EXISTENCE OF HOMOCLINIC CONNECTIONS CORRESPONDING TO BILAYER STRUCTURES IN AMPHIPHILIC POLYMER SYSTEMS

By

Li Yang

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Applied Mathematics - Doctor of Philosophy

2013

ABSTRACT

EXISTENCE OF HOMOCLINIC CONNECTIONS CORRESPONDING TO BILAYER STRUCTURES IN AMPHIPHILIC POLYMER SYSTEMS

By

Li Yang

Bilayer structures are central to amphiphilic polymer systems which possess a phase which wets two immiscible fluids. The amphiphilic component forms thin layers which separate the immiscible phases. When one of the immiscible phases and the amphiphilic material are proportional, and scarce, then the mixture can be modeled as two phase and the bilayer structures as homoclinic connections. We prove the existence of the bilayer structures (homoclinic solutions) for the functionalized Cahn-Hilliard equation, whose equilibriums support these structures. We employ two methods: a functional analytical approach and a variant of Lin's method. The functional analytical approach is based upon a Newton type contraction mapping and it gives the leading order description of the homoclinic connection in terms of a homoclinic connection of a low-order problem. The contraction mapping construction also requires a non-degeneracy condition, which we conjecture is associated to an orbit-flip bifurcation of the homoclinic connection within the higher-order system. Lin's method is an implementation of the Lyapunov-Schmidt method to prove the existence of heteroclinic chains in dynamical systems. Because of the degeneracy of the full problem, we apply Lin's method only to a more restricted parameter regime.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction to the Functionalized Cahn-Hilliard Equation

A central goal of polymer chemistry is the design of materials with novel macroscopic properties by controlling the spontaneous generation of nanoscaled, phase separated networks, [7]. One mechanism to generate such networks is through the "functionalization" of hydrophobic polymer chains by the addition of acid terminated side-chains. In the presence of a polar solvent the end groups interact exothermically, driving the spontaneous generation of polymer-solvent or nanoparticle-solvent interfaces. The resulting phase separated network structures can be exploited for charge selective conduction, and have important applications to efficient energy conversion devices such as polymer electrolyte membranes for fuel cells, [58, 54], dye sensitized solar cells, [42], and bulk-heterojunction solar cells, [51].

The phase separation of microemulsions is typically dominated by interfacial energies. The Cahn-Hilliard (CH) Energy introduced in [15] is a diffuse interface model that characterizes a binary mixture by a phase field function u that maps $\Omega \in \mathbb{R}^n$ into mixture values $[-1, 1]$. It models the free energy as a balance between entropic effects, which seek to homogenize the species, and the mixture potential, W, which assigns energies to blends

$$\mathcal{E}(u) = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\, dx, \tag{1.0.1}$$

where $\varepsilon \ll 1$ controls the width of the inner structures. The Cahn-Hilliard equation is a mass-conserving H^{-1} gradient flow of the free energy functional (1.0.1); it describes the phase separation and domain coarsening of binary mixtures of inert materials. For an appropriate choice of double well potential W, solutions of the Cahn-Hilliard equation rapidly undergo a spinodal decomposition into "pure states" or "phases" occupying respective minima of wells of W, separated by transition layers of $O(\varepsilon)$ thickness, called single-layer interfaces. These "single layer" morphologies are competitors for the minimizer of the Cahn-Hilliard energy. The considerable attention has been paid to the evolution of the spatial domains occupied by the respective phases. Pego first established the motion of the inferfaces as a Mullins-Sekerka type flow, [52]. He introduced the chemical potential as the solution of Laplace's equation on each spinodal domain with the interfacial curvature as a Dirichlet condition on the internal interfaces. For $\varepsilon \ll 1$ he showed that the leading order normal velocity of the interface of the spinodal domains can be obtained from the jump in the normal derivative of the chemical potential defined on the complementary domains. More rigorous derivations of Pego's results quickly followed, particularly [2] and later [49]. The Gamma-convergence of the Cahn-Hilliard energy to the surface area functional was rigorously established in [47, 70]. That is, for $\varepsilon \ll 1$, minimizing sequences u_ε which converge to a limit in $L^1(\Omega)$ localize their gradients on an interface $\Gamma \in \mathbb{R}^n$ while $\mathcal{E}(u_\varepsilon)$ tends to a value which is proportional to the interfacial surface area.

While mixtures of inert materials generically seek to minimize surface area, functionalized materials have embedded charged groups which interact exothermically with polar solvents., spontaneously generating polymer-solvent interface. A primary example is Nafion, a functioinalized fluorocarbon polymer frequently used as a membrane separator in polymer electrolyte membrane fuel cells. Early small angle X-ray scattering (SAXS) experiments,

[34], lead Hsu and Gierke to hypothesized that a balance between the elastic energy of the interface and the hydrophilic surface interactions among the charged functional groups and the solvent drive Nafion to generate a water-filled, "pearled" pore network comprised of small 4-5 nanometer balls interconnected by thin 1-2 nanometer cylindrical pores. Nanoscale pearling has also be studied in functionalized diblock polymers, [31, 16]. There has since been considerable controversy as numerical and experimental investigation of the microstructure of Nafion and related perfluorinated membranes have lead to the proposition of bi-layer morphologies, [69, 39], cylindrical pores, [66], an inverted pore morphology with solvent groups surrounding cylinders of crystallized backbone, [59, 58], spherical clusters, [25], as well as more complex morphologies suggested by atomistic simulation [76, 38, 50]. Central to each of these models is the recognition that the solvent and ionic groups aggregate in the perfluorinated polymer matrix to form a connected network that allows for efficient proton transport through the nanometer-scale clusters.

For these applications the morphologies we consider are not the minimizers of the Cahn-Hilliard energy but come from its vast families of saddle points structures. Indeed, it is considerably less well-known that the critical points of the Cahn-Hilliard energy encompass large classes of network structures: domains for which the characteristic width of the minority phase domain scales with ε, giving rise to long, thin, percolating structures: bilayers, pores, pearled-pores, and micelle clusters, [50]. The network structures are fundamentally distinct from the more familiar surface area minimizers, as $\varepsilon \to 0^+$: the network morphologies grow thinner, and longer, and do not approach a fixed limit in $L^1(\Omega)$.

The "functionalized" Cahn-Hilliard energy (FCH), [28], is a reformulation of the Cahn-Hilliard energy, generating a new free energy which possesses the same families of critical points as the Cahn-Hilliard but which permits a facile mechanism to select the critical points with desired attributes as its quasi-minimizers. The model affords a finite interfacial width, accommodates merging and other topological reorganization, and couples naturally to momentum balance and other macroscopic mass transport equations. We assign a negative value to interfacial energy via the Cahn-Hilliard energy, and balance the negative Cahn-Hilliard energy against the square of its own variational derivative. In general, we denote such an energy by $\mathcal{F}$, and call it the functionalization of the orginal energy, $\mathcal{E}$,

$$\mathcal{F}(u) = \int_\Omega \frac{1}{2}\left(\frac{\delta \mathcal{E}}{\delta u}\right)^2 - \tilde{\eta}\left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\right)\, dx. \tag{1.0.2}$$

Here $\mathcal{E}$ represents a free energy functional of Cahn-Hilliard type, see (1.0.1), where W is a smooth, double-well potential with equal global minima at states $u = b_\pm$, with $b_- < b_+$ and $\mu_\pm := W''(b_\pm) > 0$. The FCH remaps this paradigm, balancing the square of the first variation of the CH energy against a small multiple of itself. Viewing the square of the first variation of the CH energy as the bending energy of the interface, physical considerations suggests the constant $\tilde{\eta}$ be a small parameter that represents the strength of functionalization and u is a phase function that differentiates between the solvent and polymer backbone regions represented by two minima of a double-well potential W.

The term functionalization is borrowed from synthetic chemistry where it refers to the addition of hydrophilic (functional) groups to a hydrophobic polymer to modify its solubility. Mathematically, "functionalization" is a systematic reformulation of the original energy. Indeed for the $\tilde{\eta} = 0$ problem, all critical points of $\mathcal{E}$, that is the solutions of $\frac{\delta \mathcal{E}}{\delta u} = 0$, render $\mathcal{F}(u) = 0$ and hence are global minimizers of $\mathcal{F}$. The parameter $\tilde{\eta}$ breaks this highly degenerate situation: crucially, for $\tilde{\eta} > 0$, the perturbation term favors the critical points of $\mathcal{E}$ with more surface area. It has been demonstrated for a broad class of energies that their associated functionalized form is bounded below and possesses global minimizers over natural function spaces, [56]. An energy similar to the FCH, called the Φ^6 model, has been proposed for amphiphilic systems, in which two immiscible fluids are mixed with a surfactant forming a microemulsion at the interface, [30]. The Φ^6 model was motivated by SAXS data which can be related to the reciprocal of the Fourier transform of the second variation of the energy evaluated at a constant background state.

Higher order energies, which resemble the FCH with $\tilde{\eta} < 0$ and an untilted well W, have been proposed, [45, 71]. Indeed a De Giorgi conjecture concerning the Γ limit of the FCH energy for $\tilde{\eta} < 0$ with an untilted well has been established, [57]. Extensions of these models to address deformations of elastic vesicles subject to volume constraints, [23, 24], and multicomponent models which incorporate a variable intrinsic curvature have been investigated, [73]. However, the single-layer interface forms the essential underpinning of each of these models.

It is instructive to view the FCH energy as a diffuse-interface regularization of a Canham-Helfrich, [17, 32] sharp interface energy of the form

$$\mathcal{E}_{CH}(\Gamma) := \int_\Gamma a_1 H^2 - a_2 dS, \tag{1.0.3}$$

however this identification is potentially misleading as it is predicated on the assumption that the underlying structures are of co-dimention 1 and free of defects, such as end-caps and junctions. Over $\mathbb{R}^3$ the FCH free energy supports co-dimension one bilayer interfaces, as well as and a wide range of stable co-dimension 2 and co-dimension 3 morphologies, [21, 28], in addition to many locally stable defect structures. The structure of the problem, and the physically motivating examples, change fundamentally and dramatically with the sign of $\tilde{\eta}$.

For these reasons the FCH merits a disctinct name, see [54], which evokes the amphiphilic nature of functionalized polymers.

It is crucial to emphasize the distinction between single-layer interfaces, which separate two dissimilar phases across a co-dimension one interface, and bilayers which separate two identical phases by a thin region of a second phase. Significantly, the single-layer framework can not support perforation of the interface. In many biological processes it is essential to understand the opening and closing of pores within a vesicle, or the roll-up of a bicelle into a closed vesicle, [68]. Single layer models treat the inside and outside of a vesicle as distinct phases: they can not be merged. In contrast, the $\tilde{\eta} > 0$ perturbation supports stable, strongly incompressible bilayers which admit not only the opening of perforations, but the roll-up of the bilayer into a solid filament or its break-up into a collection of solid micelle, in a manner which naturally accounts for the competition between these morphologically distinct structures for a scarce surfactant phase. We do not address this competition within the current work, rather we prove the existence of the bilayer structures of the FCH energy.

The Cahn-Hilliard equation, the H^{-1} mass preserving gradient flow on the energy $\mathcal{E}$, drives the system to minimize interface area while preserving the volume fraction, which describes the coarsening processes of binary alloys [9, 14, 15]. Network formation in functionalized polymers is a fundamentally different process, and the FCH energy takes this into consideration by balancing the solvation energy released by formation of water-acid interface against the elastic energy required to bend the interface, and associated polymer backbones. The functionalized Cahn-Hilliard equation is a mass preserving gradient flow of the FCH energy

$$u_t = -\mathcal{G}\frac{\delta \mathcal{F}}{\delta u} = -\mathcal{G}\left((\varepsilon^2\Delta - W''(u) + \tilde{\eta})(\varepsilon^2\Delta u - W'(u))\right), \tag{1.0.4}$$

where $\mathcal{G}$ is any positive, self-adjoint operator whose only kernel is the constant factor 1. Examples include the zero-mass projection Π_0, which subtracts the average value,

$$\Pi_0 f := f - \frac{1}{|\Omega|}\int_\Omega f(x)dx, \tag{1.0.5}$$

as well as the negative Laplacian $-\Delta$.

We are interested in the critical points of (1.0.4) which are also homoclinic solutions,

$$\mathcal{G}\left((\varepsilon^2\Delta - W''(u) + \tilde{\eta})(\varepsilon^2\Delta u - W'(u))\right) = 0. \tag{1.0.6}$$

Inverting the gradient operator, (1.0.6) can be written as

$$\left((\varepsilon^2\Delta - W''(u) + \tilde{\eta})(\varepsilon^2\Delta u - W'(u))\right) = \theta, \tag{1.0.7}$$

where the constant θ can be viewed as a Lagrangian multiplier associated to the mass conservation. We look for the flat-interface co-dimension one bi-layer solutions $\Phi_m(z)$, after rescaling in transverse dimension and neglecting the tangential variation, Φ_m is the solution of

$$\left(\partial_z^2 - W''(\Phi_m) + \tilde{\eta}\right)\left(\partial_z^2 u - W'(\Phi_m)\right) = \theta, \tag{1.0.8}$$

which is homoclinic to the back-ground state b. It follows from (1.0.8) that the constants b and θ are connected via the relation

$$\theta = (W''(b) - \tilde{\eta})W'(b). \tag{1.0.9}$$

For $\theta = 0$, (1.0.8) supports single-layer heteroclinic structures which dominate the gradient

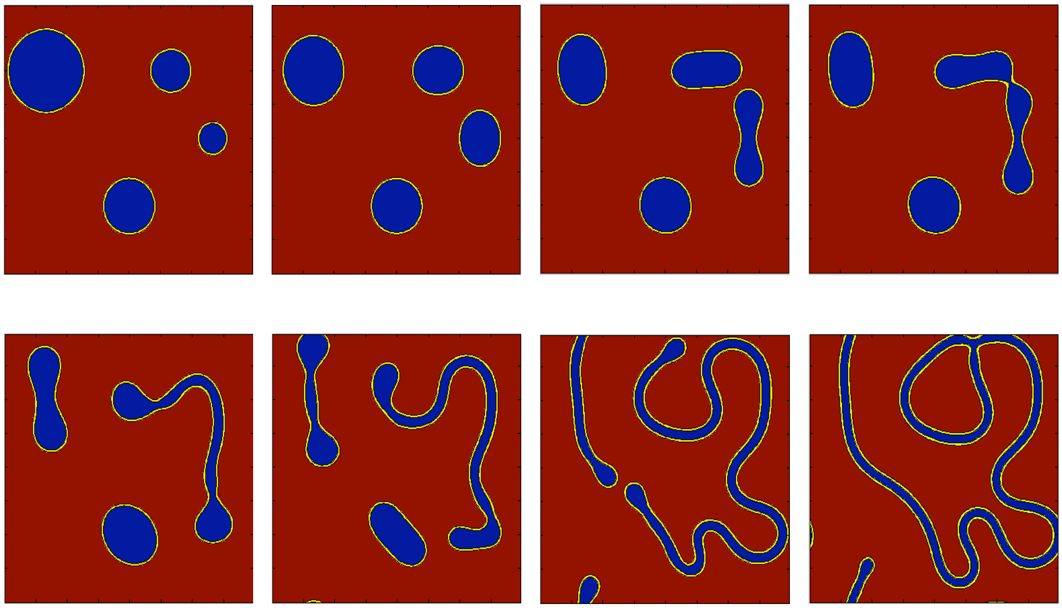

Figure 1.1: [28] Numerical evolution of (1.0.4). Reading left to right, the initial data is four circles of "water" ($u = b_+$) within a background of "polymer"($u = b_-$). The boundary between two domains is given by a front-type solution. In the early stage the high curvature circles grow at the expense of the lower curvature ones, however, the single layer interfaces are unstable to an antipodal elongation, and form bilayer structures, which lengthen and meander until achieving a quasi-equlibrium at critical bilayer width. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

flow of the Cahn-Hilliard energy. For $0 < |\theta| \ll 1$, we will show that the system has a *family* of homoclinic bi-layer solutions parameterized by bi-layer width. We demonstrate this using two distinct approaches: a functional analytical approach based upon a Newton-type contraction mapping and a dynamical systems approach based upon Lin's method.

In the next section, we will introduce the definitions and notations which will be used in the rest of paper and state our main theorems. The existence of homoclinic solution by functional analytical contraction mapping method will be illustrated in Chapter 2 and 3. In addition, Lin's method to prove the existence of homoclinic orbits in a less degenerated case is presented in Chapter 4.

1.1 Main Results

Before we state the main theorems, we introduce some definitions, assumptions and scaling which will be used throughout the paper.

(H) The well potential W is a smooth double well $W = P^2$ where P is a convex function with transverse zeros at $b_\pm$ with $b_- < b_+$, $W(b_\pm) = W'(b_\pm) = 0$ and $\mu_\pm := W''(b_\pm) > 0$, see Fig 1.2.

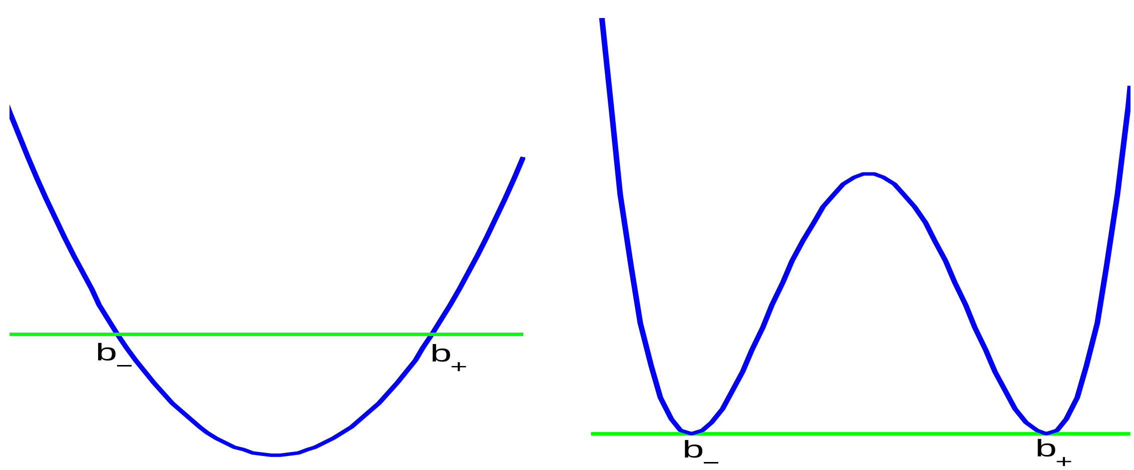

Figure 1.2: Illustration of P (left) and W (right).

The first parameter we introduce is the background state b of the homoclinic pulse profile. The goal of the functional analytical method is to characterize the leading order profile of the homoclinic solution of (1.0.8) in terms of a homoclinic solution of a second order equation. If $\theta \neq 0$, it is clear that the equilibrium state b for the homoclinic solution of (1.0.8) can not equal the minimum b_- of the potential W. To this end in Chapter 2 we introduce a parameter which shifts the minimum of the potential W to b, as well as a parameter which tilts the well potential G, yielding a new double well with one minimum at b and the other one near b_+. The tilted potential G is generically not an equal-depth well, indeed

$$G(b) = G'(b) = 0, \tag{1.1.1}$$

while $G'(b_+) = 0$ and $G'(b_+) < 0$. Consequently there exists another point $\phi_{max} < b$ such that

$$G(\phi_{max}) = 0, \quad G'(\phi_{max}) < 0, \tag{1.1.2}$$

as is depicted in Fig 1.3.

The functionalization parameter, $\tilde{\eta} > 0$, reflects the balance of the bending energy of the interface against the surface energy. To study the breaking we scale it as $\tilde{\eta} = \delta^2\eta$. Our analysis is performed in a neighborhood of $\eta = 0$, $b = b_-$ which we summarize as our scaling assumption

(S) Fix $\eta \in \mathbb{R}$ and $\beta < 0$. Then our standard scaling is

$$\tilde{\eta} = \eta\delta^2, \quad b = b_- + \delta^2\beta, \quad \text{for } 0 < \delta \ll 1. \tag{S)$-$ (1.1.3}$$

We prove the existence of the homoclinic solution under the scaling assumption (S)-(1.1.3) using the contraction mapping in Chapter 2. The dynamical system method presented in Chapter 4 yields a less concise characterization of the homoclinic solution and is presented

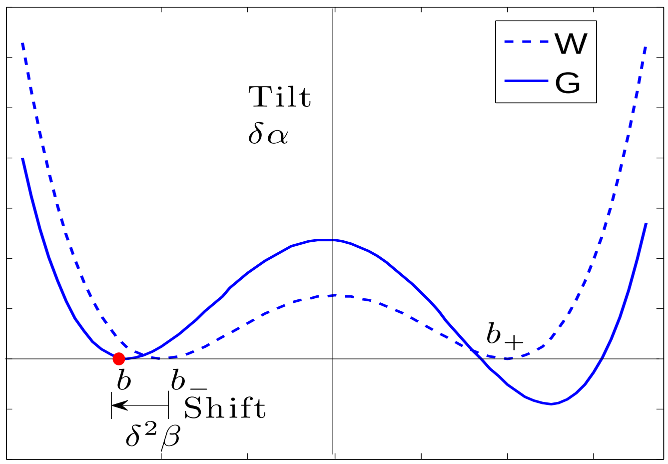

Figure 1.3: The tilted-shifted potential G.

under a less degenerate scaling assumption (S')

(S') Fix $\tilde{\eta}$, β such that $-\min\{\mu_\pm\} < \tilde{\eta} < 0$ and $\beta < 0$. Our scaling in order to apply Lin's method is

$$b = b_- + \delta^2\beta, \quad \text{for } \ 0 < \delta \ll 1. \qquad \text{(S')}- \ (1.1.4)$$

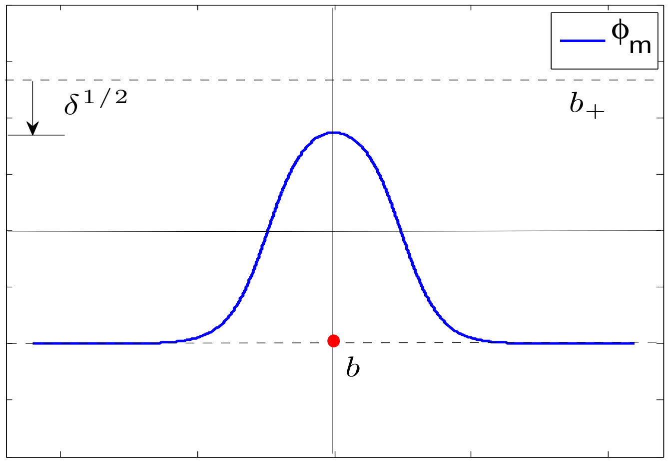

Figure 1.4: The homoclinic pulse profile of (1.1.6).

We construct the homoclinic solution in the space defined as

$$X_b := \left\{ u \middle| u - b \in H^4(\mathbb{R}) \right\}. \tag{1.1.5}$$

The contraction mapping approach seeks to construct the homoclinic solution Φ_m of (1.0.8) in the neighborhood of ϕ_m, the homoclinic solution of the second-order differential equation

$$\phi_m'' = G'(\phi_m), \tag{1.1.6}$$

which resides in X_b. Under the scaling (S)-(1.1.3) $G = G(u;\delta)$ with $G(u;\delta = 0) = W(u)$.

The difficulty in setting up the contraction mapping argument is that the linearization of the system (1.0.8) about ϕ_m is degenerate. To understand this degeneracy we first consider $\theta = 0$ in (1.0.8)

$$\left(\partial_z^2 - W''(u) + \tilde{\eta}\right)\left(\partial_z^2 u - W'(u)\right) = 0. \tag{1.1.7}$$

We denote by ϕ_h the heteroclinic solution of

$$\phi_h'' = W'(\phi_h), \tag{1.1.8}$$

which connects the two minima $b_\pm$ of W and satisfies

$$\lim_{z\to-\infty} \phi_h(z) = b_-, \quad \lim_{z\to\infty} \phi_h(z) = b_+, \tag{1.1.9}$$

and $\phi_h(0) = 0$. Linearizing the differential equation (1.1.7) around ϕ_h we obtain

$$\mathcal{L}_h := (L_h + \tilde{\eta})\, L_h, \tag{1.1.10}$$

where

$$L_h := \partial_z^2 - W''(\phi_h). \tag{1.1.11}$$

From (1.1.8) we can see that the operator L_h has a kernel spanned by ϕ_h' and hence $\mathcal{L}_h$ also has it as its kernel.

The saddle-saddle connection is broken when the well depths are unequal, $W(b_-) \neq W(b_+)$, breaking the heteroclinic chain connecting between b_- and b_+ into a homoclinic orbit and introducing a second small eigenvalue in the associated linearization. This hiddensymmetry eigenvalue makes the bifurcation of the homoclinic orbit from the heteroclinic chain a degenerate problem. *Removing this degeneracy is the main effort of the contraction mapping construction.* In particular we show that this eigenvalue + eigenvector pair has a connection to the Modica-Mortola $\sqrt{W}$ function which plays a fundamental role in Γ-convergence analysis of the Cahn-Hilliard energy [8, 43]. Indeed we show that the $\sqrt{W}$ is the optimal shape of the perturbed potential well to detune the degeneracy associated with the bifurcation of the homoclinic orbit from the heteroclinic one.

A key step of removing the degeneracy is the introduction of the tilted potential well

$$G(u;\alpha,b) = G_0(u;b) - \delta\alpha g(u;b), \tag{1.1.12}$$

where, as motivated in Chapter 2,

$$G_0 := W(u) - W(b) - W'(b)(u-b) - \frac{\tilde{\eta}}{4}(u-b)^2, \tag{1.1.13}$$

corresponds to the horizontal shift of the potential to absorb the mass constraint. The tilting function

$$g(u;b) := \int_b^u \sqrt{W_s(t;b)}\,dt, \tag{1.1.14}$$

removes the degeneracy of the heteroclinic to homoclinic bifurcation. Here $W_s(u;b) = W(u-\delta^2\beta)$ is a shifting of the double well.

A reasonable choice for g follows from an analysis of the linearized operator L_α obtained by linearizing (1.1.6) about ϕ_m,

$$L_\alpha = \partial_z^2 - G''(\phi_m;\alpha). \tag{1.1.15}$$

The point spectrum and associated eigenfunctions of $\mathcal{L}_\alpha$ play an important role in contraction mapping argument. In particular L_α has a translated eigenvalue at the origin and an $O(\delta)$ positive ground-state eigenvalue λ_0. An appropriate choice of g will yield a second order equation (1.1.6) whose homoclinic solution is the correct bi-layer Ansatz to $O(\delta^2)$. Removing the $O(\delta)$ term from the perturbation expansion require $L_\alpha g'(\phi_m) = O(\delta)$. The $\sqrt{W}$ choice for tilting perturbation in (1.1.14) is optimal since it renders

$$\sqrt{W(\phi_m)} = \psi_0 + O(\delta), \tag{1.1.16}$$

where ψ_0 is the ground state eigenfunction of operator L_α corresponding to the eigenvalue λ_0, see Fig 1.5.

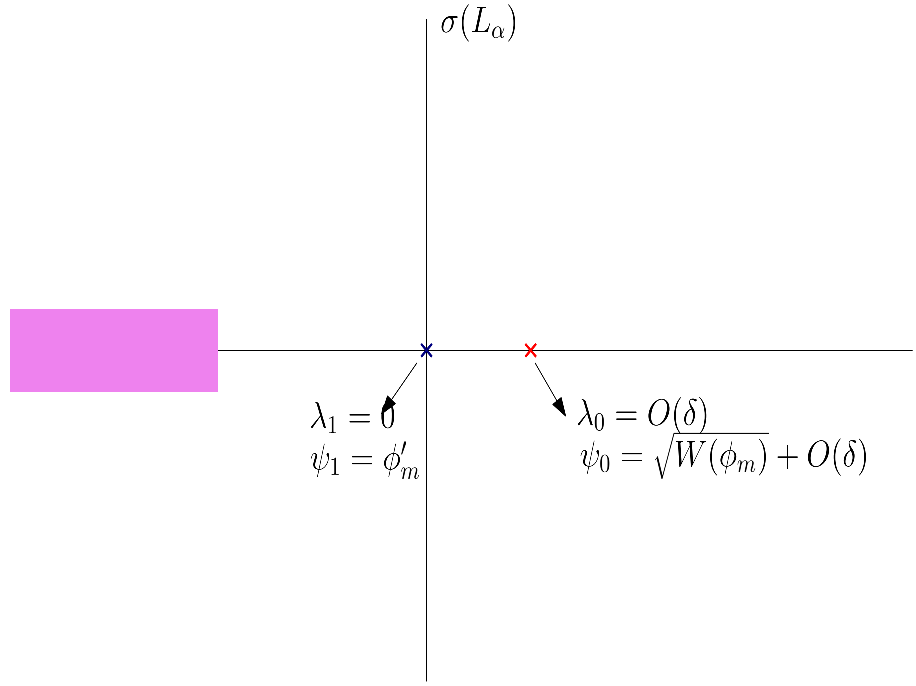

Figure 1.5: The illustration of spectrum of L_α. The colored strip denotes the essential spectrum with begins at $-W''(b) < 0$.

We also demonstrate that Lin's method applies to construct the homoclinic orbit. We adapt Lin's method [44, 61], and rewrite (1.0.8) as a one-parameter vector field,

$$\dot{x} = f(x, \theta), \tag{1.1.17}$$

where $x = (u, u', u'', u''')^T$ and $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is smooth and

$$\theta = W'(b)(W''(b) - \tilde{\eta}). \tag{1.1.18}$$

As in the case of the contraction mapping construction, the application of Lin's method requires unfolding a degeneracy. The classical application of Lin's method requires that the leading eigenvalues of $D_x f(x, \theta)|_{(p_{1,2}, 0)}$, where $p_{1,2}$ are the equilibria connected by the heteroclinic orbit, be semi-simple. In the case at hand the matrix eigenvalues form a Jordan block for the unperturbed, $\delta = 0$, case; moreover for $\delta \neq 0$ the Jordan block unfolds *smoothly* in δ forming real eigenvalues which perturb at $O(\delta^2)$. This co-dimension two situation greatly complicates our primary goal: to determine when the unfolding of the heteroclinic orbit coincides with an orbit-flip bifurcation. The contraction mapping construction realizes the leading order structure of the homoclinic as the solution of an associated second-order system. In an orbit-flip bifurcation the homoclinic solution of a fourth-order system converges to its equilibria along the fast stable or fast unstable directions within the stable or unstable manifolds respectively, see Fig 1.6. The degeneracy in our system makes the analysis of the orbit flip bifurcation particularly technical – at $\delta = 0$ the stable and unstable manifolds have a single eigenvector which perturbs into two distinct eigenvectors for $\delta \neq 0$. We must determine the projection of the homoclinic onto the fast and slow eigenvectors of the stable/unstable manifolds as they bifurcation away from each other.

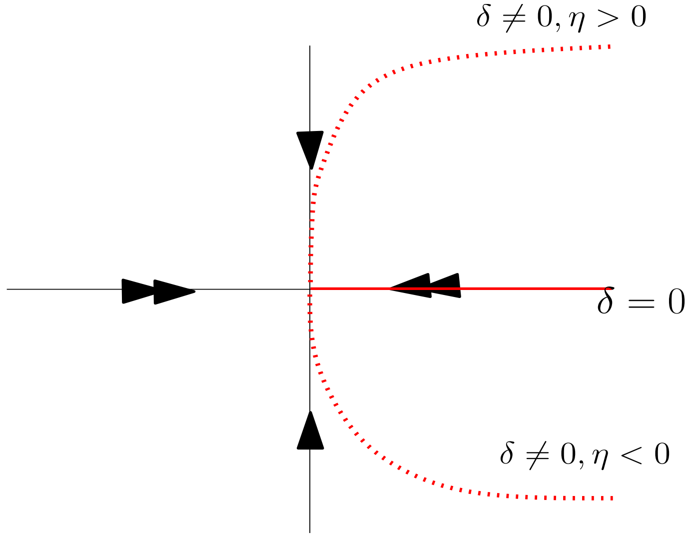

Figure 1.6: Depiction of the stable manifold of the equilibrium of a homoclinic orbit under an orbit flip bifurcation. For $\eta = 0$ the homoclinic orbit (red line) converges to the equilibrium along the fast stable direction, for $\eta > 0$ it converges to its equilibrium on one side, and flips the side for $\eta < 0$.

Our primary results, stated below, show the existence of a homoclinic orbit that bifurcates out of the heteroclinic orbit. In the contraction mapping approach we show that, up to a co-dimension one condition, characterized in terms of a Melnikov-type integral, the bifurcating homoclinic orbit can be well described by a homoclinic solution of an associated second-order dynamical system. Our primary conjecture is that this co-dimension one condition is equivalent to the orbit-flip bifurcation in the dynamical systems construction. Indeed, it is at an orbit-flip bifurcation, when the decay rate of the homoclinic switches from the fast-stable to the slow-stable rate, that a fourth dynamical system is most unlike a second order one. The full analysis of the orbit-flip bifurcation is outside the scope of this thesis, instead we use Lin's method to construct the homoclinic orbit in the non-degenerate case (S')-(1.1.4) for which $\tilde{\eta}$ is fixed at a non-zero value, yielding a heteroclinic connection at $\delta = 0$ that has simple matrix eigenvalues.

The main theorem of this paper is

Theorem 1.1. *Let the equal-depth double well W satisfying $(H1)$ be given and let ϕ_h denote the heteroclinic solution of (1.1.8). Let $\tilde{\eta}$, β be given by the scaling (S)-(1.1.3) and η, β satisfy*

$$(H_2) \qquad \left| A_1^h \beta + A_2^h \eta \right| > \nu\, \delta^{\omega}, \tag{1.1.19}$$

for some $\nu > 0$ independent of δ where $\omega > 0$ defined in (3.2.42) only depends on W. The constants A_1^h and A_2^h depend only upon the heteroclinic orbit ϕ_h,

$$A_1^h := -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + 3\left(W'''(\phi_h)(\phi_h - b_-), (\phi_h')^2\right)_2, \tag{1.1.20}$$

$$A_2^h := \left(W'''(\phi_h)(\phi_h - b_-), (\phi_h')^2\right)_2. \tag{1.1.21}$$

Then there exists a solution $\Phi_m \in X_b$ of (1.0.8) which is homoclinic to b, admits the following expansion

$$\Phi_m = \phi_m(z; \alpha_*(\delta)) + O(\delta^2), \tag{1.1.22}$$

in H^4 where $\phi_m \in X_b$ is the corresponding solution of the second-order differential equation (1.1.6), with α_* given by the smooth function $\alpha_* = \alpha_*(\delta; \beta, \eta)$ which satisfies

$$\alpha_*(\delta; \beta, \eta) := \sqrt{-\frac{\mu_+^{\frac{3}{2}}(b_+ - b_-)\beta}{\sqrt{2}\, g(b_+)}} + O(\sqrt{\delta}). \tag{1.1.23}$$

Remark 1.1. *In the neighborhood to the left of b_- depicted in Fig 1.8 there is a "bad" ray which is precisely excluded by condition (H2). Theorem 1.1 holds within the boxed domain except on this ray.*

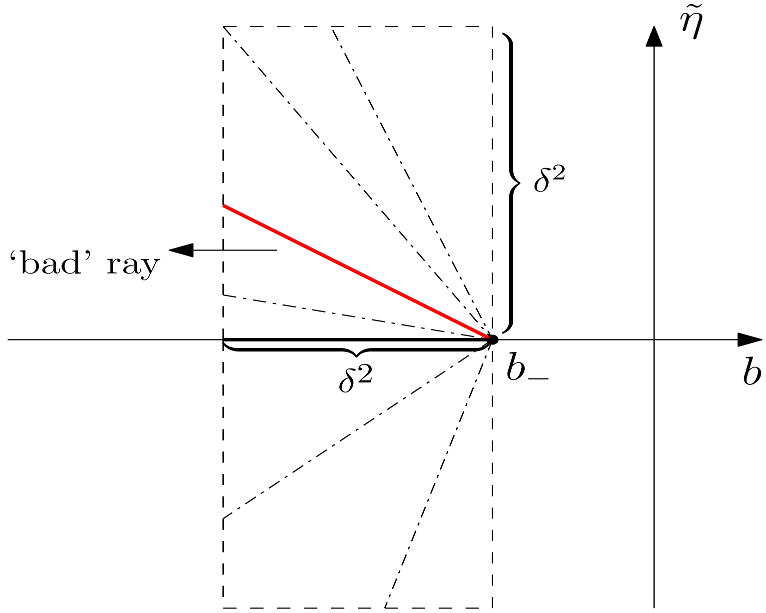

Figure 1.7: The illustration of the scaling (S)-(1.1.3) and the domain of the nondegeneracy in Theorem 1.1 in the left Figure and the illustration of the domain of applicability (S')-(1.1.4) of Theorem 1.2 in the right Figure.

