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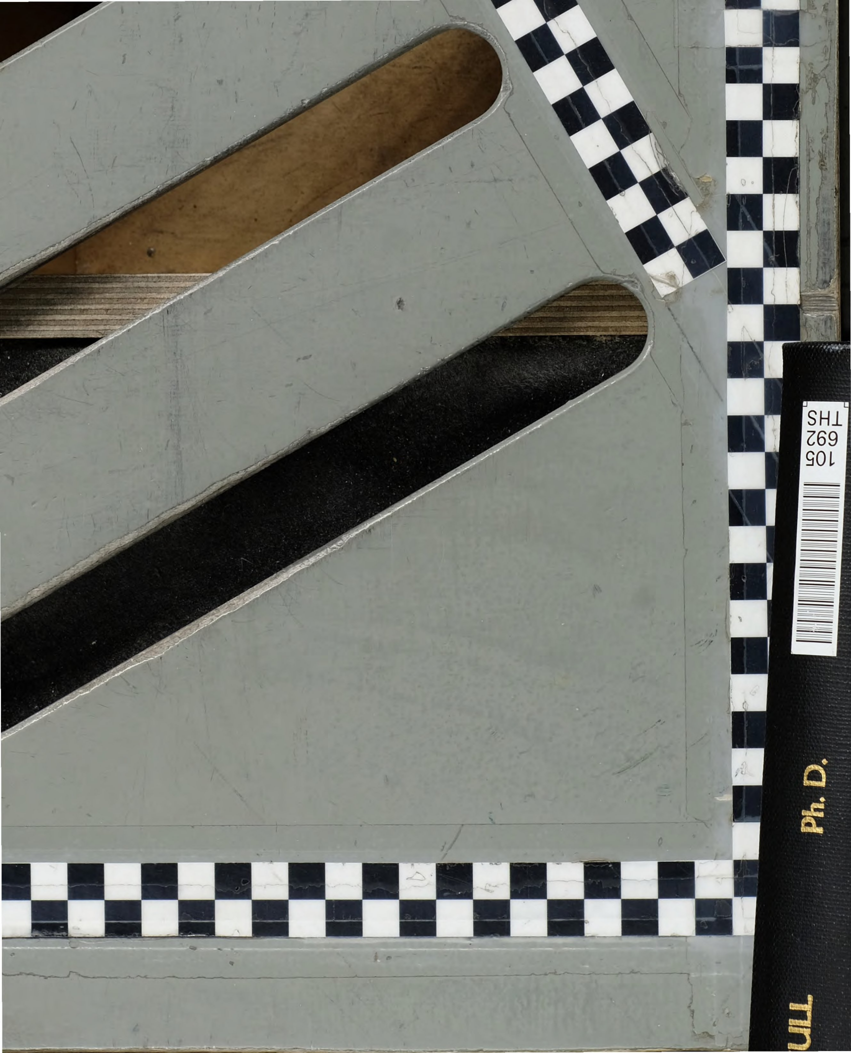
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THE RESPONSE OF A MAGNETOELASTIC PLATE TO A TRAVELING LOAD

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ABSTRACT

THE RESPONSE OF A MAGNETOELASTIC PLATE TO A TRAVELLING LOAD

by

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An unsupported, thin, elastic plate immersed in a strong, static, magnetic field is acted upon by a travelling line load. A theory is developed which describes the motion of the plate by a set of nonlinear, partial differential equations in which the displacements are coupled. Solutions to the linear problem are presented.

Seeking steady-state solutions relative to the moving load permits the equations of motion to be restated in terms of a single moving coordinate. The displacements and the load speed in the resulting set of ordinary differential equations are expanded as power series in the perturbation parameter, the load intensity. Thereby, the linear problem is defined.

The linear solution is the Green's function of the governing differential equations. For a general magnetic field, unique displacement solutions cannot be found

directly. They are recovered through a single integration of the prime solutions. (The dependent variables in the prime problem are the first derivatives of the displacements with respect to the moving coordinate.) The prime solutions for a general magnetic field are not unique either. Nevertheless, by augmenting the linear differential equations through the addition of a judiciously chosen, small term, the prime solutions to the augmented problem can be written. The prime solutions to the linear problem are found by allowing the augmentation parameter to go to zero.

The solutions consist of two parts, one of which trails the load, the other precedes; they are continuous under the load. For a general magnetic field, one tail of the displacement solution is bounded, whereas the other tail is unbounded. The unboundedness is due to a linear term. In the vicinity of the load the displacement response is composed of combinations of exponential and damped, harmonic waves. Of course, the particular form of a specific solution depends upon the magnitudes of the load speed and the magnetic field.

A short study of plane, harmonic wave propagation indicates that for certain, one dimensional, magnetic fields dilatational and distortional waves propagate as though the medium was classical, elastic. In general, however, the magnetoelastic plate is anisotropic, dissipative and dispersive.

THE RESPONSE OF A MAGNETOELASTIC PLATE TO A TRAVELLING LOAD

By

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I. INTRODUCTION

Electromagnetic fields and matter interact. This rather obvious, simple statement encompasses a great deal of physics and mathematics. For two centuries, the effect of matter on electromagnetic fields has been investigated in the discipline of electrodynamics. The opposite viewpoint--the effect of electromagnetic fields on matter--may be attributed to the discipline of mechanics. This is a much younger field of study. In fact, magnetoelasticity, which involves the interaction between an elastic* solid and a magnetic field, has a history of less than twenty years. To contribute to the topic of magnetoelasticity is the purpose of this dissertation.

Many basic questions concerning the electromagnetic-elastic system are still the subjects of conjecture. For instance, what electromagnetic loads are applied to the body? Brown [1] suggests that, in the presence of an external magnetic field $\underset{\sim}{B}_o$, the force and moment per unit volume are, respectively,

$$\underset{\sim}{f} = \underset{\sim}{J}_c \times \underset{\sim}{B}_o + (\underset{\sim}{M}\cdot\nabla)\underset{\sim}{B}_o$$

$$\underset{\sim}{C} = \underset{\sim}{r} \times (\underset{\sim}{J}_c \times \underset{\sim}{B}_o + \underset{\sim}{M}\cdot\nabla\underset{\sim}{B}_o) + \underset{\sim}{M} \times \underset{\sim}{B}_o$$

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*Unless stated otherwise, elastic always means homogeneous, isotropic elastic.

where $\underset{\sim}{J}_c$ is the conduction current, $\underset{\sim}{M}$ is the magnetization and $\underset{\sim}{r}$ is the position vector. Of course, the existence of $\underset{\sim}{C}$ advocates the use of a couple-stress continuum theory to describe the mechanical system. No one has attempted to solve a problem with such a theory. Another related question is, how the constitutive equations for an elastic body are changed by the presence of electromagnetic fields? Penfield and Haus [2] derive a lengthy stress-strain relationship which consists of the usual Hooke's law with several additional terms involving electromagnetic and mechanical quantities. Although it is generally admitted that Hooke's law is insufficient, yet it is the only constitutive law used in the literature. In short, the true physics is unsettled; the recommended laws yield a sophisticated theory, but intractable analysis; so simplifications are necessary.

Dunkin and Eringen [3] develop in a systematic way a set of field equations and boundary conditions to describe the interaction between electromagnetic fields and an elastic, electrically conducting body. Subsequently, most authors defer to this theory; so the assumptions included therein represent the current norm. In essence, they break the combined system into two parts: an electromagnetic part and a mechanical part. The mechanical system is assumed to affect the electromagnetic system in one way--the electromagnetic field quantities are altered by the

motion (velocity) of the body. Thereby the static electromagnetic constitutive equations change form. To describe the action of the electromagnetic on the mechanical system, two effects are considered to be sufficient. These consist of the Lorentz force, which neglects the magnetization term in Brown's expression, and some electromagnetic surface tractions. There are other common assumptions. Invariably in the solution of magnetoelastic boundary value problems, the magnetic field is assumed to be one-dimensional. At this point, a quotation from Duhubi [4] is relevant. After making the prior assumptions, halfway through his paper he reaches the conclusion, "Computations in the foregoing expressions are too cumbersome and tedious to carry out. In the next section we shall see that some simpler results will come out for some special cases". Whereupon he goes on to make further assumptions such as infinite conductivity. This state of affairs is prevalent.

Historically, the study of magnetoelasticity received its initial impetus from the field of geophysics when, in 1955, Knopoff [5] investigated the effect of the earth's magnetic field on seismic waves. Shortly thereafter, Banos [6], who studied the importance of magnetic field direction on the transmission of plane harmonic waves, coined the name "magnetoelasticity". It is interesting to note that the topic was eight years old before any information was

published comparing the interaction strength of the electric field to that of the magnetic field. This was done by Dunkin and Eringen [3] who showed that the contribution of the electric field was relatively insignificant. Thus its comparative neglect in magnetoelastic theory is justified. Very few mechanically significant (beams, rods, plates, etc.)* magnetoelastic studies have been undertaken; a brief statement now will be made on some of them.

Dunkin and Eringen [3] and Kaliski [7] investigated the propagation of plane, harmonic waves in a thin, free infinite plate. The transmission of torsional, harmonic waves along a free, infinite, circular cylinder was studied by Suhubi [4]. In all three papers, the magnetic field is found to act primarily as a damper. Paria [8] discussed the radial vibrations of a perfectly conducting sphere. The dynamic stability of a column was examined by Peddieson and McNitt [9]. Moon [10] [11] also studied the column stability problem. Peddieson and McNitt, who used Dunkin and Eringen's theory, found little effect other than damping. Moon concluded that the buckling load and the natural frequencies were affected significantly by the magnetic field. In a re-examination of Moon's work, Wallerstein [1]], by correcting "for field distortion due

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*The further assumption of a homogeneous, isotropic, elastic solid is implied.

to the finiteness of the beam", claims to be able to improve the theoretical and experimental correlation. Other than for two exceptions, all these papers have two things in common: they only deal with free vibrations and they only use one-dimensional magnetic fields.

The two exceptions are the forced vibrations problems which occur in [9] and [11]. Peddieson and Mcnitt took a cursory look at a beam-column acted upon by an sinusoidally distributed load which decreases exponentially with time. Excitation by a harmonically oscillating magnetic field was used in Moon's second paper.

Moon alone has presented experimental data. This work was performed for his Ph. D. dissertation and is the basis for both his papers [10], [11]. His physical model differs markedly from that of Dunkin and Eringen. He chooses to neglect the body force entirely, and claims that only the $\underset{\sim}{M} \times \underset{\sim}{B}_o$ term in Brown's body couple expression is significant. By using a strength of materials approach, he is able to include the body couple without becoming distracted by a complicated, couple-stress continuum theory. It is not surprising that his results disagree with those of Peddieson and McNitt. Apparently, Dunkin and Eringen's theory, though more sophisticated, cannot explain Moon's experimental results.

The subject of this dissertation is the response to a travelling line-load of a free, infinite plate which is

immersed in a static magnetic field. It is the first mechanically significant, boundary value problem to include either a general forcing function or a general magnetic field.

Although the study of Timoshenko beams and plates under the action of travelling loads is highly developed, the shear effect is neglected in this study. The plate theory, which can be classified as a dynamic, improved, von Karman theory, owes much to Hermann [13]. Electromagnetic aspects of the theory follow the lead of Dunkin and Eringen. However, the electromagnetic tractions on the faces of the plate are not included. This simplification is justified by the findings of both Suhubi and of Dunkin and Eringen. Their elementary beam [2] and cylinder [4] studies include no electromagnetic tractions yet they predict the effects of the magnetic field quite well.

The mechanical and electromagnetic systems are treated separately. The Lorentz body force is assumed to describe sufficiently the effect of the electromagnetic on the mechanical system. By the change in the electromagnetic quantities attributable to the particle velocity, the mechanical system affects the electromagnetic.

The theory in this present magnetoelastic investigation may not be as sophisticated as possible, yet neither is it elementary.

In Chapter 2 a first order, nonlinear, magnetoelastic plate theory is developed. Several things are accomplished in Chapter 3: the equations of motion are linearized, a moving coordinate associated with the travelling load is introduced, and a technique for describing the steady state response is presented. Chapter 4 is devoted to the details of the solution. Some conclusions are drawn in Chapter V.

II. THE EQUATIONS GOVERNING THE MOTION OF A THIN ELASTIC PLATE IN A STRONG MAGNETIC FIELD

2.1. Derivation of the Equations of Motion from Elasticity Theory

The Lagrangian description for the deformation of a continuum is given by*

$$\frac{\partial}{\partial x_j} [S_{jk} \frac{\partial y_i}{\partial x_k}] + f_i = \rho \frac{d^2 y_i}{dt^2} \quad . \qquad (2.1.1)$$

In this equation, S_{jk} is the second Piola-Kirchhoff stress tensor; f_i is the body force per unit volume; y_i are the cartesian coordinates of the current configuration; x_i are the cartesian coordinates of the initial configuration, and ρ is the mass density. A few more words of explanation will facilitate understanding of the notation. The usual summation convention applies: a repeated index in a term indicates summation on this index. Figure 2.1 shows that the relationship between the initial position vector $\underset{\sim}{x}$, the current position vector $\underset{\sim}{y}$ and the displacement vector $\underset{\sim}{U}$ is

$$y_i = U_i + x_i \; ; \qquad (2.1.2)$$

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*The equations of motion and the associated mechanics are thoroughly discussed by Malvern [14]

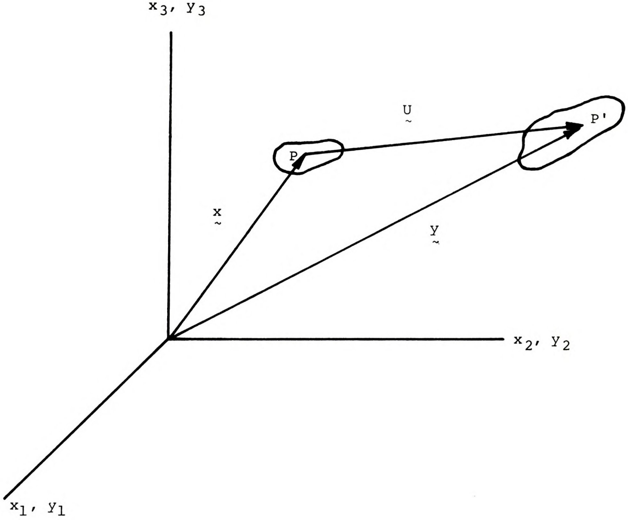

Figure 2.1. Displacement Vector $\underset{\sim}{U}$

therefore

$$\frac{\partial y_i}{\partial x_k} = \frac{\partial U_i}{\partial x_k} + \delta_{ik} \quad . \tag{2.1.3}$$

Moreover, since

$$U_i = U_i(\underset{\sim}{x}, t) \tag{2.1.4}$$

and since $\underset{\sim}{x}$ and t are independent variables, then it is clear that

$$\frac{d^2 y_i}{dt^2} = \frac{d^2 U_i}{dt^2} = \frac{\partial^2 U_i}{\partial t^2} \quad . \tag{2.1.5}$$

Substituting equations (2.1.3) and (2.1.5) into (2.1.1) introduces the displacement vector $\underset{\sim}{U}$ into the equations of motion which become

$$\frac{\partial}{\partial x_j}[S_{jk}(\delta_{ik} + \frac{\partial U_i}{\partial x_k})] + f_i = \rho\frac{\partial^2 U_i}{\partial t^2} \quad . \tag{2.1.6}$$

Define the displacement field by means of the conventional assumptions for plate deflections which are

$$U_1 = u(x_1, x_2, t) - x_3 \frac{\partial w}{\partial x_1} \quad , \tag{2.1.7}$$

$$U_2 = v(x_1, x_2, t) - x_3 \frac{\partial w}{\partial x_2} \quad , \tag{2.1.8}$$

$$U_3 = w(x_1, x_2, t) \quad . \tag{2.1.9}$$

(Figure 2.2 clarifies the notation.) Expanding the x_1-component of (2.1.6) yields

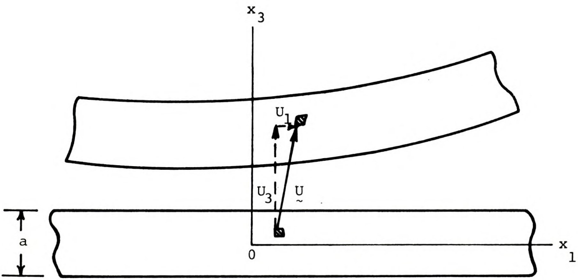

Figure 2.2. Plate Deflections

$$\frac{\partial}{\partial x_j}[S_{jk}\ (\delta_{ik} + \frac{\partial U_1}{\partial x_k})] + f_1 = \rho\frac{\partial^2 U_1}{\partial t^2} . \qquad (2.1.10)$$

Accordingly,

$$\frac{\partial}{\partial x_j}[S_{j1}\ (\delta_{11} + \frac{\partial U_1}{\partial x_1}) + S_{j2}(\delta_{12} + \frac{\partial U_1}{\partial x_2})$$

$$+ S_{j3}(\delta_{13} + \frac{\partial U_1}{\partial x_3})] + f_1 = \rho\frac{\partial^2 U_1}{\partial t^2} \qquad (2.1.11)$$

or

$$\frac{\partial}{\partial x_j}[S_{j1}\ (1 + \frac{\partial U_1}{\partial x_1}) + S_{j2}\ \frac{\partial U_1}{\partial x_2} + S_{j3}\ \frac{\partial U_1}{\partial x_3}]$$

$$+ f_1 = \rho\frac{\partial U_1}{\partial t^2} . \qquad (2.1.12)$$

At this point incorporate the assumptions (2.1.7), (2.1.8) and (2.1.9); thus,

$$\frac{\partial}{\partial x_j}[S_{j1}\ (1 + \frac{\partial u}{\partial x_1} - x_3\ \frac{\partial^2 w}{\partial x_1^2}) + S_{j2}\ (\frac{\partial u}{\partial x_2}$$

$$- x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2}) - S_{j3}\ \frac{\partial w}{\partial x_1}] + f_1$$

$$= \rho[\frac{\partial^2 u}{\partial t^2} - x_3\ \frac{\partial^3 w}{\partial x_1 \partial t^2}] \qquad (2.1.13)$$

Now perform the expansion on subscript j to find

$$\frac{\partial}{\partial x_1}[S_{11}\ (1 + \frac{\partial u}{\partial x_1} - x_3\ \frac{\partial^2 w}{\partial x_1{}^2}) + S_{12}\ (\frac{\partial u}{\partial x_2}$$

$$- x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2}) - S_{13}\ \frac{\partial w}{\partial x_1}] + \frac{\partial}{\partial x_2}[S_{21}\ (1 + \frac{\partial u}{\partial x_1}$$

$$- x_3\ \frac{\partial^2 w}{\partial x_1{}^2}) + S_{22}\ (\frac{\partial u}{\partial x_2} - x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2}) - S_{23}\ \frac{\partial w}{\partial x_1}]$$

$$+ \frac{\partial}{\partial x_3}[S_{31}\ (1 + \frac{\partial u}{\partial x_1} - x_3\ \frac{\partial^2 w}{\partial x_1{}^2}) + S_{32}\ (\frac{\partial u}{\partial x_2}$$

$$- x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2}) - S_{33}\ \frac{\partial w}{\partial x_1}] + f_1$$

$$= \rho[\frac{\partial^2 u}{\partial t^2} - x_3\ \frac{\partial^3 w}{\partial x_1 \partial t^2}] \ . \qquad (2.1.14)$$

By analogy, the x_2-component of equation (2.1.6) is

$$\frac{\partial}{\partial x_1}[S_{11}\ (\frac{\partial v}{\partial x_1} - x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2}) + S_{12}\ (1 + \frac{\partial v}{\partial x_2}$$

$$- x_3\ \frac{\partial^2 w}{\partial x_2{}^2}) - S_{13}\ \frac{\partial w}{\partial x_2}] + \frac{\partial}{\partial x_2}[S_{21}\ (\frac{\partial v}{\partial x_1}$$

$$- x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2}) + S_{22}\ (1 + \frac{\partial v}{\partial x_2} - x_3\ \frac{\partial^2 w}{\partial x_2{}^2})$$

$$- S_{23}\ \frac{\partial w}{\partial x_2}] + \frac{\partial}{\partial x_3}[S_{31}\ (\frac{\partial v}{\partial x_1} - x_3\ \frac{\partial^2 w}{\partial x_1 \partial x_2})$$

$$+ S_{32}\ (1 + \frac{\partial v}{\partial x_2} - x_3\ \frac{\partial^2 w}{\partial x_2{}^2}) - S_{33}\ \frac{\partial w}{\partial x_2}]$$

$$+ f_2 = \rho\left[\frac{\partial^2 v}{\partial t^2} - x_3 \frac{\partial^3 w}{\partial x_2 \partial t^2}\right] \quad . \tag{2.1.15}$$

The x_3-component is somewhat different from the other two. If the same procedure is followed, it is found that

$$\frac{\partial}{\partial x_j}\left[S_{jk} \left(\delta_{3k} + \frac{\partial U_3}{\partial x_k}\right)\right] + f_3 = \rho\frac{\partial^2 U_3}{\partial t^2} \tag{2.1.16}$$

or

$$\frac{\partial}{\partial x_j}\left[S_{j1} \left(\delta_{31} + \frac{\partial U_3}{\partial x_1}\right) + S_{j2} \left(\delta_{32} + \frac{\partial U_3}{\partial x_2}\right) + S_{j3} \left(\delta_{33} + \frac{\partial U_3}{\partial x_3}\right)\right] + f_3 = \rho\frac{\partial^2 U_3}{\partial t^2} \quad ; \tag{2.1.17}$$

so

$$\frac{\partial}{\partial x_j}\left[S_{j1} \frac{\partial U_3}{\partial x_1} + S_{j2} \frac{\partial U_3}{\partial x_2} + S_{j3} \left(1 + \frac{\partial U_3}{\partial x_3}\right)\right] + f_3 = \rho\frac{\partial^2 U_3}{\partial t^2} \quad . \tag{2.1.18}$$

Substitution from (2.1.9) gives

$$\frac{\partial}{\partial x_j}\left[S_{j1} \frac{\partial w}{\partial x_1} + S_{j2} \frac{\partial w}{\partial x_2} + S_{j3}\right] + f_3 = \rho\frac{\partial^2 w}{\partial t^2} \quad . \tag{2.1.19}$$

Finally, expansion on j produces the equation

$$\frac{\partial}{\partial x_1}[S_{11} \frac{\partial w}{\partial x_1} + S_{12} \frac{\partial w}{\partial x_2} + S_{13}]$$

$$+ \frac{\partial}{\partial x_2}[S_{21} \frac{\partial w}{\partial x_1} + S_{22} \frac{\partial w}{\partial x_2} + S_{23}]$$

$$+ \frac{\partial}{\partial x_3}[S_{31} \frac{\partial w}{\partial x_1} + S_{32} \frac{\partial w}{\partial x_2} + S_{33}]$$

$$+ f_3 = \rho \frac{\partial^2 w}{\partial t^2} \quad . \qquad (2.1.20)$$

Equations (2.1.14), (2.1.15), and (2.1.20) are a set of partial differential equations describing the motion of a thin plate. However, they are too complex for the present purpose. To reduce these three equations to more manageable proportions, the magnitude of each term is investigated. Only the largest terms are retained in the succeeding theory.

Reasonable assumptions for the magnitudes of the plate dimensions and for the limits of the cartesian variables are

$$a = \text{plate thickness}$$

$$x_3, \; w = O(a) \qquad (2.1.21)$$

$$x_1, \; x_2 = O(L) \qquad (2.1.22)$$

where $L >> a$.

In order to define magnitudes for the lateral displacements, appeal is made to equations (2.1.7) and (2.1.8). Assume that the four terms on the left sides

of these two equations are of the same order of magnitude. It then follows that

$$u = v = O(a^2/L) \tag{2.1.23}$$

Admittedly, the restriction of u and v to be much smaller than w is a shortcoming: large amplitude lateral motions cannot be described with this theory.

