MOVING THE COMPACT SUBSETS OF MANIFOLDS

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ABSTRACT

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By

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We make the following definition. A topological space X is said to be 1-movable when for each proper compact subset, A, of X there is a homeomorphism, h, of X onto itself with $A \cap h(A) = \emptyset$. Some results are:

1. If an open connected 2-manifold, M, imbeds in a 2-sphere with n-handles, N, then there is an imbedding of M in N that is dense in N.

2. If M^n is a closed connected n-manifold, $n \geq 2$, C a closed 0-dimensional subset of M^n, and $M^n - C$ is 1-movable, then M^n is the n-sphere and C has an invert point.

3. A counterexample to the converse of 2 is presented for $n \geq 3$.

4. If L is the (n-2)-skeleton of K (where K is a simplicial complex, not necessarily locally finite, of dimension n) then $|K| - |L|$ is 1-movable.

5. Numerous examples restricting certain possible improvements of the result of 4.

The notion of 1-movable is generalized to that of k-movable and we have results:

6. Every open connected triangulable n-manifold is n-movable.

7. For $n \geq 5$, every contractible open n-manifold is 2-movable.

8. If M is a k-movable manifold with boundary, then that boundary is not compact and has either 0 or an infinite number of compact components.

9. Let M^n be a closed n-manifold. If M^n is k-movable, then M^n is the union of $(k+1)$ open n-cells.

We consider the 1-movability of certain open 3-manifolds and one of the facts obtained is that all W-spaces (and hence all contractible domains in E^3) are 1-movable.

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INTRODUCTION

K. Borsuk made the following definition in 1934 (see R.L. Wilder [23]); a subset M of Euclidean n-space is said to be free, when for each $\epsilon > 0$ there exists an ϵ-transformation of M onto M' and $M \cap M' = \emptyset$. A topological space X is said to be invertible when for each proper closed subset A there exists a homeomorphism h of X onto itself with $A \cap h(A) = \emptyset$. This concept was created by P.H. Doyle and J.G. Hocking [6] and has been studied extensively.

Thus the following definition seems a natural one to make.

Definition 0.1: A topological space X is k-movable (for some positive integer k) if and only if for each proper compact subset B of X, there exist compact subsets $B_1, B_2, \ldots, B_k$ of B and homeomorphisms $h_1, h_2, \ldots, h_k$ of X onto itself with $B = \bigcup_{i=1}^{k} B_i$ and $B \cap h_i(B_i) = \emptyset$ for each $i = 1, 2, \ldots, k$.

Remark 0.2: A compact Hausdorff space X is 1-movable if and only if it is invertible.

Example 0.3: Both E^n and S^n are 1-movable.

A 1-movable compact manifold must be a sphere, but no such restriction is possible for non-compact manifolds as $E^1 \times M$ where M is any manifold is 1-movable (for example: $E^1 \times S^n$ and E^{n+1}).

Remark 0.4: A k-movable topological space is ℓ-movable for each integer $\ell \geq k$.

Although k-movable is defined in the topological category, results are more interesting in the category of manifolds and much of this work will concern manifolds.

The moving number of a space X is defined to be the minimum $\{k | X$ is k-movable$\}$, and $+\infty$ if $\phi = \{k | X$ is k-movable$\}$.

Chapter I of this thesis contains the result that for an open connected 2-manifold, M, that imbeds in a 2-sphere with n-handles, N, there is an imbedding of M in N that is dense in N. The main theorem of Chapter II is: if M^n is a closed connected n-manifold, $n \geq 2$, C a closed 0-dimensional subset of M^n, and $M^n - C$ is 1-movable, then M^n is the n-sphere and C has an invert point. A counterexample to the converse of this theorem is presented for $n \geq 3$.

Chapter III is an investigation of the 1-movability of $|K| - |L|$ where K is a simplicial complex and L is a subcomplex of K. We have the result that if L is the (n-2)-skeleton of K (where K is a simplicial complex, not necessarily locally finite, of dimension n) then $|K| - |L|$ is 1-movable. This chapter includes numerous examples restricting certain possible improvements of the above result.

In Chapters IV and V results are obtained concerning n-manifolds and k-movability in relation to product spaces and manifolds with boundary. Among the results are the following: Every open connected triangulable n-manifold is n-movable. For $n \geq 5$ every contractible open n-manifold is 2-movable. If M is a k-movable manifold with boundary, then that boundary is not compact and has either 0 or an infinite number of compact components. If M^n is a closed n-manifold, then M^n is k-movable implies M^n is the union of $(k+1)$ open n-cells. A connected Hausdorff space that is the union of k open n-cells is k-movable.

Chapter VI concerns the 1-movability of certain open 3-manifolds and includes the fact that all W-spaces (and hence all contractible domains in E^3) are 1-movable.

The following notation will be used for certain sets and topological spaces

$$Z = \{n \mid n \text{ is an integer}\}$$

$$E^n = \{x \mid x = (x_1, x_2, \ldots, x_n) \text{ an n-tuple of real numbers}\}$$

where E^n is given the topology determined by the Euclidean distance. A space is called a <u>n-cell</u>, an <u>open n-cell</u>, or an <u>(n-1)-sphere</u> when it is homeomorphic respectively to $\{x \in E^n \mid \text{distance } (0,x) \leq 1\}$, $\{x \in E^n \mid \text{distance } (0,x) < 1\}$, or $S^{n-1} = \{x \in E^n \mid \text{distance } (0,x) = 1\}$ with the subspace topology induced by that on E^n.

A <u>Cantor set</u> is a homeomorph of the standard Cantor set in $[0,1]$. An <u>invert point</u> of a space X is a point p such that for each neighborhood U of p, there exists a homeomorphism, h, of X onto itself with $h(X - U) \subset U$. A topological space X is said to be the <u>monotone union</u> of a sequence of topological spaces X_i, $i = 1,2,\ldots$, when X_i is a subset of X_{i+1} for $i = 1,2,\ldots$, and $X = \bigcup_{i=1}^{\infty} X_i$. A <u>pseudo-isotopy</u> of a topological space X is a homotopy that is a homeomorphism at every level except at possibly the last. For a subset, A, of a topological space, $\bar{A}$ will denote the closure of A and bdy A the boundary of A.

A <u>n-manifold</u>, M^n, is a separable metric space each point of which has a neighborhood whose closure is an n-cell. A <u>manifold</u> is a topological space that is an n-manifold for some n. All manifolds in this thesis are connected. The <u>interior of M^n</u> is the set of all points of the n-manifold M^n that have open n-cell neighborhoods, and the <u>boundary of M^n</u> is M^n - (the interior of M^n). A <u>manifold with boundary</u> is a manifold whose boundary is non-void. A <u>manifold without boundary</u> is a manifold whose boundary is the empty set. An <u>open manifold</u> is a manifold without boundary that is non-compact, and a <u>closed manifold</u> is a compact manifold without boundary.

In metric spaces, dist (x,y) will denote the distance between x and y. In a metric space M, with $(x,t) \in M \times E^1$, $S_{x,t} = \{y \in M | \text{dist } (x,y) < t\}$. For tame and locally tame Cantor sets see [19]. For locally tame in general see [18].

The end of a proof will be denoted by "$\square$".

CHAPTER I

2-MANIFOLDS

Ian Richards in [21] classified open 2-manifolds. Richards' work implies that every domain in S^2 has an imbedding which is dense in S^2, and this result is generalized by Corollary 1.5 to the 2-sphere with handles. We then prove that the open Möbius band is not 1-movable and (using a recent result of R. Jones [12] that each open 2-manifold is the union of two open 2-cells) that all open 2-manifolds are 2-movable.

We begin by proving some lemmas. In this chapter T_n will denote the closed orientable surface of genus n, namely the 2-sphere with n handles.

<u>Lemma 1.1</u>: Let M^2 be a compact 2-manifold without boundary, C be a compact 0-dimensional set, $C \subset M^2$, and U an open set in M^2, $U \neq \emptyset$.

Then there is a homeomorphism h of M^2 onto itself with $h(C) \subset U$.

<u>Proof</u>: Let $\{U_\alpha\}$ be an open covering of M^2 with U_α homeomorphic to E^2 for all α. Now let $\lambda > 0$ be the Lebesgue number of this covering. The proof of Lemma 1 in [1] ensures a finite set of pairwise disjoint, closed 2-cells each of diameter $< \lambda$, say $B_1, B_2, \ldots, B_t$, with $C \subset \bigcup_{i=1}^{t} \operatorname{int} B_i$. Choose $p_i \in B_i$ for each

$i = 1,2,\ldots,t$ and $q_i \in U$ for $i = 1,2,\ldots,t$ (with $q_i \neq q_j$ for $i \neq j$). There exists a homeomorphism f of M^2 onto itself with $f(p_i) = q_i$ for all $i = 1,2,\ldots,t$. So $f^{-1}(U)$ is an open neighborhood of $\{p_1,p_2,\ldots,p_t\}$. Since C is compact and $C \subset \bigcup_{i=1}^{t} \operatorname{int} B_i$, there is a homeomorphism g of M^2 onto itself (g is the identity on $M^2 - \bigcup_{i=1}^{t} \operatorname{int} B_i$, and in $\operatorname{int} B_i$ shrinks $B_i \cap C$ into $f^{-1}(U) \cap B_i$) with $g(C) \subset f^{-1}(U)$. Thus $h = fg$ is the desired homeomorphism. $\square$

<u>Lemma 1.2</u>: For each integer $n > 0$, let C be a compact 0-dimensional subset of T_n.

Then $T_n - C$ cannot be imbedded in T_t where $t < n$.

<u>Proof</u>: For each integer $n > 0$, let a simplicial complex of dimension 2, K, be a triangulation of T_n, and C a compact 0-dimensional subset of T_n. Let U be the interior of one of the 2-dimensional simplices of K. Then by Lemma 1.1, there is a homeomorphism, h, of T_n onto itself with $h(C) \subset U$. Thus $h(T_n - C) = T_n - h(C) \supseteq T_n - U \supset |K^{(1)}|$.

Suppose there were an imbedding, f, of $T_n - C$ onto T_t with $t < n$. then $fh^{-1}|_{|K^{(1)}|}$ would be an imbedding of $|K^{(1)}|$ in T_t. But this contradicts the theorem of 5.5 in [24], which says that the 1-skeleton of a triangulation of an orientable 2-manifold is a minimal imbedding (i.e. the 1-skeleton cannot be imbedded in a surface of lower genus).

Hence the lemma. $\square$

Lemma 1.3: Let M^2 be a surface formed, as in Theorem 3 of [21], from a sphere S by first removing a closed, totally disconnected set X from S, then removing the interiors of a finite or infinite sequence $D_1, D_2, \ldots$ of non-overlapping closed discs in $S - X$, and suitably identifying the boundaries of these discs in pairs, except perhaps the boundary of one disc may be identified with itself to produce a "cross cap"; in the finite case there is such a "cross cap" and in the infinite case, there may or may not occur such a "cross cap". The sequence $\{D_i\}$ has the property that for any open set U in S, $X \subset U$, all but a finite number of the D_i are contained in U. P will denote this last property in the proof.

Then M^2 cannot be imbedded in T_K for any integer $K \geq 0$.

Proof: Suppose M^2 is as above and can be imbedded in T_K for some integer $K \geq 0$. Let n be an integer greater than K. As M^2 imbeds as an open submanifold of orientable T_K, M^2 is orientable. Thus in the construction of M^2 from S in the hypothesis, no "cross cap" could have occurred. So we cannot have the case of a finite number of D_i. Hence we may assume that the D_i have been indexed so that in the identification to create M^2, the boundary of D_{2j-1} is identified with the boundary of D_{2j} for $j = 1,2,\ldots$.

Let g be the quotient map giving M^2 from $S - (X \cup \bigcup_{i=1}^{\infty} \text{int } D_i)$, $g: S - (X \cup \bigcup_{i=1}^{\infty} \text{int } D_i) \to M^2$. Let $B = M^2 - g(\bigcup_{i=2n+1}^{\infty} \text{boundary } D_i)$. Then B imbeds in T_K, as M^2

does. But B is the quotient under the quotient map $g|_{S-(X\cup\bigcup_{i=1}^{\infty}\text{int } D_i\cup\bigcup_{i=2n+1}^{\infty} D_i)}$. By property P of the hypotheses, for each $i \geq 2n+1$, d_i = distance $(D_i, X \cup \bigcup_{\substack{j=1\\ j\neq i}}^{\infty} D_j) > 0$. Let V_i be the open $d_i/3$ neighborhood of D_i in S. Let $p_i \in \text{int } D_i$ for each $i \geq 2n+1$. Then clearly $S - (X \cup \bigcup_{i=1}^{\infty} \text{int } D_i \cup \bigcup_{i=2n+1}^{\infty} D_i)$ is homeomorphic to $S - (X \cup \bigcup_{i=1}^{2n} \text{int } D_i \cup \bigcup_{i=2n+1}^{\infty} \{p_i\})$ by a homeomorphism ,h, which is the identity outside $\bigcup_{i=2n+1}^{\infty} V_i$, (shrink D_i to the point p_i inside V_i for each $i \geq 2n+1$).

Now, $\bigcup_{i=2n+1}^{\infty} \{p_i\}$ is a 0-dimensional set, and X is 0-dimensional (since it is compact and totally disconnected, see [11] Theorem D, p. 22), so $D = X \cup \bigcup_{i=2n+1}^{\infty} \{p_i\}$ is 0-dimensional. D is also closed.

So $gh^{-1}: S - (X \cup \bigcup_{i=2n+1}^{\infty} \{p_i\} \cup \bigcup_{i=1}^{2n} \text{int } D_i) \to B$ is actually a quotient map, and identifies only boundary D_{2j} with boundary D_{2j-1} for each $j = 1,2,\ldots,n$.

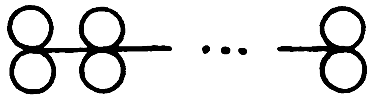
This quotient, B, is homeomorphic to a sphere with n handles less a closed 0-dimensional set.

But then $K < n$ and B imbeds in T_K, which contradicts Lemma 1.2. Hence the supposition was false, and so the lemma is proved. □

<u>Theorem 1.4</u>: Let M be an open connected 2-manifold which imbeds in T_n, k = minimum $\{t \mid M$ can be imbedded in $T_t\}$.

Then (1) If $n = k$, then M is homeomorphic to $T_n - D$ where D is a closed 0-dimensional subset of T_n.

(2) If $n > k$, then M is homeomorphic to $T_n - D$, where D is a 1-dimensional set, either (a) a finite set plus



1 2 ... (n-k)

or (b) a 0-dimensional set plus the union of $(n-k)$ pairwise disjoint homeomorphs of the 1-sphere.

<u>Proof</u>: Let M, T_n, and k be as in the hypotheses of the theorem. By Theorem 3 of [21], M is homeomorphic to either one of the surfaces of the hypotheses of Lemma 1.3 or to the complement of a closed 0-dimensional set C in T_ℓ for some integer $\ell \geq 0$. The first of these cannot occur (else by Lemma 1.3, M could not imbed in T_n, which contradicts the hypotheses of the theorem).

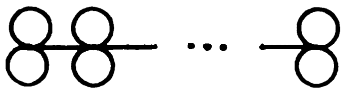
Thus M is homeomorphic to $T_\ell - C$ where C is a closed 0-dimensional subset of T_ℓ. By Lemma 1.2, M cannot imbed in T_t for $t < \ell$. But M imbeds in T_k, so $k \geq \ell$. By the minimality of k, $k \leq \ell$. Hence $k = \ell$.

(1) If $n = k = \ell$, then letting $D = C$ we have conclusion (1).

(2) If $n > k = \ell$, then $|C|$ = the cardinality of C is either $< 2(n-k)$ or $\geq 2(n-k)$.

a) In the case that $|C| < 2(n-k)$ (M is not compact, so $|C| \geq 1$) let $p \in C$, and U be an open neighborhood of p, which is homeomorphic to E^2 by homeomorphism $g: U \to E^2$ and $C \cap U = \{p\}$. In U, let $A = g^{-1}(B)$, where $B \subseteq E^2$ is the union of $2(n-k)$ pairwise disjoint closed 2-discs, $D_1, D_2, \ldots, D_{2(n-k)}$, and $2(n-k)-1$ arcs, connected in the manner of the sketch  .

$D_1 \quad D_2 \quad D_3 \quad \cdots \quad D_{2(n-k)}$

Since B is cellular in E^2, A is cellular in T_ℓ, and so there is a homeomorphism, f, from $T_\ell - \{p\}$ onto $T_\ell - A$. Thus $\bar{f} = f|_{T_\ell - C}$ is a homeomorphism from $T_\ell - C$ onto $(T_\ell - A) - f(C - \{p\})$. Let $F = \bigcup_{i=1}^{2(n-k)} \operatorname{int} D_i$, and in $T_\ell - F$ make the identification of the boundary D_{2i-1} with the boundary of D_{2i} for $i = 1,2,\ldots,n-k$. This gives T_n as the quotient space of $T_\ell - F$ (as $\ell = k$). Call the quotient map h, $h: T_\ell - F \to T_n$. So $h|_{T_\ell - A}: T_\ell - A \to T_n - h(A - F)$ is a homeomorphism and $\bar{h} = h|_{(T_\ell - A) - f(C - \{p\})}$ is a homeomorphism onto $T_n - [h(A-F) \cup hf(C - \{p\})]$. Thus $\bar{h}\,\bar{f}$ is a homeomorphism from $T_\ell - C$ onto $T_n - [h(A-F) \cup hf(C - \{p\})]$. But M is homeomorphic to $T_\ell - C$ and $hf(C - \{p\})$ if finite and $h(A-F)$ is clearly homeomorphic to  .

