_f^{(1)} - \alpha_i^{(1)} [] \underline{\alpha}^{(1)} \cdot \underline{k} \alpha_i^{(2)} \alpha^{(2)} \cdot \underline{k} \\
& - \underline{\alpha}^{(1)} \cdot \underline{k} \alpha^{(2)} \cdot \underline{k}' [] - [] \underline{\alpha}^{(1)} \cdot \underline{k}' \alpha^{(2)} \cdot \underline{k} + k_0 \alpha_f^{(2)} [] \alpha_f^{(1)} \alpha^{(2)} \cdot \underline{k} - k_0 \alpha_i^{(1)} [] \underline{\alpha}^{(1)} \cdot \underline{k}' \alpha_i^{(2)} \\
& + k_0 \alpha_i^{(1)} \alpha^{(2)} \cdot \underline{k}' [] \alpha_i^{(2)} + k_0 \underline{\alpha}^{(1)} \cdot \underline{k} \alpha_f^{(2)} [] \alpha_f^{(1)} - k_0^2 \alpha_i^{(1)} \alpha_f^{(2)} [] \alpha_f^{(1)} \alpha_i^{(2)} \\
& - k_0 \alpha_i^{(1)} \alpha_f^{(2)} [] \alpha_f^{(1)} \alpha_i^{(2)} \alpha^{(2)} \cdot \underline{k} + k_0 \alpha_i^{(1)} \alpha_f^{(2)} \alpha^{(2)} \cdot \underline{k}' [] \alpha_f^{(1)} \alpha_i^{(2)} \\
& \left. \left. + \alpha_i^{(1)} \alpha_f^{(2)} \alpha^{(2)} \cdot \underline{k}' [] \alpha_f^{(1)} \alpha_i^{(2)} \alpha^{(2)} \cdot \underline{k} \right] \{w.f.\} \right\rangle
\end{aligned}$$

(4D.22)

where $\{w.f.\} = \{D\varphi + F(\varphi) \underline{\alpha}^{(1)} \cdot \underline{p} + \overline{F}(\varphi) \underline{\alpha}^{(2)} \cdot \underline{p} + G(\varphi) \underline{\alpha}^{(1)} \cdot \underline{p} \underline{\alpha}^{(2)} \cdot \underline{p}\}$

The spin-spin contribution is extracted using identical techniques to those discussed earlier. In all of the following we need consider only spin averages of dimension $[\text{momentum}]^2$ or less as the k_0 integrations have dimension $[\text{momentum}]^{-5}$, $[\text{momentum}]^{-4}$, ..., and we want an integrand of dimension $[\text{momentum}]^{-2}$.

The $-\underline{\alpha}^{(2)} \cdot \underline{k}' [] \underline{\alpha}^{(2)} \cdot \underline{k} \{w.f.\}$ and $-\underline{\alpha}^{(1)} \cdot \underline{k} [] \underline{\alpha}^{(2)} \cdot \underline{k}' \{w.f.\}$ terms: No contribution from these terms as spin averages are at least of dimension $[\text{momentum}]^4$.

The $-\alpha_f^{(2)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \alpha_f^{(1)} \alpha^{(2)} \cdot \underline{k} \{w.f.\}$ and $+\alpha_i^{(1)} \underline{\alpha}^{(2)} \cdot \underline{k} [] \alpha_i^{(2)} \alpha^{(2)} \cdot \underline{k} \{w.f.\}$ terms: All spin averages are of too high a dimension to contribute to our order.

The $-\alpha_i^{(1)} \underline{k} \alpha_j^{(2)} \underline{k}' [] \alpha_j^{(1)} \{ \omega, f. \}$ term: we consider only spin averages of dimension [momentum]²:

$$\frac{-1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2' + m_2) D(p) \tilde{D} \tilde{I}_{++} + (E_1 - m_1)(E_2' - m_2) D(p) \tilde{D} \tilde{I}_{+-} \right. \\ \left. + (E_1 + m_1)(E_2' + m_2) D(p) \tilde{D} \tilde{I}_{-+} + (E_1 + m_1)(E_2' - m_2) D(p) \tilde{D} \tilde{I}_{--} \right\}$$

and the $-\alpha_i^{(1)} [] \alpha_i^{(1)} \underline{k} \alpha_j^{(2)} \underline{k}' \{ \omega, f. \}$ term:

$$\frac{-1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2' + m_2) D(p) \tilde{D}' \tilde{I}_{++} + (E_1 - m_1)(E_2' - m_2) D(p) \tilde{D}' \tilde{I}_{+-} \right. \\ \left. + (E_1 + m_1)(E_2' + m_2) D(p) \tilde{D}' \tilde{I}_{-+} + (E_1 + m_1)(E_2' - m_2) D(p) \tilde{D}' \tilde{I}_{--} \right\}$$

and the $\alpha_i^{(1)} \underline{k} \alpha_j^{(2)} \underline{k}' [] \{ \omega, f. \}$ and $\alpha_i^{(1)} \underline{k} \alpha_j^{(2)} \underline{k}' [] \{ \omega, f. \}$ terms:

There is no $\alpha^6 \ln \alpha^{-1}$ contribution because spin averages are of too high a dimension.

The $k_0 \alpha_j^{(2)} [] \alpha_j^{(1)} \alpha_i^{(1)} \underline{k} \{ \omega, f. \}$ term: Spin averages are of at least dimension [momentum]², so need to examine I_{++}' only:

$$\frac{1}{4E_1 E_2} \left\{ (E_1 + m_1)(E_2' - m_2) F(p) \tilde{E} - (E_2' - m_2) D(p) \tilde{F} - (E_1 + m_1) G(p) \tilde{H} + F(p) \tilde{I} \right\} \tilde{I}_{++}'$$

The $-k_0 \alpha_i^{(1)} [] \alpha_i^{(1)} \underline{k}' \alpha_j^{(2)} \{ \omega, f. \}$ term

$$\frac{-1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2' + m_2) F(p) \tilde{E}' - (E_2' + m_2) G(p) \tilde{F}' - (E_1 - m_1) D(p) \tilde{H}' + F(p) \tilde{I}'' \right\} \tilde{I}_{++}'$$

The $k_0 \alpha_i^{(1)} \alpha_j^{(2)} \underline{k}' [] \alpha_i^{(1)} \{ \omega, f. \}$ term

$$\frac{1}{4E_1 E_2} \left\{ (E_1 - m_1)(E_2' - m_2) F(p) \tilde{E} - (E_2' - m_2) D(p) \tilde{F} - (E_1 - m_1) G(p) \tilde{H} + F(p) \tilde{I} \right\} \tilde{I}_{++}'$$

The $k_0 \alpha_i^{(1)} \underline{k} \alpha_j^{(2)} [] \alpha_j^{(1)} \{ \omega, f. \}$ term

$$\frac{1}{4E_1E_2} \left\{ (E_1 - m_1)(E_2' - m_2) \bar{F}(q) \tilde{B} - (E_2' - m_2) G(q) \tilde{M} - (E_1 - m_1) D(q) \tilde{D} + F(q) \tilde{L}' \right\} \tilde{I}_{++}'$$

The $-k_0 \alpha_i'' \alpha_j^{(2)} [] \alpha_j'' \alpha_i^{(2)} \{ \omega.f. \}$ term: Since I_{++}'' is of dimension [momentum]⁻³ the only spin average which could possibly contribute is $\tilde{A}$:

$$\frac{-\tilde{A}}{4E_1E_2} \left\{ (E_1 - m_1)(E_2' - m_2) \tilde{I}_{++}'' + (E_1 - m_1)(E_2' + m_2) \tilde{I}_{+-}'' + (E_1 + m_1)(E_2' - m_2) \tilde{I}_{-+}'' + (E_1 + m_1)(E_2' + m_2) \tilde{I}_{--}'' \right\}$$