We also apply Lin's method to construct the homoclinic orbit under the scaling (S')-(1.1.4). Since $b = b_- + \beta\delta^2$, $W(b_-) = W'(b_-) = 0$ and $\mu_- = W''(b_-) > 0$ we observe that

$$\begin{aligned} \theta &= W''(b_-)\left(W''(b_-) - \tilde{\eta}\right)\beta\delta^2 + O(\delta^3), \\ &= \mu_-(\mu_- - \tilde{\eta})\beta\delta^2 + O(\delta^3). \end{aligned} \tag{1.1.24}$$

For the case $\theta = 0$, the vector system (1.1.17) has two hyperbolic equilibria $p_1 := (b_+, 0, 0, 0)^T$ and $p_2 := (b_-, 0, 0, 0)^T$ satisfying $f(p_i; \theta = 0) = 0$. For $\theta = 0$, (1.1.17) has two symmetric heteroclinic connections $q_1(z)$, $q_2(z)$ between the bi-saddle hyperbolic equilibrium points p_1

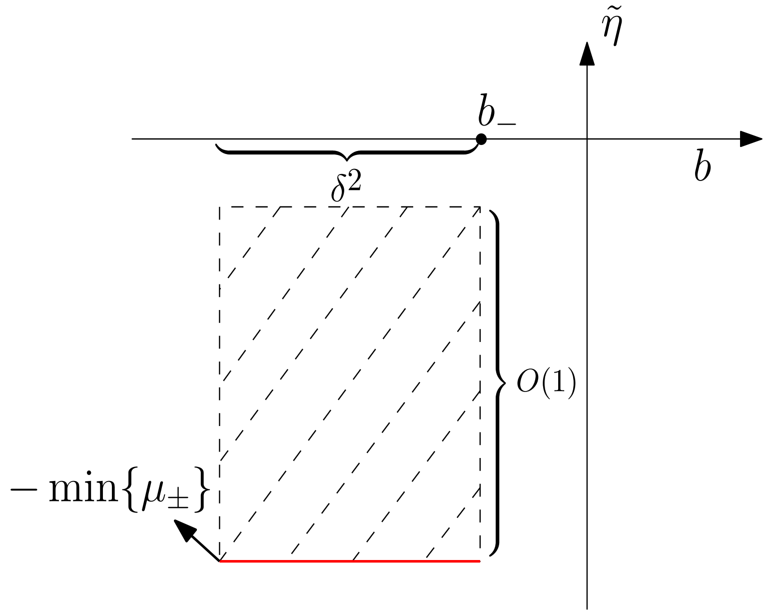

Figure 1.8: The illustration of the scaling (S)-(1.1.3) and the domain of the nondegeneracy in Theorem 1.1 in the left Figure and the illustration of the domain of applicability (S')-(1.1.4) of Theorem 1.2 in the right Figure.

and p_2, see Figure 1.9

$$\lim_{z\to-\infty} q_1(z) = p_1, \quad \lim_{z\to\infty} q_1(z) = p_2, \tag{1.1.25}$$

$$\lim_{z\to-\infty} q_2(z) = p_2, \quad \lim_{z\to\infty} q_2(z) = p_1. \tag{1.1.26}$$

The system (1.1.17) is reversible, that is symmetric under the transformation $z \mapsto -z$ and has two hyperbolic equilibria p_1 and p_2 at $\theta = 0$.

For $\delta = 0$, then $\theta = 0$ and the spectrum of $D_x\, f(x;\delta)\big|_{(p_i;0)}$ $i = 1, 2$ is given by

$$\sigma(D_x\, f(x;\delta)\big|_{(p_1;0)}) \;=\; \{\pm\sqrt{\mu_-}\}, \tag{1.1.27}$$

$$\sigma(D_x\, f(x;\delta)\big|_{(p_2;0)}) \;=\; \{\pm\sqrt{\mu_+}\}. \tag{1.1.28}$$

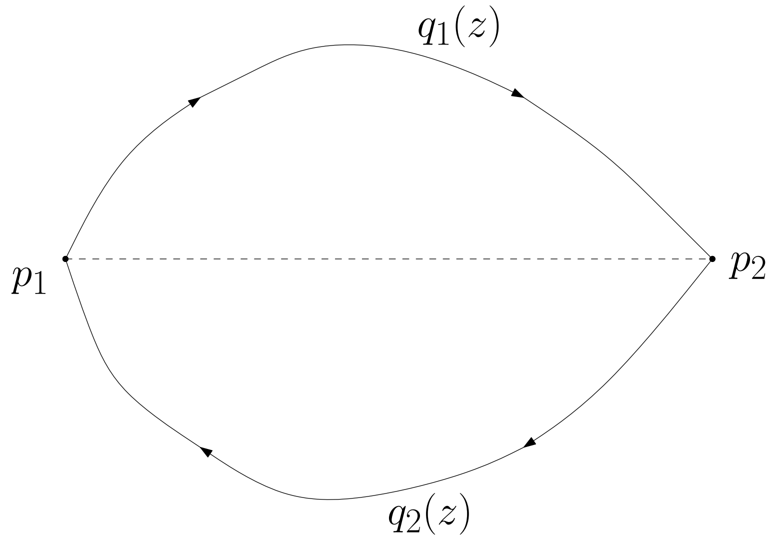

Figure 1.9: heteroclinic connection $\theta = 0$.

with $\mu_{\pm} > 0$. Moreover, they are not the simple eigenvalues, having a 2-dimensional Jordan block structure, which does not satisfies the hypothesis of Lin's method, [44, 61], that the leading eigenvalues be semisimple. For the Jordan block case, the $\delta = 0$ structure does not readily contain the full information. In this thesis we apply Lin's method to construct the homoclinic orbit in the non-degenerate case (S')-(1.1.4) for which $\tilde{\eta}$ is fixed at a non-zero value, yielding a heteroclinic connection at $\delta = 0$ that has simple matrix eigenvalues. For $\delta = 0$, then $\theta = 0$ and the spectrum of $D_x f(x;\delta)\big|_{(p_i;0)}$ is given by

$$\sigma(D_x f(x;\delta)\big|_{(p_1;0)}) = \{\pm\sqrt{\mu_-}, \pm\sqrt{\mu_- - \tilde{\eta}}\}, \tag{1.1.29}$$

$$\sigma(D_x f(x;\delta)\big|_{(p_2;0)}) = \{\pm\sqrt{\mu_+}, \pm\sqrt{\mu_+ - \tilde{\eta}}\}. \tag{1.1.30}$$

which satisfies the hypothesis of Lin's method.

We construct the section planes Σ_i, $i = 1, 2$, which are transverse to $q_i(z)$ at some point, say(without loss of generality) at $q_i(0)$, see Fig 1.13. For $\theta = 0$, the stable and unstable manifolds $W^s(p_i)$ and $W^u(p_i)$, $i = 1, 2$ for our system (1.1.17) are two-dimensional.

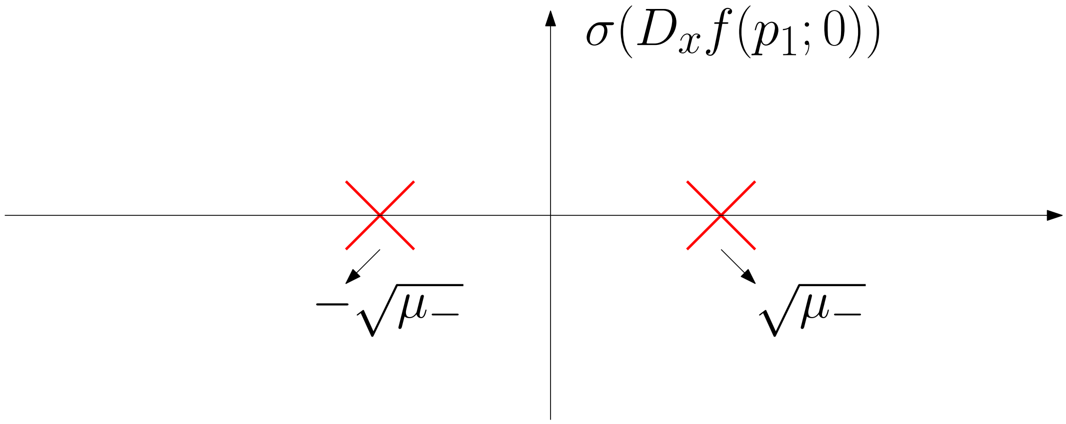

Figure 1.10: Under the assumption (S)-(1.1.3) the illustration of the eigenvalue of $D_x\, f(x;\delta)\big|_{(p_1;0)}$ for $\delta = 0$.

Moreover, our system (1.1.17) admits that the intersection between of the stable and unstable manifolds along the heteroclinic connection is of dimension one,

$$T_{q_1(0)}W^u(p_1) \cap T_{q_2(0)}W^s(p_2) \;=\; span\{\dot{q}_1(0)\}, \tag{1.1.31}$$

$$T_{q_2(0)}W^u(p_2) \cap T_{q_1(0)}W^s(p_1) \;=\; span\{\dot{q}_2(0)\}. \tag{1.1.32}$$

where T_qM denotes the tangent space of the manifold M at q. We introduce the subspace Z_i, $i = 1,2$ such that

$$\mathbb{R}^4 \;=\; Z_1 \oplus \left(T_{q_1(0)}W^u(p_1) + T_{q_1(0)}W^s(p_2)\right), \tag{1.1.33}$$

$$\mathbb{R}^4 \;=\; Z_2 \oplus \left(T_{q_2(0)}W^u(p_2) + T_{q_2(0)}W^s(p_1)\right). \tag{1.1.34}$$

Under the assumption (S')-(1.1.4) we show that the heteroclinic connecting orbit $\Gamma_1 := \{q_1(z)|z \in \mathbb{R}\}$ does not lie in the strong stable manifold of p_2 or the strong unstable manifold of p_1. This condition excludes the orbit flip for Γ_1, [61]. Due to the reversibility property of our system (1.1.17) it also excludes the orbit flip for $\Gamma_2 := \{q_2(z)|z \in \mathbb{R}\}$.

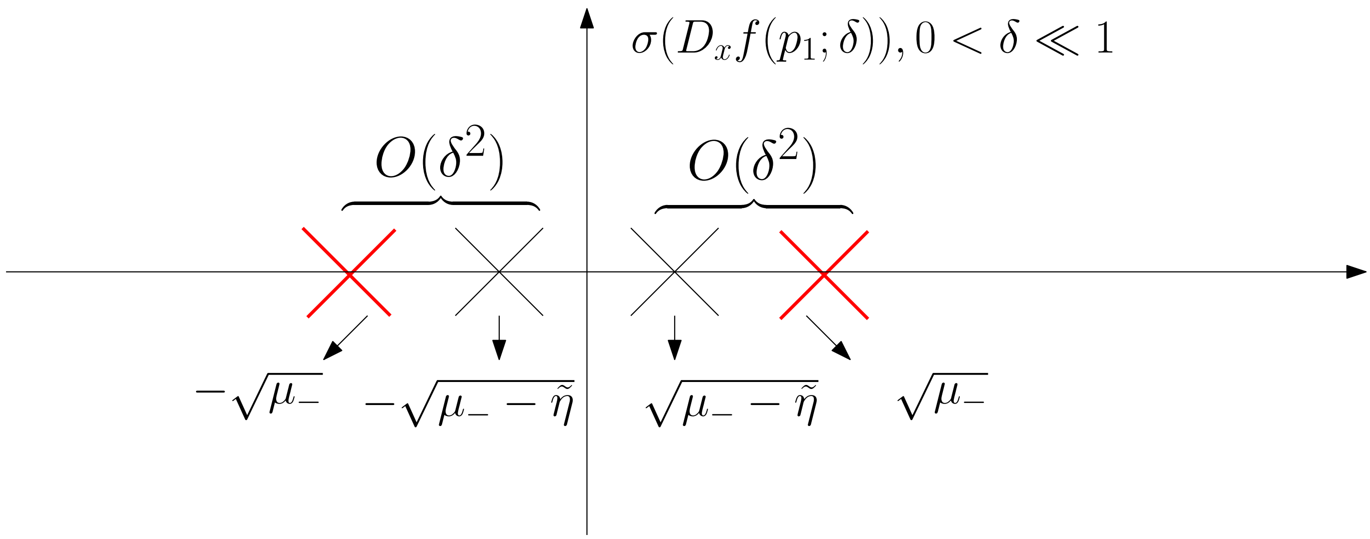

Figure 1.11: Under the assumption (S)-(1.1.3) the illustration of the eigenvalue of $D_x\, f(x;\delta)|_{(p_1;\delta)}, 0 < \delta \ll 1$ for $\delta \neq 0$.

Now we are ready to apply Lin's method [44, 61, 37] to demonstrate the bifurcation of the homoclinic solution from the heteroclinic connection for $\theta \neq 0$. To start we construct the perturbed heteroclinic connection $q_i^{\pm}(z)$, which converges to $p_{1,2}$ as $z \to \pm\infty$ and has a possible discontinuity jump in $Z_{1,2}$, see Figure 1.12. The discontinuity, called the jump $\xi_i^{\infty} = \xi_i^{\infty}(\theta)$, depends on the parameter θ and gives a bifurcation equation for the existence of such perturbed connections for $\theta \neq 0$. The solvability condition for the bifurcation equation is

$$\int_{\mathbb{R}} \psi_i(s)\, D_\theta f(q_i(s), 0)\, ds \neq 0, \qquad i = 1, 2, \tag{1.1.35}$$

where ψ_i spans the subspace Z_i. We show in Lemma 4.1 that this condition (1.1.35) is satisfied for our system. The second step of Lin's method is to construct the 'Lin's orbits'. These piecewise continuous orbits $x_i^{\pm}(z)$ are solutions to the system (1.1.17) and lie in a neighborhood of the perturbed heteroclinic orbits $q_i^{\pm}(z)$ and these orbits have the prescribed flying times $2\omega_2$ from Σ_1 to Σ_2. Moreover, they also satisfy

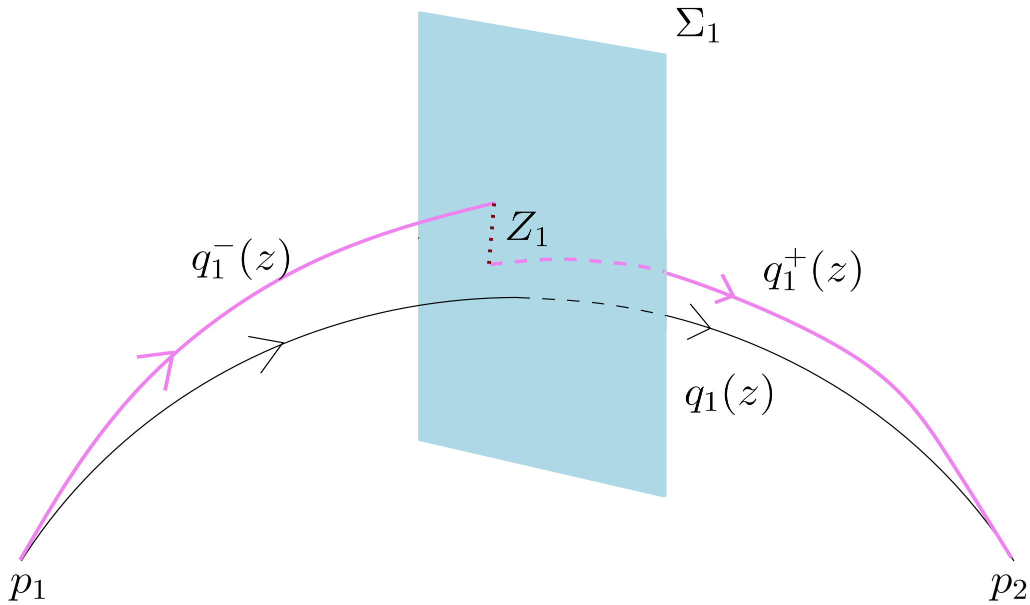

Figure 1.12: Perturbed heteroclinic connection for Γ_1.

- $x_i^{\pm}(0) \in \Sigma_i$, $i = 1, 2$,
- $x_i^+(0) - x_i^-(0) \in Z_i$, $i = 1, 2$,
- $x_1^-(-\infty) = x_2^+(\infty)$ and $x_1^+(\omega_2) = x_2^-(-\omega_2)$,

see Fig 1.13. We prove the existence and uniqueness of those 'Lin's orbits' $x_i^{\pm}(\theta, \omega_2)(z)$ and derive an expression for the jump

$$\begin{aligned} \xi_i(\theta, \omega_2) &:= \; < \psi_i, x_i^{+}(\theta, \omega_2)(0) - x_i^{-}(\theta, \omega_2)(0) >, \\ &= \; \xi_i^{\infty}(\theta) + \xi_i^{\omega_2}(\theta), \quad i = 1, 2. \end{aligned} \tag{1.1.36}$$

To obtain the homoclinic orbit, we require the jumps to be zero, i.e., $\xi_1(\theta, \omega_2) = 0$ which by the symmetry property of the system (1.1.17) also implies $\xi_2 = 0$. We also derive the leading order term of $\xi_1(\omega_2, \theta)$

$$\xi_1(\omega_2, \theta) = M_1\theta + c^u(\theta) e^{-2\omega_2 \lambda_2^u(\theta)} + o(e^{-2\omega_2 \lambda_2^u(\theta)}), \tag{1.1.37}$$

where $\lambda_2^u(\theta) = \sqrt{\mu_+}$ and the function $c^u(\cdot)$ is smooth and $c^u(0) \neq 0$. The following theorem, which is the main result of our application of Lin's method, is a consequence of the implicit function theorem.

Theorem 1.2. *Let $\tilde{\eta}$, δ and double well W be given and satisfy (H1) and (S')-(1.1.4). Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists a homoclinic solution of (1.0.8) denoted by Φ_m which is homoclinic to b where b satisfies (1.0.9).*

Remark 1.2. *Both Theorems 1.1 and 1.2 give the existence of a homoclinic solution Φ_m of (1.0.8). Theorem 1.1 provides for a sharp characterization of the homoclinic solution Φ_m in terms of a homoclinic solution of a corresponding second-order problem (1.1.6). On the other hand, Theorem 1.2 gives a perturbation expansion of the homoclinic orbit but does not classify the leading order expression in terms of a lower-order problem. The homoclinic solution is only shown to exist in the neighborhood of the heteroclinic solution that connects*

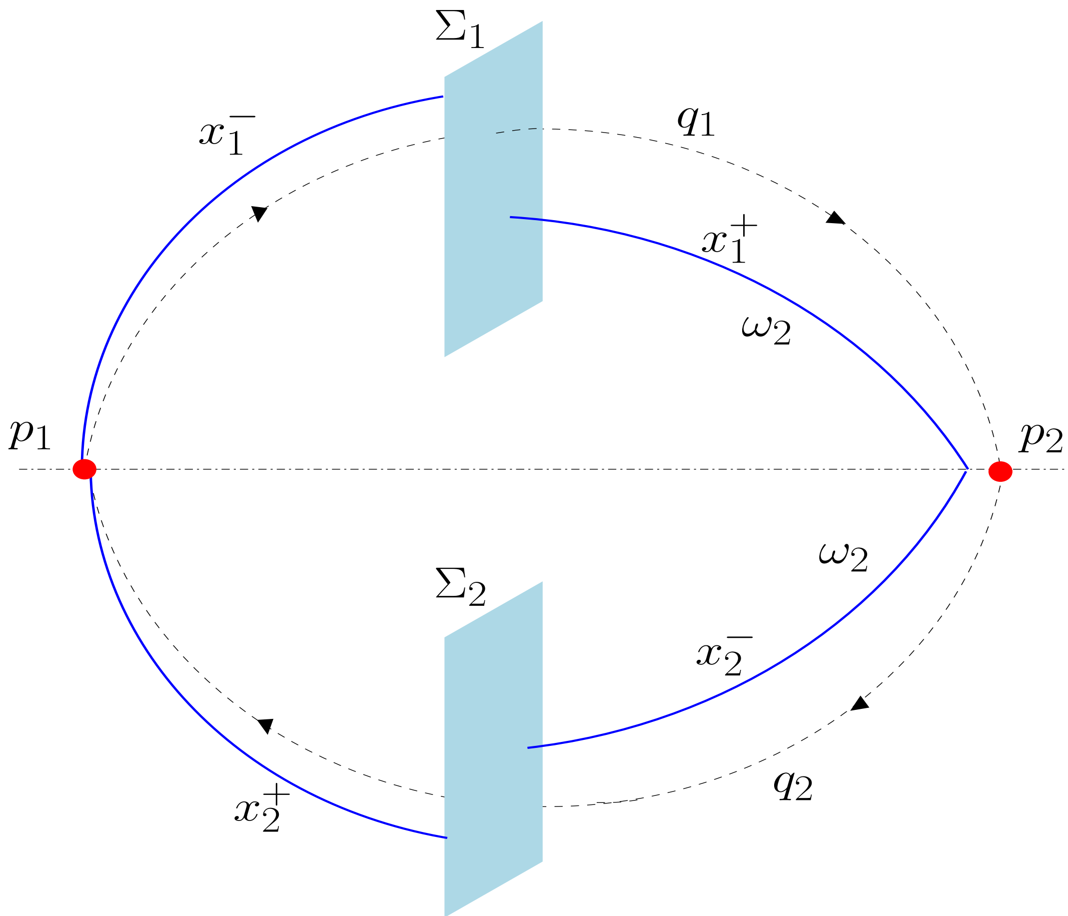

Figure 1.13: Lin's orbit.

the two minimums of W. While Theorem 1.1 requires the additional condition (H2) and the restriction $b > b_-$, Theorem 1.2 does not permit $\tilde{\eta}$ to scale with δ.

Chapter 2

Functional Analytical Method

In this chapter, we use a contraction mapping argument to construct a homoclinic solution Φ_m of (1.0.7). A key issue is that the natural contraction mapping arising from Newton's method is poorly scaled; we identify a tuning parameter corresponding to a tilting of the double well, see (1.1.14) and (2.2.3), which conditions the mapping.

2.1 Introduction

We are interested in the critical point of (1.0.4) which is also a homoclinic solution and satisfies

$$\Pi_0 \frac{\delta \mathcal{F}}{\delta u} = 0, \tag{2.1.1}$$

where Π_0 is the zero mass projection. Observing that the kernel of Π_0 is comprised of spatial constants, so if the shifted energy $\mathcal{H}$ satifies

$$\frac{\delta \mathcal{H}}{\delta u} = \frac{\delta \mathcal{F}}{\delta u} + c, \tag{2.1.2}$$

for some constant c, then u will satisfy (2.1.1) exactly when it satisfies (2.1.2). In the sequel we look for the homoclinic solution of the reformulated problem

$$\Pi_0 \frac{\delta \mathcal{F}}{\delta u} = \frac{\delta \mathcal{H}}{\delta u} = 0. \tag{2.1.3}$$

As a first step we rewrite the energy (1.0.2). Integrating by parts we find

$$\mathcal{F}(u) = \int\limits_{\Omega} \frac{1}{2}\varepsilon^4 (\Delta u)^2 + \varepsilon^2 f_1(u) |\nabla u|^2 + f_2(u)\, dx, \tag{2.1.4}$$

where we have introduced the functions

$$\begin{aligned} f_1(u) &= W''(u) - \frac{\tilde{\eta}}{2}, && (2.1.5)\\ f_2(u) &= \frac{1}{2}(W'(u))^2 - \tilde{\eta} W(u). && (2.1.6) \end{aligned}$$

We modify the potential f_2 to have a double zero at $u = b$,

$$f_3(u) = f_2(u) - f_2'(b)(u - b) - f_2(b). \tag{2.1.7}$$

From (2.1.7) it is easy to see that for our case $c = -f_2'(b)$. Define the shifted energy

$$\mathcal{H}(u) = \int_{\Omega} \frac{1}{2}\varepsilon^4 (\Delta u)^2 + \varepsilon^2 f_1(u) |\nabla u|^2 + f_3(u)\; dx, \tag{2.1.8}$$

whose variational derivative differs from that of $\mathcal{F}$ by a constant. We rewrite the shifted energy in terms of the potential

$$G_0(u) \equiv \int\limits_b^u \int\limits_{s_1=b}^{s_2} f_1(s_1)\, ds_1\, ds_2 = W_T(u;b) - \frac{\tilde{\eta}}{4}(u-b)^2, \tag{2.1.9}$$

where $W_T(u;b) = W(u) - W(b) - W'(b)(u-b)$ is the result of shifting the double zero of W at b_- to b by subtracting the Taylor polynomial. Integrating by parts on f_1 in (2.1.8) and completing the square we obtain

$$\mathcal{H}(u) = \int_\Omega \frac{1}{2}\left(\varepsilon^2 \Delta u - G_0'(u)\right)^2 + p(u)\; dx, \tag{2.1.10}$$

where $p(u) = f_3(u) - \frac{1}{2}\left(G_0'(u)\right)^2$. Analysis of the potential p performed in [28] yields the following expression

Lemma 2.1. *The potential p takes the form*

$$\begin{aligned} p(u) \;&=\; \left(W_T'(u) - W''(b)(u-b)\right) W'(b) - \tilde{\eta} W_T(u) + \\ &\quad + \frac{\tilde{\eta}}{2}\left(W_T'(u) - \frac{\tilde{\eta}}{4}(u-b)\right)(u-b), \end{aligned} \tag{2.1.11}$$

with a double zero at $u = b$. Under the scaling (S)-(1.1.3) then $p(u) = \delta^2 p_2(u)$ where

$$\begin{aligned} p_2(u) \;&=\; \overbrace{\mu_-(W'(u) - W'(b) - \mu_-(u-b))}^{p_{20}(u)}\beta + \\ &\quad + \overbrace{\left(\frac{1}{2}W'(u)(u-b) - W(u) + W(b)\right)}^{p_{21}(u)} + O(\delta), \end{aligned} \tag{2.1.12}$$

in $L^\infty([b,-b])$, and $\mu_- = W''(b_-)$. Moreover, if $uW''(u) > 0$, then both p_{20} and p_{21} are non-positive on $[b,-b]$, and zero at $z = b$.

2.2 Construction of Critical points by Contraction Mapping

Restricting to one space dimension and rescaling the transverse direction by ε the Euler-Lagrange equation associated to the shifted energy becomes

$$\frac{\delta \mathcal{H}}{\delta u}(u) = (\partial_z^2 - G_0''(u))(u_{zz} - G_0'(u)) + p'(u) = 0. \tag{2.2.1}$$

Recalling the scaling (S)-(1.1.3), i.e., $\tilde{\eta} = \delta^2 \eta$ and $b = b_- + \delta^2 \beta$, (2.2.1) becomes

$$F(u) := \left(\partial_z^2 - G''(u) - \alpha\delta g''(u)\right)\left(u_{zz} - G'(u) - \alpha\delta g'(u)\right) + p'(u) = 0, \tag{2.2.2}$$

where we have introduced the tilted well

$$G(u) := G_0(u) - \delta\alpha g(u), \tag{2.2.3}$$

g is defined in (1.1.14), and α is the δ-scaled parameter that tunes the shape of the potential.

We will show that for δ small enough, we can generate a solution of the full Euler-Lagrange equation in a small neighborhood of $\phi_m(z;\alpha,b)$ via a modified Newton's method. Here $\phi_m(z;\alpha,b)$ is the solution of the second order equation (1.1.6) associated to G. To this end we define the Newton map

$$N(u) = u - \mathcal{L}_\alpha^{-1}\left(F(u)\right), \tag{2.2.4}$$

where the self-adjoint linear operator

$$\mathcal{L}_\alpha = \left(L_\alpha - \delta\alpha g''(\phi_m)\right)^2 + \alpha\delta G'''(\phi_m)g'(\phi_m) + \delta^2\left(\alpha^2 g'''(\phi_m)g'(\phi_m) + p_2''(\phi_m)\right), \quad (2.2.5)$$

is the linearization of F about ϕ_m. Here L_α is defined in (1.1.15). Expanding $\mathcal{L}_\alpha$ we may write it as

$$\begin{aligned}\mathcal{L}_\alpha &= L_\alpha^2 + \alpha\delta\left(G'''(\phi_m)g'(\phi_m) - g''(\phi_m)L_\alpha - L_\alpha g''(\phi_m)\right) + \\ &\quad + \delta^2\left(\alpha^2 g'''(\phi_m)g'(\phi_m) + \alpha^2 g''(\phi_m)^2 + p_2''(\phi_m)\right). \end{aligned} \quad (2.2.6)$$

Spectrum analysis of L_α is performed in [28].

Lemma 2.2. *Let g be as in (1.1.14) with $\delta \ll 1$. Then there exists $\nu_0 > 0$, independent of δ, such that the spectrum of the linear operator L_α given in (1.1.15) consists of two point eigenvalues*

$$\sigma_p(L_\alpha) = \{\lambda_1 = 0, \lambda_0(\alpha, b)\}, \quad (2.2.7)$$

and a remainder contained in $(-\infty, -\nu_0]$. The ground state eigenvalue is given by the formula

$$\lambda_0 = \delta\hat{\lambda}_0 + O\left(\delta^2\right) > 0, \quad (2.2.8)$$

where

$$\hat{\lambda}_0 := \frac{4\alpha\gamma_0}{\|\phi_m'\|_2^2}. \quad (2.2.9)$$

Here

$$\begin{aligned}\gamma_0 &:= -\left(g'''(\phi_m)g(\phi_m), g'(\phi_m)\right)_2 = -\sqrt{2}g''(\phi_{max})g(\phi_{max}) + O\left(\delta^{\frac{1}{2}}\right), \\ &= \sqrt{\mu_+}g(b_+) + O\left(\delta^{\frac{1}{2}}\right) > 0. \end{aligned} \tag{2.2.10}$$

The corresponding normalized eigenfunctions take the form

$$\psi_1 = \frac{\phi_m'}{\|\phi_m'\|_2}, \tag{2.2.11}$$

$$\psi_0 = \frac{g'(\phi_m)}{\|g'(\phi_m)\|_2} + O(\delta), \tag{2.2.12}$$

with the equality on ψ_0 holding in L^2. Moreover we may relate $g'(\phi_m)$ to the translational eigenvalue

$$g'(\phi_m) = \frac{1}{\sqrt{2}}|\phi_m'| + O\left(\delta^{\frac{1}{2}}\right), \tag{2.2.13}$$

in the L^1 norm.

Proof. The operator L_α is a self-adjoint, 2nd-order, Sturn-Liouville operator and its spectrum is real. Taking the z derivative of (1.1.6) shows, from the translational symmetry, that ϕ_m' is in the kernel of L_α. This kernel has one node so the Sturn-Liouville theory implies that the ground state eigenvalue, λ_0, must be positive. To develop an expansion for λ_0 and the ground state $\psi_0 > 0$, from definition (2.2.3) of G we may rewrite the potential G as

$$G(s) = \left(g'(s)\right)^2 - \delta\alpha g(s) + \delta^2 h(s), \tag{2.2.14}$$

where the higher order term

$$
\begin{aligned}
h(s) &= \Big(W(s) - W(b) - W'(b)(s-b) - W(s-\beta\delta^2)\Big) - \frac{\eta}{4}(u-b)^2 \\
&= \big(W'(s) - \mu_-(s-b)\big)\,\beta - \frac{\eta}{4}(u-b)^2 + O(\delta),
\end{aligned}
\tag{2.2.15}
$$

has a double zero at $s = b$, is $O(1)$ over the range of ϕ_m and is smooth. The definition (1.1.14) of g implies that g is only C^1 with discontinuities in its second derivative at $s = b$, $b_+ + \beta\delta^2$. However g is piece-wise C^∞, and is C^∞ on the range of ϕ_m, since $\phi_{max} < b_+ + \beta\delta^2$. Applying L_α to $g'(\phi_m)$ and using (1.1.6) and its first integral

$$
\frac{1}{2}\left(\phi_m'\right)^2 = G(\phi_m), \tag{2.2.16}
$$

to eliminate derivative of ϕ_m, we obtain

$$
L_\alpha g'(\phi_m) = 2g'''(\phi_m)G(\phi_m) + g''(\phi_m)G'(\phi_m) - G''(\phi_m)g'(\phi_m). \tag{2.2.17}
$$

Taking derivative of (2.2.14) with respect to s we may eliminate G for g and h and their derivatives. The leading order terms cancel, yielding

$$
L_\alpha g'(\phi_m) = -2\delta\alpha g'''(\phi_m)g(\phi_m) + \delta^2 r(\phi_m), \tag{2.2.18}
$$

where the second order term

$$
r(s) = 2g'''h + g''h' - g'h'', \tag{2.2.19}
$$

is zero at $s = b$ and hence $r(\phi_m)$ is $O(1)$ in $\|\cdot\|_{L^2}$. Since the RHS of (2.2.18) is even, it is orthogonal to the kernel of L_α and we may invert, yielding

$$g'(\phi_m) = -\frac{2\delta\alpha\left(g'''(\phi_m)g(\phi_m) + \delta r(\phi_m), \psi_0\right)_2}{\lambda_0}\psi_0 + \delta g^{\perp}, \tag{2.2.20}$$

in L^2, where $g^{\perp} \perp \psi_0$ is $O(1)$. Taking the L^2 norm of both sides of (2.2.20), we deduce that $\|g'(\phi_m)\|_2 = O(1)$ since ϕ_m ranges from b to ϕ_{max} with $O(1)$ derivatives. From this we may infer that $\lambda_0 = O(\delta)$, which further yields the asymptotic expression

$$\lambda_0 = \frac{2\delta\alpha\left|\left(g'''(\phi_m)g(\phi_m), \psi_0\right)_2\right|}{\|g'(\phi_m)\|_2} + O(\delta^2). \tag{2.2.21}$$

Dividing both sides of (2.2.20) by their L^2 norms yields (2.2.12), and using this to substitute for ψ_0 in (2.2.21) yields (2.2.8). Using (2.2.14) and (2.2.16) we can obtain

$$\left(g'(\phi_m)\right)^2 = \frac{1}{2}|\phi_m'|^2 + \delta\alpha g(\phi_m) - \delta^2 h(s). \tag{2.2.22}$$

We may isolate $g'(\phi_m)$,

$$g'(\phi_m) = \sqrt{\frac{1}{2}\,|\phi_m'|^2 + \delta\alpha g(\phi_m) - \delta^2 h(s)}. \tag{2.2.23}$$

Expanding this expression yields (2.2.13), except for a neighborhood of $z = 0$ where $\phi_m'(z) = \phi''(0)z + O(z^3)$ where $\phi_m''(0) = G'(\phi_{max}) = O(\sqrt{\delta})$. These estimates yield the $O(\sqrt{\delta})$ error bound in (2.2.13). To simplify the expression for γ_0 we use (2.2.13) to write

$$
\begin{aligned}
\gamma_0 &= -\frac{1}{\sqrt{2}}\int_{\mathbb{R}} g'''(\phi_m)g(\phi_m)|\phi_m'|dz + O\left(\delta^{\frac{1}{2}}\right), \\
&= \sqrt{2}\int_0^{\infty} \partial_z\left(g''(\phi_m)\right)g(\phi_m)dz + O\left(\delta^{\frac{1}{2}}\right), \\
&= -\frac{1}{\sqrt{2}}\left(\left(g'(\phi_m)\right)^2 - 2g''(\phi_m)g(\phi_m)\right)\Big|_0^{\infty}. \qquad (2.2.24)
\end{aligned}
$$

However $g'(b) = 0$ and $g'(\phi_m(0)) = g'(\phi_{max}) = O(\sqrt{\delta})$ so the first term is $O(\delta)$. Similarly, $g(b) = 0$, and the expression reduces to

$$
\gamma_0 = -\sqrt{2}g''(\phi_m)g(b_+) + O\left(\delta^{\frac{1}{2}}\right). \qquad (2.2.25)
$$

Finally, from definition (1.1.14) of g we obtain

$$
\begin{aligned}
g''(\phi_{max}) &= \frac{W'(\phi_{max} - \beta\delta^2)}{2\sqrt{W(\phi_{max} - \beta\delta^2)}} \\
&= -\sqrt{\frac{\mu_+}{2}} + O\left(\delta^{\frac{1}{2}}\right). \qquad (2.2.26)
\end{aligned}
$$

□

From Lemma 2.2 we see that L_α has two small eigenvalues $\lambda_0 > 0$ and $\lambda_1 = 0$ with positive, even groundstate $\psi_0 > 0$ and $\psi_1 = \phi_m'/\|\phi_m'\|_2$. Since $\mathcal{L}_\alpha$ is an $O(\delta)$, relatively compact perturbation of the operator L_α^2, we deduce that it has two small eigenvalues, which we denote by $\Lambda_0 = \lambda_0^2 + O(\delta)$ and $\Lambda_1 = O(\delta)$. It is important to indentify the leading order behavior in Λ_0, which comes from the perturbations, since $\lambda_0^2 = O(\delta^2)$ is formally lower order. We introduce the spectral projections used in the rest of the paper. For $i = 0, 1$

we define the spectral projections π_i associated to the eigenvector ψ_i of L_α

$$\pi_i f := (f, \psi_i)_2 \psi_i, \tag{2.2.27}$$

and their sum $\pi = \pi_0 + \pi_1$ and their complement $\tilde{\pi}_i = I - \pi_i$.