To find the order of magnitude of the stress terms in the equations of motion, assume the constitutive law

$$S_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij}, \tag{2.1.24}$$

in which λ and μ are the Lamé constants and E_{ij} is the Lagrangian strain tensor

$$E_{ij} = \frac{1}{2}\left[\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} - \delta_{ij}\right] .$$

Eliminating $\underset{\sim}{y}$ in favor of $\underset{\sim}{U}$ with the aid of equation (2.1.2) puts the strain tensor in the more useful form

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right] \tag{2.1.25}$$

Substitute (2.1.7), (2.1.8), and (2.1.9) into (2.1.25) and expand each element of E_{ij}. Now use the assumptions (2.1.21), (2.1.22), and (2.1.23). The final step is to reject all terms of magnitude less than $O(a^2/L^2)$. The results are

$$E_{11} = \frac{\partial u}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2 ,$$

$$E_{22} = \frac{\partial v}{\partial x_2} - x_3 \frac{\partial^2 w}{\partial x_2^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2 ,$$

$$E_{33} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2\right] ,$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} - 2x_3 \frac{\partial^2 w}{\partial x_1 \partial x_2} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right) ,$$

$$E_{31} = \frac{1}{2}\left(- \frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_2} + x_3 \frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial w}{\partial x_2} - \frac{\partial u}{\partial x_1}\frac{\partial w}{\partial x_1} + x_3 \frac{\partial^2 w}{\partial x_1^2}\frac{\partial w}{\partial x_1}\right) ,$$

$$E_{23} = \frac{1}{2}\left(- \frac{\partial u}{\partial x_2}\frac{\partial w}{\partial x_1} + x_3 \frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial w}{\partial x_1} - \frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2} + x_3 \frac{\partial^2 w}{\partial x_2^2}\frac{\partial w}{\partial x_2}\right) . \qquad (2.1.26)$$

Note that E_{23} and E_{31} are recorded here although each is of the order $O(a^3/L^3)$.

The orders of magnitude of the various stresses, found by substituting (2.1.26) into (2.1.24), are

$$S_{11} = S_{22} = S_{33} = S_{12} = O(Na^2/L^2)$$

$$S_{23} = S_{31} = O(Na^3/L^3) \qquad (2.1.27)$$

where N is a quantity of the same order as the Lamé constants. However, conventional plate theory requires that S_{33}, as well as S_{23} and S_{31}, be vanishingly small. In other words, S_{33} cannot be greater than S_{23} or S_{31}; so

$$S_{33} \leq O(Na^3/L^3) \quad . \tag{2.1.28}$$

To remove the discrepancy in S_{33} contained in (2.1.27) and (2.1.28), appeal directly to the x_3-component of the equations of motion, equation (2.1.20).

The expansion of (2.1.20) shows that each term contains a stress component. All the terms that do not contain S_{33} are of the order $O(Na^3/L^4)$. Reasonably, the S_{33}-term should be neither superior nor inferior to the other terms. Consequently,

$$\frac{\partial S_{33}}{\partial x_3} = O(\frac{Na^3}{L^4})$$

which implies that

$$S_{33} = O(Na^4/L^4) \quad . \tag{2.1.29}$$

Thus, the orders of magnitude of the six stresses are assumed to be

$$S_{11} = S_{22} = S_{12} = O(Na^2/L^2) \quad ,$$

$$S_{31} = S_{23} = O(Na^3/L^3) \quad ,$$

$$S_{33} = O(Na^4/L^4) \quad . \tag{2.1.30}$$

By means of assumptions (2.1.21), (2.1.22), (2.1.23), and (2.1.30), it is now possible to re-examine the

equations of motion, (2.1.14), (2.1.15), and (2.1.20) term by term, thereby identifying the high order terms. Retaining terms of order $O(Na^2/L^3)$ from (2.1.14) and (2.1.15), and terms of order $O(Na^3/L^4)$ from (2.1.20), reduces the equations of motion ot the following forms:

$$\frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{21}}{\partial x_2} + \frac{\partial S_{31}}{\partial x_3} + f_1 = \rho\left[\frac{\partial^2 u}{\partial t^2} - x_3 \frac{\partial^3 w}{\partial x_1 \partial t^2}\right] , \tag{2.1.31}$$

$$\frac{\partial S_{12}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{32}}{\partial x_3} + f_2 = \rho\left[\frac{\partial^2 v}{\partial t^2} - x_3 \frac{\partial^3 w}{\partial x_2 \partial t^2}\right. , \tag{2.1.32}$$

$$\frac{\partial}{\partial x_1}\left[S_{11} \frac{\partial w}{\partial x_1} + S_{12} \frac{\partial w}{\partial x_2} + S_{13}\right] + \frac{\partial}{\partial x_2}\left[S_{12} \frac{\partial w}{\partial x_1} + S_{22} \frac{\partial w}{\partial x_2} + S_{23}\right] + \frac{\partial}{\partial x_3}\left[S_{13} \frac{\partial w}{\partial x_1} + S_{23} \frac{\partial w}{\partial x_2} + S_{33}\right] + f_3 = \rho \frac{\partial^2 w}{\partial t^2} . \tag{2.1.33}$$

Subsequently, these equations are called the first-order equations.

2.2 Formulation of the Equations of Motion in Terms of Displacements and Applied Loads

The most common form for plate equations is in terms of the plate stresses:

$$M_{ij} = \int_{-a/2}^{a/2} x_3 S_{ij} dx_3 \tag{2.2.1}$$

$$N_{ij} = \int_{-a/2}^{a/2} S_{ij} dx_3 \tag{2.2.2}$$

Integrating equations (2.1.21), (2.1.32), and (2.1.33) over the plate thickness yields the following equations.

For example, consider the integration of (2.1.31)

$$\int_{-a/2}^{a/2} \left(\frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{21}}{\partial x_2} + \frac{\partial S_{31}}{\partial x_3} + f_1\right) dx_3$$

$$= \int_{-a/2}^{a/2} \rho\left(\frac{\partial^2 u}{\partial t^2} - x_3 \frac{\partial^3 w}{\partial x_1 \partial t^2}\right) dx_3$$

but,

$$\int_{-a/2}^{a/2} \frac{\partial S_{11}}{\partial x_1} dx_3 = \frac{\partial}{\partial x_1} \int_{-a/2}^{a/2} S_{11} dx_3 = \frac{\partial N_{11}}{\partial x_1} ,$$

$$\int_{-a/2}^{a/2} \frac{\partial S_{21}}{\partial x_2} dx_3 = \frac{\partial}{\partial x_2} \int_{-a/2}^{a/2} S_{21} dx_3 = \frac{\partial N_{21}}{\partial x_2} ,$$

$$\int_{-a/2}^{a/2} \frac{\partial S_{31}}{\partial x_3} dx_3 = [S_{31}]_{-a/2}^{a/2} ,$$

$$\int_{-a/2}^{a/2} f_1 \, dz = F_1 \quad ,$$

$$\int_{-a/2}^{a/2} \rho\frac{\partial^2 u}{\partial t^2} dx_3 = \rho\frac{\partial^2 u}{\partial t^2} [x_3]_{-a/2}^{a/2} = \rho a\frac{\partial^2 u}{\partial t^2} \quad ,$$

$$\int_{-a/2}^{a/2} \rho x_3\frac{\partial^3 w}{\partial x_1 \partial t^2} dx_3 = \rho\frac{\partial^3 w}{\partial x_1 \partial t^2} \left[\frac{x_3^{\,2}}{2}\right]_{-a/2}^{a/2} = 0 \; ; \tag{2.2.3}$$

so the resulting equation is

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{21}}{\partial x_2} + [S_{31}]_{-a/2}^{a/2} + F_1$$

$$= \rho a\frac{\partial^2 u}{\partial t^2} \quad . \tag{2.2.4}$$

Similarly, (2.1.32) becomes

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + [S_{32}]_{-a/2}^{a/2} + F_2$$

$$= \rho a\frac{\partial^2 v}{\partial t^2} \tag{2.2.5}$$

and (2.1.33) changes to

$$\frac{\partial}{\partial x_1}(N_{11}\frac{\partial w}{\partial x_1} + N_{12}\frac{\partial w}{\partial x_2} + N_{13})$$

$$+ \frac{\partial}{\partial x_2}(N_{21}\frac{\partial w}{\partial x_1} + N_{22}\frac{\partial w}{\partial x_2} + N_{23})$$

$$+ \left[S_{31}\frac{\partial w}{\partial x_1} + S_{32}\frac{\partial w}{\partial x_2} + S_{33}\right]_{-a/2}^{a/2} + F_3 = \rho a\frac{\partial^2 w}{\partial t^2} \quad . \tag{2.2.6}$$

Next, multiply each term of (2.1.31) and (2.1.32) by x_3, then integrate with respect to x_3 over the plate thickness to find the two equations

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} + [x_3 S_{31}]_{-a/2}^{a/2} - N_{13} + m_1 = -\rho\frac{a^3}{12}\frac{\partial^3 w}{\partial x_1 \partial t^2} \tag{2.2.7}$$

and

$$\frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} + [x_3 S_{32}]_{-a/2}^{a/2} - N_{23} + m_2 = -\rho\frac{a^3}{12}\frac{\partial^3 w}{\partial x_2 \partial t^2} \tag{2.2.8}$$

where

$$m_i = \int_{-a/2}^{a/2} x_3 f_i \, dx_3$$

Solve (2.2.7) and (2.2.8) for N_{13} and N_{23}; substitute these results into (2.2.6), and rearrange the terms. This leads to the equation

$$\frac{\partial^2 M_{11}}{\partial x_1^{\,2}} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^{\,2}} + \frac{\partial}{\partial x_1}(N_{11}\frac{\partial w}{\partial x_1})$$

$$+ \frac{\partial}{\partial x_2}(N_{22}\frac{\partial w}{\partial x_2}) + \frac{\partial}{\partial x_1}(N_{12}\frac{\partial w}{\partial x_2}) + \frac{\partial}{\partial x_2}(N_{12}\frac{\partial w}{\partial x_1})$$

$$+ \frac{\partial}{\partial x_1}\{[x_3 S_{13}]_{-a/2}^{a/2}\} + \frac{\partial}{\partial x_2}\{[x_3 S_{23}]_{-a/2}^{a/2}\}$$

$$+ [S_{13}\frac{\partial w}{\partial x_1} + S_{23}\frac{\partial w}{\partial x_2} + S_{33}]_{-a/2}^{a/2} + \frac{\partial m_1}{\partial x_1} + \frac{\partial m_2}{\partial x_2}$$

$$+ F_3 = \rho a\frac{\partial^2 w}{\partial t^2} - \rho\frac{a^3}{12}\frac{\partial^4 w}{\partial x_1^{\,2}\partial t^2} - \rho\frac{a^3}{12}\frac{\partial^4 w}{\partial x_2^{\,2}\partial t^2} \quad . \tag{2.2.9}$$

The first-order equations of motion in terms of plate stresses are (2.2.4), (2.2.5), and (2.2.9).

The advantage of the plate-stress equations is immediately noticeable--the boundary conditions on stress are incorporated into the equations of motion.

In the initial configuration, the upper and lower faces ($x_3 = \pm a/2$) are free from shearing effects. However, there are loads which are normal to the plane of the plate.

$$[S_{31}]_{-a/2}^{a/2} = [S_{32}]_{-a/2}^{a/2} = [S_{31}\frac{\partial w}{\partial x_1} + S_{32}\frac{\partial w}{\partial x_2}]_{-a/2}^{a/2} = 0$$

and

$$[S_{33}]_{-a/2}^{a/2} = P \quad .$$

Accordingly, the plate stress equations reduce to

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + F_1 = \rho a \frac{\partial^2 u}{\partial t^2} \quad , \tag{2.2.10}$$

$$\frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + F_2 = \rho a \frac{\partial^2 v}{\partial t^2} \quad , \tag{2.2.11}$$

$$\frac{\partial^2 M_{11}}{\partial x_1{}^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2} + \frac{\partial}{\partial x_1}(N_{11}\frac{\partial w}{\partial x_1})$$
$$+ \frac{\partial}{\partial x_2}(N_{22}\frac{\partial w}{\partial x_2}) + \frac{\partial}{\partial x_1}(N_{12}\frac{\partial w}{\partial x_2}) + \frac{\partial}{\partial x_2}(N_{12}\frac{\partial w}{\partial x_1}) + P$$
$$+ \frac{\partial m_1}{\partial x_1} + \frac{\partial m_2}{\partial x_2} + F_3 = \rho a \frac{\partial^2 w}{\partial t^2} - \rho \frac{a^3}{12}(\frac{\partial^4 w}{\partial x_1{}^2 \partial t^2}$$
$$+ \frac{\partial^4 w}{\partial x_2{}^2 \partial t^2}) \tag{2.2.12}$$

The next step is to eliminate the plate stresses in favor of the displacements. This goal is achieved through the use of the constitutive relations (2.1.24), the strain displacement equations (2.1.26) and the plate stress definitions (2.2.1) and (2.2.2). First, change E_{33}, (2.1.26), to bring it into agreement with assumption (2.1.30). From equation (2.1.24),

$$S_{33} = \lambda(E_{11} + E_{22}) + (\lambda + 2\mu)E_{33} \quad . \qquad (2.2.13)$$

Therefore, following the assumption for S_{33} in (2.1.30),

$$E_{33} \cong - \frac{\lambda}{\lambda + 2\mu}(E_{11} + E_{22}) \qquad (2.2.14)$$

The three required stresses, found by substituting (2.2.14) into (2.1.24), are

$$S_{11} = \frac{E}{1 - \nu^2} E_{11} + \frac{\nu E}{1 - \nu^2} E_{22} \quad ,$$

$$S_{22} = \frac{\nu E}{1 - \nu^2} E_{11} + \frac{E}{1 - \nu^2} E_{22} \quad ,$$

$$S_{12} = \frac{E}{1 + \nu} E_{12} \quad . \qquad (2.2.15)$$

where E is Young's modulus and ν is Poisson's ratio. Furthermore, substituting (2.1.26) into (2.2.15) defines the three significant stresses in terms of the displacements

$$S_{11} = \frac{E}{1 - \nu^2}\left[\frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]$$

$$+ \frac{\nu E}{1 - \nu^2}\left[\frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right] \quad ,$$

$$S_{22} = \frac{E}{1 - \nu^2}\left[\frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right]$$

$$+ \frac{\nu E}{1 - \nu^2}\left[\frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right] \quad ,$$

$$S_{12} = \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} - 2x_3\frac{\partial^2 w}{\partial x_1 \partial x_2}\right.$$

$$\left. + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right] ; \qquad (2.2.16)$$

whereupon the plate stresses (2.2.1) and (2.2.2) can be defined as

$$N_{11} = \frac{Ea}{1-\nu^2}\left\{\left[\frac{\partial u}{\partial x_1} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right] + \nu\left[\frac{\partial v}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right]\right\},$$

$$N_{22} = \frac{Ea}{1-\nu^2}\left\{\left[\frac{\partial v}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right] + \nu\left[\frac{\partial u}{\partial x_1} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]\right\},$$

$$N_{12} = \frac{Ea}{2(1+\nu)}\left[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right] ,$$

$$M_{11} = -\frac{Ea^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x_1^2} + \nu\frac{\partial^2 w}{\partial x_2^2}\right] ,$$

$$M_{22} = -\frac{Ea^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x_2^2} + \nu\frac{\partial^2 w}{\partial x_1^2}\right] ,$$

$$M_{12} = -\frac{Ea^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x_1 \partial x_2} . \qquad (2.2.17)$$

Using (2.2.17) in (2.2.10), (2.2.11), and (2.2.12) leads to the displacement equations of motion

$$R\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial x_2^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\right.$$

$$\left. + \frac{\partial w}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2 w}{\partial x_2^2}\right) + \frac{1+\nu}{2}\frac{\partial w}{\partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]$$

$$+ F_1 = \rho a \frac{\partial^2 u}{\partial t^2} , \qquad (2.2.18)$$

$$R\left[\frac{\partial^2 v}{\partial x_2^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x_1^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right.$$

$$\left. + \frac{\partial w}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_2^2} + \frac{1-\nu}{2}\frac{\partial^2 w}{\partial x_1^2}\right) + \frac{1+\nu}{2}\frac{\partial w}{\partial x_1}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]$$

$$+ F_2 = \rho a \frac{\partial^2 v}{\partial t^2} , \qquad (2.2.19)$$

$$-D\left(\frac{\partial^4 w}{\partial x_1^4} + \frac{2\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4}\right) + \frac{\partial}{\partial x_1}\left\{R\left[\frac{\partial u}{\partial x_1}\right.\right.$$

$$\left.\left. + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2 + \nu\left(\frac{\partial u}{\partial x_2}\right) + \frac{\nu}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right]\frac{\partial w}{\partial x_1}\right\}$$

$$+ \frac{\partial}{\partial x_2}\left\{R\left[\frac{\partial v}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2 + \nu\frac{\partial u}{\partial x_1} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]\frac{\partial w}{\partial x_2}\right\}$$

$$+ \frac{\partial}{\partial x_1}\left\{\left(\frac{1-\nu}{2}\right)R\left[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2}\frac{\partial w}{\partial x_1}\right]\frac{\partial w}{\partial x_2}\right\}$$

$$+ \frac{\partial}{\partial x_2}\left\{\left(\frac{1-\nu}{2}\right)R\left[\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right]\frac{\partial w}{\partial x_1}\right\} + \frac{\partial m_1}{\partial x_1}$$

$$+ \frac{\partial m_2}{\partial x_2} + F_3 + P = \rho a \frac{\partial^2 w}{\partial t^2} - \rho \frac{a^3}{12} \left(\frac{\partial^4 w}{\partial x_1^{\,2} \partial t^2} + \frac{\partial^4 w}{\partial x_2^{\,2} \partial t^2} \right) , \tag{2.2.20}$$

in which $R = Ea/(1 - \nu^2)$ and $D = Ea^3/12(1 - \nu^2)$. These three equations govern the motions of the plate.

In the following section, the body-force will be defined and each of the body-force-terms used above will be evaluated.

2.3. Magnetoelastic Plate Equations

Details on the electromagnetic theory of moving bodies can be found in many texts. Here Sommerfeld [15], from which a few ideas are transcribed, is used as a reference.

To an observer moving with the body, and to one who is fixed, different values are witnessed for the same electromagnetic quantity. The relationships between fixed-frame and moving-frame values for the several items are

$$\begin{aligned}
\underset{\sim}{E}' &= \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}, \\
\underset{\sim}{B}' &= \underset{\sim}{B} - \underset{\sim}{v} \times \underset{\sim}{E}/c^2, \\
\underset{\sim}{D}' &= \underset{\sim}{D} + \underset{\sim}{v} \times \underset{\sim}{H}/c^2, \\
\underset{\sim}{H}' &= \underset{\sim}{H} - \underset{\sim}{v} \times \underset{\sim}{D}, \\
\underset{\sim}{J}' &= \underset{\sim}{J} - \rho \underset{\sim}{v}, \\
\rho_e' &= \rho_e .
\end{aligned} \tag{2.3.1}$$

Equations (2.3.1) are the non-relativistic forms of those quoted by Sommerfeld ($v^2/c^2 \ll 1$). Primes denote quantities measured in the moving frame. The symbols are conventional: $\underset{\sim}{E}$ is the electric field, $\underset{\sim}{D}$ the electric displacement, $\underset{\sim}{B}$ the magnetic induction, $\underset{\sim}{H}$ the magnetic field, $\underset{\sim}{J}$ the conduction current density, ρ_e the charge density, and $\underset{\sim}{v}$ the particle velocity.

A fundamental supposition is that the electromagnetic constitutive equations evaluated in the moving body are identical to the familiar static body-equations; so

$$\underset{\sim}{D}' = \varepsilon \underset{\sim}{E}'$$

$$\underset{\sim}{B}' = \kappa \underset{\sim}{H}'$$

$$\underset{\sim}{J}' = \sigma \underset{\sim}{E}' \qquad (2.3.2)$$

In these relations, ε, κ, and σ are scalar constants* and they respectively represent the permittivity, permeability, and conductivity of the body. Substituting (2.3.1) into (2.3.2) produces the constitutive equations applicable to the fixed frame of reference,

$$\underset{\sim}{D} = \varepsilon \underset{\sim}{E} + (\varepsilon\kappa - \varepsilon_o\kappa_o)\underset{\sim}{v} \times \underset{\sim}{H},$$

$$\underset{\sim}{B} = \kappa \underset{\sim}{H} - (\varepsilon\kappa - \varepsilon_o\kappa_o)\underset{\sim}{v} \times \underset{\sim}{E},$$

$$\underset{\sim}{J} = \rho_e \underset{\sim}{v} + \sigma(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}), \qquad (2.3.3)$$

where ε_o and κ_o are the free-space values of ε and κ.

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*The assumptions of homogeneity and isotropy apply to electromagnetic properties also.

Maxwell's equations are unchanged in transforming from the moving to the fixed frame of reference. They are

$$-\nabla \times \underset{\sim}{E} = \frac{\partial \underset{\sim}{B}}{\partial t} \quad ,$$

$$\nabla \times \underset{\sim}{H} = \frac{\partial \underset{\sim}{D}}{\partial t} + \underset{\sim}{J} \quad ,$$

$$\nabla \cdot \underset{\sim}{D} = \rho_e \quad ,$$

$$\nabla \cdot \underset{\sim}{B} = 0 \quad . \qquad (2.3.4)$$

In magnetoelasticity, the magnetic and electric fields may be expressed as

$$\underset{\sim}{H} = \underset{\sim}{H}_o + \underset{\sim}{h} \quad ,$$

$$\underset{\sim}{E} = \underset{\sim}{e} \quad . \qquad (2.3.5)$$

$\underset{\sim}{H}_o$ is the large, static, uniform field which exists when the body is stationary. The motion of the body induces fields $\underset{\sim}{h}$ which is presumed to be small with respect to $\underset{\sim}{H}_o$, and $\underset{\sim}{e}$ which is presumed to be small in the sense defined in section 2.4.

The purpose of this section, to introduce the effect of the electromagnetic system on the mechanical system, can now be accomplished. Recall from the introduction that this interaction is taken to be represented sufficiently by the Lorentz body-force,

$$\underset{\sim}{f} = \rho_e \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B} \qquad (2.3.6)$$

Substitute (2.3.3) and (2.3.5) together with the velocity determined from (2.1.7), (2.1.8), and (2.1.9) into (2.3.6).