The $-k_0 \alpha_i'' \alpha_j^{(2)} [] \alpha_j'' \alpha_i^{(2)} \{ \omega.f. \}$ term:

$$\frac{-1}{4E_1E_2} \left\{ (E_1 - m_1)(E_2' - m_2) \bar{F}(q) \tilde{F} - (E_2' - m_2) G(q) \tilde{L} - (E_1 - m_1) D(q) \tilde{D} + F(q) \tilde{M} \right\} \tilde{I}_{++}'$$

The $k_0 \alpha_i'' \alpha_j^{(2)} \alpha_i^{(2)} k' [] \alpha_j'' \alpha_i^{(2)} \{ \omega.f. \}$ term:

$$\frac{1}{4E_1E_2} \left\{ (E_1 - m_1)(E_2' + m_2) \bar{F}(q) \tilde{B}' - (E_2' + m_2) G(q) \tilde{C} - (E_1 - m_1) D(q) \tilde{G} + F(q) \tilde{N} \right\} \tilde{I}_{++}'$$

Finally the $\alpha_i'' \alpha_j^{(2)} \alpha_i^{(2)} k' [] \alpha_j'' \alpha_i^{(2)} \{ \omega.f. \}$ term:

$$\frac{1}{4E_1E_2} \left\{ (E_1 - m_1)(E_2' + m_2) D(q) \tilde{D} \tilde{I}_{++} + (E_1 - m_1)(E_2' - m_2) D(q) \tilde{D} \tilde{I}_{+-} + (E_1 + m_1)(E_2' + m_2) D(q) \tilde{D} \tilde{I}_{-+} + (E_1 + m_1)(E_2' - m_2) D(q) \tilde{D} \tilde{I}_{--} \right\}$$

Collecting all the underlined terms which will potentially lead to an $\alpha^6 \ln \alpha^{-1}$ contribution, we have:

$$\Delta E_c^{dd} = \frac{-i \alpha^3 |\varphi(0)|^2}{4 m_2^2 M (2\pi)^4} \int \frac{d^3 p \, d^3 k \, d^3 k' \, \delta(\underline{k}' - \underline{k} - \underline{p})}{(p^2 + \gamma^2)^2 E_1(k) E_2(k')} \times$$

$$\times \left\{ (E_1 + m_1)(E_2' + m_2) D(p) [\tilde{D} - \tilde{D}' - \tilde{Z}'] \tilde{I}_{-+} \right\}$$
(4D.23)

Using Table VI we find:

$$\Delta E_c^{dd} = -K^2 \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2} \left(\frac{2}{3} \frac{M}{m_2} \right)$$
(4D.24)

Thus the total crossed diagram contribution is (see (4D.11), (4D.21), and (4D.24)):

$$\Delta E_c = -\eta \frac{M}{\mu} - K \eta \left(\frac{6}{3} \frac{M}{m_2} + \frac{2}{3} \frac{M}{m_1} \right) + K^2 \eta \left(\frac{2}{3} \frac{M}{m_2} \right)$$
(4D.25)

where $\eta \equiv \frac{\pi \alpha^3 |\varphi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2}$

E. Non-Contributing Diagrams

Radiative corrections to the one photon exchange represent a potential source of $\alpha^6 \ln \alpha^{-1}$ contributions, but detailed calculations⁽¹⁴⁾ have shown that in fact they do not contribute to our order.

Since terms involving three photons are at least of order α^6 , we conclude that the $\alpha^6 \ln \alpha^{-1}$ terms are those calculated in this Chapter and give by (4B.59), (4C.30), and (4D.25):

$$\Delta E = \eta \, 3 - \kappa \, \eta \left(\frac{13}{3} + \frac{7}{3} \frac{m_1}{m_2} \right) + \kappa^2 \eta \left(\frac{7}{3} - \frac{14}{3} \frac{m_1}{m_2} \right) \quad (4E.1)$$

where $\eta \equiv \frac{\pi \alpha^3 |\psi(0)|^2 \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \ln \alpha^{-1}}{M^2}$

V. SUMMARY AND CONCLUSIONS

The $(m_e/m_p)\alpha^2 \ln \alpha^{-1}$ recoil correction for each of the components is listed below, diagram by diagram. η represents the quantity $\pi \alpha^3 |\phi(0)|^2 \langle \sigma^1 \cdot \sigma^2 \rangle \ln \alpha^{-1} / M^2$ which can be readily factored (see (5.4)) into the Fermi frequency times a term with the "correct" mass dependence (i.e. doesn't diverge for $m_2 \rightarrow \infty$). It should be noted in Tables IX, X, XI, terms with the "wrong" mass dependence (e.g., $\eta(M/\mu)$) cancel out so that we are left with only terms with the "correct" mass dependence.

The total $\alpha^6 \ln \alpha^{-1}$ contribution is thus

$$\Delta E = \frac{\pi}{3} \left\{ 9 - K \left(13 + 7 \frac{m_1}{m_2} \right) + K^2 \left(7 - 14 \frac{m_1}{m_2} \right) \right\} \quad (5.1)$$

$$\text{where, again, } \eta = \pi \alpha^3 |\phi(0)|^2 \langle \sigma^1 \cdot \sigma^2 \rangle \ln \alpha^{-1} M^{-2} \quad (5.2)$$

Using $M = (m_1 + m_2)$, $|\phi(0)|^2 = \gamma^3 \pi^{-1}$, and $R_{\infty} = \alpha^2 m_1 / 2$ we can write (5.2) as

$$\eta = \frac{8 \left(\frac{m_1}{m_2} \right)^2 \alpha^4 \ln \alpha^{-1} R_{\infty}}{\left(1 + \frac{m_1}{m_2} \right)^5} \quad (5.3)$$

This can then be rewritten with the Fermi coefficient as a factor

$$\eta = \left\{ \frac{16}{3} \alpha^4 R_{\infty} \frac{\mu_P}{\mu_B} \left(1 + \frac{m_1}{m_2} \right)^{-3} \right\} \left(\frac{3}{2} \frac{m_1}{m_2} \left(\frac{1}{1+K} \right) \alpha^2 \ln \alpha^{-1} \left(1 + \frac{m_1}{m_2} \right)^{-2} \right) \quad (5.4)$$

Table IX. Charge component

One Photon Diagram	Two Photon Ladder Diagram	Two Photon Crossed Diagram
$\eta \frac{2}{3} \frac{M}{\mu}$	$\eta \left(3 + \frac{1}{3} \frac{M}{\mu} \right)$	$-\eta \frac{M}{\mu}$

Table X. Linear dipole component

One Photon Diagram	Two Photon Ladder Diagram	Two Photon Crossed Diagram
$\kappa \eta \left(\frac{2}{3} \frac{M}{\mu} + \frac{1}{3} - \frac{1}{3} \frac{m_1}{m_2} \right)$	$-\kappa \eta \left(\frac{10}{3} + \frac{2}{3} \frac{m_1}{m_2} \right)$	$-\kappa \eta \left(\frac{2}{3} \frac{M}{\mu} + \frac{4}{3} + \frac{4}{3} \frac{m_1}{m_2} \right)$

Table XI. Quadratic dipole component

Two Photon Ladder Diagram	Two Photon Crossed Diagram
$\kappa^2 \eta \left(3 - 4 \frac{m_1}{m_2} \right)$	$-\kappa^2 \eta \left(\frac{2}{3} + \frac{2}{3} \frac{m_1}{m_2} \right)$

where the later factor in parentheses has the value

1.070×10^{-4} MHz (where we have used $\alpha^{-1} = 137.03608$).

Thus the frequency shift corresponding to (5.1) is

$$\nu_H = 2.899 \times 10^{-4} \text{ MHz} \quad (0.2 \text{ ppm}). \quad (5.5)$$

It is interesting to note that, numerically, the linear dipole term and the quadratic dipole term almost exactly cancel each other (the linear dipole term being -8.324×10^{-4} MHz, and the quadratic dipole term being $+7.913 \times 10^{-4}$ MHz). The present theoretical value⁽¹⁵⁾, including this work, of the hydrogen ground state hyperfine splitting is

$$(\nu_H)_{th} = 1420.4026 (1 + \delta'_p) \pm 0.0057 \text{ MHz} \quad (4.0 \text{ ppm}) \quad (5.6)$$

where the quoted uncertainty is the root sum square of the following errors: 3.8 ppm in α_{WQED}^2 , 0.6 ppm due to the uncertainty for the uncalculated terms in ρ , and 0.9 ppm in δ_p ⁽¹⁵⁾. Hence the uncertainty in the determination of the fundamental constant, α , represents the bulk of the uncertainty in $(\nu_H)_{th}$.