Lemma 2.3. *The operator $\mathcal{L}_\alpha$ is self-adjoint, with real spectrum. Moreover there exists $\nu > 0$, independent of δ, such that*

$$\sigma_p(\mathcal{L}_\alpha) \cap \{|\lambda| < \nu\} = \{\Lambda_0, \Lambda_1\}, \tag{2.2.28}$$

with associated eigenfunctions Ψ_0 and Ψ_1,

$$\begin{aligned} \Psi_0 &= \psi_0 + O(\delta), &(2.2.29)\\ \Psi_1 &= \psi_1 + O(\delta), &(2.2.30) \end{aligned}$$

in L^2. Here ψ_0 and ψ_1 are the eigenfunctions of L_α corresponding to two small eigenvalues λ_0 and λ_1. Moreover

$$\Lambda_0 = \delta^2 \left(\hat{\lambda}_0^2 + \frac{\left(p_2''(\phi_m), (\phi_m')^2\right)_2 + 6\sqrt{2}\alpha^2 g(\phi_{max})(g''(\phi_{max}))^2}{\|\phi_m'\|_2^2} \right) + O\left(\delta^{\frac{5}{2}}\right), \tag{2.2.31}$$

where α is the tuning parameter introduced in (2.2.3), p_2 is defined in (2.1.12) and the scaling $\hat{\lambda}_0$ of λ_0 is defined in (2.2.9).

Proof. The operator $\mathcal{L}_\alpha$ is a relatively compact, $O(\delta)$ perturbation of L_α^2, and as such the point spectrum of $\mathcal{L}_\alpha$ is the square of that of L_α, up to $O(\delta)$. From lemma (2.2), L_α^2 has two small eigenvalues $\{0, \lambda_0^2\}$, with the rest of its spectrum an $O(1)$ distance from the origin, hence we may find a $\nu > 0$ which verifies (2.2.28). We obtain formulas for the eigenvalues via a regular perturbation expansion. We expand the operator $\mathcal{L}_\alpha = L_\alpha^2 + \delta\mathcal{L}_1 + \delta^2\mathcal{L}_2 + O(\delta^3)$ where

$$\mathcal{L}_1 = \alpha\left(G'''(\phi_m)g'(\phi_m) - g''(\phi_m)L - Lg''(\phi_m)\right), \tag{2.2.32}$$

$$\mathcal{L}_2 = \left(\alpha^2 g'''(\phi_m)g'(\phi_m) + \alpha^2 g''(\phi_m)^2 + p_2''(\phi_m)\right). \tag{2.2.33}$$

Similarly the eigenfunction and and eigenvalue have expansions

$$\Psi_0 = \psi_0 + \delta\Psi_{0,1} + \delta^2\Psi_{0,2} + O(\delta^3), \tag{2.2.34}$$

$$\Lambda_0 = \lambda_0^2 + \delta\Lambda_{0,1} + \delta^2\Lambda_{0,2} + O(\delta^3). \tag{2.2.35}$$

In light of these expansions, the eigenvalue equation

$$\mathcal{L}_\alpha\Psi_0 = \Lambda_0\Psi_0, \tag{2.2.36}$$

becomes

$$\begin{aligned}&(L^2 + \delta\mathcal{L}_1 + \delta^2\mathcal{L}_2)(\psi_0 + \delta\Psi_{0,1} + \delta^2\Psi_{0,2})\\ &= (\lambda_0^2 + \delta\Lambda_{0,1} + \delta^2\Lambda_{0,2})(\psi_0 + \delta\Psi_{0,1} + \delta^2\Psi_{0,2}) + O(\delta^3).\end{aligned} \tag{2.2.37}$$

From the relation $L^2\psi_0 = \lambda_0^2\psi_0$, the $O(1)$ terms cancel and at $O(\delta)$ we find,

$$(L^2 - \lambda_0^2)\Psi_{0,1} = \Lambda_{0,1}\psi_0 - \mathcal{L}_1\psi_0. \tag{2.2.38}$$

Focusing on the second term, $\mathcal{L}_1\psi_0$, in the right hand side of (2.2.38), by the definition of $\mathcal{L}_1$ (2.2.32) we have

$$\mathcal{L}_1\psi_0 = \alpha\left(G'''(\phi_m)g'(\phi_m)\psi_0 - Lg''(\phi_m)\psi_0\right) + O(\delta). \tag{2.2.39}$$

From (2.2.12) and (2.2.22) we have the expansion

$$\begin{aligned} G'''(\phi_m)g'(\phi_m)\psi_0 &= \frac{1}{\|g'(\phi_m)\|_2}G'''(\phi_m)\left(g'(\phi_m)\right)^2 + O(\delta), \\ &= \frac{1}{2\|g'(\phi_m)\|_2}G'''(\phi_m)\left(\phi_m'\right)^2 + O(\delta). \end{aligned} \tag{2.2.40}$$

Applying (3.1.4) from lemma 3.2 we further simplify this expression

$$G'''(\phi_m)g'(\phi_m)\psi_0 = \frac{1}{2\|g'(\phi_m)\|_2}L\phi_m'' + O(\delta). \tag{2.2.41}$$

This identity permits us to factor the operator L out of $\mathcal{L}_1\psi_0$, yielding the expression

$$\mathcal{L}_1\psi_0 = \alpha L\left(\frac{1}{2\|g'(\phi_m)\|_2}\phi_m'' - g''(\phi_m)\psi_0\right) + O(\delta). \tag{2.2.42}$$

However from (2.2.12) and (3.1.25) we find a leading-order cancelation yielding the result $\mathcal{L}_1\psi_0 = O(\delta)$, and hence (2.2.38) reduces to

$$(L^2 - \lambda_0^2)\Psi_{0,1} = \Lambda_{0,1}\psi_0. \tag{2.2.43}$$

The operator $L^2 - \lambda_0^2$ has kernel spanned by ψ_0 and an $O(\delta^2)$ eigenspace spanned by ψ_1. However by parity considerations the right-hand side of (2.2.43) is orthogonal to ψ_1 and the solvability of (2.2.38) requires that,

$$\Lambda_{0,1} = 0. \tag{2.2.44}$$

Since we require $\Psi_{0,1} \perp \psi_0 = 0$, we deduce that $\Psi_{0,1} = 0$ and (2.2.37) simplifies to

$$(L^2 + \delta \mathcal{L}_1 + \delta^2 \mathcal{L}_2)(\psi_0 + \delta^2 \Psi_{0,2}) = (\lambda_0^2 + \delta^2 \Lambda_{0,2})(\psi_0 + \delta^2 \Psi_{0,2}) + O(\delta^3). \tag{2.2.45}$$

Collecting the $O(\delta^2)$ terms in (2.2.45), and recalling $\mathcal{L}_1 \psi_0 = O(\delta)$ we obtain

$$(L^2 - \lambda_0^2)\Psi_{0,2} = -\frac{1}{\delta}\mathcal{L}_1 \psi_0 - \mathcal{L}_2 \psi_0 + \Lambda_{0,2}\psi_0. \tag{2.2.46}$$

Again, the right-hand side is even about $z = 0$ and hence orthogonal to ψ_1, we choose $\Lambda_{0,2}$ to make it orthogonal to ψ_0, obtaining

$$\Lambda_{0,2} = \frac{\left(\frac{1}{\delta}\mathcal{L}_1 \psi_0 + \mathcal{L}_2 \psi_0, \psi_0\right)_2}{||\psi_0||_2^2} = \frac{1}{\delta}(\mathcal{L}_1 \psi_0, \psi_0)_2 + (\mathcal{L}_2 \psi_0, \psi_0)_2, \tag{2.2.47}$$

where the second equality follows since $\|\psi_0\|_2 = 1$.

We analyze the first term of (2.2.47), substituting the expression (2.2.32) for $\mathcal{L}_1$ obtaining the equality

$$(\mathcal{L}_1 \psi_0, \psi_0)_2 = \alpha\big((G'''(\phi_m) g'(\phi_m)\psi_0, \psi_0)_2 - (g''(\phi_m) L\psi_0, \psi_0)_2 - (Lg''(\phi_m)\psi_0, \psi_0)_2\big). \tag{2.2.48}$$

Since L is self-adjoint and $L\psi_0 = \lambda_0 \psi_0$, we have the reduction

$$(\mathcal{L}_1 \psi_0, \psi_0)_2 = \alpha \left(G'''(\phi_m) g'(\phi_m)\psi_0, \psi_0\right)_2 - 2\alpha\lambda_0 (g''(\phi_m)\psi_0, \psi_0)_2. \tag{2.2.49}$$

From (3.1.22) we obtain an expression for ψ_0 which we substitute into (2.2.49), yielding

$$\begin{aligned}(\mathcal{L}_1\psi_0,\psi_0)_2 &= \frac{\alpha}{\rho_g^2}\left(\overbrace{\int_{\mathbb{R}}\frac{1}{2}G'''(\phi_m)g'(\phi_m)(\phi_m')^2dz}^{T_1}+\right.\\ &\quad+\overbrace{\int_{\mathbb{R}}\sqrt{2}G'''(\phi_m)g'(\phi_m)|\phi_m'|\left(\sqrt{\delta}(h_1-h_2)-\delta h_3\right)dz}^{T_2}+\\ &\quad\left.-\overbrace{\int_{\mathbb{R}}\delta\hat{\lambda}_0 g''(\phi_m)(\phi_m')^2dz}^{T_3}\right)+O\left(\delta^{\frac{3}{2}}\right). \end{aligned} \tag{2.2.50}$$

We examine the terms T_1, T_2 and T_3 in the integral above one-by-one. Using (3.1.4) to eliminate $(\phi_m')^2$ in T_1 we obtain

$$T_1=\frac{1}{2}\left(L\phi_m'', g'(\phi_m)\right)_2. \tag{2.2.51}$$

Using (3.1.9) and (2.2.9) to eliminate $g'(\phi_m)$ we obtain the reduction,

$$T_1=\frac{\delta}{2}\left(\rho_g\hat{\lambda}_0(\psi_0,\phi_m'')_2-2\alpha(g'''(\phi_m)g(\phi_m),\tilde{\pi}\phi_m'')_2\right)+O(\delta^2). \tag{2.2.52}$$

From (3.1.6) we see that the first inner product in (2.2.52) is $O(\delta^2)$. For the second inner product we observe that $\tilde{\pi}\phi_m''=\phi_m''+O(\delta)$ and hence

$$T_1 \;=\; -\delta\alpha\left(g'''(\phi_m)g(\phi_m),\phi_m''\right)_2+O(\delta^2).$$

Applying (3.1.28) from the appendix we obtain,

$$T_1 = \delta\alpha \left(\frac{1}{4} ||\phi_m''||_2^2 - \sqrt{2} g(\phi_{max}) (g''(\phi_{max}))^2 \right) + O(\delta^2). \tag{2.2.53}$$

Addressing the second term on the right-hand side of equality (2.2.50), we split it into three parts

$$\begin{aligned} T_2 &= \overbrace{\int_{\mathbb{R}} \sqrt{2\delta} G'''(\phi_m) g'(\phi_m) |\phi_m'| h_1 \, dz}^{T_{21}} + \\ &\quad - \overbrace{\left(\int_{\mathbb{R}} \sqrt{2\delta} G'''(\phi_m) g'(\phi_m) |\phi_m'| h_2 \, dz + \int_{\mathbb{R}} \delta\sqrt{2} G'''(\phi_m) g'(\phi_m) |\phi_m'| h_3 \, dz \right)}^{T_{22}}. \end{aligned} \tag{2.2.54}$$

Addressing T_{21} term of (2.2.50), we substitute the expression (2.2.13) for $g'(\phi_m)$ to obtain

$$T_{21} = \sqrt{\delta} \int_{\mathbb{R}} G'''(\phi_m) (\phi_m')^2 h_1 \, dz + O(\delta^2). \tag{2.2.55}$$

From (3.1.13) we eliminate $|\phi_m'| h_1$ to obtain

$$T_{21} = \frac{\delta\alpha}{\sqrt{2}} \int_{\mathbb{R}} G'''(\phi_m) |\phi_m'| g(\phi_m) \, dz + O(\delta^2). \tag{2.2.56}$$

Since ϕ_m is even and ϕ_m' is odd, we may break the integral into twice the half-line value

$$T_{21} = -\delta\alpha\sqrt{2} \int_0^\infty G'''(\phi_m) \phi_m' g(\phi_m) \, dz + O(\delta). \tag{2.2.57}$$

From (1.1.6) we deduce that $\phi_m''' = G''(\phi_m)\phi_m'$, , integrating by parts we find

$$\begin{aligned} T_{21} &= \delta\alpha\sqrt{2}\left(G''(\phi_m(0))g(\phi_m(0)) - \frac{1}{\sqrt{2}}\int_0^\infty \phi_m'''\phi_m'\,dz\right) + O(\delta^2), \\ &= \delta\alpha\sqrt{2}\left(G''(\phi_m(0))g(\phi_m(0)) + \frac{1}{2\sqrt{2}}||\phi_m''||_2^2\right) + O(\delta^2). \end{aligned} \tag{2.2.58}$$

Using (3.1.29) we can re-write the equality as

$$T_{21} = \delta\alpha(2\sqrt{2}g(\phi_m(0))(g''(\phi_m(0)))^2 + \frac{1}{2}||\phi_m''||_2^2) + O(\delta^2). \tag{2.2.59}$$

Turning to T_{22} and using (3.1.14) on T_{22} we find

$$T_{22} = \int_{\mathbb{R}} G'''(\phi_m)(\phi_m')^2\left(\sqrt{\delta}h_2 + \delta h_3\right)\,dz + O(\delta^2). \tag{2.2.60}$$

However from (3.1.4) and the self-adjointness of L we obtain

$$T_{22} = \int_{\mathbb{R}} \phi_m'' L\left(\sqrt{\delta}h_2 + \delta h_3\right)\,dz + O(\delta^2). \tag{2.2.61}$$

From (3.1.11) we obtain that

$$T_{22} = \int_{\mathbb{R}} \delta\phi_m'' L h_3\,dz + O(\delta^{\frac{3}{2}}). \tag{2.2.62}$$

From (3.1.12) we deduce that

$$Lh_3 = -2\alpha\tilde{\pi}g'''(\phi_m)g(\phi_m) + O(\delta), \tag{2.2.63}$$

and hence

$$
\begin{aligned}
T_{22} &= -2\delta\alpha \int_{\mathbb{R}} \phi_m'' \tilde{\pi} g'''(\phi_m) g(\phi_m)\, dz + O(\delta^{\frac{3}{2}}), \\
&= -2\delta\alpha \int_{\mathbb{R}} \phi_m'' g'''(\phi_m) g(\phi_m)\, dz + O(\delta^{\frac{3}{2}})
\end{aligned} \tag{2.2.64}
$$

where we again used that $\tilde{\pi}\phi_m'' = \phi_m'' + O(\delta)$. Finally, applying (3.1.28) yields,

$$
T_{22} = \delta\alpha \left(\frac{1}{2} ||\phi_m''||_2^2 - 2c_3 \right) + O(\delta^{\frac{3}{2}}). \tag{2.2.65}
$$

Finally, returning to (2.2.50) we consider the third term and using (3.1.8) show that

$$
\begin{aligned}
T_3 &= \delta\hat{\lambda}_0 \int_{\mathbb{R}} g'(\phi_m)\phi_m'' dz \\
&= \frac{\delta\hat{\lambda}_0}{\sqrt{2}} \int_{\mathbb{R}} |\phi_m'|\phi_m'' dz + O\left(\delta^{\frac{3}{2}}\right) = O\left(\delta^{\frac{3}{2}}\right).
\end{aligned} \tag{2.2.66}
$$

Combining the expressions (2.2.53), (2.2.66), (2.2.59) and (2.2.65) for T_1, T_{21}, T_{22}, T_3 we obtain

$$
\frac{1}{\delta}(\mathcal{L}_1\psi_0, \psi_0)_2 = \frac{\alpha^2}{\|\phi_m'\|_2^2} \left(\frac{1}{2}\|\phi_m''\|_2^2 + 6\sqrt{2} g(\phi_{max})(g''(\phi_{max}))^2 \right) + O\left(\delta^{\frac{1}{2}}\right). \tag{2.2.67}
$$

Turning our attention to the second term of $\Lambda_{0,2}$ in (2.2.47) we use the definition (2.2.33) of $\mathcal{L}_2$ to write this term as

$$
\begin{aligned}
(\mathcal{L}_2\psi_0, \psi_0) &= \int_{\mathbb{R}} \left(\alpha^2 g'''(\phi_m) g'(\phi_m) + \alpha^2 (g''(\phi_m))^2 + p_2''(\phi_m) \right) \psi_0^2 dz, \\
&= \frac{1}{\|\phi_m'\|_2^2} \int_{\mathbb{R}} \left(\alpha^2 g'''(\phi_m) g'(\phi_m) + \alpha^2 (g''(\phi_m))^2 + p_2''(\phi_m) \right) (\phi_m')^2 dz + O\left(\delta^{\frac{1}{2}}\right),
\end{aligned} \tag{2.2.68}
$$

where the second equality comes from the relation $\psi_0 = \frac{|\phi_m'|}{\|\phi_m'\|_2} + O(\delta^{\frac{1}{2}})$ which is a consequence of (2.2.12) and (2.2.13). Integrating by parts on the second term in the integrand we obtain

$$(\mathcal{L}_2\psi_0, \psi_0) = \frac{1}{\|\phi_m'\|_2^2} \int_{\mathbb{R}} -\alpha^2 g''(\phi_m)g'(\phi_m)\phi_m'' + p_2''(\phi_m)(\phi_m')^2 dz + O\left(\delta^{\frac{1}{2}}\right). \quad (2.2.69)$$

Finally, applying (3.1.25) to the first term in the integrand we obtain,

$$(\mathcal{L}_2\psi_0, \psi_0) = \frac{\left(p_2''(\phi_m), (\phi_m')^2\right)_2 - \frac{1}{2}\alpha^2\|\phi_m''\|_2^2}{\|\phi_m'\|_2^2} + O\left(\delta^{\frac{1}{2}}\right). \quad (2.2.70)$$

Combining the equalities (2.2.67) and (2.2.70) we simplify (2.2.47) to

$$\Lambda_{0,2} = \frac{\left(p_2''(\phi_m), (\phi_m')^2\right)_2 + 6\sqrt{2}\alpha^2 g(\phi_{max})(g''(\phi_{max}))^2}{\|\phi_m'\|_2^2} + O\left(\delta^{\frac{1}{2}}\right). \quad (2.2.71)$$

Inserting this expression into (2.2.35) and recalling the scaling (2.2.9) of λ_0, we obtain (2.2.31).

□

2.3 Conditioning of the Newton map

As a first step towards understanding the conditioning of the Newton map, we Introduce the spectral projections $\Pi_0(\alpha)$, $\Pi_1(\alpha)$ associated to the operator $\mathcal{L}_\alpha$

$$\Pi_0(\alpha)\, u \;:=\; (u, \Psi_0(\cdot, \alpha))_2\, \Psi_0, \quad (2.3.1)$$

$$\Pi_1(\alpha)\, u \;:=\; (u, \Psi_1(\cdot, \alpha))_2\, \Psi_1, \quad (2.3.2)$$

where Ψ_0, Ψ_1 are the two small eigenvalues associated to the operator $\mathcal{L}_\alpha$. Their sum is defined

$$\Pi(\alpha) := \Pi_1(\alpha) + \Pi_2(\alpha). \tag{2.3.3}$$

Their complements are

$$\begin{aligned} \tilde{\Pi}_0(\alpha) &:= I - \Pi_0(\alpha), && (2.3.4)\\ \tilde{\Pi}_1(\alpha) &:= I - \Pi_1(\alpha). && (2.3.5) \end{aligned}$$

Their sum is defined

$$\tilde{\Pi}(\alpha) := \tilde{\Pi}_0(\alpha) + \tilde{\Pi}_1(\alpha). \tag{2.3.6}$$

Because of the small eigenvalues Λ_0 and Λ_1 of the linearization $\mathcal{L}_\alpha$ the Newton map (2.2.4) is ill-conditioned. We also observe that $\tilde{\Pi}(\alpha)\mathcal{L}_\alpha^{-1} : L^2 \to H^4$ is uniformly bounded. We may "remove" Λ_1 by restricting $\mathcal{L}_\alpha$ to act upon even functions. For the second small eigenvalue Λ_0 in order to condition the Newton map we may tune $\alpha = \alpha(\delta; \beta, \eta)$ to eliminate Λ_0 for fixed β and η. Here we recall the scaling $b = b_- + \delta^2\beta$, $\tilde{\eta} = \delta^2\eta$, and $p = \delta^2 p_2$ and determine $\alpha = \alpha(\delta; \beta, \eta)$ for which the homoclinic solution $\phi_m^* = \phi_m^*(z; \alpha, \beta, \eta)$ of (1.1.6) satisfies

$$(F(\phi_m^*), \Psi_0(\cdot, \alpha))_2 = 0. \tag{2.3.7}$$

Lemma 2.4. *For b, $\tilde{\eta}$ given in (S)-(1.1.3) with $\beta < 0$, there exists a C^1 function $\alpha_* = \alpha_*(\delta; \beta, \eta)$ such that $\phi_m^* := \phi_m(\cdot, \alpha_*)$, the homoclinic solution of (1.1.6) satisfies*

$$(F(\phi_m^*), \Psi_0(\cdot, \alpha_*))_2 = 0, \tag{2.3.8}$$

where Ψ_0 is the eigenfunction corresponding to the smallest eigenvalue of $\mathcal{L}_\alpha$. Moreover, the function α_* enjoys the asymptotic expansion

$$\alpha_* = \sqrt{-\frac{\mu_+^2(b_+ - b_-)\beta}{\sqrt{2}\gamma_0}} + O(\sqrt{\delta}). \tag{2.3.9}$$

Remark 2.1. *The form of the leading order term of α_* requires that β be negative. Although α_* depends on β and η, it follows from (2.3.9) that the leading order expression of α_* depends only on β, indeed for fix $\beta > 0$*

$$\frac{\partial \alpha_*}{\partial \eta} = O(\sqrt{\delta}) \ll 1. \tag{2.3.10}$$

In the sequel, quantities denoted with "" are evaluated at $\alpha = \alpha_*(\delta; \beta, \eta)$. We will subsequently have a separate function, $\alpha = \alpha(u)$, which will be unadorned.*

Proof. We fix β and η and introduce the function

$$h(\alpha, \delta; \beta, \eta) := (F(\phi_m(\cdot, \alpha), \Psi_0(\cdot, \alpha)))_2 . \tag{2.3.11}$$

For a given value of β and η we wish to find $\alpha_* = \alpha_*(\delta)$ such that

$$h(\alpha_*, \delta) = 0. \tag{2.3.12}$$

Substituting $u = \phi_m$ into the expression (2.2.2) for F, recalling the definition of L_α and using (1.1.6) we can rewrite h as

$$h(\alpha, \delta) = \Big((L - \delta\alpha g''(\phi_m))(-\delta\alpha g'(\phi_m)) + \delta^2 p_0'(\phi_m), \Psi_0(\alpha) \Big)_2 . \tag{2.3.13}$$

We apply (2.2.18) and recall that $\Psi_0(\alpha) = \psi_0(\alpha) + O(\delta^2)$ to obtain

$$\begin{aligned} h(\alpha,\delta) &= \left(\delta^2\alpha^2 g''(\phi_m)g'(\phi_m) - \delta\alpha L g'(\phi_m) + \delta^2 p_2'(\phi_m), \psi_0(\alpha)\right)_2 + O(\delta^3), \\ &= \delta^2 \left(\alpha^2 g''(\phi_m)g'(\phi_m) + 2\alpha^2 g'''(\phi_m)g(\phi_m) + p_2'(\phi_m), \psi_0(\alpha)\right)_2 + O(\delta^3). \end{aligned} \tag{2.3.14}$$

We apply (3.1.22) to rewrite ψ_0

$$\begin{aligned} h(\alpha,\delta) &= \frac{\delta^2}{\sqrt{2}\rho_g}\left(\alpha^2 g''(\phi_m)g'(\phi_m) + 2\alpha^2 g'''(\phi_m)g(\phi_m) + p_2'(\phi_m), |\phi_m'|\right)_2 + O\left(\delta^{\frac{5}{2}}\right), \\ &= \frac{2\delta^2}{\sqrt{2}\rho_g}\tilde{h}(\alpha) + O\left(\delta^{\frac{5}{2}}\right), \end{aligned} \tag{2.3.15}$$

where, since ϕ_m is even and $\phi_m' < 0$ for $z > 0$, we have introduced

$$\begin{aligned} \tilde{h}(\alpha,\delta) &= -\alpha^2 \overbrace{\int_0^\infty g''(\phi_m)g'(\phi_m)\phi_m' dz}^{\tilde{h}_1} - 2\alpha^2 \overbrace{\int_0^\infty g'''(\phi_m)g(\phi_m)\phi_m' dz}^{\tilde{h}_2} + \\ &\quad - \overbrace{\int_0^\infty p_2'(\phi_m)\phi_m' dz}^{\tilde{h}_3}. \end{aligned} \tag{2.3.16}$$

Recalling $\phi_m(0) = \phi_{max}$ we obtain

$$\tilde{h}_1 = \frac{1}{2}(g'(\phi_m(\infty)))^2 - \frac{1}{2}(g'(\phi_{max}))^2 = O\left(\delta^{\frac{1}{2}}\right). \tag{2.3.17}$$

Integrating by parts on the second term $\tilde{h}_2$ and recalling that $\phi_m(0) = \phi_{max}$we have

$$\begin{aligned} \tilde{h}_2 &= -4\int_0^\infty g'''(\phi_m)g(\phi_m)\phi_m' dz, \\ &= 4g''(\phi_{max})g(\phi_{max}) + 4\int_0^\infty g''(\phi_m)g'(\phi_m)\phi_m' dz. \end{aligned} \tag{2.3.18}$$

The second integral of $\tilde{h}_2$ equals to $4\tilde{h}_1$ and hence

$$\tilde{h}_2 = 4g''(\phi_{max})g(\phi_{max}) + O\left(\delta^{\frac{1}{2}}\right). \tag{2.3.19}$$

Integrating by parts on $\tilde{h}_3$ and recalling $\phi_m(0) = \phi_{max}$ yields

$$\tilde{h}(\alpha, \delta) = 2p_2(\phi_{max}) + 4\alpha^2 g''(\phi_{max})g(\phi_{max}) + O\left(\delta^{\frac{1}{2}}\right). \tag{2.3.20}$$

Therefore we can choose

$$\alpha = \sqrt{-\frac{p_2(\phi_{max})}{2g''(\phi_{max})g(\phi_{max})}} + O\left(\delta^{\frac{1}{2}}\right), \tag{2.3.21}$$

such that

$$\tilde{h}(\alpha, \delta = 0) = 0. \tag{2.3.22}$$

By (3.1.3), (2.2.10) from Lemma 2.1 and 3.1, we can simplify the expression (2.3.21),

$$\alpha = \sqrt{-\frac{\mu_+^2(b_+ - b_-)\beta}{\sqrt{2}\gamma_0}} + O\left(\delta^{\frac{1}{2}}\right). \tag{2.3.23}$$

Since $\gamma_0 > 0$ and $b_+ - b_- > 0$, we choose $\beta < 0$ to render α is real. If we establish

$$\frac{\partial \tilde{h}}{\partial \alpha}(\alpha, \delta = 0) = \sqrt{-\frac{\mu_+^2 (b_+ - b_-)\beta}{\sqrt{2}\gamma_0}} > 0, \tag{2.3.24}$$

and

$$\sqrt{-\frac{\mu_+^2 (b_+ - b_-)\beta}{\sqrt{2}\gamma_0}} = O(1), \tag{2.3.25}$$

then by the implicit function theorem, there exists a C^1 function $\alpha_* = \alpha(\delta)$ such that $\tilde{h}(\alpha_*, \delta) = 0$, thereby completing the proof for this lemma. We first observe that

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial \alpha}(\alpha, \delta) \;\; &= \;\; \partial_\alpha \left(-2\alpha^2 \int_0^\infty g''(\phi_m) g'(\phi_m) \phi_m' dz - 4\alpha^2 \int_0^\infty g'''(\phi_m) g(\phi_m) dz + \right. \\ & \qquad \left. -2 \int_0^\infty p_2'(\phi_m)\phi_m' dz \right) + O\left(\delta^{\frac{1}{2}}\right). \end{aligned} \tag{2.3.26}$$

The first term of (2.3.26)

$$\begin{aligned} \partial_\alpha \left(-2\alpha^2 \int_0^\infty g''(\phi_m) g'(\phi_m) \phi_m' dz \right) \;\; &= \;\; -4\alpha \int_0^\infty g''(\phi_m) g'(\phi_m)\phi_m' dz + \\ & \quad -2\alpha^2 \int_0^\infty g'''(\phi_m) g'(\phi_m) \phi_m' \partial_\alpha \phi_m \, dz + \\ & \quad -2\alpha^2 \int_0^\infty (g''(\phi_m))^2 \phi_m' \partial_\alpha \phi_m \, dz + \\ & \quad -2\alpha^2 \int_0^\infty g''(\phi_m) g'(\phi_m) \partial_\alpha \phi_m' \, dz, \end{aligned} \tag{2.3.27}$$

by (3.1.24) from appendix we observe that

$$\partial_\alpha \phi_m = -\frac{1}{\sqrt{2}\hat{\lambda}_0}\phi_m' + O\left(\delta^{\frac{1}{2}}\right), \tag{2.3.28}$$

$$\partial_\alpha \phi_m' = -\frac{1}{\sqrt{2}\hat{\lambda}_0}\phi_m'' + O\left(\delta^{\frac{1}{2}}\right), \tag{2.3.29}$$

in $\|\cdot\|_{L^2}$ norm so that (2.3.27) can be simplified

$$\begin{aligned}
&\partial_\alpha\left(2\alpha^2\int_0^\infty g''(\phi_m)g'(\phi_m)\phi_m' dz\right),\\
&= -4\alpha\int_0^\infty g''(\phi_m)g'(\phi_m)\phi_m' dz + \frac{\sqrt{2}\alpha^2}{\hat{\lambda}_0}\int_0^\infty \left(g''(\phi_m)g'(\phi_m)\phi_m'\right)'\, dz + O\left(\delta^{\frac{1}{2}}\right),\\
&= -2\alpha(g''(\phi_m))^2\Big|_0^\infty + \frac{\sqrt{2}\alpha^2}{\hat{\lambda}_0}g''(\phi_m)g'(\phi_m)\phi_m'\Big|_0^\infty + O\left(\delta^{\frac{1}{2}}\right),\\
&= O(\delta^{\frac{1}{2}}).
\end{aligned} \tag{2.3.30}$$

Addressing the second term of (2.3.26), we expand it as

$$\begin{aligned}
&\partial_\alpha\left(-4\alpha^2\int_0^\infty g'''(\phi_m)g(\phi_m)\phi_m' dz\right)\\
&= -8\alpha\int_0^\infty g'''(\phi_m)g(\phi_m)\phi_m' dz - 4\alpha^2\int_0^\infty g''''(\phi_m)g(\phi_m)\phi_m'\partial_\alpha\phi_m\, dz +\\
&-4\alpha^2\int_0^\infty g''(\phi_m)g'(\phi_m)\phi_m'\partial_\alpha\phi_m\, dz - 4\alpha^2\int_0^\infty g'''(\phi_m)g(\phi_m)\partial_\alpha\phi_m'\, dz.
\end{aligned} \tag{2.3.31}$$

Integrating the first term of this expansion by parts and applying (2.3.29) to the remaining three terms, we obtain the reduction

$$
\begin{aligned}
&\partial_\alpha\left(-4\alpha^2\int_0^\infty g'''(\phi_m)g(\phi_m)\phi_m' dz\right)\\
&= 8\alpha g''(\phi_m(0))g(\phi_m(0)) + 8\alpha\int_0^\infty g''(\phi_m)g'(\phi_m)\phi_m' dz +\\
&\qquad + \frac{2\sqrt{2}\alpha^2}{\hat{\lambda}_0}\int_0^\infty \left(g'''(\phi_m)g(\phi_m)\phi_m'\right)' dz + O\left(\delta^{\frac{1}{2}}\right)\\
&= 8\alpha g''(\phi_{max})g(\phi_{max}) + O\left(\delta^{\frac{1}{2}}\right). \qquad (2.3.32)
\end{aligned}
$$

The third term of (2.3.26) enjoys the expansion

$$
\partial_\alpha\left(-\int_0^\infty p_2'(\phi_m)\phi_m' dz\right) = -2\int_0^\infty p_2''(\phi_m)\phi_m'\partial_\alpha\phi_m\, dz - 2\int_0^\infty p_2'(\phi_m)\partial_\alpha\phi_m'\, dz, \quad (2.3.33)
$$

and using (2.3.29) we find

$$
\begin{aligned}
\partial_\alpha\left(-\int_0^\infty p_2'(\phi_m)\phi_m' dz\right) &= \frac{\sqrt{2}}{\hat{\lambda}_0}\int_0^\infty \left(p_2'(\phi_m)\phi_m'\right)' dz + O\left(\delta^{\frac{1}{2}}\right),\\
&= O\left(\delta^{\frac{1}{2}}\right). \qquad (2.3.34)
\end{aligned}
$$

Combining the results of (2.3.30), (2.3.32) and (2.3.34), the expression (2.3.26) reduces to

$$
\frac{\partial\tilde{h}}{\partial\alpha}(\alpha,\delta) = 8\alpha g''(\phi_{max})g(\phi_{max}). \qquad (2.3.35)
$$

The relation (2.2.10) allows us to rewrite (2.3.35) as

$$
\frac{\partial\tilde{h}}{\partial\alpha}(\alpha,\delta=0) = -4\sqrt{2}\gamma_0. \qquad (2.3.36)
$$

Since γ_0 is $O(1)$ and non-zero, the lemma follows. $\square$

We have established the existence of a C^1 function $\alpha_* = \alpha_*(z;\beta,\eta) = O(1)$ such that the homoclinic solution $\phi_m^*(z) = \phi_m(z;\alpha_*)$ satisfies

$$(F(\phi_m^*), \Psi_0(.;\alpha_*))_2 = 0. \tag{2.3.37}$$

From the definition (2.2.2) of $F(u)$ we may expand $F(\phi_m^*)$

$$F(\phi_m^*) = \left(\partial_z^2 - G''(\phi_m^*) - \alpha\delta g''(\phi_m^*)\right)\left((\phi_m^*)'' - G'(\phi_m^*) - \alpha\delta g'(\phi_m^*)\right) + p'(\phi_m^*). \tag{2.3.38}$$

Recalling the definition (1.1.15) of $\mathcal{L}_{\alpha_*}$ and that ϕ_m^* is the homoclinic solution of (1.1.6) we may further deduce

$$F(\phi_m^*) = \alpha_*^2\delta^2 g''(\phi_m^*)g'(\phi_m^*) - \alpha_*\delta L_{\alpha_*}\left(g'(\phi_m^*)\right) + \delta^2 p_2'(\phi_m^*). \tag{2.3.39}$$

From (2.2.18) of we see that

$$L_{\alpha_*}\left(g'(\phi_m^*)\right) = -2\delta\alpha_* g'''(\phi_m^*)g(\phi_m^*) + \delta^2 r(\phi_m^*), \tag{2.3.40}$$

where $r(\phi_m^*) = O(1)$ in $\|\cdot\|_{L^2}$. So the residual of F at ϕ_m^* takes the form

$$F(\phi_m^*) = \delta^2\left[\alpha_*^2\Big(g''(\phi_m^*)g'(\phi_m^*) + 2g'''(\phi_m^*)g(\phi_m^*)\Big) + p_2'(\phi_m^*)\right] + \delta^3 r(\phi_m^*). \tag{2.3.41}$$

Since ϕ_m^* is even, and hence to Ψ_1 we may introduce the function ξ_* defined by

$$\xi_* := \tilde{\Pi}(\alpha_*)\mathcal{L}_{\alpha_*}^{-1}F(\phi_m^*) = O(\delta^2). \tag{2.3.42}$$

In particular $\phi_m^* \mapsto \xi_*$ is the first iteration of Newton map. We fix δ, η and β, then define $H_e^k(\mathbb{R})$ to be the subspace of H^k comprised of even functions. And for $\rho > 0$ introduce

$$B_\rho^* = \left\{ u \in H_e^4(\mathbb{R}) \Big| \|u - (\phi_m^* - \xi_*)\|_{H^4} \leq \rho\delta^{\frac{5}{2}} \right\}. \tag{2.3.43}$$

For $u \in B_\rho^*$ we may decompose u as

$$u = \phi_m^* - \xi_* + v_0 \tag{2.3.44}$$

and also as

$$u = \phi_m(z; \alpha) - \xi_* + v, \tag{2.3.45}$$

where in the second decomposition α, and hence v, are to be determined. Using the second decomposition we expand

$$F(u) = F(\phi_m) + \mathcal{L}_\alpha(-\xi_* + v) + \mathcal{N}(-\xi_* + v), \tag{2.3.46}$$

where $\|\mathcal{N}(v)\|_{H^4} \leq c\|v\|_{H^4}^2$ represents nonlinear terms. The Newton map can be expressed as

$$\begin{aligned} N(u) &= u - \mathcal{L}_\alpha^{-1}\left(F(\phi_m) + \mathcal{L}_\alpha(-\xi_* + v) + \mathcal{N}(-\xi_* + v)\right), && (2.3.47) \\ &= \phi_m - \mathcal{L}_\alpha^{-1}\tilde{F}(u), && (2.3.48) \end{aligned}$$

where we have introduced

$$\tilde{F}(u) := F(\phi_m(\alpha)) + \mathcal{N}(-\xi_* + v). \tag{2.3.49}$$

The function $F(\phi_m^*)$ admits the expansion

$$F(\phi_m^*) = F(\phi_m) + \mathcal{L}_\alpha(\phi_m^* - \phi_m) + \mathcal{N}(\phi_m^* - \phi_m), \tag{2.3.50}$$

which, when re-arranged takes the form

$$F(\phi_m) = F(\phi_m^*) + \mathcal{L}_\alpha(\phi_m - \phi_m^*) - \mathcal{N}(\phi_m^* - \phi_m). \tag{2.3.51}$$

Substituting (2.3.51) into the definition (2.3.49) of $\tilde{F}$ we obtain

$$\tilde{F}(u) = F(\phi_m^*) + \mathcal{L}_\alpha(\phi_m^* - \phi_m) - \mathcal{N}(\phi_m^* - \phi_m) + \mathcal{N}(-\xi_* + v). \tag{2.3.52}$$

In fact for a small v we can further decompose the nonlinear term as

$$\mathcal{N}(v) = Q(v, v) + R(v), \tag{2.3.53}$$

where the term

$$Q(v, w) := -\frac{1}{2} L_\alpha \left(G'''(\phi_m)\, vw \right) - G'''(\phi_m) v L_\alpha(w) + O(\delta \|v\| \|w\|), \tag{2.3.54}$$

is quadratic in v, while R represents terms that are cubic or higher. Recalling the two decompositions of u (2.3.44), (2.3.45), the v-terms are related by

$$v = v_0 + (\phi_m^* - \phi_m), \tag{2.3.55}$$

so that

$$
\begin{aligned}
Q(-\xi_* + v) &= Q(-\xi_* + v_0 + \phi_m^* - \phi_m), \\
&= Q(\phi_m^* - \phi_m) + Q_s(-\xi_* + v_0, \phi_m^* - \phi_m) + Q(-\xi_* + v_0), \quad (2.3.56)
\end{aligned}
$$

where $Q_s(v, w) := Q(v, w) + Q(w, v)$. With these manipulations we may expand $\tilde{F}(u)$ as

$$
\begin{aligned}
\tilde{F}(u) &= F(\phi_m^*) + \mathcal{L}_\alpha(\phi_m^* - \phi_m) + Q_s(-\xi_* + v_0, \phi_m^* - \phi_m) + \\
&\quad + Q(-\xi_* + v_0) - R(\phi_m^* - \phi_m) + R(-\xi_* + v). \quad (2.3.57)
\end{aligned}
$$

Lemma 2.5. *Let α_* and ϕ_m^* be as defined in lemma 2.4. Under the assumption (H2) of Theorem 1.1, there exist $\rho_1, \rho_2 > 0$ such that for any $u \in B^*_{\rho_1}$, there is a unique $\alpha = \alpha(u; \beta, \eta)$ satisfying*

$$|\alpha - \alpha_*| < \rho_2 \delta^2, \quad (2.3.58)$$

for which

$$\tilde{F}(u, \alpha(u)) \perp \Psi_0(\cdot; \alpha(u)). \quad (2.3.59)$$

Here Ψ_0 is the ground state eigenfunction of $\mathcal{L}_\alpha$ given in (1.1.15) and $\tilde{F}$ is as given in (2.3.49). In particular, u may be written in the form

$$u = \phi_m(z; \alpha) - \xi_* + v, \quad (2.3.60)$$

where $\|v\|_{H^4} = O\left(\delta^{\frac{5}{2}}\right)$.