$1 \quad 2 \qquad\qquad (n-k)$

b) In the case that $|C| \geq 2(n-k)$, let $p_1, p_2, \ldots, p_{2(n-k)}$ be $2(n-k)$ distinct elements of C. So clearly there exist pair-wise disjoint open sets $U_1, U_2, \ldots, U_{2(n-k)}$ of T_ℓ with $p_i \in U_i$ for $i = 1,2,\ldots,2(n-k)$ and U_i homeomorphic to E^2 for $i = 1,2,\ldots,2(n-k)$. For each $i = 1,2,\ldots,2(n-k)$ let D_i be a closed disk, cellular in U_i, $D_i \subset U_i$. Then there exists a homeomorphism h_i from $U_i - \{p_i\}$ onto $U_i - D_i$, with h_i equal to the identity outside a compact set which is a neighborhood of D_i in U_i. Define a homeomorphism h from

$T_\ell - \{p_1, p_2, \ldots, p_{2(n-k)}\}$ onto $T_\ell - \bigcup_{i=1}^{2(n-k)} D_i$ by $h|_{U_i} = h_i$ for $i = 1,2,\ldots,2(n-k)$ and $h|_{T_\ell - \bigcup_{i=1}^{2(n-k)} U_i}$ = the identity. Let $B = \bigcup_{i=1}^{2(n-k)} \operatorname{int} D_i$, and in $T_\ell - B$ make the identification of the boundary of D_{2i-1} with the boundary of D_{2i}, for $i = 1,2,\ldots,n-k$. This gives T_n as the quotient space of $T_\ell - B$ (as $\ell = k$). Call the quotient map g, $g: T_\ell - B \to T_n$. $\bar{g} = g|_{T_\ell - \bigcup_{i=1}^{2(n-k)} D_i}$ is clearly a homeomorphism onto $T_n - X$ where X is the pairwise disjoint union of $(n-k)$ homeomorphs of the 1-sphere. Let $\bar{h} = h|_{T_\ell - C}$, then $\bar{g}\,\bar{h}$ is a homeomorphism from $T_\ell - C$ onto $T_n - [X \cup gh(C - \{p_1, p_2, \ldots, p_{2(n-k)}\})]$. But M is homeomorphic to $T_\ell - C$ and $gh(C - \{p_1, p_2, \ldots, p_{2(n-k)}\})$ is a 0-dimensional set with X as above, so we have (2) (b). □

<u>Corollary 1.5</u>: Let M be an open connected 2-manifold which imbeds in T_n. Then there exists $N \subset T_n$ such that N is homeomorphic to M and $\bar{N} = T_n$.

<u>Proof</u>: Let $N = T_n - D$ of Theorem 1.4. D clearly has void interior, so $\bar{N} = \overline{T_n - D} = T_n$. □

<u>Theorem 1.6</u>: Let M^2 be a compact 2-manifold without boundary and C a 0-dimensional compact subset of M^2. If $M^2 - C$ is 1-movable, then M^2 is a 2-sphere.

<u>Proof</u>: Let $M^2 - C$ be 1-movable, and U the interior of D, a closed 2-cell in M^2. By Lemma 1.1, there exists a homeomorphism, h, of M^2 onto itself with $h(C) \subset U$. Then using D as the

initial 2-cell in the proof of Theorem 1 of [7], there is a standard decomposition of M^2, $M^2 = P^2 \cup A$ (with P^2 homeomorphic to E^2 and $U \subset P^2$). But A is compact in M^2, so $h^{-1}(A)$ is compact in $M^2 - C$. $M^2 - C$ is 1-movable, so there exists a homeomorphism g of $M^2 - C$ onto itself with $g(h^{-1}(A)) \cap h^{-1}(A) = \emptyset$. Thus $h(gh^{-1}(A)) \cap hh^{-1}(A) = \emptyset$. By Lemma 2.1, there exists a homeomorphism $\hat{g}$ of M^2 onto itself with $\hat{g}|_{M^2-C} = g$. $h\hat{g}h^{-1}$ is a homeomorphism of M^2 onto itself. But $h\hat{g}h^{-1}(A) \cap A = hgh^{-1}(A) \cap hh^{-1}(A) = \emptyset$. Thus $(hgh^{-1})(A) \subset P^2$. So by Corollary 1 of [7], M^2 is a 2-sphere. □

Theorem 1.7: The open Möbius band is not 1-movable.

Proof: Let the open Möbius band M be given by the quotient map $g: [0,1] \times (0,1) \to M$ by the identification of $(0,t)$ with $(1,1-t)$ for each $t \in (0,1)$. $A = g([0,1] \times [\frac{1}{3}, \frac{2}{3}])$ is a compact subset of M and $A \supset U = g([0,1] \times (\frac{1}{3}, \frac{2}{3}))$ which is an open Möbius band and so not orientable. If M were 1-movable there would exist a homeomorphism h of M onto itself with $h(A) \subset M - A$. But $M - A$ is homeomorphic to $S^1 \times (0,1)$ and so is orientable, while $h(U) \subset M - A$ and so $h(U)$ is orientable. Hence U is orientable. This is a contradiction. □

Theorem 1.8: All open 2-manifolds are 2-movable.

Proof: By a theorem of Jones [12], an open 2-manifold, M, is the union of two open 2-cells, and so M is 2-movable by Theorem 5.5. □

CHAPTER II

1-MOVABILITY AND 0-DIMENSIONAL SETS IN MANIFOLDS

Theorem 1.6 tells us that if the complement in a closed 2-manifold of a closed 0-dimensional set is 1-movable, then that manifold is a 2-sphere. In this chapter we consider the complement of closed 0-dimensional subsets of closed n-manifolds. The main result of this chapter is Corollary 2.5, if M^n is a closed n-manifold, C a closed 0-dimensional subset of M^n, and $M^n - C$ is 1-movable, then M^n is the n-sphere and C has an invert point. We end this chapter with some examples. For each integer $n \geq 3$ there are two Cantor sets in S^n such that the complement of one of them is 1-movable while that of the other is not 1-movable. But a Cantor set clearly has an invert point and so the converse of Corollary 2.5 is not true for $n \geq 3$. We begin with a lemma.

Lemma 2.1: Let M^n be a compact n-dimensional manifold without boundary $(n \geq 2)$ and C a closed 0-dimensional subset of M^n.

Then for each homeomorphism h of $M^n - C$, there exists a homeomorphism $\hat{h}$ of M^n such that $\hat{h}|_{M^n-C} = h$.

Proof: Given the homeomorphism h of $M^n - C$ as above, define $\hat{h}$ as follows in i) and ii).

i) $\hat{h}(x) = h(x) \ \forall\, x \in M^n - C$, so $\hat{h}|_{M^n-C} = h$.

ii) Given $e \in C \subseteq M^n$, there is an open neighborhood U of e with a homeomorphism $g\colon E^n \xrightarrow[\text{onto}]{} U$ with $g(0,\ldots,0) = e$. Let $U_i = g(S_{(0,\ldots,0),\ 1/i})$ for $i = 1,2,\ldots$, then U_i is homeomorphic to E^n and $\bigcap_{i=1}^{\infty} U_i = e$. By Theorem IV 4. of [11], p. 48, E^n cannot be disconnected by a subset of dimension $\leq n - 2$, $0 \leq n - 2$, so $U_i \cap (M^n - C) \neq \phi$ is connected. Thus $h(U_i \cap (M^n - C)) \neq \phi$ is connected and so $\overline{h(U_i \cap (M^n - C))}$ is connected and closed, thus compact in M^n. Let $x \in M^n - C$. Then $h^{-1}(x) \in M^n - C$ and $h^{-1}(x) \neq e$. Clearly there exists in $M^n - C$ an open neighborhood V of $h^{-1}(x)$ and integer j_o so that $V \cap U_{j_o} = \phi$, and so $V \cap U_i = \phi$ for all $i \geq j_o$. Thus $h(V)$ is an open neighborhood of $h(h^{-1}(x)) = x$ which is disjoint from $\overline{h(U_i \cap (M^n - C))}$ as h is a homeomorphism. Thus $x \notin \overline{h(U_i \cap (M^n - C))}$ for all $i \geq j_o$ and so $x \notin \bigcap_{i=1}^{\infty} \overline{h(U_i \cap (M^n - C))}$. Thus

$\bigcap_{i=1}^{\infty} \overline{h(U_i \cap (M^n - C))} \subset C$, and is a compact non-void connected set. The components of C are points. Hence

$\bigcap_{i=1}^{\infty} \overline{h(U_i \cap (M^n - C))} = \{t\}$ for some $t \in C$.

Now define $\hat{h}(e) = t$ and denote U_i by ${}_eU_i$ $i = 1,2,\ldots$.

iii) To show that $\hat{h}$ is continuous.

$\hat{h}$ is clearly continuous at each point of $M^n - C$, open in M^n. Let $a \in C$ and $\hat{h}(a) \in \theta$ where θ is open in M^n. Let $V_i = {}_aU_i$ of ii) for $i = 1,2,\ldots$. Then $\{\hat{h}(a)\}$ is the intersection of a decreasing family of non-empty compact sets, $\overline{h(V_i \cap (M^n - C))}$. Since $\{\hat{h}(a)\} \subset \theta$, Theorem 1.6 b) of [8], p. 226 implies there is a positive integer i_o so that $\overline{h(V_{i_o} \cap (M^n - C))} \subset \theta$.

Consider V_{i_o} (an open neighborhood of a). Then $\hat{h}(V_{i_o} \cap (M^n - C)) = h(V_{i_o} \cap (M^n - C)) \subset \theta$. Let $b \in V_{i_o} \cap C$. Now let $W_i = {}_bU_i$ of ii) for $i = 1,2,\ldots$, so the W_i, $i = 1,2,\ldots$, form a neighborhood basis for b in M_n. Thus $\exists\, j_o$ so that $W_{j_o} \subset V_{i_o}$. So $\underline{W_{j_o} \cap (M^n - C)} \subset \underline{V_{i_o} \cap (M^n - C)}$ and $\hat{h}(b) \in h(W_{j_o} \cap (M^n - C)) \subset h(V_{i_o} \cap (M^n - C)) \subset \theta$.

Thus $\hat{h}(V_{i_o}) \subset \theta$ and so $\hat{h}$ is continuous at a. But a was an arbitrary element of C, hence $\hat{h}$ is continuous.

iv) To show that $\hat{h}: M^n \to M^n$ is surjective, we observe that as $\hat{h}$ is continuous and M^n is compact, $\hat{h}(M^n)$ is compact, hence closed in the T_2 space M^n. But M^n is connected and C is 0-dimensional thus $\overline{M^n - C} = M^n$. Now $\hat{h}(M^n) \supset \hat{h}(M^n - C)$ $= h(M^n - C) = M^n - C$ with $\hat{h}(M^n)$ closed, so $\hat{h}(M^n) \supset \overline{M^n - C} = M^n$.

v) $\hat{h}$ is injective by the following.
As h is a homeomorphism of $M^n - C$, so is h^{-1}, and so by i), ii) and iii) there is a continuous map $\widehat{(h^{-1})}$ from M^n to M^n so that $\widehat{(h^{-1})}|_{M^n-C} = h^{-1}$. Note that $\widehat{(h^{-1})}\hat{h}|_{M^n-C} = h^{-1}h = 1_{M^n-C}$, the identity map on $M^n - C$. But $1_{M^n}|_{M^n-C} = 1_{M^n-C}$ also, and as M^n is a T_2 space with $M^n - C$ dense in M^n we have (by 1.5(2), p. 140 of [8]) $\widehat{(h^{-1})}\hat{h}: M^n \to M^n$ is the identity map on M^n. Thus $\hat{h}$ is injective.

vi) Since $\hat{h}$ is a continuous bijective map from the compact space M^n to the T_2 space M^n, $\hat{h}$ is a homeomorphism (by Theorem 8, p. 141 [14]). $\square$

<u>Theorem 2.2</u>: Let M^n be a closed n-manifold which is the carrier of a finite simplicial complex, T, $C \subset M^n$, C is 0-dimensional and closed, $n \geq 2$ and $M^n - C$ is 1-movable. Then either C has an invert point or there exists $p, c \in C$, $p \neq c$, where each homeomorphism h of C has $h(c) = c$ and $h(p) = p$, and given $\varepsilon > 0$, there exist (open in C) neighborhoods θ_p of p and θ_c of c with $\theta_p \subset S_{p,\varepsilon}$, $\theta_c \subset S_{c,\varepsilon}$ and a homeomorphism h of C such that $h(C - \theta_p) \subseteq \theta_c$ (thus p and c are the only fixed points of C). This h is the restriction to C of a homeomorphism of M^n onto itself with the above property for θ_p and θ_c open in M^n.

<u>Proof: i)</u> There exists a sequence K_i, $i = 1,2,\ldots$ of compact connected sets $\subset M^n - C$ such that $M^n - K_i \subset \{x \in M^n | \text{dist } (x,C) \leq \lambda_i\}$ and $\lambda_i \to 0$. To construct the $\{K_i\}$, let $T^{(0)}=T$, let $T^{(i)}$ be the barycentric subdivision of $T^{(i-1)}$ for $i = 1,2,\ldots,$ and λ_i be the mesh of $T^{(i)}$. Then $\lambda_i \to 0$ (Theorem 5-20, [10]). Let $L_i = \{s \in T^{(i)} |\ |s| \cap C = \emptyset\}$ and thus the number of components of $|L_i|$ is finite, say $\ell_1,\ell_2,\ldots,\ell_{t_i}$. $|L_i| \subset M^n - C$ which is path connected (connected by Corollary 1, p. 48 of [11], and each point has a path connected neighborhood so Theorem 5.5, p. 116 of [8] gives path connected), thus there exists arcs p_j in $S^n - C$ from a point of ℓ_1 to a point of ℓ_j for $j = 2,\ldots,t_i$. Let $K_i = (\bigcup_{j=1}^{t_i} \ell_j) \cup (\bigcup_{j=2}^{t_i} p_j)$ which is clearly connected, $\subseteq M^n - C$ and compact and $K_i \supseteq |L_i|$. If $s \in M^n - K_i$ then s is a point of simplex of $T^{(i)}$ which also contains a point of C and has diameter $\leq \lambda_i$; so dist $(s,C) \leq \lambda_i$.

ii). For each positive integer t there exists a compact connected set M_t such that $M_t \subset M^n - C$ and the diameter of each component of $M^n - M_t$ is $< \frac{1}{t}$, and $M^n - M_t \subset \{x \in M^n |\ \text{dist}(x,C) < \frac{1}{t}\}$. Construct M_t as follows for each positive integer t.

As C is 0-dimensional, for each $b \in C$, there is an open neighborhood, θ_p, of b in M^n with (boundary $\theta_b) \cap C = \emptyset$ and $\theta_b \subseteq \{x \in M^n | \text{dist}(x,b) < \frac{1}{2t}\}$. C is compact so a finite number of these suffice, say $\theta_{b_1}, \ldots, \theta_{b_\ell}$, to cover C. Let $N = (M^n - \bigcup_{j=1}^{\ell} \theta_{b_j}) \cup (\bigcup_{j=1}^{\ell} \text{boundary } \theta_{b_j})$. Then $N \subset M^n - C$, N is clearly compact, and each component of $M^n - N$ is a subset of one of the θ_{b_j} and thus has diameter $<$ the diameter of $\theta_{b_j} < \frac{1}{t}$. As $N \cap C = \emptyset$, we have $d = \text{distance } (N,C) > 0$, so there is a λ_i from i) so that $\lambda_i < \text{minimum } (d,\frac{1}{t})$. Thus $N \subset K_i$ and each component of $M^n - K_i$ is thus a subset of one in $M^n - N$ and has diameter $< \frac{1}{t}$. Let $M_t = K_i$, then clearly $M^n - M_t \subseteq \{x \in M^n |\ \text{dist}(x,C) \leq \lambda_i\} \subseteq \{x \in M^n | \text{dist}(x,C) < \frac{1}{t}\}$.

iii) As $M^n - C$ is 1-movable, there exist homeomorphisms h_t of $M^n - C$ onto itself such that $h_t(M_t) \cap M_t = \emptyset$. But $h_t(M_t)$ is connected and so a subset of one of the connected components of $(M^n - M_t) - C$, which must be a subset of one of the components of $M^n - M_t$, all of which have diameter $< \frac{1}{t}$. So $h_t(M_t)$ has diameter $< \frac{1}{t}$. Now, for all t, $M_t \neq \emptyset$, so there exists a sequence $\langle a_t \rangle_{t=1}^{\infty}$ with $a_t \in h_t(M_t)$. But this sequence is in a compact space, M^n, and so has a subsequence which converges in M^n, say $\langle a_{t_j} \rangle_{j=1}^{\infty} \to c \in M^n$.

Thus the sequence of compact connected sets $\{h_{t_j}(M_{t_j})\}_{j=1}^{\infty}$ converges to c.