The experimental value for the hydrogen hyperfine splitting of the ground state is⁽¹⁶⁾

$$(\nu_H)_{exp} = 1420.4057517864 (17) \text{ MHz} \quad (1.2/10^{12})$$

which is over six orders of magnitude more precise than the theoretical calculations.

The uncertainty in α does not affect the ratio of muonium to hydrogen hyperfine splittings (see (1.4)) as this ratio

is insensitive to the value of α :

$$\frac{\nu_\mu}{\nu_H} = \frac{\mu_\mu}{\mu_P} \frac{\left(1 + \frac{m_e}{m_P}\right)^3}{\left(1 + \frac{m_e}{m_\mu}\right)^3} \left\{ 1 + \delta_\mu - \delta_P - \delta_P' \right\} \quad (1.4)$$

The greatest difficulty in using (1.4) is the uncertainty involved in the determination of the ratio μ_μ/μ_P . At present there are two determinations of the ratio, a University of Washington/Lawrence Radiation Laboratory (UW/LRL) collaboration⁽¹⁷⁾ which measured the muon and proton precession rates in a liquid, and a University of Chicago result⁽¹⁸⁾ which measured Zeeman transitions in muonium.

The two results are

$$(\mu_\mu/\mu_P)_{\text{UW/LRL}} = 3.183\,346\,7 \quad (82) \quad (2.6 \text{ ppm})$$

$$(\mu_\mu/\mu_P)_{\text{Chi}} = 3.183\,338 \quad (13) \quad (4.1 \text{ ppm})$$

From these values we can put an upper limit on δ_P' using (1.4). The present value (including this work) of the proton recoil correction, δ_P , is $(-34.4 \pm 0.9) \text{ ppm}^{(15)}$, the uncertainty coming primarily from uncertainties in the form factors in the $(m_e/m_P)\alpha$ correction. The muonium recoil correction, δ_μ , is $(-174.0 \text{ ppm})^{(19)}$, and the muonium hyperfine frequency, ν_m , is $4463.3023 \text{ (35)} \quad (0.7 \text{ ppm})^{(17)}$. These numbers then lead to an upper limit of δ_P' :

$$(\delta_P')_{\text{UW/LRL}} \leq (6.5 \pm 2.8) \text{ ppm}$$

$$(\delta_P')_{\text{Chi}} \leq (3.8 \pm 4.3) \text{ ppm}$$

The uncertainty in each case was computed as the root sum square of the component uncertainties listed above. Clearly, the measurement of the μ_μ/μ_p ratio affords a most accurate determination of the proton polarizability.

The calculation presented here is the logical culmination of a series of $\alpha^6 \ln \alpha^{-1}$ recoil correction calculations that have included the positronium calculation⁽¹²⁾, the muonium calculation⁽¹⁹⁾, and finally this calculation which has been the most involved of all.

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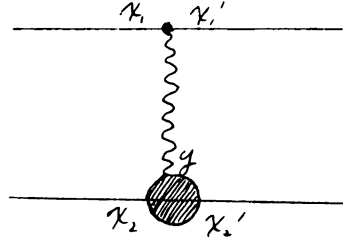
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APPENDICES

APPENDIX A

DERIVATION OF KERNELS

1. Single Photon Exchange



Using well known Feynman diagram techniques, the above diagram is written:

$$\begin{aligned}
 I(x_1, x_2; x_1', x_2') &= -e^2 \int d^4 y d(x_1 - x_1') \delta_\mu^{(3)}(-i D_{\mu\lambda, F}(x_1 - y)) \frac{\partial}{\partial \lambda} (x_2 - y, y - x_2') \\
 &= ie^2 \int \frac{d^4 y d^4 p d^4 p' d^4 k d^4 p''}{(2\pi)^{16}} e^{ip(x_1 - x_1')} e^{ik(x_1 - y)} e^{ip'(x_2 - y)} \\
 &\quad \times e^{ip''(y - x_2')} \delta_\mu^{(3)} D_{\mu\lambda, F}(k) F_\lambda(p', p'')
 \end{aligned}
 \tag{A.1}$$

We introduce the center of mass and relative coordinate system (see (2.7)):

$$\begin{aligned}
 I(x, x', X, X') &= \frac{ie^2}{(2\pi)^{16}} \int d^4 y d^4 p d^4 k d^4 p' d^4 p'' \delta_\mu^{(3)} D_{\mu\lambda, F}(k) F_\lambda(p', p'') \\
 &\quad \times e^{ipx_1(x - x')} e^{ik(x_2 - y)} e^{-ip'(x_1 + y)} e^{ip''(y + x_2')} \\
 &\quad \times e^{i(p + k + p')X} e^{-i(p + p'')X''}
 \end{aligned}
 \tag{A.2}$$

But $I(x, x') \equiv \int d^4 X' e^{-iK(X-X')} I(x, x', X, X')$ (2.8c)

so performing the indicated integrations on (A.2) we obtain:

$$I(x, x') = \{A.2\} e^{i(p+k+p'-K)X} (2\pi)^4 \delta(p+p''-K)$$

Now do the trivial p integration. Then do the y integration obtaining $(2\pi)^4 \delta(k+p'-p'')$. Then do the p'' integration:

$$I(x, x') = \frac{ie^2}{(2\pi)^8} \int d^4 k d^4 p' y_\mu'' D_{\mu\lambda, F}(k) F_\lambda(k) e^{ikx} e^{i(p'-K)(x-x')} \quad (A.3)$$

Now change variable $p' \rightarrow p' + K$ and do the p' integration:

$$I(x, x') = \frac{ie^2}{(2\pi)^4} \int d^4 k y_\mu'' D_{\mu\lambda, F}(k) F_\lambda(k) e^{ikx'} \delta(x'-x) \quad (A.4)$$

Insert $D_F = \delta_{\mu\nu} \lambda/k^2$ and let $F_\lambda(k)$ include a phenomenological magnetic moment for particle 2:

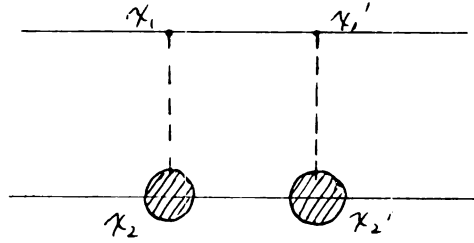
$$F_\lambda(k) \equiv F_1(k^2) \gamma_\lambda^{(2)} - \frac{F_2(k^2)}{2m_2} \sigma_{\lambda\nu}^{(2)} k_\nu \quad (A.5)$$

Thus:

$$I(x, x') = -\frac{i(4\pi\alpha)}{(2\pi)^4} \delta(x-x') \int \frac{d^4 k}{k^2} e^{ikx} \left\{ F_1(k^2) y_\mu'' y_\mu^{(2)} + \frac{F_2(k^2)}{2m_2} y_\mu'' \sigma_{\mu\nu}^{(2)} k_\nu \right\} \quad (A.6)$$



2. Ladder Diagram



As was seen in Chapter II, the two photon ladder kernel is a second order perturbation expansion of the one photon interaction kernel, K'_K (see (2.21)). I'_K is that part of the one photon interaction kernel (A.6) that remains when the Coulomb interaction is separated out, namely:

$$I''(x, x') = -i 4\pi\alpha \delta(x - x') \int \frac{d^4 k}{(2\pi)^4} e^{ikx} \left\{ F_1(k^2) \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \underline{x}^{(1)} \cdot \underline{x}^{(2)} \right) + \frac{F_2(k^2)}{2m_2} \gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu \right\}$$