Proof. We want to show from the implicit function theorem that given $u \in B^*_{\rho_1}$ there is a unique $\alpha = \alpha(u)$ such that

$$H(\alpha, u; \beta, \eta, \delta) := \left(\tilde{F}(u), \Psi_0(\alpha)\right)_2 = 0, \tag{2.3.61}$$

and that α is smooth in u in H^4. We first remark that Lemma 2.4 establishes the fact that

$$H(\alpha_*, \phi_m^* - \xi_*; \beta, \eta, \delta) = 0. \tag{2.3.62}$$

and α is smooth in u. By the expansion of $\tilde{F}(u)$ given in (2.3.57), we may write $H(\alpha, u; \beta, \eta, \delta)$ as

$$\begin{aligned} H(\alpha, u; \beta, \eta, \delta) &= (F(\phi_m^*) + \mathcal{L}_\alpha(\phi_m^* - \phi_m) + Q_s(-\xi_* + v_0, \phi_m^* - \phi_m) + Q(-\xi_* + v_0) + \\ &\quad -R(\phi_m^* - \phi_m) + R(-\xi_* + v), \Psi_0(\alpha))_2 \, . \end{aligned} \tag{2.3.63}$$

The first three terms on the right hand side of (2.3.63) are linear in $\alpha - \alpha_*$. The remaining terms are quadratic in $\alpha - \alpha_*$ or higher order terms. In addition the leading order inhomogeneous term is $(Q(\xi_*, \xi_*), \Psi_0(\alpha_*))_2$. We conclude that $H(\alpha, u; \beta, \eta, \delta)$ can be rewritten as

$$H(\alpha, u; b, \eta, \delta) = (\alpha - \alpha_*)B - (Q(\xi_*, \xi_*), \Psi_0(\alpha_*))_2 + O\left(|\alpha - \alpha_*|^2 \|v\|_{H^4}\right), \tag{2.3.64}$$

where the coefficient B comes from the first three terms. The remaining of the proof requires that we compute B explicitly to determine conditions under which it is non-zero at leading order. We focus on the first term on the RHS of (2.3.63), using (2.3.8) we may rewrite it as

$$
\begin{aligned}
(F(\phi_m^*), \Psi_0(\alpha))_2 &= (F(\phi_m^*), \Psi_0(\alpha) - \Psi_0(\alpha_*))_2\,, && (2.3.65)\\
&= \left(F(\phi_m^*), \frac{\partial \psi_0}{\partial \alpha}(\alpha_*)\right)_2 (\alpha - \alpha_*) + O(|\alpha_* - \alpha|^2). && (2.3.66)
\end{aligned}
$$

The first contribution to B arises from

$$
B_1 := \left(F(\phi_m^*), \frac{\partial \psi_0}{\partial \alpha}(\alpha_*)\right)_2 . \tag{2.3.67}
$$

From (2.3.41) we may expand B_1

$$
B_1 = \delta^2 \left(\alpha_*^2((g''(\phi_m^*)g'(\phi_m^*) + 2g'''(\phi_m^*)g(\phi_m^*)) + p_2'(\phi_m^*), \frac{\partial \psi_0}{\partial \alpha}(\alpha_*)\right)_2 + O(\delta^3). \tag{2.3.68}
$$

Applying (3.1.24) and (3.1.25) we further simplify B_1

$$
B_1 = -\frac{\delta^2}{\sqrt{2}\|(\phi_m^*)'\|_2 \hat{\lambda}_0^*} \left(\frac{\alpha_*^2}{2}(\phi_m^*)'' + 2\alpha_* g'''(\phi_m^*)g(\phi_m^*) + p_2'(\phi_m^*), (\phi_m^*)''\right)_2 + O(\delta^3), \tag{2.3.69}
$$

where $\hat{\lambda}_0^* := \hat{\lambda}_0(\alpha_*)$. By (3.1.28) we may combine the first two terms,

$$
B_1 = -\frac{\delta^2}{\sqrt{2}\|(\phi_m^*)'\|_2 \hat{\lambda}_0^*} \left(2\sqrt{2}\alpha_*^2 g(\phi_{max}^*)(g''(\phi_{max}^*))^2 + \left(p_2'(\phi_m^*), (\phi_m^*)''\right)\right) + O(\delta^3), \tag{2.3.70}
$$

Addressing the second term of (2.3.63) we obtain

$$
(\mathcal{L}_\alpha(\phi_m^* - \phi_m), \Psi_0(\alpha))_2 = \Lambda_0^*\,(\alpha_* - \alpha)B_2 + O(|\alpha_* - \alpha|^2), \tag{2.3.71}
$$

where $\Lambda_0^* := \Lambda_0(\alpha_*)$ and

$$B_2 := \Lambda_0^* \left(\frac{\partial \phi_m}{\partial \alpha}(\alpha_*), \Psi_0(\alpha_*) \right)_2 . \tag{2.3.72}$$

Applying (3.1.24) and (2.2.29) we further simplify

$$B_2 = \frac{\Lambda_0^*}{\sqrt{2}\hat{\lambda}_0^*} \|(\phi_m^*)'\|_2 + O(\delta^3). \tag{2.3.73}$$

The remaining terms that are linear in $\alpha - \alpha_*$ are $(Q_s(\phi_m^* - \phi_m, v_0), \Psi_0(\alpha))_2$ and $(Q_s(-\xi_*, \phi_m^* - \phi_m), \Psi_0(\alpha))_2$. The first of these terms is

$$(Q_s(\phi_m^* - \phi_m, v_0), \Psi_0(\alpha))_2 = (\alpha_* - \alpha) B_3 + O(|\alpha_* - \alpha|^2), \tag{2.3.74}$$

where the coefficient

$$B_3 := \left(Q_s \left(\frac{\partial \phi_m}{\partial \alpha}(\alpha_*), v_0 \right), \Psi_0(\alpha) \right)_2 . \tag{2.3.75}$$

Using the definition of Q_s and Q, (2.3.54), this term has the expansion

$$\begin{aligned} Q_s(\frac{\partial \phi_m}{\partial \alpha}(\alpha_*), v_0) &= Q(\frac{\partial \phi_m}{\partial \alpha}(\alpha_*), v_0) + Q(v_0, \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)), \\ &= -\frac{1}{2} L_\alpha \left(G'''(\phi_m^*) \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) v_0 \right) - G'''(\phi_m^*) \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) L_\alpha \, v_0 + \\ &\quad -\frac{1}{2} L_\alpha \left(G'''(\phi_m^*) v_0 \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) \right) - G'''(\phi_m^*) v_0 L_\alpha (\frac{\partial \phi_m}{\partial \alpha}(\alpha_*)) + \\ &\quad + O\left(\delta ||v_0||_{H^4} \right) . \end{aligned} \tag{2.3.76}$$

Substituting this term into B_3 we obtain

$$\begin{aligned}B_3 &= -\frac{1}{2}\left(L_\alpha\left(G'''(\phi_m^*)(v_0\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)+\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)v_0)\right), \Psi_0(\alpha)\right)_2 + \\ &\quad -\left(G'''(\phi_m^*)v_0\, L_\alpha\frac{\partial\phi_m}{\partial\alpha}(\alpha_*), \Psi_0(\alpha)\right)_2 + \\ &\quad -\left(G'''(\phi_m^*)\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)\, L_\alpha v_0, \Psi_0(\alpha)\right)_2 + \\ &\quad +O\left(\delta\|v_0\|_{H^4}\right).\end{aligned} \tag{2.3.77}$$

Using the self-adjointness of L_α and (2.2.29) to replace $\Psi_0(\alpha)$ by $\psi_0(\alpha)$ yields the form

$$\begin{aligned}B_3 &= -\frac{\lambda_0}{2}\left(G'''(\phi_m^*)(v_0\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)+\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)v_0), \psi_0(\alpha)\right)_2 + \\ &\quad -\lambda_0\left(G'''(\phi_m^*)v_0\,\psi_0(\alpha_*), \psi_0(\alpha)\right)_2 + \\ &\quad -\left(G'''(\phi_m^*)\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)\, L_\alpha v_0, \psi_0(\alpha)\right)_2 + \\ &\quad +O\left(\delta||v_0||_{H^4}\right).\end{aligned} \tag{2.3.78}$$

Recalling that $\lambda_0 = O(\delta)$ and $v_0 = O\left(\delta^{\frac{5}{2}}\right)$ the leading order term is

$$B_3 = -\left(G'''(\phi_m^*)\frac{\partial\phi_m}{\partial\alpha}(\alpha_*)\, L_\alpha v_0, \psi_0(\alpha)\right)_2 + O\left(\delta^{\frac{7}{2}}\right). \tag{2.3.79}$$

Using (2.2.12) and (2.2.13) we obtain

$$\psi_0 = \frac{|(\phi_m^*)'|}{\|(\phi_m^*)'\|_2} + O\left(\delta^{\frac{1}{2}}\right), \tag{2.3.80}$$

in L^2 norm. Combining (2.3.80), (3.1.24) and (3.1.4) we may further simplify B_3

$$B_3 = -\frac{1}{\sqrt{2}\hat{\lambda}_0\|(\phi_m^*)'\|_2}\left(L_\alpha(\phi_m^*)'', L_\alpha v_0\right)_2 + O\left(\delta^{\frac{7}{2}}\right). \tag{2.3.81}$$

From the definition (2.2.8) of $\hat{\lambda}_0$, the form (2.3.9) of α_* from Lemma 2.4 and the definition (2.2.10) of γ_0 and identity (3.1.26), the coefficient of B_3 may be reduced to

$$2\|(\phi_m^*)'\|_2^2(\hat{\lambda}_0^*)^2 = 4\mu_+\alpha_*^2\|(\phi_m^*)'\|_2^2 = -8\mu_+^{\frac{5}{2}}(b_+ - b_-)\beta + O(\sqrt{\delta}). \tag{2.3.82}$$

Substituting this term into B_3 we obtain

$$B_3 = \frac{1}{8\mu_+^{\frac{5}{2}}(b_+ - b_-)\beta}\left(L_\alpha(\phi_m^*)'', L_\alpha v_0\right)_2 + O(\delta^3). \tag{2.3.83}$$

Using the fact that $v_0 = O\left(\delta^{\frac{5}{2}}\right)$ we conclude

$$B_3 = O\left(\delta^{\frac{5}{2}}\right). \tag{2.3.84}$$

The final term that is linear in $\alpha - \alpha_*$ takes the form

$$(Q_s(-\xi_*, \phi_m^* - \phi_m), \Psi_0(\alpha))_2 = (\alpha_* - \alpha)B_4 + O(|\alpha_* - \alpha|^2), \tag{2.3.85}$$

where the coefficient

$$B_4 := \left(Q_s(-\xi_*, \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)), \Psi_0(\alpha_*)\right)_2. \tag{2.3.86}$$

From the definition of Q_s and Q (2.3.54), we have

$$
\begin{aligned}
Q_s(-\xi_*, \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)) &= Q(-\xi_*, \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)) + Q(\frac{\partial \phi_m}{\partial \alpha}(\alpha_*), -\xi_*), \\
&= \frac{1}{2} L_\alpha \left(G'''(\phi_m^*)\xi_* \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) \right) + G'''(\phi_m^*)\xi_* L_\alpha \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) + \\
&\quad + \frac{1}{2} L_\alpha \left(G'''(\phi_m^*) \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)\xi_* \right) + G'''(\phi_m^*) \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) L_\alpha \xi_* + \\
&\quad + O\left(\delta ||\xi_*||_{L^2} \right). \qquad (2.3.87)
\end{aligned}
$$

Substituting back into B_4 we obtain

$$
\begin{aligned}
B_4 &= \frac{1}{2} \left(L_\alpha \left(G'''(\phi_m^*)(\xi_* \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) + \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)\xi_*) \right), \Psi_0(\alpha_*) \right)_2 + \\
&\quad + \left(G'''(\phi_m^*)\xi_* L_\alpha \frac{\partial \phi_m}{\partial \alpha}(\alpha_*), \Psi_0(\alpha_*) \right)_2 + \\
&\quad + \left(G'''(\phi_m^*) \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) L_\alpha \xi_*, \Psi_0(\alpha_*) \right)_2 + \\
&\quad + O\left(\delta ||\xi_*||_{L^2} \right). \qquad (2.3.88)
\end{aligned}
$$

Using (3.1.24) to replace $\partial \phi_m / \partial_\alpha$ by ψ_0 and using the self-adjointness of L_α and (2.2.29), we obtain the simplified form

$$
\begin{aligned}
B_4 &= \frac{\lambda_0}{2} \left(G'''(\phi_m^*)(\xi_* \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) + \frac{\partial \phi_m}{\partial \alpha}(\alpha_*)\xi_*), \psi_0(\alpha_*) \right)_2 + \\
&\quad + \lambda_0 \left(G'''(\phi_m^*)\xi_* \psi_0(\alpha_*), \psi_0(\alpha_*) \right)_2 + \\
&\quad + \left(G'''(\phi_m^*) \frac{\partial \phi_m}{\partial \alpha}(\alpha_*) L_\alpha \xi_*, \psi_0(\alpha) \right)_2 + \\
&\quad + O\left(\delta ||\xi_*||_{L^2} \right). \qquad (2.3.89)
\end{aligned}
$$

Recalling that $\|\xi_*\| = O(\delta^2)$ and $\lambda_0 = O(\delta)$, the leading order term is

$$B_4 = \left(G'''(\phi_m^*)\frac{\partial \phi_m}{\partial \alpha}(\alpha_*)L_\alpha \xi_*, \psi_0(\alpha_*)\right)_2 + O(\delta^3). \tag{2.3.90}$$

Using the definition (2.3.43) of ξ_* and (3.1.24), we obtain

$$B_4 = \int_{\mathbb{R}} G'''(\phi_m^*)\frac{\partial \phi_m}{\partial \alpha}(\alpha_*)\,\psi_0(\alpha_*)\,L_\alpha \tilde{\Pi}_0(\alpha_*)\mathcal{L}_{\alpha*}^{-1}F(\phi_m^*)dz + O(\delta^3). \tag{2.3.91}$$

Using (3.1.24), (3.1.22), we obtain

$$B_4 = \frac{1}{\sqrt{2}\|(\phi_m^*)'\|_2\hat{\lambda}_0^*}\int_{\mathbb{R}} G'''(\phi_m^*)((\phi_m^*)')^2 L_\alpha \tilde{\Pi}_0(\alpha_*)L_{\alpha*}^{-1}F(\phi_m^*)dz + O(\delta^3), \tag{2.3.92}$$

Applying the identity (3.1.4) and recalling from (2.2.6) that $\mathcal{L}_{\alpha*} = L_{\alpha*}^2 + O(\delta)$, we obtain

$$B_4 = \frac{1}{\sqrt{2}\|(\phi_m^*)'\|_2\hat{\lambda}_0^*}\int_{\mathbb{R}} \left[\tilde{\Pi}_0(\alpha_*)F(\phi_m^*)\right](\phi_m^*)''dz + O\left(\delta^{\frac{5}{2}}\right). \tag{2.3.93}$$

Substituting (2.3.41) into the expression for $F(\phi_m^*)$ and using (3.1.25) and (3.1.29) to simplify the resulting expression we obtain

$$B_4 = \frac{\delta^2}{\sqrt{2}\|(\phi_m^*)'\|_2\hat{\lambda}_0^*}\left(2\sqrt{2}\alpha_*^2 g(\phi_{max}^*)(g''(\phi_{max}^*))^2 + (p_2'(\phi_m^*),(\phi_m^*)'')_2\right) + O\left(\delta^{\frac{5}{2}}\right). \tag{2.3.94}$$

We have decomposed the quantity B in (2.3.64) as

$$B = B_1 - B_2 - B_3 - B_4. \tag{2.3.95}$$

Combining the expression (2.3.70), (2.3.73), (2.3.84), (2.3.94) for B_1, B_2, B_3 and B_4 respectively, we obtain the expression

$$
\begin{aligned}
B &= -\frac{\sqrt{2}\delta^2}{\|(\phi_m^*)'\|_2 \hat{\lambda}_0^*}\left(2\sqrt{2}\alpha_*^2 g(\phi_{max}^*)(g''(\phi_{max}^*))^2 + (p_2'(\phi_m^*), (\phi_m^*)'')_2\right) + \\
&\quad -\frac{\Lambda_0^*}{\sqrt{2}\hat{\lambda}_0^*}\|(\phi_m^*)'\|_2 + O\left(\delta^{\frac{5}{2}}\right). \qquad (2.3.96)
\end{aligned}
$$

Using the expression (2.2.31) of Lemma 2.3 to replace Λ_0^* we obtain

$$
\begin{aligned}
B &= -\frac{\sqrt{2}\delta^2}{\hat{\lambda}_0^*\|(\phi_m^*)'\|_2}\left(\frac{1}{2}\|(\phi_m^*)'\|_2^2(\hat{\lambda}_0^*)^2 + 5\sqrt{2}\alpha_*^2 g(\phi_{max}^*)(g''(\phi_{max}^*))^2 + \right. \\
&\quad \left. +\frac{3}{2}(p_2'(\phi_m^*), (\phi_m^*)'')_2\right) + O\left(\delta^{\frac{5}{2}}\right). \qquad (2.3.97)
\end{aligned}
$$

Our goal is to establish conditions under which the leading order term of B is non-zero. Using the form (2.3.9) of α_*, the definition (2.2.10) of γ_0, and identities (3.1.26) and (3.1.27) the second term in B reduces to

$$
5\sqrt{2}\alpha_*^2 g(\phi_{max}^*)(g''(\phi_{max}^*))^2 = -\frac{5}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-)\beta + O(\sqrt{\delta}). \qquad (2.3.98)
$$

Finally the expression (2.1.12) for p_2 allows us to reduce the third term of B to the explicit form

$$
\begin{aligned}
(p_2''(\phi_m^*), ((\phi_m^*)')^2)_2 &= \beta\mu_-\left(W'''(\phi_m^*), ((\phi_m^*)')^2\right)_2 + \\
&\quad +\frac{\eta}{2}\left(W'''(\phi_m^*)(\phi_m^* - b), ((\phi_m^*)')^2\right)_2 + O(\delta). \qquad (2.3.99)
\end{aligned}
$$

Combining these expressions and collecting terms in β and η, we express $B = B(\beta, \eta, \delta)$ in the form

$$B = -\frac{\sqrt{2}\delta^2}{\hat{\lambda}_0^* \|(\phi_m^*)'\|_2}(A_1\beta + A_2\eta) + O\left(\delta^{\frac{5}{2}}\right), \tag{2.3.100}$$

where we have introduced

$$\begin{aligned} A_1 &:= -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + \frac{3}{2}\mu_-\left(W'''(\phi_m^*), ((\phi_m^*)')^2\right)_2, &(2.3.101)\\ A_2 &:= \frac{3}{4}\left(W'''(\phi_m^*)(\phi_m^* - b), ((\phi_m^*)')^2\right)_2. &(2.3.102)\end{aligned}$$

So long as the leading order term $(A_1\beta + A_2\eta)$ in B is non-zero (at least to $O(\sqrt{\delta})$) we may solve (2.3.64) to obtain

$$\alpha = \alpha_* - \frac{(Q(\xi_*, \xi_*), \Psi_0(\alpha_*))_2}{B} + O\left(\delta^{\frac{5}{2}}\right). \tag{2.3.103}$$

In particular we remark that $|\alpha - \alpha_*| = O(\delta^2)$ which follows from $\|\xi_*\|_{H^2} = O(\delta^2)$ and $\|Q(\xi_*, \xi_*)\|_2 = O(\delta^4)$. Note that A_1, A_2 depend on η and β. To examine this dependence we define

$$F(\eta, \beta, \delta) := A_1(\eta, \beta, \delta)\beta + A_2(\eta, \beta, \delta)\eta. \tag{2.3.104}$$

As $\delta \to 0$ the homoclinic solution ϕ_m^* bifurcates out the heteroclinic solution ϕ_h; this bifurcation is not smooth in $\|\cdot\|_2$. However the inner products in A_1 and A_2 are smooth as $\delta \to 0$, as we establish in Lemma 3.7. Consequently applying (3.2.33) and (3.2.34) of Lemma 3.7 we may conclude

$$\begin{aligned} F(\eta, \beta, \delta) &= F(\eta, \beta, 0) + O(\delta^\omega) \\ &= A_1^h\beta + A_2^h\eta + O(\delta^\omega), &(2.3.105)\end{aligned}$$

where $\omega = \min\{C, \frac{1}{2}\}$ and C is some positive constant depending upon the exponential dichotomies of the second order differential equation(1.1.8) and hence is independent of η, β and δ. Here we have introduced

$$A_1^h := -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + 3\mu_- \left(W'''(\phi_h), (\phi_h')^2\right)_2, \tag{2.3.106}$$

$$A_2^h := \frac{3}{2}\left(W'''(\phi_h)(\phi_h - b_-), (\phi_h')^2\right)_2. \tag{2.3.107}$$

In particular A_1^h and A_2^h do not depend on η, β or δ. They only depend on the heteroclinic solution ϕ_h of the second order differential equation (1.1.8) which is fully determined by the double well potential W.

As long as β and η satisfies the condition (H_2)

$$\left|A_1^h\beta + A_2^h\eta\right| > \nu\,\delta^\omega, \tag{2.3.108}$$

for some $\nu > 0$ independent of δ, then $F(\eta, \beta, \delta)$ is bounded away from zero by an $O(\delta^\omega)$ and hence (2.3.103) holds.

$\square$

Lemma 2.6. *Let $\alpha = \alpha(u; \eta, \beta, \delta)$ be given in Lemma 2.5. Then for every $\rho > 0$ there exists $\delta_0 > 0$ such that the Newton map*

$$\mathcal{N}(u) = \phi_m - \mathcal{L}_\alpha^{-1}\tilde{F}(u), \tag{2.3.109}$$

defined in (2.2.4) maps B_ρ^* into B_ρ^*.

Remark 2.2. *In (2.3.109) α in the subscript where we emphasize that $\alpha = \alpha(u, \delta)$ is a* *function of u as given in Lemma 2.5. However, $\alpha_* = \alpha(\delta;\beta,\eta)$ is as defined in (2.3.9).*

Proof. We show that the Newton map takes B_ρ^* into B_ρ^*. With $\alpha = \alpha(u)$ the Newton map becomes

$$\begin{aligned} N(u) &= \phi_m - \left(\tilde{\Pi}_\alpha \mathcal{L}_\alpha\right)^{-1} \Big(F(\phi_m^*) + \mathcal{L}_\alpha(\phi_m^* - \phi_m) + Q_s(\xi_* + v_0, \phi_m^* - \phi_m) + \\ &\quad + Q(\xi_* + v_0) - R(\phi_m^* - \phi_m) + R(\xi_* + v)\Big), \end{aligned} \tag{2.3.110}$$

where $\tilde{\Pi}_\alpha \mathcal{L}_\alpha$ maps L^2 into H^4 with an $O(1)$ norm. We perform a Neumann expansion on $(\tilde{\Pi}_\alpha \mathcal{L}_\alpha)^{-1}$ in terms of $\tilde{\Pi}_* \mathcal{L}_* := \tilde{\Pi}_{\alpha*} \mathcal{L}_{\alpha*}$, and obtain

$$\begin{aligned} N(u) &= \phi_m^* - \left(\tilde{\Pi}_* \mathcal{L}_*\right)^{-1} F(\phi_m^*) + \\ &\quad - \left(\tilde{\Pi}_* \mathcal{L}_*\right)^{-1} \left[Q_s(\xi_* + v_0, \phi_m^* - \phi_m) + Q(\xi_* + v_0)\right] + O(\delta^4). \end{aligned} \tag{2.3.111}$$

Observe that $F(\phi_m^*) = O(\delta^2)$ and $|\alpha - \alpha_*| = O(\delta^2)$. While $\|\xi_*\|_{H^4} = O(\delta^2)$ implies that

$$\begin{aligned} &\left(\tilde{\Pi}_* \mathcal{L}_*\right)^{-1} \left(Q_s(\xi_* + v_0, \phi_m^* - \phi_m) + Q(\xi_* + v_0)\right) \\ &= \left(\tilde{\Pi}_* \mathcal{L}_*\right)^{-1} \left((\alpha_* - \alpha) Q_s(\xi_* + v_0, \partial_\alpha \phi) + Q(\xi_* + v_0)\right), \\ &= O(\delta^4). \end{aligned} \tag{2.3.112}$$

In particular, since $N(\phi_m^*) = \phi_m^* - \left(\tilde{\Pi}_* \mathcal{L}_*\right)^{-1} F(\phi_m^*)$, we have

$$\|N(u) - N(\phi_m^*)\|_{H^4} = O(\delta^4). \tag{2.3.113}$$

Since ρ is fixed, taking δ sufficiently small implies that $N(u) \in B_\rho^*$. □

Lemma 2.7. *Let $\alpha = \alpha(u;\beta,\eta,\delta)$ be as defined in Lemma 2.5. Then for every $\rho > 0$ there exists $\delta_0 > 0$ such that the operator N is an asymptotically strong contraction on B^*_ρ for all $0 < \delta < \delta_0$. That is, there exists $\gamma = O(1) > 0$ such that*

$$||\mathcal{N}(u_1) - \mathcal{N}(u_2)||_{H^4} \le \gamma\delta^{\frac{1}{2}}||u_1 - u_2||_{H^4}, \tag{2.3.114}$$

for all $u_1,\ u_2 \in\ B^*_\rho$.

Proof. Recall the two expansions of u defined in (2.3.44) and (2.3.45)

$$\begin{aligned} u_i &= \phi^*_m + \xi_* + v^i_0 & (2.3.115)\\ u_i &= \phi^i_m(z;\alpha^i) + \xi_* + v^i,\ i = 1,2. & (2.3.116)\end{aligned}$$

Correspondingly from (2.3.111) we have

$$\begin{aligned}\mathcal{N}(u_1) - \mathcal{N}(u_2) = (\tilde{\Pi}_*\mathcal{L}_*)^{-1}\Big(&Q_s(\xi_* + v^1_0, \phi^*_m - \phi^1_m) - Q_s(\xi_* + v^2_0, \phi^*_m - \phi^2_m) + \\ &+Q(\xi_* + v^1_0) - Q(\xi_* + v^2_0)\Big) + h.o.t. \quad . \end{aligned} \tag{2.3.117}$$

From the definition (2.3.54) of Q we expand

$$\begin{aligned} Q(\xi_* + v^i_0) &= -\frac{1}{2}L_{\alpha^i}G'''(\phi^i_m)(\xi_* + v^i_0)^2 - G'''(\phi^i_m)(\xi_* + v^i_0)L_{\alpha^*}(\xi_0 + v_0) + \\ &\qquad +O(\delta||\xi_*||_{L^2}\,||v_0||_{H^4}),\\ &= Q(\xi_*) + Q(v^i_0) - L_{\alpha^i}G'''(\phi^i_m)\xi_* v^i_0 - G'''(\phi^i_m)\xi_* L_{\alpha^i}v_0 + \\ &\qquad -G'''(\phi^i_m)v^i_0 L_{\alpha^i}\xi_* + O(\delta||\xi_*||_{L^2}\,||v^i_0||_{H^4}), \end{aligned} \tag{2.3.118}$$

from which we deduce that

$$
\begin{aligned}
\left\|\tilde{\Pi}_* \mathcal{L}_*^{-1}\left(Q(\xi_*+v_0^1)-Q(\xi_*+v_0^2)\right)\right\|_{L^2} \leq\ & \left\|\tilde{\Pi}_* \mathcal{L}_*^{-1}\right\|\left(\left\|Q(v_0^1)-Q(v_0^2)\right\|_{L^2}+\right. \\
& +\left\|L_{\alpha^1} G'''(\phi_m^1)\xi_* v_0^1-L_{\alpha^2} G'''(\phi_m^2)\xi_* v_0^2\right\|_{L^2}+ \\
& +\left\|G'''(\phi_m^1)\xi_* L_{\alpha^1} v_0^1-G'''(\phi_m^2)\xi_* L_{\alpha^2} v_0^2)\right\|_{L^2}+ \\
& \left.+\left\|(G'''(\phi_m^1) v_0^1 L_{\alpha^1}-G'''(\phi_m^1) v_0^2 L_{\alpha^2})\xi_*\right\|_{L^2}\right).
\end{aligned} \tag{2.3.119}
$$

Recalling that $\|\xi_*\|_{H^4}=O(\delta^2)$ and $\|v_0^i\|_{H^4}=O(\delta^{\frac{5}{2}})$, we deduce that

$$
\begin{aligned}
\left\|\tilde{\Pi}_* \mathcal{L}_*^{-1}\left(Q(\xi_*+v_0^1)-Q(\xi_*+v_0^2)\right)\right\|_{L^2} \leq\ & \left\|\tilde{\Pi}_* \mathcal{L}_*^{-1}\right\|\left(\left\|Q(v_0^1)-Q(v_0^2)\right\|_{L^2}+\right. \\
& \left.+\gamma_1 \delta\left\|v_0^1-v_0^2\right\|_{L^2}\right).
\end{aligned} \tag{2.3.120}
$$

From the definition (2.3.54) of Q we expand

$$
\begin{aligned}
\left\|Q(v_0^1)-Q(v_0^2)\right\|_{L^2} \leq\ & \frac{1}{2}\left\|L_{\alpha^1} G'''(\phi_m^1)(v_0^1)^2-L_{\alpha^2} G'''(\phi_m^2)(v_0^2)^2)\right\|_{L^2}+ \\
& +\left|G'''(\phi_m^1) v_0^1 L_{\alpha^1} v_0^1-G'''(\phi_m^2) v_0^2 L_{\alpha^2} v_0^2\right\|_{L^2}, \\
\leq\ & \gamma_2 \delta\left\|v_0^1-v_0^2\right\|_{H^4}.
\end{aligned} \tag{2.3.121}
$$

In particular the first term from RHS of (2.3.117) satisfies the estimate

$$
\left\|\tilde{\Pi}_* \mathcal{L}_*^{-1}\left(Q(\xi_*+v_0^1)-Q(\xi_*+v_0^2)\right)\right\|_{L^2} \leq \gamma_3 \delta\left\|v_0^1-v_0^2\right\|_{H^4}, \tag{2.3.122}
$$

where we have introduced the constant

$$\gamma_3 := \left\| \tilde{\Pi}_* \mathcal{L}_*^{-1} \right\|_{L^2 \to H^4} (\gamma_1 + \gamma_2). \tag{2.3.123}$$

Addressing the second term on the RHS of (2.3.117), we recall that $Q_s(v, w) = Q(v, w) + Q(w, v)$ and definition (2.3.54) of Q to obtain

$$\begin{aligned}
&\left\| Q_s(\xi_* + v_0^1, \phi_m^* - \phi_m^1) - Q_s(\xi_* + v_0^2, \phi_m^* - \phi_m^2) \right\|_{L^2} \\
&\leq \overbrace{\left\| L_{\alpha^1} G'''(\phi_m^1)(\xi_* + v_0^1)(\phi_m^* - \phi_m^1) - L_{\alpha^2} G'''(\phi_m^2)(\xi_* + v_0^2)(\phi_m^* - \phi_m^2) \right\|_{L^2}}^{T_1} + \\
&+ \overbrace{\left\| G'''(\phi_m^1)(\xi_* + v_0^1) L_{\alpha^1}(\phi_m^* - \phi_m^1) - G'''(\phi_m^2)(\xi_* + v_0^2) L_{\alpha^2}(\phi_m^* - \phi_m^2) \right\|_{L^2}}^{T_2} + \\
&+ \overbrace{\left\| G'''(\phi_m^1)(\phi_m^* - \phi_m^1) L_{\alpha^1}(\xi_* + v_0^1) - G'''(\phi_m^2)(\phi_m^* - \phi_m^2) L_{\alpha^2}(\xi_* + v_0^2) \right\|_{L^2}}^{T_3}.
\end{aligned} \tag{2.3.124}$$

The first term T_1 on the RHS of (2.3.124) satisfies

$$\begin{aligned}
T_1 \quad = \quad & \Big\| L_{\alpha^1} G'''(\phi_m^1)(\xi_* + v_0^1) \partial_\alpha \phi_m (\alpha_* - \alpha^1) + \\
& - L_{\alpha^2} G'''(\phi_m^2)(\xi_* + v_0^2) \partial_\alpha \phi_m (\alpha_* - \alpha^2) \Big\|_{L^2}.
\end{aligned} \tag{2.3.125}$$

However

$$|\alpha_* - \alpha^i| = O(\delta^2), \quad i = 1, 2 \tag{2.3.126}$$

$$\left\| \xi_0 + v_0^i \right\|_{H^4} = O(\delta^2), \quad i = 1, 2 \tag{2.3.127}$$

and we deduce that

$$T_1 \leq \gamma_4 \delta^{\frac{1}{2}} \|v_0^1 - v_0^2\|_{H^4}, \tag{2.3.128}$$

for some $\gamma_4 > 0$, independent of δ. Similar estimates apply to T_2, T_3 and we conclude that

$$\left\| Q_s(\xi_* + v_0^1, \phi_m^* - \phi_m^1) - Q_s(\xi_* + v_0^2, \phi_m^* - \phi_m^2) \right\|_{L^2} \leq \gamma_5 \delta^{\frac{1}{2}} \left\| v_0^1 - v_0^2 \right\|_{H^4}, \tag{2.3.129}$$

for a constant $\gamma_5 > 0$, independent of $\delta < \delta_0$. Combining the inequalities (2.3.122) and (2.3.129) with (2.3.117) we deduce that

$$\begin{aligned} \|\mathcal{N}(u_1) - \mathcal{N}(u_2)\|_{H^4} &\leq \left\| \tilde{\Pi}_* \mathcal{L}_*^{-1} (Q_s(\xi_* + v_0^1, \phi_m^* - \phi_m^1) - Q_s(\xi_* + v_0^2, \phi_m^* - \phi_m^2)) \right\|_{H^4} + \\ &\quad + \left\| \tilde{\Pi}_* \mathcal{L}_*^{-1} (Q(\xi_* + v_0^1) - Q(\xi_* + v_0^2)) \right\|_{H^4}, \\ &\leq \gamma \delta^{\frac{1}{2}} \left\| v_0^1 - v_0^2 \right\|_{L^2} = \gamma \delta^{\frac{1}{2}} \|u_1 - u_2\|_{H^4}, \end{aligned} \tag{2.3.130}$$

for some constant $\gamma > 0$, independent of $\delta < \delta_0$. □

The proof of Theorem 1.1 follows by applying Lemma 2.6, Lemma 2.7 and the contraction mapping principle to derive the existence of a unique fixed point Φ_m of N in $B^*_{\rho_0}$.