Reduce the resulting expression by dropping all terms which are products of $\underset{\sim}{e}$, $\underset{\sim}{h}$, $\underset{\sim}{u}$, and their derivitives; whereupon the body-force becomes

$$\underset{\sim}{f} = \kappa\sigma \underset{\sim}{e} \times \underset{\sim}{H}_o + \sigma\kappa^2(\underset{\sim}{v}\cdot\underset{\sim}{H}_o)\underset{\sim}{H}_o - \sigma\kappa^2(\underset{\sim}{H}_o\cdot\underset{\sim}{H}_o)\underset{\sim}{v}. \tag{2.3.7}$$

Now the various body-force-terms can be evaluated; they are

$$F_1 = a\kappa\sigma\{(e_2H_{03} - e_3H_{02}) + \kappa[(H_{01}^2 - H_0^2)\frac{\partial u}{\partial t} + H_{02}H_{01}\frac{\partial v}{\partial t} + H_{03}H_{01}\frac{\partial w}{\partial t}]\},$$

$$F_2 = a\kappa\sigma\{(e_3H_{01} - e_1H_{03}) + \kappa[H_{01}H_{02}\frac{\partial u}{\partial t} + (H_{02}^2 - H_0^2)\frac{\partial v}{\partial t} + H_{03}H_{02}\frac{\partial w}{\partial t}]\},$$

$$F_3 = a\kappa\sigma\{(e_1H_{02} - e_2H_{01}) + \kappa[H_{01}H_{03}\frac{\partial u}{\partial t} + H_{02}H_{03}\frac{\partial v}{\partial t} + (H_{03}^2 - H_0^2)\frac{\partial w}{\partial t}]\},$$

$$m_1 = -\frac{\kappa^2\sigma a^3}{12}[(H_{01}^2 - H_0^2)\frac{\partial^2 w}{\partial x_1\partial t} + H_{02}H_{01}\frac{\partial^2 w}{\partial x_2\partial t}],$$

$$m_2 = -\frac{\kappa^2\sigma a^3}{12}[H_{01}H_{02}\frac{\partial^2 w}{\partial x_1\partial t} + (H_{02}^2 - H_0^2)\frac{\partial^2 w}{\partial x_2\partial t}]. \tag{2.3.8}$$

The substitution of (2.3.8) into the equations of motion (2.2.18), (2.2.19) and (2.2.20) yields the first order magnetoelastic plate equations:

$$R\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial x_2^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial w}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2 w}{\partial x_2^2}\right) + \frac{1+\nu}{2}\frac{\partial w}{\partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]$$

$$+ a\kappa\sigma(e_2 H_{03} - e_3 H_{02}) + a\kappa^2\sigma\left[(H_{01}^2 - H_0^2)\frac{\partial u}{\partial t} + H_{02}H_{01}\frac{\partial v}{\partial t} + H_{03}H_{01}\frac{\partial w}{\partial t}\right] = \rho a \frac{\partial^2 u}{\partial t^2} , \qquad (2.3.9)$$

$$R\left[\frac{\partial^2 v}{\partial x_2^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x_1^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial w}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_2^2} + \frac{1-\nu}{2}\frac{\partial^2 w}{\partial x_1^2}\right) + \frac{1+\nu}{2}\frac{\partial w}{\partial x_1}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]$$

$$+ a\kappa\sigma(e_3 H_{01} - e_1 H_{03}) + a\kappa^2\sigma\left[H_{01}H_{02}\frac{\partial u}{\partial t} + (H_{02}^2 - H_0^2)\frac{\partial v}{\partial t} + H_{03}H_{02}\frac{\partial w}{\partial t}\right] = \rho a \frac{\partial^2 v}{\partial t^2} , \qquad (2.3.10)$$

$$-D\left(\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4}\right) + \frac{\partial}{\partial x_1}\left\{R\left[\frac{\partial u}{\partial x_1} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right.\right.$$

$$+ \nu\frac{\partial v}{\partial x_2} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\Big]\frac{\partial w}{\partial x_1}\Big\} + \frac{\partial}{\partial x_2}\Big\{R\Big[\frac{\partial v}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2$$

$$+ \nu\frac{\partial u}{\partial x_1} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\Big]\frac{\partial w}{\partial x_2}\Big\} + \frac{\partial}{\partial x_1}\Big\{\left(\frac{1-\nu}{2}\right)R\Big[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}$$

$$+ \frac{\partial w}{\partial x_2}\frac{\partial w}{\partial x_1}\Big]\frac{\partial w}{\partial x_2}\Big\} + \frac{\partial}{\partial x_2}\Big\{\left(\frac{1-\nu}{2}\right)R\Big[\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}$$

$$+ \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\Big]\frac{\partial w}{\partial x_1}\Big\} - \frac{a^3\kappa^2\sigma}{12}\Big[(H_{01}^2 - H_0^2)\frac{\partial^3 w}{\partial x_1^2\partial t}$$

$$+ 2H_{01}H_{02}\frac{\partial^3 w}{\partial x_1\partial x_2\partial t} + (H_{02}^2 - H_0^2)\frac{\partial^3 w}{\partial x_2^2\partial t}\Big]$$

$$+ a\kappa\sigma(e_1H_{02} - e_2H_{01}) + a\kappa^2\sigma\Big[H_{01}H_{03}\frac{\partial u}{\partial t}$$

$$+ H_{02}H_{03}\frac{\partial v}{\partial t} + (H_{03}^2 - H_0^2)\frac{\partial w}{\partial t}\Big] + P = \rho a\frac{\partial^2 w}{\partial t^2}$$

$$- \rho\frac{a^3}{12}\left(\frac{\partial^4 w}{\partial x_1^2\partial t^2} + \frac{\partial^4 w}{\partial x_2^2\partial t^2}\right) \quad . \qquad (2.3.11)$$

These equations show that the displacement components are coupled not only to each other, but also to the electromagnetic fields. Thus, a complete description of the behavior must include the plate equations (2.3.9), (2.3.10), and (2.3.11) together with Maxwell's equations (2.3.4)

suitably altered by (2.3.5). As it stands, the first order theory remains intractable for the present purpose.

2.4. Simplification of the First Order Theory

It is assumed in (2.3.5) that the motion of a conductor through a strong, static, magnetic field induces two very much smaller fields $\tilde{e}$ and $\tilde{h}$. Equation (2.3.7) indicates that the only feedback from the electromagnetic system to the plate is attributable to the $\tilde{e}$-field through the eH_0-term. The $\tilde{e}$-field will now be assumed to be small so that this term may be dropped. Justification for the omission is based upon a conclusion reached by Dunkin and Eringen, "that the large, static, electric field as studied here does not introduce a significant coupling between dynamic, electromagnetic effects and elastic effects". Thereby, Maxwell's equations are uncoupled from the plate equations, and the first-order, magnetoelastic plate theory is completely contained in the equations of motion.

$$R\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial x_2^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial w}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2 w}{\partial x_2^2}\right) + \frac{1+\nu}{2}\frac{\partial w}{\partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right] + a\kappa^2\sigma\left[H_{01}^2 - H_0^2\right)\frac{\partial u}{\partial t} + H_{01}H_{02}\frac{\partial v}{\partial t} + H_{01}H_{03}\frac{\partial w}{\partial t}\right] = \rho a\frac{\partial^2 u}{\partial t^2}, \quad (2.3.12)$$

$$R\left[\frac{\partial^2 v}{\partial x_2^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x_1^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right.$$

$$\left. + \frac{\partial w}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_2^2} + \frac{1-\nu}{2}\frac{\partial^2 w}{\partial x_1^2}\right) + \frac{1+\nu}{2}\frac{\partial w}{\partial x_1}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]$$

$$+ a\kappa^2\sigma\left[H_{02}H_{01}\frac{\partial u}{\partial t} + (H_{02}^2 - H_0^2)\frac{\partial v}{\partial t} + H_{02}H_{03}\frac{\partial w}{\partial t}\right]$$

$$= \rho a\frac{\partial^2 v}{\partial t^2} , \qquad (2.3.13)$$

$$-D\left(\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4}\right) + \frac{\partial}{\partial x_1}\left\{R\left[\frac{\partial u}{\partial x_1} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right.\right.$$

$$\left.\left. + \nu\frac{\partial v}{\partial x_2} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right]\frac{\partial w}{\partial x_1}\right\} + \frac{\partial}{\partial x_2}\left\{R\left[\frac{\partial v}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right.\right.$$

$$\left.\left. + \nu\frac{\partial u}{\partial x_1} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]\frac{\partial w}{\partial x_2}\right\} + \frac{\partial}{\partial x_1}\left\{\left(\frac{1-\nu}{2}\right)R\left[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right.\right.$$

$$\left.\left. + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right]\frac{\partial w}{\partial x_2}\right\} + \frac{\partial}{\partial x_2}\left\{\left(\frac{1-\nu}{2}\right)R\left[\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\right.\right.$$

$$\left.\left. + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right]\frac{\partial w}{\partial x_1}\right\} - \frac{a^3\kappa^2\sigma}{12}\left[(H_{01}^2 - H_0^2)\frac{\partial^3 w}{\partial x_1^2 \partial t}\right.$$

$$\left. + 2H_{01}H_{02}\frac{\partial^3 w}{\partial x_1 \partial x_2 \partial t} + (H_{02}^2 - H_0^2)\frac{\partial^3 w}{\partial x_2^2 \partial t}\right]$$

$$+ a\kappa^2\sigma\left[H_{01}H_{03}\frac{\partial u}{\partial t} + H_{02}H_{03}\frac{\partial v}{\partial t} + (H_{03}^2 - H_0^2)\frac{\partial w}{\partial t}\right]$$

$$+ P = \rho a\frac{\partial^2 w}{\partial t^2} - \rho\frac{a^3}{12}\left(\frac{\partial^4 w}{\partial x_1^2 \partial t^2} + \frac{\partial^4 w}{\partial x_2^2 \partial t^2}\right) . \quad (2.3.14)$$

In the ensuing chapters, this theory is used to study the response of the plate to a particular excitation P.

III. RESPONSE OF A LINEAR MAGNETOELASTIC PLATE TO A TRAVELLING LOAD--GENERAL CONSIDERATIONS

3.1. One-Dimensional Theory

Assume that the load acting on the lateral surfaces of the plate is a function of x_1 and t only:

$$P = P(x_1, t). \qquad (3.1.1)$$

Consequently, all the x_2-derivatives in equations (2.3.12), (2.3.13), and (2.3.14) may be omitted. The resulting equations of motion are

$$R\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial w}{\partial x_1}\frac{\partial^2 w}{\partial x_1^2}\right) + a\kappa^2\sigma\left[(H_{01}^2 - H_0^2)\frac{\partial u}{\partial t} + H_{02}H_{01}\frac{\partial v}{\partial t} + H_{01}H_{03}\frac{\partial w}{\partial t}\right] = \rho a\frac{\partial^2 u}{\partial t^2}, \qquad (3.1.2)$$

$$\left(\frac{1-\nu}{2}\right) R\frac{\partial^2 v}{\partial x_1^2} + a\kappa^2\sigma\left[H_{01}H_{02}\frac{\partial u}{\partial t} + (H_{02}^2 - H_0^2)\frac{\partial v}{\partial t} + H_{03}H_{02}\frac{\partial w}{\partial t}\right] = \rho a\frac{\partial^2 v}{\partial t^2}, \qquad (3.1.3)$$

$$-D\frac{\partial^4 w}{\partial x_1^4} + R\frac{\partial}{\partial x_1}\left\{\left[\frac{\partial u}{\partial x_1} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]\frac{\partial w}{\partial x_1}\right\}$$

$$- \frac{a^3\kappa^2\sigma}{12}\left[(H_{01}^2 - H_0^2)\frac{\partial^3 w}{\partial x_1^2\partial t}\right] + a\kappa^2\sigma\left[H_{01}H_{03}\frac{\partial u}{\partial t}\right.$$

$$+ H_{02}H_{03}\frac{\partial v}{\partial t} + (H_{03}{}^2 - H_0{}^2)\frac{\partial w}{\partial t}] + P = \rho a\frac{\partial^2 w}{\partial t^2}$$

$$- \rho\frac{a^3}{12}\frac{\partial^4 w}{\partial x_1{}^2 \partial t^2} \quad . \qquad (3.1.4)$$

Assume further that load P is a travelling load which is moving in the x_1-direction with speed S. Specifically, define

$$P(x_1, t) = P(x_1 - St). \qquad (3.1.5)$$

For steady state solutions relative to the moving load, it is possible to eliminate the two independent variables x_1 and t in favor of a single moving coordinate

$$\tau = x_1 - St \quad . \qquad (3.1.6)$$

Accordingly, the various differentiation operations become

$$\frac{\partial^m}{\partial x_1{}^m} = \frac{d^m}{d\tau^m} ,$$

$$\frac{\partial^n}{\partial t^n} = (-S)^n \frac{d^n}{d\tau^n} ,$$

$$\frac{\partial^{(m+n)}}{\partial x_1{}^m \partial t^n} = (-S)^n \frac{d^{(m+n)}}{d\tau^{(m+n)}} \quad . \qquad (3.1.7)$$

The equations of motion are no longer partial differential equations, but ordinary differential equations; they take the form:

$$\left(1 - \frac{S^2}{C_p^2}\right)u'' + w'w'' - \frac{\kappa^2\sigma S}{\rho C_p^2}[(H_{01}^2 - H_0^2)u'$$

$$+ H_{02}H_{01}v' + H_{03}H_{01}w'] = 0 , \qquad (3.1.8)$$

$$\left(\frac{C_s^2 - S^2}{C_p^2}\right)v'' - \frac{\kappa^2\sigma S}{\rho C_p^2}[H_{01}H_{02}u' + (H_{02}^2 - H_0^2)v'$$

$$+ H_{03}H_{02}w'] = 0, \qquad (3.1.9)$$

$$-\left(1 - \frac{S^2}{C_p^2}\right)\frac{w''''}{12} - \frac{S^2}{a^2C_p^2}w'' + \frac{1}{a^2}\{[u' + \frac{1}{2}(w')^2]w'\}'$$

$$- \frac{\kappa^2\sigma S}{\rho C_p^2 a^2}[-\left(\frac{H_{01}^2 - H_0^2}{12}\right)a^2 w''' + H_{01}H_{03}u'$$

$$+H_{02}H_{03}v' + (H_{03}^2 - H_0^2)w'] + \frac{P}{\rho a^3 C_p^2} = 0 \qquad (3.1.10)$$

in which the classical wave speeds,

$$C_p^2 = E/\rho(1 - \nu^2),$$

$$C_s^2 = E/2\rho(1 + \nu) \qquad (3.1.11)$$

are used to redefine the constants

$$D = (\rho a^3/12)C_p^2,$$

$$R = \rho a C_p^2 = [2\rho a/(1 - \nu)]C_s^2 . \qquad (3.1.12)$$

Before proceeding, it is reasonable to rid the theory of any forthcoming dimensional difficulties by nondimensionalizing the variables in the following manner. Let

$$U = u/a \ ,$$

$$V = v/a \ ,$$

$$W = w/a \ ,$$

$$H_i = H_{oi}/\Xi \ ,$$

$$\zeta = \tau/a \ . \qquad (3.1.13)$$

This last equality defines the differentiation operation

$$\frac{d^n}{d\tau^n} = \frac{1}{a^n}\frac{d^n}{d\zeta^n} \ .$$

The force-term P can be represented dimensionally as

$$[\frac{F}{L^2}] = [\frac{M}{LT^2}] = [\frac{M}{L^3}]/[\frac{T^2}{L^2}] \ ;$$

therefore, the nondimensionalizing quantity chosen for the force term is $\rho C_p^{\,2}$, whereupon

$$\Pi = P/\rho C_p^{\,2} \ .$$

Now the first order magnetoelastic theory can be written in its final form. The motion of a thin plate, subjected to a moving load and a strong magnetic field, is governed by the three coupled nonlinear ordinary differential equations:

$$(1 - \frac{S^2}{C_p^{\,2}})U'' + W'W'' - \frac{\kappa^2\sigma S a \Xi^2}{\rho C_p^{\,2}}[(H_1^{\,2} - H^2)U'$$

$$+ H_2H_1V' + H_3H_1W'] = 0 \ , \qquad (3.1.14)$$

$$(\frac{C_s^{\,2} - S^2}{C_p^{\,2}})V'' - \frac{\kappa^2\sigma S a \Xi^2}{\rho C_p^{\,2}}[H_1H_2U' + (H_2^{\,2} - H^2)V'$$

$$+ H_3H_2W'] = 0 \ , \qquad (3.1.15)$$

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$$-\left(1 - \frac{s^2}{C_p^2}\right)\frac{W''''}{12} - \frac{s^2}{C_p^2}W'' + \{[U' + \tfrac{1}{2}(W')^2]W'\}'$$

$$- \frac{\kappa^2\sigma Sa\Xi^2}{\rho C_p^2} \left[-(H_1^2 - H^2)\frac{W'''}{12} + H_1H_3U' + H_2H_3V'\right.$$

$$\left. + (H_3^2 - H^2)W'\right] + \Pi = 0 \ . \qquad (3.1.16)$$

3.2. Perturbation Expansions

Subsequently, the nondimensionalized travelling load Π in (3.1.16) will be taken as

$$\Pi = -\varepsilon\delta(\zeta) \qquad (3.2.1)$$

where $\delta(\zeta)$ is the Dirac delta function, and ε, the dimensional load intensity, is a small perturbation parameter. The minus sign indicates that the load acts in the negative x_3-direction.

Equations (3.1.14), (3.1.15), and (3.1.16) suggest that each of the deflections U, V, and W is of the order ε; thus the following power series expansions are introduced

$$U = \sum_{i=1}^{\infty} U_{i-1}\varepsilon^i \ ,$$

$$V = \sum_{i=1}^{\infty} V_{i-1}\varepsilon^i \ ,$$

$$W = \sum_{i=1}^{\infty} W_{i-1}\varepsilon^i \ .$$

Furthermore, there also may be the necessity, which will be explained shortly, to expand the load speed as the power series

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$$S = \sum_{i=0}^{\infty} S_i \varepsilon^i \quad . \tag{3.2.3}$$

Substitute these series into the three equations of motion, (3.1.14), (3.1.15) and (3.1.16), and collect terms of similar power in ε. The resulting equations are

$$\varepsilon^1: \quad L_u(U_0) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{H_1 H_2 S_0 V_0{}' + H_1 H_3 S_0 W_0{}'\} = 0 \quad ,$$

$$L_v(V_0) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{H_2 H_1 S_0 U_0{}' + H_2 H_3 S_0 W_0{}'\} = 0 \quad ,$$

$$L_w(W_0) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{H_3 H_1 S_0 U_0{}' + H_3 H_2 S_0 V_0{}'\}$$

$$- \delta = 0 \quad , \tag{3.2.4}$$

$$\varepsilon^2: \quad L_u(U_1) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{H_1 H_2 S_0 V_1{}' + H_1 H_3 S_0 W_1{}'\}$$

$$= \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{(H_1^{\,2} - H^2) S_1 U_0{}' + H_1 H_2 S_1 V_0{}'$$

$$+ H_1 H_3 S_1 W_0{}'\} + \frac{2 S_0 S_1}{C_p^{\,2}} U_0{}'' - W_0{}' W_0{}'' \quad ,$$

$$L_v(V_1) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{H_2 H_1 S_0 U_1{}' + H_2 H_3 S_0 W_1{}'\}$$

$$= \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\,2}} \{(H_2^{\,2} - H^2) S_1 V_0{}' + H_2 H_3 S_1 W_0{}'$$

$$+ H_2H_1S_1U_0'\} + \frac{2S_oS_1}{C_p^2}V_0'' \quad ,$$

$$L_w(W_1) - \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^2}\{H_3H_1S_0U_1' + H_3H_2S_0V_1'\}$$

$$= \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^2}\{-(\frac{H_1^2 - H^2}{12})S_1W_0''' + H_3H_2S_1V_0'$$

$$+ (H_3^2 - H^2)S_1W_0' + H_3H_1S_1U_0'\} - (U_0''W_0'$$

$$+ U_0'W_0'') + \frac{2S_0S_1}{C_p^2}W_0'' - \frac{S_0S_1}{6C_p^2}W_0'''' \quad , \qquad (3.2.5)$$

$$\varepsilon^3: \quad L_u(U_2) - \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^2}\{H_1H_2S_0V_2' + H_1H_3S_0W_2''\}$$

$$= \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^2}\{H_1^2 - H^2)(S_1U_1' + S_2U_0') + H_1H_2(S_1V_1'$$

$$+ S_2V_0') + H_1H_3(S_1W_1' + S_2W_0')\} + \frac{2S_0S_1}{C_p^2}U_0''$$

$$- W_0'W_1'' - W_1'W_0'' \quad ,$$

$$L_v(V_2) - \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^2}\{H_2H_1S_0U_2' + H_2H_3S_0W_2'\}$$

$$= \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^2}\{H_2H_1(S_1U_1' + S_2U_0') + (H_2^2 - H^2)(S_1V_1'$$

$$+ S_2V_0') + H_2H_3(S_1W_1' + S_2W_0')\} + \frac{1}{C_p^2}(V_0''S_1^2$$

$$+ 2V_0''S_1S_2 + 2V_1''S_0S_1) \quad ,$$

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$$L_w(W_2) - \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^{\,2}}\{H_3H_1S_0U_2' + H_3H_2S_0V_2'\}$$

$$= \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^{\,2}}\{- \frac{H_1^{\,2} - H^2}{12}(S_1W_1''' + S_2W_0''')$$

$$+ H_3H_1(S_1U_1' + S_2U_0') + H_3H_2(S_1V_1' + S_2V_0')$$

$$+ (H_3^{\,2} - H^2)(S_1W_1' + S_2W_0')\} + \frac{1}{C_p^{\,2}}[W_0''S_1^{\,2}$$

$$+ 2W'' S_0S_2 + 2W_1'' S_0S_1] - \frac{1}{12C_p^{\,2}}(W_0'''' S_1^{\,2}$$

$$+ 2W_0'''' S_0S_2 + 2W_1'''' S_0S_1) - [W_1'U_0'' + W_0'U_1''$$

$$+ U_0'W_1'' + U_1'W_0'' + \frac{3}{2}(W_0')^2W_0''] \qquad (3.2.6)$$

ε^4: ....

where

$$L_u = (1 - \frac{S_0^{\,2}}{C_p^{\,2}})\frac{d^2}{d\zeta^2} - \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^{\,2}}(H_1^{\,2} - H^2)S_0\frac{d}{d\zeta} \qquad (3.2.7)$$

$$L_v = (\frac{C_s^{\,2} - S_0^{\,2}}{C_p^{\,2}})\frac{d^2}{d\zeta^2} - \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^{\,2}}(H_2^{\,2} - H^2)S_0\frac{d}{d\zeta} \qquad (3.2.8)$$

$$L_w = - \frac{1}{12}(1 - \frac{S_0^{\,2}}{C_p^{\,2}})\frac{d^4}{d\zeta^4} - \frac{S_0^{\,2}}{C_p^{\,2}}\frac{d^2}{d\zeta^2} + \frac{\kappa^2\sigma a\Xi^2}{\rho C_p^{\,2}}$$

$$\{(\frac{H_1^{\,2} - H^2}{12})S_0\frac{d^3}{d\zeta^3} - (H_3^{\,2} - H^2)S_0\frac{d}{d\zeta}\} \qquad (3.2.9)$$

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The above equations are simplified if the load speed is not expanded. These simpler equations are achieved by setting $S_i = 0$ for $i \geq 1$ and $S_0 = s$ in (3.2.4)-(3.2.9):

$$\varepsilon^1: \quad L_u(U_0) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_1 H_2 s V_0' + H_1 H_3 s W_0'\} = 0$$

$$L_v(V_0) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_2 H_1 s U_0' + H_2 H_3 s W_0'\} = 0$$

$$L_w(W_0) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_3 H_1 s U_0' + H_3 H_2 s V_0'\} - \delta = 0$$

(3.2.10)

$$\varepsilon^2: \quad L_u(U_1) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_1 H_2 s V_1' + H_1 H_3 s W_1'\} = -W_0' W_0''$$

$$L_v(V_1) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_2 H_1 s U_1' + H_2 H_3 s W_1'\} = 0$$

$$L_w(W_1) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_3 H_1 s U_1' + H_3 H_2 s V_1'\}$$

$$= -(U_0'' W_0' + U_0' W_0'') \qquad (3.2.11)$$

$$\varepsilon^3: \quad L_u(U_2) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_1 H_2 s V_2' + H_1 H_3 s W_2'\}$$

$$= -W_0' W_1'' - W_1' W_0''$$

$$L_v(V_2) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_2 H_1 s U_2' + H_2 H_3 s W_2'\} = 0$$

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$$L_w(W_2) - \frac{\kappa^2 \sigma a \Xi^2}{\rho C_p^{\ 2}} \{H_3 H_1 s U_2' + H_3 H_2 s V_2'\}$$

$$= -[W_1'U_0'' + W_0'U_1'' + U_0'W_1'' + U_1'W_0''$$

$$+ \frac{3}{2}(W_0')^2 W_0''] \qquad (3.2.12)$$

ε^4: ....

L_u, L_v and L_w are unchanged.

Through the investigation of the linear theory, encompassed by both (3.2.4) or (3.2.10), certain load speeds (critical speeds) may be found for which the displacements become unbounded. For load speeds which lie outside the neighborhood of a critical speed, equations (3.2.10), (3.2.11) and (3.2.12) adequately describe the motion. However, if a solution is to be continued up to and perhaps beyond a critical speed, recourse must be taken to the more complicated equations (3.2.4) to (3.2.9).*

3.3. The Linear Solutions as a Green's Function

The linear theory expressed in (3.2.10) is not in its most convenient form. The notation is simplified somewhat by making the following substitutions:

$$A = 1 - s^2/C_p^{\ 2} ,$$

$$B_1 = \kappa^2 \sigma a \Xi^2/\rho C_p^{\ 2} ,$$

$$C_i = H_i^{\ 2} - H^2 ,$$

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*Yen and Tang developed this procedure [19].

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$$C_{ij} = H_i H_j \ ,$$

$$D = (C_s^{\,2} - s^2)/C_p^{\,2} \ ,$$

$$S = s/C_p^{\,2} \ ; \qquad (3.3.1)$$

whereupon the three equations of motion become

$$AU'' - B_1 S(C_1 U' + C_{12} V' + C_{13} W') = 0 \ , \qquad (3.3.2)$$

$$DV'' - B_1 S(C_{12} U' + C_2 V' + C_{23} W') = 0 \ , \qquad (3.3.3)$$

$$- \frac{A}{12} W'''' + \frac{B_1 S C_1}{12} W''' + (A-1) W'' - B_1 S(C_{13} U'$$

$$+ C_{23} V' + C_3 W') = \delta(\zeta) \ . \qquad (3.3.4)$$

Equations (3.3.2), (3.3.3) and (3.3.4) can be viewed in two ways, as equations either in U', V' and W' or in U, V and W. Subsequently, any problem for which solutions in the primed variables are sought, will be identified by the prefix, prime, otherwise the prefix, displacement, will be used. The absence of nonprimed terms in the three equations of motion creates difficulties, which shortly will become evident, that prevent the solution-technique from being used directly to find the displacement-solutions. However, any prime solution can be integrated to yield a corresponding displacement-solution; so, hereinafter, emphasis is directed towards finding the prime solutions.