We now show that $c \in C$. Assume $c \notin C$. So $d =$ distance $(c,C) > 0$. Then there exists j_1, j_2, positive integers, so that dist $(c,a_t) < \frac{d}{4}$ for all $j \geq j_1$, and $\frac{1}{t_{j_2}} < \frac{d}{4}$, and diameter $h_{t_j}(M_{t_j}) < \frac{d}{4}$ for all $j \geq j_2$. Thus for all $j \geq j_0 =$ maximum (j_1, j_2), $h_{t_j}(M_{t_j}) \subset \{x \in M^n | \text{dist } (x,c) < \frac{d}{2}\} = S_{c,d/2}$. So for each $x \in h_{t_{j_0}}(M_{t_{j_0}})$ we have dist $(x,C) \geq d/2$ (else dist $(c,C) < d$, a contradiction) and so dist $(x,C) \geq \frac{d}{2} > \frac{d}{4} > \frac{1}{t_{j_2}} > \frac{1}{t_{j_0}}$. By ii) we have $x \notin M^n - M_{t_{j_0}}$ (if not, dist $(x,C) < \frac{1}{t_{j_0}}$) thus $x \in M_{t_{j_0}}$. But then $h_{t_{j_0}}(M_{t_{j_0}}) \subseteq M_{t_{j_0}}$ which contradicts the first sentence of iii), i.e. $h_t(M_t) \cap M_t = \phi$, as $M_{t_{j_0}} \neq \phi$. Thus the assumption $c \notin C$ is false and so $c \in C$.

iv) Note Lemma 2.1 says that any homeomorphism of $M^n - C$ extends to a homeomorphism of M^n. Thus we have for each j a homeomorphism g_{t_j} which is the extension to M^n of h_{t_j}, and so $\{g_{t_j}(M_{t_j})\}_{j=1}^{\infty}$ converges to c.

v) Assume C has no invert point (x is an invert point of the topological space X means for each open neighborhood θ of x, there exists a homeomorphism h of X so that $h(X-\theta) \subseteq \theta$). For each $j = 1,2,\ldots,$ let U_j be the component of $M^n - M_{t_j}$ that has $c \in U_j$. There is a k so that $g_{t_k}(M_{t_k}) \subset U_j$ and $M_{t_j} \subset M_{t_k}$. So $g_{t_k}(M_{t_j}) \subset U_j$; denote this g_{t_k} by $\hat{h}_{t_j}$. Let θ_j be that component of $M^n - M_{t_j}$ such that $\hat{h}_{t_j}(\theta_j) \supseteq M_{t_j}$ and hence $\hat{h}_{t_j}(\theta_j) \supseteq M^n - U_j$ which is connected and contains M_{t_j}. C has more than one element (or else it would have an invert point) so let $z \in C$, $z \neq c$. Let $x =$ distance (z,c). Then there is a positive integer j_0

so that for each integer $j \geq j_0$, $1/j < \frac{x}{2}$. But U_j has diameter $< \frac{1}{t_j} \leq \frac{1}{j} < \frac{x}{2}$ and $c \in U_j$, so $z \notin U_j$. Thus $z \in M^n - U_j \subset \hat{h}_{t_j}(\theta_j)$ and hence $\hat{h}_{t_j}^{-1}(z) \in \theta_j$. But $\hat{h}_{t_j}|_C$ is a homeomorphism of C onto itself, so $\hat{h}_{t_j}^{-1}(z) \in C \cap \theta_j$. Let $b_j = \hat{h}^{-1}_{t_{(j+j_0)}}(z)$. Then $<b_j>_{j=1}^{\infty}$ is a sequence in compact C, and so has a subsequence $<b_{j_\ell}>_{\ell=1}^{\infty}$ (with $j_1 \geq j_0$) which converges to a point, say p, of C. Thus the sequence $<\theta_{j_\ell}>_{\ell=1}^{\infty}$ converges to p.

So, given $\epsilon > 0$, there exists an ℓ such that $\theta_{j_\ell} \subset S_{p,\epsilon}$ and $\frac{1}{t_{j_\ell}} \leq \frac{1}{j_\ell} \leq \frac{1}{\ell} < \epsilon$, so $U_{j_\ell} \subset S_{c,\epsilon}$ (as $c \in U_{j_\ell}$ which has diameter $< \frac{1}{t_{j_\ell}} < \epsilon$) and $\hat{h}_{t_{j_\ell}}(\theta_{j_\ell}) \supseteq M^n - U_{j_\ell}$. Denote $\hat{h}_{t_{j_\ell}}|_C$ by h_ϵ and $\theta_{j_\ell} \cap C$ by θ_ϵ and $U_{j_\ell} \cap C$ by U_ϵ. Then for $h = h_\epsilon$, $\theta_p = \theta_\epsilon$, $\theta_c = U_\epsilon$, we have $h(\theta_p) \supseteq C - \theta_c$ and so $h(C - \theta_p) \subseteq \theta_c$. Note $\theta_\epsilon \subset S_{p,\epsilon}$, $U_\epsilon \subset S_{c,\epsilon}$, $h^{-1}(C - \theta_c) \subset \theta_p$. It remains to show that $p \in \theta_p = \theta_\epsilon \ \forall \epsilon$, $p \neq c$, and that p and c are fixed points under homeomorphisms of C.

vi) $p \neq c$. Suppose $p = c$. Let θ be any open neighborhood of p. Then for some positive number ϵ, $S_{p,\epsilon} = S_{c,\epsilon} \subset \theta$. Now by v) there exist open sets θ_ϵ and U_ϵ neighborhoods of p and c respectively and a homeomorphism h of C so that $h(C - \theta_\epsilon) \subseteq U_\epsilon$ and $\theta_\epsilon \subseteq S_{p,\epsilon}$ and $U_\epsilon \subseteq S_{c,\epsilon}$. But $C - \theta \subseteq C - S_{p,\epsilon} \subseteq C - \theta_\epsilon$ so $h(C - \theta) \subseteq h(C - \theta_\epsilon) \subseteq U_\epsilon \subseteq S_{c,\epsilon} \subseteq \theta$. Thus $p = c$ is an invert point of C. This is a contradiction.

vii) There does not exist a homeomorphism h of C onto itself such that $h(p) = c$. Suppose there were a homeomorphism h of C such that $h(p) = c$. Let θ be an open neighborhood of c. Then for some $\beta > 0$, $S_{c,\beta} \subset \theta$. Let $\alpha = 1/2 \min(\beta, \text{dist}(c,p)) > 0$. So $S_{c,\alpha} \cap S_{p,\alpha} = \phi$ and $S_{c,\alpha} \subset \theta$, h is continuous so there exists $\delta > 0$ such that $h(S_{p,\delta}) \subset S_{c,\alpha}$ and $\delta < \alpha$. As C is 0-dimensional there exists a clopen (closed and open) neighborhood B of p, $B \subset S_{p,\delta}$. So $h(B) \subset S_{c,\alpha}$ is a clopen neighborhood of c. Since B is a neighborhood of p, there exists a $\lambda > 0$ such that $S_{p,\lambda} \subset B$ and $\lambda < \alpha$. By v) there exist open sets θ_λ and U_λ and a homeomorphism f of C so that $\theta_\lambda \subset S_{p,\lambda}$, $U_\lambda \subset S_{c,\lambda}$ and $f(C - \theta_\lambda) \subset U_\lambda$. Define a homeomorphism μ of C by letting $\mu|_B = h|_B$, $\mu|_{h(B)} = h^{-1}|_{h(B)}$ and $\mu|_{C-(B \cup h(B))}$ = the identity on $C - (B \cup h(B))$ (this is a homeomorphism as B and $h(B)$ are clopen, so $C - (B \cup h(B))$ also is clopen, and h and h^{-1} are homeomorphisms). Consider the homeomorphism $f\mu$ on C. $\mu(C - S_{c,\beta}) \subset \mu(C - h(B)) \subset C - \mu h(B) = C - B \subset C - S_{p,\lambda}$. So $f\mu(C - \theta) \subset f\mu(C - S_{c,\beta}) \subset f(C - S_{p,\lambda}) \subset f(C - \theta_\lambda) \subset U_\lambda \subset S_{c,\lambda} \subset S_{c,\alpha} \subset \theta$. Thus c is an invert point of C. But this is a contradiction.

viii) $p[c]$ is a fixed point of C; i.e. both are fixed. Suppose not, then there exists a homeomorphism f of C so that $f(p) \neq p$ $[f(c) \neq c]$, then $f(p) \notin \{p,c\}$ $[f(c) \notin \{p,c\}]$, as no homeomorphism of C can send p to c or c to p by vii). Given any neighborhood θ of $c[p]$ there is a $\beta > 0$ so that $S_{c,\beta} \subset \theta$ $[S_{p,\beta} \subset \theta]$. Let $\alpha = \frac{1}{2}\min(\beta, \text{dist}(p,c), \text{dist}(f(p),c), \text{dist}(f(p),p))$ $[\alpha = 1/2 \min(\beta, \text{dist}(p,c), \text{dist}(f(c),c), \text{dist}(f(c),p))]$. As C is 0-dimensional, there exists a clopen neighborhood B of $p[c]$ with $B \subset S_{p,\delta}$ $[B \subset S_{c,\delta}]$ where $\delta > 0$ is that given by the continuity of f so that

$f(S_{p,\delta}) \subset S_{f(p),\alpha}$, $\delta < \alpha$ $[f(S_{c,\delta}) \subset S_{f(c),\alpha}$, $\delta < \alpha]$.

Thus we have $f(B) \subset S_{f(p),\alpha}$ and so $f(B) \subset C - S_{p,\alpha}$ $[f(B) \subset S_{f(c),\alpha}$ and so $f(B) \subset C - S_{c,\alpha}]$.

But B is a clopen neighborhood of $p[c]$ so there is a $\lambda > 0$ with $\lambda < \alpha$ so that $S_{p,\lambda} \subset B$ $[S_{c,\lambda} \subset B]$. By v) there exist open sets θ_λ, U_λ and a homeomorphism h_λ so that $h_\lambda(C - \theta_\lambda) \subset U_\lambda$ and $h_\lambda^{-1}(C - U_\lambda) \subset \theta_\lambda$ and $\theta_\lambda \subset S_{p,\lambda}$, $U_\lambda \subset S_{c,\lambda}$.
Thus $h_\lambda f(B) \subset h_\lambda(C - S_{p,\alpha}) \subset h_\lambda(C - S_{p,\lambda}) \subset h_\lambda(C - \theta_\lambda) \subset U_\lambda \subset S_{c,\lambda}$ $[h_\lambda^{-1}f(B) \subset h_\lambda^{-1}(C - S_{c,\alpha}) \subset h_\lambda^{-1}(C - S_{c,\lambda}) \subset h_\lambda^{-1}(C - U_\lambda) \subset \theta_\lambda \subset S_{p,\lambda}]$ and $h_\lambda f(B)[h_\lambda^{-1}f(B)]$ is clopen. $B \cap h_\lambda f(B) = \emptyset[B \cap h_\lambda^{-1}f(B) = \emptyset]$.

Define a homeomorphism μ of C by $\mu|_B = h_\lambda f|_B$ $\mu|_{h_\lambda f(B)} = (h_\lambda f)^{-1}|_{h_\lambda f(B)}$ and $\mu|_{C-(B \cup h_\lambda f(B))}$ = identity on $C - (B \cup h_\lambda f(B))$
$[\mu|_B = h_\lambda^{-1}f|_B$, $\mu|_{h_\lambda^{-1}f(B)} = (h_\lambda^{-1}f)^{-1}|_{h_\lambda^{-1}f(B)}$, and $\mu|_{C-(B \cup h_\lambda^{-1}f(B))}$ = identity on $C - (B \cup h_\lambda^{-1}f(B))]$.
Consider the homeomorphism $h_\lambda\mu$ of C $[h_\lambda^{-1}\mu$ of $C]$.
$\mu(C - \theta) \subset \mu(C - h_\lambda f(B)) = C - \mu(h_\lambda f(B)) = C - B \subset C - S_{p,\lambda} \subset C - \theta_\lambda$ so $h_\lambda\mu(C - \theta) \subset h_\lambda(C - \theta_\lambda) \subset U_\lambda \subset S_{c,\lambda} \subset S_{c,\alpha} \subset S_{c,\beta} \subseteq \theta$. Hence c is an invert point of C, a contradiction.
$[\mu(C - \theta) \subset \mu(C - h_\lambda^{-1}f(B)) = C - \mu(h_\lambda^{-1}f(B)) = C - B \subset C - S_{c,\lambda} \subset C - U_\lambda$ so $h_\lambda^{-1}\mu(C - \theta) \subset h_\lambda^{-1}(C - U_\lambda) \subset \theta_\lambda \subset S_{p,\lambda} \subset S_{p,\alpha} \subset S_{p,\beta} \subseteq \theta$.

Hence p is an invert point of C, a contradiction.]

ix) If $p \notin \theta_{\epsilon_0}$ for some ϵ_0, then let $\alpha = 1/2$ minimum $(\epsilon_0$, dist $(p,c))$. So there exist open sets θ_α and U_α and a homeomorphism h_α of C such that $h_\alpha(C - \theta_\alpha) \subset U_\alpha$, $c \in U_\alpha$, $\theta_\alpha \subset S_{p,\alpha}$, $U_\alpha \subset S_{c,\alpha}$. If $p \in \theta_\alpha$, replace θ_ϵ, U_ϵ, and h, by

θ_α, U_α, and h_α respectively. If $p \notin \theta_\alpha$. Then as $\theta_\alpha \cap U_\alpha = \phi$ we have $h_\alpha(p) \subset h_\alpha(C - \theta_\alpha) \subset U_\alpha$ and $p \notin U_\alpha$. Thus p is not a fixed point after all, a contradiction. Hence $p \in \theta_\alpha$ as above. And for the new choice of $\theta_p = \theta_\alpha$ and $\theta_c = U_\alpha$, and $h = h_\alpha$ we have $p \in \theta_p$. Making this change for all ϵ_0 such that $p \notin \theta_{\epsilon_0}$ gives us the theorem. □

Theorem 2.3: The non-invert point case in the conclusion of Theorem 2.2 cannot occur.

Proof: Assume as in the non-invert point case in the conclusion of Theorem 2.2, that there exist $p, c \in C$, C a compact 0-dimensional metric space with $h(p) = p$ and $h(c) = c$ for all homeomorphisms h of C onto itself. Also given any $\epsilon > 0$, there exist open neighborhoods θ_p of p and θ_c of c, with $\theta_p \subset S_{p,\epsilon}$, $\theta_c \subset S_{c,\epsilon}$ and a homeomorphism h of C onto itself such that $h(C - \theta_p) \subseteq \theta_c$ (note that h^{-1} is thus a homeomorphism of C onto itself such that $h^{-1}(C - \theta_c) \subseteq \theta_p$).

Hence we have that $h(C - S_{p,\epsilon}) \subset h(C - \theta_p) \subset \theta_c \subset S_{c,\epsilon}$ and $h^{-1}(C - S_{c,\epsilon}) \subset h^{-1}(C - \theta_c) \subset \theta_p \subset S_{p,\epsilon}$.

Now let $d = 1/3$ distance (p,c), so $d > 0$ and since C is 0-dimensional there exist clopen sets D_0 and E_0 with $p \in D_0 \subseteq S_{p,d}$ and $c \in E_0 \subseteq S_{c,d}$. Note $D_0 \cap E_0 = \phi$. As D_0 and E_0 are open, there exists an $\alpha_0 > 0$ so that $S_{p,\alpha_0} \subset D_0$ and $S_{c,\alpha_0} \subset E_0$, so $\alpha_0 \leq d$. So by the hypotheses there exists a homeomorphism h_0 of C onto itself such that $h_0(C - S_{c,\alpha_0}) \subset S_{p,\alpha_0}$. So $h_0(C - E_0) \subseteq h_0(C - S_{c,\alpha_0}) \subseteq S_{p,\alpha_0} \subset D_0$. Let $G_0 = C - (D_0 \cup E_0) \subset C - E_0$, then $h_0(G_0) \subset D_0$. Note G_0 and $h_0(G_0)$

are clopen sets. Let d_1 = minimum $\{\frac{d}{2}, \frac{\alpha_0}{2}, \frac{1}{2}$ distance $(\{p,c\}, G_0 \cup h_0(G_0))\}$. Then there exist clopen sets D_1 and E_1 with $p \in D_1 \subseteq S_{p,d_1}$ and $c \in E_1 \subseteq S_{c,d_1}$. As D_1 and E_1 are open, there exists an $\alpha_1 > 0$ so that $S_{p,\alpha_1} \subset D_1$ and $S_{c,\alpha_1} \subset E_1$, so $\alpha_1 \leq d_1$. By the hypotheses there exists a homeomorphism h_1 of C onto itself such that $h_1(C - S_{p,\alpha_1}) \subseteq S_{c,\alpha_1}$. So $h_1(C - D_1) \subseteq h_1(C - S_{p,\alpha_1}) \subseteq S_{c,\alpha_1} \subseteq E_1$. Let $G_1 = D_0 - (h_0(G_0) \cup D_1) \subset C - D_1$, then $h_1(G_1) \subset E_1$.