The two photon ladder kernel is then given by:

$$I_K(x, x') = I''(x, y) G_K(y, y') I''(y', x') \quad (\text{A.7})$$

$$\text{where } G_K(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-x')} S_F^{(1)}(p + \eta_1 K) S_F^{(2)}(-p + \eta_2 K)$$

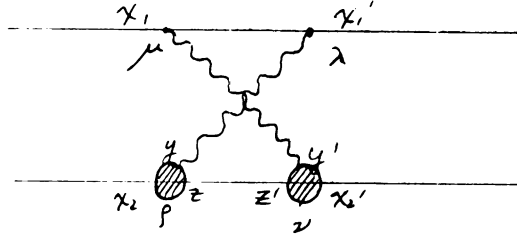
Thus

$$I_K(x, x') = - \frac{(4\pi\alpha)^2}{(2\pi)^8} \int \frac{d^4 k d^4 k'}{k^2 k'^2} e^{ikx} e^{ik'x'} \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \underline{x}^{(1)} \cdot \underline{x}^{(2)} \right) + \frac{F_2(k^2)}{2m_2} \gamma_\mu^{(1)} \sigma_{\mu\nu}^{(2)} k_\nu \right\} G_K(x, x') \left\{ \gamma_4^{(1)} \gamma_4^{(2)} \left(\frac{k_0^2}{|k|^2} - \underline{x}^{(1)} \cdot \underline{x}^{(2)} \right) + \frac{F_2(k'^2)}{2m_2} \gamma_\lambda^{(1)} \sigma_{\lambda\rho}^{(2)} k'_\rho \right\}$$

(A.8)

which is the desired result.

3. Crossed Diagram Kernel



The indicated diagram is written:

$$\begin{aligned} \mathbf{I}(x_1, x_2, x_1', x_2') &= e^2 \int d^4 y d^4 z d^4 y' d^4 z' \gamma_\mu^{(1)} \Gamma_p^{(2)}(x_2 - y; y - z) S_F^{(1)}(x_1 - x_1') \\ &\quad S_F^{(2)}(z - z') \gamma_\lambda^{(1)} \Gamma_\nu^{(2)}(z' - y'; y' - x_2') D_{\mu\nu, F}(x_1 - y') D_{\lambda\rho, F}(y - x_1') \end{aligned} \quad (\text{A.9})$$

We set $\Gamma_p^{(2)}(x_2 - y; y - z) = \int \frac{d^4 \xi d^4 \xi'}{(2\pi)^8} e^{i\xi(x_2 - y)} e^{i\xi'(y - z)} \Gamma_p^{(2)}(\xi, \xi')$

and take the appropriate Fourier transforms:

$$\begin{aligned} \mathbf{I}(x_1, x_2, x_1', x_2') &= \frac{4\pi\alpha}{(2\pi)^{32}} \int d^4 y d^4 y' d^4 z d^4 z' d^4 \xi d^4 \xi' d^4 p d^4 p' d^4 \eta d^4 \eta' d^4 k d^4 k' \\ &\quad \times \gamma_\mu^{(1)} \Gamma_p^{(2)}(\xi, \xi') S_F^{(1)}(p) S_F^{(2)}(p') \gamma_\lambda^{(1)} \Gamma_\nu^{(2)}(\eta, \eta') D_{\mu\nu, F}(k) D_{\lambda\rho, F}(k') \\ &\quad e^{i\xi(x_2 - y)} e^{i\xi'(y - z)} e^{ip(x_1 - x_1')} e^{ip'(z - z')} e^{i\eta(z' - y')} \\ &\quad e^{i\eta'(y' - x_2')} e^{ik(x_1 - y')} e^{ik'(y - x_1')} \end{aligned} \quad (\text{A.10})$$

The y, y', z, z' integrations yield:

$$(2\pi)^6 \delta(-\xi + \xi' + k') \delta(-\eta + \eta' - k) \delta(-\xi' + p') \delta(-p' + \eta)$$

Doing the now trivial ξ, ξ', η, η' integrations, we have:

$$\begin{aligned}
I(x_1, x_2, x_1', x_2') &= \frac{4\pi\alpha}{(2\pi)^6} \int d^4p d^4p' d^4k d^4k' \gamma_\mu^{(n)} \Gamma_p^{(2)}(p'+k', p') S_F^{(1)}(p) S_F^{(2)}(p') \gamma_\lambda^{(n)} \\
&\quad \Gamma_\nu^{(2)}(p'; p'+k) D_{\mu\nu, F}(k) D_{\lambda\rho, F}(k') e^{i(k+p)x_1} \\
&\quad e^{i(p+k')x_1'} e^{i(p'+k')x_2} e^{i(p'+k)x_2'} \quad (A.11)
\end{aligned}$$

We transform to the center of mass using (2.7) and then integrate over X' according to (2.8c):

$$\begin{aligned}
I(x, x') &= \frac{4\pi\alpha}{(2\pi)^6} \int d^4p d^4p' d^4k d^4k' \gamma_\mu^{(n)} \Gamma_p^{(2)}(p'+k', p') S_F^{(1)}(p) S_F^{(2)}(p') \gamma_\lambda^{(n)} \\
&\quad \Gamma_\nu^{(2)}(p'; p'+k) D_{\mu\nu, F}(k) D_{\lambda\rho, F}(k') e^{i(k+p+p'+k'-K)X} \\
&\quad e^{i(q_1 p - q_1 p' + q_2 k - q_2 k')x} e^{i(q_1 p' - q_1 p + q_2 k - q_2 k')x'} \\
&\quad (2\pi)^4 \delta(p'+p+k'+k-K) \quad (A.12)
\end{aligned}$$

Do the p' integration and change variables $p \rightarrow p + q(K - k - k')$

$$\begin{aligned}
I(x, x') &= \frac{4\pi\alpha}{(2\pi)^6} \int d^4p d^4k d^4k' \gamma_\mu^{(n)} \Gamma_p^{(2)}(k') S_F^{(1)}(p+q, \tilde{K}) S_F^{(2)}(-p+q, \tilde{K}) \\
&\quad \gamma_\lambda^{(n)} \Gamma_\nu^{(2)}(-k) D_{\mu\nu, F}(k) D_{\lambda\rho, F}(k') e^{ip(x-x')} e^{i(q \cdot k - q_2 k')x} \\
&\quad e^{i(q \cdot k - q_2 k')x'} \quad (A.13)
\end{aligned}$$

where $\tilde{K} \equiv K - k - k'$

We now insert the appropriate vertex functions:

$$\Gamma_{\lambda}^{(2)}(k) = -ie(F_1(k^2)\gamma_{\lambda}^{(1)} - \frac{F_2(k^2)}{2m_2}\sigma_{\lambda\omega}^{(2)}k_{\omega})$$

and obtain:

$$\begin{aligned} \mathcal{I}(x, x') = & -\frac{(4\pi\alpha)^2}{(2\pi)^8} \int \frac{d^4k d^4k'}{k^2 k'^2} e^{i(q_2 k - q_1 k')} e^{i(q_1 k - q_2 k')x'} \\ & \times \left\{ F_1(k'^2)F_1(k^2) \gamma_{\mu}^{(1)}\gamma_{\nu}^{(2)} G_{\widetilde{H}}(x-x') \gamma_{\nu}^{(1)}\gamma_{\mu}^{(2)} \right. \\ & - \frac{F_1(k'^2)F_2(k^2)}{2m_2} \gamma_{\mu}^{(1)}\gamma_{\nu}^{(2)} G_{\widetilde{H}}(x-x') \gamma_{\nu}^{(1)}\sigma_{\mu\alpha}^{(2)}k_{\alpha} \\ & + \frac{F_2(k'^2)F_1(k^2)}{2m_2} \gamma_{\mu}^{(1)}\sigma_{\nu\beta}^{(2)}k'_{\beta} G_{\widetilde{H}}(x-x') \gamma_{\nu}^{(1)}\gamma_{\mu}^{(2)} \\ & \left. - \frac{F_2(k'^2)F_2(k^2)}{4m_2^2} \gamma_{\mu}^{(1)}\sigma_{\nu\beta}^{(2)}k'_{\beta} G_{\widetilde{H}}(x-x') \gamma_{\nu}^{(1)}\sigma_{\mu\alpha}^{(2)}k_{\alpha} \right\} \end{aligned} \quad (\text{A.14})$$

where

$$G_{\widetilde{H}}(x-x') = \int \frac{d^4p'}{(2\pi)^4} e^{ip'(x-x')} S_F^{(1)}(p'+q_1, \widetilde{H}) S_F^{(2)}(-p'+q_2, \widetilde{H})$$