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To complete the description of the prime problem, require the solutions to be bounded over the entire range of the independent variable, that is

$$U', V', W' < M \text{ in } -\infty < \zeta < \infty \qquad (3.3.5)$$

where M is some finite number.

In Appendix I, the details for finding the solution to the equations of motion are developed; the linear solution constitutes the Green's function for the governing equations. It is shown there that U', V' and W' are the solutions to the homogeneous differential equations:

$$AU'' - B_1S(C_1U' + C_{12}V' + C_{13}W'') = 0 , \qquad (3.3.6)$$

$$DV'' - B_1S(C_{12}U' + C_2V' + C_{23}W') = 0 , \qquad (3.3.7)$$

$$-\frac{A}{12}W'''' + \frac{B_1SC_1}{12}W''' + (A-1)W'' - B_1S(C_{13}U'$$

$$+ C_{23}V' + C_3W') = 0 , \qquad (3.3.8)$$

which satisfy both the boundary conditions (3.3.5) and the five continuity and jump conditions at $\zeta = 0$:

$$[U'] = [V'] = [W'] = 0 , \qquad (3.3.9)$$

$$[W''] = 0 , \qquad (3.3.10)$$

$$[W'''] = -\frac{12}{A} . \qquad (3.3.11)$$

The bracket-notation is explained in Appendix I.

Substituting the solution

$$\begin{pmatrix} U' \\ V' \\ W' \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} e^{\lambda\zeta} \qquad (3.3.12)$$

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into equations (3.3.6), (3.3.7) and (3.3.8) yields a set of algebraic equations in the unknowns r_1, r_2, r_3 and λ. These equations can be written in the matrix form

$$\begin{pmatrix} A\lambda - B_1SC_1, & -B_1SC_{12}, & -B_1SC_{13} \\ -B_1SC_{12}, & D\lambda - B_1SC_2, & -B_1SC_{23} \\ -B_1SC_{13}, & -B_1SC_{23}, & \begin{matrix} -A\lambda^3/12 + B_1SC_1\lambda^2/12 \\ + (A-1)\lambda - B_1SC_3 \end{matrix} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = 0. \qquad (3.3.13)$$

Nontrivial solutions for the amplitude vector, (r_1, r_2, r_3), exist if the value of the determinant of the coefficient matrix is equal to zero. The result is the fifth-order polynomial equation in λ,

$$\lambda(b_0\lambda^4 + 4b_1\lambda^3 + 6b_2\lambda^2 + 4b_3\lambda + b_4) = 0 \qquad (3.3.14)$$

in which

$$\begin{aligned} b_0 &= -A^2D/12, \\ 4b_1 &= -BSA[2(m^2 + n^2)D + (\ell^2 + n^2)A]/12 , \\ 6b_2 &= -S^2\{AD + B^2[\ell^2m^2 + 2n^2)A + (m^2 + n^2)^2D]/12\}, \\ 4b_3 &= BS[-B^2S^2(m^2 + n^2)n^2/12 + (\ell^2 + m^2)DA \\ &\quad + (\ell^2 + n^2)A(A-1) + (m^2 + n^2)(A-1)D] , \\ b_4 &= B^2S^2[\ell^2A + m^2D + n^2(A-1)] . \end{aligned} \qquad (3.3.15)$$

The substitution $B = B_1H^2$ has been made in these expressions. The direction cosines, ℓ, m, and n, of the magnetic field vector also have been introduced. Note that a comple

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complete quintic would contain the term, b_5. Here b_5 unexpectedly is zero--an important fact, since it assures the existence of at least one zero root. Presently, this zero root will prove to be a hindrance to the analysis.

The components of the amplitude vector can be determined from (3.3.13). For example, the algebraic expressions for r_1/r_3 and r_2/r_3, found by solving the first two equations, are

$$r_1/r_3 = BS\ \ell n(\lambda D + BS)/d$$
$$r_2/r_3 = BS\ mn(\lambda A + BS)/d \qquad (3.3.16)$$

where

$$d = AD\lambda^2 + BS\ [(\ell^2 + n^2)A + (m^2 + n^2)D]\lambda + B^2S^2n^2$$

Of course, each distinct value of λ generates a distinct amplitude vector.

At this point, the conventional procedure halts and some rather less conventional steps are taken. For the sake of clarity, these will be illustrated now.

Suppose all five values of λ are nonzero and distinct. Let λ_1 be real and positive; let λ_2 and λ_3 be complex congugates with positive real parts and let λ_4 and λ_5 be real and negative.

The solutions for U', V', and W' consist of two parts. One portion applies to the range $-\infty < \zeta < 0$ and the other portion applies to $0 < \zeta < \infty$. In other words, there is a positive portion and a negative portion to each solution.

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The boundedness-condition, (3.3.5), assigns each exponential factor to one portion or the other; all factors with exponents which have negative, real parts belong to the positive portion; the remaining factors, all of which have exponents with positive, real parts, make up the negative portion. Thereby, it is certain that U', V', and W' are bounded within each range, and they are

$$W' = \begin{cases} F_1 e^{\lambda_1 \zeta} + F_2 3^{\lambda_2 \zeta} + F_3 e^{\lambda_3 \zeta} & \zeta<0 \\ F_4 e^{\lambda_4 \zeta} + F_5 e^{\lambda_5 \zeta} & \zeta>0 \end{cases} \qquad (3.3.17)$$

$$U' = \begin{cases} r_{11} F_1 e^{\lambda_1 \zeta} + r_{12} F_2 e^{\lambda_2 \zeta} + r_{13} F_3 e^{\lambda_3 \zeta} & \zeta<0 \\ r_{14} F_4 e^{\lambda_4 \zeta} + r_{15} F_5 e^{\lambda_5 \zeta} & \zeta>0 \end{cases} \qquad (3.3.18)$$

$$V' = \begin{cases} r_{21} F_1 e^{\lambda_1 \zeta} + r_{22} F_2 e^{\lambda_2 \zeta} + r_{23} F_3 e^{\lambda_3 \zeta} & \zeta<0 \\ r_{24} F_4 e^{\lambda_4 \zeta} + r_{25} F_5 e^{\lambda_5 \zeta} & \zeta>0 \end{cases} \qquad (3.3.19)$$

The F's remain to be determined.

Application of the continuity and jump conditions yield the five equations

$$F_1 + F_2 + F_3 = F_4 + F_5 \quad ,$$

$$r_{11}F_1 + F_{12}F_2 + r_{13}F_3 = r_{14}F_4 + r_{15}F_5 \; ,$$

$$r_{21}F_1 + r_{22}F_2 + r_{23}F_3 = r_{24}F_4 + r_{25}F_5 \; ,$$

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = \lambda_4 F_4 + \lambda_5 F_5,$$

$$\lambda_4{}^2 F_4 + \lambda_5{}^2 F_5 - \lambda_1{}^2 F_1 - \lambda_2{}^2 F_2 - \lambda_3{}^2 F_3 = -12/A \qquad (3.3.20)$$

from which the five unknown F-coefficients can be solved. The solution is complete.

The solution-technique just illustrated is quite straightforward. Unfortunately, for a true set of λ-roots, difficulties arise at one of the steps: the step at which each exponential factor is assigned to either the positive or the negative portion of the solution. This assignment can be made only if the factor has an exponent with a non-zero, real part. When the exponent is zero or purely imaginary,* the factor is bounded and so valid in both ranges. The result is more unknown F-coefficients than can be determined from the five jump and continuity conditions. Consequently, a unique solution cannot be found.

Each zero valued λ-root creates two F-coefficients which is one too many. This conclusion justifies the decision reached earlier: not to pursue the displacement-solution. The displacement-problem has eight λ-roots, four of which are zero. Values for the resulting twelve F-coefficients cannot be determined because the corresponding jump and continuity conditions supply only eight equations. Clearly, a displacement-solution cannot be written.

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*Later, it is proved that purely imaginary λ-roots do not occur.

Even with the aid of the artifice introduced below, such a task would be exceedingly difficult.

In general, the prime λ-polynomial has only one zero root. It is true that there are special circumstances in which multiple, zero roots arise, but these are not important just now. The following ruse is used to circumvent the zero-root-difficulty. Judiciously choose a certain term and add it to the initial differential equations; then analyze this new problem, henceforth called the augmented problem, as was just illustrated. The purpose of the added term is to remove the zero root. Consider the technique as it is applied to the prime problem.

Augment (3.3.4) with the addition of a damping force that is velocity-dependent; the differential equation becomes

$$- \frac{A}{12} W'''' \quad \frac{B_1 S C_1}{12} W''' + (A-1)W'' - B_1 S (C_{13} U' + C_{23} V' + C_3 W') + \eta W' = \delta(\zeta) \qquad (3.3.21)$$

in which η is an arbitrarily small, positive number. The augmented problem consists of (3.3.2), (3.3.3), (3.3.21) and (3.3.5). There is only one alteration to the homogeneous differential equations: (3.3.8) contains the extra term, $+ \eta W'$. The jump and continuity conditions remain unchanged. Since the augmented λ-polynomial contains no zero roots, the augmented solution follows the

same format as that outlined in the illustration. The same conclusion is reached; bounded solutions are found.

Solutions to the prime problem can now be retrived. They are the limits of the augmented solutions as η tends to zero. However, in practice, the augmented solution is never completed. Sufficient information is supplied by the augmented λ-polynomial to return to and to finish the prime problem.

The augmented λ-polynomial is just the prime λ-polynomial perturbed by a small amount. Four of the roots are virtually identical in both problems and the fifth, small root of the augmented problem can be made as close to zero as desired. In fact, this fifth root is used to allot a sign to the zero root. As η approaches zero, it defines a set of either positive or negative numbers, the limit of which is the zero root. Whatever the limit, whether 0+ or 0-, the zero root now can be assigned to only one portion of the solution; thereby, the zero-root-difficulty is overcome.

Although an overview of the solution-technique has been presented above, all the details are missing. To supply the details for the prime solution is the purpose of the next chapter.

IV. SOLUTION OF THE PRIME-PROBLEM

4.1. General Comments

The solution technique outlined in Chapter III is deceptively simple. One step, the evaluation of the λ-roots, was taken for granted. Its tortuous course will occupy the bulk of this chapter. Most effort will be expended in analyzing the numerical solutions of the λ-quartic, the quartic part of (3.3.14).

Although general expressions for the four roots of the λ-quartic are available, they are so unwieldy as to preclude their use. For instance, one step in the procedure is to assign each exponential factor to either the negative or the positive portion of the solution which requires that the sign of the real part of each root be established. To this end, the general expressions are inefficacious.

The load speed S and the magnetic quantity B are the two primary parameters. Together they define the operating regime which can be viewed as the space encompassing all positive values of S and B. The operating regime is divided into different regions and within each region, the character of the solution is different. Since the character of the solution depends upon the behavior of the

λ-roots, then the determination of their behavior becomes the pivotal task of the entire procedure.

Most of the attendant algebraic manipulations are relegated to the Appendices.

4.2. Zero Root--Augmented Problem

Before delving into the analysis of the λ-quartic, first examine the zero root. Some of the changes in notation may appear to be capricious, but they all will be justified eventually.

The behavior of the zero root is deduced from the augmented λ-polynomial of which the relevant portion is

$$B_1^2 S^2 [C_2 C_3 A + C_3 C_1 D + C_1 C_2 (A-1) - C_{23}^2 A$$

$$- C_{12}^2 (A-1) - C_{13}^2 D] \lambda + B_1^3 S^3 (C_1 C_{23}^2$$

$$+ C_3 C_{12}^2 + C_2 C_{13}^2 - 2C_{12} C_{13} C_{23}$$

$$- C_1 C_2 C_3) - \eta B_1^2 S^2 (C_{12}^2 - C_1 C_2) = 0 \ . \qquad (4.2.1.)$$

Make the C-substitutions defined in (3.3.1). Recognize that the second term is the product of the roots of the original quintic equation, one of which is zero; so this term is zero. The remaining two terms give the expression

$$\lambda = \frac{-n^2 \eta}{A\ell^2 + Dm^2 + (A-1)n^2} \ . \qquad (4.2.2)$$

Reduce the denominator with the definitions for A and D, again taken from (3.3.1). In addition, introduce the terminology

$$X = S^2 ,$$
$$X_2 = \ell^2 + m^2 C_s^{\,2}/C_p^{\,2} = \ell^2 + m^2 \mathbb{C} \qquad (4.2.3)$$

where the symbol $\mathbb{C} = C_s^{\,2}/C_p^{\,2}$.

In its final form, the expression for λ is

$$\lambda = -n^2\, \eta/(X_2 - X) \qquad (4.2.4)$$

Taking the limit in (4.2.4) as $\eta \to 0$ specifies the sign of the zero root at any point in the operating regime. Since only the direction of the magnetic field enters, and since $X_2 \leq 1$, the limit is easily interpreted. For $X < X_2$, the zero root is 0- and for $X > X_2$, the zero root is 0+. Subsequently, this information will be used.

4.3. Discriminant and Type of Roots

In the theory of algebraic equations [16], the discriminant of a quartic is identified as

$$I^3 - 27J^2 \quad . \qquad (4.3.1)$$

From (3.3.14), the symbols I and J are

$$I = b_0 b_4 - 4b_1 b_3 + 3b_2^{\,2} \quad , \qquad (4.3.2)$$
$$b_0^{\,3} J = b_0^{\,2} HI - G^2 - 4H^3 \quad , \qquad (4.3.3)$$

and the symbols G and H are

$$G = b_0^{\,2} b_3 - 3b_0 b_1 b_2 + 2b_1^{\,3} \quad , \qquad (4.3.4)$$

$$H = b_0 b_2 - b_1^{\,2} \qquad (4.3.5)$$

After I and J are further expanded* by using (3.3.15), it is found that

$$I = I_0 + I_1 Y + I_2 Y^2 , \tag{4.3.6}$$

$$b_0{}^3 J = J_0 + J_1 Y + J_2 Y^2 + J_3 Y^3 ; \tag{4.3.7}$$

so the discriminant can be written as the sixth order polynomial in Y,

$$C_6 Y^6 + C_5 Y^5 + C_4 Y^4 + C_3 Y^3 + C_2 Y^2$$

$$C_1 Y + C_0 . \tag{4.3.8}$$

All the coefficients in (4.3.6), (4.3.7), and (4.3.8) are functions of X only. These two variables

$$X = S^2 , \qquad Y = B^2 \tag{4.3.9}$$

define a convenient coordinate system in which to view the behavior of the λ-roots and thereby, the solutions. Note that the operating regime is the upper quadrant of the X - Y plane.

The several regions in the operating regime are delimited by a set of curves, the equal-root loci, which are defined by the equation

$$I^3 - 27J^2 = 0 \tag{4.3.10}$$

Replace the left side of (4.3.10) with (4.3.8) and solve it numerically. From the resulting set of values (X, Y) are

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*See Appendix II.

constructed the equal-root loci, a representative example of which is shown in Figure 4.1.* The Roman numerals identify the three major regions each of which is divided by the equal-root locus, $X = 1$; the part above $X = 1$ is denoted by the letter b; the part below $X = 1$ by the letter a. Subsequently, if the numeral is not accompanied by a letter, then reference is being made to the total region. The signs are those of the discriminant. From the definition (4.3.10), it is obvious that the sign of the discriminant can change only by crossing an equal-root locus; therefore, the sign is constant within each region.

The sign of the discriminant is insufficient to determine the type of roots. Nevertheless, it is possible to identify the type of roots without solving the quartic equation, but to do so, the signs of other two quantities, H and $3b_0J-2HI$, are needed. The signs of these three quantities can be combined in several ways. Obviously, the λ-quartic can have real and complex roots which also can be combined in several ways. There is a correlation between the sign combinations and the type of root combinations. Although the procedure being implied is not used, an idea of this correlation is noted in Table 4.1.

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*Throughout the remainder of the chapter, this example is used for illustration.

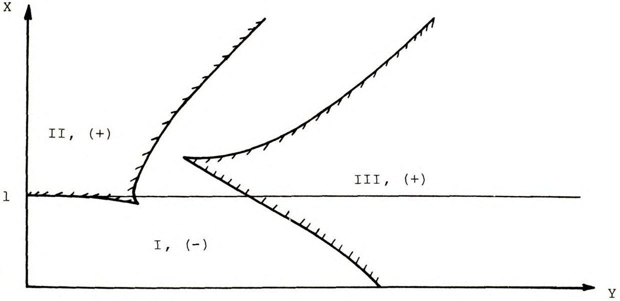

Figure 4.1. Equal-root Loci ($\ell = m = n$)

Table 4.1. Table of Roots

Sign of ---			
I^3-27J^2	H	$3b_0J-2HI$	Type of roots
+	+	+ or -	4 complex
+	-	-	4 complex
+	-	+	4 real
= 0	+ or -	+ or -	At least two equal
-	+ or -	+ or -	2 real plus 2 complex

Based upon the sign of the discriminant, region II contains either four complex or four real roots, perhaps both choices coexist in adjacent, as yet unrecognized, subdivisions; region I contains two real and two complex roots, and region III has the same possibilities as region II. To resolve the dichotomy in regions II and III is quite easy.

Suppose both choices coexist in region II(b), for example; then there must be a curve in II(b) across which the change takes place. At all points on this curve, there must exist two pairs of equal, real roots. In other words, the curve is an equal-root locus. But all the equal-root loci are identified on Figure 4.1; so this extra curve cannot exist. It can be concluded that the four roots are either real everywhere or complex

everywhere within region II(b). To determine which choice is valid, it is sufficient to solve the quartic equation at one point. Calculation proves that four complex roots do not occur. Similar reasoning applies throughout regions II and III; thus, everywhere in these two regions, the four roots are real.

It has been proven that only two of the three possible combinations of root-types occur. Furthermore, only one combination holds true in each region; these are summarized in the following, short table.

Table 4.2. Type of Roots

Region	Type of roots
II, III	4 real
I	2 real, 2 complex

4.4. Root-Signs

The type of roots in each region have just been identified, but the root-signs are still unknown. Unless these signs are specified, there will result a plethora of solutions. Consider, by way of example, region III. Throughout this region, the λ-quartic has four real roots. These four, signed numbers can be combined in five different ways and each arrangement corresponds to a different solution. Altogether region III, which consists of region III(b) and the three, subsequently defined

subdivisions of region III(a), has twenty feasible solutions. Obviously, the total number of plausible solutions in the operating regime could become unwieldy. Conversely, with the root-signs specified, only one solution exists in each region or subdivision. The value of the root-signs is quite evident. In their determination, the sum and the product of the roots can be used to advantage.

The general quartic

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4) = 0 \qquad (4.4.1)$$

can also be written in the form

$$\lambda^4-\Sigma\lambda^3 + \Sigma\Sigma\lambda^2 - \Sigma\Sigma\Sigma\lambda + \pi = 0 \qquad (4.4.2)$$

where the symbols Σ, $\Sigma\Sigma$, $\Sigma\Sigma\Sigma$ and π are defined as

$$\Sigma = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 ,$$

$$\Sigma\Sigma = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 ,$$

$$\Sigma\Sigma\Sigma = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 ,$$

$$\pi = \lambda_1\lambda_2\lambda_3\lambda_4 .$$

Comparison of (4.4.2) with (3.3.14) shows that

$$\Sigma = -4b_1/b_0 \qquad (4.4.3)$$

and

$$\pi = b_4/b_0 \qquad (4.4.4)$$

The other two relationships are neglected because they introduce more complications than simplifications. Substitution from (3.3.15) into the right sides of (4.4.3) and (4.4.4), and incorporation of the definitions for X and Y from (4.3.9) together with $\mathfrak{C}$ and X_2 from (4.3.3), yield

$$\Sigma = \frac{-X^{\frac{1}{2}}Y^{\frac{1}{2}}(X_1-X)}{(1-X)(\mathbb{C}-X)}\,[2(m^2+n^2) + (n^2+\ell^2)] \qquad (4.4.5)$$

where

$$1 \geq X_1 = \frac{2(m^2+n^2)\mathbb{C} + (\ell^2+n^2)}{2(m^2+n^2) + (\ell^2+n^2)} \geq \mathbb{C} \qquad (4.4.6)$$

and

$$\pi = -\frac{12XY(X_2-X)}{(1-X)^2(C-X)} \qquad (4.4.7)$$

where

$$1 \geq X_2 \geq 0 \quad . \qquad (4.4.8)$$

The useful information gained from (4.4.5) and (4.4.7) is SGN(π) and SGN(Σ). From these two relations, two lines, $X=\mathbb{C}$ and $X=X_2$, prove to be important. Along the former, one root is infinite, whereas along the latter, one root is zero. Across both lines a change in the sign of one root occurs; so different solutions exist above and below each line. On the other hand, although the sum of the roots is zero along $X=X_1$, the solutions on either side of it have the same form. Figure 4.2 illustrates the bounds within which different solutions are applicable.

On Figure 4.3, the variations in SGN(Σ), SGN(π), and SGN(zero) which is the sign of the zero root from (4.2.4) are noted.

The information on Figure 4.3 reduces the number of possible solutions. To illustrate this fact, consider region II(b) in which there are four real roots. Suppose

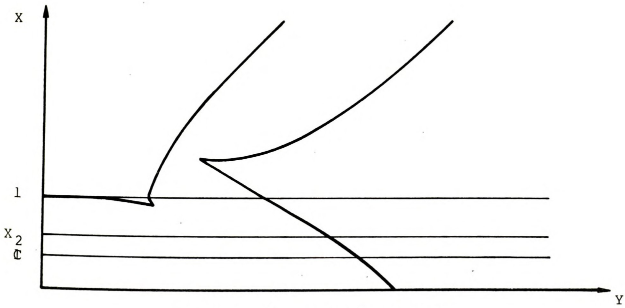

Figure 4.2. Solution Bounds

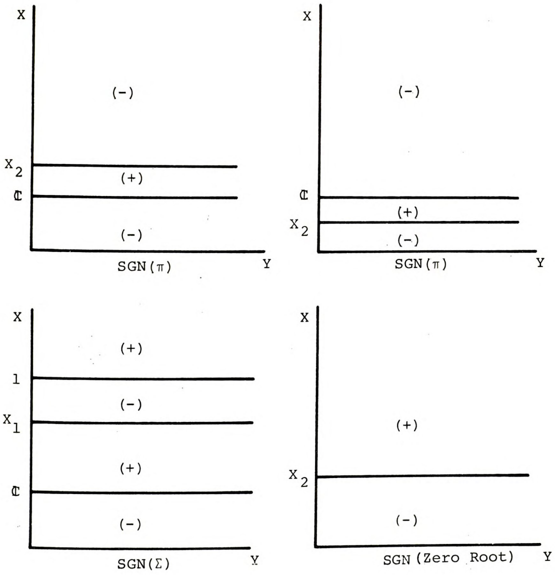

Figure 4.3. SGN- Information

$$\lambda_1 = \begin{cases} +A \\ -A \end{cases}, \quad \lambda_2 = \begin{cases} +B \\ -B \end{cases}, \quad \lambda_3 = \begin{cases} +C \\ -C \end{cases},$$

$$\lambda_4 = \begin{cases} +D \\ -D \end{cases} \qquad (4.4.9)$$

where A, B, C, D >0.

From Figure 4.3, SGN(Σ) >0 and SGN(π) <0. Of the five possible ways for combining the four roots, only two satisfy the SGN- requirements. The two acceptable combinations consist of three positive roots and one negative root or three negative and one positive. Thus the number of plausible solutions is reduced from five to two. A similar analysis can be made in each region and subdivision. The results are collected in Figure 4.4 where the symbol CC(±) gives the sign of the real part of a complex root.

4.5. Correct Choice of Root-Signs

The goal of assigning one root-sign-choice to each region or subdivision is achieved in this section. Several fundamental ideas are required. Regions with four, real roots are treated first.

Figure 4.4 shows that in each region or subdivision with four, real roots there are two feasible choices for the root-signs. By adroit use of the equal-root loci, all of which are identified on Figure 4.1, one choice can be eliminated. To illustrate the logic involved, consider the subdivision in region III(a) where $X_2 > X > \mathbb{C}$. Here the two

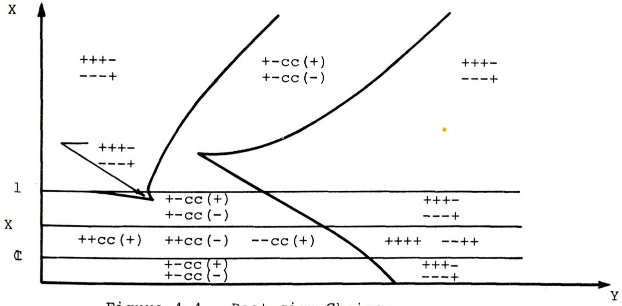

Figure 4.4. Root-sign Choices

choices are ++-- and ++++. If both coexist, a curve of demarcation must separate the two parts of the subdivision. On such a curve, two λ-roots must be equal; in fact, they are both zero. In other words, the curve is an equal-root locus: an untenable conclusion. Thus, only one choice is valid throughout the subdivision and it is found by solving the λ-quartic at one point. A similar argument holds everywhere in regions II and III. The correct choice of root signs for these two regions is indicated on Figure 4.5.