Define inductively d_i, D_i, E_i, α_i, h_i, and G_i as follows for each integer $i \geq 2$. Let d_i = minimum $\{\frac{d}{2^i}, \frac{\alpha_{i-1}}{2}, \frac{1}{2}$ distance $(\{p,c\}, \bigcup_{j=0}^{i-1} (G_j \cup h_j(G_j)))\}$, then there exist clopen sets D_i and E_i with $p \in D_i \subset S_{p,d_i}$ and $c \in E_i \subset S_{c,d_i}$. As D_i and E_i are open, there exists an $\alpha_i > 0$ so that $S_{p,\alpha_i} \subset D_i$ and $S_{c,\alpha_i} \subset E_i$. Note that $D_i \subset D_{i-1}$ and $E_i \subset E_{i-1}$.

Now if i is even, by the hypotheses there exists a homeomorphism h_i of C onto itself such that $h_i(C - S_{c,\alpha_i}) \subset S_{p,\alpha_i}$. So $h_i(C - E_i) \subset h_i(C - S_{c,\alpha_i}) \subset S_{p,\alpha_i} \subset D_i$. Let $G_i = E_{i-2} - (h_{i-1}(G_{i-1}) \cup E_i) \subset C - E_i$, then $h_i(G_i) \subset D_i$.

If i is odd, by the hypotheses there exists a homeomorphism h_i of C onto itself such that $h_i(C - S_{p,\alpha_i}) \subset S_{c,\alpha_i}$. So $h_i(C - D_i) \subset h_i(C - S_{p,\alpha_i}) \subset S_{c,\alpha_i} \subset E_i$. Let $G_i = D_{i-2} - (h_{i-1}(G_{i-1}) \cup D_i) \subset C - D_i$, then $h_i(G_i) \subset E_i$.

Define the function f of C onto itself by $f|_{G_i} = h_i|_{G_i}$, $f|_{h_i(G_i)} = h_i^{-1}|_{h_i(G_i)}$, $f(p) = c$, and $f(c) = p$. Since E_i and D_i are clopen for all i, the G_i and $h_i(G_i)$ are clopen for all i (inductively as G_0 is clopen and all the h_i are homeomorphisms), and the family of sets $\{H | H = G_i$ or $h_i(G_i)$ for some non-negative integer $i\}$ is easily seen to be pairwise disjoint, we have that f is a bijective function of C onto itself which is continuous on $\bigcup_{i=0}^{\infty} (G_i \cup h_i(G_i)) = C - \{p,c\}$. As $d_i \leq \frac{d}{2^i}$, the D_i's converge to p and the E_i's converge to c, thus for $i = 2k+1$, $D_i = \{p\} \cup \bigcup_{j=k+1}^{\infty} G_{2j+1} \cup \bigcup_{j=k+1}^{\infty} h_{2j}(G_{2j})$, while for $i = 2k$, $E_i = \{c\} \cup \bigcup_{j=k+1}^{\infty} G_{2j} \cup \bigcup_{j=k}^{\infty} h_{2j+1}(G_{2j+1})$. So for i even $f(E_i) \subset D_{i-1}$ and for i odd $f(D_i) \subset E_{i-1}$. By this, f is easily seen to be continuous also at p and c. Hence f, a bijective continuous function from a compact space to a Hausdorff space, is a homeomorphism.

But $f(p) \neq p$ (and $f(c) \neq c$) which contradicts our hypotheses. Hence our assumption of the existence of a space satisfying the properties of the non-invert point case in the conclusion of Theorem 2.2 was false.□

Theorem 2.4: If M^n is a closed n-manifold, C is a compact 0-dimensional subset of M^n, and $M^n - C$ is 1-movable, then M^n is an n-sphere.

Proof: Let M^n and C be as in the hypothesis and suppose M^n were not an n-sphere. A result of D. Galewski, Corollary 1.14 of [9] is that C has an open neighborhood, U, which imbeds in E^n. So $M^n - U$ is a compact subset of $M^n - C$. d = distance $(C, M^n - U) > 0$. Let c, $\{M_{t_j}\}$, g_{t_j} be as in the proof of Theorem 2.2. $c \in U$, which is open, and M^n is a n-manifold without boundary, so there is an open set V which is homeomorphic to E^n and $c \in V \subset U$. Since the $g_{t_j}(M_{t_j})$ converge to c, there is a positive integer y such that $g_{t_j}(M_{t_j}) \subset V$ for all integers $j > y$. There is an integer $b > y$ so that $\frac{1}{b} < d$. Now, $t_b \geq b$ so $\frac{1}{t_b} \leq \frac{1}{b} < d$. Hence $M_{t_b} \supseteq (M^n - U)$, so $g_{t_b}(M^n - U) \subset g_{t_b}(M_{t_b}) \subset V$. So $g_{t_b}(M^n - U)$ is contained in the interior of a closed n-cell, D, with bicollared boundary in V. Hence $M^n - U$ is contained in a closed n-cell $(g_{t_b})^{-1}(D)$ with bicollared boundary, which shall be taken as the initial n-cell in the proof of Theorem 1 of [7]. Thus the "standard decomposition" of [7] which is obtained, $P^n \cup R = M^n$, has $M^n - U \subset P^n$ so $R \subset U$. This says R has a neighborhood, U, which can be imbedded in E^n. But this contradicts Theorem 4 of [7].□

Theorems 2.2, 2.3, and 2.4 together imply immediately the following corollary.

Corollary 2.5: If M^n is a closed n-manifold, C a closed 0-dimensional subset of M^n, $M^n - C$ is 1-movable and $n \geq 2$, then M^n is the n-sphere and C has an invert point.

Corollary 2.6: Let C be any Cantor set in E^n, $n \geq 2$. Then $E^n - C$ is not 1-movable.

Proof: Let h be the inverse of the stereographic projection homeomorphism, $h: E^n \to S^n - \{p\} \subseteq S^n$, $p \in S^n$. So $h|_{E^n - C}$ is a homeomorphism of $E^n - C$ onto $S^n - (\{p\} \cup h(C))$ and $H = \{p\} \cup h(C)$ is clearly 0-dimensional and closed. Suppose $E^n - C$ were 1-movable, then by Corollary 2.5, H has an invert point. But $\{p\}$ is the only isolated point of H and so H has no invert point. This is a contradiction.□

Because of Corollary 2.5, let us now look at the complements of some 0-dimensional subsets of S^n.

Notation for Lemma 2.7 is as follows: let "Cantor set" mean a 0-dimensional compact, perfect metric space, let

$$x = (x_1, x_2, \ldots, x_{n+2}) \in E^{n+2}$$

$$S^{n+1} = \{x \in E^{n+2} \mid |x| = 1\}$$

$$S^n = \{x \in S^{n+1} \mid x_{n+2} = 0\}$$

$$S^1 = \{x \in S^{n+1} \mid x_2 = x_3 = \ldots = x_{n+1} = 0\}$$

$$p = (1, 0, \ldots, 0) \in E^{n+2} .$$

Lemma 2.7: Let C be a 0-dimensional compact set, $C \subseteq S^n - \{p\}$, $n \geq 1$, $A \subset S^1 - \{p\}$, and α a homeomorphism from C onto A. Then $S^{n+1} - C$ is homeomorphic to $S^{n+1} - A$ by the restriction to $S^{n+1} - C$ of an extension of α to S^{n+1}. There exists such an A and α for each such C.

Proof: i) The stereographic projection, g from $S^{n+1} - \{p\}$ to $E^{n+1} = E^n \times E^1$ can be taken so that $g(S^n - \{p\}) = E^n \times \{0\}$ and $g(S^1 - \{p\}) = \{(0, \ldots, 0)\} \times E^1$.

ii) Thus C and A are compact and so $g(C)$ and $g(A)$ are homeomorphic by $g \alpha g^{-1}|_{g(C)}$ and compact, and so closed. Also $g(C) \subseteq E^n \times \{0\}$ and $g(A) \subseteq \{(0, \ldots, 0)\} \times E^1$. So by Klee's Theorem 3.3 [15], $\exists$ a homeomorphism $h \circlearrowleft E^{n+1} = E^n \times E^1$ such that $h(g(C)) = g(A)$ and $h|_{g(C)} = g\alpha g^{-1}|_{g(C)}$. Extend $g^{-1}hg$ to $\hat{h}$, a homeomorphism of S^{n+1} onto itself, by $\hat{h}(p) = p$. Then $\hat{h}$ is a homeomorphism of S^{n+1} and $\hat{h}(C) = A$. Note: $\hat{h}|_C = g^{-1}hg|_C = g^{-1}h|_{g(C)} = g^{-1}(h|_{g(C)}) = g^{-1}g\alpha g^{-1}|_{g(C)} = g^{-1}g\alpha g^{-1}g|_C = \alpha|_C = \alpha$. So $\hat{h}|_{S^{n-1}-C}$ is the desired homeomorphism.

iii) $C \subseteq \underline{C}$ (a "Cantor set") $\subseteq S^n$. C has at most a countable number of isolated points (as S^n is $2^{\underline{nd}}$ countable) say $\{a_i\}_{i=1}^{\infty}$. For each i, there clearly exists a "Cantor set", $T_i \subset S^n$ with $a_i \in T_i$ and diameter $T_i < \frac{1}{i}$.

Let $\underline{C} = C \cup (\bigcup_{i=1}^{\infty} T_i) \subseteq S^n$. Now, the T_i are compact, thus closed, and 0-dimensional in $\underline{C}$, so that Theorem II 2. of [11] applies to give $\underline{C}$ is a 0-dimensional metric space. $\underline{C}$ clearly has no isolated points and so is a perfect space. Let c be a point of $\overline{\underline{C}}$. If $c \in C$, then $c \in \underline{C}$. If $c \notin C$, then $d = \text{distance } (c,C) > 0$. So there is a positive integer n with $\frac{1}{n} < \frac{d}{2}$. Thus the neighborhood $N_\epsilon = \{x \in S^n \,|\, \text{distance } (c,x) < \epsilon\}$ of c for $\epsilon = \frac{d}{2}$ is disjoint from T_K for all $K \geq n$ (else $d \leq \text{distance } (c,a_K) < \frac{d}{2} + \frac{d}{2} = d$). Thus $N_\epsilon \cap (\bigcup_{K\geq n} T_K) = \phi$ for all $\epsilon \leq \frac{d}{2}$. But $N_\epsilon \cap \underline{C} \neq \phi$ for any $\epsilon > 0$, so for each $0 < \epsilon < \frac{d}{2}$, $N_\epsilon \cap (C \cup \bigcup_{K=1}^{n-1} T_K) \neq \phi$. Thus $c \in (C \cup \bigcup_{K=1}^{n-1} T_K) = C \cup \bigcup_{K=1}^{n-1} T_K \subseteq \underline{C}$. Thus $\overline{\underline{C}} \subseteq \underline{C}$ and so $\underline{C}$ is closed, therefore compact. Hence $\underline{C}$ is a "Cantor set".

iv) It is well known that any 2 "Cantor sets" are homeomorphic (reference Corollary 2-98 of [10]). Since there is clearly a homeomorphic image B of the standard Cantor set in $S^1 - \{p\}$, there is a homeomorphism f from $\underline{C}$ onto B, and so, $f(C) \subseteq B \subseteq S^1 - \{p\}$ is the desired set A. $\square$

<u>Lemma 2.8</u>: Let C be a Cantor set $\subseteq S^n - \{p\}$ (with notation as in Lemma 2.7) $n \geq 1$. Then $S^{n+1} - C$ is 1-movable.

<u>Proof</u>: Using the notation of Lemma 2.7 and of i) in its proof, let D be the standard Cantor set in $[0,1] \subset E^1$ and $F = \{x \in E^1 | x + \frac{1}{2} + t \in D$ for some integer $t\}$ and consider $H = \{(0,\ldots,0)\} \times F \subset \{(0,\ldots,0)\} \times E^1 \subset E^n \times E^1$. Then $T = E^n \times E^1 - H$ is 1-movable. Since if M is a compact set in $T = E^n \times E^1 - H$, then $S = \{y \in E^1 | (x,y) \in M \subset E^n \times E^1 - H \subset E^n \times E^1$ for some $x \in E^n\}$ is compact. So there exists an integer $L > 0$ with $|y| < L$ for all $y \in S$. Define $h: T \to T$ by $h(x,y) = (x,y + 2L)$. h is clearly a homeomorphism of T onto itself with $M \cap h(M) = \phi$.

Now $g^{-1}|_T$ is a homeomorphism from T onto $S^{n+1} - (\{p\} \cup g^{-1}(H))$ and $f: S^{n+1} \to S^{n+1}$ defined by $f(x_1,x_2,\ldots,x_{n+2}) = (-x_1,x_2,\ldots,x_{n+2})$ is a homeomorphism of S^{n+1} onto itself with $f(S^1) = S^1$.

Let $A = f(\{p\} \cup g^{-1}(H))$ which is a Cantor set since $\{p\} \cup g^{-1}(H)$ clearly is a Cantor set.

Now $0 \notin F$, so $\{(0,\ldots,0)\} \times \{0\} \notin H$. Thus $p = f(-1,0,\ldots,0) = f\,g^{-1}(\{0,\ldots,0\} \times \{0\}) \notin f\,g^{-1}(H)$. Also $f(p) \neq p$. So $p \notin A$ and $A \subset S^1 - \{p\}$. Since any two Cantor sets are homeomorphic there exists a homeomorphism α from C onto A. So Lemma 2.7 applies and $S^{n+1} - C$ is homeomorphic to $S^{n+1} - A$, which by $f = f^{-1}|_{S^{n+1}-A}$ is homeomorphic to $S^{n+1} - (\{p\} \cup g^{-1}(H))$ which is homeomorphic by g to T which is 1-movable. Thus $S^{n+1} - C$ is 1-movable. $\square$

Lemma 2.8 followed by Corollary 2.5 proves that a Cantor set has an invert point (this could of course be shown directly). The following theorem is a counterexample to the converse of Corollary 2.5, in dimension ≥ 3; that is, $C \subset S^n$, $n \geq 3$, and C

has an invert point does not imply that $S^n - C$ is 1-movable.

Theorem 2.9: For each integer $n \geq 3$, there exists a Cantor set C in S^n with $S^n - C$ not 1-movable.

Proof: Let B be a wild Cantor set in S^n (such exist by [3]), then where $p \in S^n - B$ and $g: S^n - p \to E^n$ is the stereographic projection homeomorphism, B is wild in $S^n - p$ and $g(B)$ is wild in E^n. By Corollary 3 of [19], $g(B)$ contains a Cantor set A which is wild at each of its points in E^n. There exists a translation homeomorphism f of E^n with $f(A) \subset E^2 \times (0,+\infty) \times E^{n-3} \subset E^2 \times E^1 \times E^{n-3} = E^n$ (as A is compact). Let L' be the standard Cantor set in E^1. Then $L = L' \times \{(0,-1,0,0,\ldots,0)\} \subset E^1 \times E^{n-1} = E^n$ is a tame Cantor set in E^n.

Clearly $f(A) \cup L$ is a Cantor set in E^n which is locally wild at each point of $f(A)$ and locally tame at each point of L.

Let $C = g^{-1}(f(A) \cup L)$. There is a constant, K, large enough so that $T = \{(x_1,\ldots,x_n) \in E^n | \, x_1^2 + x_2^2 + \ldots + x_n^2 = K\} \cup \{(x_1,\ldots,x_n) \in E^n | \, x_1^2 + x_2^2 \leq K$ and $x_3 = x_4 = \ldots = x_n = 0\}$ gives three connected components for $E^n - T$, the two bounded ones are u' and v', containing respectively $f(A)$ and L.

So $S^n - g^{-1}(T)$ has three connected components, with two of them $u = g^{-1}(u')$ and $v = g^{-1}(v')$ containing respectively $g^{-1}(f(A))$ and $g^{-1}(L)$.

Suppose $S^n - C$ were 1-movable. Let $\ell = 1/4 \min \{\mathrm{dist}\,(g^{-1}f(A), g^{-1}(L)), \mathrm{dist}\,(g^{-1}f(A), g^{-1}(T)), \mathrm{dist}\,(g^{-1}(L), g^{-1}(T)), \mathrm{diam}\,(g^{-1}f(A)), \mathrm{dist}\,(p,C), \mathrm{diam}\, g^{-1}(L)\}$.

In the proof of Theorem 2.2, there exists a t_{j_0} with $1/t_{j_0} < \ell$ and $h_{t_{j_0}}(M_{t_{j_0}}) \subset S_{c,\ell}$ with $h_{t_{j_0}}$ a homeomorphism of $S^n - C$. From Lemma 2.1, $h_{t_{j_0}}$ extends to a homeomorphism h of S^n onto itself. ℓ was chosen so small that no component of $S^n - M_{t_{j_0}}$ contains points of both $g^{-1}f(A)$ and $g^{-1}(L)$ (as $g^{-1}(T)$ separates $g^{-1}f(A)$ and $g^{-1}(L)$, while $g^{-1}(T) \subset M_{t_{j_0}}$). But $S^n - S_{c,\ell}$ is connected and so contained in a single connected component of $S^n - h(M_{t_{j_0}})$. But ℓ was chosen so small that $S^n - S_{c,\ell}$ contains an open neighborhood of a point from each of $g^{-1}f(A)$ and $g^{-1}(L)$, say $a \in g^{-1}f(A)$, $b \in g^{-1}(L)$ and $a \in \theta_a$, $b \in \theta_b$, θ_a and θ_b are open in S^n, and $\theta_a \cup \theta_b \subseteq S^n - S_{c,\ell}$.