APPENDIX B SPHERICAL AVERAGING

We outline the proof⁽¹⁴⁾ of the statement $\langle \underline{\sigma}^{(n)} \cdot \underline{A} \underline{\sigma}^{(n)} \cdot \underline{B} \rangle \rightarrow \frac{1}{3} \underline{A} \cdot \underline{B} \langle \underline{\sigma}^{(n)} \cdot \underline{\sigma}^{(n)} \rangle$ which was used extensively in the calculations. Consider an integral of the type we encountered in Chapter IV:

$$\mathcal{J} = \int d^3A d^3B f(A^1, B^1, \underline{A} \cdot \underline{B}, \dots) \underline{\sigma}^{(n)} \cdot \underline{A} \underline{\sigma}^{(n)} \cdot \underline{B} \quad (\text{B.1})$$

where $\underline{A}$ and $\underline{B}$ are momentum vectors, and f an arbitrary function bilinear in $\underline{A}$ and $\underline{B}$. If we subject all momentum vectors to a simultaneous rotation, $R(\phi, \psi, \theta)$, where ϕ, ψ, θ are the usual Euler angles, remains invariant. Thus:

$$\mathcal{J} = \int d^3A d^3B f(A^1, B^1, \underline{A} \cdot \underline{B}, \dots) (\underline{\sigma}^{(n)} \cdot R \underline{A}) (\underline{\sigma}^{(n)} \cdot R \underline{B}) \quad (\text{B.2})$$

We integrate both sides of (B.2) over the invariant volume for Euler angles, $d\Omega_R = \sin\theta d\theta d\phi d\psi$

$$\begin{aligned} \mathcal{J} \int_R d\Omega_R &= \int d^3A d^3B f(A^1, B^1, \underline{A} \cdot \underline{B}, \dots) \int d\Omega_R (\underline{\sigma}^{(n)} \cdot R \underline{A}) (\underline{\sigma}^{(n)} \cdot R \underline{B}) \\ &= 8\pi^2 \mathcal{J} = \int d^3A d^3B f(A^1, B^1, \underline{A} \cdot \underline{B}, \dots) \sigma_i^{(n)} \sigma_j^{(n)} \int d\Omega_R (R A)_i (R B)_j \end{aligned} \quad (\text{B.3})$$

The integral over R can be written as a general second rank tensor:

$$\sigma_i^{(n)} \sigma_j^{(n)} \{ C_{ij} \delta_{ij} + C_{ij} A_i B_j \} \quad (\text{B.4})$$

where C_1 and C_2 are constants. However C_2 is zero which can be shown by the following argument:

$$\int d\Omega_R (R\mathbf{A})_i (R\mathbf{B})_j = C_1 \delta_{ij} + C_2 A_i B_j \quad (\text{B.5})$$

Now perform a further rotation R' such that

$$\mathbf{A} \rightarrow R'\mathbf{A} \quad (\text{B.6})$$

$$\mathbf{B} \rightarrow R'\mathbf{B}$$

Then (B.5) becomes:

$$\begin{aligned} \int (R R' \mathbf{A})_i (R R' \mathbf{B})_j d\Omega_R &= \int (R R' \mathbf{A})_i (R R' \mathbf{B})_j d\Omega_{R R'} \\ &= C_1 \delta_{ij} + C_2 A_i B_j \end{aligned} \quad (\text{B.7})$$

The second term in (B.7) is not invariant under (B.6), so C_2 must be zero.

Thus (B.4) becomes:

$$\sigma_i^{(1)} \sigma_j^{(2)} C_1 \delta_{ij} = C_1 \tilde{\sigma}^{(1)} \cdot \tilde{\sigma}^{(2)} \quad (\text{B.8})$$

To evaluate C_1 we take the trace of (B.5):

$$\begin{aligned} \text{Tr} \int (R\mathbf{A})_i (R\mathbf{B})_j d\Omega_R &= \text{Tr} C_1 \delta_{ij} = 3C_1 \\ &= \int (R\mathbf{A}) \cdot (R\mathbf{B}) d\Omega_R \\ &= \mathbf{A} \cdot \mathbf{B} \int d\Omega_R \\ &= \mathbf{A} \cdot \mathbf{B} (8\pi) \\ \Rightarrow C_1 &= \frac{8\pi}{3} \mathbf{A} \cdot \mathbf{B} \end{aligned}$$

(B.9)

Thus using (B.1) and (B.3) we obtain:

$$\langle \sigma^{(1)} \underline{A} \underline{\alpha}^{(2)} \underline{B} \rangle \rightarrow \frac{1}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \quad (\text{B.10})$$

The remainder of this section lists some general spherical averages used in computing the values listed in Tables II, IV, and VII. Anticommutation relations can be applied to find related spin averages.

$$\begin{aligned} \langle \underline{\alpha}^{(1)} \underline{A} \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \underline{B} \rangle &\rightarrow \frac{2}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}^{(1)} \underline{A} \underline{\alpha}^{(2)} \underline{B} \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{C} \rangle &\rightarrow -\frac{1}{3} (\underline{A} \times \underline{C}) \cdot (\underline{B} \times \underline{D}) \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}^{(1)} \underline{A} \underline{\alpha}^{(2)} \underline{B} \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \underline{C} \rangle &\rightarrow \left\{ -\frac{2}{3} (\underline{A} \cdot \underline{B}) (\underline{C} \cdot \underline{D}) - \frac{1}{3} (\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) \right. \\ &\quad \left. + \frac{1}{3} (\underline{A} \cdot \underline{C}) (\underline{B} \cdot \underline{D}) - \frac{1}{3} (\underline{A} \cdot \underline{D}) (\underline{B} \cdot \underline{C}) \right\} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \end{aligned}$$

$$\begin{aligned} \langle \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{\alpha}^{(1)} \underline{A} \underline{\alpha}^{(2)} \underline{B} \rangle &\rightarrow -\frac{2}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \underline{A} \underline{\alpha}^{(2)} \underline{B} \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \rangle &\rightarrow -\frac{6}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}_f^{(2)} \underline{\alpha}^{(1)} \underline{A} \underline{\alpha}_f^{(1)} \underline{\alpha}^{(2)} \underline{B} \rangle &\rightarrow \frac{2}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}_i^{(1)} \underline{\alpha}^{(1)} \underline{A} \underline{\alpha}_i^{(2)} \underline{\alpha}^{(2)} \underline{B} \rangle &\rightarrow -\frac{2}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}_i^{(1)} \underline{\alpha}_f^{(2)} \underline{\alpha}^{(2)} \underline{A} \underline{\alpha}_f^{(1)} \underline{\alpha}_i^{(2)} \underline{\alpha}^{(2)} \underline{B} \rangle &\rightarrow -\frac{2}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}_i^{(1)} \underline{\alpha}_f^{(2)} \underline{\alpha}^{(2)} \underline{A} \underline{\alpha}^{(2)} \underline{B} \underline{\alpha}_f^{(1)} \underline{\alpha}_i^{(2)} \rangle &\rightarrow \frac{6}{3} \underline{A} \cdot \underline{B} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \\ \langle \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{\alpha}^{(2)} \underline{A} \underline{\alpha}^{(1)} \underline{B} \underline{\alpha}^{(2)} \underline{\alpha}^{(1)} \underline{\alpha}^{(2)} \underline{C} \rangle &\rightarrow \left\{ \frac{2}{3} (\underline{A} \cdot \underline{B}) (\underline{C} \cdot \underline{D}) + \frac{1}{3} (\underline{A} \times \underline{C}) \cdot (\underline{B} \times \underline{D}) \right. \\ &\quad \left. + \frac{2}{3} (\underline{A} \cdot \underline{D}) (\underline{B} \cdot \underline{C}) - \frac{2}{3} (\underline{A} \cdot \underline{C}) (\underline{B} \cdot \underline{D}) \right\} \langle \sigma^{(1)}, \sigma^{(2)} \rangle \end{aligned}$$

APPENDIX C

INTEGRALS

In this Appendix the derivation of the k_0 integrations and the momentum integrations used in this calculation will be given. We start with the k_0 integrations.