Chapter 3

Analysis of the Second-Order System

In chapter 2 we use the contraction mapping argument to prove the existence of the homoclinic solution of full system (1.0.8), describing it as a perturbation of the homoclinic solution of the associated second-order differential equation (1.1.6),

$$\phi_m'' = G'(\phi_m), \tag{3.0.1}$$

where G is defined in (2.2.3).

3.1 Analysis of the second order eigenvalue problem

Here we establish some of the properties of spectrum of the second order system used in Chapter 2. Recall that $\phi_m(z; \alpha, b)$ is the homoclinic solution of the second order equation (1.1.6).

Lemma 3.1. *Under the scaling (S)-(1.1.3), where $b = b_- + \delta^2 \beta$ and $\tilde{\eta} = \delta^2 \eta$ there exists a smooth function*

$$\alpha_*(b) = -\frac{\delta}{g(b_+)} \left(\mu_- \beta (b_+ - b_-) + \frac{\eta}{4} (b_+ - b_-)^2 \right) + O(\delta^2), \tag{3.1.1}$$

for which $G(\cdot; \alpha_*(b), b)$ has a double zero at $u = \phi^*_{max}(b)$. For $\alpha > \alpha_*$, this double zero breaks *into two zeros, the smaller of which takes the form*

$$\phi_{max}(\alpha, b) = b_+ - \sqrt{\frac{2\delta\alpha g(b_+)}{\mu_+}} + O\left(\delta^{\frac{3}{2}}\right). \tag{3.1.2}$$

In particular, the value $\phi_{max}(\alpha)$ *is the maximum of* ϕ_m *over* $z \in \mathbb{R}$, *and*

$$p_2(\phi_{max}) = -\mu_+^2(b_+ - b_-)\beta - \sqrt{\frac{2\alpha g(b_+)}{\mu_+}} \left(\mu_-(\mu_+ - \mu_-)\beta + \frac{\mu_+}{2}(b_+ - b_-)\eta\right)\delta^{\frac{1}{2}} + O(\delta), \tag{3.1.3}$$

which is positive so long as $\beta < 0$.

Proof. The expression (3.1.2) follows from a Taylor expression of G near b_+, and (3.1.3) results from substitution of (3.1.2) into (2.1.12). □

Lemma 3.2. *Let* $\phi_m(z; \alpha, b)$ *is the homoclinic solution of the second order equation (1.1.6), then the following equalities hold*

$$L_\alpha \phi_m'' = G'''(\phi_m)(\phi_m')^2, \tag{3.1.4}$$

$$L_\alpha \psi_0' = G'''(\phi_m)\phi_m'\psi_0 + \lambda_0\psi_0', \tag{3.1.5}$$

$$L_\alpha(z\phi_m') = 2\phi_m'', \tag{3.1.6}$$

$$L_\alpha(z\psi_0) = 2\psi_0' + \lambda_0\psi_0, \tag{3.1.7}$$

where $L_\alpha = \partial_z^2 - G''(\phi_m)$ *is the linearization of (1.1.6) about* ϕ_m *and* λ_0 *is the ground-state eigenvalue of* L_α *with eigenfunction* ψ_0.

Proof. The first equality follows from taking partial derivatives of (1.1.6) with respect to z while the second equality arises from taking partial derivative of the eigenvalue equation for ψ_0 with respect to z. The last two equalities follow from distributing the action of L_α.

$\square$

Lemma 3.3. *The function $g'(\phi_m)$ admits the following expansions*

$$g'(\phi_m) = \frac{1}{\sqrt{2}}|\phi_m'| + \sqrt{\delta}h_1 + O(\delta^2), \tag{3.1.8}$$

where $\|h_1\|_{L^1} + \|h_1\|_{L^2} = O(1)$. In addition we have

$$g'(\phi_m) = \frac{\|\phi_m'\|_2}{\sqrt{2}}\psi_0 + \sqrt{\delta}h_2 + \delta h_3 + O(\delta^2), \tag{3.1.9}$$

in L^2 norm, where the correction terms take the form

$$\begin{aligned} h_1 &= \frac{1}{\sqrt{\delta}}\left(\sqrt{\frac{1}{2}|\phi_m'|^2 + \delta\alpha g(\phi_m)} - \frac{1}{\sqrt{2}}|\phi_m'|\right), && (3.1.10) \\ h_2 &= C\psi_0, && (3.1.11) \\ h_3 &= -2\alpha(\tilde{\pi}L)^{-1}g'''(\phi_m)g(\phi_m) + \frac{(r(\phi_m), \psi_0)_2}{\hat{\lambda}_0}\psi_0. && (3.1.12) \end{aligned}$$

Moreover

$$|\phi_m'|h_1 = \sqrt{\frac{\delta}{2}}\alpha g(\phi_m) + O(\delta), \tag{3.1.13}$$

and

$$|\phi_m'|g'(\phi_m) = \frac{1}{\sqrt{2}}\left(|\phi_m'|^2 + \delta\alpha g(\phi_m)\right) + O(\delta^2). \tag{3.1.14}$$

Proof. From (1.1.6) we observe its first integral form

$$\frac{1}{2}(\phi_m')^2 = G(\phi_m). \tag{3.1.15}$$

Applying the identity (3.1.15) to (2.2.14) we obtain

$$\frac{1}{2}(\phi_m')^2 = (g'(\phi_m))^2 - \delta\alpha g(\phi_m) + \delta^2 h(\phi_m). \tag{3.1.16}$$

Solving for $g'(\phi_m)$ we obtain

$$\begin{aligned} g'(\phi_m) &= \sqrt{\frac{1}{2}|\phi_m'|^2 + \delta\alpha g(\phi_m) + O(\delta^2)} \\ &= \frac{1}{\sqrt{2}}|\phi_m'| + \left(\sqrt{\frac{1}{2}|\phi_m'|^2 + \delta\alpha g(\phi_m)} - \frac{1}{\sqrt{2}}|\phi_m'|\right) + O(\delta^2). \end{aligned} \tag{3.1.17}$$

Expanding the expression (3.1.17) yields (3.1.8), except for a neighborhood of $z = 0$ with $\phi_m' = \phi_m''(0)z + O(z^2)$ where $\phi_m''(0) = G'(\phi_m(0)) = O(\sqrt{\delta})$. These estimates yield the $O(\sqrt{\delta})$ error bound in L^2 and L^1, which recovers the equality (3.1.8). From (3.1.10) it is easy to derive (3.1.13) and plug it into (3.1.8) yielding (3.1.14). Recalling (2.2.18)

$$Lg'(\phi_m) = -2\delta\alpha g'''(\phi_m)g(\phi_m) + \delta^2 r(\phi_m), \tag{3.1.18}$$

where $r(s) = 2g'''h + g''h' - g'h''$ and $r(\phi_m)$ is $O(1)$ in L^2. Since the right hand side of (3.1.18) is even, it is orthogonal to the kernel of L and we can invert L, yielding

$$\begin{aligned} g'(\phi_m) &= \underbrace{-\frac{2\delta\alpha(g'''(\phi_m)g(\phi_m), \psi_0)_2}{\lambda_0}}_{H_1}\psi_0 + \frac{\delta^2(r(\phi_m), \psi_0)_2}{\lambda_0}\psi_0 +, \\ &\quad -2\delta\alpha L^{-1}\tilde{\Pi}g'''(\phi_m)g(\phi_m) + O(\delta^2), \end{aligned} \tag{3.1.19}$$

in L^2. Combining (2.2.8), (2.2.10), (2.2.12) and (2.2.13) we may further simplify the leading coefficient of ψ_0 in (3.1.19) yielding

$$\begin{aligned} H_1 &:= -\frac{2\delta\alpha(g'''(\phi_m)g(\phi_m),\psi_0)_2}{\lambda_0}, \\ &= \frac{||\phi_m'||_2}{\sqrt{2}} + O(\delta^{\frac{1}{2}}). \end{aligned} \tag{3.1.20}$$

Therefore

$$\begin{aligned} g'(\phi_m) &= \frac{\|\phi_m'\|_2}{\sqrt{2}}\psi_0 + \delta^{\frac{1}{2}}\underbrace{C\psi_0}_{:=h_2} + \\ &+\delta\underbrace{\left(\frac{(r(\phi_m),\psi_0)_2}{\hat{\lambda}_0}\psi_0 - 2\alpha L^{-1}\tilde{\Pi}g'''(\phi_m)g(\phi_m)\right)}_{:=h_3} + O(\delta^2), \end{aligned} \tag{3.1.21}$$

where C is some constant.

$\square$

Remark 3.1. *From the results of Lemma 3.3 we may refine (3.1.8) to obtain the equality*

$$\psi_0 = \frac{\sqrt{2}}{\|\phi_m'\|_2}\left(\frac{1}{\sqrt{2}}|\phi_m'| + \sqrt{\delta}(h_1 - h_2) - \delta h_3\right) + O(\delta^2), \tag{3.1.22}$$

in L^2 norm, where h_1, h_2 and h_3 are defined in (3.1.10), (3.1.11) and (3.1.12).

Lemma 3.4. *Let $\phi_m(z;\beta,\eta,\delta)$ be the homoclinic solution of (1.1.6). Then the following important identities hold*

$$\frac{\partial\psi_0}{\partial\alpha} = -\frac{\phi_m''}{\sqrt{2}\hat{\lambda}_0||\phi_m'||_2} + O\left(\delta^{\frac{1}{2}}\right), \tag{3.1.23}$$

in L^2 norm, while

$$\partial_\alpha \phi_m = \frac{1}{\sqrt{2}\hat{\lambda}_0}|\phi_m'| + O\left(\delta^{\frac{1}{2}}\right), \tag{3.1.24}$$

in L^2 norm.

$$g''(\phi_m)g'(\phi_m) = \frac{1}{2}\phi_m'' + O(\delta), \tag{3.1.25}$$

in L^∞ norm. In addition

$$g(\phi_m(0)) = g(b_+) + O(\delta^{\frac{1}{2}}) = \frac{1}{2\sqrt{2}}\|\phi_m'\|_2^2 + O\left(\delta^{\frac{1}{2}}\right), \tag{3.1.26}$$

$$g''(\phi_m(0)) = -\sqrt{\frac{\mu_+}{2}} + O\left(\delta^{\frac{1}{2}}\right), \tag{3.1.27}$$

$$\begin{aligned}(\phi_m'', g(\phi_m)g'''(\phi_m))_2 &= \sqrt{2}g(\phi_m(0))(g''(\phi_m(0)))^2 + \\ &\qquad -\frac{1}{4}||\phi_m''||_2^2 + O\left(\delta^{\frac{1}{2}}\right),\end{aligned} \tag{3.1.28}$$

where

$$g(\phi_m(0))(g''(\phi_m(0)))^2 = \frac{1}{2}g(\phi_m(0))G''(\phi_m(0)) + O\left(\delta^{\frac{1}{2}}\right). \tag{3.1.29}$$

Proof. From the definition (1.1.14) of $g(\phi_m)$ we can obtain

$$\begin{aligned} g'(\phi_m) &= \sqrt{W_s(\phi_m)}, \\ g''(\phi_m) &= \frac{1}{2}\frac{W_s'(\phi_m)}{\sqrt{W_s(\phi_m)}}, \end{aligned} \tag{3.1.30}$$

from which we deduce

$$
\begin{aligned}
g''(\phi_m)g'(\phi_m) &= \frac{1}{2}W_s'(\phi_m), \\
&= \frac{1}{2}G'(\phi_m) + O(\delta), \\
&= \frac{1}{2}\phi_m'' + O(\delta),
\end{aligned} \tag{3.1.31}
$$

which recovers (3.1.27). From (1.1.6) and the definition of $G(\phi)$ (2.2.3) we have

$$
\begin{aligned}
\phi_m'' &= G'(\phi_m), \\
&= G_0'(\phi_m) - \delta\alpha g'(\phi_m).
\end{aligned} \tag{3.1.32}
$$

Then taking derivative with respect to α we obtain

$$
\begin{aligned}
(\partial_\alpha \phi_m)'' &= G_0''(\phi_m)\partial_\alpha\phi_m - \delta\alpha g''(\phi_m)\partial_\alpha\phi_m - \delta g'(\phi_m), \\
&= G''(\phi_m)\partial_\alpha\phi_m - \delta g'(\phi_m),
\end{aligned} \tag{3.1.33}
$$

so from the definition of L (1.1.15) we obtain

$$
\begin{aligned}
L\partial_\alpha\phi_m &= -\delta g'(\phi_m), \\
&= -\delta(g'(\phi_m), \psi_0)_2\psi_0 + \delta\,\tilde{\pi}_0\, g'(\phi_m).
\end{aligned} \tag{3.1.34}
$$

Inverting L yields

$$
\partial_\alpha\phi_m = -\frac{\delta(g'(\phi_m), \psi_0)_2}{\lambda_0}\psi_0 - \delta L^{-1}\tilde{\pi}_0 g'(\phi_m), \tag{3.1.35}
$$

in L^2 norm. From (2.2.12) we observe that $\tilde{\pi}_0 g'(\phi_m) = O(\delta)$. It yields

$$\partial_\alpha \phi_m = -\frac{(\psi_0, g'(\phi_m))_2}{\hat{\lambda}_0}\psi_0 + O(\delta^2). \tag{3.1.36}$$

Combining the results of (3.1.9), (2.2.12) and (2.2.13) we may further simplify

$$\begin{aligned} \partial_\alpha \phi_m &= -\frac{||\phi_m'||_2}{\sqrt{2}\hat{\lambda}_0}\psi_0 + O\left(\delta^{\frac{1}{2}}\right), \\ &= \frac{1}{\sqrt{2}\hat{\lambda}_0}|\phi_m'| + O\left(\delta^{\frac{1}{2}}\right), \end{aligned} \tag{3.1.37}$$

in L^2 which covers (3.1.24). In order to prove (3.1.23) it follows from (2.2.12)

$$\frac{\partial \psi_0}{\partial \alpha} = \frac{g''(\phi_m)\,\partial_\alpha \phi_m\,||g'(\phi_m)||_2 - g'(\phi_m)\,\partial_\alpha\left(||g'(\phi_m)||_2\right)}{||g'(\phi_m)||_2^2} + O(\delta), \tag{3.1.38}$$

in L^2 norm. Observing that

$$\begin{aligned} \partial_\alpha\left(||g'(\phi_m)||_2\right) &= \partial_\alpha\left(\int_{\mathbb{R}} (g'(\phi_m))^2 dz\right)^{1/2}, \\ &= \frac{\int_{\mathbb{R}} g'(\phi_m) g''(\phi_m)\partial_\alpha \phi_m dz}{2||g'(\phi_m)||_2}, \end{aligned} \tag{3.1.39}$$

from (3.1.24) and (3.1.25) we have

$$\begin{aligned} \partial_\alpha ||g'(\phi_m)||_2 &= \frac{\int_{\mathbb{R}} g'(\phi_m) g''(\phi_m)|\phi_m'| dz}{2\sqrt{2}\hat{\lambda}_0 ||g'(\phi_m)||_2} + O\left(\delta^{\frac{1}{2}}\right), \\ &= -\frac{\int_0^\infty g'(\phi_m) g''(\phi_m), \phi_m' dz}{\sqrt{2}\hat{\lambda}_0 ||g'(\phi_m)||_2} + O\left(\delta^{\frac{1}{2}}\right), \\ &= -\frac{\int_0^\infty \phi_m'' \phi_m' dz}{2\sqrt{2}\hat{\lambda}_0 ||g'(\phi_m)||_2} + O\left(\delta^{\frac{1}{2}}\right), \\ &= O\left(\delta^{\frac{1}{2}}\right). \end{aligned} \tag{3.1.40}$$

So (3.1.38) becomes

$$\frac{\partial\psi_0}{\partial\alpha} = \frac{g''(\phi_m)\,\partial_\alpha\phi_m}{||g'(\phi_m)||_2} + O\left(\delta^{\frac{1}{2}}\right). \tag{3.1.41}$$

Combining the results of (3.1.24), (3.1.8) and (3.1.25) we may further simplify

$$\begin{aligned}\frac{\partial\psi_0}{\partial\alpha} &= -\frac{g''(\phi_m)g'(\phi_m)}{\hat{\lambda}_0||g'(\phi_m)||_2} + O\left(\delta^{\frac{1}{2}}\right),\\ &= -\frac{\phi_m''}{2\hat{\lambda}_0||g'(\phi_m)||_2} + O\left(\delta^{\frac{1}{2}}\right),\\ &= -\frac{\phi_m''}{\sqrt{2}\hat{\lambda}_0||\phi_m'||_2} + O\left(\delta^{\frac{1}{2}}\right), \qquad (3.1.42)\end{aligned}$$

in L^2 norm which covers (3.1.23). To prove (3.1.27) we use (3.1.25) and (3.1.8) to obtain

$$\begin{aligned}(\phi_m'', g(\phi_m)g'''(\phi_m))_2 &= \int_{\mathbb{R}} 2g'''(\phi_m)g''(\phi_m)g'(\phi_m)g(\phi_m)dz + O\left(\delta^{\frac{1}{2}}\right),\\ &= \int_{\mathbb{R}} \sqrt{2}g'''(\phi_m)g''(\phi_m)|\phi_m'|g(\phi_m)dz + O\left(\delta^{\frac{1}{2}}\right), \qquad (3.1.43)\end{aligned}$$

by even-odd symmetry of ϕ_m and ϕ_m' and integration by parts we obtain

$$\begin{aligned}(\phi_m'', g(\phi_m)g'''(\phi_m))_2 &= -\int_0^\infty 2\sqrt{2}g'''(\phi_m)g''(\phi_m)\phi_m' g(\phi_m)dz + O\left(\delta^{\frac{1}{2}}\right), \qquad (3.1.44)\\ &= \sqrt{2}g(\phi_m(0))(g''(\phi_m(0)))^2 +\\ &\quad + \int_0^\infty \sqrt{2}(g''(\phi_m))^2 g'(\phi_m)\phi_m' dz + O\left(\delta^{\frac{1}{2}}\right). \qquad (3.1.45)\end{aligned}$$

Applying (3.1.8) and (3.1.25) again we obtain

$$\begin{aligned}(\phi_m'', g(\phi_m)g'''(\phi_m))_2 &= \sqrt{2}g(\phi_m(0))(g''(\phi_m(0)))^2 + \\ &\quad -\int_0^\infty 2(g''(\phi_m))^2(g'(\phi_m))^2 dz + O\left(\delta^{\frac{1}{2}}\right), \\ &= \sqrt{2}g(\phi_m(0))(g''(\phi_m(0)))^2 + \\ &\quad -\int_0^\infty \frac{1}{2}(\phi_m'')^2 dz + O\left(\delta^{\frac{1}{2}}\right), \end{aligned} \tag{3.1.46}$$

which recovers (3.1.28). In order to prove (3.1.26) we use (3.1.16) and the fact that $\phi_m'(0) = 0$ to obtain

$$g'(\phi_m(0)) = \sqrt{\delta\alpha g(\phi_m(0))} + O(\delta) = O\left(\delta^{\frac{1}{2}}\right). \tag{3.1.47}$$

From (2.2.14) and the equality (3.1.47) we obtain

$$\begin{aligned}G''(\phi_m(0)) &= 2\left((g''(\phi_m(0)))^2 + g'''(\phi_m(0))g'(\phi_m(0))\right) + O(\delta), \\ &= 2(g''(\phi_m(0)))^2 + O\left(\delta^{\frac{1}{2}}\right), \end{aligned} \tag{3.1.48}$$

so

$$G''(\phi_m(0))g(\phi_m(0)) = 2(g''(\phi_m(0)))^2 g(\phi_m(0)) + O\left(\delta^{\frac{1}{2}}\right), \tag{3.1.49}$$

which finishes the proof of (3.1.29).

□

3.2 Heteroclinic limit of the second-order problem

The differential equation (1.1.6) reduces to (1.1.8) in the limit as δ approaches to 0. In this section we study the convergence of the homoclinic ϕ_m of (1.1.6) to the heteroclinic ϕ_h of (1.1.8).

We perform the analysis in the dynamical systems framework. We first write (1.1.6) as a system, introducing

$$u = \phi, v = \phi'. \tag{3.2.1}$$

Then we can rewrite (1.1.6) as

$$\dot{\begin{pmatrix} u \\ v \end{pmatrix}} = \begin{pmatrix} v \\ G'(u) \end{pmatrix}, \tag{3.2.2}$$

here $\cdot$ means $\partial/\partial z$. Define

$$x = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.2.3}$$

then (3.2.2) becomes

$$\dot{x} = F(x, \delta), \tag{3.2.4}$$

where $F(x, \delta)$ is defined by the right hand side of (3.2.2). It is also easy to see that $p_+ := (b_+, 0)^T$ is a saddle point equilibrium point of (3.2.2) in case $\delta = 0$. The spectrum of $D_x F(x, \delta)\big|_{(p_+,0)}$ consists of two matrix eigenvalues

$$\sigma(D_x F(x, \delta)\big|_{(p_+,0)}) = \lambda^s \cup \lambda^u, \tag{3.2.5}$$

where $\lambda^s = -\sqrt{\mu_+} < 0$ and $\lambda^u = \sqrt{\mu_+} > 0$. Since $\mu_+ = W''(b_+) > 0$ it follows that p_+ is saddle equilibrium point. For $\delta \neq 0$ the matrix eigenvalues of

$$\sigma(D_x h(x,\delta)\big|_{(p_+,\delta)}) = \lambda^s(\delta) \cup \lambda^u(\delta), \tag{3.2.6}$$

where $\lambda^s(\delta) < 0$ and $\lambda^u(\delta) > 0$ perturb smoothly from λ^s and λ^u upon changing δ, respectively. We denote the stable and unstable eigenspaces of $D_x h(x,\delta)\big|_{(p_+,\delta)}$ by E_0^s and E_0^u respectively and choose local coordinates $x = (u,v) \in E_0^s \oplus E_0^u$. By an appropriate smooth coordinate change near b_+ seen in Lemma 6.1 in [33], (3.2.4) takes the normal form

$$\dot{u} = \lambda^s(\delta)u + f^s(u,v), \tag{3.2.7}$$

$$\dot{v} = \lambda^u(\delta)v, \tag{3.2.8}$$

where $f^s(u,v)$ satisfies

$$f^s(u,v) = O(|u|^2), \tag{3.2.9}$$

as $|u|, |v| \to 0$. We also define the local section Σ_{in} as

$$\Sigma_+ = \left\{(u,v)\big|u = \nu, |v| \leq \nu\right\} \tag{3.2.10}$$

where $\nu > 0$ is a small constant. It is easy to check that (3.2.4) is a reversible system with a symmetry plane. We can also define the symmetry plane in (u,v) coordinates as

$$\Sigma_{sym} = \left\{(u,v)\big|u = K(v)\right\}, \tag{3.2.11}$$

where K is a smooth function satisfying that $K(v) \to 0$ as $v \to 0$ and $K'(0) \neq 0$.

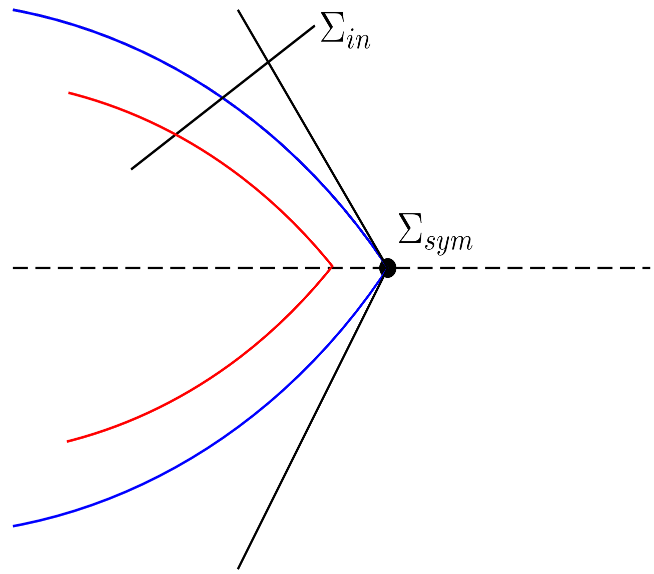

Figure 3.1: The homoclinic solution of (1.1.6) in red color and the heteroclinic solution of (1.1.8) in blue color. At $z = 0$, ϕ_m and ϕ_h are on the cross section Σ_{in}.

Lemma 3.5. *For all sufficiently small ν and ξ such that $0 < \xi < \nu$ then there exists $\tau > 0$ and a unique corresponding solution of (3.2.8) $(u, v)(z) = (u, v)(z; \nu, \xi; \delta)$ satisfying the conditions*

- $$u(0) = \nu, \qquad |v(0)| \leq \nu \tag{3.2.12}$$

- $$u(\tau) = K(v(\tau)) = \xi. \tag{3.2.13}$$

Moreover

$$|v(0)| \leq C\,\xi\,K^{-1}(\xi), \quad \tau \sim \frac{\ln(\xi)}{|\lambda_s|}. \tag{3.2.14}$$

where C is a positive constant independent of ξ and ν.

Proof. By rescaling we may assume $\nu = 1$ and $\xi < 1$, then the normal form (3.2.8) with $f^s(u,v) = \nu O(|u|^2)$. Then from (3.2.7) and (3.2.13) we have

$$v(z) = K^{-1}(\xi)\, e^{\lambda u(z-\tau)}, \tag{3.2.15}$$

and

$$u(z) = \nu e^{\lambda s z} + \int_0^z e^{\lambda s(z-t)} f^s(u,v)dt. \tag{3.2.16}$$

We define the weighted norm $\|u\| := \sup_{z\in[0,\infty)} e^{-\lambda s z}|u(z)|$ and denote by U the Banach space which it forms. A ball in U is defined

$$B_R = \left\{u \middle| \|u\| \leq R\right\}. \tag{3.2.17}$$

We define $\Gamma : U \to U$ as the map given by the right hand side of (3.2.16). We claim that there is a ball $B_R \subset U$ for which $\Gamma(B_R) \subset B_R$. Let $u \in B_R$ then by the definition of B_R it follows that $|u| \leq R\, e^{\lambda s z}$. Consequently, from the definition of $\Gamma(u)$ in (3.2.16) we have the point-wise estimate

$$\begin{aligned} |\Gamma(u)| &= \left|\nu e^{\lambda s z} + \int_0^z e^{\lambda s(z-t)} f^s(u(t), v(t))dt\right|, \\ &\leq \nu e^{\lambda s z} + C\int_0^z e^{\lambda s(z-t)}|u(t)|^2 dt, \end{aligned} \tag{3.2.18}$$

where $C > 0$ is a constant independent of $u \in B_R$, and the last inequality results from

(3.2.9). Using the fact that $|u| \le R\,e^{\lambda_s z}$ we obtain

$$\begin{aligned} \Gamma(u) &\le \nu e^{\lambda_s z} + CR^2 e^{\lambda_s z} \int_0^z e^{\lambda_s t} dt, \\ &\le e^{\lambda_s z}\left(\nu + CR^2\right). \end{aligned} \tag{3.2.19}$$

We deduce that for ν small enough, there exists $R > 0$ such that $\|\Gamma(u)\| \le R$ for all $u \in B_R$. Applying the contraction mapping theorem to B_R we deduce the existence of a unique solution u of (3.2.8)-(3.2.7) subject to $u(0) = \nu$. Moreover, since $u(0) = \nu > \xi$ and $u(z)$ approaches 0 as z approaches infinity, it follows from the intermediate value theorem that there exists $\tau > 0$ such that $u(\tau) = \xi$. From (3.2.15) and $\lambda_u > 0$ we deduce that $|v(0)| < K^{-1}(\xi) < \nu$. Since $u \in B_R$ we have the point-wise estimate $|u(z)| \le R\,e^{\lambda_s z}$ from which we obtain the bound

$$\left|\int_0^\tau e^{\lambda_s(\tau-t)} f^s(u,v)dt\right| \sim \frac{1}{|\lambda_s|} e^{2\lambda_s \tau}, \tag{3.2.20}$$

Since $u(\tau) = \xi$, we may use (3.2.16) to solve for τ which satisfies the asymptotic relation

$$\tau \sim \frac{\ln(\xi)}{\lambda_s} = \frac{\ln(b_+ - \phi_{max})}{\lambda_s}. \tag{3.2.21}$$

From (3.2.15) it follows

$$|v(0)| \le C\,\xi\,K^{-1}(\xi). \tag{3.2.22}$$

□

Remark 3.2. *ν can be chosen to be $O(1)$ but small enough that $\|\Gamma(u)\| \le R$ which implies that $\nu < 1/4C$.*

Remark 3.3. *The above lemma is very similar to Shil'nikov variable method which deals well with local flow near the saddle equilibrium point.*

Under the assumption (H) we know that the double well potential W actually has the following form $W = P^2$ and P is a parabola that opens upward with two zeros $b_\pm$, see Fig 1.2. We translate the homoclinic and heteroclinic orbits so that $\phi_m(0) = \phi_{max}$ and $\phi_h(0) = 0$. We choose τ_m and τ_h are chosen so that $\phi_m(\tau_m)$ and $\phi_h(\tau_h)$ are on the cross section Σ_+ and note that $\tau_m < 0$ and $\tau_h > 0$. From lemma 3.5 we can see that τ_m depends on δ since $\xi = b_+ - \phi_{max}$ and ϕ_{max} depends on δ. However, τ_h is independent of δ.

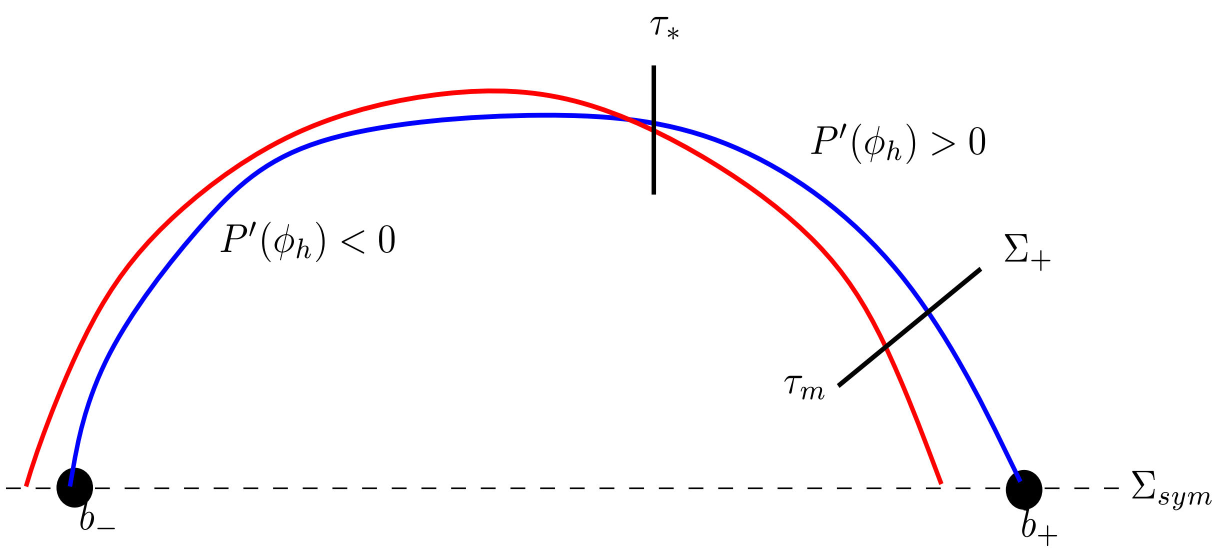

Figure 3.2: The homoclinic solution of (1.1.6) in red color and the heteroclinic solution of (1.1.8) in blue color. At $z = \tau_m$, ϕ_m and ϕ_h are on the cross section Σ_{in}. At $z = \tau_*$, $P'(\phi_h(\tau_*)) = 0$.

Lemma 3.6. *The homoclinic and heteroclinic solutions, translated so that $\phi_m(0) = \phi_{max}$ and $\phi_h(0) = 0$ are proximal in the following sense,*

$$\left\| \phi_m - \phi_h(\cdot - \tau_m + \tau_h) - \delta^2 \beta \right\|_{H^1(-\infty,0]} \leq C\, \delta^{\frac{1}{2}}, \tag{3.2.23}$$

where the translations τ_m and τ_h are chosen such that $\phi_m(\tau_m)$ and $\phi_h(\tau_h)$ are on the cross section Σ_+.

Proof. From the local coordinates used in Lemma 3.5 and Lemma 3.1 it follows $\xi = b_+ - \phi_{max} = O(\sqrt{\delta})$. From remark 3.2 we can choose ν in Lemma 3.5 independent of δ. So from (3.2.14) we have

$$\left\|\phi_m - \phi_h(\cdot - \tau_m + \tau_h) - \delta^2\beta\right\|_{H^1[\tau_m,0]} \le C\,\delta^{\frac{1}{2}}. \tag{3.2.24}$$

The time τ_* is chosen so that $P'(\phi_h(\tau_* - \tau_m + \tau_h)) = 0$. From (3.1.15) and using the fact that $G(\phi_m) > 0$ for $-\infty \le z \le \tau_m$ we have

$$\phi_m' = \sqrt{2G(\phi_m)}. \tag{3.2.25}$$

Similarly we can also obtain

$$\phi_h' = \sqrt{2W(\phi_h)} = -\sqrt{2}P(\phi_h). \tag{3.2.26}$$

Let $y = \phi_m - \phi_h(\cdot - \tau_m + \tau_h) - \delta^2\beta$. From (3.2.25) and (3.2.26) we have

$$\begin{aligned}
\Big(\phi_m - \phi_h(\cdot - \tau_m + \tau_h) - \delta^2\beta\Big)' &= \sqrt{2G(\phi_m)} - \sqrt{2W(\phi_h(\cdot - \tau_m + \tau_h))}, \\
&= \sqrt{2W(\phi_m)} - \sqrt{2W(\phi_h(\cdot - \tau_m + \tau_h))} + \\
&\quad -\sqrt{2W(\phi_m)} + \sqrt{2G(\phi_m)}, \\
&= \sqrt{2}(P(\phi_h(\cdot - \tau_m + \tau_h)) - P(\phi_m)) + \\
&\quad -\sqrt{2W(\phi_m)} + \sqrt{2G(\phi_m)}.
\end{aligned} \tag{3.2.27}$$

From the definition of G (2.2.3) we observe that $\sqrt{2W(\phi_m)} - \sqrt{2G(\phi_m)} = O(\sqrt{\delta})$. So

$$y' = \sqrt{2}P'(\phi_h(\cdot - \tau_m + \tau_h))y + O(\sqrt{\delta}, y^2). \tag{3.2.28}$$

Note that $P'(\phi_h(\cdot - \tau_m + \tau_h)) < 0$ for $z \in (\tau_*, \tau_m]$ and $P'(\phi_h(\cdot - \tau_m + \tau_h)) > 0$ for $z \in (-\infty, \tau_*)$. At the time τ_m $y = O(\sqrt{\delta})$ which follows from (3.2.24). Since ν is $O(1)$, then it is easy to see that the flight time $\tau_* - \tau_m$ is uniformly bounded, independent of δ here. By using (3.2.28) we have

$$y = O\left(\delta^{\frac{1}{2}}\right), \quad \tau_* - 1 \leq z \leq \tau_m. \tag{3.2.29}$$

Using the fact that $\tau_* - \tau_m + 1$ is bounded independent of δ and (3.2.29) it yields

$$\left\|\phi_m - \phi_h(\cdot - \tau_m + \tau_h) - \delta^2\beta\right\|_{H^1[\tau_*-1,\tau_m]} \leq C\,\delta^{\frac{1}{2}}. \tag{3.2.30}$$

On the interval $(-\infty, \tau_* - 1)$ we have $P'(\phi_h) < -C_0 < 0$ for some $C_0 > 0$ independent of δ. From (3.2.28) we have

$$y \leq e^{C_0(z-\tau_*+1)}y(\tau_* - 1) + O\left(\delta^{\frac{1}{2}}\right), \quad -\infty \leq z \leq \tau_* - 1, \tag{3.2.31}$$

Moreover, since y tends to zero at an exponential rate as $z \to -\infty$ we obtain the bound

$$\left\|\phi_m - \phi_h(\cdot - \tau_m + \tau_h) - \delta^2\beta\right\|_{H^1[-\infty,\tau_*-1]} \leq C\,\delta^{\frac{1}{2}}. \tag{3.2.32}$$

Combining the results of (3.2.24), (3.2.30), and (3.2.32) we recovers (3.2.23). □

Remark 3.4. *Lemma 3.6 is applied in the proof of Lemma 2.5. In Lemma 2.5 the implicit function theorem is applied which requires that ϕ_m approaches ϕ_h as δ goes to zero.*

Lemma 3.7. *There exists $\omega > 0$ for which the quantities A_1 and A_2 which are defined by (2.3.101) and (2.3.102) in Lemma 2.5 admit the the asymptotic expansion*

$$A_1(\eta, \beta; \delta) = A_1^h(\eta, \beta) + O(\delta^\omega), \tag{3.2.33}$$

$$A_2(\eta, \beta; \delta) = A_2^h(\eta, \beta) + O(\delta^\omega), \tag{3.2.34}$$

point-wise, where we have introduced the heteroclinic Melnikov integrals

$$A_1^h := -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + 3\mu_- \int_{\mathbb{R}} W'''(\phi_h)\,(\phi_h')^2\,dz \tag{3.2.35}$$

$$A_2^h := \frac{3}{2}\Big(W'''(\phi_h)(\phi_h - b_-), (\phi_h')^2\Big)_2. \tag{3.2.36}$$

Proof. Since the terms involving β and η contains factor δ, then β, $\eta \to 0$ precisely when $\delta \to 0$. Recalling the definition, (2.3.101), of A_1 we have

$$\begin{aligned} A_1 &= -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + \frac{3}{2}\mu_- \int_{\mathbb{R}} W'''(\phi_m^*)\,((\phi_m^*)')^2\,dz \\ &= -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + 3\mu_- \int_{-\infty}^{0} W'''(\phi_m^*)\,((\phi_m^*)')^2\,dz, \end{aligned} \tag{3.2.37}$$

where the second equality follows from the fact that ϕ_m^* is even. Using (3.2.23) from lemma 3.6 and remarking that $W'''(\phi_h)$ is uniformly bounded in $\|\cdot\|_{L^\infty(\mathbb{R})}$ yields

$$A_1 = -\frac{9}{2}\mu_+^{\frac{5}{2}}(b_+ - b_-) + 3\mu_- \int_{-\infty}^{-\tau_m+\tau_h} W'''(\phi_h)\,(\phi_h')^2\,dz + O(\sqrt{\delta}). \tag{3.2.38}$$

Using the definition (3.2.35) of A_1^h we may further simplified A_1

$$A_1 = A_1^h - 3\mu_- \int_{-\tau_m+\tau_h}^{\infty} W'''(\phi_h)\,(\phi_h')^2\,dz + O(\sqrt{\delta}). \tag{3.2.39}$$

Choosing $\xi = O(\sqrt{\delta})$ in (3.2.14) we have the estimate

$$-\tau_m + \tau_h = -C_1 \ln \delta, \tag{3.2.40}$$

where C_1 is a positive constant of order 1. Substituting (3.2.40) into A_1 we obtain

$$A_1 \;=\; A_1^h - 3\mu_- \int_{-C_1 \ln(\delta)}^{\infty} W'''(\phi_h)\,(\phi_h')^2\,dz + O(\sqrt{\delta}). \tag{3.2.41}$$

Since the heteroclinic solution $\phi_h(z)$ of (1.1.8) exponentially approaches the equilibrium b_+ as $z \to \infty$ and $W'''(\phi_h)$ is bounded in $\|\cdot\|_{L^\infty(\mathbb{R})}$ we obtain

$$\begin{aligned} A_1 &= A_1^h - C_2 \int_{-C_1 \ln \delta}^{\infty} e^{-C_3 z}\,dz + O(\sqrt{\delta}), \\ &= A_1^h + O(\delta^{\omega}), \end{aligned} \tag{3.2.42}$$

where $\omega = \min\{C_1 C_3, \frac{1}{2}\}$. The derivation of (3.2.34) is similar and omitted here.