Thus $h^{-1}(\theta_a \cup \theta_b) \subset$ some connected components of $S^n - M_{t_{j_0}}$ and so $\{h^{-1}(a), h^{-1}(b)\}$ is a subset of either $g^{-1}f(A)$ or $g^{-1}(L)$. In the first case let $\alpha = h^{-1}(b) \in g^{-1}f(A)$ and $\varphi = h$, and in the second caae let $\alpha = a$ and $\varphi = h^{-1}$. Then φ is a homeomorphism of S^n and $\alpha \in g^{-1}f(A)$ with $\varphi(\alpha) \in g^{-1}(L)$. As h is an extension of $h_{t_{j_0}}$ and $p \in M_{t_{j_0}}$ (by the choice of ℓ) $\{p,\varphi(p)\} \subseteq \{p,h(p),h^{-1}(p)\} \subseteq S^n - C$.

Since $S^n - C$ is a connected manifold, for any two points (and so for p and $\varphi(p)$ in particular) of $S^n - C$ there is a homeomorphism Ψ' of $S^n - C$ onto $S^n - C$ which is the identity outside of and on the boundary of an n-cell in $S^n - C$, and $\Psi'(\varphi(p)) = p$. Extend Ψ' to a homeomorphism Ψ of S^n onto itself by $\Psi|_{S^n-C} = \Psi'$ and $\Psi|_C =$ identity on C.

Thus $\Psi\varphi$ is a homeomorphism of S^n with $\Psi\varphi(p) = p$ and so $\Psi\varphi|_{S^n - p}$ is a homeomorphism of $S^n - p$ onto itself. Hence $\mu = g(\Psi\varphi|_{S^n-p})g^{-1}$ is a homeomorphism of E^n onto itself which sends $g(\alpha) \in f(A)$ to an element β of L.

Then there exists an open neighborhood θ of $g(\alpha)$ with $\theta \subset E^2 \times (0,+\infty) \times E^{n-3}$ and $\mu(\theta) \subset E^2 \times (-\infty,0) \times E^{n-3}$. So $\mu(\theta \cap (f(A) \cup L)) = \mu(\theta \cap f(A)) \subset L$ which is tame in E^n and so $f(A) \cup L$ is locally tame at $g(\alpha) \in f(A)$. This is a contradiction as $f(A) \cup L$ is wild at each point of $f(A)$. Thus $S^n - C$ is not 1-movable. □

There is perhaps a chance of proving the following converse of Corollary 2.5 when $n = 2$, because of Lemma 2.11 and Theorem 2.13.

<u>Conjecture 2.10</u>: If C is a closed 0-dimensional subset of S^2 and C has an invert point, then $S^2 - C$ is 1-movable.

<u>Lemma 2.11</u>: If C is a 0-dimensional compact subset of S^2 and h is a homeomorphism of C onto itself, then there exists a homeomorphism $\hat{h}$ of S^2 onto itself so that $\hat{h}|_C = h$.

<u>Proof</u>: i) C is a subset of a 0-dimensional, compact, perfect subset A of S^2. First C is a second countable metric space (as S^2 is such), and hence has at most a countable number of isolated points, say $\{x_i | i \in I\}$ for some $I \subset \{1,2,\dots\}$. For each $i \in I$, x_i has a open neighborhood N with $N \subset S_{x_i,\frac{1}{i}}$ and a homeomorphism $g: N \to E^2$ with $g(x_i) = (0,0)$. Let $A_{x_i} = g^{-1}(B)$ where B is the standard Cantor set in $[0,1] \times \{0\} \subset E^2$. Then A_{x_i} is a 0-dimensional, compact, perfect subset

of S^2 with $x \in A_{x_i} \subset S_{x_i, \frac{1}{i}}$. Let $A = C \cup \bigcup_{i \in I} A_{x_i}$. Then by Theorem II 2, of [11], A is 0-dimensional, A clearly has no isolated points, and as diameter $A_{x_i} < \frac{2}{i}$ with A_{x_i} and C closed for all $i \in I$ it is quite easy to show that A is closed. Thus A is also perfect and compact in S^2.

ii) Let $x \in S^2 - A$. Then there is the stereographic projection homeomorphism $s: S^2 - \{x\} \to E^2$. Let B' be the standard Cantor set in $[0,1] \times \{0\}$, thus a perfect, compact, 0-dimensional set in E^2, as is $s(A)$. As components of 0-dimensional sets are points, Theorem I of [1] applies and there is a homeomorphism t of E^2 onto itself with $t(s(A)) = B'$. Now let $B = t(s(C))$. If it were true that for every homeomorphism g of $B \subset E^2$ onto itself, there is a homeomorphism $\hat{g}$ of E^2 onto itself with $\hat{g}|_B = g$, then we would be through. For given a homeomorphism h of C onto itself, $g = ts\, hs^{-1}t^{-1}|_B$ is a homeomorphism of B onto itself, and so extends to homeomorphism $\hat{g}$ of E^2 onto itself with $\hat{g}|_B = g$. Thus $f = s^{-1}t^{-1}\hat{g}ts$ is a homeomorphism of $S^2 - \{x\}$ onto itself and so extends to a homeomorphism $\hat{h}$ of S^2 onto itself by $\hat{h}(x) = x$ and $\hat{h}|_{S^2-\{x\}} = f$. Now $\hat{h}|_C = f|_C = s^{-1}t^{-1}\hat{g}\, ts|_C$ (and as $ts(C) = B) = s^{-1}t^{-1}gts|_C = s^{-1}t^{-1}(tshs^{-1}t^{-1})ts|_C = (s^{-1}t^{-1}ts)h(s^{-1}t^{-1}ts)|_C = h|_C = h$.

iii) Thus it remains only to show that if g is a homeomorphism of B, a closed subset of the standard Cantor set $\subset [0,1] \times \{0\} \subset E^2$, onto B then there is a homeomorphism $\hat{g}$ of E^2 onto itself with $\hat{g}|_B = g$. So suppose such a B and g are given. Define a homeomorphism r from B onto a closed subset of $\{0\} \times E^1 \subset E^2$ by $r(y,0) = (0,z)$ where $g(y,0) = (z,0)$. Then by Klee's Theorem 3.3 [15], there exists a homeomorphism $\hat{r}$ of E^2 onto itself with $\hat{r}|_B = r$. $\ell: E^2 \to E^2$, defined by $\ell(x,y) = (y,x)$, is a homeomorphism of E^2 onto itself and so let $\hat{g} = \ell \cdot \hat{r}$. $\square$

Corollary 2.12: Let $C \subset S^n - \{p\} \subset S^{n+1} - \{p\}$ as in Lemma 2.7. Then any homeomorphism of C onto itself extends to a homeomorphism of S^{n+1} onto itself.

Proof: Let $h: C \to C$ be a homeomorphism of C onto itself. By Lemma 2.7 there exists a homeomorphism $\alpha: C \to A$ (of C onto a set $A \subseteq S^1 - \{p\}$) and this homeomorphism extends to a homeomorphism $\bar{\alpha}$ of S^{n+1} onto itself. Then $\alpha h: C \to A$ is a homeomorphism satisfying the hypotheses of Lemma 2.7, so there is a homeomorphism β of S^{n+1} onto itself and $\beta|_C = \alpha h$.

Consider the homeomorphism $(\bar{\alpha})^{-1}\beta$ of S^{n+1} onto itself, $(\bar{\alpha})^{-1}\beta|_C = (\bar{\alpha})^{-1}(\beta|_C) = (\bar{\alpha})^{-1}\alpha h = ((\bar{\alpha})^{-1}|_A)\alpha h = \alpha^{-1}\alpha h = h$. $\square$

Theorem 2.13: $S^2 - C$ is 1-movable when C is a "Cantor set".

Proof: Let us use the notation for Lemma 2.7 with $n = 1$. If $p \in C$, then there is $q \in S^2 - C$ and there is a homeomorphism h of S^2 onto itself with $h(q) = p$. Thus without loss of generality we may assume $p \notin C$. So let g be the stereographic projection of $S^2 - \{p\}$ onto E^2. Then $g(C)$ is a "Cantor set" in E^2 and $g(S^1 - \{p\})$ is a line in E^2. There exists a "Cantor set", say B, on $g(S^1 - \{p\})$. So by Theorem I of [1] there is a homeomorphism f of E^2 onto itself with $f|_{g(C)}$ a homeomorphism from $g(C)$ onto B. Thus $\Phi = g^{-1} f\, g$ is a homeomorphism of $S^2 - \{p\}$ onto itself and so has an extension, say Ψ, to S^2 by $\Psi|_{S^2-\{p\}} = \Phi$ and $\Psi(p) = p$. Then letting the C and n of Lemma 2.8 be $g^{-1}(B)$ and 1 respectively, Lemma 2.8 tells us that $S^2 - g^{-1}(B)$ is 1-movable. But $\Psi(C) = g^{-1}(B)$ so $\Psi|_{S^2-C}$ is a homeomorphism from $S^2 - C$ onto $S^2 - g^{-1}(B)$. Hence $S^2 - C$ is 1-movable. $\square$

CHAPTER III

SIMPLICIAL COMPLEXES AND 1-MOVABILITY

In this chapter we will first prove some lemmas and a theorem to enable us to prove the main result of this chapter, which is that $|K| - |K^{(n-2)}|$ is 1-movable for each n-dimensional abstract simplicial complex K, where $K^{(q)}$ is the q-skeleton of K and $n \geq 3$. We then observe that a 1-movability condition and engulfing theorems trivially give characterizations of the n-sphere using a theorem of Doyle and Hocking in [7]. We will then give many examples showing that even when K is a compact combinatorial n-manifold we cannot claim $|K| - |L|$ is 1-movable, when L is a q-dimensional subcomplex of K, $0 \leq q \leq n$ or even when $L = K^{(q)}$ for $0 \leq q \leq [\frac{n-2}{2}]$.

When σ is simplex, $\alpha \in |\sigma|$, $B \subset |\sigma|$, $\alpha \cdot B$ will denote the set $\{t\alpha + (1-t)\beta | t \in [0,1], \beta \in B\}$, σ^t will denote the t-skeleton of σ considered as the simplicial complex consisting of all the faces of σ, and $<\sigma>$ will denote the set $|\sigma| - |\dot{\sigma}|$ as in [22] with $\dot{\sigma} = \partial\sigma = \sigma^{(n-1)}$.

<u>Lemma 3.1</u>: Let σ be an n-simplex, C a compact set, $C \neq \emptyset$, $C \subset |\sigma| - |\sigma^{(n-2)}|$, U a neighborhood of $|\sigma^{(n-2)}|$ in $|\partial\sigma|$ with $\bar{U} \cap C = \emptyset$. Let $\hat{\sigma}$ be the barycenter of σ. Then there is a pseudo-isotopy $\{H_t\}$ of $|\sigma|$ onto $|\sigma|$ which is fixed on $|\sigma^{(n-1)}|$ and has $H_1(C) \subset \hat{\sigma} \cdot (|\partial\sigma| - \bar{U})$.

Proof: We may consider $|\sigma|$ to be linearly embedded in E^n and B^n to be the n - ball with center $\hat{\sigma}$ and radius large enough, say R, so that $|\sigma| \subseteq B^n$. There is a homeomorphism h of $|\sigma|$ onto B^n induced by the radial projection of $|\sigma^{(n-1)}|$ onto bdy B^n.

Now, $h(\bar{U}) \cap h(C) = \phi$ (as $\bar{U} \cap C = \phi$) so $d = \text{distance}(h(\bar{U}), h(C)) > 0$. Let D^n be the n-ball with center $\hat{\sigma}$ and radius $R - d/2$. Then $(\hat{\sigma} \cdot h(\bar{U})) - (\hat{\sigma} \cdot$ (radial projection from $\hat{\sigma}$ of $h(\bar{U})$ onto bdy D^n)) is disjoint from $h(C)$.

Let $\{G_t\}$ be the obvious pseudo-isotopy that shrinks D^n radially down to $\hat{\sigma}$ and sends each ray from $\hat{\sigma}$ onto itself.

$$G_t(x) = \begin{cases} (1-t)(x-\hat{\sigma}) + \hat{\sigma} & \text{for } |x-\hat{\sigma}| \le R - d/2 \\ \dfrac{(x-\hat{\sigma})}{|x-\hat{\sigma}|}\left\{R + \dfrac{(|x-\hat{\sigma}|-R)[R-(1-t)(R-d/2)]}{d/2}\right\} + \hat{\sigma} & \\ & \text{for } R \ge |x-\hat{\sigma}| \ge R - d/2 \end{cases}$$

where the points x and $\hat{\sigma}$ are considered as vectors from the origin and

$$|(y_1, y_2, \ldots, y_n)| = (y_1^2 + y_2^2 + \ldots + y_n^2)^{\frac{1}{2}} .$$

Thus $G_1(h(C)) \subset \hat{\sigma} \cdot (\text{bdy } B^n - h(U))$ and so $\{H_t\}$ where $H_t = h^{-1}G_t h$ is the desired pseudo-isotopy on σ as h also preserves rays from $\hat{\sigma}$. $\square$

Lemma 3.2: Let σ be an n-simplex, $\hat{\sigma}$ its barycenter, and F a compact subset of $|\sigma| - |\sigma^{(n-1)}|$.

Then there is a pseudo-isotopy $\{H_t\}$ of $|\sigma|$ onto $|\sigma|$ which is fixed on $|\sigma^{(n-1)}|$ and has $H_1(F) = \hat{\sigma}$.

Proof: Construct B^n and h as in Lemma 3.1 and let d = distance (bdy B^n,$h(F)$). Then $d > 0$. Let D^n be the n-ball of radius $R - d/2$ and center $\hat{\sigma}$ with $\{G_t\}$ as in Lemma 3.1. Then as $h(F) \subset D^n$, $G_1(h(F)) = \hat{\sigma}$, so letting

$$H_t = h^{-1}G_t h,\ \{H_t\} \text{ is the desired pseudo-isotopy on } |\sigma| . \square$$

Lemma 3.3: Let σ be an n-simplex with barycenter $\hat{\sigma}$ and $\tau_1,\tau_2,\ldots,\tau_{n+1}$ the (n-1)-faces of σ. Let ${}_iG_t$ be a pseudo-isotopy of $|\tau_i|$ onto $|\tau_i|$ fixed on $|\tau_i^{(n-2)}|$ for each $i = 1,2,\ldots,n+1$. Then there is a pseudo-isotopy $\{H_t\}$ of $|\sigma|$ onto $|\sigma|$ with $H_t|_{|\tau_i|} = {}_iG_t$ for $t \in [0,1]$ and $i = 1,2,\ldots,n+1$.

Proof: Let $\{G_t\}$ be the pseudo-isotopy of $|\sigma^{(n-1)}|$ defined by $G_t|_{|\tau_i|} = {}_iG_t$ for $i = 1,2,\ldots,n+1$ and $t \in [0,1]$. Using polar coordinates in $|\sigma|$ (as in [22], p. 117), $\{G_t\}$ extends to the pseudo-isotopy $\{H_t\}$ by

$$H_t([\alpha,\lambda]) = [G_t(\alpha),\ \lambda].$$

$\{H_t\}$ clearly has the desired properties. $\square$

Theorem 3.4: Let K be a simplicial complex (not necessarily locally finite) of dimension n, A be a subset of $|K| - |K^{(n-2)}|$ which has $A \cap |\sigma|$ compact for each $\sigma \in K$. Then there are two pseudo-isotopies, $\{F_t\}$ and $\{G_t\}$, of $|K|$ onto $|K|$ which are fixed on $|K^{(n-2)}|$ and have

$$G_1 F_1(A) \subseteq T = \bigcup_{\substack{\sigma\in K-K^{(n-1)} \\ \tau \text{ is } (n-1)\text{-face of } \sigma \\ \text{and } A \cap <\sigma> \neq \emptyset}} \hat{\sigma}\cdot\hat{\tau} \cup \bigcup_{\substack{\tau\in K^{(n-1)}-K^{(n-2)} \\ \text{and } A \cap <\tau> \neq \emptyset}} \{\hat{\tau}\}$$

Proof: For each $(n-1)$ simplex τ of K, obtain, from the construction in Lemma 3.2(with $\emptyset \neq F = A \cap |\tau| = A \cap |\tau| - |\tau^{(n-2)}|$, τ as the σ of Lemma 3.2) $U_\tau = h^{-1}(B^n - D^n)$ a closed neighborhood of $|\tau^{(n-2)}|$ in $|\tau|$ with $\emptyset = U_\tau \cap F = U_\tau \cap A \cap |\tau| = U_\tau \cap A$ and let $\{{}_\tau H_t\} = \{H_t\}$ (the pseudo-isotopy of Lemma 3.2 for the above F) when $A \cap |\tau| \neq \emptyset$; when $A \cap |\tau| = \emptyset$, let ${}_\tau H_t$ = identity on $|\tau|$ and $U_\tau = |\tau|$. ${}_\tau H_1(|\tau| - U_\tau) \subseteq \{\hat{\tau}\}$. Now, for each n-simplex σ of K, let $\tau_1, \tau_2, \ldots, \tau_{n+1}$ be its $(n-1)$-faces. Then $U = \bigcup_{i=1}^{n+1} U_{\tau_i}$ is a closed neighborhood of $\bigcup_{i=1}^{n+1} |\tau_i^{(n-2)}| = |\sigma^{(n-2)}|$ in $|\partial\sigma|$. By Lemma 3.1 (with $\sigma = \sigma$, $U = U$, and $C = A \cap |\sigma|$) there is a pseudo-isotopy $\{{}_\sigma F_t\}$ (the pseudo-isotopy $\{H_t\}$ of Lemma 3.1) of $|\sigma|$ onto $|\sigma|$ which is fixed on $|\sigma^{(n-1)}|$ and has

$${}_\sigma F_1(A \cap |\sigma|) \subseteq \hat{\sigma}\cdot(|\sigma^{(n-1)}| - \bar{U})$$

when $A \cap <\sigma> \neq \emptyset$. When $A \cap <\sigma> = \emptyset$, let $\{{}_\sigma F_t\}$ be the pseudo-isotopy which is fixed on $|\sigma|$, ${}_\sigma F_t$ = identity on $|\sigma|$ for all $t \in [0,1]$. Hence there is a pseudo-isotopy $\{F_t\}$ of $|K|$ onto $|K|$ defined by $F_t|_{|\tau|}$ = identity on $|\tau|$ for each $\tau \in K^{(n-1)}$ and $F_t|_{|\sigma|} = {}_\sigma F_t$ for each $\sigma \in K - K^{(n-1)}$. Now by Lemma 3.3 (letting $\sigma = \sigma$, $\hat{\sigma} = \hat{\sigma}$, $\tau_i = \tau_i$ for $i = 1,2,\ldots,n+1$, ${}_iG_t = {}_\tau H_t$) we obtain the pseudo-isotopy $\{{}_\sigma G_t\}$ = the $\{H_t\}$ of Lemma 3.3. Hence, there is a pseudo-isotopy $\{G_t\}$ of $|K|$ onto $|K|$ defined by

$$G_t\big|_{|\sigma|} = {}_{\sigma}G_t \quad \text{for each} \quad \sigma \in K - K^{(n-1)}$$

$$G_t\big|_{|\tau|} = {}_{\tau}H_t \quad \text{for each} \quad \tau \in K^{(n-1)} - K^{(n-2)}$$

$$G_t\big|_{|\tau|} = \text{identity on } |\tau| \quad \text{for} \quad \tau \in K^{(n-2)}$$

$\{G_t\}$ and $\{F_t\}$ are both clearly fixed on $|K^{(n-2)}|$ and $G_1F_1(A) \subseteq T$ is easily verified.$\square$

<u>Lemma 3.5</u>: Let A be a compact subset of a normal space X, and $\{H_t\}$ a pseudo-isotopy of X.