The basic k_0 integration for the ladder diagram is of the following form:

$$I_{++} = \int_{-\infty}^{\infty} dk_0 \frac{1}{k_0^2 - |k|^2} \frac{1}{k_0^2 - |k'|^2} \left(\frac{-\kappa \Lambda^2}{k_0^2 - (|k|^2 + \Lambda^2)} \right) \frac{-1}{[k_0 - (\eta_1 \kappa_0 - \epsilon, \eta_2 + i\epsilon)]} \frac{1}{[k_0 + (\eta_1 \kappa_0 - \epsilon, \eta_2 + i\epsilon)]} \quad (C.1)$$

where we have included one form factor of the type $\kappa \Lambda^2 / (k^2 + \Lambda^2)$. We will show that, in fact, any form factors are superfluous to the $\alpha^6 \ln \alpha^{-1}$ calculation because terms with the form factor parameters (in this case Λ^2) are not singular enough to yield a logarithmic contribution.

The integrand in (C.1) has poles in the upper half plane at $k_0 = -|k| + i\epsilon$, $k_0 = -|k'| + i\epsilon$, $k_0 = -(k^2 + \Lambda^2)^{1/2} + i\epsilon$, and $k_0 = \eta_1 K_0 - E_1 + i\epsilon$. We then close the integration contour in the upper half plane to obtain

$$\begin{aligned}
I_{++} = \kappa(2\pi i) \Big\{ & \frac{\Lambda^2}{-2|k|(k^2 - k'^2)(k^2 - k'^2 - \Lambda^2)(-k - (\eta_1 K_0 - E_1))(-k + \eta_2 K_0 - E_2)} \\
& + \frac{\Lambda^2}{-2|k'| (k'^2 - k^2)(-\Lambda^2)(-k' - (\eta_1 K_0 - E_1))(-k' + \eta_2 K_0 - E_2)} \\
& + \frac{\Lambda^2}{-2(k^2 + \Lambda^2)^{1/2} (k'^2 - k^2 + \Lambda^2)\Lambda^2 \left(-(k^2 + \Lambda^2)^{1/2} - (\eta_1 K_0 - E_1) \right) \left((k^2 + \Lambda^2)^{1/2} + (\eta_2 K_0 - E_2) \right)} \\
& + \frac{\Lambda^2}{\left[(\eta_1 K_0 - E_1)^2 - k^2 \right] \left[(\eta_1 K_0 - E_1)^2 - k'^2 \right] \left[(\eta_1 K_0 - E_1)^2 - k^2 - \Lambda^2 \right] \left[\eta_1 K_0 - E_1 + \eta_2 K_0 - E_2 \right]} \Big\}
\end{aligned}
\tag{C.2}$$

Since all logarithmic contributions come from the low momentum limit of momentum integrands of dimension $[\text{momentum}]^{-2}$, we can set $\eta_1 K_0 - E_1 \simeq m_1 - E_1 - \mu\alpha^2/2m_1 \simeq -k^2/2m_1 - \gamma^2/2m_2$. Moreover, Λ is at least as large as m_2 so that we can expand in powers of k^2/Λ^2 to obtain the leading low momentum behavior of I_{++} .

$$\begin{aligned}
I_{++} = \kappa(2\pi i) \Big\{ & \frac{2\mu}{|k'|^2 |k|^2 (|k|^2 + \gamma^2)} - \frac{|k'|^2 + |k|/|k| + |k|^2}{2|k'|^3 |k|^3 (|k'| + |k|)} \\
& - \frac{2\mu}{\Lambda^2 |k|^2 (|k|^2 + \gamma^2)} + \frac{1}{2\Lambda^2 |k|^3} \Big\}
\end{aligned}
\tag{C.3}$$

The important point in (C.3) is that the form factor

dependent terms (i.e. the Λ dependent terms) are two powers of momentum less singular than the Λ independent terms. Explicit calculation has shown this to be true for all k_0 integrations encountered in this thesis. If the momentum integrands are never more singular than dimension [momentum] $^{-3}$, then the most singular Λ dependent terms will always be of dimension [momentum] $^{-1}$, and hence will never give a logarithmic contribution. This is in fact the case. The [momentum] $^{-3}$ terms in the two photon diagram give part of the α^5 contribution, but there are no terms more singular than [momentum] $^{-3}$ in either of the two photon diagrams. Thus the $\alpha^6 \ln \alpha^{-1}$ contribution from the two photon diagrams will be totally independent of the proton form factors. This is a valuable result.

A few examples of other k_0 integrations where special care must be taken to isolate the two most singular momentum terms will be given below.

$$I''_{++} = \int_{-\infty}^{\infty} dk_0 \frac{k_0^2}{k_0^2 - |k|^2} \frac{1}{k_0^2 - |k'|^2} \frac{-1}{[k_0 - (\eta_1 k_0 - E_1 + i\epsilon)]} \frac{1}{[k_0 + (\eta_1 k_0 - E_1 + i\epsilon)]}$$

Poles in the upper half plane are at $k_0 = -|k| + i\epsilon$,

$k_0 = -|k'| + i\epsilon$, and $\eta_1 K_0 - E_1 + i\epsilon$.

$$\begin{aligned}
I_{++}'' = 2\pi i \Big\{ & \frac{-|k|^2}{2|k| (k^2 - k'^2) (k + E_1 - \eta_1 \mu_0) (k - (E_1 - \eta_1 \mu_0))} \\
& - \frac{|k'|^2}{2|k| (k^2 - k'^2) (k' + E_1 - \eta_1 \mu_0) (k' - (E_1 - \eta_1 \mu_0))} \\
& - \frac{(\eta_1 \mu_0 - E_1)^2}{(\eta_1 \mu_0 - E_1)^2 - k^2 ((\eta_1 \mu_0 - E_1)^2 - k'^2) (\eta_1 \mu_0 - E_1 + \eta_2 \mu_0 - E_2)} \Big\}
\end{aligned}$$

but $\frac{1}{k + E_1 - \eta_1 \mu_0} = \frac{1}{k} \left(1 - \frac{E_1 - \eta_1 \mu_0}{k + E_1 - \eta_1 \mu_0} \right) \approx \frac{1}{k} - \frac{1}{2m_1}$

using $\eta_1 \mu_0 = m_1 - \frac{Y_1^2}{2m_1}$; $E_1 - m_1 \approx k^2/2m_1$,

Keeping only the two most singular terms of I_{++}'' , we obtain

$$I_{++}'' = 2\pi i \left\{ \frac{1}{2|k||k'| (|k| + |k'|)} + \frac{\frac{Y_1^2}{2m_1} |k|^2}{|k|^2 (|k|^2 + Y_1^2)} + O(k^{-1}) \right\} \quad (C.4)$$

The calculation for the crossed diagram is similar.

$$\tilde{I}_{++}'' = \int_{-\infty}^{\infty} d\ell_0 \frac{\ell_0^2}{\ell_0^2 - |k|^2} \frac{\ell_0^2 - |k'|^2}{\ell_0^2 - |k'|^2} \frac{1}{[\ell_0 - (\eta_1 \mu_0 - E_1 + i\epsilon)]} \frac{1}{[\ell_0 - (\eta_1 \mu_0 - E_1' + i\epsilon)]}$$