In region I, where there are two real and two complex roots, the equal-root loci do not help. Instead, appeal is made to the Michailov criterion [17] which is a theory by which the existence of imaginary roots can be predicted. The details are developed in Appendix III. The criterion reduces to the following test. Imaginary roots exist along the curve

$$b_1^2 b_4 - 6 b_1 b_2 b_3 + b_0 b_3^2 = 0 \qquad (4.5.1)$$

providing

$$b_3/b_1 > 0 \quad . \qquad (4.5.2)$$

Numerically, it is found that the two conditions are mutually exclusive throughout the operating regime--no imaginary roots exist. Consideration of region I(b), in which the root-sign-choices are +-CC(-) and +-CC(+), illustrates the value of this result.

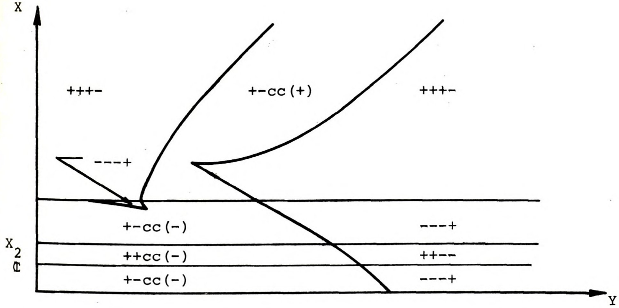

Figure 4.5. Correct Root Choices I

Suppose both choices coexist in separate parts of the region. Along the curve separating the two parts, there must be either two CC(0) or two CC(∞) roots. Immediately, the latter choice may be rejected because the λ-quartic clearly indicates that two infinite roots occur simultaneously only along X=1. The former choice is precisely two, imaginary roots; by the Michailov criterion, this possibility is also rejected; thus one of the choices for the root-signs is incorrect. Again, it is identified by solving the λ-quartic at one point in the region. Region I(b) and the other subdivisions in region I(a) may be analyzed analogously. The correct choices are noted in Figure 4.5.

In each distinct region and subdivision, the one correct choice for the types and signs of the roots has been identified. So far, these conclusions pertain only to a single member, first introduced in Figure 4.1, of an infinite family of λ-quartics. In Appendix IV, where an overview of the entire family is presented, a cursory glance at any of the figures is sufficient to reveal the drastic changes in shape that the equal root-loci undergo in progressing from member to member. Nevertheless, the conclusions summarized on Figure 4.5, together with those on companion Figure 4.6, cover the entire family.

This claim is based upon the fact that the equal-root loci shown on Figure 4.1 can be continuously deformed

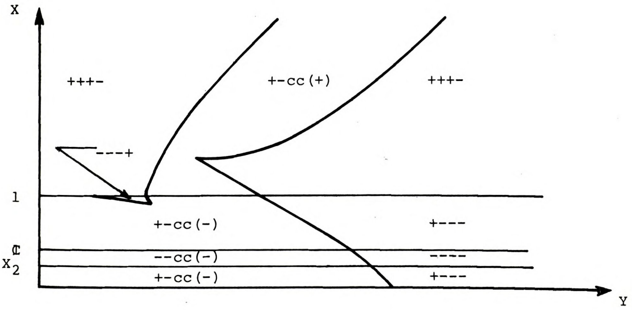

Figure 4.6. Correct Root Choices II

into the equal-root loci of any other member. In other words, the original regions and subdivisions can always be recognized. However, there is one situation in which care must be exercised.

In regions I(a) and III(a), it is possible for the subdivision between $X=X_2$ and $X=\mathbb{C}$ to collapse into $X=\mathbb{C}$ then to reappear on the opposite side of $X=\mathbb{C}$. When this occurs, a test must be made to determine if the prior choice of root-signs remains valid. In fact, it does not. The alternative choices are stated on Figure 4.6.

Together, Figures 4.5 and 4.6 contain all, possible choices of root-types and signs. Figure 4.5 applies to all λ-quartics for which $X_2>\mathbb{C}$ and Figure 4.6 applies to all λ-quartics for which $X_2<\mathbb{C}$. At last, knowledge of the λ-roots is sufficient to enable one prime-solution to be assigned to each distinct region and subdivision. At most, this amounts to eight different solutions for a particular member of the family. A representative prime solution is outlined in the next section.

4.6. Sample Prime Solution

The solution written below applies to region III(a) in the subdivision $X_2>X>\mathbb{C}$. It follows the procedure developed in section 3.3.

From Figure 4.5, the λ-quartic has four, real roots with signs, ++--. From Figure 4.3, the zero root is 0^-. Prescribe the roots as follows:

$$\lambda_1 = 0^-$$
$$\lambda_2 = \alpha$$
$$\lambda_3 = -\beta$$
$$\lambda_4 = -\gamma$$
$$\lambda_5 = \omega \qquad (4.6.1)$$

with α, β, γ, ω all positive. The solutions have the forms

$$U' = \begin{cases} r_{12}F_2e^{\alpha\zeta} + r_{15}F_5e^{\omega\zeta} \\ r_{11}F_1 + r_{13}F_3e^{-\beta\zeta} + r_{14}F_4e^{-\gamma\zeta} \end{cases}, \qquad (4.6.3)$$

$$V' = \begin{cases} r_{22}F_2e^{\alpha\zeta} + r_{25}F_5e^{\omega\zeta} \\ r_{21}F_1 + r_{23}F_3e^{-\beta\zeta} + r_{24}F_4e^{-\gamma\zeta} \end{cases}, \qquad (4.6.3)$$

$$W' = \begin{cases} F_2e^{\alpha\zeta} + F_5e^{\omega\zeta} \\ F_1 + F_3e^{-\beta\zeta} + F_4e^{-\gamma\zeta} \end{cases}, \qquad (4.6.4)$$

where in each case the upper line holds for $\zeta<0$ and the lower line holds for $\zeta>0$.

The jump and continuity conditions give the five equations

$$r_{12}F_2 + r_{15}F_5 = r_{11}F_1 + r_{13}F_3 + r_{14}F_4\ , \qquad (4.6.5)$$

$$r_{22}F_2 + r_{25}F_5 = r_{21}F_1 + r_{23}F_3 + r_{24}F_4\ , \qquad (4.6.6)$$

$$F_2 + F_5 = F_1 + F_3 + F_4 \quad , \qquad (4.6.7)$$

$$\alpha F_2 + \omega F_5 = - \beta F_3 - \gamma F_4 \quad , \qquad (4.6.8)$$

$$\beta^2 F_3 + \gamma^2 F_4 - \alpha^2 F_2 - \omega^2 F_5 = -12/A \quad , \qquad (4.6.9)$$

which can be solved for the five F's in terms of the λ-roots. These expressions may be written directly from Appendix V:

$$r_{1i} = BS\ell n(D\lambda_i + BS)/D_i \quad , \qquad (4.6.10)$$

$$r_{2i} = BSmn(A\lambda_i + BS)/D_i \quad , \qquad (4.6.11)$$

where

$$D_i = AD\lambda_i^2 + BS[(\ell^2+n^2)A+(m^2+n^2)D]\lambda i$$
$$+B^2S^2n^2. \qquad (4.6.12)$$

In the last three definitions, $i = 1,..,5$. The F coefficients are

$$F_1 = \frac{B^2S^2n^2(\text{NUMR1})}{\alpha\beta\gamma\omega(\text{DENR})} \quad , \qquad (4.6.13)$$

$$F_2 = \frac{-D_2(DT_2)}{\alpha(\text{DENR})} \quad , \qquad (4.6.14)$$

$$F_3 = \frac{D_3(DT_3)}{\beta(\text{DENR})} \quad , \qquad (4.6.15)$$

$$F_4 = \frac{-D_4(DT_4)}{\gamma(\text{DENR})} \quad , \qquad (4.6.16)$$

$$F_5 = \frac{D_5(DT_5)}{\omega(\text{DENR})} \quad , \qquad (4.6.17)$$

where

$$12(DENR) = A^2D[\alpha^3(DT_2)+\beta^3(DT_3)$$

$$-\lambda^3(DT_4)-\omega^3(DT_5)] \qquad (4.6.18)$$

$$NUMR1 = \beta\gamma\omega(DT_2)+\alpha\gamma\omega(DT_3)-\alpha\beta\omega(DT_4)$$

$$-\alpha\beta\gamma(DT_5) \qquad (4.6.19)$$

in which

$$DT_2 = \beta^2(-\gamma-\omega)+\gamma^2(\omega+\beta)+\omega^2(-\beta+\gamma) \qquad (4.6.20)$$

$$DT_3 = \alpha^2(-\gamma-\omega)+\gamma^2(\omega-\alpha)+\omega^2(\alpha+\gamma) \qquad (4.6.21)$$

$$DT_4 = \alpha^2(-\beta-\omega)+\beta^2(\omega-\alpha)+\omega^2(\alpha+\beta) \qquad (4.6.22)$$

$$DT_5 = \alpha^2(-\beta+\gamma)+\beta^2(-\gamma-\alpha)+\gamma^2(\alpha+\beta) \qquad (4.6.23)$$

Appendix V contains a complete compilation of all prime solutions by means of some general formulas.

4.7. The Displacement Solution

Why is the prime solution determined rather than the displacement-solution? At this point, it is worthwhile to review the answer to this question.

The λ-polynomial for both the prime problem and the displacement-problem has zero roots. In general, there is one zero root in the prime problem and four in the displacement-problem. As already mentioned, it is difficult to construct a unique solution to a problem whose λ-polynomial has one or more zero root; therefore, the technique of augmenting the differential equations by

adding some small term(s) was devised. All augmented problems have well defined solutions which are bounded everywhere. The limits of these solutions, as the augmentation parameters go to zero, provide solutions to the original problems. When the λ-polynomial has only one zero root the procedure works well, but when it has multiple zero roots, the technique may become intractable. This is the one reason for solving the prime problem.

Once the prime porblem has been solved, the displacement-solution is determined by integration. For continuity of the displacement components, each displacement solution is known within an additive constant. Moreover, if the displacements are required to approach some steady state value as the independent variable ζ tends to $+\infty$ or $-\infty$, the integration constants may be evaluated. In any event, the integration constants can be interpreted as rigid-body translations; thereby, they are unimportant insofar as the deformation-response is concerned.

The following example illustrates the determination of a displacement-solution from a prime solution. Integrating (4.6.4) with respect to ζ yields

$$W = \int W' d\zeta = \begin{cases} \dfrac{F_2 e^{\alpha\zeta}}{\alpha} + \dfrac{F_5 e^{\omega\zeta}}{\omega} + C_w^- & \zeta<0 \\ F_1\zeta - \dfrac{F_3 e^{-\beta\zeta}}{\beta} - \dfrac{F_4 3^{-\gamma\zeta}}{\gamma} + C_w^+ & \zeta>0 \end{cases} \quad (4.7.1)$$

in which C_w^- and C_w^+ are integration constants. Since W must be continuous at $\zeta=0$, then

$$C_w^+ = \frac{F_2}{\alpha} + \frac{F_3}{\beta} + \frac{F_4}{\gamma} + \frac{F_5}{\omega} + C_w^- \quad . \qquad (4.7.2)$$

Eliminate C_w^+ from (4.7.1) with (4.7.2). Both portions of the solution now contain the constant C_w^-: the rigid-body translation mentioned above.

Note also in (4.7.1) that the displacement-solutions for a general magnetic field always contain a linear term. It may occur in either the trailing or the leading portion of the response. Thus, for a general magnetic field, the displacement-solution is unbounded. Nevertheless, displacement solutions which are bounded everywhere do exist in some special situations. A few examples to illustrate this point will be presented shortly.

The existence or nonexistence of a bounded displacement-solution depends upon the physical makeup of the system. Loosely speaking, if the system has a mechanism which is capable of either storing or dissipating the energy associated with the load, then a displacement-solution which is bounded everywhere is expected. On the other hand, if such a mechanism is absent, it is reasonable to anticipate an unbounded displacement-solution

To illustrate these observations, two problems involving one-dimensional magnetic fields are considered. When the magnetic field is one-dimensional, the equations

for the displacement components U, V and W uncouple - a great simplification. For the present purpose only the W-component is used. A compilation of all the W-solutions for the three, one-dimensional magnetic field cases is contained in Appendix VI. The W-solutions for the zero-field case are developed in Appendix VII.

Consider first the problem with the magnetic field oriented in the x_1-direction. From (3.3.4) the W-equation of the motion is

$$-\frac{A}{12}W'''' + (A-1)W'' - B_1SC_3W' = \delta(\zeta) \qquad (4.7.3)$$

By the procedures outlined in Chapters III and IV, the solution to (4.7.3) can be found. The following differences in detail exist. In this case, the displacement-solution can be obtained directly which leads to a quartic λ-polynomial equation with one zero root; so the nonzero roots are determined from a cubic equation, not a quartic. The sign of the zero root is determined by augmenting (4.7.3) with the term $-kW$. In Figure 4.7, the equal root loci are shown along with the correct root choices for each region. In region II, for example, the displacement-solution is

$$W = \frac{12}{A}\begin{cases} E + Fe^{\beta\zeta} + Ge^{\gamma\zeta} & \zeta<0 \\ He^{-\xi\zeta} & \zeta>0 \end{cases} \qquad (4.7.4)$$

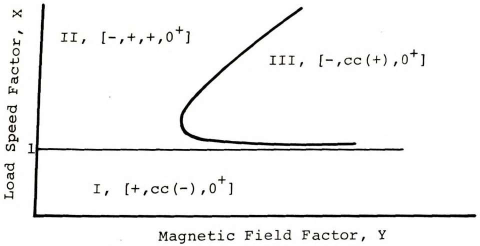

Figure 4.7. Correct Root Choices III, $\underset{\sim}{H} = (H_1,0,0)$

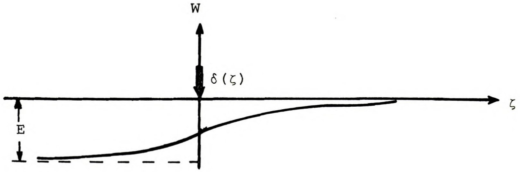

Figure 4.8. W-displacement, Region II, $\underset{\sim}{H} = (H_1,0,0)$

where

$$E = \frac{1}{\beta\gamma\xi} \quad ,$$

$$F = \frac{1}{\beta(\xi+\beta)(\beta-\gamma)} \quad ,$$

$$G = \frac{1}{\gamma(\xi+\gamma)(\beta-\gamma)} \quad ,$$

$$H = \frac{1}{\xi(\xi+\beta)(\xi+\gamma)} \quad , \qquad (4.7.5)$$

and β, γ, and ξ are positive numbers. Figure 4.8 is a sketch of this W-displacement.

Clearly, even when the elastic support is removed, the system described by (4.7.3) still contains a mechanism which is sufficient to absorb the energy associated with the load: the solution is bounded everywhere.

Consider now the problem in which the magnetic field is oriented in the x_3-direction. The corresponding W-equation of motion is

$$-\frac{A}{12}W'''' + \frac{B_1SC_1}{12}W''' + (A-1)W'' = \delta(\zeta) \qquad (4.7.6)$$

Again, the same general procedure is used to find the displacement-solution. In this case, it is evident that the λ-polynomial can be reduced to a quadratic. In other words, it has two zero roots. Thus the augmentation procedure requires the addition of two terms, $\varepsilon W'$ and $-kW$ to (4.7.6). In this case there is a further difference in detail from other solutions. Here the augmented displacement-solution is completely determined, whereupon the displacement-solution to (4.7.6) is evaluated by letting k and ε go to zero. Figure 4.9 contains a sketch of the equal root loci and the correct root-choices. For example, in region III the displacement solution is

$$W = \frac{12}{A}\begin{cases} \dfrac{(\alpha-\beta)-\alpha\beta\zeta}{\alpha^2\beta^2} + \dfrac{e^{\alpha\zeta}}{\alpha^2(\beta+\alpha)} & \zeta<0 \\ \dfrac{e^{-\beta\zeta}}{\beta^2(\beta+\alpha)} & \zeta>0 \end{cases} \qquad (4.7.7)$$

where α and β are positive real numbers. This solution is sketched on Figure 4.10.

Evidently, upon removal of the elastic and viscous supports, the system represented in (4.7.6) no longer contains an adequate mechanism to absorb the energy associated with the load; therefore, unbounded-displacements, in the form of a linear term in ζ, occur in the trailing portion of the response.

The existence of both bounded and unbounded displacement solutions has just been illustrated for some simplified problems. It is noteworthy that for the two examples just considered the displacement-solution may be obtained in two ways: either by seeking the displacement-solution directly, or by seeking a prime solution and then integrating the result. Although the details have been omitted, both approaches yield identical solutions. However, when a more general magnetic field is involved, the first

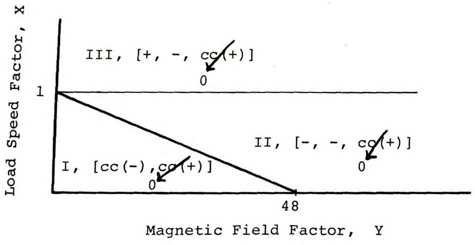

Figure 4.9. Correct Root Choices IV, $\underset{\sim}{H} = (0,0,H_3)$

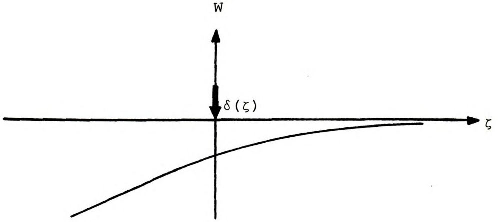

Figure 4.10. W-displacement, Region III, $\underset{\sim}{H} = (0,0,H_3)$

approach of obtaining displacement-solutions directly may become extremely complicated. The following explanation is offered for this conclusion.

All augmented displacement-solutions are bounded everywhere. In fact, they all tend to zero as $\zeta \to \pm\infty$. However, the actual displacement solutions are known to be unbounded. So great care must be exercised not only in taking the limits of the augmented displacement-solutions as the augmentation parameters go to zero, but also in interpreting such limits. Similar complications do not arise in prime problems since all the prime solutions are bounded. This is another reason for solving the prime problem.

V. SOME DISCUSSION AND CONCLUSIONS

5.1. Differential Equations

For this chapter and for Appendix VIII the equations of motion are rewritten; they take the form

$$c_p^2 \frac{\partial^2 u}{\partial x_1^2} + \chi[-(m^2+n^2)\frac{\partial u}{\partial t} + \ell m \frac{\partial v}{\partial t}$$

$$+ \ell n \frac{\partial w}{\partial t}] = \frac{\partial^2 u}{\partial t^2} , \qquad (5.1.1)$$

$$c_s^2 \frac{\partial^2 v}{\partial x_1^2} + \chi[\ell m \frac{\partial u}{\partial t} - (\ell^2+n^2)\frac{\partial v}{\partial t}$$

$$+ mn \frac{\partial w}{\partial t}] = \frac{\partial^2 v}{\partial t^2} , \qquad (5.1.2)$$

$$- \frac{a^2}{12} c_p^2 \frac{\partial^4 w}{\partial x_1^4} + \frac{a^2 \chi (m^2+n^2)}{12} \frac{\partial^3 w}{\partial x_1^2 \partial t} + \chi[\ell n \frac{\partial u}{\partial t}$$

$$+ mn \frac{\partial v}{\partial t} - (\ell^2+m^2)\frac{\partial w}{\partial t}] = \frac{\partial^2 w}{\partial t^2} - \frac{a^2}{12} \frac{\partial^4 w}{\partial x_1^2 \partial t^2} . \qquad (5.1.3)$$

These three equations are found by linearizing equations (3.1.2), (3.1.3) and (3.1.4). The load P is dropped. The material parameters, together with the magnitude of the

magnetic field, are lumped together in the symbol $\chi = K^2\sigma Ho^2/\rho$. The magnetic field is $\underset{\sim}{Ho} = Ho(\ell,m,n)$.

5.2. Lorentz Force

In equations (5.1.1), (5.1.2) and (5.1.3) the various components of the Lorentz force are those terms which contain the direction cosines ℓ, m and n. To interpret each term physically requires a little recapitulation.

The successive simplifications imposed upon the Lorentz force in section 2.3 reduce it to the expression

$$\underset{\sim}{f} = \underset{\sim}{J} x \underset{\sim}{B} = \sigma K^2 (\underset{\sim}{v} x \underset{\sim}{Ho}) x \underset{\sim}{Ho} \qquad (5.2.1)$$

in which the velocity $\underset{\sim}{v}$ is determined by differentiating with respect to time the displacements (2.1.7), 2.1.8) and (2.1.9) and the magnetic field is $\underset{\sim}{Ho} = Ho(\ell,m,n)$. Substituting for $\underset{\sim}{v}$ and $\underset{\sim}{Ho}$ in (5.2.1) identifies the three components of $\underset{\sim}{f}$ to be

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \underset{\sim}{M} \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{pmatrix} - \underset{\sim}{M} x_3 \begin{pmatrix} \frac{\partial^2 w}{\partial x_1 \partial t} \\ \frac{\partial^2 w}{\partial x_2 \partial t} \\ 0 \end{pmatrix} \qquad (5.2.2)$$

where

$$\underset{\sim}{M} = \rho\chi \begin{pmatrix} -(m^2+n^2), & \ell m, & \ell n \\ m\ell, & -(\ell^2+n^2), & mn \\ n\ell, & nm, & -(\ell^2+m^2) \end{pmatrix}$$

Already, it is possible to draw parallels between the Lorentz force terms in (5.1.1), (5.1.2) and (5.1.3), and the definition of the Lorentz force contained in (5.2.2). However, one additional recollection will be made to make the correlation clearer.

Refer to equations (2.2.18), (2.2.19) and (2.2.20). The Lorentz force enters the theory in these equations through the three F_i-terms and the two $\partial m_i/\partial x_i$-terms. F_i and m_i are defined as

$$F_i = \int_{-a/2}^{a/2} f_i dx_3, \qquad i = 1,2,3$$

$$m_i = \int_{-a/2}^{a/2} f_i x_3 dx_3, \qquad i = 1,2 .$$

Clearly, F_i is a body force and m_i is a moment. F_i depends only upon the first matrix product on the right side of (5.2.2) and m_i depends only upon the second matrix product. Since m_i is a moment, then $\partial m_i/\partial x_i$ is a shearing force. Note that only $\partial m_1/\partial x_1$ remains in the equations of motion after section 3.1 since all x_2-dependence was eliminated there.

The physical meaning of each of the Lorentz force terms in (5.1.1), (5.1.2) and (5.1.3) can now be identified. All the terms within the square brackets evolve from the F_i and so they are body forces. The remaining term, the third derivative of w in (5.1.3), develops from $\partial m_1/\partial x_1$; therefore it is a shearing force.

Thus the Lorentz force produces a body force with components in the three coordinate directions plus a shearing force in the x_3-direction.

5.3. Coupling and Magnetic Field

Coupling between the displacement components u, v and w are due entirely to terms which include the products of the direction cosines: ℓm, ℓn, mn. For this reason it is a simple matter to catalogue any uncoupled motion against the appropriate magnetic field. When the magnetic field acts along one of the coordinate directions the u, v and w-motions are uncoupled. However, if the magnetic field is two-dimensional the displacement component orthogonal to the plane of the magnetic field is uncoupled from the other two. For example, for $\underset{\sim}{H}o = Ho(\ell,m,o)$ the w-motion is uncoupled from the u and v-motions. In Appendix VIII these uncoupled motions are used to provide further insight into some magnetoelastic interactions.

5.4. Displacement Solutions

In section 4.7 and Appendix VI, displacement solutions for the magnetoelastic plate are discussed. The displacement response is found to consist of two parts one of which precedes the load and the other trails. For a general magnetic field one tail of the solution is always unbounded--a linear function of ζ. Nevertheless, bounded solutions do exist when the magnetic field acts either in the x_1 or the x_2-direction.