If there is a homeomorphism, g, of X onto itself with $g(H_1(A)) \cap H_1(A) = \phi$, then there is a homeomorphism, f, of X onto itself with $f(A) \cap A = \phi$.

<u>Proof</u>: As X is normal, $g(H_1(A))$ and $H_1(A)$ closed, we have disjoint open sets U and V with $U \supseteq g(H_1(A))$ and $V \supseteq H_1(A)$. $\theta = g^{-1}(U) \cap V$ is open and $H_1(A) \subset \theta \subset V$, $g(\theta) \subset U$, and $U \subset X - \theta$. Let $H\colon X \times [0,1] \to X$ be the pseudo-isotopy $\{H_t\}$. Then $H^{-1}(\theta)$ is open and $H^{-1}(\theta) \supset A \times \{1\}$ which is compact in $X \times [0,1]$; so there is a $s \in [0,1]$ with $H^{-1}(\theta) \supset A \times (s,1]$. Let $p = (1+s)/2$. Then H_p is a homeomorphism of X onto itself and $H_p(A) \subset \theta$. Let $f = H_p^{-1} g\, H_p$. Then $f(A) = H_p^{-1} g\, H_p(A) \subseteq H_p^{-1} g(\theta) \subset H_p^{-1}(U) \subset H_p^{-1}(X-\theta) \subset H_p^{-1}(X-H_p(A)) \subseteq H_p^{-1}(H_p(X-A)) = X-A$. So $f(A) \cap A = \phi$.$\square$

<u>Lemma 3.6</u>: Given an n-simplex σ, $n \geq 3$, with its vertices ordered as indicated $\sigma = \{a_1,a_2,\ldots,a_{n+1}\}$, let τ_i, $i = 1,2,\ldots,n+1$ be the (n-1)-faces of σ with their vertices ordered as induced and indicated by the ordering of the vertices of σ, $\tau_i = \{a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{n+1}\}$. Then there is a well defined

set $\{\alpha,\beta_1,\beta_2,\ldots,\beta_{n+1}\} \subset |\sigma|$ dependent only on the ordering of the vertices of σ with $\alpha \in \langle\sigma\rangle$, $\beta_i \in \langle\tau_i\rangle$ for $i = 1,2,\ldots,n+1$, and $\left(\bigcup_{i=1}^{n+1} \alpha\cdot\beta_i\right) \cap \left(\bigcup_{i=1}^{n+1} \hat{\sigma}\cdot\hat{\tau}_i\right) = \phi$.

Proof: We will use barycentric coordinates throughout. $u \in \hat{\sigma}\cdot\hat{\tau}_i$ for some i, when $u = \left(\sum_{\substack{j=1\\ j\neq i}}^{n+1} \left[\frac{(1-t)}{(n+1)} + \frac{t}{n}\right] a_j\right) + \frac{1-t}{n+1} a_i$ for some $t \in [0,1]$, and so each coordinate of u is one of at most two possible values, $\left(\frac{1-t}{n+1} + \frac{t}{n}\right)$ and $\frac{1-t}{n+1}$. Let $\alpha = \sum_{j=1}^{n+1} \frac{2j}{(n+1)(n+2)} a_j \in \langle\sigma\rangle$ and for each $i = 1,2,\ldots,n+1$ let $\beta_i = \left(\sum_{j=1}^{i-1} \frac{2j}{n(n+1)} a_j\right) + \sum_{j=i+1}^{n+1} \frac{2(j-1)}{n(n+1)} a_j \in \langle\tau_i\rangle$. Then $v \in \alpha\cdot\beta_i$ for some i, when for some $s \in [0,1]$

$$v = \left(\sum_{j=1}^{i-1}\left(\frac{2js}{n(n+1)} + \frac{2(1-s)j}{(n+1)(n+2)}\right) a_j\right) + \frac{2(1-s)i}{(n+1)(n+2)} a_i + \sum_{j=i+1}^{n+1}\left(\frac{2(s)(j-1)}{n(n+1)} + \frac{2(1-s)j}{(n+1)(n+2)}\right) a_j .$$

Observe the coefficients of $a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{n+1}$ in the barycentric coordinates of v, are all different values. But this is n different values and $n \geq 3$, so

$$\left(\bigcup_{i=1}^{n+1} \hat{\sigma}\cdot\hat{\tau}_i\right) \cap \left(\bigcup_{i=1}^{n+1} \alpha\cdot\beta_i\right) = \phi . \square$$

Theorem 3.7: Let K be an n-dimensional simplicial complex (not necessarily locally finite), $n \geq 3$.

Then $|K| - |K^{(n-2)}|$ is 1-movable.

Proof: Let $A \subset |K| - |K^{(n-2)}|$, A compact. We need only show a homeomorphism, f, of $|K| - |K^{(n-2)}|$ onto itself with $f(A) \cap A = \phi$. Since A is compact, $A \cap \langle\sigma\rangle \neq \phi$ for at most a finite number of simplices $\sigma \in K$ (by Corollary 19 of [22], p. 113), also A and K satisfy the hypotheses of Theorem 3.4. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_p\} = \{\sigma \in K - K^{(n-1)} \mid \langle\sigma\rangle \cap A \neq \emptyset\}$ and

$$\{\lambda_1, \lambda_2, \ldots, \lambda_q\} = \left\{\tau \in K^{(n-1)} - K^{(n-2)} \,\middle|\, \begin{array}{l} A \cap \langle\tau\rangle \neq \emptyset \text{ or } \tau \text{ is a face of } \sigma_i \\ \text{for some } i \in \{1,2,\ldots,p\} \end{array}\right\}$$

with $\sigma_i \neq \sigma_j$ if $i \neq j$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Now let

$$T = \left(\bigcup_{\substack{i=1,2,\ldots,p \\ \lambda_k \text{ is a face of } \sigma_i}} \hat{\sigma}_i \cdot \hat{\lambda}_k\right) \cup \left(\bigcup_{k=1}^{q} \{\hat{\lambda}_k\}\right) .$$

Then by Theorem 3.4, followed by Lemma 3.5 applied twice, the existence of a homeomorphism, h, of $|K| - |K^{(n-2)}|$ onto itself with $h(T) \cap T = \emptyset$ would imply the existence of the required f, and hence the theorem.

Let $S = \{v \in K \mid v$ is a vertex of either σ_i for some $i = 1,2,\ldots,p$ or some λ_k for some $k = 1,2,\ldots,q\}$.

S has only a finite number of elements, so they can be ordered by counting them, $v_1, v_2, v_3, \ldots, v_\ell$. This ordering of S induces an ordering of the vertices of each σ_i, $i = 1,2,\ldots,p$ and of each λ_k, $k = 1,2,\ldots,q$.

Now, for each $k = 1,2,\ldots,q$, either λ_k is not the face of any σ_i, $i = 1,2,\ldots,q$, in which case choose δ_k to be any element of $\langle\lambda_k\rangle - \{\hat{\lambda}_k\}$, or λ_k is an $(n-1)$-face of σ_i for an $i \in \{1,2,\ldots,p\}$. In the latter case let σ_i be the σ of Lemma 3.6, then $\lambda_k = \tau_j$ for some $j \in \{1,2,\ldots,n+1\}$. Define δ_k = the β_j of Lemma 3.6. Note that δ_k is well defined for each k since the ordering of the vertices of λ_k induced by S is the same as that induced by the ordering of the vertices of any σ_t which has λ_k as a face, $t \in \{1,2,\ldots,p\}$.

As $\hat{\lambda}_k$ and δ_k are both elements of $<\lambda_k>$, it may be easily shown using Lemma 2 of [22], p. 116, that there is a homeomorphism g_k of $|\lambda_k|$ onto itself which is fixed on $|\lambda_k^{(n-2)}|$ and $g_k(\hat{\lambda}_k) = \delta_k$. Define a homeomorphism g of $|K^{(n-1)}|$ onto itself by

$$g|_{|\lambda_k|} = g_k \quad \text{for each} \quad k = 1,2,\ldots,q$$

and $g|_{|K^{(n-1)}| - \bigcup_{k=1}^{q}|\lambda_k|}$ = identity on $|K^{(n-1)}| - \bigcup_{k=1}^{q}|\lambda_k|$.

For each n-simplex, $\sigma \in K$, let ${}_\sigma g$ be the homeomorphism of $|\sigma|$ onto $|\sigma|$ which has ${}_\sigma g|_{|\sigma^{(n-1)}|} = g|_{|\sigma^{(n-1)}|}$, defined by ${}_\sigma g([a,t]) = [g(a),t]$ (using polar coordinates in $|\sigma|$ as in [22], p. 117). Then define the homeomorphism, $\bar{g}$, of $|K|$ onto itself by $\bar{g}|_{|K^{(n-1)}|} = g$ and $\bar{g}|_{|\sigma|} = {}_\sigma g$ for each $\sigma \in K - K^{(n-1)}$.

Now for each $\sigma \in K - K^{(n-1)}$, let $\alpha_\sigma = \hat{\sigma}$ if $\sigma \notin \{\sigma_1,\sigma_2,\ldots,\sigma_p\}$ and α_{σ_i} = the α of Lemma 3.6 for $\sigma = \sigma_i$ under the ordering of the vertices of σ_i induced by that of S. Now define the homeomorphism ${}_\sigma h$ of $|\sigma|$ onto itself given by (again using polar coordinates)

$${}_\sigma h([a,t]) = t\,\alpha_\sigma + (1-t)\,a\ .$$

Note ${}_\sigma h$ is the identity on $|\sigma^{(n-1)}|$. Define the homeomorphism,

$\bar{h}$, of $|K|$ onto itself by

$$\bar{h}\big|_{|\sigma|} = {}_{\sigma}h \quad \text{for each} \quad \sigma \in K - K^{(n-1)}$$

$$\text{and} \quad \bar{h}\big|_{|K^{(n-1)}|} = \text{identity on} \quad |K^{(n-1)}| .$$

Now let $h = \bar{h}\bar{g}$. h is clearly a homeomorphism of $|K|$ onto itself and $h|_{|\sigma|}$ is a homeomorphism of $|\sigma|$ onto itself and $h|_{<\sigma>}$ is a homeomorphism of $<\sigma>$ onto itself for each $\sigma \in K$.

Thus to show $h(T) \cap T = \phi$, we need only show

(*) $\quad h(T \cap <\sigma>) \cap (T \cap <\sigma>) = \phi$ for each $\sigma \in K$.

This is clearly true for each $\sigma \in K$ with $T \cap <\sigma> = \phi$.

<u>Case 1</u>. $\sigma \in K^{(n-2)}$. $T \cap |K^{(n-2)}| = \phi$ so $T \cap <\sigma> = \phi$. (*) is true.

<u>Case 2</u>. $\sigma \in K^{(n-1)} - K^{(n-2)}$ and $T \cap <\sigma> \neq \phi$. Then $\sigma = \lambda_k$ for some $k \in \{1,2,\ldots,q\}$. But $T \cap <\lambda_k> = \hat{\lambda}_k$ and $h(\hat{\lambda}_k) = \bar{h}\bar{g}(\hat{\lambda}_k) = \bar{h}(g(\hat{\lambda}_k)) = \bar{h}(g_k(\hat{\lambda}_k)) = \bar{h}(\delta_k) = \delta_k$ as $\delta_k \in |K^{(n-1)}|$. But $\delta_k \neq \hat{\lambda}_k$, so $h(T \cap <\sigma>) \cap (T \cap <\sigma>) = \{\delta_k\} \cap \{\hat{\lambda}_k\} = \phi$. (*) is true.

<u>Case 3</u>. $\sigma \in K - K^{(n-1)}$ and $T \cap <\sigma> \neq \phi$. Then $\sigma = \sigma_i$ for some $i \in \{1,2,\ldots,p\}$. So $T \cap <\sigma_i> \subseteq T \cap |\sigma_i|$

$= \bigcup_{\lambda_k \text{ is face of } \sigma_i} \hat{\sigma}_i \cdot \hat{\lambda}_k$. Now for λ_k a face of σ_i

$h(\hat{\sigma}_i \cdot \hat{\lambda}_k) = \bar{h}\bar{g}(\hat{\sigma}_i \cdot \hat{\lambda}_k) = \bar{h}\bar{g}\big|_{|\sigma_i|}(\hat{\sigma}_i \cdot \hat{\lambda}_k)$

$= \bar{h}({}_{\sigma_i}g(\hat{\sigma}_i \cdot \hat{\lambda}_k)) = \bar{h}(\hat{\sigma}_i \cdot g(\hat{\lambda}_k)) = \bar{h}(\hat{\sigma}_i \cdot g_k(\hat{\lambda}_k))$

$= \bar{h}(\hat{\sigma}_i \cdot \delta_k) = \bar{h}\big|_{|\sigma_i|}(\hat{\sigma}_i \cdot \delta_k) = {}_{\sigma_i}h(\hat{\sigma}_i \cdot \delta_k) = \alpha_{\sigma_i} \cdot \delta_k$

$= \alpha\cdot\delta_k$ (where α is the α of Lemma 3.6 for our ordering of vertices)

$= \alpha\cdot\beta_j$ (where β_j is the β_j of Lemma 3.6 for our ordering of vertices and $\sigma_i = \sigma$ and $\lambda_k = \tau_j$).

So $h(T \cap <\sigma_i>) \subseteq \bigcup_{j=1}^{n+1} \alpha\cdot\beta_j$ (the α and β_j being those of Lemma 3.6 with our ordering of vertices and $\sigma_i = \sigma$)

and $T \cap <\sigma_i> \subseteq \bigcup_{\substack{\lambda_k \text{ is a face of } \sigma_i \\ \text{and } k \in \{1,2,\ldots,q\}}} \hat{\sigma}_i\cdot\hat{\lambda}_k = \bigcup_{j=1}^{n+1} \hat{\sigma}\cdot\hat{\tau}_j$. (the σ and τ_j's being those of Lemma 3.6 with our ordering of vertices and $\sigma_i = \sigma$).

But Lemma 3.6 gives us $(\bigcup_{j=1}^{n+1} \alpha\cdot\beta_j) \cap (\bigcup_{j=1}^{n+1} \hat{\sigma}\cdot\hat{\tau}_j) = \phi$, hence $h(T \cap <\sigma_i>) \cap (T \cap <\sigma_i>) = \phi$. (*) is true. □

Let us consider open n-manifolds that have been obtained by the removal of a closed set from a closed n-manifold. The following theorem is trivially given by the sphere characterization theorem of Doyle and Hocking [7].

<u>Theorem 3.8</u>: Let M^n be a closed n-manifold, A be a closed subset of M^n, $A \subset V \subset M^n$ with V homeomorphic to E^n, and $M^n - A$ be 1-movable. Then M^n is an n-sphere.