$\square$

Chapter 4

Lin's Method

4.1 Introduction

Homoclinic and heteroclinic orbits play an important role in applications. Lin's method is regarded as of great importance in proving existence of heteroclinic and homoclinic solutions of dynamical systems. This method was first proposed by X. Lin, [44], to construct solutions that stay close to a finite or infinite chain of heteroclinic connections. Lin's method is based upon Melnikov's method, an implementation of the Lyapunov-Schmidt method. In [61] Sandstede developed Lin's method and improved it by giving an better representation of the bifurcation equation. In this section, we follow Sandstede's way and adopt Lin's method to prove the existence of homoclinic solutions of (1.0.8). Reconsider the differential equation (1.0.8)

$$\left(\partial_z^2 - W''(u) + \tilde{\eta}\right)\left(\partial_z^2 u - W'(u)\right) = \theta. \tag{4.1.1}$$

Let $\mathcal{L}_h$ be the linearization of (4.1.1) around the heteroclinic solution ϕ_h for $\theta = 0$, then

$$\mathcal{L}_h = (L_h + \tilde{\eta})L_h, \tag{4.1.2}$$

where

$$L_h = \partial_z^2 - W''(\phi_h). \tag{4.1.3}$$

It is easy to see that ϕ_h' is the eigenvector of L_h corresponding to eigenvalue 0. From Sturn-Liouville Theory, ϕ_h' is the lowest eigenstate. There is a basic fact based on assumption (S')-(1.1.4) and we will use it through this Chapter:

Lemma 4.1. 0 *is a simple eigenvalue of operator* $\mathcal{L}_h$*, i.e.,* $Null(\mathcal{L}_h) = span\{\phi_h'\}$.

Proof. Assume that there exists $\varphi \notin span\{\phi_h'\}$ such that

$$\mathcal{L}_h\varphi = (L_h + \tilde{\eta})L_h\varphi = 0 \tag{4.1.4}$$

By Sturn-Liouville theory, 0 is the simple eigenvalue of L_h associated with eigenvector ϕ_h' and the other eigenvalues are $O(1)$ away from the 0. By using spectrum mapping theorem for self-adjoint operator and the assumption (S')-(1.1.4) $\tilde{\eta} + \lambda_2^h \neq 0$, we conclude that $L_h + \tilde{\eta}$ doesn't have kernel, which implies that

$$L_h\varphi = 0. \tag{4.1.5}$$

Therefore $\varphi = C\,\phi_h'$, which contradict to the definition of φ. □

Let

$$u_1 = u,\ u_2 = \dot{u_1},\ u_3 = \dot{u_2},\ u_4 = \dot{u_3}, \tag{4.1.6}$$

where $\dot{}$ denotes $\partial/\partial z$, we can rewrite (4.1.1) obtaining

$$\begin{pmatrix} \dot{u_1} \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ \theta + \tilde{\eta} W'(u_1) - W'(u_1)W''(u_1) + W'''(u_1)u_2^2 + (2W''(u_1) - \tilde{\eta})u_3 \end{pmatrix}. \quad (4.1.7)$$

We further define

$$x = (u_1, u_2, u_3, u_4)^T, \quad (4.1.8)$$

and write (4.1.7) in the compact form

$$\dot{x} = f(x; \theta), \quad (4.1.9)$$

where $f(x; \theta)$ represents the right-hand side of (4.1.7). Recall that (1.0.9)

$$\theta = W'(b)\left(W''(b) - \tilde{\eta}\right), \quad (4.1.10)$$

and under the scaling (S')-(1.1.4) $b = b_- + \beta\delta^2$ and since $W(b_-) = W'(b_-) = 0$, $\mu_- = W''(b_-)$

$$\begin{aligned} \theta &= W''(b_-)\left(W''(b_-) - \tilde{\eta}\right)\beta\delta^2 + O(\delta^3), \\ &= \mu_-(\mu_- - \tilde{\eta})\beta\delta^2 + O(\delta^3). \end{aligned} \quad (4.1.11)$$

So θ is very small $\theta = O(\delta^2)$.

For the case $\theta = 0$, the system (4.1.9) has equilibriums $p_1 := (b_-, 0, 0, 0)^T$ and $p_2 :=$

$(b_+, 0, 0, 0)^T$ satisfying $f(p_i; \theta = 0) = 0$. For $\theta \neq 0$ the equilibriums are described by the following lemma.

Lemma 4.2. *There exist $\theta_0 > 0$ and a smooth function $p_i(\theta)$ such that for all $|\theta| < \theta_0$ the points $p_1(\theta) = (b_-(\theta), 0, 0, 0)^T$ and $p_2(\theta) = (b_+(\theta), 0, 0, 0)^T$ are the equilibriums of the system (4.1.9).*

Proof. Note that $f(p_i; 0) = 0$ and f is smooth in x and θ. Moreover the Jacobian matrix

$$D_x f(x; \theta)\big|_{(p_i;0)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu_\pm \tilde{\eta} - \mu_\pm^2 & 0 & 2\mu_\pm - \tilde{\eta} & 0 \end{pmatrix}, \tag{4.1.12}$$

is nonsingular. Here $\mu_\pm = W''(b_\pm)$. The result follows from the implicit function theorem.

□

Remark 4.1. *From Lemma 4.2 we know that for each small $|\theta| < \theta_0$ there exists the corresponding equilibriums $p_i(\theta)$ satisfying $f(p_i(\theta), \theta) = 0$. Without lose of generality we can assume that $f(p_i, \theta) = 0$ for small $|\theta| < \theta_0$.*

It is easy to see that the spectrum of $D_x f(x; \theta)\big|_{(p_i;0)}$ is comprised of four eigenvalues,

$$\sigma(D_x f(x; \theta)\big|_{(p_i;0)}) = \{\pm\sqrt{\mu_\pm}, \pm\sqrt{\mu_\pm - \tilde{\eta}}\}. \tag{4.1.13}$$

Then for p_1 $\lambda_1^s := -\sqrt{\mu_+}$, $\lambda_1^{ss} := -\sqrt{\mu_+ - \tilde{\eta}}$, $\lambda_1^u := \sqrt{\mu_+}$, and $\lambda_1^{uu} := \sqrt{\mu_+ - \tilde{\eta}}$. Under the assumption (S')-(1.1.4) we have $\tilde{\eta} < 0$ so

$$\lambda_1^{ss} < \lambda_1^{s} < 0 < \lambda_1^{u} < \lambda_1^{uu}. \tag{4.1.14}$$

Similarly for p_2 $\lambda_2^s := -\sqrt{\mu_-}$, $\lambda_2^{ss} := -\sqrt{\mu_- - \tilde{\eta}}$, $\lambda_2^u := \sqrt{\mu_-}$, and $\lambda_2^{uu} := \sqrt{\mu_- - \tilde{\eta}}$.

$$\lambda_2^{ss} < \lambda_2^{s} < 0 < \lambda_2^{u} < \lambda_2^{uu}. \tag{4.1.15}$$

Remark 4.2. *By the Hypotheses (H) and (S')-(1.1.4) $\mu_\pm = W''(b_\pm) > 0$ and $|\tilde{\eta}| < \mu_\pm$ so $\mu_\pm$, $\mu_\pm - \tilde{\eta}$ are strictly positive and $(p_i, 0)^T$ are saddle points. Furthermore, $\theta \neq 0$ but small, the eigenvalues for the fixed point $p_{1,2}$ be the same as (4.1.13).*

The system (4.1.9) has some symmetry property in geometric sense which is very important for the later analysis. We call the dynamical system (4.1.9) S-reversible if there exists a 4×4 matrix S with $S^2 = I$ satisfying

$$f(S\,x, \theta) = -S\,f(x, \theta), \quad \forall x \in \mathbb{R}^4. \tag{4.1.16}$$

It is easy to see that if $x(z)$ is a solution of S-reversible system (4.1.9) then $S\,x(-z)$ is also a solution. Define $Fix\,S = \{x \in \mathbb{R}^4 \,|\, Sx = x\}$ and $Fix(-S) = \{x \in \mathbb{R}^4 \,|\, Sx = -x\}$. Since $x = \frac{1}{2}(I - S)x + \frac{1}{2}(I + S)$ then $x = x_1 + x_2$ where $x_1 \in Fix\,S$ and $x_2 \in Fix(-S)$.

Lemma 4.3. *The system (4.1.9) is reversible under the map $S : \mathbb{R}^4 \to \mathbb{R}^4$ defined by*

$$S(u_1, u_2, u_3, u_4) = (u_1, -u_2, u_3, -u_4). \tag{4.1.17}$$

Proof. It is easy to verify that $S\,f(x; \theta) = -f(S\,x; \theta)$. $\square$

Remark 4.3. *for our system it is easily checked* $S\,q_1(0) = q_2(0)$ *where* S *is defined in (4.1.17).*

4.2 Lin's Orbit

We will construct Lin's orbit in this section. The first step of construction of Lin's orbit is to study the splitting of the manifolds with respect to the parameter θ. Recall that ϕ_h is the heteroclinic solution of (1.1.8). Since ϕ_h is the heteroclinic connection of (1.0.8) between b_- and b_+ it follows that

$$q_1(z) = (\phi_h(z), \phi_h'(z), \phi_h''(z), \phi_h'''(z))^T, \tag{4.2.1}$$

is the heteroclinic connection of (4.1.9) for $\theta = 0$ between $p_1 = (b_-, 0, 0, 0)^T$ and $p_2 = (b_+, 0, 0, 0)^T$, that is

$$\lim_{z\to-\infty} q_1(z) = p_1, \quad \lim_{z\to\infty} q_1(z) = p_2. \tag{4.2.2}$$

Correspondingly there is another heteroclinic solution q_2 of (1.0.8) and

$$q_2(z) = (\phi_h(-z), -\phi_h'(-z), \phi_h''(-z), -\phi_h'''(-z))^T, \tag{4.2.3}$$

which connects b_+ and b_- satisfying

$$\lim_{z\to-\infty} q_2(z) = p_2, \quad \lim_{z\to\infty} q_2(z) = p_1. \tag{4.2.4}$$

By the invariant manifold theorem, there is an unstable manifold W_-^s and a stable manifold W_-^u generated by the saddle point p_2. Similarly there are stable manifold W_+^s and unstable manifold W_+^u associated to p_1. Consider the differential equation obtained by the linearization of (4.1.9) about Φ_h

$$\dot{v} = A(z)v, \tag{4.2.5}$$

where $A(z) := D_x f(\Phi_h(z), 0)$. We denote by $T(t, s)$ the fundamental transition operator for (4.2.5). It has semigroup properties: $T(t, t) = I$; $T(t, w) = T(t, s)T(s, w)$ for $t \geq s \geq w$. Since $\lim_{z \to \pm\infty} A(z) = D_x\, f(x; \theta)\big|_{(p_{1,2}; 0)}$ is hyperbolic, it has exponential dichotomies $Q_i^-(z)$, $P_i^-(z)$ for $z \leq 0$ and $P_i^+(z)$, $Q_i^+(z)$ for $z \geq 0$ [18]. This means that there exist constants $C \geq 1$ and $\nu > 0$, and continuous mappings $Q_i^-(\cdot) : \mathbb{R}_- \to \mathcal{L}(\mathbb{R}^4)$ and $P_i^+(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathbb{R}^4)$ where $\mathcal{L}(\mathbb{R}^4)$ represents the bounded linear operator on $\mathbb{R}^4$. The following properties hold:

- $Q_i^-(z)$ are projections on ${\mathbb{R}_-}^4$, and $P_i^+(z)$ are projections on ${\mathbb{R}_+}^4$,

$$\begin{aligned} (P_i^+)^2 &= P_i^+, && (4.2.6) \\ (Q_i^-)^2 &= Q_i^-. && (4.2.7) \end{aligned}$$

-

$$\begin{aligned} P_i^+(t)T_i(t,s) &= T_i(t,s)P_i^+(s), \quad t, s \in \mathbb{R}_+, && (4.2.8) \\ Q_i^-(t)T_i(t,s) &= T_i(t,s)Q_i^-(s), \quad t, s \in \mathbb{R}_-. && (4.2.9) \end{aligned}$$

-

$$\left\|T_i(t,s)\,P_i^+(s)\right\| \;\le\; C\,e^{-\nu(t-s)}, \quad 0\le s\le t, \tag{4.2.10}$$

$$\left\|T_i(t,s)\,Q_i^+(s)\right\| \;\le\; C\,e^{-\nu(s-t)}, \quad 0\le t\le s, \tag{4.2.11}$$

$$\left\|T_i(t,s)\,P_i^-(s)\right\| \;\le\; C\,e^{-\nu(t-s)}, \quad s\le t\le 0, \tag{4.2.12}$$

$$\left\|T_i(t,s)\,Q_i^-(s)\right\| \;\le\; C\,e^{-\nu(s-t)}, \quad t\le s\le 0, \tag{4.2.13}$$

where $Q_i\pm := I - P_i^\pm$. In fact

$$R\,P_1^+(z) = T_{q_1(z)}W^s(p_1), \quad R\,Q_1^-(z) = T_{q_1(z)}W^u(p_2). \tag{4.2.14}$$

Similarly for P_2^+ and Q_2^-. Let $T_i^*(t,s)$ be the adjoint operator of $T_i(t,s)$, whose definition is given by

$$<\psi, T_i(t,s)v> = < T_i^*(t,s)\psi, v>, \quad v\in\mathbb{R}^4, \psi\in\mathbb{R}^{4*}. \tag{4.2.15}$$

The operator $T_i^*(t,s)$ has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projections $P_i^{\pm,*}(z)$, $Q_i^{\pm,*}(z)$ being the adjoint operators of the projections for $T_i(t,s)$.

- $Q_i^-(z)$ are projections on ${\mathbb{R}_-}^4$, and $P_i^+(z)$ are projections on ${\mathbb{R}_+}^4$,

$$(P_i^{+,*})^2 \;=\; P_i^{+,*}, \tag{4.2.16}$$

$$(Q_i^{-,*})^2 \;=\; Q_i^{-,*}. \tag{4.2.17}$$

-

$$P_i^{+,*}(t)T_i^*(t,s) \;=\; T_u^*(t,s)P_i^{+,*}(s), \quad t,s\in\mathbb{R}_+, \tag{4.2.18}$$

$$Q_i^{-,*}(t)T_i^*(t,s) \;=\; T_i^*(t,s)Q_i^{-,*}(s), \quad t,s\in\mathbb{R}_-. \tag{4.2.19}$$

•

$$\left\| T_i^*(t,s)\, P_i^{+,*}(s) \right\| \;\leq\; C\, e^{-\nu(t-s)}, \quad 0 \leq s \leq t, \tag{4.2.20}$$

$$\left\| T_i^*(t,s)\, Q_i^{+,*}(s) \right\| \;\leq\; C\, e^{-\nu(s-t)}, \quad 0 \leq t \leq s, \tag{4.2.21}$$

$$\left\| T_i^*(t,s)\, P_i^{-,*}(s) \right\| \;\leq\; C\, e^{-\nu(t-s)}, \quad s \leq t \leq 0, \tag{4.2.22}$$

$$\left\| T_i^*(t,s)\, Q_i^{-,*}(s) \right\| \;\leq\; C\, e^{-\nu(s-t)}, \quad t \leq s \leq 0, \tag{4.2.23}$$

where $Q_i^{+,*} := I - P_i^{+,*}$ and $P_i^{-,*} := I - Q_i^{-,*}$.

Let R denote by the range of the operator.

Corollary 1. *Under the assumptions (H) and (S')-(1.1.4) we have the following non-degeneracy condition*

$$R\, Q_i^-(0) \cap R\, P_i^+(0) = span\ \dot{q}_i(0). \tag{4.2.24}$$

Proof. By theorem 4.1 in [64] and lemma (4.1) we know that the only bounded solution of (4.2.5) is given by $\dot{q}_i(z) = (\phi_h', \phi_h'', \phi_h''', \phi_h'''')^T$, up to constant scalar multiples, which proves the corollary.

□

From corollary 1 we have

$$R\, Q_i^-(0) = span\ \dot{q}_i(0) \oplus Y_i^u, \quad R\, P_i^+(0) = span\ \dot{q}_i(0) \oplus Y_i^s, \tag{4.2.25}$$

where $dim\, Y_i^u = dim\, Y_i^s = 1$. Define $Y_i := Y_i^u + Y_i^s$. Then

$$\mathbb{R}^4 = span\ \dot{q}_i(0) \oplus Y_i \oplus Z_i \tag{4.2.26}$$

where $dim\, Z_i = 1$. Finally we are ready to define a cross section Σ_1 of the heteroclinic orbit q_1 by

$$\Sigma_i := q_i(0) + Y_i. \tag{4.2.27}$$

In particular we remark that $R\, P_1^+(0) = R\, Q_2^-(0)$ and $R\, Q_1^-(0) = R\, P_2^+(0)$. Then we conclude

$$\dot{q}_2(0) = S\, \dot{q}_1(0), \quad Z_2 = S\, Z_1, \tag{4.2.28}$$

$$Y_2^s = S\, Y_1^u, \quad Y_2^u = S\, Y_1^s. \tag{4.2.29}$$

In order to study of the splitting manifolds we look for the solutions $q_i^+(z;\theta)$ and $q_i^-(z;\theta)$ defined on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively satisfying the conditions:

- (Q1) $q_i^\pm(z;\theta)$ are close to $q_i(z)$.
- (Q2) $\lim_{z\to\infty} q_1^+(z;\theta) = p_2$, $\lim_{z\to-\infty} q_1^-(z;\theta) = p_1$.
- (Q3) $\lim_{z\to\infty} q_2^+(z;\theta) = p_1$, $\lim_{z\to-\infty} q_2^-(z;\theta) = p_2$.
- (Q4) $q_i^\pm(0;\theta) \in \Sigma_i$.
- (Q5) $\xi_i^\infty \psi_i := q_i^+(0;\theta) - q_i^-(0;\theta) \in Z_i$.

We are looking for the solutions $q_i^+(z) = q_i(z) + r_i^+(z)$ and $q_i^-(z) = q_i(z) + r_i^-(z)$ defined on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively such that $q_i^\pm(z;\theta)$ satisfies conditions (Q1)-(Q4) with the norm $\|r_i\|_\infty = \max\{\sup_{z\in\mathbb{R}^+} |r_i^+(z)|, \sup_{z\in\mathbb{R}^-} |r_i^-(z)|\}$. Define $C_{b+}^0 := \{r \in C^0(\mathbb{R}^+) \,|\, \|r\|_\infty < \infty\}$ and $C_{b-}^0 := \{r \in C^0(\mathbb{R}^-) \,|\, \|r\|_\infty < \infty\}$. If r_i is finite in norm $\|\cdot\|_\infty$ and close to zero, then by the theory of stable and unstable manifolds, $q_i^\pm(z) = q_i(z) + r_i^\pm(z)$ is on the desired stable or unstable manifold.

Lemma 4.4. (Perturbed Heteroclinic Orbit) *Given θ be small enough there is a unique pair $(q_i^+(z;\theta), q_i^-(z;\theta))$ solutions of (4.1.9) satisfying (Q1)-(Q5). Moreover, the mappings $(q_i^+(z;\cdot), q_i^-(z;\cdot)) : \mathbb{R} \to C^0_{b+} \times C^0_{b-}$ are smooth.*

Proof. Plug $q_i^\pm(z) = q_i(z) + r_i^\pm(z)$ into (4.1.9) we obtain

$$\dot{r}_i^\pm = A_i^\pm(z) r_i^\pm + h_i^\pm(z, r_i^\pm, \theta), \tag{4.2.30}$$

where $A_i^\pm(z) := D_x f(q_i^\pm(z), 0)$ and

$$h_i^\pm(z, r_i^\pm, \theta) = f(x,\theta) - f(q_i, 0) - A_i^\pm(z) r_i^\pm. \tag{4.2.31}$$

Let's focus on case $+$. From the variation of constants formula for (4.2.30) and exponential dichotomy (4.2.20) we have

$$\begin{aligned} r_i^+(z) &= T_i(t,0)\nu_i + \int_0^z T_i(z,s) P_i^+(s) h_i(s, r_i^+, \theta) ds - \int_z^\infty T_i(z,s) Q_i^+(s) h_i(s, r_i^+, \theta) ds, \quad (4.2.32) \\ &:= \mathbb{S}(r_i^+, \nu_i, \theta). \quad (4.2.33) \end{aligned}$$

where $\nu_i = P_i^+(0) z_i^+(0)$. From the definition (4.2.31) of h_i and the fact that the orbits of $\Gamma_i := \{q_i(t) | t \in \mathbb{R}\}$ are bounded one can easily show that the RHS of (4.2.32) $\mathbb{S}(r_i^+, \nu_i, \theta)$ defined for each $(\nu_i, \theta) \in R\, P_i^+(0) \times \mathbb{R}$ is a smooth mapping from C^0_{b+} into itself. So we can consider equation (4.2.30) as an equation in C^0_{b+}. Note that for $(\nu_i, \theta) = (0,0)$ $r_i^+ = 0$ is the solution of (4.2.30). From the definition (4.2.31) of h_i it is easy to check that

$$D_{r_i^+} \mathbb{S}(r_i^+, \nu_i, \theta)\Big|_{(r_i^+,\nu_i,\theta)=(0,0,0)} = 0. \tag{4.2.34}$$

Hence by the implicit function theorem we can solve the equation (4.2.32) for (ν_i, θ) close enough to $(0, 0)$ which means

$$r_i^+(z) = r_i^+(\nu_i, \theta)(z), \tag{4.2.35}$$

for small $\nu_i \in R\,P_i^+(0)$. We go through the similar proof for $-$. From the variation of constants formula and exponential dichotomy (4.2.22) equation (4.2.30) becomes

$$r_i^-(z) = T_i(t,0)\zeta_i - \int_z^0 T_i(z,s)Q_i^-(s)h_i(s, r_i^-(s), \theta)ds + \int_{-\infty}^z T_i(z,s)P_i^-(s)h_i(s, r_i^-, \theta)ds, \tag{4.2.36}$$

where $\zeta_i = Q_i^-(0)z_i^-(0)$. Similarly we can solve (4.2.36) for (ζ_i, θ) close to $(0, 0)$ which means

$$r_i^-(z) = r_i^-(\zeta_i, \theta)(z), \tag{4.2.37}$$

for small $\zeta_i \in R\,Q_i^-(0)$. From (4.2.32) and (4.2.36) we can obtain

$$r_i^+(0) = \nu_i - \underbrace{\int_0^\infty Q_i^+(0)T_i(0,s)h_i(s, r_i^+(\nu_i,\theta)(s), \theta)ds,}_{T^+(\nu_i,\theta)} \tag{4.2.38}$$

$$r_i^-(0) = \zeta_i + \underbrace{\int_{-\infty}^0 P_i^-(0)T_i(0,s)h_i(s, r_i^-(\zeta_i,\theta)(s), \theta)ds\,.}_{T^-(\zeta_i,\theta)} \tag{4.2.39}$$

Condition (Q4) indicates that $r_i^{\pm}(0) \in Y_i \oplus Z_i$. Then it follows

$$r_i^+(0) = \underbrace{\nu_i}_{\in Y_i^s} - \overbrace{\underbrace{T_i^{+,y}(\nu_i,\theta)}_{\in Y_i^u} - \underbrace{T_i^{+,z}(\nu_i,\theta)}_{\in Z}}^{=T^+(\nu_i,\theta)}, \tag{4.2.40}$$

$$r_i^-(0) = \underbrace{\zeta_i}_{\in Y_i^u} + \overbrace{\underbrace{T_i^{-,y}(\zeta_i,\theta)}_{\in Y_i^s} + \underbrace{T_i^{-,z}(\zeta_i,\theta)}_{\in Z}}^{=T^-(\zeta_i,\theta)}. \tag{4.2.41}$$

From (4.2.40), (4.2.41) and Condition (Q5) we obtain

$$\nu_i = -T_i^{-,y}(\xi_i,\theta), \tag{4.2.42}$$

$$\zeta_i = T_i^{+,y}(\nu_i,\theta). \tag{4.2.43}$$

Note that $(\nu_i,\zeta_i,\theta) = (0,0,0)$ solve (4.2.42) and (4.2.43). From the definition (4.2.31) of h_i it is easy to check that

$$D_{\nu_i} T_i^{+,y}(\nu_i,\theta)\big|_{(\nu_i,\theta)=(0,0)} = 0, \tag{4.2.44}$$

$$D_{\zeta_i} T_i^{-,y}(\zeta_i,\theta)\big|_{(\zeta_i,\theta)=(0,0)} = 0. \tag{4.2.45}$$

By the implicit function theorem we can solve (4.2.42)-(4.2.43) near $(\zeta_i,\nu_i,\theta) = (0,0,0)$ and

$$\zeta_i = \zeta_i(\theta), \nu_i = \nu_i(\theta). \tag{4.2.46}$$

. Combining the results of (4.2.35), (4.2.37), (4.2.46) we obtain

$$r_i^{\pm}(z) = r_i^{\pm}(\theta)(z). \tag{4.2.47}$$

Hence $q_i^{\pm}(z) := q_i(z)+r_i^{\pm}(z)$ are the unique heteroclinic connections close to $q_i(z)$ satisfying the conditions (Q1)-(Q5). $\square$

Remark 4.4. *Lemma 4.4 indicates that we can construct the unique perturbed heteroclinic orbits $q_i^{\pm}(z;\theta) = q_i(z) + r_i^{\pm}(z)$ which comprising the stable and unstable manifolds of p_i. In particular the proof of Lemma 4.4 also shows that $q_i^{\pm}$ are S symmetric with respect to each other, i.e., $q_2^{\pm} = S\,q_1^{\pm}$.*

Remark 4.5. *From definition we know that $\xi_i^{\infty}(\theta) =< \psi_i, r_i^{+}(0) - r_i^{-}(0) >$. From (4.2.32) and (4.2.36) we have*

$$\begin{aligned}
\xi_i^{\infty}(\theta) &= < \psi_i, \int_0^{\infty} Q_i^{+}(0)T_i(0,s)h_i(s,r_i^{+},\theta)ds + \int_{-\infty}^{0} P_i^{-}(0)T_i(0,s)h_i(s,r_i^{-},\theta)ds >, \\
&= < \psi_i, \int_{-\infty}^{\infty} T_i(0,s)h_i(s,r_i,\theta) > ds, \\
&= \int_{-\infty}^{\infty} < \psi_i(s), h_i(s,r_i,\theta) > ds. \qquad (4.2.48)
\end{aligned}$$

where

$$\psi_i(s) := T_i(0,s)^{*}\psi_i, \qquad (4.2.49)$$

and $T_i(0,s)^{*}$ denotes by the transpose of $T_i(0,s)$. Note that $\xi_i^{\infty}(\theta = 0) = 0$ and

$$\begin{aligned}
D_\theta\, \xi_i^{\infty}\big|_{\theta=0} &= \int_{-\infty}^{\infty} < \psi_i(s), D_\theta\, h_i(s,0,0) > ds, \\
&= \int_{-\infty}^{\infty} < \psi_i(s), D_\theta\, f(q_i(s),0) > ds. \qquad (4.2.50)
\end{aligned}$$

By Lemma 4.1 we know that 0 is the simple eigenvalue of $\mathcal{L}_h$ with corresponding eigenvector ϕ_h. It follows that there doesn't exist $\hat{\phi}$ such that $\mathcal{L}_h\hat{\phi} = \phi_h$, which implies that $\phi_h \perp$

$Null(\mathcal{L}_h^\dagger)$. So

$$< \phi_h, \psi^\dagger > \neq 0, \tag{4.2.51}$$

where $\psi^\dagger$ spans the null space of $\mathcal{L}_h^\dagger$. By rewriting in vector forms, it is easy to check that (4.2.51) is equivalent to

$$\int_{-\infty}^{\infty} < \psi_i(s), D_\theta f(q_i(s), 0) > ds \neq 0. \tag{4.2.52}$$

By the implicit function theorem

$$\xi_i^\infty(\theta) = M_i\, \theta + O(\theta^2), \tag{4.2.53}$$

where

$$M_i := \int_{-\infty}^{\infty} < \psi_i(s), D_\theta\, f(q_i(s), 0) > ds. \tag{4.2.54}$$

Now we are ready to construct the Lin's orbit $x_i^\pm(z)$ which are the perturbations of $q_i^\pm(z)$ and could be discontinuous on the cross sections Σ_i. To be precise, we look for the solutions of (4.1.9) which have the form

$$x_i^\pm(z) = q_i^\pm(z; \theta) + v_i^\pm(z). \tag{4.2.55}$$

Then plug into (4.1.9) and we have

$$\dot{v}_i^\pm = A_i^\pm(z; \theta) v_i^\pm + h_i^\pm(z, v_i^\pm, \theta), \tag{4.2.56}$$

where $A_i^{\pm}(z;\theta) := D_x f(q_i^{\pm}(z;\theta),\theta)$ and

$$h_i^{\pm}(z, v_i^{\pm}, \theta) = f(x,\theta) - f(q_i^{\pm},\theta) - A_i^{\pm}(z;\theta)v_i^{\pm}. \tag{4.2.57}$$

Moreover we also want the solutions $x_i^{\pm}(0)$ be on the cross sections Σ_i which implies that $v_i^{\pm}(0) \in \Sigma_i$ since Lemma 4.4 indicates $q_i^{\pm}(0) \in \Sigma_i$.

Let ω_1, $\omega_2 \in \mathbb{R}^+$. Define

$$\begin{aligned} X_\omega \;\; := \;\; & \Big\{ v = (v_1^-, v_1^+, v_2^-, v_2^+) \,\Big|\, v_1^- \in C^0([-\omega_1, 0], \mathbb{R}^4), v_1^+ \in C^0([0,\omega_2], \mathbb{R}^4), \\ & v_2^- \in C^0([-\omega_2, 0], \mathbb{R}^4), v_1^+ \in C^0([0,\omega_1], \mathbb{R}^4) \Big\}. \end{aligned} \tag{4.2.58}$$

Here is the main theorem in this section:

Theorem 4.1. *There exists constants $\omega_0 > 0$ and $c_0 > 0$. Then for any $\omega_i > \omega_0$, $|\theta| < c_0$, there exists a unique solution $v = v(\omega,\theta) \in X_\omega$ of (4.2.56) satisfying*

- *(L1) $v_i^{\pm}(0) \in Y_i \oplus Z_i$,*
- *(L2) $v_i^+(0) - v_i^-(0) \in Z_i$,*
- *(L3) $v_2^+(\omega_1) - v_1^-(-\omega_1) = d_1$ and $v_1^+(\omega_2) - v_2^-(-\omega_2) = d_2$ where*

$$d_1 \;\; := \;\; q_1^-(\theta)(-\omega_1) - q_2^+(\theta)(\omega_1), \tag{4.2.59}$$

$$d_2 \;\; := \;\; q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2). \tag{4.2.60}$$

From Lemma 4.4 and Theorem 4.1 we can construct Lin's orbit.

Corollary 2. (Lin's Orbit) *There exists constants $\omega_0 > 0$ and $c_0 > 0$. Then for any $\omega_i > \omega_0$, $|\theta| < c_0$, there exists a unique solution $x = x(\omega, \theta) \in X_\omega$ of (4.1.9) satisfying*

- $x_i^{\pm}(0) \in Y_i \oplus Z_i$,
- $x_i^{+}(0) - x_i^{-}(0) \in Z_i$,
- $x_1^{-}(-\omega_1) = x_2^{+}(\omega_1)$ *and* $x_1^{+}(\omega_2) = x_2^{-}(-\omega_2)$.

Remark 4.6. *Since the equation (4.1.9) is S-reversible, $\bar{x}(z) := S\, x(\omega, \theta)(-z)$ is also a solution of (4.1.9) satisfying*

- $\bar{x}_i^{\pm}(0) \in S\, Y_i \oplus S\, Z_i$,
- $\bar{x}_i^{+}(0) - \bar{x}_i^{-}(0) \in S\, Z_i$,
- $\bar{x}_1^{-}(-\omega_1) = \bar{x}_2^{+}(\omega_1)$ *and* $\bar{x}_1^{+}(\omega_2) = \bar{x}_2^{-}(-\omega_2)$.

Note that $S\, Y_1 = Y_2$ and $S\, Z_1 = Z_2$. So $\bar{x}(\omega, \theta)$ is another Lin's orbit corresponding to the same θ and ω. By the uniqueness of Lin's orbit from Corollary 2 we conclude that the orbit of $x(\omega, \theta)$ is the same with that of $\bar{x}(\omega, \theta)$. So

$$x_1^{-}(z) = S\, x_2^{+}(-z), \tag{4.2.61}$$

$$x_1^{+}(z) = S\, x_2^{-}(-z). \tag{4.2.62}$$

From the definition of symmetry of S-reversible system we obtain that Lin's orbit $x(\omega, \theta)$ is symmetric. So we have the following lemma:

Lemma 4.5. (Symmetry) *Lin's orbit $x(\omega, \theta)$ in Corollary 2 is symmetric and*

$$\xi_1 = S\, \xi_2, \tag{4.2.63}$$

where

$$\xi_1(0) \quad := \quad x_1^-(0) - x_1^+(0), \tag{4.2.64}$$

$$\xi_2(0) \quad := \quad x_2^-(0) - x_2^+(0). \tag{4.2.65}$$

Remark 4.7. ξ_i *are the jumps of Lin's orbit. In order to construct the homoclinic orbit we need to merge the jumps* $\xi_i = 0$*. From (4.2.63) it is sufficient to make* ξ_1 *to be zero.*

Now we will illustrate the main steps of proof for Theorem 4.1. Note that (4.2.56) is a nonlinear differential equation. We will prove this theorem in several steps. Before dealing with this nonlinear equation, we consider the linear version first,

$$\dot{v}_i^{\pm} = A_i^{\pm}(z;\theta)v_i^{\pm} + g_i^{\pm}(z), \tag{4.2.66}$$

where $g := (g_1^-, g_1^+, g_2^-, g_2^+) \in X_\omega$. Let $T(\cdot,\cdot;\theta)$ be the fundamental transition operator of the homogeneous equation (4.2.66)

$$\dot{v}_i^{\pm} = A_i^{\pm}(z;\theta)v_i^{\pm}. \tag{4.2.67}$$

As before these equations also have exponential dichotomies on $\mathbb{R}^{\pm}$ with projections $P_i^{\pm}(\cdot;\theta)$. This means that there exists constants $C > 0$ and $\nu > 0$ such that

$$\left\|T_i(t,s;\theta)\,P_i^{+}(s;\theta)\right\| \;\leq\; C\,e^{-\nu(t-s)}, \quad 0 \leq s \leq t, \tag{4.2.68}$$

$$\left\|T_i(t,s;\theta)\,Q_i^{+}(s;\theta)\right\| \;\leq\; C\,e^{-\nu(s-t)}, \quad 0 \leq t \leq s, \tag{4.2.69}$$

$$\left\|T_i(t,s;\theta)\,P_i^{-}(s;\theta)\right\| \;\leq\; C\,e^{-\nu(t-s)}, \quad t \leq s \leq 0, \tag{4.2.70}$$

$$\left\|T_i(t,s;\theta)\,Q_i^{-}(s;\theta)\right\| \;\leq\; C\,e^{-\nu(s-t)}, \quad t \leq s \leq 0, \tag{4.2.71}$$

where $Q_i\pm := I - P_i^{\pm}$.