Four types of motion can be identified. The correspondence between the algebraic term and the motion is as follows:

(I) constant = rigid body translation

(II) linear term = rigid body rotation

(III) real exponential term = exponential wave

(IV) complex exponential term = damped harmonic wave

The linear and constant terms arise from integrating the prime solution. As mentioned earlier, the linear term may either precede or trail the load. Insofar as deformation response is concerned these two rigid body motions are unimportant. In the regions II and III delimited by the equal-root loci the travelling waves are all exponential in form. In Region I both exponential and damped harmonic waves exist: if the load speed is less than C_p, the harmonic waves lead the load; if the load speed is greater than C_p, the harmonic waves trail.

5.5. Free Waves

In order to get a feeling for the interaction effects and their magnitudes, some problems in free wave propagation are considered in Appendix VIII. The investigation is quite restricted. It pertains only to damped, harmonic, plane waves propagating in the beam-plate in the direction of the travelling load. Furthermore, the magnetic field is limited to the three, one-dimensional cases. Notwithstanding these limitations, many effects are revealed.

The presence of the magnetic field makes an otherwise isotropic material anisotropic. However, plane polarized waves exist when the magnetic field is aligned either parallel or perpendicular to the direction of propagation. Specifically, distortional waves are unaffected by a magnetic field orthogonal to the direction of propagation, and dilatational waves are oblivious to a magnetic field parallel to the direction of propagation. In general, the magnetic field introduces dispersion. Its damping effect reduces the phase velocities below their isotropic-plate values.

The quantity χ indicates the manner in which the physical properties of a material enter the magnetoelastic interaction. A large value signifies a strong interaction. Above some critical value harmonic waves cannot propagate.

Although the effects reported in Appendix VIII appear to be strong, it must be noted that the flux density is high: $0.01(KH)_{cr}$ for aluminum is greater than 0.05 Wb/m^2.

5.6. Conclusions

A magnetoelastic-plate theory is presented. Solutions to the linearized theory encompassed by equations (3.3.2), (3.3.3) and (3.3.4) have been found for all combinations of load speed and magnetic field.

The solution is the Green's function for the equations of motion. Thereby the solutions for any arbitrary travelling load may be found directly by well established methods.

The effect of the magnetic field is twofold. It makes the medium anisotropic and its effect upon the motion of the plate is that of a damper. However, the field strengths required for significant interactions is large.

APPENDICES

APPENDIX I

GREEN'S FUNCTION

GREEN'S FUNCTION

Consider the set of differential equations* with variable coefficients

$$A_1U'' + A_2U' + A_3V' + A_4W' = f_1(\zeta) \quad , \qquad \text{(A1.1)}$$

$$B_1V'' + B_2V' + B_3U' + B_4W' = f_2(\zeta) \quad , \qquad \text{(A1.2)}$$

$$C_1W'''' + C_2W''' + C_3W'' + C_4W' + C_5U'$$

$$+ C_6V' = f_3(\zeta) \quad , \qquad \text{(A1.3)}$$

in which the functions f_1, f_2, and f_3 are continuous and tend to zero as ζ becomes infinite. In order that the solutions can be expressed in the form

$$\underset{\sim}{d}(\zeta) = \int_{-\infty}^{\infty} \underset{\approx}{G}(\zeta,\xi)\underset{\sim}{f}(\xi)d\xi \quad , \qquad \text{(A1.4)}$$

where

$$\underset{\sim}{d} = (U,\ V,\ W) \quad , \qquad \text{(A1.5)}$$

$$\underset{\sim}{f} = (f_1,\ f_2,\ f_3) \quad , \qquad \text{(A1.6)}$$

$$\underset{\approx}{G} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} \qquad \text{(A1.7)}$$

---

*Throughout this section, primes indicate differentiation with respect to ζ.

the Green's functions, the elements of matrix $\underset{\approx}{G}$, must have special properties. To determine these properties is the purpose of this Appendix. However, before proceeding, a commonly used motivation for a Green's function-solution is now presented.

For the purpose of illustration, consider equation (A1.3). Think of it as describing the transverse displacement of a plate due to the applied force f_3. First, suppose f_3 is a single, discrete force acting at the point $\zeta = \xi$. If G (ζ,ξ) is the deflection at any point due to a discrete, unit-force applied at $\zeta = \xi$; then $G(\zeta,\xi)\ f(\xi)$ is the deflection at any point due to $f(\xi)$. Take this idea one step further. Let f_3 consist of N discrete forces $f_3(\xi_1)$, $f_3(\xi_2)$---$f_3(\xi_n)$, each of which acts at a different point, $\zeta = \xi_1$---ξ_n. Since (A1.3) is linear, simple superposition gives the resulting displacement,

$$W(\zeta) = \sum_{n=1}^{N} G(\zeta,\xi_n)\, f_3(\xi_n) \quad . \qquad \text{(A1.8)}$$

Logically, the next step is to permit the point forces to become so numerous that the force distribution f_3 becomes continuous. Now the displacement at any point is

$$W(\zeta) = \int_{-\infty}^{\infty} G(\zeta,\xi)\ f_3(\xi)\ d\xi \quad . \qquad \text{(A1.9)}$$

$G(\zeta,\xi)$ must be continuous since it represents the displacement of a plate. However, it is created by a unit, point force action as $\zeta = \xi$; so perhaps one of its derivatives

$$\frac{d^nG(\xi,\zeta)}{d\zeta^n} \qquad n = 1, 2, 3,... \qquad \text{(A1.10)}$$

may be discontinuous at this point. This is the justification for interrupting the integration when calculating derivatives in the succeeding development.

The substitution of (A1.4) into (A1.1) gives

$$A_1\frac{d^2}{d\zeta^2}\{\int_{-\infty}^{\infty}[G_{11}(\zeta,\xi)f_1(\xi) + G_{12}(\zeta,\xi)f_2(\xi)$$

$$+ G_{13}(\zeta,\xi)f_3(\xi)]d\xi\} + A_2\frac{d}{d\zeta}\{\int_{-\infty}^{\infty}[G_{11}(\zeta,\xi)f_1(\xi)$$

$$+ G_{12}(\zeta,\xi)f_2(\xi) + G_{13}(\zeta,\xi)f_3(\xi)]d\xi\}$$

$$+ A_3\frac{d}{d\zeta}\{\int_{-\infty}^{\infty}[G_{21}(\zeta,\xi)f_1(\xi) + G_{22}(\zeta,\xi)f_2(\xi)$$

$$+ G_{23}(\zeta,\xi)f_3(\xi)]d\xi\} + A_4\frac{d}{d\zeta}\{\int_{-\infty}^{\infty}[G_{31}(\zeta,\xi)f_1(\xi)$$

$$+ G_{32}(\zeta,\xi)f_2(\xi) + G_{33}(\zeta,\xi)f_3(\xi)]d\xi\} = f_1(\zeta)$$

(A1.11)

The meaning of the derivatives must be established.

A representative example of a first order differentiation is

$$A_2\frac{d}{d\zeta}[\int_{-\infty}^{\infty} G_{11}(\zeta,\xi)f_1(\xi)d\xi] = \lim_{\varepsilon\to 0} A_2\frac{d}{d\zeta}$$

$$[\int_{-\infty}^{\zeta-\varepsilon} G_{11}(\zeta,\xi)f_1(\xi)d\xi + \int_{\zeta+\varepsilon}^{\infty} G_{11}(\zeta,\xi)f_1(\xi)d\xi]$$

(Al.12)

in which the integration has been interrupted at $\xi = \zeta$. After differentiation, the expression on the right becomes

$$A_2\{\lim_{\varepsilon\to 0}[G_{11}(\zeta,\zeta-\varepsilon)f_1(\zeta-\varepsilon) - G_{11}(\zeta,\zeta+\varepsilon)f_1(\zeta+\varepsilon)] + \int_{-\infty}^{\infty} f_1(\xi)\frac{dG_{11}(\zeta,\xi)}{d\zeta} d\xi\} . \quad \text{(Al.13)}$$

Since G_{11} and f_1 are continuous, then

$$\lim_{\varepsilon\to 0}[G_{11}(\zeta,\zeta-\varepsilon)f_1(\zeta-\varepsilon) - G_{11}(\zeta,\zeta+\varepsilon)\ f_1(\zeta+\varepsilon)] = 0 ;$$

(Al-14)

therefore,

$$A_2\frac{d}{d\zeta}[\int_{-\infty}^{\infty} G_{11}(\zeta,\xi)f_1(\xi)d\xi]$$

$$= A_2 \int_{-\infty}^{\infty} f_1(\xi)\frac{dG_{11}(\zeta,\xi)}{d\zeta} d\xi \quad . \quad \text{(Al.15)}$$

Using the same procedure, but allowing for a jump in the first order derivative, yields the expression for second order derivatives, an example of which is

$$A_1 \frac{d^2}{d\zeta^2}\left[\int_{-\infty}^{\infty} G_{11}(\zeta,\xi) f_1(\xi) d\xi\right]$$

$$= A_1 \{ \lim_{\varepsilon \to 0} [f_1(\xi) \frac{dG_{11}(\zeta,\xi)}{d\zeta}]_{\xi=\zeta+\varepsilon}^{\xi=\zeta-\varepsilon}$$

$$+ \int_{-\infty}^{\infty} f_1(\xi) \frac{d^2 G_{11}(\zeta,\xi)}{d\zeta^2} d\xi \} \qquad \text{(A1.16)}$$

All other first and second derivatives are defined by analogous expressions.

Substitute for all first and second derivatives, according to (A1.15) and (A1.16) in equation (A1.11). The result is

$$\int_{-\infty}^{\infty} \{ f_1(\xi) [A_1 G_{11}'' + A_2 G_{11}' + A_3 G_{21}' + A_4 G_{31}']$$

$$+ f_2(\xi) [A_1 G_{12}'' + A_2 G_{12}' + A_3 G_{22}' + A_4 G_{32}']$$

$$+ f_3(\xi) [A_1 G_{13}'' + A_2 G_{13}' + A_3 G_{23}' + A_4 G_{33}'] \} d\xi$$

$$+ A \lim_{\varepsilon \to 0} [f_1(\xi) G_{11}' + f_2(\xi) G_{12}' + f_3 G_{13}']_{\xi=\zeta+\varepsilon}^{\xi=\zeta-\varepsilon}$$

$$= f_1(\zeta) \quad . \qquad \text{(A1.17)}$$

This equation is true if the Green's functions satisfy the homogeneous differential equations

$$A_1 G_{11}'' + A_2 G_{11}' + A_3 G_{21}' + A_4 G_{31}' = 0 \ , \qquad \text{(A1.18)}$$

$$A_1 G_{12}'' + A_2 G_{12}' + A_3 G_{22}' + A_4 G_{32}' = 0 \ , \qquad \text{(A1.19)}$$

$$A_1 G_{13}'' + A_2 G_{13}' + A_3 G_{23}' + A_4 G_{33}' = 0 \ , \qquad \text{(A1.20)}$$

everywhere except at $\xi = \zeta$ and if at this point G_{11}, G_{12}, and G_{13} satisfy the conditions

$$\lim_{\varepsilon\to 0} G_{11}'(\zeta,\xi)\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = -\frac{1}{A_1(\zeta)} , \qquad \text{(A1.21)}$$

$$\lim_{\varepsilon\to 0} G_{12}'(\zeta,\xi)\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = 0 , \qquad \text{(A1.22)}$$

$$\lim_{\varepsilon\to 0} G_{13}'(\zeta,\xi)\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = 0 . \qquad \text{(A1.23)}$$

The name, jump condition, is applied to expressions such as (A1.21); the name, continuity condition, is applied to expressions such as (A1.22) and (A1.23).

The substitution of (A1.4) into (A1.2) yields a similar set of conditions on the Green's functions G_{21}, G_{22}, and G_{23}. They must satisfy the homogeneous differential equations

$$B_1G_{21}'' + B_2G_{21}' + B_3G_{11}' + B_4G_{31}' = 0 , \qquad \text{(A1.24)}$$

$$B_1G_{22}'' + B_2G_{22}' + B_3G_{12}' + B_4G_{32}' = 0 , \qquad \text{(A1.25)}$$

$$B_1G_{23}'' + B_2G_{23}' + B_3G_{13}' + B_4G_{33}' = 0 , \qquad \text{(A1.26)}$$

everywhere except at $\xi = \zeta$, and at this point, they must satisfy the continuity and jump conditions

$$\lim_{\varepsilon\to 0} G_{21}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = 0 , \qquad \text{(A1.27)}$$

$$\lim_{\varepsilon\to 0} G_{22}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = -\frac{1}{B_1(\zeta)} , \qquad \text{(A1.28)}$$

$$\lim_{\varepsilon\to 0} G_{23}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = 0 \quad . \tag{A1.29}$$

The third and fourth order derivatives which arise from substituting (A1.4) into (A1.3) have expressions similar to those of a second derivative. For example,

$$C_1\frac{d^4}{d\zeta^4}\left[\int_{-\infty}^{\infty} G_{31}(\zeta,\xi)f_1(\xi)d\xi\right]$$

$$= C_1\{\lim_{\varepsilon\to 0}[f_1(\xi)G_{31}'''(\zeta,\xi)]_{\xi=\zeta+\varepsilon}^{\xi=\zeta-\varepsilon}$$

$$+ \int_{-\infty}^{\infty} f(\xi)G''''(\zeta,\xi)d\xi\} \quad , \tag{A1.30}$$

and

$$C_2\frac{d^3}{d\zeta^3}\left[\int_{-\infty}^{\infty} G_{31}(\zeta,\xi)f_1(\xi)d\xi\right]$$

$$= C_2\{\lim_{\varepsilon\to 0}[f_1(\xi)G_{31}''(\zeta,\xi)]_{\xi=\zeta+\varepsilon}^{\xi=\zeta-\varepsilon}$$

$$+ \int_{-\infty}^{\infty} f(\xi)G'''(\zeta,\xi)d\xi\} \quad . \tag{A1.31}$$

After using these definitions, it is found that the Green's functions G_{31}, G_{32} and G_{33} must satisfy the homogeneous equations

$$C_1G_{31}'''' + C_2G_{31}''' + C_3G_{31}'' + C_4G_{31}'$$

$$+ C_5G_{11}' + C_6G_{21}' = 0 \quad , \tag{A1.32}$$

$$C_1G_{32}'''' + C_2G_{32}''' + C_3G_{32}'' + C_4G_{32}'$$

$$+ C_5G_{12}' + C_6G_{22}' = 0 \quad , \tag{A1.33}$$

$$C_1G_{33}'''' + C_2G_{33}''' + C_3G_{33}'' + C_4G_{33}'$$

$$+ C_5G_{13}' + C_6G_{23}' = 0 \quad . \tag{A1.34}$$

The appropriate continuity and jump conditions are

$$\lim_{\varepsilon\to 0}\{G_{31}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} , \quad G_{32}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} , \quad G_{33}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon}\} = 0 \quad , \tag{A1.35}$$

$$\lim_{\varepsilon\to 0}\{G_{31}''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} , \quad G_{32}''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} , \quad G_{33}''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon}\} = 0 \quad , \tag{A1.36}$$

$$\lim_{\varepsilon\to 0}\{G_{31}'''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} , \quad G_{32}'''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon}\} = 0 \quad ; \tag{A1.37a}$$

$$G_{33}'''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = -\frac{1}{C_1(\zeta)} \quad . \tag{A1.37b}$$

In summary, a solution of form (A1.4) to equations (A1.1), (A1.2) and (A1.3) can be found providing that the Green's functions satisfy both the associated homogeneous equations and the appropriate continuity and jump conditions at $\xi = \zeta$. How this result relates to the solutions of (3.3.2), (3.3.3) and (3.3.4) is the next consideration.

By taking $\underset{\sim}{f} = (0, 0, 0, f_3)$, equations (A1.1), (A1.2) and (A1.3) correspond directly to equations (3.3.2), (3.3.3) and (3.3.4). This reduced expression for $\underset{\sim}{f}$ permits considerable simplification in the properties of the Green's functions since only G_{13}, G_{23} and G_{33} are involved

in the solution. These three Green's functions must satisfy the homogeneous differential equations

$$A_1G_{13}'' + A_2G_{13}' + A_3G_{23}' + A_4G_{33}' = 0 \quad , \tag{A1.38}$$

$$B_1G_{23}'' + B_2G_{23}' + B_3G_{13}' + B_4G_{33}' = 0 \quad , \tag{A1.39}$$

$$C_1G_{33}'''' + C_2G_{33}''' + C_3G_{33}'' + C_4G_{33}' + C_5G_{13}' + C_6G_{23}' = 0 \quad , \tag{A1.40}$$

together with the continuity and jump conditions at $\xi = \zeta$

$$\lim_{\varepsilon\to 0} \left\{ G_{13}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} \; , \; G_{23}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} \; , \; G_{33}'\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} \right\} = 0 \; , \tag{A1.41}$$

$$\lim_{\varepsilon\to 0} G_{33}''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = 0 \; , \tag{A1.42}$$

$$\lim_{\varepsilon\to 0} G_{33}'''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = - \frac{1}{C_1(\zeta)} \; . \tag{A1.43}$$

To be consistent, the variable in the jump and continuity conditions is now changed from ξ to ζ. After their roles are reversed, G_{13}', G_{23}', G_{33}' and G_{33}'' remain continuous at $\zeta = \xi$. The jump in G_{33}''' also remains, but the minus sign disappears. This sign change is demonstrated as follows. Let

$$G_{33}''(\zeta,\xi) = \begin{cases} G_A(\zeta,\xi) & -\infty<\xi<\zeta \\ G_B(\zeta,\xi) & \zeta<\xi<\infty \; , \end{cases} \tag{A1.44}$$

where

$$G_A(\zeta,\zeta) = G_B(\zeta,\zeta) \; . \tag{A1.45}$$

With this definition (A1.43) becomes

$$\lim_{\varepsilon\to 0} G_{33}'''\Big|_{\xi=\zeta-\varepsilon}^{\xi=\zeta+\varepsilon} = G_B'\Big|_{\xi=\zeta} - G_A'\Big|_{\xi=\zeta} = -\frac{1}{C_1(\zeta)} \quad . \tag{A1.46}$$

However, if the roles of ζ and ξ are reversed by making ξ the parameter, (A1.44) becomes

$$G_{33}''\ (\zeta,\xi) = \begin{cases} G_A(\zeta,\xi) & \xi<\zeta<\infty \\ G_B(\zeta,\xi) & -\infty<\zeta<\xi \end{cases} \tag{A1.47}$$

where

$$G_A(\xi,\xi) = G_B(\xi,\xi) \quad . \tag{A1.48}$$

So the jump condition is

$$\lim_{\varepsilon\to 0} G_{33}'''\Big|_{\zeta=\xi-\varepsilon}^{\zeta=\xi+\varepsilon} = G_A'\Big|_{\zeta=\xi} - G_B'\Big|_{\zeta=\xi} \quad . \tag{A1.49}$$

Comparison of (A1.49) with (A1.46) shows that

$$\lim_{\varepsilon\to 0} G_{33}'''\Big|_{\zeta=\xi-\varepsilon}^{\zeta=\xi+\varepsilon} = \frac{1}{C_1(\xi)} \quad . \tag{A1.50}$$

In the body of the thesis, a simpler notation is adopted. The symbol [] is introduced. Its meaning is readily evident from the example

$$[G_{33}] = \lim_{\varepsilon\to 0} G_{33}\Big|_{\zeta=\xi-\varepsilon}^{\zeta=\xi+\varepsilon} \quad . \tag{A1.51}$$

With this symbolism, the complete set of jump and continuity conditions are

$$[G_{13}] = [G_{23}] = [G_{33}] = 0 \quad , \tag{A1.52}$$

$$[G_{13}'] = [G_{23}'[= [G_{33}'] = 0 \quad , \qquad \text{(A1.53)}$$

$$[G_{33}''] = 0 \quad , \qquad \text{(A1.54)}$$

$$[G_{33}'''] = 1/C_1(\xi) \quad . \qquad \text{(A1.55)}$$

One final simplification exists. Since the function $\underset{\sim}{f}$ in (3.3.2), (3.3.3) and (3.3.4) is

$$\underset{\sim}{f}(\zeta) = [0,\ 0,\ \delta(\zeta)] \qquad \text{(A1.56)}$$

and since by definition

$$\int_{-\infty}^{\infty} h(\xi)\,\delta(\xi-a)\,d\xi = h(a)$$

then (A1.4) yields

$$U = \int_{-\infty}^{\infty} G_{13}(\zeta,\xi)\,\delta(\xi)\,d\xi = G_{13}(\zeta,0) \quad , \qquad \text{(A1.57)}$$

$$V = \int_{-\infty}^{\infty} G_{23}(\zeta,\xi)\,\delta(\xi)\,d\xi = G_{23}(\zeta,0) \quad , \qquad \text{(A1.58)}$$

$$W = \int_{-\infty}^{\infty} G_{33}(\zeta,\xi)\,\delta(\xi)\,d\xi = G_{33}(\zeta,0) \quad . \qquad \text{(A1.59)}$$

Thus the Green's functions for (3.3.2), (3.3.3) and (3.3.4) are the displacement solutions. Now the problem can be restated

Displacement-solutions to (3.3.2), (3.3.3) and (3.3.4) must satisfy the associated, homogeneous differential equations

$$AU'' - B_1 S(C_1 U' + C_{12} V' + C_{13} W') = 0 \quad , \qquad \text{(A1.60)}$$

$$DV'' - B_1S(C_{12}U' + C_2V' + C_{23}W') = 0 \ , \qquad \text{(A1.61)}$$

$$-\frac{A}{12}W'''' + \frac{B_1SC_1}{12}W''' + (A-1)W'' - B_1S(C_{13}U'$$

$$+C_{23}V' + C_{33}W') = 0 \ , \qquad \text{(A1.62)}$$

for all values of ζ except $\zeta = \xi = 0$, at which point they must satisfy the continuity and jump conditions

$$[U] = [V] = [W] = 0 \ , \qquad \text{(A1.63)}$$

$$[U'] = [V'] = [W'] = 0 \ , \qquad \text{(A1.64)}$$

$$[W''] = 0 \ , \qquad \text{(A1.65)}$$

$$[W'''] = -12/A \ . \qquad \text{(A1.66)}$$

Prime solutions to (3.3.2), (3.3.3) and (3.3.4) must satisfy the same homogeneous differential equations, continuity and jump conditions with one exception--(A1.63) is deleted.

APPENDIX II

EXPANSION OF I^3-27J^2

EXPANSION OF I^3-27J^2

The discriminant of the quartic is I^3-27J^2. If the definitions for I, J, G and H are expanded by using the definitions of $b_0,...,b_4$ from (3.3.15), it is found that

$$I = I_0 + I_1Y + I_2Y^2 \quad , \tag{A2.1}$$

where

$$I_0 = AD^2X^2/12 \tag{A2.2}$$

$$I_1 = ADX^2[(\ell^2m^2 + 2n^2)A + (m^2 + n^2)^2D]/72$$
$$-A^2DX(\ell^2A + m^2D - n^2X)/12 + AX[2(m^2 + n^2)D$$
$$+ (\ell^2 + n^2)A][(\ell^2 + m^2)DA - (\ell^2 + m^2)AX$$
$$- (m^2 + n^2)DX]/48, \tag{A2.3}$$

and

$$I_2 = X^2[(\ell^2m^2 + 2n^2)A + (m^2 + n^2)D]^2/(12)^3$$
$$- AX^2n^2(m^2 + n^2)[2(m^2 + n^2)D + (\ell^2 + n^2)A]4(12)^2. \tag{A2.4}$$

Also,

$$b_0{}^3J = J_0 + J_1Y + J_2Y^2 + J_3Y^3 \tag{A2.5}$$

where

$$J_0 = b_0^{\,2} H_0 I_0 - 4H_0^{\,3} \quad , \tag{A2.6}$$

$$J_1 = b_0^{\,2}(H_0 I_1 + H_1 I_0) - 12H_0^{\,2}H_1 - G_1^{\,2} \quad , \tag{A2.7}$$

$$J_2 = b_0^{\,2}(H_0 I_2 + H_1 I_1) - 12H_0 H_1^{\,2}$$

$$- 2G_1 G_2 \quad , \tag{A2.8}$$

$$J_3 = b_0^{\,2} H_1 I_2 - 4H_1^{\,3} - G_2^{\,2} \quad , \tag{A2.9}$$

in which

$$H_0 = A^3 D^2 X/72 \quad , \tag{A2.10}$$

$$H_1 = A^2 DX[(\ell^2 m^2 + 2n^2)A + (m^2 + n^2)^2 D]/6(12)^2$$

$$- A^2 X[2(m^2 + n^2)D + (\ell^2 + n^2)A]^2/(4)^2(12)^2 \quad , \tag{A2.11}$$

$$G_1 = A^4 D^2 \{2[(\ell^2 + m^2)\ DA - (\ell^2 + n^2)A$$

$$- (m^2 + n^2)DX]X^{\frac{1}{2}} + [2(m^2 + n^2)D$$

$$+ (\ell^2 + n^2)A]X^{\frac{3}{2}}\}/8(12)^2 \quad , \tag{A2.12}$$

$$G_2 = A^3 X^{\frac{3}{2}}\{4D[2(m^2 + n^2)D + (\ell^2 + n^2)A]$$

$$[(\ell^2 m^2 + 2n^2)A + (m^2 + n^2)^2 D] - 8AD^2 n^2 (m^2 + n^2)$$

$$- [2(m^2 + n^2)D + (\ell^2 + n^2)A]^3\}/2(4)^3(12)^3 . \tag{A2.13}$$

The factors A and D are

$$A = 1 - X$$

$$D = \mathbb{C} - X$$

Of course, ℓ, m and n are the direction cosines of the magnetic field vector.