<u>Proof</u>: Let M^n, A, and V be as in the hypothesis. Then A is compact and so contained in the interior of an n-ball, $B \subset V$, with bicollared boundary. Let B be the initial n-ball with bicollared boundary in the proof of Theorem 1 of [7] for the manifold M^n. Then $M^n = U \cup R$ with U homeomorphic to E^n and $U \cup R$ a "standard decomposition" as in [7] and $B \subseteq U$. Now

$M^n - A = (U - A) \cup R$ and R is compact in M^n, hence compact in $M^n - A$. $M^n - A$ is 1-movable so there is a homeomorphism h of $M^n - A$ onto itself with $h(R) \cap R = \phi$. Thus $h^{-1}(U - A)$ is a neighborhood of R and $U - A$ is a Euclidean domain (can be imbedded in E^n), so $h^{-1}(U - A)$ is also a Euclidean domain. But then Theorem 4 of [7] says that M^n is homeomorphic to S^n. □

Thus any engulfing theorem that says a closed subset A of a closed n-manifold M^n is engulfed by an open n-cell tells us that if $M^n - A$ is 1-movable then M^n is an n-sphere, also if M^n is not an n-sphere then $M^n - A$ is not 1-movable. So the engulfing theorems of Newman in [18] and Zeeman in [25] give Corollaries 3.9 and 3.10 respectively.

Corollary 3.9: Let X be a locally tame closed set of dimension $\leq n-3$ in M, a p-connected closed topological n-manifold without boundary (thus $n \geq 3$). Then $M - X$ is 1-movable implies that M is an n-sphere.

Proof: Let X and M be as in the hypothesis. Then since there is an open set V, homeomorphic to E^n and so (p-1)-connected, in M and $M - V$ is compact the main theorem of [18] applies to insure the existence of a homeomorphism, h, of M onto itself with $X \subset h(V)$. But $h(V)$ is homeomorphic to E^n and so Theorem 3.8 implies that M is an n-sphere. □

Corollary 3.10: Let M^n be a connected closed combinatorial n-manifold, $n \geq 3$. Let $q \leq n-3$ and $\pi_i(M^n) = 0$ for $i = 0,1,\ldots,q$. If $M^n - |L|$ is 1-movable where L is some subcomplex of dimension q of some triangulation of M^n, then M^n is homeomorphic to S^n.

Proof: Assume the hypotheses. The result may then be obtained as a corollary to Corollary 3.9. An alternate proof is to observe that Theorem 1 of [25] is that $|L|$ lies in the interior of an open n-ball in M^n. As $|L|$ is closed in M^n and an open n-ball is homeomorphic to E^n, Theorem 3.8 implies that M^n is an n-sphere.□

We now have some examples that restrict the possible improvement of Theorem 3.7 in certain ways.

Corollary 3.11: Let q and n be integers such that $0 \le q \le n-3$ and $q \le [\frac{n-2}{2}]$. Then there exists a finite n-dimensional simplicial complex K (that is even a combinatorial n-manifold without boundary) having $|K| - |K^{(q)}|$ not 1-movable.

Proof: There is a finite triangulation, K, of $S^{q+1} \times S^{n-q-1}$ that is a combinatorial n-manifold. As $n-q-1 \ge q+1$, $\pi_i(|K|) = \pi_i(S^{q+1}) \times \pi_i(S^{n-q-1}) = 0$ for $i = 0,1,2,\ldots,q$. Suppose $|K| - |K^{(q)}|$ were 1-movable. Then by Corollary 3.10, $|K|$ is homeomorphic to the n-sphere. But $|K|$ is homeomorphic to $S^{q+1} \times S^{n-q-1}$, which is not homeomorphic to the n-sphere. This is a contradiction, so $|K| - |K^{(q)}|$ is not 1-movable.□

For an n-dimensional simplicial complex K, we have the question of whether or not $|K| - |K^{(q)}|$ must be 1-movable. The answer is affirmative for $q = n$ (since ϕ is 1-movable), $q = n-1$ (since $|K| - |K^{(n-1)}|$ is the disjoint union of open n-simplexes, each homeomorphic to E^n, and so clearly 1-movable), and for $q = n-2$ (by Theorem 3.7). The answer is negative when $n \ge 3$, and $0 \le q \le [\frac{n-2}{2}]$ (by Corollary 3.11). When $[\frac{n-2}{2}] < q < n-2$, the answer is unknown at present.

Remark 3.12: Observe that when the codimension, n-q, is 0,1, or 2 we have $|K| - |K^q|$ is 1-movable. However when $n-q \geq 3$, if we let $n = 2q + 2$, then Corollary 3.11 gives an example where $|K| - |K^{(q)}|$ is not 1-movable. Hence for each $p > 2$ we cannot have the theorem that $|K| - |K^{(n-p)}|$ is 1-movable, so Theorem 3.7 has the best fixed codimension possible.

Example 3.13: There is a n-dimensional, infinite, locally finite simplicial complex, K, that is a combinatorial n-manifold and a triangulation of $E^n = E^1 \times E^{n-1}$ (so that those simplices whose carrier is a subset of $E^1 \times ([0,+\infty) \times E^{n-2})$ form a triangulation of $[0,+\infty) \times E^{n-1}$) with the following property. Every homeomorphism h that is a translation along the first factor by some integer d (i.e. $h(x,y) = (x + d,y)$) is a simplicial map (that is, $h(v)$ is a vertex of K for each vertex of K and whenever $v_1, v_2, \ldots, v_k$ are the vertices of a simplex of K, $h(v_1), h(v_2), \ldots, h(v_k)$ are vertices of a simplex of K). Thus such an h has $h(|K^{(q)}|) = |K^{(q)}|$ for each integer q, $0 \leq q \leq n$, and so $h|_{|K|-|K^{(q)}|}$ is a homeomorphism of $|K| - |K^{(q)}|$ onto itself.

Let q be an integer, $0 \leq q \leq n-1$, and A be a compact subset of $|K| - |K^{(q)}|$, then $A \subset [-a,a] \times E^{n-1}$ for some integer $a > 0$. So the homeomorphism h that is a translation by the integer 3a along the first factor has $h|_{|K|-|K^{(q)}|}$ a homeomorphism of $|K| - |K^{(q)}|$ onto itself and $A \cap h(A) = \emptyset$. Hence $|K| - |K^{(q)}|$ is 1-movable, and so examples of 1-movability occur for all integers q and n with $0 \leq q \leq n-1$.

Suppose we consider (as a relaxation of the previous question) the possible 1-movability of $|K| - |L|$ where K is an n-dimensional finite simplicial complex that is even a compact combinatorial manifold, L is a subcomplex of $K^{(q)}$, $L \neq K^{(q)}$. There are three basic possible relationships possible for K and L as follows:

a) K has no boundary

b) K has non-void boundary, ∂K, and $L \cap \partial K = \phi$

c) K has non-void boundary, ∂K, and $L \cap \partial K \neq \phi$.

For each of a), b), and c) we have an example with $|K| - |L|$ 1-movable and another one with $|K| - |L|$ not 1-movable (except for b), where no 1-movable example could exist), for each pair of integers n and q, $n \geq 2$, $0 \leq q \leq n$.

1) For a) 1-movable, let $K = \dot{\sigma}$, for an $(n+1)$-simplex σ, L be a q-simplex of K, with its faces. Then $|L|$ is cellular in $|K|$, so $|K| - |L|$ is homeomorphic to $|K|$ less a point, so to S^n less a point, so to E^n and so 1-movable.

2) For a) not 1-movable, let K be a triangulation of $S^1 \times S^{n-1}$ and L be the faces of a fixed q simplex of K. Then $|L|$ is a subset of an open subset of $|K|$ that is homeomorphic to E^n, and $S^1 \times S^{n-1}$ is not homeomorphic to S^n. Thus Theorem 3.8 would be contradicted if $|K| - |L|$ were 1-movable, and so $|K| - |L|$ is not 1-movable.

3) For b) not 1-movable, let K be a complex that triangulates $S^1 \times |D^{n-1}|$ where D^{n-1} is a $(n-1)$-dimensional simplex and K has an n-simplex σ with $\sigma \cap \partial K = \phi$. Let L be the faces of a fixed q-dimensional face of σ. Then $|K| - |L|$

is a n-manifold with boundary homeomorphic to $S^1 \times S^{n-2}$, but then the boundary is compact so Theorem 5.1 implies that $|K| - |L|$ is not 1-movable.

4) For c) 1-movable, let K be the first barycentric subdivision of the faces of a fixed n-simplex. When $q \leq n-1$ let L be the faces of a q-simplex of boundary K, and when $q = n$ let L be the faces of a n-simplex of K that has exactly n of its vertices in boundary K. In either case $|K| - |L|$ is homeomorphic to $|K|$ less a single point, and so homeomorphic to $[0,+\infty) \times E^{n-1}$. Thus $|K| - |L|$ is 1-movable.

5) For c) not 1-movable, let K be the second barycentric subdivision of a complex that triangulates $S^1 \times |D^{n-1}|$ where D^{n-1} is a (n-1)-dimensional simplex. For $q \leq n-1$ let L be the faces of a cellular q-simplex of boundary K, and for $q = n$ let L be the faces of a n-simplex with exactly n of its vertices in boundary K and $L \cap$ boundary K cellular in $|\partial K|$. In either case, Remark 5.2 tells us that if $|K| - |L|$ is 1-movable, then so is B = boundary $(|K| - |L|)$ = $|\partial K| - |L|$, which is homeomorphic to $|\partial K| - \{p\}$ for a point $p \in |\partial K|$. But p has an open neighborhood homeomorphic to E^{n-1}. When $n > 2$, Theorem 3.8 says that $|\partial K|$ is homeomorphic to S^{n-1}. But $|\partial K|$ is homeomorphic to $S^1 \times S^{n-2}$, thus not homeomorphic to S^{n-1}. Hence $|K| - |L|$ is not 1-movable. When $n = 2$, $|\partial K| - \{p\}$ is homeomorphic to $(S^1 \times \{0,1\}) - \{x\}$ (for $x \in S^1 \times \{0,1\}$), which has exactly one compact component. This contradicts Theorem 5.1 and so $|\partial K| - \{p\}$, hence B, hence $|K| - |L|$ is not 1-movable.

CHAPTER IV

PRODUCTS AND k-MOVABILITY

In this chapter we prove a theorem about products and then give some examples showing that the non-compactness hypothesis of the theorem cannot be dropped and that even under fairly strong restrictions the converse of the theorem does not hold.

Theorem 4.1: If X is a non-compact k-movable topological space and all compact subsets of X are closed, and Y is any topological space, then $X \times Y$ is k-movable (k any positive integer).

Proof: Let X and Y be as in the hypotheses. Given a compact subset, K, of $X \times Y$, $L = p_1(K)$ is compact in X (where p_i is the projection of $X \times Y$ onto the i^{th} factor). So there exist compact subsets of L, say $L_1, L_2, \ldots, L_k$, and homeomorphisms of X onto itself, say $h_1, h_2, \ldots, h_k$, so that $L = \bigcup_{i=1}^{k} L_i$ and $L \cap h_i(L_i) = \emptyset$ for $i = 1,2,\ldots,k$.

Now for each $i = 1,2,\ldots,k$ we have L_i is compact, thus closed, and so $p_1^{-1}(L_i)$ is closed and $K_i = K \cap p_1^{-1}(L_i)$ is closed in K and so compact. Define the homeomorphism f_i of $X \times Y$ onto itself by $f_i(x,y) = (h_i(x),y)$.

It is now easily seen that $K = \bigcup_{i=1}^{k} K_i$, the K_i are compact, and $f_i(K_i) \subseteq f_i(p_1^{-1}(L_i)) \subseteq f_i(L_i \times Y) \subseteq h_i(L_i) \times Y$. But $K \subset L \times Y$, so $K \cap f_i(K_i) \subseteq (L \times Y) \cap (h_i(L_i) \times Y) =$ $(L \cap h_i(L_i)) \times Y = \emptyset \times Y = \emptyset$. □

Remark 4.2: The non-compactness of X when $k = 1$ in Theorem 4.1 cannot be dropped (even if we require X and Y to be manifolds) as S^m is 1-movable and $S^m \times S^m$ is not 1-movable. Note that a closed n-manifold is 1-movable if and only if it is invertible (this is easily seen by the definitions), and invertible if and only if it is a n-sphere (see [5] Theorem 1, and in [6] the statement on p. 959), and $S^m \times S^m$ is not homeomorphic to S^{2m}.

Remark 4.3: The converse of Theorem 4.1 with $k = 1$ does not hold as there exists a non-compact non-1-movable metric space G with $G \times G$ homeomorphic to E^6, and E^6 is 1-movable.

Let G be the decomposition space G of E^3 of R.H. Bing in section 2 of [2]. The union of the family of non-degenerate inverses of points G is a subset of the set A (which is a topological solid sphere with two handles) of p. 485 of [2]. A is clearly compact. But G is a quotient space of E^3, say by the quotient map g, so $g(A)$ is a compact subset of X. Assume G is 1-movable, then there exists a homeomorphism h of G onto itself such that $A \cap h(A) = \emptyset$. Let z be one of the points of G which has a non-degenerate inverse. Then $h(z)$ has a degenerate inverse and so $h(z)$ has a neighborhood homeomorphic to E^3, and hence z has a neighborhood homeomorphic to E^3, so G is locally E^3 at z. But this contradicts Theorem 13 of [2].

Now, A. Boals has shown in [4] that $G \times G$ is homeomorphic to E^6.

<u>Remark 4.4</u>: The converse of Theorem 4.1 does not hold for any k even if we require X and Y to be manifolds, either one of which may be specified to be compact and either one of which may be specified to be 1-dimensional.

This may be seen by using the following facts.

a) If an n-manifold M^n has compact boundary then it is not k-movable for any positive integer k (see Theorem 5.1).

b) $(0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$.

Now $(0,1) \times ([0,1) \times S^n)$ is homeomorphic to $((0,1) \times [0,1)) \times S^n$, which is homeomorphic (using b)) to $([0,1] \times [0,1)) \times S^n$, which is homeomorphic to both $[0,1] \times ([0,1) \times S^n)$ and $[0,1) \times ([0,1] \times S^n)$.

For each positive integer k, $(0,1)$ is k-movable, so Theorem 4.1 tells us that $(0,1) \times ([0,1) \times S^n)$ is k-movable. But $[0,1]$, $[0,1)$, $[0,1) \times S^n$, and $[0,1] \times S^n$ are all manifolds with compact boundary, and so they are not k-movable by a).

Thus counterexamples to the converse of Theorem 4.1 are $X = [0,1]$, $Y = [0,1) \times S^n$ and $X = [0,1)$, $Y = [0,1] \times S^n$.

CHAPTER V

MANIFOLDS AND k-MOVABILITY

In this chapter we prove that if M is a 1-movable n-manifold with boundary then the boundary of M is not compact and number of compact components of the boundary of M is either zero or infinite. Further statements concerning k-movability and manifolds with boundary are followed by the theorem that a connected Hausdorff space that is the union of k open subsets homeomorphic to E^n must be k-movable. This theorem has the corollaries that every open connected triangulable n-manifold is n-movable and that for $n \geq 5$, every contractible open n-manifold is 2-movable. A closed k-movable n-manifold is the union of $(k+1)$ open n-cells.

<u>Theorem 5.1</u>: If M is a k-movable manifold with boundary then the boundary of M is not compact and the number of compact components of the boundary is either zero or infinite.

<u>Proof</u>: Let M be a k-movable manifold with boundary and suppose that the boundary of M, B, is compact. Then $\emptyset \neq B = \bigcup_{i=1}^{k} B_i$ with the B_i compact, $B_1 \neq \emptyset$, and there exist homeomorphisms $h_1, h_2, \ldots, h_k$ of M onto itself with $B \cap h_i(B_i) = \emptyset$ for $i = 1,2,\ldots,k$. But then $B \cap h_1(B_1) = \emptyset$ and this cannot happen since a homeomorphism of a manifold onto itself has the boundary as the image of the boundary.

Now let M be a k-movable manifold with boundary and suppose that the boundary of M, B, has a finite non-zero number of compact components. Let D be the union of the compact components of B. Then $\phi \neq D = \bigcup_{i=1}^{k} D_i$ with the D_i compact and $D_1 \neq \phi$, and there exist homeomorphisms $h_1, h_2, \ldots, h_k$ of M onto itself with $D \cap h_i(D_i) = \phi$. So $D \cap h_1(D_1) = \phi$. But $h_1|_B$, the restriction to the boundary, of a homeomorphism of a manifold onto itself, is a homeomorphism of B onto itself, and as such, sends a compact component of B onto a compact component of B. Hence $\phi \neq h_1(D_1) = h_1|_B(D_1) \subseteq D$ and $D \cap h_1(D) \neq \phi$. This is a contradiction. □

Each homeomorphism of a manifold onto itself when restricted to the interior is a homeomorphism onto the interior and when restricted to the boundary is a homeomorphism onto the boundary. Thus we have the following remark.

Remark 5.2: If M is a k-movable n-manifold with boundary then both the interior of M and the boundary of M are k-movable spaces.

Remark 5.3: If we let M be the n-ball, then the interior of M is homeomorphic to E^n and so k-movable while the boundary of M is S^n and so k-movable. But M has a compact boundary and so (by Theorem 5.1) M is not k-movable. Hence the converse of Remark 5.2 is not true.