The first step is to construct the 'Lin's orbit' for the linear inhomogeneous equation (4.2.66). Consider the case $i = 1$ first. For the brevity of the notation we omit θ in $T_1(z,s;\theta)$ and $P_1^{\pm}(z;\theta)$ and $Q_1^{\pm}(z;\theta)$ in the following Lemma.

Lemma 4.6. *For any $\omega_i \in \mathbb{R}^+$, $a_1 = (a_1^-, a_1^+) \in R\,P_1^-(-\omega_1) \times R\,Q_1^+(\omega_2)$ and $g_1 = (g_1^-, g_1^+) \in C^0([-\omega_1,0],\mathbb{R}^4)\times C^0([0,\omega_2],\mathbb{R}^4)$, there exists a solution $v_1 = v_1(z;\omega_1,\omega_2,a_1^{\pm},g_1^{\pm},\theta)$ with $v_1^- \in C^0([-\omega_1,0],\mathbb{R}^4)$ and $v_1^+ \in C^0([0,\omega_2],\mathbb{R}^4)$ satisfying (4.2.66)*

$$\dot{v_1}^{\pm} = A_1^{\pm}(z;\theta)v_1^{\pm} + g_1^{\pm}(z), \tag{4.2.72}$$

with the jump conditions

$$v_1^{\pm}(0) \;\in\; Z_1 \oplus Y_1, \tag{4.2.73}$$

$$v_1^{+}(0) - v_1^{-}(0) \;\in\; Z_1. \tag{4.2.74}$$

Moreover

$$\begin{aligned}
\|v_1\| &\leq C(|a_1^-| + |a_1^+| + \|g_1\|), && (4.2.75)\\
|Q_1^-(-\omega_1)v_1^-(-\omega_1)| + |P_1^+(\omega_2)v_1^+(\omega_2)| &\leq C(\|g_1\| +, \\
&\quad + e^{-2\nu(\omega_1+\omega_2)}(|a_1^-| + |a_1^+|)). && (4.2.76)
\end{aligned}$$

where $\|\cdot\|$ *denotes the sup norm.*

Proof. From the variation of constant formula we can obtain

$$\begin{aligned}
v_1^-(z) &= \int_{-\omega_1}^{z} T_1(z,s)P_1^-(s)g_1^-(s)ds - \int_{z}^{0} T_1(z,s)Q_1^-(s)g_1^-(s)ds + \\
&\quad + T(z,0)b_1^- + T_1(z,-\omega_1)a_1^-, \qquad -\omega_1 < z \leq 0; && (4.2.77)\\
v_1^+(z) &= \int_{0}^{z} T_1(z,s)P_i^+(s)g_1^+(s)ds - \int_{z}^{\omega_2} T_1(z,s)Q_1^+(s)g_1^+(s)ds + \\
&\quad + T(z,0)b_1^+ + T_1(z,\omega_2)a_1^+, \qquad 0 \leq z \leq \omega_2, && (4.2.78)
\end{aligned}$$

where $b_1^- = R\,Q_1^-(0)$, $b_1^+ = R\,P_1^+(0)$, $a_1^- = R\,P_1^-(-\omega_1)$ and $a_1^+ = R\,Q_1^+(\omega_2)$. Then

$$\begin{aligned}
v_1^-(0) &= \underbrace{b_1^-}_{\in Y_1^u} + \underbrace{T_1(0,-\omega_1)a_1^-}_{\in Y_1^s \oplus Z_1} + \underbrace{P_1^-(0)\int_{-\omega_1}^{0} T_1(0,s)g_1^-(s)ds,}_{\in Y_1^s \oplus Z_1}\\
&:= \underbrace{b_1^-}_{Y_1^u} + \underbrace{H_s(\omega_1, a_1^-, g_1^-, \theta)}_{\in Y_1^s} + \underbrace{H_z(\omega_1, a_1^-, g_1^-, \theta)}_{\in Z_1}. && (4.2.79)\\
v_1^+(0) &= \underbrace{b_1^+}_{\in Y_1^s} + \underbrace{T_1(0,\omega_2)a_1^+}_{\in Y_1^u \oplus Z_1} - \underbrace{Q_1^+(0)\int_{0}^{\omega_2} T_1(0,s)g_1^+(s)ds,}_{\in Y_1^u \oplus Z_1}\\
&:= \underbrace{b_1^+}_{Y_1^s} + \underbrace{G_s(\omega_2, a_1^+, g_1^+, \theta)}_{\in Y_1^u} + \underbrace{G_z(\omega_2, a_1^+, g_1^+, \theta)}_{\in Z_1}. && (4.2.80)
\end{aligned}$$

From Condition (4.2.74) we can solve that

$$
\begin{aligned}
b_1^- &= G_s(\omega_2, a_1^-, g_1^+, \theta), && (4.2.81)\\
b_1^+ &= H_s(\omega_1, a_1^+, g_1^-, \theta). && (4.2.82)
\end{aligned}
$$

So combining with (4.2.77) and (4.2.78) we conclude that

$$
v_1^{\pm}(z) = v_1^{\pm}(\omega_1, \omega_2, g_1^-, g_1^+, \theta)(z), \tag{4.2.83}
$$

and it is easy to see that $v_1^- \in C^0([-\omega_1, 0], \mathbb{R}^4)$ and $v_1^+ \in C^0([0, \omega_2], \mathbb{R}^4)$. From (4.2.81) and the definition of G_s we obtain

$$
\begin{aligned}
|b_1^-| &= |G_s(\omega_2, a_1^-, g_1^+, \theta)|,\\
&\leq C(\left|T_1(0, \omega_2)a_1^+\right| + \int_0^{\omega_2} \left|Q_1^+(0)T_1(0, s)g_1^+(s)\right| ds). && (4.2.84)
\end{aligned}
$$

Using the exponential dichotomies properties (4.2.68), (4.2.69) it yields

$$
\begin{aligned}
|b_1^-| &\leq Ce^{-\nu\omega_2}|a_1^+| + C\|g_1^+\| \int_0^{\omega_2} e^{-\nu s} ds,\\
&\leq C(e^{-\nu\omega_2}|a_1^+| + \|g_1^+\|). && (4.2.85)
\end{aligned}
$$

From (4.2.77) and using the exponential dichotomies properties (4.2.70), (4.2.71) we obtain

$$
\begin{aligned}
\|v_1^-\| &\leq C\|g_1^-\| \int_{-\omega_1}^{z} e^{-\nu(z-s)} ds + \|g_1^-\| \int_z^0 e^{\nu(z-s)} ds+,\\
&\qquad +C|b_1^-| + C|a_1^-|,\\
&\leq C(\|g_1^-\| + |b_1^-| + |a_1^-|). && (4.2.86)
\end{aligned}
$$

Combining (4.2.85) and (4.2.86) it yields

$$\|v_1^-\| \leq C(e^{-\nu\omega_2}|a_1^+| + |a_1^-| + \|g_1^-\| + \|g_1^+\|). \tag{4.2.87}$$

Similarly we can also prove

$$\|v_1^+\| \leq C(e^{-\nu\omega_1}|a_1^-| + |a_1^+| + \|g_1^-\| + \|g_1^+\|). \tag{4.2.88}$$

Combining (4.2.87) and (4.2.88) we recover the estimate (4.2.95). From (4.2.77) we have

$$Q_1^-(-\omega_1)v_1^-(-\omega_1) = -\int_{-\omega_1}^0 T_1(0,s)Q_1^-(s)g_1^-(s)ds + b_1^-. \tag{4.2.89}$$

From (4.2.68) and (4.2.85) we obtain

$$\begin{aligned} |Q_1^-(-\omega_1)v_1^-(-\omega_1)| &\leq C\|g_1^-\|\int_{-\omega_1}^0 e^{\nu s}ds + e^{-\nu\omega_1}|b_1^-|, \\ &\leq C(e^{-\nu(\omega_1+\omega_2)}|a_1^+| + \|g_1^-\| + \|g_1^+\|) \end{aligned} \tag{4.2.90}$$

Similarly we can also prove that

$$|P_1^+(\omega_2)v_1^+(\omega_2)| \leq C(e^{-\nu(\omega_1+\omega_2)}|a_1^-| + \|g_1^-\| + \|g_1^+\|). \tag{4.2.91}$$

(4.2.90) and (4.2.91) recovers the estimate (4.2.76).

□

Remark 4.8. *During the proof of Lemma 4.6 it is easy to see that* $v_1^{\pm}(\omega_1, \omega_2, a_1^-, a_1^+, g_1^-, g_1^+, \theta)(z)$ *depends on* $(a_1^-, a_1^+, h_1^-, h_1^+)$ *linearly.*

Similarly we can also prove it for the case $i = 2$.

Lemma 4.7. *For any $\omega_i \in \mathbb{R}^+$, $a_2 = (a_2^-, a_2^+) \in R\,P_2^-(-\omega_2) \times R\,Q_2^+(\omega_1)$ and $g_2 = (g_2^-, g_2^+) \in C^0([-\omega_2, 0], \mathbb{R}^4) \times C^0([0, \omega_1], \mathbb{R}^4)$, there exists a solution $v_2 = v_2(z; \omega_1, \omega_2, a_2^\pm, g_2^\pm, \theta)$ with $v_2^- \in C^0([-\omega_2, 0], \mathbb{R}^4)$ and $v_2^+ \in C^0([0, \omega_1], \mathbb{R}^4)$ satisfying (4.2.66)*

$$\dot{v}_2^\pm = A_2^\pm(z;\theta)v_2^\pm + g_2(z), \tag{4.2.92}$$

with the jump conditions

$$\begin{aligned} v_2^\pm(0) &\in Z_2 \oplus Y_2, && (4.2.93) \\ v_2^+(0) - v_2^-(0) &\in Z_2. && (4.2.94) \end{aligned}$$

Moreover

$$\begin{aligned} \|v_2\| &\leq C(|a_2^-| + |a_2^+| + \|g_2\|), && (4.2.95) \\ |Q_2^-(-\omega_2)v_2^-(-\omega_2)| + |P_2^+(\omega_1)v_2^+(\omega_1)| &\leq C(\|g_2\|+, \\ & \quad +e^{-2\nu(\omega_1+\omega_2)}(|a_2^-| + |a_2^+|)). && (4.2.96) \end{aligned}$$

where $\|\cdot\|$ denotes the sup norm.

Remark 4.9. *Similarly during the proof of Lemma 4.7 it is easy to see that the solutions $v_2^\pm(\omega_1, \omega_2, a_2^-, a_2^+, g_2^-, g_2^+, \theta)(z)$ depends on $(a_2^-, a_2^+, h_2^-, h_2^+)$ linearly.*

The next step is to couple the two solutions

$$x_i(\omega_1, \omega_2)(z) := q_i^\pm(z;\theta) + v_i^\pm(\omega_1, \omega_2, a_i^-, a_i^+, g_i^-, g_i^+)(z). \tag{4.2.97}$$

near the equilibrium point q_1. For the preparation for the next step we need the following technical lemma [61].

Lemma 4.8. *Suppose the fixed point q_i has the leading stable eigenvalue $\lambda_i^s(\theta)$ and the leading unstable eigenvalue $\lambda_i^u(\theta)$. Then we introduce constants α_i^{ss}, α_i^s, α_i^u such that*

$$\{\lambda_i^{ss}(\theta)\} < \alpha_i^{ss} < \lambda_i^s(\theta) < \alpha_i^s < 0 < \alpha_i^u < \lambda_i^u(\theta) < \{\lambda_i^{uu}(\theta)\}, \tag{4.2.98}$$

where $\{\lambda^{ss}(\theta)\}$ and $\{\lambda_i^{uu}(\theta)\}$ denotes the strong stable and strong unstable eigenvalues of the fixed point respectively.There exists constants $\omega_0 > 0$, $c_0 > 0$ and $M > 0$ such that for any $\omega_i \geq \omega_0$ and $|\theta| < c_0$

$$\mathbb{R}^4 = R\,Q_1^+(\omega_2;\theta) \oplus R\,P_2^-(-\omega_2;\theta), \tag{4.2.99}$$

$$\mathbb{R}^4 = R\,Q_2^+(\omega_1;\theta) \oplus R\,P_1^-(-\omega_1;\theta). \tag{4.2.100}$$

Furthermore

$$\|\bar{Q}_{i,\omega_i}(\theta)\| \leq M, \tag{4.2.101}$$

$$\|\bar{P}_{i,\omega_i}(\theta)\| \leq M, \tag{4.2.102}$$

where $\bar{P}_{i,\omega_i}(\theta)$ denotes the projection on $R\,P_i^-(-\omega_i;\theta)$ and $\bar{Q}_{i,\omega_i}(\theta) := I - \bar{P}_{i,\omega_i}(\theta)$.

$$\|\bar{Q}_{i,\omega} - Q_i\| \leq Ce^{-\min\{-\alpha_i^s,\alpha_i^u\}\omega_i}. \tag{4.2.103}$$

$$\|\bar{P}_{i,\omega} - P_i\| \leq Ce^{-\min\{-\alpha_i^s,\alpha_i^u\}\omega_i}. \tag{4.2.104}$$

Now we are ready to couple the two solutions x_i :

Lemma 4.9. (Lin's orbit for linear equation) *There exists constants $\omega_0 > 0$ and $c_0 > 0$. Then for any $\omega_i > \omega_0$, $|\theta| < c_0$, $g \in X_w$ and $d = (d_1, d_2) \in \mathbb{R}^2$, then there exist a unique solution $v(z) = v(\omega_1, \omega_2, d, g, \theta)(z)$ with $v = (v_1^-, v_1^+, v_2^-, v_2^+) \in X_\omega$ satisfying the equation (4.2.66),*

$$\dot{v_i}^{\pm} = A_i^{\pm}(z;\theta)v_i^{\pm} + g_i^{\pm}(z),$$

with the jump conditions

$$v_i^{\pm}(0) \in Z_i \oplus Y_i, \tag{4.2.105}$$

$$v_i^{+}(0) - v_i^{-}(0) \in Z_i, \tag{4.2.106}$$

and the boundary condition

$$v_1^{+}(\omega_2) - v_2^{-}(-\omega_2) = d_1, \tag{4.2.107}$$

$$v_2^{+}(\omega_1) - v_1^{-}(-\omega_1) = d_2. \tag{4.2.108}$$

And

$$\|v_i^{\pm}\| \leq C(|d| + \|g\|). \tag{4.2.109}$$

Proof. From Lemma 4.6 and Lemma 4.7 we have the solution $v_i^{\pm}(z) = v(\omega_1, \omega_2, a_i^{\pm}, g_i^{\pm}, \theta)(z)$ of (4.2.66) denoted by $\bar{v}_i^{\pm}$. In particular Lemma 4.6 and Lemma 4.7 also indicate that $\bar{v}_i^{\pm}$

satisfy the two jump conditions (4.2.105) and (4.2.106). From (4.2.107) we can obtain

$$\begin{aligned} a_1^+ - a_2^- &= d_1 + \underbrace{Q_2^-(-\omega_2;\theta)\bar{v}_2^-(-\omega_2)}_{:=b_2^- \in R\,Q_2^-(-\omega_2;\theta)} - \underbrace{P_1^+(\omega_2;\theta)\bar{v}_1^+(\omega_2)}_{:=b_1^+ \in R\,P_1^+(\omega_2;\theta)}, && (4.2.110) \\ a_1^- - a_2^+ &= d_2 + \underbrace{P_2^+(\omega_1;\theta)\bar{v}_2(\omega_1)}_{:=b_2^+ \in R\,P_2^+(\omega_1;\theta)} - \underbrace{Q_1^-(-\omega_1;\theta)\bar{v}_1^-(-\omega_1)}_{:=b_1^- \in R\,Q_1^-(-\omega_1;\theta)}. && (4.2.111) \end{aligned}$$

Applying projections $P_{i,\omega_i}(\theta)$ and $Q_{i,\omega_i}(\theta)$ defined in Lemma 4.8 we obtain

$$\begin{aligned} a_1^- &= P_{1,\omega_1}(d_2 + b_2^+ - b_1^-), && (4.2.112) \\ a_1^+ &= Q_{2,\omega_2}(d_1 + b_2^- - b_1^+), && (4.2.113) \\ a_2^- &= P_{2,\omega_2}(-d_1 - b_2^- + b_1^+), && (4.2.114) \\ a_2^+ &= Q_{1,\omega_1}(-d_2 - b_2^+ + b_1^-). && (4.2.115) \end{aligned}$$

From the remarks 4.8, 4.9 we know that $b_i^\pm$ are linear in $(a_i^-, a_i^+, g_i^-, g_i^+)$. Then we may write

$$a = L_1(\theta)a + L_2(\theta)g + L_3(\theta)d, \qquad (4.2.116)$$

where $a = (a_1^-, a_1^+, a_2^-, a_2^+)$, $g = (g_1^-, g_1^+, g_2^-, g_2^+)$ and $d = (d_2, d_1, d_1, d_2)$. Here $L_i(\theta)$ are linear operators smoothly depending on θ. From (4.2.76) we know that if ω_i be sufficiently large then $\|L_1(\theta)\| < 1$. So the operator $I - L_1(\theta)$ is invertible and we could uniquely solve (4.2.116) and $a = a(g,\theta,d)$. Hence $v_i^\pm(z) = \bar{v}_i^\pm(\omega_1,\omega_2,a_i^\pm(\omega_1,\omega_2,g,d,\theta),g,\theta)(z)$ solve the equation (4.2.66) and it is clearly in X_ω. We denote these solutions by $\hat{v}_i^\pm$. The estimate (4.2.109) can be obtained from (4.2.116) and (4.2.95).

□

Now we are ready to construct Lin's orbits for the nonlinear equation (4.2.56)

$$\dot{v}_i^{\pm} = A_i^{\pm}(z;\theta)v_i^{\pm} + h_i^{\pm}(z, v_i^{\pm}, \theta),$$

where $A_i^{\pm}(z;\theta) := D_x f(q_i^{\pm}(z;\theta), \theta)$ and

$$h_i^{\pm}(z, v_i^{\pm}, \theta) = f(x, \theta) - f(q_i^{\pm}, \theta) - A_i^{\pm}(z;\theta)v_i^{\pm}.$$

We also require the solutions $v_i^{\pm}$ satisfy the conditions (L1)-(L3).

Proof of Theorem 4.1. From Lemma 4.9 it follows that there exist solutions

$$\begin{aligned} v &= \hat{v}(\omega, h(v,\theta), d(\omega,\theta)), && (4.2.117) \\ &:= \mathcal{S}(v, \omega, \theta). && (4.2.118) \end{aligned}$$

satisfying the equations (4.2.56) and the conditions (L1)-(L2). Here $v = (v_1^-, v_1^+, v_2^-, v_2^+)$, $h = (h_1^-, h_1^+, h_2^-, h_2^+)$, $\omega = (\omega_1, \omega_2)$, and $d = (d_1, d_2)$. This is a fixed point problem and it can be solved by contraction mapping theorem. Denote the RHS of (4.2.117) by $\mathcal{S}$ and we will show that $\mathcal{S}(v, \omega, \theta)$ is a contraction map on v from $B_r \to B_r$ in the space X_ω. Here B_r denotes the ball centered at 0 with radius r.

From the definition (4.2.57) of $h(v,\theta)$ it follows that $|h(v,\theta)| \leq C(v^2 + |\theta| + v|\theta|)$, $|D_v h(z,v,\theta)| \leq C(v + |\theta|)$ for $|v| \leq r$ and $|\theta| \leq c_0$, then combining with (4.2.109) it yields

$$\begin{aligned} \|\mathcal{S}(v,h,\theta)\|_\omega &\leq C\left(\|h(v,\theta)\|_\omega + |d|\right), && (4.2.119) \\ &\leq C\left(\|v\|_\omega^2 + |d| + |\theta|\right), && (4.2.120) \\ &\leq C\left(r^2 + c_0 + |d|\right). && (4.2.121) \end{aligned}$$

From the definition (4.2.59) (4.2.60) of d and the conditions (Q2) and (Q3) of $q_i^{\pm}$ we can choose ω_i be sufficiently large so that $|d| \leq c_0$. So

$$\|\mathcal{S}(v,h,\theta)\|_\omega \leq C(r^2 + c_0). \tag{4.2.122}$$

We can also choose r be small enough and $c_0 < r$ so that

$$\|\mathcal{S}(v,h,\theta)\|_\omega \leq r. \tag{4.2.123}$$

Moreover, from (4.3.39) and the definition (4.2.57) of h we obtain

$$\begin{aligned} \|D_v\mathcal{S}(v,h,\theta)\|_\omega &\leq C(\|D_v\, h(v,\theta)\|_\omega), && (4.2.124) \\ &\leq C(\|v\|_\omega + |d|), && (4.2.125) \\ &\leq C\, r. && (4.2.126) \end{aligned}$$

We choose r small enough such that

$$\|D_v\mathcal{S}(v,h,\theta)\|_\omega < 1. \tag{4.2.127}$$

Applying the Banach fixed point theorem we can deduce the existence and uniqueness of the solution $v(\omega, \theta)$. $\square$

4.3 Estimates for the Jump

By Corollary 2 we already construct 'Lin's orbit'. However, 'Lin's orbit' is actually not the homoclinic orbit that we are looking for since 'Lin's orbit' has the jumps ξ_i at Σ_i. If we could merge those jumps, i.e., $\xi_i = 0$, then we can construct the expected homoclinic orbit. Moreover, from remark 4.7 it is sufficient to merge only one of those jumps to zero.

Theorem 4.2. *Under the assumptions (H) and (S')-(1.1.4) there exists constants $\omega_0 > 0$ and $c_0 > 0$. Then for any $\omega_i > \omega_0$, $|\theta| < c_0$, there exists a unique solution $v = v(\omega, \theta) \in X_\omega$ of (4.2.56) satisfying (L1)-(L3). The jump ξ_i can be written as*

$$\begin{aligned} \xi_1(\omega, \theta) &= M_1\, \theta + c^s(\theta) e^{2\omega_1 \lambda_1^s} + c^u(\theta) e^{-2\omega_2 \lambda_2^u} + \\ &\qquad + o(e^{-2\omega_2 \lambda_2^s}) + o(e^{-2\omega_1 \lambda_1^u}). \end{aligned} \tag{4.3.1}$$

where $\xi_1 =< \psi_1, v_1^+(0) - v_1^-(0) >$ and $M_1 = \int_{\mathbb{R}} < \psi_1(s), D_\theta f(q_1(s), 0) > ds \neq 0$. Here λ_i^s and λ_i^u are the leading stable and unstable eigenvalues at equilibrium point p_i. The functions $c^u(\theta)$ and $c^s(\theta)$ are smooth in θ.

Proof. From the definition of ξ_1 then we have

$$\begin{aligned}\xi_1 &= <\psi_1, Q_1^+(0;\theta)x_1^+(0) - P_1^-(0;\theta)x_1^-(0)>, \\ &= \underbrace{<\psi_1, Q_1^+(0;\theta)q_1^+(\theta)(0) - P_1^-(0;\theta)q_2^-(\theta)(0)>}_{=\xi_1^\infty} + \\ &\qquad + \underbrace{<\psi_1, Q_1^+(0;\theta)v_1^+(\theta)(0) - P_1^-(0;\theta)v_2^-(\theta)(0)>}_{:=\xi_1^\omega}. \end{aligned} \tag{4.3.2}$$

From (4.2.53) we have

$$\xi_1^\infty = M_1\,\theta. \tag{4.3.3}$$

Addressing the first term of ξ_1^ω, from (4.2.80) we obtain

$$v_1^+(0) = T_1(0,\omega_2;\theta)a_1^+ - \int_0^{\omega_2} T_1(0,s;\theta)Q_1^+(s;\theta)h_1^+(s, v_1^+(s;\omega,\theta),\theta)ds. \tag{4.3.4}$$

From (4.2.110) and (4.2.60) it yields

$$\begin{aligned} a_1^+ - a_2^- &= q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2) + \\ &\quad + Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2). \end{aligned} \tag{4.3.5}$$

So

$$\begin{aligned} a_1^+ &= \bar{Q}_{2,\omega_2}(\theta)\left(q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2) + \right. \\ &\quad \left. + Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2)\right). \end{aligned} \tag{4.3.6}$$

Plugging into (4.3.4) it yields

$$\begin{aligned} Q_1^+(0;\theta)v_1^+(0) &= T_1(0,\omega_2;\theta)Q_1^+(\omega_2;\theta)\bar{Q}_{2,\omega_2}(\theta)\left(q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2) + \right. \\ &\quad \left. + Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2)\right) + \\ &\quad - \int_0^{\omega_2} T_1(0,s;\theta)Q_1^+(s;\theta)h_1^+(s, v_1^+(s;\omega,\theta),\theta)ds \end{aligned} \tag{4.3.7}$$

Similarly we can also compute the expression for $P_1^-(0;\theta)v_1^-(0)$

$$\begin{aligned} P_1^-(0;\theta)v_1^-(0) &= T_1(0,-\omega_1;\theta)P_1^-(-\omega_1;\theta)\bar{P}_{1,\omega_1}(\theta)\left(q_1^-(\theta)(-\omega_1) - q_2^+(\theta)(\omega_1) + \right. \\ &\quad \left. + P_2^+(\omega_1;\theta)v_2^+(\omega_1) - Q_1^-(-\omega_1;\theta)v_1^-(-\omega_1)\right) + \\ &\quad + \int_{-\omega_1}^{0} T_1(0,s;\theta)P_1^-(s;\theta)h_1^-(s, v_1^-(s;\omega,\theta),\theta)ds. \end{aligned} \tag{4.3.8}$$

From (4.3.7) and (4.3.8) we can obtain the expression for ξ_1^ω

$$\begin{aligned} \xi_1^\omega &= \underbrace{< T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1, \bar{Q}_{2,\omega_2}(\theta)\left(q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2)\right) >}_{H_1} + \\ &\quad + \underbrace{< T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1, \bar{Q}_{2,\omega_2}(\theta)\left(Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2)\right) >}_{H_2} + \\ &\quad - \underbrace{< \psi_1, \int_0^{\omega_2} T_1(0,s;\theta)Q_1^+(s;\theta)h_1^+(s, v_1^+(s;\omega,\theta),\theta)ds >}_{H_3} + \\ &\quad + \underbrace{< T_1^*(0,-\omega_1;\theta)P_1^{-,*}(0;\theta)\psi_1, \bar{P}_{1,\omega_1}(\theta)\left(q_1^-(\theta)(-\omega_1) - q_2^+(\theta)(\omega_1)\right) >}_{H_4} + \\ &\quad + \underbrace{< T_1^*(0,-\omega_1;\theta)P_1^{-,*}(0;\theta)\psi_1, \bar{P}_{1,\omega_1}(\theta)\left(P_2^+(\omega_1;\theta)v_2^+(\omega_1) - Q_1^-(-\omega_1;\theta)v_1^-(-\omega_1)\right) >}_{H_5} + \\ &\quad + \underbrace{< \psi_1, \int_{-\omega_1}^{0} T_1(0,s;\theta)P_1^-(s;\theta)h_1^-(s, v_1^-(s;\omega,\theta),\theta)ds >}_{H_6}, \end{aligned} \tag{4.3.9}$$

where $*$ denotes the adjoint operator with respect to the inner product $< \cdot, \cdot >$. In the following we need to analyze the estimate for each term of (4.3.9). We will show later that H_1 and H_4 account for the leading order of ξ_1^ω and the rest terms of ξ_1^ω are higher order terms. We will show the proof of these estimates later. From (4.3.31), (4.3.30) we have

$$
\begin{aligned}
T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1 &= \Psi_{p_1}(\omega_2,0;\theta)\eta^+(\psi_1,\theta) + h.o.t. &\quad (4.3.10)\\
T_1^*(0,-\omega_1)P_1^{-,*}(0;\theta)\psi_1 &= \Psi_{p_2}(-\omega_1,0;\theta)\eta^-(\psi_1,\theta) + h.o.t. &\quad (4.3.11)\\
\bar{Q}_{2,\omega_2}(\theta)q_2^-(\theta)(-\omega_2) &= \Phi_{p_1}(-\omega_2,0;\theta)\eta^u(q_2^-(0),\theta) + h.o.t. &\quad (4.3.12)\\
\bar{P}_{1,\omega_1}(\theta)q_2^+(\theta)(\omega_1) &= \Phi_{p_2}(\omega_1,0;\theta)\eta^s(q_2^+(0),\theta) + h.o.t. &\quad (4.3.13)
\end{aligned}
$$

where the $h.o.t$ denote the terms of higher order than $e^{-\lambda^u\omega}$. $\Psi_{p_i}(t,s;\theta)$ is the transition matrix of the equation $\dot{\psi} = -(D_x f(p_i,\theta))^*\psi$ and $\Phi_{p_i}(t,s;\theta)$ is the transition matrix of the equation $\dot{\phi} = D_x f(p_i,\theta)\phi$. $\eta^+(\psi_1,\theta)$ (resp. $\eta^-(\psi_1,\theta)$) is the eigenvector of $-(D_x f(p_1,\theta))^*$ (resp. $-D_x f(p_2,\theta)^*$) with respect to the eigenvalue $-\lambda^u(\theta)$ (resp. $-\lambda^s(\theta)$). Similarly $\eta^u(q_2^-(0),\theta)$ (resp. $\eta^s(q_2^-(0),\theta)$) is an eigenvector of $D_x f(p_1,\theta)$ (resp. $D_x f(p_2,\theta)$) with respect to the eigenvalue $\lambda^u(\theta)$ (resp. $\lambda^s(\theta)$). Then we have

$$\Psi_{p_1}(\omega_2,0;\theta)\eta^+(\psi_1,\theta) = G_\theta\, e^{-\Lambda^*\omega_2}\, G_\theta^{-1}\, \eta^+(\psi_1,\theta), \tag{4.3.14}$$

where G_θ is a nonsingular matrix and $-\Lambda^*$ is the block diagonalization of the $-(D_x f(p_1,\theta))^*$. We also have

$$\Phi_{p_1}(-\omega_2,0;\theta)\eta^u(q_2^-(0),\theta) = (G_\theta^*)^{-1}e^{-\Lambda\omega_2}\, G_\theta^*\, \eta^u(q_2^-(0),\theta), \tag{4.3.15}$$

where Λ is the block diagonalization of $D_x f(p_1,\theta)$. By construction we know that $G_\theta^{-1}\,\eta^+(\psi_1,\theta)$ and $G_\theta^*\,\eta^u(q_2^-(0),\theta)$ are in the stable space of $e^{-\Lambda^* z}$ and unstable space of $e^{\Lambda z}$ respectively. Then combining the results of (4.3.10), (4.3.12), (4.3.14) and (4.3.15) we may obtain the expression for the first term in H_1

$$
\begin{aligned}
& < T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1, \bar{Q}_{2,\omega_2}(\theta)q_2^-(\theta)(-\omega_2) > \\
= \; & < G_\theta\, e^{-\Lambda^*\omega_2}\, G_\theta^{-1}\,\eta^+(\psi_1,\theta), (G_\theta^*)^{-1} e^{-\Lambda\omega_2}\, G_\theta^*\,\eta^u(q_2^-(0),\theta) > +h.o.t., \\
= \; & < e^{-\Lambda^*\omega_2}\, G_\theta^{-1}\,\eta^+(\psi_1,\theta), e^{-\Lambda\omega_2}\, G_\theta^*\,\eta^u(q_2^-(0),\theta) > +h.o.t., \\
= \; & e^{-2\lambda_2^u(\theta)\omega_2} < G_\theta^{-1}\,\eta^+(\psi_1,\theta), G_\theta^*\,\eta^u(q_2^-(0),\theta) > +h.o.t., \\
= \; & e^{-2\lambda_2^u(\theta)\omega_2} \underbrace{< \eta_2^+(\psi,\theta), \eta_2^u(q_2^-(0),\theta) >}_{:=c^u(\theta)} +h.o.t.
\end{aligned}
\tag{4.3.16}
$$

From (4.3.16) and (4.3.34) we obtain that

$$
H_1 = e^{-2\lambda_2^u(\theta)\omega_2} c^u(\theta) + h.o.t. \tag{4.3.17}
$$

Similarly we can also compute the second term in H_4 and obtain

$$
H_4 = e^{2\lambda_1^s(\theta)\omega_1} c^s(\theta) + h.o.t. \tag{4.3.18}
$$

Combining (4.3.17) and (4.3.18) we can recover the estimate (4.3.1). $\square$

Before we calculate the estimates for H1-H6. We will introduce two important lemma. They are concerned with the leading order terms of an orbit of the system $\dot{x} = f(x,\theta)$ that approaches the equlibrium in the weak stable manifold.

Lemma 4.10. *Let $x = 0$ be an asymptotically stable fixed point of a C^2 vector field*

$$\dot{x} = f(x, \theta), \tag{4.3.19}$$

where $f(\cdot, \theta) : \mathbb{R}^k \to \mathbb{R}^k$, $\theta \in \mathbb{R}^p$. Suppose the spectrum of the fixed point be $\sigma(D_x f(0, \theta) = \lambda^s(\theta) \cup \lambda^{ss}(\theta))$. Then we introduce constants α^{ss}, α^s such that

$$\{\lambda^{ss}(\theta)\} < \alpha^{ss} < \lambda^s(\theta) < \alpha^s < 0, \tag{4.3.20}$$

where $\{\lambda^{ss}(\theta)\}$ denotes the strong stable eigenvalues of the fixed point. Let $E^s(\theta)$, $E^{ss}(\theta)$ be the stable, strong stable subspace of $A(\theta)$ respectively. Also $P_s(\theta)$ denotes the projection onto $E^s(\theta)$ along $E^{ss}(\theta)$. Then there exists a constant $\delta > 0$ such that for all solutions $x(\cdot)$ of (4.3.19) with $|x(0)| < \delta$ the limit

$$\eta(x(0), \theta) := \lim_{z \to \infty} \Phi(0, z; \theta) P_s(\theta) x(z) \tag{4.3.21}$$

exists, where $\Phi(0, z; \theta)$ is the transition matrix of

$$\dot{x} = D_x f(0, \theta) x, \tag{4.3.22}$$

from z to 0. Furthermore, there exists a constant C such that

$$|x(z) - \Phi(z, 0; \theta)\eta(x(0), \theta)| \leq C\, e^{-\min\{|\alpha^{ss}|, 2|\alpha^s|\} z}. \tag{4.3.23}$$

Remark 4.10. *In Lemma 4.10 we assume that* 0 *is an asymptotically stable fixed point. If* 0 *is a hyperbolic fixed point then Lemma 4.10 describes the behavior of solutions in the stable manifold. By reversing 'time' we can obtain a similar lemma for solutions in the unstable manifold.*

Remark 4.11. *Let the assumptions of Lemma 4.10 hold. Then $\eta(x, \theta) \neq 0$ if and only if $x \notin W_\theta^{ss}(0)$*

We will also need the following lemma that make the same assertion as previous lemma 4.10 for non-automomous perturbed linear system.