Poles in the lower half plane are at $k_0 = |k| - i\epsilon$, and

$$k_0 = |k'| - i\epsilon$$

$$\begin{aligned}
\tilde{I}_{++}'' &= -(2\pi i) \left\{ \frac{k^2}{2k(k^2-k'^2)(k+E_1-m_1)(k+E_2-m_2)} + \frac{k'^2}{2k'(k'^2-k^2)(k'+E_1-m_1)(k'+E_2-m_2)} \right\} \\
&= -(2\pi i) \left\{ \frac{-k}{2(k'^2-k^2)k^2} \left(1 - \frac{E_1-m_1}{k+E_1-m_1} \right) \left(1 - \frac{E_2'-m_2}{k+E_2'-m_2} \right) \right. \\
&\quad \left. + \frac{k'}{2(k'^2-k^2)k'^2} \left(1 - \frac{E_1-m_1}{k'+E_1-m_1} \right) \left(1 - \frac{E_2'-m_2}{k'+E_2'-m_2} \right) \right\}
\end{aligned}$$

Keeping only the two most singular terms at low momentum, we have,

$$\tilde{I}_{++}'' = 2\pi i \left\{ \frac{1}{2|k'|k|(1k'|+1k|)} - \frac{1}{4} \left(\frac{1}{k'^2 m_1} + \frac{1}{k^2 m_2} \right) + O(k') \right\} \quad (C.5)$$

And in like manner,

$$\tilde{I}_{--} = \int_{-\infty}^{\infty} dk_0 \frac{k_0^2}{k_0^2 - |k|^2} \frac{1}{k_0^2 - |k'|^2} \frac{1}{[k_0 - (m_1 k_0 + E_1 - i\epsilon)]} \frac{1}{[k_0 - (m_2 k_0 + E_2 - i\epsilon)]}$$

Poles are in the upper half plane at $k_0 = -|k| + i\epsilon$, and

$$k_0 = -|k'| + i\epsilon$$

$$\begin{aligned}
\tilde{I}_{--} &= 2\pi i \left\{ \frac{1}{-2k(k^2-k'^2)(-k-(E_1+m_1))(-k-(E_2'+m_2))} \right. \\
&\quad \left. + \frac{1}{-2k'(k'^2-k^2)(-k'-(E_1+m_1))(-k'-(E_2'+m_2))} \right\}
\end{aligned}$$

but

$$\frac{1}{k+E_1+m_1} = \frac{1}{E_1+m_1} \left(1 - \frac{k}{k+E_1+m_1}\right)$$

$$\simeq \frac{1}{E_1+m_1} - \frac{k}{(E_1+m_1)^2}, \text{ etc.}$$

so

$$\tilde{I}_{--} = 2\pi i \left\{ \frac{1}{2k(k'^2-k^2)(E_1+m_1)(E_2'+m_2)} \left(1 - \frac{k}{E_1+m_1}\right) \left(1 - \frac{k}{E_2'+m_2}\right) \right.$$

$$\left. + \frac{1}{-2k'(k'^2-k^2)(E_1+m_1)(E_2'+m_2)} \left(1 - \frac{k'}{E_1+m_1}\right) \left(1 - \frac{k}{E_2'+m_2}\right) \right\}$$

which for small momentum becomes

$$\tilde{I}_{--} = 2\pi i \left\{ \frac{1}{8m_1m_2|k'| |k| (|k'| + |k|)} - \frac{1}{8m_1m_2(k'^2-k^2)} \left(\frac{1}{E_1+m_1} + \frac{1}{E_2'+m_2} \right) \right.$$

$$\left. + \frac{1}{8m_1m_2(k'^2-k^2)} \left(\frac{1}{E_1+m_1} + \frac{1}{E_2'+m_2} \right) + O(k^{-1}) \right\}$$

$$= 2\pi i \left\{ \frac{1}{8m_1m_2|k'| |k| (|k'| + |k|)} + O(k^{-1}) \right\} \quad (\text{C.6})$$

The rest of the k_0 integrations are listed in Table V and in Table VIII.

We turn now to the momentum integrals (Tables III and VI). The general form of the momentum integrals encountered

in this calculation is

$$\int \frac{d^3 p d^3 p' F(p, p')}{(p^2 + \delta^2)^2 (p'^2 + \delta^2)^2 E(p) E(p') |p - p'|^2} \quad (i=1,2) \quad (C.7)$$

where F is a function involving the spin averages.

The first thing to note is that for both p and p' $\gg \gamma (= \mu\alpha)$, (C.7) is simply of order α^6 because the integral itself is independent of α . Thus the logarithmic behavior arises from the low momentum region. We also note that by putting on of the wave functions at the origin in the two photon calculation, we have forced the other wave function momentum, p , to be small to insure a logarithmic contribution. Hence we are justified in setting $E(k+p) \approx E(k)$ in the denominators of the two photon momentum integrals. Similarly we are justified in setting $m_i/E_i(p) \approx 1$ as was done repeatedly in the use of the static wave function.

The momentum integrations depend upon the use of the formula⁽¹²⁾

$$\frac{1}{\sqrt{a} \, b} = \frac{1}{2} \int_0^1 \frac{dv}{(1-v)^{3/2}} [a + (b-a)v]^{-3/2} \quad (C.8)$$

which is used in conjunction with the usual Feynman parameter integrals to obtain the following expressions:

$$\frac{1}{a^{1/2} b^{1/2} c} = \frac{15}{8} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \int_0^1 \frac{du (1-u)}{[a + (b-a)v + (c-a)u - (b-a)uv]^{7/2}} \quad (C.9)$$

$$\frac{1}{a^{1/2} b^{1/2} c^2} = \frac{105}{16} \int_0^1 \frac{v dv}{(1-v)^{3/2}} \int_0^1 \frac{du u(1-u)}{[a + (b-a)v + (c-a)u - (b-a)uv]^{9/2}} \quad (C.10)$$

Equations (C.9) and (C.10) are used to expand the denominators of the momentum integrals. For example, ($i=1,2$)

$$\begin{aligned} I_{\Phi} &= \int \frac{d^3 p d^3 p'}{(p^2 + \gamma^2)^2 (p'^2 + \gamma^2)^2} \frac{p^2 p'^2}{|p' - p|^2 \epsilon_i'} \\ &= \int \frac{d^3 p}{(p^2 + \gamma^2)^2} \int d^3 p' p^2 p'^2 \frac{15}{8} \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} [(p' - up)^2 + \Lambda_i]^{-7/2} \end{aligned}$$

$$\text{where } \Lambda_i = (1-u)[p^2 u + m_i^2 + (\gamma^2 - m_i^2)v]$$

$$= (1-u)[p^2 u + \Delta_i] \geq 0.$$

Let $p' \rightarrow (p' + up)$ and integrate over p' .

$$I_{\Phi} = \left(\frac{15}{8} \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \int \frac{d^3 p p^2}{(p^2 + \gamma^2)^2} \right) 4\pi \left\{ \frac{1}{5\Lambda_i} + \frac{2u^2 p^2}{15\Lambda_i^2} \right\}$$

The integral over p is then done, yielding

$$I_{\odot} = \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \left\{ \frac{3\pi}{2} \frac{4\pi}{(1-u)u} \left[\frac{u^{1/2}}{\Delta_i^{1/2}} \frac{\pi}{2} \right] + \frac{4\pi}{(1-u)^2} \frac{u^{1/2}}{2\Delta_i^{1/2}} \frac{\pi}{2} \right\}$$

The parameter integration over u is trivial. The v integration is

$$\int_0^1 \frac{v dv}{(1-v)^{1/2}} = -\frac{m_i}{m_i^2 - \gamma^2} +$$

$$+ \frac{2m_i^2 - \gamma^2}{2m_i^2} \left\{ \frac{2}{(m_i^2 - \gamma^2)^{1/2}} \left[\log |-y| - \log |(m_i^2 - \gamma^2)^{1/2} - m_i| \right] \right\}$$

expanding in powers of $(\gamma/m_i)^2$, since $\gamma \ll m_1$, the leading logarithmic behavior is

$$\frac{2}{m_i} \left[\log \frac{2m_i}{\mu} + \log \alpha^{-1} \right]$$

Since the coefficient of the momentum integrals contains an α^6 , we need keep only the $\ln \alpha^{-1}$ term. All other terms of the v integration contribute to order α^6 or higher. Hence we can write

$$\int_0^1 \frac{v dv}{(1-v)^{1/2} \Delta_i^{1/2}} = \frac{2}{m_i} \ln \alpha^{-1} \quad (i=1,2) \quad (C.11)$$