Remark 5.4: By Theorem 5.1, no compact manifold can be the boundary of a k-movable manifold. However if N is any non-compact k-movable n-manifold, then N is homeomorphic to the boundary of the $(n + 1)$-manifold $N \times [0,1)$. $N \times [0,1)$ is k-movable by Theorem 4.1.

<u>Theorem 5.5</u>: A connected Hausdorff space X that is the union of k open subsets homeomorphic to E^n is k-movable.

<u>Proof</u>: Let B be a proper non-void compact subset of $X = \bigcup_{i=1}^{k} X_i$, where for each i, X_i is open and homeomorphic to E^n by a homeomorphism $g_i: X_i \to E^n$. For each $b \in B$, $b \in X_{j_b}$ for some $j_b \in \{1,2,\ldots,k\}$ and so $b \in g_{j_b}^{-1}(S_{0,t_b})$ for some real $t_b > 0$. $B \subseteq \bigcup_{b \in B} g_{j_b}^{-1}(S_{0,t_b})$, $g_{j_b}^{-1}(S_{0,t_b})$ is open in X for all $b \in B$, and B is compact implies that $B \subseteq \bigcup_{\ell=1}^{N} g_{j_{b_\ell}}^{-1}(S_{0,t_{b_\ell}})$ and $b_\ell \in B$ for $\ell = 1,2,\ldots,N$, and N some positive integer. For each $i = 1,2,\ldots,k$ let $\lambda = \text{maximum}\,\{t_{b_\ell} \mid \ell \in \{1,2,\ldots,N\}\}$ and $B_i = B \cap g_i^{-1}(S_{0,\lambda}) \subset X_i$. Then $B = \bigcup_{i=1}^{k} B_i$. Since B is a proper subset of X, there exists an $x \in X - B$, and there exists for each $i = 1,2,\ldots,k$, a homeomorphism f_i of X onto itself with $f_i(g_i^{-1}(0)) = x$. But $X - B$ is open and $f_i g_i^{-1}$ are continuous functions for $i = 1,2,\ldots,k$ so there is a real number $t > 0$ with $f_i g_i^{-1}(S_{0,t}) \subset X - B$ for $i = 1,2,\ldots,k$. Let μ be a homeomorphism of E^n onto itself that is the identity outside some compact set and $\mu(S_{0,\lambda}) \subset S_{0,t}$. Then for each $i = 1,2,\ldots,k$ $\bar{\mu}_i = g_i^{-1}\mu g_i$ is a homeomorphism of X_i onto itself that is the identity outside some compact subset of X_i. So

$$\mu_i(x) = \begin{cases} \bar{\mu}_i(x) & \text{for } x \in X_i \\ x & \text{for } x \in X - X_i \end{cases}$$

is a homeomorphism of X onto itself. Let $h_i = f_i\mu_i$ a homeomorphism of X onto itself. Then $h_i(B_i) = f_i\mu_i(B_i) = f_i\bar{\mu}_i(B_i) = f_i g_i^{-1}\mu g_i(B_i) \subset f_i g_i^{-1}\mu g_i(g_i^{-1}(S_{0,\lambda})) \subset f_i g_i^{-1}\mu(S_{0,\lambda}) \subset f_i g_i^{-1}(S_{0,t}) \subset X - B$. Hence $B \cap h_i(B_i) = \emptyset$ for

$i = 1,2,\ldots,k.$ □

<u>Corollary 5.6</u>: Every open connected triangulable n-manifold is n-movable.

<u>Proof</u>: The theorem of [13] says that every open connected triangulable n-manifold is a union of n open n-cells. The corollary follows immediately by applying Theorem 5.5. □

<u>Corollary 5.7</u>: If M^n is a k-connected n-manifold without boundary and q is the minimum of k and $n - 3$ then M^n is $([\frac{n}{q+1}] + 1)$-movable.

<u>Proof</u>: Theorem 1 of [20] (also Theorem 2 of [16]) gives the result that M^n can be covered by $[\frac{n}{q+1}] + 1$ open n-cells. The corollary follows immediately by applying Theorem 5.5. □

<u>Corollary 5.8</u>: For $n \geq 5$, a contractible open n-manifold is 2-movable.

<u>Proof</u>: Let M^n be a contractible open n-manifold with $n \geq 5$. So M^n is n-connected and Corollary 5.7 (with $q = n - 3$) says that M^n is $([\frac{n}{n-2}] + 1)$-movable. As $n \geq 5$, $[\frac{n}{n-2}] = 1.$ □

<u>Theorem 5.9</u>: Let M^n be a closed n-manifold. If M^n is k-movable, then M^n is the union of $(k+1)$ open n-cells.

<u>Proof</u>: Let M^n be a k-movable closed n-manifold. Let $U \subset M^n$ and U be homeomorphic to an open n-cell. Then $R = M^n - U$ is a proper compact subset of M^n. So there exist compact subsets $R_1, R_2, \ldots, R_k$ of R and homeomorphisms, $h_1, h_2, \ldots, h_k$, of M^n onto itself with $R = \bigcup_{i=1}^{k} R_i$ and $R \cap h_i(R_i) = \emptyset$ for $i = 1,2,\ldots,k$. So for each $i = 1,2,\ldots,k$, $h_i(R_i) \subset U$, and so $R_i \subset h_i^{-1}(U)$. But then

$U, h_1^{-1}(U), h_2^{-1}(U), \ldots, h_k^{-1}(U)$ are $(k+1)$ open n-cells and clearly $M^n = U \cup \bigcup_{j=1}^{k} h_j^{-1}(U)$. $\square$

CHAPTER VI

CERTAIN OPEN 3-MANIFOLDS AND 1-MOVABILITY

The 1-movability of certain classes of open 3-manifolds is considered in this chapter. The complexity of the proof of Theorem 6.1 is to insure that the homeomorphism obtained has compact support. This gives the result that the monotone union of 3-dimensional non-trivial products of domains is 1-movable if E^1 is one of the factors an infinite number of times. Examples both of a 1-movable union and of a non-1-movable union exist when E^1 appears at most a finite number of times. We then prove that every W-space (contractible open 3-manifold that can be triangulated by a countable, locally finite simplicial complex which is a combinatorial manifold without boundary and each compact subset of which can be imbedded in S^3) is 1-movable. This has a corollary that all contractible domains in E^3 are 1-movable.

<u>Theorem 6.1</u>: Let A be a proper non-void compact subset of $E^1 \times B$, where B is an open manifold. Then there is a homeomorphism h of $E^1 \times B$ onto itself such that $A \cap h(A) = \emptyset$ and h is the identity map outside a compact subset of $E^1 \times B$.

<u>Proof</u>: Let A and B be as in the hypothesis, and for $i = 1,2$ let p_i be the projection onto the i^{th} factor from $E^1 \times B$. Now $p_1(A)$ is compact in E^1, so for some real number $a > 0$, $p_1(A) \subset [-a,a]$. $p_2(A)$ is compact in B and so has an open

neighborhood N with compact closure, $\bar{N}$, in B. Let λ = distance $(p_2(A), B - N) > 0$, and $\ell = \underset{b\in B}{\text{maximum}}$ distance $(b, B - N)$. Let $\mu(b) = \frac{\text{distance } (b, B-N)}{\lambda}$, a continuous function from B to the real numbers. Then $0 \le \mu(b) \le \frac{\ell}{\lambda}$ and $1 \le \mu(b)$ for $b \in p_2(A)$ while $\mu(b) = 0$ for $b \in B - N$. Define a homeomorphism g_b of E^1 onto itself by

$$g_b(t) = \begin{cases} \text{for } t \le -2a & t \\ \text{for } -2a \le t \le -a & \left(\frac{t+2a}{a}\right)\cdot(3a\mu(b) + a) - 2a \\ \text{for } -a \le t \le a & t + 3a\mu(b) \\ \text{for } a \le t \le \frac{3a\ell}{\lambda} + 2a & \left(\frac{t-a}{\frac{3a\ell}{\lambda} + a}\right)\left(\frac{3a\ell}{\lambda} + a - 3a\mu(b)\right) + 3a\mu(b) + a \\ \text{for } \frac{3a\ell}{\lambda} + 2a \le t & t \end{cases}$$

Define the function h from $E^1 \times B$ onto itself by $h(t,b) = (g_b(t),b)$. h is clearly continuous and has as inverse the continuous function f given by $f(t,b) = (g_b^{-1}(t),b)$.

If $(t,b) \in A$ then $b \in p_2(A)$ and so $\mu(b) \ge 1$, while $t \in p_1(A)$ and so $-a < t < a$. Hence $p_1(h(t,b)) = g_b(t) = t + 3a\mu(b) > -a + 3a\mu(b) \ge -a + 3a = 2a$. Then $h(t,b) \notin A$.

If $(t,b) \in E^1 \times B - [-2a, \frac{3a\ell}{\lambda} + 2a] \times (\bar{N})$, then either $t \in (-\infty,-2a) \cup (\frac{3a\ell}{\lambda} + 2a, +\infty)$ or $b \in B - N$. In the first case $g_b(t) = t$ for all $b \in B$ and in the second case $\mu(b) = 0$ so $g_b(t) = t$ for all $t \in E^1$. Hence in either case $h(t,b) = (g_b(t),b) = (t,b)$, and h is the identity outside the compact set $[-2a, \frac{3a\ell}{\lambda} + 2a] \times \bar{N}$. $\square$

Corollary 6.2: Let $X = \bigcup_{i=1}^{\infty} X_i$ where X is a manifold, and for each i, X_i is open in X and homeomorphic to $E^1 \times B_i$ with B_i

an open manifold, $X_i \subset X_{i+1}$. Then X is 1-movable.

Proof: Let X, the X_i's, and the B_i's be as in the hypothesis with $g_i \colon X_i \to E^1 \times B_i$ a homeomorphism for each i. Let A be a proper non-void subset of X. We have $A \subset X_j$ for some j. Applying Theorem 6.1 to $g_j(A) \subset E^1 \times B_j$, there is a homeomorphism h of $E^1 \times B_j$ onto itself with $g_j(A) \cap h(g_j(A)) = \emptyset$ and h is the identity outside a compact subset, L, of $E^1 \times B_j$. So $g_j^{-1} h\, g_j$ is a homeomorphism of X_j onto itself that is the identity outside the compact subset $g_j^{-1}(L)$ and has $A \cap (g_j^{-1} h\, g_j)(A) = \emptyset$. But X_j and $X - g_j^{-1}(L)$ are open in X and so $g_j^{-1} h\, g_j$ extends to a homeomorphism f of X onto itself, defined by $f(x) = \begin{cases} x & \text{for } x \in X - g_j^{-1}(L) \\ g_j^{-1} h\, g_j(x) & \text{for } x \in X_i \end{cases}$. $A \cap f(A) = \emptyset$. Thus X is 1-movable. □

Let D be a domain in E^3 which is the monotone union of open non-trivial products of domains $A_i \times B_i$ for $i = 1,2,\ldots$. For each i let dimension $A_i \le$ dimension B_i, then as $1 \le$ dimension $A_i \le$ dimension $B_i \le 2$ and dimension A_i + dimension $B_i = 3$, it must be true that dimension $A_i = 1$ and dimension $B_i = 2$. The only connected 1-dimensional manifolds are E^1 and S^1, so either $A_i = E^1$ for an infinite number of i or $A_i = S^1$ for all but a finite number of i. In the first case, Theorem 6.2 tells us that D is 1-movable. In the second case, there is an example where D is 1-movable and another where D is not 1-movable (Examples 6.3 and 6.4 respectively).

Example 6.3: Let $D = (-\infty,\infty) \times B$ where B is any open manifold of dimension 2. D is 1-movable by Theorem 4.1 and

$D = \bigcup_{i=1}^{\infty} (-i,i) \times B.$

Example 6.4: Let $D = \bigcup_{i=1}^{\infty} S^1 \times B_i = S^1 \times (E^2 - \{(1,0),(-1,0)\}) \subset S^1 \times E^2$, where $B_i = E^2 - \overline{(S_{(1,0),(\frac{1}{2})^i} \cup S_{(-1,0),(\frac{1}{2})^i})}$. To show that D is not 1-movable, consider the compact set A of D given by $A = S^1 \times T$ where $T = (\{-2,0,2\} \times [-2,2]) \cup ([-2,2] \times \{-2,2\})$.

If D is 1-movable, there exists a homeomorphism h of D onto itself with $A \cap h(A) = \phi$. There is a strong deformation retraction, r, of D onto A, so the inclusion map, i, of A into D induces an isomorphism, i_*, from $\pi_1(A,a)$ onto $\pi_1(D,a)$ for $a \in A$. $h|_A$ is a homeomorphism from A onto $h(A)$ and so induces an isomorphism $(h|_A)_*$ from $\pi_1(A,a)$ onto $\pi_1(h(A),h(a))$. $\pi_1(A,a)$ is isomorphic to $\pi_1(S^1,s) \times \pi_1(T,t) = Z \times G$ where $(s,t) = a$, G is a free group on two generators and Z is the group of the integers under addition.

Now, A is connected and so $h(A)$ is a subset of U, one of the components of $D - A$. Observe that each component of $D - A$ is $S^1 \times V$ where V is a component of $E^2 - (\{(-1,0), (1,0)\} \cup T)$. Let V be that component of $E^2 - (\{(-1,0), (1,0)\} \cup T)$ for which $U = S^1 \times V$. Let $h(a) = (u,v) \in S^1 \times (E^2 - \{(-1,0),(1,0)\})$. But then V has a strong deformation retraction onto F where $v \in F$ and F is homeomorphic to S^1. So there exists a strong deformation retraction, λ, of $S^1 \times V$ onto $S^1 \times F$. Then the induced homomorphism, λ_*, from $\pi_1(S^1 \times V,(u,v))$ onto $\pi_1(S^1 \times F,(u,v))$ is an isomorphism. But $\pi_1(U,(u,v)) = \pi_1(S^1 \times F,(u,v)) = \pi_1(S^1,u) \times \pi_1(F,v) = Z \times Z$.

Let j be the inclusion map from $h(A)$ into D.

$$\begin{tikzcd} D \arrow[r, "h"] & D \\ A \arrow[u, "i"] \arrow[r, "h|_A"'] & h(A) \arrow[u, "j"'] \end{tikzcd}$$ commutes.

So

$$\begin{tikzcd} \pi_1(D,a) \arrow[r, "h_*"] & \pi_1(D,h(a)) \\ \pi_1(A,a) \arrow[u, "i_*"'] \arrow[r, "(h|_A)_*"'] & \pi_1(h(A),h(a)) \arrow[u, "j_*"'] \end{tikzcd}$$ commutes.

Since the other three homomorphisms are isomorphisms, so is j_*, the homomorphism of fundamental groups induced by j.

Where the homomorphisms α and β are those induced by set inclusion, the following diagram commutes.

$$\begin{tikzcd} \pi_1(D,h(a)) & \\ & \pi_1(U,h(a)) \arrow[ul, "\beta"'] \\ \pi_1(h(A),h(a)) \arrow[uu, "j_*"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

But j_* is an isomorphism, so α is injective. So α is an injection of $Z \times G$ into $Z \times Z$. But $Z \times G$ is not abelian and $Z \times Z$ is abelian. This is a contradiction. Hence D is not 1-movable.

Let us now consider the 1-movability of a certain class of 3-manifolds, the W-spaces.

<u>Definition 6.5</u>: A <u>W-space</u> is a contractible open 3-manifold that can be triangulated by a countable, locally finite simplicial complex

which is a combinatorial manifold without boundary, and each compact subset of the open 3-manifold can be imbedded in S^3.

<u>Theorem 6.6</u>: Every W-space is 1-movable.

<u>Proof</u>: Let U be a W-space. Theorem 1 of [17] is the result that $U = \bigcup_{i=1}^{\infty} H_i$ where each H_i is a cube with handles, $H_i \subseteq$ interior H_{i+1}. Let A be a compact subset of U, then there is a positive integer j so that $A \subset$ interior H_j. There exists a pesudo-isotopy, $\{G_t\}$ $t \in [0,1]$, of H_j so that $G_1(A) \subset T \subset H_j$ and a N so that for $t \in [0,1]$ G_t is the identity map outside, N, the compact closure of a neighborhood of $A \cup T$ (where T is the union of a finite number of line segments and is homeomorphic to $\bigvee_{k=1}^{\ell} S^1$; ℓ is the number of handles on the cube with handles that is H_j). So this pseudo-isotopy extends to one of U by defining $G_t(x) = x$ for all $x \in U - H_j$ and $t \in [0,1]$. But since T is the union of a finite number of line segments in H_j, there is a homeomorphism h of U onto itself so that $T \cap h(T) = \emptyset$. Now, $G_1(A) \cap h(G_1(A)) \subseteq T \cap h(T) = \emptyset$ so $G_1(A) \cap h(G_1(A)) = \emptyset$. Hence Theorem 3.5 applies and there exists a homeomorphism, f, of U onto itself with $f(A) \cap A = \emptyset$. Therefore U is 1-movable. □

<u>Corollary 6.7</u>: All contractible domains in E^3 are 1-movable.

<u>Proof</u>: Let D be a contractible domain in E^3. It is well known that any domain in a Euclidean space can be triangulated by a countable, locally finite simplicial complex which is a combinatorial manifold without boundary. D, and so each compact subset of D, can be imbedded in S^3. Thus D is a W-space, and Theorem 6.6 applies. □

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