Lemma 4.11. *Let $x = 0$ be a hyperbolic fixed point of a C^2 vector field*

$$\dot{x} = A(\theta)x + B(z, \theta)x, \tag{4.3.24}$$

where $A(\theta)$, $B(\cdot, \theta) \in GL(k, \mathbb{R})$, $\theta \in \mathbb{R}^p$. Suppose the fixed point has the leading stable eigenvalue $\lambda^s(\theta)$ and the leading unstable eigenvalue $\lambda^u(\theta)$. Then we introduce constants α^{ss}, α^s, α^u such that

$$\{\lambda^{ss}(\theta)\} < \alpha^{ss} < \lambda^s(\theta) < \alpha^s < 0 < \alpha^u < \lambda^u(\theta), \tag{4.3.25}$$

where $\{\lambda^{ss}(\theta)\}$ denotes the strong stable eigenvalues of the fixed point. Let $E^s(\theta)$, $E^{ss}(\theta)$, $E^u(\theta)$ be the leading stable, strong stable, leading unstable subspace of $A(\theta)$ respectively. Suppose that there is a constant $\beta < 0$ such that $|B(z, \theta)| < e^{\beta z}$ and $|\alpha^s + \beta| > |\lambda^s(\theta)|$ for small θ. Also $P_s(\theta)$ denotes the projection onto $E^s(\theta)$ along $E^{ss}(\theta) \oplus E^u(\theta)$. Then there exists a constant $\delta > 0$ such that for all solutions $x(\cdot)$ of (4.3.24) with $|x(0)| < \delta$ the limit

$$\eta(x(0), \theta) := \lim_{z \to \infty} \Phi(0, z; \theta) P_s(\theta) x(z) \tag{4.3.26}$$

exists, where $\Phi(0,z;\theta)$ *is the transition matrix of*

$$\dot{x} = A(\theta)x, \tag{4.3.27}$$

from z *to 0. Furthermore, there exists a constant* C *such that*

$$|x(z) - \Phi(z,0;\theta)\eta(x(0),\theta)| \leq C\, e^{-\min\{|\alpha^{ss}|,|\alpha^{s}+\beta|\}z}. \tag{4.3.28}$$

Estimate of $T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1$

From the definition of ψ_1 we have that $T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1 = T_1^*(0,\omega_2;\theta)\psi_1$ solving $\dot{\psi} = -(A_1^{\pm}(z,\theta))^*\psi$. This equation has the exponential dichotomies on $\mathbb{R}_-$, $\mathbb{R}_+$ with projections $P_1^{\pm,*}(z;\theta)$, $Q_1^{\pm,*}(z;\theta)$. Also

$$\{w : \sup_{z\in\mathbb{R}_+} |\Psi(z,0;\theta)w| < \infty\} = (T_{q_1^+(\theta)(0)}W_\theta^s(p_1))^{\perp}, \tag{4.3.29}$$

and $R\,Q_1^{+,*}(z;\theta) = (T_{q_1^+(\theta)(0)}W_\theta^s(p_1))^{\perp}$. Note that $\sigma(-(D_x\, f(p_1,\theta))^*) = -\sigma(D_x\, f(p_1,\theta))$. Then by Lemma 4.11 we obtain

$$\begin{aligned} T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1 &= T_1^*(0,\omega_2;\theta)\psi_1, \\ &= \Psi_{p_1}(\omega_2,0;\theta)\eta^+(\psi_1,\theta) + O(e^{-\min\{|\alpha_1^{uu}|,2\alpha_1^u\}\omega_2}), \end{aligned} \tag{4.3.30}$$

where $\Psi_{p_1}(z,0;\theta)$ is the transition matrix of the equation $\dot{\psi} = -(D_x\, f(p_1,\theta))^*\psi$ and $\eta^+(\psi_1,\theta)$

is defined in (4.3.26). Similarly we also have

$$\begin{aligned} T_1^*(0,-\omega_1;\theta)P_1^{-,*}(0;\theta)\psi_1 &= T_1^*(0,-\omega_1;\theta)\psi_1, \\ &= \Psi_{p_2}(-\omega_1,0;\theta)\eta^-(\psi_1,\theta) + O(e^{-\min\{|\alpha_2^{ss}|,2\alpha_2^s\}\omega_1}), \end{aligned} \tag{4.3.31}$$

where $\Psi_{p_2}(z,0;\theta)$ is the transition matrix of the equation $\dot{\psi} = D_x\, f(p_2,\theta)\psi$ and $\eta^+(\psi_1,\theta)$ is defined in (4.3.26).

Estimate of $\bar{Q}_{2,\omega_2}(\theta)\left(q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2)\right)$

From Lemma 4.10 we obtain

$$\begin{aligned} q_1^+(\theta)(z) &= \Phi_{p_1}(z,0;\theta)\eta_1^s(q_1^+(\theta)(0),\theta) + \\ &\quad +O(e^{-\min\{|\alpha_1^{ss}|,2|\alpha_1^s|\}z}), \end{aligned} \tag{4.3.32}$$

$$\begin{aligned} q_2^-(\theta)(-z) &= \Phi_{p_1}(-z,0;\theta)\eta_1^s(q_2^-(\theta)(0),\theta) + \\ &\quad +O(e^{-\min\{|\alpha_1^{uu}|,2|\alpha_1^u|\}z}), \end{aligned} \tag{4.3.33}$$

where $\Phi_{p_i}(\theta,\cdot,\cdot)$ is the transition matrix of the equation $\dot{x} = D_x\, f(p_i,\theta)x$ and $\eta_1^s(q_1^+(\theta)(0),\theta)$ (resp. $\eta_1^u(q_2^-(\theta)(0),\theta)$)is in the leading unstable subspace of $D_x\, f(p_1,\theta)$. Combining the results of (4.3.32), (4.3.33) and (4.2.103) it yields

$$\bar{Q}_{2,\omega_2}(q_1^+(\theta)(\omega_2)) = O(e^{-(\min\{-\alpha_1^s,\alpha_1^u\}+\min\{|\alpha_1^{ss}|,2|\alpha_1^s|\})\omega_2}), \tag{4.3.34}$$

$$\begin{aligned} \bar{Q}_{2,\omega_2}(q_2^-(\theta)(-\omega_2)) &= \Phi_{p_1}(-\omega_2,0;\theta)\eta_1^u(q_2^-(\theta)(0),\theta) + \\ &\quad +O(e^{-(\min\{-\alpha_1^s,\alpha_1^u\}+\min\{|\alpha_1^{ss}|,2|\alpha_1^s|\})\omega_2}). \end{aligned} \tag{4.3.35}$$

Similarly we also have

$$\begin{aligned}\bar{P}_{1,\omega_1}(q_2^+(\theta)(\omega_1)) &= \Phi_{p_2}(-\omega_1, 0;\theta)\eta_2^s(q_2^+(\theta)(0),\theta) + \\ &\qquad +O(e^{-(\min\{-\alpha_2^s,\alpha_2^u\}+\min\{|\alpha_2^{ss}|,2|\alpha_2^s|\})\omega_1}), \qquad (4.3.36)\\ \bar{P}_{1,\omega_1}(q_1^-(\theta)(-\omega_1)) &= O(e^{-(\min\{-\alpha_2^s,\alpha_2^u\}+\min\{|\alpha_2^{ss}|,2|\alpha_2^s|\})\omega_1}). \qquad (4.3.37)\end{aligned}$$

Estimate of $\bar{Q}_{2,\omega_2}(\theta)\left(Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2)\right)$

From (4.2.76), (4.2.96) and (4.2.116) we obtain

$$\begin{aligned}|Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2)| + |P_1^+(\omega_2;\theta)v_1^+(\omega_2)| &\leq C\left(e^{-2\min\{\alpha_1^u,-\alpha_1^s\}\omega_2}|a| + \right.\\ &\qquad \left. + \|h_1^+\| + \|h_2^-\|\right),\\ &\leq C\left(e^{-2\min\{\alpha_1^u,-\alpha_1^s\}\omega_2}\|d\| + \right.\\ &\qquad \left. + \|h_1^+\| + \|h_2^-\|\right).\end{aligned} \qquad (4.3.38)$$

From the definition we know that $h_i^\pm = O(\|v_i^\pm\|^2)$ and hence for $\|v\|$ small enough, from (4.2.109) we obtain

$$\|v_i^\pm\| \leq C\|d_i\|. \qquad (4.3.39)$$

Using (4.3.32) and (4.3.33) we have

$$\|d_i\| = O(e^{-\min\{-\alpha_1^s,\alpha_1^u\}\omega_i}). \qquad (4.3.40)$$

So

$$\|v_i^{\pm}\| = O(e^{-\min\{-\alpha_1^s,\alpha_1^u\}\omega_i}), \tag{4.3.41}$$

and

$$\|h_i^{\pm}\| = O(e^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_i}). \tag{4.3.42}$$

Combining with (4.3.38) we obtain

$$|Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2)| + |P_1^+(\omega_2;\theta)v_1^+(\omega_2)| = O(e^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_i}). \tag{4.3.43}$$

Applying (4.2.101) it yield

$$\left|\bar{Q}_{2,\omega_2}(\theta)\left(Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2)\right)\right| = O(e^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_i}). \tag{4.3.44}$$

Similarly we could also have

$$\left|\bar{P}_{1,\omega_1}(\theta)\left(P_2^+(\omega_1;\theta)v_2^+(\omega_1) - Q_1^-(-\omega_1;\theta)v_1^-(-\omega_1)\right)\right| = O(e^{-2\min\{-\alpha_2^s,\alpha_2^u\}\omega_i}). \tag{4.3.45}$$

Estimate of $\int_0^{\omega_2} T_1(0,s;\theta)Q_1^+(s;\theta)h_1^+(s,v_1^+(s;\omega,\theta),\theta)ds$

Decompose v_1^+ into $v_1^+ = v_1^{+,u} + v_1^{+,s}$, where

$$v_1^{+,s} := P_1^+(z;\theta)v_1^+(z), \tag{4.3.46}$$

$$v_1^{+,u} := Q_1^+(z;\theta)v_1^+(z). \tag{4.3.47}$$

Following the similar proof in, we have

$$|Q_1^+(z;\theta)h_1^+(z,v_1^+(z;\omega,\theta))| \leq C|v_1^{+,u}(z)|\left(|v_1^{+,s}(z)|+|v_1^{+,u}(z)|\right). \tag{4.3.48}$$

Exploiting the properties of exponential dichotomies and (4.3.48) it yields

$$\begin{aligned}
&\overbrace{\left|\int_0^{\omega_1} T_1(0,s;\theta)Q_1^+(s;\theta)h_1^+(s,v_1^+(s),\theta)ds\right|}^{:=H_7} \\
&\leq C\omega_2 e^{-\alpha_1^u\omega_2}\sup_{s\in[0,\omega_2]}\{e^{\alpha_1^u(\omega_2-s)}\left|Q_1^+(z;\theta)h_1^+(z,v_1^+(z;\omega,\theta))\right|\}, \\
&\leq C\omega_2 e^{-\alpha_1^u\omega_2}\|v_1^+\|\sup_{s\in[0,\omega_2]}\{e^{\alpha_1^u(\omega_2-s)}|v_1^{+,u}(s)|\}.
\end{aligned} \tag{4.3.49}$$

Choose $\delta<0$ such that $\delta+\alpha_1^u>0$ and ω_2 so large that $e^{\delta\alpha_2}\omega_2<1$. Then

$$H_7 \leq Ce^{-(\delta+\alpha_1^u)\omega_2}\|v_1^+\|\sup_{s\in[0,\omega_2]}\{e^{\alpha_1^u(\omega_2-s)}|v_1^{+,u}(s)|\}. \tag{4.3.50}$$

From (4.3.41) it follows

$$H_7 \leq Ce^{-(\delta+\alpha_1^u)\omega_2}e^{-\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}\underbrace{\sup_{s\in[0,\omega_2]}\{e^{\alpha_1^u(\omega_2-s)}|v_1^{+,u}(s)|\}}_{:=H_8}. \tag{4.3.51}$$

Addressing H_8 term we want to obtain the estimate for it. From (4.2.78) it yields

$$v_1^{+,u}(s) = -\int_s^{\omega_2} T_1(z,\tau)Q_1^+(\tau;\theta)h_1^+(\tau,v_1^+(\tau),\theta)d\tau + T_1(z,\omega_2)Q_1^+(\omega_2;\theta)a_1^+. \tag{4.3.52}$$

From the property of exponential dichotomies and (4.3.48) we obtain

$$\begin{aligned}
|v_1^{+,u}(s)| &\leq Ce^{-\alpha_1^u(\omega_2-s)}|a_1^+| + \\
&+C\int_s^{\omega_2} e^{-\alpha_1^u(\tau-s)}|v_1^{+,u}(\tau)|\left(|v_1^{+,s}(\tau)\| + \|v_1^{+,u}(\tau)|\right)d\tau, \\
&\leq Ce^{-\alpha_1^u(\omega_2-s)}|a_1^+| + \\
&+C\omega_2 \sup_{\tau\in[s,\omega_2]}\left\{e^{-\alpha_1^u(\tau-s)}|v_1^{+,u}(\tau)|\left(|v_1^{+,s}(\tau)\| + \|v_1^{+,u}(\tau)|\right)\right\}. \quad (4.3.53)
\end{aligned}$$

Then

$$\begin{aligned}
H_8 &\leq C|a_1^+| + K\omega_2 \sup_{\tau\in[0,\omega_2]}\left\{e^{-\alpha_1^u(\tau-\omega_2)}|v_1^{+,u}(\tau)|\right\}\cdot \\
&\sup_{\tau\in[0,\omega_2]}\left\{|v_1^{+,u}(\tau)| + |v_1^{+,s}(\tau)|\right\}. \qquad (4.3.54)
\end{aligned}$$

From (4.3.41) we choose ω_2 be sufficiently large such that

$$\sup_{\tau\in[0,\omega_2]}\left\{|v_1^{+,u}(\tau)| + |v_1^{+,s}(\tau)|\right\} < \frac{1}{2C}. \qquad (4.3.55)$$

Applying (4.3.55) to (4.3.54) we obtain

$$H_8 \leq 2C|a_1^+|. \qquad (4.3.56)$$

Combining the results of (4.2.116), (4.3.40) and (4.3.42) we conclude

$$\begin{aligned}
|a_1^+| &\leq C(\|d\| + \|h\|), \\
&\leq Ce^{-\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}. \qquad (4.3.57)
\end{aligned}$$

Then

$$H_8 \le Ce^{-\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}. \tag{4.3.58}$$

Plugging (4.3.58) into (4.3.51) we obtain

$$H_7 \le Ce^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}. \tag{4.3.59}$$

Similarly we could also get the estimate

$$\left|\int_{-\omega_1}^{0} T_1(0,s;\theta)P_1^-(s;\theta)h_1^-(s,v_1^-(s),\theta)ds\right| \le Ce^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}. \tag{4.3.60}$$

Let's summarize what we have found in the last few sections.

- $$T_1^*(0,\omega_2;\theta)Q_1^{+,*}(0;\theta)\psi_1 = \Psi_{p_1}(\omega_2,0;\theta)\eta^+(\psi_1,\theta) + O(e^{-\min\{|\alpha_1^{uu}|,2\alpha_1^u\}\omega_2}), \tag{4.3.61}$$

- $$\begin{aligned}\bar{Q}_{2,\omega_2}(\theta)\left(q_2^-(\theta)(-\omega_2) - q_1^+(\theta)(\omega_2)\right) \;&=\; \Phi_{p_1}(-\omega_2,0;\theta)\eta_1^u(q_2^-(\theta)(0),\theta) + \\ &\quad +O(e^{-(\min\{-\alpha_1^s,\alpha_1^u\}+\min\{|\alpha_1^{ss}|,2|\alpha_1^s|\})\omega_2}),\end{aligned} \tag{4.3.62}$$

- $$\bar{Q}_{2,\omega_2}(\theta)\left(Q_2^-(-\omega_2;\theta)v_2^-(-\omega_2) - P_1^+(\omega_2;\theta)v_1^+(\omega_2)\right) = O(e^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_i}), \tag{4.3.63}$$

- $$\int_0^{\omega_2} T_1(0,s;\theta)Q_1^+(s;\theta)h_1^+(s,v_1^+(s;\omega,\theta),\theta)ds = O(e^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}), \tag{4.3.64}$$

- $$T_1^*(0,-\omega_1;\theta)P_1^{-,*}(0;\theta)\psi_1 = \Psi_{p_2}(-\omega_1,0;\theta)\eta^-(\psi_1,\theta) + O(e^{-\min\{|\alpha_2^{ss}|,2\alpha_2^s\}\omega_1}), \tag{4.3.65}$$

- $$\begin{aligned}\bar{P}_{1,\omega_1}(q_2^+(\theta)(\omega_1) - q_1^-(\theta)(-\omega_1)) &= \Phi_{p_2}(-\omega_1,0;\theta)\eta_2^s(q_2^+(\theta)(0),\theta) + \\ &+ O(e^{-(\min\{-\alpha_2^s,\alpha_2^u\}+\min\{|\alpha_2^{ss}|,2|\alpha_2^s|\})\omega_1}),\end{aligned} \tag{4.3.66}$$

- $$\bar{P}_{1,\omega_1}(\theta)\left(P_2^+(\omega_1;\theta)v_2^+(\omega_1) - Q_1^-(-\omega_1;\theta)v_1^-(-\omega_1)\right) = O(e^{-2\min\{-\alpha_2^s,\alpha_2^u\}\omega_i}), \tag{4.3.67}$$

- $$\int_{-\omega_1}^0 T_1(0,s;\theta)P_1^-(s;\theta)h_1^-(s,v_1^-(s),\theta)ds = O(e^{-2\min\{-\alpha_1^s,\alpha_1^u\}\omega_2}). \tag{4.3.68}$$

Remark 4.12. *The above estimates are sufficient to obtain the estimates of H_1-H_6 which have been used in Theorem 4.2. It is easy to see that only H_1 and H_4 account for the leading order term of ξ_1^ω and the rest of terms are all high-order term.*

Remark 4.13. *Since we are looking for the homoclinic solution for (4.1.9), then $\omega_1 = \infty$ and ξ_1 in Theorem simplifies to*

$$\xi_1(\omega_2, \theta) = M_1\theta + c^u(\theta)e^{-2\omega_2\lambda_2^u(\theta)} + o(e^{-2\omega_2\lambda_2^u(\theta)}). \tag{4.3.69}$$

4.4 Solving the bifurcation equation

In this section we will discuss how to solve the bifurcation equation (4.3.69),

$$M_1\theta + c^u(\theta)e^{-2\omega_2\lambda_2^u(\theta)} + o(e^{-2\omega_2\lambda_2^u(\theta)}) = 0. \tag{4.4.1}$$

Before we solve (4.4.1) directly we need some preliminary lemmas to compute M_1 and $c^u(0)$ first. From the definition (4.2.1) of q_1 and the fact that ϕ_h is the heteroclinic solution of the second order differential equation (1.1.8)

$$q_1(z) = \begin{pmatrix} \phi_h' \\ \phi_h'' \\ \phi_h''' \\ \phi_h'''' \end{pmatrix} \doteq \begin{pmatrix} -\sqrt{\mu_+} \\ \mu_+ \\ -(\mu_+)^{\frac{3}{2}} \\ \mu_+^2 \end{pmatrix} e^{-\sqrt{\mu_+}z}, \tag{4.4.2}$$

as $z \to \infty$ and

$$q_1(z) = \begin{pmatrix} \phi_h' \\ \phi_h'' \\ \phi_h''' \\ \phi_h'''' \end{pmatrix} \doteq \begin{pmatrix} \sqrt{\mu_-} \\ \mu_- \\ (\mu_-)^{\frac{3}{2}} \\ \mu_-^2 \end{pmatrix} e^{\sqrt{\mu_-}z}, \tag{4.4.3}$$

as $z \to -\infty$. Similarly we also have

$$q_2(z) = \begin{pmatrix} -\phi_h'(-z) \\ \phi_h''(-z) \\ -\phi_h'''(-z) \\ \phi_h''''(-z) \end{pmatrix} \doteq \begin{pmatrix} -\sqrt{\mu_-} \\ \mu_- \\ -(\mu_-)^{\frac{3}{2}} \\ \mu_-^2 \end{pmatrix} e^{-\sqrt{\mu_-}z}, \tag{4.4.4}$$

as $z \to \infty$ and

$$q_2(z) = \begin{pmatrix} -\phi_h'(-z) \\ \phi_h''(-z) \\ -\phi_h'''(-z) \\ \phi_h''''(-z) \end{pmatrix} \doteq \begin{pmatrix} \sqrt{\mu_+} \\ \mu_+ \\ (\mu_+)^{\frac{3}{2}} \\ \mu_+^2 \end{pmatrix} e^{\sqrt{\mu_+}z}, \tag{4.4.5}$$

as $z \to -\infty$. Non-degeneracy condition (4.2.24) implies that the adjoint variational equation

$$\dot{w} = -\left(D_x f(q_1(s), 0)\right)^* w, \tag{4.4.6}$$

$$= \begin{pmatrix} 0 & 0 & 0 & H_{1,4} \\ -1 & 0 & 0 & -2\phi' W'''(\phi_h) \\ 0 & -1 & 0 & -2W''(\phi_h) + \tilde{\eta} \\ 0 & 0 & -1 & 0 \end{pmatrix} w, \tag{4.4.7}$$

has a unique, up to constant multiples, bounded solution $\psi_1(z)$ and

$$H_{1,4} = \tilde{\eta}W''(\phi_h) - \big(W''(\phi_h)\big)^2 - W''''(\phi_h)(\phi_h')^2 + 2W'''(\phi_h)\phi_h''. \tag{4.4.8}$$

From (4.4.7) the bounded solution $\psi_1(z)$ of (4.4.6) is given by

$$\begin{aligned} \psi_1 \;&=\; \big(-\Psi''' + 2W'''(\phi_h)\phi_h'\Psi - \big(-2W''(\phi_h) + \tilde{\eta}\big)\,\Psi' - 2\phi_h'W'''(\phi_h)\Psi, \\ &\quad\; \Psi'' + (-2W''(\phi_h) + \tilde{\eta})\Psi, -\Psi', \Psi\big)^T . \end{aligned} \tag{4.4.9}$$

In order to obtain $\psi_1(z)$ it is sufficient to look for Ψ. Recall that the linearization of (4.1.1) about ϕ_h for $\theta = 0$ is

$$(L_h + \tilde{\eta})\, L_h\, u = 0, \tag{4.4.10}$$

with the adjoint equation given by

$$L_h\, (L_h + \tilde{\eta})\, v = 0, \tag{4.4.11}$$

which Ψ should satisfy. It is easy to see that $\Psi = \phi_h'$ is the solution of (4.4.11). Plugging it into (4.4.9) and using (4.4.2) we obtain

$$\psi_1 \doteq \begin{pmatrix} (\mu_+ - \tilde{\eta})\mu_+ \\ -\sqrt{\mu_+}(\tilde{\eta} - \mu_+) \\ -\mu_+ \\ -\sqrt{\mu_+} \end{pmatrix} e^{-\sqrt{\mu_+}z}. \tag{4.4.12}$$

as $z \to \infty$. From the definition (4.3.16) of $c^u(\theta)$, (4.4.5) and (4.4.12) we deduce

$$\begin{aligned} c^u(0) &= < \begin{pmatrix} \sqrt{\mu_+} \\ \mu_+ \\ (\mu_+)^{\frac{3}{2}} \\ \mu_+^2 \end{pmatrix}, \begin{pmatrix} (\mu_+ - \tilde{\eta})\mu_+ \\ -\sqrt{\mu_+}(\tilde{\eta} - \mu_+) \\ -\mu_+ \\ -\sqrt{\mu_+} \end{pmatrix} >, \\ &= -2\mu_+^{\frac{3}{2}}\tilde{\eta}. \end{aligned} \tag{4.4.13}$$

Combining the results of the definition (4.2.54) of M_1, (4.4.9) and (4.1.7) we obtain

$$\begin{aligned} M_1 &= \int_{\mathbb{R}} < \psi_1(s), D_\theta f(q_1(s), 0) > ds, \\ &= \int_{\mathbb{R}} \phi_h' ds, \\ &= b_+ - b_-. \end{aligned} \tag{4.4.14}$$

We summarize what we have proved above:

Lemma 4.12. *Under the assumptions (H) and (S')-(1.1.4) we have*

$$M_1 = b_+ - b_-, \tag{4.4.15}$$

$$c^u(0) = -2\mu_+^{\frac{3}{2}}\tilde{\eta}. \tag{4.4.16}$$

Now we are ready to solve the bifurcation equation (4.4.1).

Lemma 4.13. *Under the assumptions (H) and (S')-(1.1.4) there exists constants* $c_0 > 0$

such that for any $-c_0 < \theta < 0$ *there exists a* $\omega_2 = \omega_2(\theta)$ *satisfying*

$$\xi_1(\omega_2(\theta), \theta) = 0. \tag{4.4.17}$$

Proof. Solving the leading order term in (4.4.1) we have

$$\omega_2 = -\frac{\ln\left(\frac{-M_1\theta}{c^u(0)}\right)}{2\lambda_2^u(0)}. \tag{4.4.18}$$

Combining the results of (4.1.13), (4.4.15) and (4.4.16) we can further simplify (4.4.18)

$$\omega_2 = -\frac{\ln\left(\frac{\frac{(b_+-b_-)\theta}{3}}{2\mu_+^{\frac{3}{2}}\tilde{\eta}}\right)}{2\sqrt{\mu_+}}. \tag{4.4.19}$$

In particular we need to choose $\theta < 0$ to make sure

$$\frac{(b_+ - b_-)\theta}{2\mu_+^{\frac{3}{2}}\tilde{\eta}} > 0. \tag{4.4.20}$$

Remark 4.14. *From (4.1.11) and the fact that* $\theta < 0$ *we conclude that* β *has to be negative, which is stated in assumption (S')-(1.1.4).*

Theorem 4.3. *Let* $\tilde{\eta}$, δ *and double well* W *be given and satisfy (H1) and (S')-(1.1.4). Then there exists* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$ *there exists a homoclinic solution of (1.0.8) denoted by* Φ_m *which is homoclinic to* b *where* b *satisfies (1.0.9).*

Proof. By Theorem 4.1 and Lemma 4.5 we prove that there exists a unique 'Lin's orbit' with the jumps satisfying $\xi_1 = S\,\xi_2$. Lemma 4.13 indicates that we can solve the bifurcation equation $\xi_1 = 0$ so $\xi_2 = 0$, which finish the proof of the theorem. $\square$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] N. Alikakos. A new proof for the existence of an equivariant entire solution connecting the minima of the potential for the system $\Delta u - W_u(u) = 0$. preprint.

[2] N. Alikakos, P. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Archive for Rational Mechanics and Analysis*, 128: 165-205, 1994.

[3] N. Alikakos and G. Fusco. The spectrum of the Cahn-Hilliard operator for generic interface in higher space dimensions. *Indiana Univ. Math. J.*, 42(2):165-205, 1994.

[4] N. Alikakos and G. Fusco. On the connection problem for potentials with several global minima. *Indiana Univ. Math. J.*, 57:1871-1906, 2008.

[5] N. Alikakos and G. Fusco. On an elliptic system with symmetric potential possessing two global minima. Bull. Geek. Math. Soc. To appear, 2011.

[6] N. Alikakos and N. Katzourakis. Heteroclinic travelling waves of gradient diffusion systems. *Trans. Amer. Math. Soc.*, 363:1362-1397, 2011.

[7] B. Ameduri. From Vinyidene Fluoride (VDF) to the applications of VDF-containing polymers and co-polymers: Recent developments and future trends. *Chem. Rev.*, 109: 6632-6686, 2009.

[8] A. Bertozzi, S. Esedoglu, and A. Gillette. Analysis of a two-scale Cahn-Hilliard model for binary image inpainting. *SIAM Multiscale Model. Simul.*, 6, No.3: 913-936, 2007.

[9] P. Bates and P. Fife. Spectral comparison principles for the Cahn-Hilliard and phase-field equations, and time scales for coarsening. *Physica D: Nonlinear Phenomena*, 43:334-348, 1990.

[10] P. Bates and P. Fife. The dynamics of nucleation for the Cahn-Hilliard equation. *SIAM J. Appl. Math.*, 53:990-1008, 1993.

[11] D. Blomker and A. Jentzen. Galerkin Approximations for the Stochastic Burgers Equation. 2009. preprint.

[12] L. Bronsard and R. Kohn. On the slowness of phase boundary motion in one space dimension. *Comm. Pure Appl. Math.*, 43:983-997, 1990.

[13] L. Bronsard and R. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Dff. Eqns.*, 90:211-237, 1991.

[14] J. Cahn. On the spinodal decomposition. *Acta Metall.*, 9:795-801, 1961.

[15] J. Cahn and J. Hilliard. Free energy of a nonuniform system. I. Interfacial energy. *J. Chem. Phys.*, 28: 258-266, 1958.

[16] F. Campelo and A. Hernandez-Machado, Model of curvature-driven pearling instability in membranes. *Physical Review Letters* 99:88-101, 2007.

[17] P. Canham. Minimum energy of bending as a possible explanation of biconcave shape of human red blood cell. *J. Theor. Biol.* 26:61-81, 1970.

[18] W. Coppel. Dichotomies in Stability Theory. *Springer-Verlag*, 1978.

[19] G. Da Prato and A. Debussche. Stochastic Cahn-Hilliard Equation. *Nonlinear Anal.*, 26:241-263, 1996.

[20] G. Da Prato and J. Zabczyk. Stochastic Equations in Infinite Dimensions. *Cambridge University Press.* 1992.

[21] A. Doelman and K. Promislow. Equilibria of the functionalized Cahn-Hilliard energy: stability and competition, preprint.

[22] Q. Du, C. Liu, R. Ryham, and X. Wang. A phase field formulation of the Willmore problem. *Nonlinearity*, 18:1249-1267, 2005.

[23] Q. Du, C. Liu, X. Wang. A phase field approach in the numerical study of the elastic bending energy for vesicle membranes. *J. Comp. Physics* 198:450-468, 2004.

[24] Q. Du, C. Liu, and X. Wang. Simulating the deformation of vesicle membranes under elastic bending energy in three dimensions. *J. Comp. Physics*, 212:757-777, 2006.

[25] J.A. Elliott, S. Hannna, A.M.S. Elliott, and G. E. Cooley. Interpretation of the small-angle X-ray scattering from swollen and oriented perfluorinated ionomer membranes. *Macromolecules*, 33:4161-4171, 2000.

[26] P. Fife. Models for phase separation and their mathematics. *Electronic Journal of Differential Equations*, 48:1-26, 2000.

[27] A.C. Fowler, C.T. Sparrow. Bifocal homoclinic orbits in four dimensions. *Nonlinearity* 4:1159-1182, 1991.

[28] Nir Gavish, Gurgen Hayrapetyan, Keith Promislow, Li Yang, Curvature Driven Flow of Bi-Layer Interfaces, *Physica D: Nonlinear Phenomena*, 240(7):675-693, 2011.

[29] Nir Gavish, Jaylan Jones, Zhengfu Xu, Andrew Christlieb, Keith Promislow, Variational Models of Network Formation and Ion Transport: Applications to Perfluorosulfonate Ionomer Membranes. *Polymers*, submitted.

[30] G. Gompper and M.Schick. Correlation between structural and interfacial properties of amphillic systems. *Phys. Rev. Lett.*, 65:1116-1119, 1990.

[31] G.M. Grason and C.D. Santangelo. Undulated cylinders of charged diblock copolymers. *European Physics Journal E*. 20:335-346, 2006.

[32] W. Helfrich. Elastic properties of lipid bilayers-theory and possible experiments. *Zeitschcrift fur naturforschung C*. 28:693-703, 1973.

[33] A.J. Homburg. Notes on homoclinic bifurcation theory. 1-21,2005.

[34] W. Hsu and T. Gierke. Ion transport and clustering in nafion perfluorinated membranes. *J. Membrane Science*, 13:307-326, 1983.

[35] A.J. Homburg and B. Sandstede. Homoclinic and heteroclinic bifurcations in vector fields. sl Handbook of Dynamical System III, 379-524, 2010.

[36] A. Jentzen, P. Kloeden, G. Winkel. Efficient Simulation of Nonlinear Parabolic SPDEs with Additive Noise. *The Annals of Applied Probability.*, 21(3):908-950, 2011.

[37] J. Knobloch, Lin's method for discrete and continuous dynamical systems and applications, thesis, TU Ilmenau, 2004.

[38] K.D. Kreuer, S.J. Paddison, E. Spohr, and M. Schuster. Transport in proton conductors for fuel-cell applications: simulation, elementary reactions, and phenomenology. *Chemical Reviews*. 104: 4637-4678, 2004.

[39] A.V. Krivandin, A.B. Solovena, N.N. Glagolev, O.V. Shatalova, and S.L. Kotova. Structure alternations of perfluorinated sulfocationic membranes under the actoni of ethylene glycol (SAXS and WAXS) studies. *Polymer* 44:5789-5796, 2003.

[40] C. Laing, P. Glendinning. Bifocal homoclinic bifurcations, *Phsica D.*, 102:1-14, 1997.

[41] S. Larsson, A. Mesforush. Finite Element Approximation of the Linearized Cahn-Hilliard-Cook Equation. *IMA Numer. Anal.*, published online 2011 May.

[42] K. Lee, C. Hsu, W. Chiu, M. Tsui, Y. Tung, S. Tsai, and K. Ho. Dye-sensitized solar cells with mircoporous TiO2 electrode and gel polymer electrolytes prepared by in situ cross-link reaction. *Solar Energy Materials & Solar cells*, 93:2003-2007, 2009.

[43] N. Q. Le, A Gamma-Convergence approach to the Cahn-Hilliard equation, *Calc. Var. Partial Differential Equations* 32 no. 4: 499-522, 2008.

[44] X. B. Lin. Using Melnikov's method to solve Silnikov's problems, *Proc. Roy. Soc. Edinburgh* 116A: 295-325, 1990.

[45] P. Loreti and R. March. Propagation of fronts in a nonlinear fourth order equation. *Euro. J. of Applied Mathematics* 11:203-213, 2000.

[46] D. Liu. Convergence of the Spectral Method for Stochastic Ginzburg-Landau Equation driven by Space-Time White Noise. *Comm. Math. Sci.*, 1(2):361-375, 2003.

[47] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. Arch. Rational. Mech. Anal., 98:123-142, 1987.

[48] R. Moore and K. Promislow. Renormalization group reduction of pulse dynamics in thermally loaded optical parametric oscillators. *Physica D*, 206:62-81, 2005.

[49] P. De Mottoni and M. Schatzman. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.*, 347:1533-1589, 1995.

[50] S.J. Paddison and K.S. Promislow, eds., Device and materials modeling of polymer electrolyte membrane fuel cells. *Springer, New York*, 2009.

[51] J. Peet, A. Heeger, G. Bazan. "Plastic" solar cells: self-assembly of bulk heterojunction nanomaterials by spontaneous phase separation. *Acc. Chem. Res.* 42:1700-1708, 2009.

[52] R. Pego. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. R. Soc. Lond.*, 422, 1989.

[53] K. Promislow. A renormalization method for modulational stability of quasi-steady patterns in dispersive systems. *SIAM J. Math. Anal*, 33:1455-1482, 2002.

[54] K. Promislow and B. Wetton. PEM fule cells: a mathematical overview. *SIAM J. Math. Anal.*, 70 369-409, 2009.

[55] K. Promislow, L. Yang. Existence of Homoclinic solutions of the Functionalized Cahn-Hilliard Equation (2011), preprint.

[56] K. Promislow, H. Zhang. Critical points of functionalized Lagranagians (submitted for publication).

[57] M. Roger and R. Schatzle. On a modified conjecture of De Giorgi. *Math. Z.*, 254:675-714, 2006.

[58] L. Rubatat, G. Gebel, and O. Diat. Fibriallar structure of nafion: Matching fourier and real space studies of corresponding films and solutions. *Macromolecules*, 7772-7783, 2004.

[59] L. Rubatat, A.L. Rollet, G. Gebel, and O. Diat. Evidence of elongated polymeric aggregates in Nafion. *Macromolecules*, 35:4050-4055, 2002.

[60] J. Rubinstein and P. Sternberg. Nonlocal reaction-diffusion equations and nucleation. *IMA Journal of Aplied Mathematics*, 48:249-264, 1992.

[61] B. Sandstede, Verzweigungstheorie homokliner Verdopplungen, Ph.D. thesis, University of Stuttgart, 1993.

[62] B. Sandstede. Constructing dynamical systems having homoclinic bifurcation points of codimension two. *J. Diff. Eqns.* 218:390-443. 2005.

[63] B. Sandstede. Stability of multi-pulse solutions. *Trans. Amer. Math. Soc.*, 350:429-472, 1998.

[64] B. Sandstede. Stability of travelling waves. *Handbook of Dynamical System II*, 2002.

[65] B. Sandstede and A. Scheel. Glueing unstable fronts and backs together can produce stable pulses. *Nonlinearity*, 13:233-277, 2000.

[66] K. Schmidt-Rohr and Q. Chen. Parallel cylindrical water nanochannels in Nafion fuel cell membranes. *Nature Materials*, 7:75-83, 2008.

[67] L. P. Shil'nikov, A.L. Shil'nikov, D. V. Turaev and L. O. Chua. Methods of Qualitative Theory in Nonlinear Dynamics I II. *World Scientific*, 2001.

[68] W. Sinoda, T. Nakamura, S. Nielsen. Free energy analysis of vesicle-to-bicelle transformation. *Soft Matter.* 7:9-12, 2011.

[69] H.W. Starkweather. Crystallinity in perfluorosulfonic acid ionomers and related polymers. *Macromolecules* 15:320-323, 1982.

[70] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems. *Arch. Ration. Mech. Anal.* 101:209-260, 1988.

[71] S. Torabi, J. Lowengrub, A. Voigt and S. Wise. A new phase-field model for strongly aniostropic systems. *Proc. Royal Society A* 465:1337-1359, 2009.

[72] A. Vanderbauwhede and B. Fiedler, Homoclinic period blow-up in reversible and conservative systems, *Z Angew Math Phys* 43:292-318, 1992.

[73] X. Wang and Q. Du. Modeling and simulation of multi-component lipid membranes and open membranes via diffuse interface approaches. *J. Math. Bio.* 56(3): 347-371, 2008.

[74] C.C. Weber. Implicit Approximation Scheme for the Cahn-Hilliard Stochastic Equation. 2000.

[75] C.C. Weber. Cahn-Hilliard Stochastic Equation: Existence of the Solution and of its Density. *Bernoulli* 7(5):777-816, 2001.

[76] D.S. Wu, S.J. Paddison and J.A. Elliott. A compparative study of hydrated morphologies of perfluorosulfonic acid fuel cell membranes with mescopic simulations. *Energy and Environmental Science*, 1:284-293, 2008.

[77] Li Yang and Yanzhi Zhang. Convergence of Spectral Method for Stochastic Cahn-Hilliard Equation. in progress 2011.