The other momentum integrals for the one photon exchange (Table III) are done in a similar manner. The general method is to integrate over p' (using the Feynman parameters), integrate then over p , and finally do the parameter integrals using (C.10). Rather than go through all of the details of the remaining integrals in Table III, we list a few helpful p' integrations that are necessary to the remaining derivations.

$$\int \frac{d^3 p'}{(p'^2 + \gamma^2)^2} \frac{p'^2}{|k - p'|^2 E_i'} = \pi \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \left[\frac{-3/2}{\Lambda_i^2} + \frac{u^2 p^2}{\Lambda_i^4} \right]$$

$$\int \frac{d^3 p'}{(p'^2 + \gamma^2)^2} \frac{|k \times p'|^2}{|k - p'|^4 E_i'} = \pi \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \frac{u p^2}{\Lambda_i^2}$$

$$\int \frac{d^3 p'}{(p'^2 + \gamma^2)^2} \frac{k \cdot p'}{|k - p'|^2 E_i'} = \pi \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \frac{u p^2}{\Lambda_i^2}$$

$$\int \frac{d^3 p'}{(p'^2 + \gamma^2)^2} \frac{1}{|k - p'|^2 E_i'} = \pi \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \frac{1}{\Lambda_i^2}$$

$$\int \frac{d^3 p'}{(p'^2 + \gamma^2)^2} \frac{|k \times p'|^2}{|k - p'|^4 E_i'} = \pi \int_0^1 du (1-u)^{3/2} \int_0^1 \frac{v dv}{(1-v)^{1/2}} \frac{p^2}{\Lambda_i^2}$$

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where $\Lambda_i = (1-u)[p^+u + m_i^2 + (y^2 - m_i^2)/u] = (1-u)[p^+u + \Delta_i] \geq 0$

Similar methods are applied to the momentum integrals for the two photon diagrams (see Table VI). For example,

$$\begin{aligned} & \int \frac{d^3p d^3k}{(p^2+y^2)^2} \frac{p^2}{E_1(k) E_2(k) k^2 |k+p|^2} \\ &= \int \frac{d^3k}{E_1(k) E_2(k) k^2} \left\{ \frac{\pi^2 k^2}{y^2 (k^2+y^2)} - \frac{\pi^2}{y^2} + \frac{2\pi^2}{k} \tan^{-1}\left(\frac{k}{y}\right) \right\} \\ &= \frac{4\pi^3}{y^4} \int_0^\infty \frac{k^2 dk}{E_1 E_2 (k^2+y^2)} - \frac{4\pi^3}{y^4} \int_0^\infty \frac{dk}{E_1(k) E_2(k)} + 8\pi^3 \int_0^\infty \frac{dk \tan^{-1}(k/y)}{E_1(k) E_2(k) k} \end{aligned}$$

The first two terms contain no logarithmic dependence, while the later integral is evaluated by integration in the complex plane.

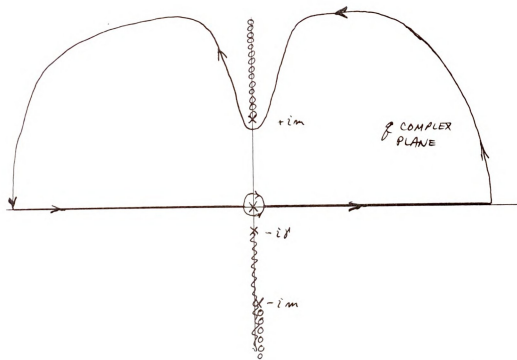
$$\begin{aligned} I &\equiv \int_0^\infty d\varrho \frac{\tan^{-1} \varrho/y}{E(\varrho) |\varrho|} = \frac{i}{2} \int_0^\infty d\varrho \frac{\ln\left(\frac{1-i\varrho/y}{1+i\varrho/y}\right)}{\varrho (\varrho^2+m^2)^{1/2}} \\ &= \frac{i}{2} \int_0^\infty d\varrho \frac{\ln(1-i\varrho/y)}{\varrho (\varrho^2+m^2)^{1/2}} - \frac{i}{2} \int_0^\infty d\varrho \frac{\ln(1+i\varrho/y)}{\varrho (\varrho^2+m^2)^{1/2}} \end{aligned}$$

Change variables in the later integral, $q \rightarrow -q$.

$$I = \frac{i}{2} \int_{-\infty}^{\infty} d\varrho \frac{\ln(1-i\varrho/y)}{\varrho (\varrho^2+m^2)^{1/2}}$$

In the complex plane we have poles at $q = 0$, and $q = \pm im$.

The square root branch cut is indicated by $\circ \circ \circ \circ \circ \circ \circ \circ \circ$
 while the logarithmic branch cut is indicated by $\sim \sim \sim \sim \sim$
 and there is a logarithmic singularity at $q = -iy$



We close the integration contour in the upper half plane,
 and use the residue theorem.

$$\begin{aligned} \frac{i}{2} \oint &= \frac{i}{2} \int_{-\infty}^{\infty} dq \frac{\ln(1 - i\delta/y)}{f(q^2 + m^2)^{1/2}} + \frac{i}{2} \int_{i\infty}^{im} dq \frac{\ln(1 - i\delta/y)}{f(q^2 + m^2)^{1/2}} \Big|_{\text{QUAD ①}} + \frac{i}{2} \int_{im}^{i\infty} dq \frac{\ln(1 - i\delta/y)}{f(q^2 + m^2)^{1/2}} \Big|_{\text{QUAD ②}} \\ &= \frac{2i\pi i}{2} \sum \text{Res.} = 0 \end{aligned}$$

We note that the function just to the right of the branch cut is the negative of the function just to the left of the branch cut. so we have

$$\frac{i}{2} \int_{-\infty}^{\infty} d\varphi \frac{\ln(1 - i\varphi/r)}{\varphi (\varphi^2 + m^2)^{1/2}} - i \int_{im}^{i\infty} d\varphi \frac{\ln(1 - i\varphi/r)}{\varphi (\varphi^2 + m^2)^{1/2}} = 0$$

Now let $y = -iq$ in the later integral

$$\frac{i}{2} \int_{-\infty}^{\infty} d\varphi \frac{\ln(1 - i\varphi/r)}{\varphi (\varphi^2 + m^2)^{1/2}} = \int_m^{\infty} dy \frac{\ln(1 + y/r)}{y (y^2 - m^2)^{1/2}}$$

or

$$I = \int_m^{\infty} dy \frac{\ln(1 + y/r)}{y (y^2 - m^2)^{1/2}} = \int_m^{\infty} dy \frac{\{\ln(y/r) + \ln(1 + \frac{r}{y})\}}{y (y^2 - m^2)^{1/2}}$$

$$\approx \underbrace{\int_m^{\infty} dy \frac{\ln(y/r)}{y (y^2 - m^2)^{1/2}} + \int_m^{\infty} dy \frac{r}{y^2 (y^2 - m^2)^{1/2}}}_{\text{CONTRIBUTES TO } O(\alpha^2)}$$

$$\rightarrow \int_m^{\infty} dy \frac{\ln \alpha^{-1}}{y (y^2 - m^2)^{1/2}} + \underbrace{\int_m^{\infty} dy \frac{\ln(y/r)}{y (y^2 - m^2)^{1/2}}}_{\text{CONTRIBUTES TO } O(\alpha^6)}$$

$$\rightarrow \frac{\pi}{2m_i} \ln \alpha^{-1}$$

Thus to order $\alpha^6 \ln \alpha^{-1}$ we have

$$I = \int_0^{\infty} d\varphi \frac{\tan^{-1}(\varphi/r)}{E_i(\varphi) |\varphi|} = \frac{\pi}{2m_i} \ln \alpha^{-1} \quad (i=1,2) \quad (C.12)$$

An identical procedure is followed in performing the remaining integrations listed in Table VI.





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