In equation (4.3.8), the discriminant is written as a sixth order polynomial in Y. The C- coefficients are:

$$C_0 = I_0^3 - B_0 J_0^2 , \quad \text{(A2.14)}$$

$$C_1 = 3I_0^2 I_1 - 2B_0 J_0 J_1 , \quad \text{(A2.15)}$$

$$C_2 = 3(I_0^2 I_2 + I_0 I_1^2) - B_0(2J_0 J_2 + J_1^2), \quad \text{(A2.16)}$$

$$C_3 = 6I_0 I_1 I_2 + I_1^3 - 2B_0(J_0 J_3 + J_1 J_2), \quad \text{(A2.17)}$$

$$C_4 = 3(I_0 I_2^2 + I_1^2 I_2) - B_0(2J_1 J_3 + J_2^2), \quad \text{(A2.18)}$$

$$C_5 = 3I_1 I_2^2 - 2B_0 J_2 J_3 , \quad \text{(A2.19)}$$

$$C_6 = I_2^3 - B_0 J_3^2 , \quad \text{(A2.20)}$$

where

$$B_0 = 27/b_0^6 .$$

Note that C_0 is identically zero.

APPENDIX III

MICHAILOV CRITERION

MICHAILOV CRITERION

The Michailov Criterion can be stated as follows: if the polynomial

$$F(\lambda) = c_0\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 \qquad \text{(A3.1)}$$

has real coefficients, then two additional polynomials

$$K(\phi) = c_n - c_{n-2}\phi^2 + \ldots \qquad \text{(A3.2)}$$

$$O(\phi) = c_{n-1}\phi - c_{n-3}\phi^3 + \ldots \qquad \text{(A3.3)}$$

can be defined which combined form a fourth polynomial

$$Q(\phi) = K(\phi) + iO(\phi) \; . \qquad \text{(A3.4)}$$

Let ϕ vary from 0 to ∞. Each time that $Q(\phi) = 0$, $F(\lambda)$ has two imaginary roots, $\pm i\phi$, unless $\phi = 0$.

This statement is clarified by developing the details for the quartic

$$F(\lambda) = c_0\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 \; . \qquad \text{(A3.5)}$$

Suppose $\lambda = i\phi$;

then $F(i\phi) = (c_4 - c_2\phi^2 + c_0\phi^4) + i(c_3\phi - c_1\phi^3)$;

so $K(\phi) = c_4 - c_2\phi^2 + c_0\phi^4$

and $O(\phi) = c_3\phi - c_1\phi^3$.

In order for $Q(\phi) = 0$, two conditions must be satisfied:

$$\phi(c_3 - c_1\phi^2) = 0 \quad \text{(A3.6)}$$

and $$c_4 - c_2\phi^2 + c_0\phi^4 = 0 \quad . \quad \text{(A3.7)}$$

From equation (A3.6), either

$$\phi = 0$$

or $$\phi^2 = c_3/c_1$$

Since the first choice does not lead to imaginary values for λ, it is considered no further. The second choice when substituted into (A3.7) yields the equation

$$c_1^2c_4 - c_1c_2c_3 + c_0c_3^2 = 0 \quad \text{(A3.8)}$$

Equation (A3.8) defines the locus of the purely imaginary roots of (A3.1) if ϕ is real. This last provision is equivalent to the inequality

$$(c_3/c_1) > 0 \quad \text{(A3.9)}$$

Numerical analysis shows that (A3.8) and (A3.9) cannot be satisfied simultaneously within the operating regime. Thus, purely imaginary roots do not occur in regions I(a) or I(b).

APPENDIX IV

EQUAL-ROOT LOCI

EQUAL-ROOT LOCI

The family of curves which individually and collectively are called the equal-root loci are represented most simply by the equation

$$I^3 - 27J^2 = 0 \qquad \text{(A4.1)}$$

In Chapter III, it was decided to plot these curves in a two dimensional space with the coordinates $X = S^2$ and $Y = B^2$; one axis corresponds to the load speed and the other to the strength of the magnetic field. Two factors remain to be used as parameters, the material property

$$\mathbb{C} = C_s^{\,2}/C_p^{\,2} = (1 - \nu)/2 \qquad \text{(A4.2)}$$

and the direction cosines of the magnetic field (ℓ, m, n). Poisson's ratio for the material is arbitrarily set at $\nu = 0.29$; so $\mathbb{C} = 0.355$. The direction cosines are restricted in three different ways; each generates a distinct group of curves. The idea is to start each group with the common loci for $\ell = m = n$ and to complete each group with the different unidirectional loci. The three cases may be described as

Case ℓ -- $1/\sqrt{3} \le \ell \le 1$, $m = n$,

Case m -- $1/\sqrt{3} \le m \le 1$, $n = \ell$,

Case n -- $1/\sqrt{3} \le n \le 1$, $\ell = m$.

Notwithstanding these several restrictions, Figures A4.1, A4.2 and A4.3 present a quite general description of the behavior of the equal-root loci.

The numerical procedure for finding the equal-root loci is straightforward. Since the coefficient C_0 in (4.3.8) is always zero, then the equation to be solved is

$$C_6Y^5 + C_5Y^4 + C_4Y^3 + C_3Y^2 + C_2Y + C_1 = 0. \quad (A4.3)$$

Recall that all of the coefficients are functions of X only. For any value of X, (A4.3) can be solved for its five roots. Thereby, the coordinates of five points on the equal-root loci are established. This procedure is repeated a sufficient number of times, for values of X within the operating regime, so that a well defined set of curves can be drawn.

Some caution must be exercised in interpreting the graphs; Figures A4.1, A4.2 and A4.3 are log-log plots, whereas Figures A4.4 and A4.5 are not. In the latter two figures, magnified views of two interesting, small regions are presented; the behavior is best communicated with natural scales. Recourse to logarithmic scales is necessary to foreshorten the range of the variable Y, yet retain the many details of the transforming shapes.

Each curve is designated by a number which is the angle in degrees associated with the dominant direction cosine. The dominant direction cosine is identified easily; for example, it is ℓ in Case ℓ.

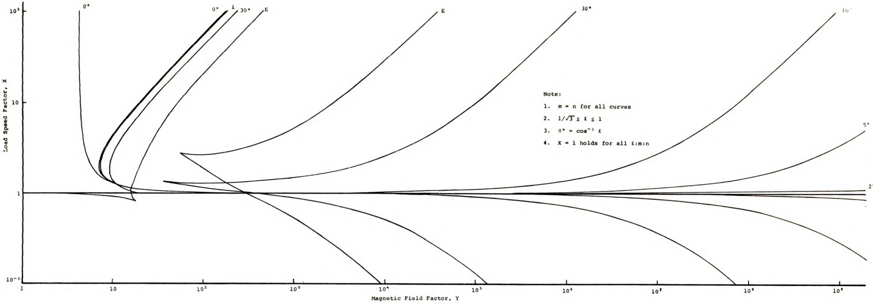

Figure A4.1. Equal Root Loci, Case ℓ

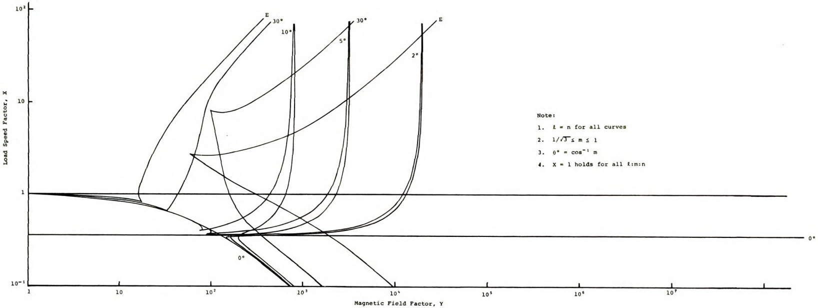

Figure A4.2. Equal Root Loci, Case m

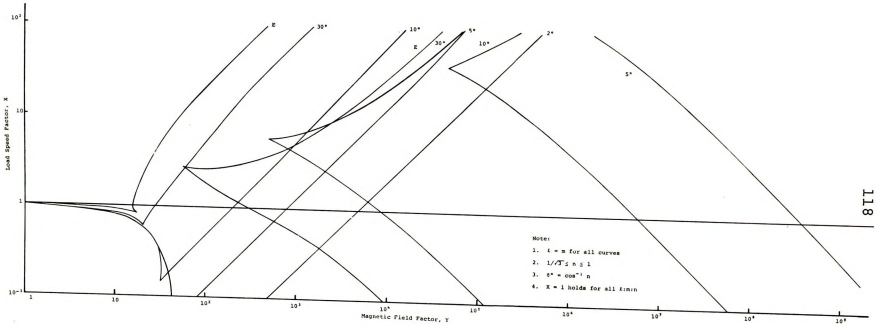

Figure A4.3. Equal Root Loci, Case n

Generally, the trends in each case can be followed without difficulty; so word descriptions are superfluous. Nevertheless, a few comments about some unexpected developments now are made:

Case ℓ

The curve, 0, bifurcates when $\ell = 1$. This idiosyncracy can be justified. Usually, (A4.3) has five roots. For $X>1$, they consist of two positive and one negative real roots in addition to a complex, conjugate pair. As $\ell \to 1$, the negative and one positive root assume very large values while the imaginary part of the complex roots acquires a very small value. In the limit when $\ell = 1$, the complex pair becomes the real, equal pair which defines the curve, 0'. Thereby the sudden appearance of a new equal-root locus is explained.

Case m

The region designated as I(b) in Chapter IV displays interesting behavior. It simultaneously narrows and rotates clockwise. Eventually it collapses onto the line $X = \mathfrak{C}$ when $m = 1$. Figure A4.4 illustrates the terminal stages of this development. That $X = \mathfrak{C}$ is indeed an equal-root locus when $m = 1$ can be proved. In this limit, both C_6 and C_5 are zero and the quantity, $D = \mathfrak{C} - X$, is common to each of the remaining coefficients $C_4, \ldots, C_1$.

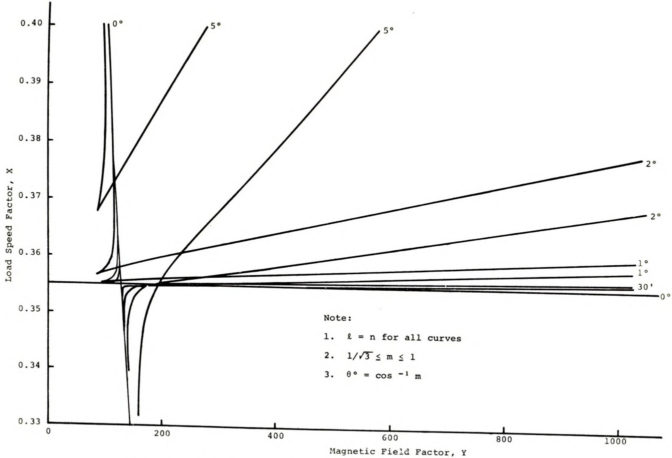

Figure A4.4. Equal Root Loci, Case m

Case n

The interesting feature of Case n is a portion of the equal-root loci for n = 1; it becomes the straight line

$$48X + (Y - 48) = 0 \qquad (A4.4)$$

Figure A4.5 portrays various stages in the process of reaching this final state.

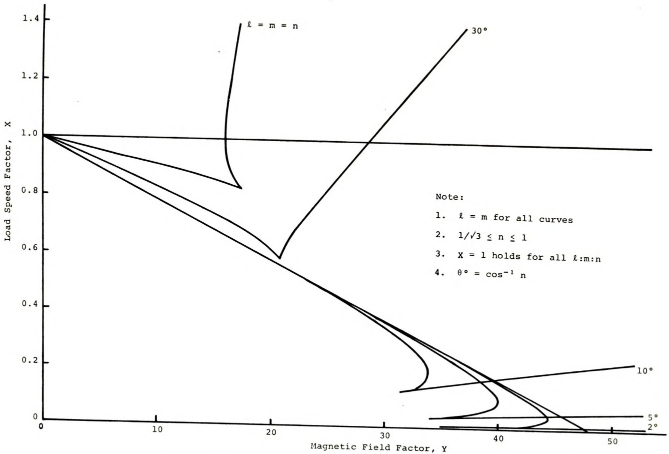

Figure A4.5. Equal Root Loci, Case n

APPENDIX V

COMPILATION OF PRIME SOLUTIONS

COMPILATION OF PRIME SOLUTIONS

In section 4.6, a sample prime solution is developed. Since it is a straightforward, though onerous, algebraic exercise, there is no need for further amplification here.

Altogether, counting from Figures 4.5 and 4.6, there are ten different prime solutions. Rather than list each separately, a general formula is given below, from which each one can be determined. Each solution has the form

$$\begin{pmatrix} U' \\ V' \\ W' \end{pmatrix} = \sum_{i=1}^{5} F_i \begin{pmatrix} r_{1i} \\ r_{2i} \\ 1 \end{pmatrix} e^{\lambda_i \zeta} \qquad \text{(A5.1)}$$

where the λ_i are the roots of the λ-polynomial (3.3.14). Note that λ_1 is always the zero root. From (3.3.16),

$$r_{1i} = \frac{BS\ell n(\lambda_i D+BS)}{D_i} , \qquad \text{(A5.2)}$$

$$r_{2i} = \frac{BSmn(\lambda_i A+BS)}{D_i} , \qquad \text{(A5.3)}$$

$$D_i = AD\lambda_i^2 + BS[(\ell^2 + n^2)A + (m^2 + n^2)D]\lambda_i + B^2S^2n^2 . \qquad \text{(A5.4)}$$

All the F- coefficients are defined by the two expressions

$$F_1 = \frac{-B^2 S^2 n^2 (\text{NUMR1})}{\lambda_2 \lambda_3 \lambda_4 \lambda_5 (\text{DENR})} \; \text{sgn}(\lambda_1) \; , \qquad (A5.5)$$

$$F_i = \frac{(-1)^{i+1} D_i (\text{DT}_i)}{\lambda_i (\text{DENR})} \; \text{sgn}[\text{Re}(\lambda_i)] \quad i=2,..,5 \; . \qquad (A5.6)$$

The new symbols introduced in (A5.5) and A5.6) are

$$\text{NUMR1} = \lambda_3 \lambda_4 \lambda_5 (\text{DT}_2) - \lambda_2 \lambda_4 \lambda_5 (\text{DT}_3)$$

$$+ \lambda_2 \lambda_3 \lambda_5 (\text{DT}_4) - \lambda_2 \lambda_3 \lambda_4 (\text{DT}_5) \; , \qquad (A5.7)$$

$$12(\text{DENR}) = A^2 D[\lambda_2{}^3 (\text{DT}_2) - \lambda_3{}^3 (\text{DT}_3)$$

$$+ \lambda_4{}^3 (\text{DT}_4) - \lambda_5{}^3 (\text{DT}_5)] \; , \qquad (A5.8)$$

in which

$$\text{DT}_2 = \lambda_3{}^2 (\lambda_4 - \lambda_5) + \lambda_4{}^2 (\lambda_5 - \lambda_3)$$

$$+ \lambda_5{}^2 (\lambda_3 - \lambda_4) \; , \qquad (A5.9)$$

$$\text{DT}_3 = \lambda_2{}^2 (\lambda_4 - \lambda_5) + \lambda_4{}^2 (\lambda_5 - \lambda_2)$$

$$+ \lambda_5{}^2 (\lambda_2 - \lambda_4) \; , \qquad (A5.10)$$

$$\text{DT}_4 = \lambda_2{}^2 (\lambda_3 - \lambda_5) + \lambda_3{}^2 (\lambda_5 - \lambda_2)$$

$$+ \lambda_5{}^2 (\lambda_2 - \lambda_3) \; , \qquad (A5.11)$$

$$\text{DT}_5 = \lambda_2{}^2 (\lambda_3 - \lambda_4) + \lambda_3{}^2 (\lambda_4 - \lambda_2)$$

$$+ \lambda_4{}^2 (\lambda_2 - \lambda_3) \; . \qquad (A5.12)$$

APPENDIX VI

W-SOLUTIONS FOR ONE-DIMENSIONAL MAGNETIC FIELDS

W-SOLUTIONS FOR ONE-DIMENSIONAL MAGNETIC FIELDS

In this appendix are recorded all the W-solutions for problems with one-dimensional magnetic fields. Several pieces of information are gathered in Figures A6.1, A6.2 and A6.3. Displayed are the equal root-loci, the correct root-choices in each region and sketches of the corresponding solutions. In the solutions, the exponents β, γ and ξ are positive real numbers; the subscripts just refer to the region in which the respective solutions hold. For ease of identification, the solutions are listed before each figure.

W-SOLUTIONS FOR $\underset{\sim}{H} = (H_1, 0, 0)$

Region I.

$$W = \begin{cases} E_1 + F_1 e^{\beta_1 \zeta} & \zeta<0 \\ e^{-\gamma_1 \zeta} (G_1 e^{i\xi\zeta} + H_1 e^{-i\xi\zeta}) & \zeta>0 \end{cases}$$

$$E_1 = -12/A\beta_1 (\gamma_1^{\ 2} + \xi_1^{\ 2})$$

$$F_1 = 12/A\beta_1 [(\beta_1 + \gamma_1)^2 + \xi_1^{\ 2}]$$

$$G_1 = -6/Ai\xi_1 (-\gamma + i\xi)(\gamma + \beta - i\xi)$$

$$H_1 = \tilde{G}_1 \ . \qquad \text{(A6.1)}$$

Region II.

$$W = \begin{cases} E_2 + F_2 e^{\beta_2 \zeta} + G_2 e^{\gamma_2 \zeta} & \zeta<0 \\ H_2 e^{-\xi_2 \zeta} & \zeta>0 \end{cases}$$

$$E_2 = 12/A\beta_2 \gamma_2 \xi_2$$

$$F_2 = 12/A\beta_2 (\xi_2 + \beta_2)(\beta_2 - \gamma_2)$$

$$G_2 = -12/A\gamma_2 (\xi_2 + \gamma_2)(\beta_2 - \gamma_2)$$

$$H_2 = 12/A\xi_2 (\xi_2 + \beta_2)(\xi_2 + \gamma_2) \qquad \text{(A6.2)}$$

Region III.

$$W = \begin{cases} E_3 + e^{\gamma_3 \zeta}(G_3 e^{i\xi_3 \zeta} + H_3 e^{-i\xi_3 \zeta}) & \zeta<0 \\ F_3 e^{-\beta_3 \zeta} & \zeta>0 \end{cases}$$

$$E_3 = 12/A\beta_3(\gamma_3^{\ 2} + \xi_3^{\ 2})$$

$$F_3 = 12/A\beta_3[(\gamma_3 + \beta_3)^2 + \xi_3^{\ 2}]$$

$$G_3 = 6/Ai\xi_3(\gamma_3 + i\xi_3)(\gamma_3 + \beta_3 + i\xi_3)$$

$$H_3 = \tilde{G}_3 \qquad \text{(A6.3)}$$

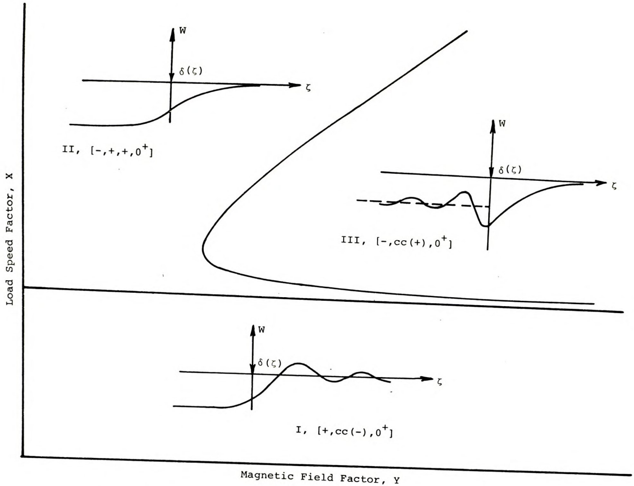

Figure A6.1. W-Solutions for $\underset{\sim}{H} = (H_1,0,0)$

W-SOLUTIONS FOR $\underset{\sim}{H} = (0,\ H_2,\ 0)$

Region I.

$$W = \begin{cases} E_1 + F_1 e^{\beta_1 \zeta} & \zeta<0 \\ G_1^{-\gamma_1 \zeta}(G_1 e^{i\xi_1 \zeta} + H_1 e^{-i\xi_1 \zeta}) & \zeta>0 \end{cases}$$

$$E_1 = -12/A\beta_1(\gamma_1^{\ 2} + \xi_1^{\ 2})$$

$$F_1 = 12/A\beta_1[(\beta_1 + \gamma_1)^2 + \xi_1^{\ 2}]$$

$$G_1 = -6/Ai\xi_1(-\gamma_1 + i\xi_1)(\gamma_1 + \beta_1 - i\xi_1)$$

$$H_1 = \tilde{G}_1 \tag{A6.4}$$

Region II.

$$W = \begin{cases} E_2 + F_2 e^{\beta_2 \zeta} & \zeta<0 \\ G_2 e^{-\gamma_2 \zeta} + H_2 e^{-\xi_2 \zeta} & \zeta>0 \end{cases}$$

$$E_2 = -12/A\beta_2\gamma_2\xi_2$$

$$F_2 = 12/A\beta_2(\gamma_2 + \beta_2)(\xi_2 + \beta_2)$$

$$G_2 = 12/A\gamma_2(\gamma_2 + \beta_2)(\gamma_2 - \xi_2)$$

$$H_2 = -12/A\xi_2(\xi_2 + \beta_2)(\gamma_2 - \xi_2) \tag{A6.5}$$

Region III.

$$W = \begin{cases} E_3 + F_2 e^{\beta_3 \zeta} + G_3 e^{\gamma_3 \zeta} & \zeta<0 \\ H_3 e^{-\xi_3 \zeta} & \zeta>0 \end{cases}$$

$$E_3 = 12/A\beta_3\gamma_3\xi_3$$

$$F_3 = 12/A\beta_3(\xi_3 + \beta_3)(\beta_3 - \gamma_3)$$

$$G_3 = -12/A\gamma_3(\xi_3 + \gamma_3)(\beta_3 - \gamma_3)$$

$$H_3 = 12/A\xi_3(\xi_3 + \beta_3)(\xi_3 + \gamma_3) \qquad \text{(A6.6)}$$

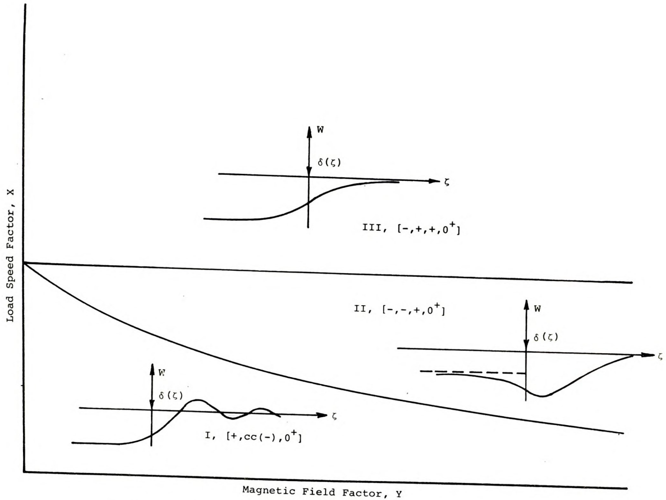

Figure A6.2. W-Solutions for $\underset{\sim}{H} = (0, H_2, 0)$

W-SOLUTIONS FOR $\underset{\sim}{H} = (0,0,H_3)$

Region I.

$$W = \begin{cases} \dfrac{-12[2\alpha_1 - (\alpha_1^2 + \beta_1^2)\zeta]}{A(\alpha_1^3 + \beta_1^2)^2} & \zeta<0 \\ \dfrac{6e^{-\alpha_1\zeta}[-(\alpha_1 + i\beta_1)^2 e^{i\beta_1\zeta} + (\alpha_1 - i\beta_1)^2 e^{-i\beta_1\zeta}}{Ai\beta_1(\alpha_1^2 + \beta_1^2)^2} & \zeta>0 \end{cases} \qquad (A6.7)$$

Region II.

$$W = \begin{cases} \dfrac{-12[(\alpha_2 + \beta_2) - \alpha_2\beta_2\zeta]}{A\alpha_2^2\beta_2^2} & \zeta<0 \\ \dfrac{-12[\beta_2^2 e^{-\alpha_2\zeta} - \alpha_2^2 e^{-\beta_2\zeta}]}{A\alpha_2^2\beta_2^2(\beta_2 - \alpha_2)} & \zeta>0 \end{cases} \qquad (A6.8)$$

Region III.

$$W = \begin{cases} \dfrac{12[(\alpha_3 - \beta_3) - \alpha_3\beta_3]}{A\alpha_3^2\beta_3^2} + \dfrac{12e^{\alpha_3\zeta}}{A\alpha_3^2(\beta_3 + \alpha_3)} & \zeta<0 \\ \dfrac{12e^{-\beta_3\zeta}}{A\beta_3^2(\beta_3 + \alpha_3)} & \zeta>0 \end{cases}$$

(A6.9)

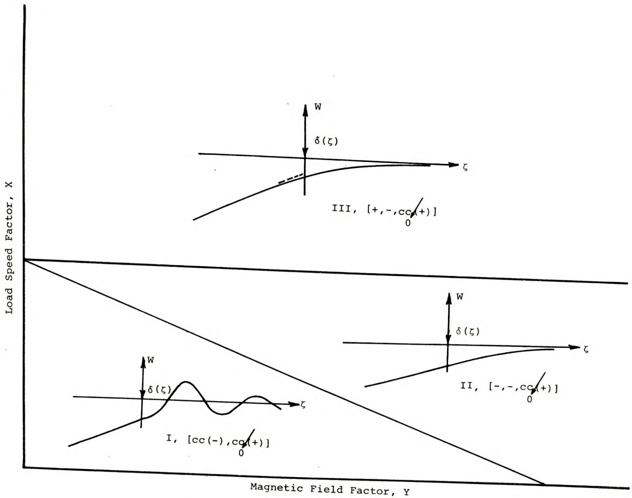

Figure A6.3. W-Solutions for $\underset{\sim}{H} = (0,0,H_3)$

APPENDIX VII

RAYLEIGH BEAM WITH A MOVING LOAD

RAYLEIGH BEAM WITH A MOVING LOAD

A Rayleigh beam is a Euler beam with the effects of rotatory inertia included. The transverse motion of such a beam under the effect of a moving load is described by the differential equation

$$-(A/12)W'''' + (A-1)W'' = \delta(\zeta) \qquad (A7.1)$$

which obviously is determined from (3.3.4). Using the Green's function approach to find the prime solution requires that the solution satisfy the homogeneous differential equation

$$-(A/12)W'''' + (A-1)W'' = 0 \qquad (A7.2)$$

together with the jump and continuity conditions at $\zeta = 0$

$$[W'] = 0 \ , \qquad (A7.3)$$

$$[W''] = 0 \ , \qquad (A7.4)$$

$$[W'''] = -12/A \ . \qquad (A7.5)$$

Assume that the prime solution to (A7.2) has the form

$$W' = e^{\lambda\zeta} \ ; \qquad (A7.6)$$

substitution leads to the λ-polynomial

$$-(A/12)\lambda^3 + (A-1)\lambda = 0 \ , \qquad (A7.7)$$

or

$$\lambda[-(A/12)\lambda^2 + (A-1)] = 0. \qquad (A7.8)$$

Besides the zero value, the λ-polynomial (A7.8) has the two roots

$$\lambda = \pm \sqrt{12(A-1)/A} = \pm\sqrt{12X/(X-1)} \qquad (A7.9)$$

The two roots are real for X>1 and imaginary for X<1:

$$\lambda = \begin{cases} \pm\sqrt{12X/(X-1)} & X>1 \\ \pm i\sqrt{12X/(1-X)} & x<1 \end{cases} \qquad (A7.10)$$

Obviously X=1 is important; two distinct solutions exist: one is valid when X<1 and the other when X>1. Some additional information must be found before these solutions can be determined.

It is necessary to establish the sign of the real part of the imaginary roots. To accomplish this end, augment (A7.2) with the term, $-\theta W'''$, where θ is a small, positive number; the sign is the same as that of the W'''-term in (3.3.8). The augmented equation of motion is

$$-(A/12)W'''' - \theta W''' + (A-1)W'' = 0 \quad . \qquad (A7.11)$$

The augmented λ-polynomial is

$$\lambda[(1-X)\lambda^2/12 + \theta\lambda + X] = 0 \quad ; \qquad (A7.12)$$

the quadratic portion has the roots

$$\lambda = \left[-\theta \pm\sqrt{\theta^2 - 4X(1-X)/12}\right] \div (1-X)/6 \quad . \qquad (A7.13)$$

Thus the real part of the complex roots associated with the augmented problem is

$$Re(\lambda) = -6\theta/(1-X) \quad , \qquad (A7.14)$$

from which the complex roots in (A7.10) are recognized to be

$$\lambda = 0^- \pm i\sqrt{12X/(1-X)} \qquad X<1 \quad . \qquad (A7.15)$$

To complete the description of the three roots in (A7.8), the behavior of the zero root must be determined. This is accomplished by augmenting (A7.2) with the term, $+\eta W'$, which was identified previously in Chapter III; therefore,

$$-(A/12)W'''' + (A-1)W'' + \eta W' = 0 \quad . \qquad (A7.16)$$

The corresponding λ-polynomial is

$$-(A/12)\lambda^3 + (A-1)\lambda + \eta = 0 \quad . \qquad (A7.17)$$

For small values of λ the λ^3-term may be overlooked, so

$$\lambda = -\eta/(A-1) = \eta/X \quad . \qquad (A7.18)$$

Accordingly, the zero root in (A7.8) is

$$\lambda = 0^+ \qquad (A7.19)$$

The two sets of roots of the λ-polynomial (A7.8) are identified in Table A7.1

Table A7.1. Type of Roots

Region	Roots
X<1	$0^+, \pm\sqrt{12X/(X-1)}$
X>1	$0+, 0^- \pm i\sqrt{12X/(1-X)}$

Following the solution technique outlined in Chapter III leads to the bounded prime solutions:

X>1

$$W' = \frac{6}{A\lambda_0^2}\begin{cases} -2 + e^{\lambda_0\zeta} & \zeta<0 \\ -e^{-\lambda_0\zeta} & \zeta>0 \end{cases} \qquad (A7.20)$$

where

$$\lambda_0 = \sqrt{12X/(X-1)}$$

X<1

$$W' = \frac{12}{A\lambda_0^2}\begin{cases} 1 & \zeta<0 \\ \cos \lambda_0\zeta & \zeta>0 \end{cases} \qquad (A7.21)$$

where

$$\lambda_0 = \sqrt{12X/(1-X)}$$

The integrated prime solution for X>1 is

$$W = \frac{6}{A\lambda_0^3}\begin{cases} -2\lambda_0\zeta + e^{\lambda_0\zeta} + C^- & \zeta<0 \\ e^{-\lambda_0\zeta} + C^+ & \zeta>0 \end{cases} \qquad (A7.22)$$

where C^- and C^+ are integration constants. Since the deflection of a real beam must be continuous, then

$$W(0^-) = W(0^+) \qquad (A7.23)$$

so

$$C^- = C^+$$

In other words, the sole function of the integration constants is to shift the deflection curve up or down--this rigid body motion is not too significant. The same conclusion is reached for the integrated prime solution in X<1.

Note that it is possible to find the displacement-solution for the Rayleigh beam directly; these displacement-solutions are identical to the integrated prime solutions. Neglecting the rigid body motion, the Rayleigh beam deflections are:

X>1

$$W = \frac{6}{A\lambda_0^3}\begin{cases} -2\lambda_0\zeta + e^{\lambda_0\zeta} & \zeta<0 \\ e^{-\lambda_0\zeta} & \zeta>0 \end{cases} \qquad \text{(A7.24)}$$

where

$$\lambda_0 = \sqrt{12X/(X-1)} \quad .$$

X<1

$$W = \frac{12}{A\lambda_0^3}\begin{cases} \lambda_0\zeta & \zeta<0 \\ \sin \lambda_0\zeta & \zeta>0 \end{cases} \qquad \text{(A7.25)}$$

where

$$\lambda_0 = \sqrt{12X/(1-X)} \quad .$$

Figure A7.1 contains a sketch of both solutions

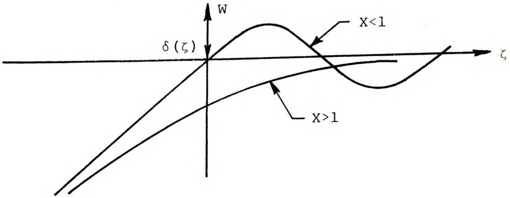

Figure A7.1. Rayleigh Beam Displacement

APPENDIX VIII

FREE, DAMPED, HARMONIC WAVES

FREE, DAMPED, HARMONIC WAVES

Since the magnetic field serves to damp the motion, it is reasonable to investigate the validity of a damped, harmonic wave solution to equations (5.1.1), (5.1.2) and (5.1.3). Assume a solution in the form

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} e^{-dt} e^{ik(x_1 - st)} \qquad \text{(A8.1)}$$

in which d is the damping coefficient, s is the phase velocity and k is the wave number; all three are positive quantities. Substituting this solution into the differential equations leads to the matrix equation

$$(\mathbb{R} + iks\,\mathbb{I})\,\mathbb{r} = 0 \qquad \text{(A8.2)}$$

where

$$R = \begin{pmatrix} R_{11} & -\chi \ell m d\,, & -\chi \ell n d \\ -\chi \ell m d\,, & R_{22}\,, & -\chi m n d \\ -\chi \ell n d\,, & -\chi m n d\,, & R_{33} \end{pmatrix}$$

with

$$R_{11} = k^2(s^2 - C_p^{\,2}) + \chi(m^2+n^2)d - d^2 \;,$$

$$R_{22} = k^2(s^2 - C_s^{\,2}) + \chi(\ell^2+n^2)d - d^2 \;,$$

$$R_{33} = k^2[(\frac{12+a^2k^2}{12})s^2 - \frac{a^2k^2C_p^2}{12}] + \frac{\chi a^2k^2(m^2+n^2)d}{12}$$

$$+ \chi(\ell^2+m^2)d - (\frac{12+a^2k^2}{12})d^2 ,$$

and

$$\mathbb{I} = \begin{pmatrix} I_{11} , & -\chi\ell m , & -\chi\ell n \\ -\chi\ell m , & I_{22} , & -\chi mn \\ -\chi\ell n & -\chi mn , & I_{33} \end{pmatrix}$$

with

$$I_{11} = \chi(m^2+n^2) - 2d$$

$$I_{22} = \chi(\ell^2+n^2) - 2d$$

$$I_{33} = \chi\frac{a^2k^2(m^2+n^2)}{12} + \chi(\ell^2+m^2) - 2(\frac{12+a^2k^2}{12})d$$

and $\mathbb{r}$ is the transpose of (r_1,r_2,r_3). For the amplitude vector $\mathbb{r}$ to be non-trivial,

$$|\mathbb{R} + iks\ \mathbb{I}| = 0 \qquad \text{(A8.3)}$$

This equation defines the phase velocity in terms of the wave number and the damping coefficient. Clearly, the velocities are functions of ℓ, m and n: the medium is anisotropic. Some simple cases which describe various magnetoelastic interactions are now considered.

A. Rayleigh Plate

Remove the magnetic field. The resulting set of differential equations describe the undamped motion of a plate, the Rayleigh Plate. Equation (A8.2) reduces to

$$\begin{pmatrix} s^2-C_p^{\ 2} & , 0 & , 0 \\ 0 & , s^2-C_s^{\ 2} & , 0 \\ 0 & , 0 & , s^2(1+\frac{a^2k^2}{12}) - \frac{a^2k^2C_p^{\ 2}}{12} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = 0 \tag{A8.4}$$

The three modes of propagation are readily identified. Dilatational and distortional waves travel undispersed at phase velocityes C_p and C_s, respectively. The medium is dispersive to flexural motion; the flexural wave speed is

$$s^2 = (a^2k^2/12+a^2k^2)C_p^{\ 2} = C_f^{\ 2}$$

Some interesting motions are predicted by (A8.4). For example, suppose $k\to 0$; then (A8.4) approaches

$$\begin{pmatrix} s^2-C_p^{\ 2} & , 0 & , 0 \\ 0 & , s^2-C_s^{\ 2} & , 0 \\ 0 & , 0 & , s^2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = 0 \tag{A8.5}$$

Whereupon, if $s\to 0$, then the amplitude vector is $\mathbb{r} = (0,0,r_3)$. In words, for small wave number and phase velocity the flexural motion deteriorates into rigid body translation. On the other hand, if $k\to\infty$ then (A8.4) approaches

$$\begin{pmatrix} s^2-C_p^{\ 2} & , 0 & , 0 \\ 0 & , s^2-C_s^{\ 2} & , 0 \\ 0 & , 0 & , s^2-C_p^{\ 2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = 0 \tag{A8.6}$$

Whereupon if $s \to C_p$, the solution is $\mathbb{r} = (r_1, 0, r_3)$. Evidently, a flexural and a dilatational wave propagate together, but uncoupled. Similarly, there exists an intermediate wave number at which a flexural and a shear wave travel uncoupled at the shear wave velocity, C_s.

It is expected that the three modes of propagation existing in the Rayleigh Plate will not be augmented by the reintroduction of the magnetic field. Although the magnetic field may not introduce new modes, the differential equations (5.1.1), (5.1.2) and (5.1.3) indicate that it can couple the three modes and it can create dispersion of the dilatational and distortional waves.

B. Magnetic Field Acts in the Direction of Propagation: $\underset{\sim}{H} = H(1,0,0)$

Alter the differential equations by restricting the magnetic field to be $\underset{\sim}{H} = H(1,0,0)$. Obviously, there are three uncoupled motions which consist of an undamped dilatational wave, a damped distortional wave and a damped flexural wave. The effects of the magnetic field on each mode can be evaluated from (A8.4) in which

$$\mathbb{R} = \begin{pmatrix} R_{11}, & 0, & 0 \\ 0, & R_{22}, & 0 \\ 0, & 0, & R_{33} \end{pmatrix}$$

where

$$R_{11} = k^2(s^2-C_p^{\ 2}) - d^2$$

$$R_{22} = k^2(s^2-C_s^{\ 2}) + \chi d - d^2$$

$$R_{33} = k^2[(\frac{12+a^2k^2}{12})s^2 - \frac{a^2k^2}{12}C_p^{\ 2}] + \chi d - \frac{12+a^2k^2}{12}d^2$$

and

$$\Pi = \begin{pmatrix} -2d, & 0, & 0 \\ 0, & \chi-2d, & 0 \\ 0, & 0, & \chi-2(\frac{12+a^2k^2}{12})d \end{pmatrix}$$

By selecting d=0, the dilatational wave is seen to propagate with the phase velocity $s=C_p$. For $d=\chi/2$, a distortional wave propagates with the phase velocity $s^2=C_s^{\ 2}-\chi^2/4k^2$. The flexural wave exhibits the damping coefficient $d = (\chi/2)(12/12+a^2k^2)$ and the velocity $s^2 = C_f^{\ 2} - (\chi^2/4k^2)(12/12+a^2k^2)^2$.

C. Magnetic Field in the Plane of the Plate but Orthogonal to the Direction of Propagation: $\underset{\sim}{H} = H(0,1,0)$

The differential equations suitably adjusted by the assumption that $\underset{\sim}{H} = H(0,1,0)$ identify three modes of propagation each of which is again uncoupled. Now, the dilatational waves are damped, the distortional waves are undamped and the flexural waves experience two types of damping.

The various combinations of d and s are listed in Table A8.1. The most interesting aspect of the response in this magnetic field is that the two different types of damping which occur in the flexural motion can be compared. The w-motion is described by the differential equation

$$- \frac{a^2 c_p^2}{12} \frac{\partial^4 w}{\partial x_1^4} + \frac{a^2 \chi}{12} \frac{\partial^3 w}{\partial x_1^2 \partial t} - \chi \frac{\partial w}{\partial t}$$

$$= \frac{\partial^2 w}{\partial t^2} - \frac{a^2}{12} \frac{\partial^4 w}{\partial x_1^2 \partial t^2} \qquad \text{(A8.7)}$$

The two damping terms, $(a^2\chi/12)(\partial^3 w/\partial x_1^2 \partial t)$ and $\chi \partial w/\partial t$ are given the names solid and viscous, respectively. By arbitrarily setting one, then the other, to zero, two different relationships for the damping coefficient and the phase velocity result. The two curves on Figure A8.1. show that the two effects complement each other. The solid damping, curve 1, increases in value with increasing wave number while the viscous camping, curve 2, exhibits the opposite effect. It is interesting to note that the sum of the two effects, curve 3, is a constant. So their combined effect creates the same damping on flexural waves as the viscous damping alone produces on dilatational and distortional waves.

Table A8.1

Expressions for Magnetoelastic Interactions

$\underset{\sim}{H}$	Mode	Damping Factor d	Phase Velocity s^2	Critical Flux Density KH
1,0,0	dilatation	0	C_p^2	None
	distortion	$\chi/2$	$C_s^2-\chi^2/4k^2$	$(2k\rho C_s/\sigma)^{1/2}$
	flexure	$(\chi/2)(12/12+a^2k^2)$	$C_f^2-(\chi^2/4k^2)(12/12+a^2k^2)^2$	$[(2k\rho C_f/\sigma)(12+a^2k^2/12)]^{1/2}$
0,1,0	dilatation	$\chi/2$	$C_p^2-\chi^2/4k^2$	$(2k\rho C_p/\sigma)^{1/2}$
	distortion	0	C_s^2	None
	flexure	$\chi/2$	$C_f^2-\chi^2/4k^2$	$(2k\rho C_f/\sigma)^{1/2}$
	flexure (solid only)	$(\chi/2)(a^2k^2/12+a^2k^2)$	$C_f^2-(\chi^2/4k^2)(a^2k^2/12+a^2k^2)^2$	$(2k\rho C_f/\sigma)(12+a^2k^2/a^2k^2)^{1/2}$
	flexure (viscous only)	$(\chi/2)(12/12+a^2k^2)$	$C_f^2-(\chi^2/4k^2)(12/12+a^2k^2)^2$	$(2k\rho C_f/\sigma)(12+a^2k^2/12)^{1/2}$
0,0,1	dilatation	$\chi/2$	$C_p^2-\chi^2/4k^2$	$(2k\rho C_p/\sigma)^{1/2}$
	distortion	$\chi/2$	$C_s^2-\chi^2/4k^2$	$(2k\rho C_s/\sigma)^{1/2}$
	flexure	$(\chi/2)(a^2k^2/12+a^2k^2)$	$C_f^2-(\chi^2/4k^2)(a^2k^2/12+a^2k^2)^2$	$(2k\rho C_f/\sigma)(12+a^2k^2/a^2k^2)^{1/2}$

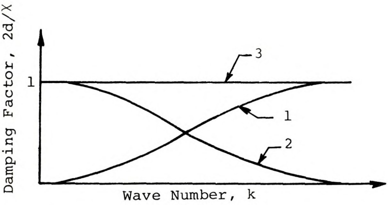

Figure A8.1. Viscous and Solid Damping

D. Magnetic Field Normal to the Plane of the Plate: $\underset{\sim}{H} = H(0,0,1)$

When the magnetic field is $\underset{\sim}{H} = H(0,0,1)$, the three, uncoupled modes are again easily recognizable from the equations of motion. In this case, all three motions are damped. Moreover, it is the only instance in which solid damping occurs alone. This suggests an experiment to discover at what wave length and magnetic field strength solid damping becomes significant. Table A8.1 contains the expressions for d and s corresponding to each mode.

E. Magnetoelastic Interactions from Free Wave Propagation

Three interactions between the magnetic field and the material are listed on Table A8.1. Two, the damping coefficient and the phase velocity, have already been mentioned. The third factor, $(KH)_{cr}$, is the critical flux density. These interactions are represented graphically on Figures A8.2 and A8.3. The damping coefficient d is a measure of the rapidity with which a wave decays. Figure A8.2 on which the information from Table A8.1 is sketched identifies the linear relationship between the damping coefficient and the magnetic field strength: the stronger the magnetic field, the more rapid the decay. In this figure, the letters p, s and f refer respectively to the dilatational, distortional and flexural modes; the bracketted quantity obviously identifies the associated magnetic field. It is seen that in addition to the two,

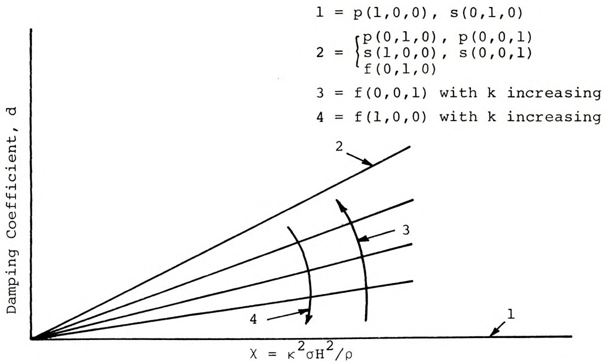

Figure A8.2. Damping Coefficient

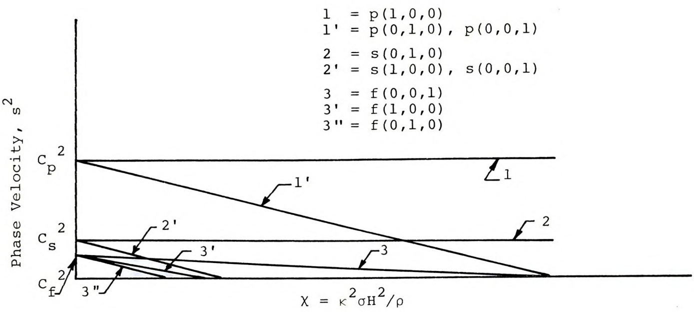

Figure A8.3. Phase Velocity

previously identified, undamped modes, p(1,0,0) and s(0,1,0), the flexural motion is also undamped in the two limiting cases, f(1,0.0) as $k \to \infty$ and f(0,0,1) as $k \to 0$. Shortly, some numerical values for d for the different modes and fields will be presented.

Except for the dilatational mode in the x_1-field and the distortional mode in the x_2-field, all the wave speeds are affected in the following way: the phase velocity is reduced by the magnetoelastic interaction. The effect on each mode is sketched on Figure A8.3. The dilatational mode 1' and the flexural mode 3 are least affected; the flexural mode 3" is most strongly affected. All the other cases lie in between these two extremes.

Obviously, there exists a critical value of the magnetic field strength above which oscillatory motion is not possible. The expressions for $(KH)_{cr}$, listed on Table A8.1, correspond to the value of χ on Figure A8.3 at which the phase velocity line intersects the χ-axis.

Note that specific values for the wave number and the plate thickness are used in drawing Figure A8.3. They are $k=1(\text{cm})^{-1}$ and $a = \sqrt{3}$ cm. Thus the wave length selected is $\sqrt{2}\pi/3$ times the plate thickness.

To get some feeling for the magnitudes involved in the magnetoelastic interactions, some calculations have been made for two dissimilar materials. The two materials

chosen were ATJ graphite and aluminum; the symbols GR and Al are used for identification in Tables A8.2 and A8.3. Obviously the aluminum is much more severely affected than the graphite.

Table A8.2

Properties of Al and Gr

	ρ in $Kg(m)^{-3}$	σ in $(\Omega m)^{-1}$	E in $N(M)^{-2}$	ν	C_p in $m(sec)^{-1}$	C_s in $m(sec)^{-1}$	C_f in $m(sec)^{-1}$
Al	2700	3.54×10^7	6.3×10^{10}	0.330	5360	3100	2400
Gr	1810	1.11×10^5	8.41×10^9	0.08	2160	1530	965

$\underset{\sim}{H}$	Mode	d^{-1} in (sec) *†		S in $m(sec)^{-1}$ †		$(KH)_{cr}$ in $Wb(m)^{-2}$ †	
		Al	Gr	Al	Gr	Al	Gr
(1,0,0)	dilatation	∞	∞	5.36×10^3	2.16×10^3	∞	∞
	distortion	0.529×10^{-3}	0.13	2.46×10^3	1.53×10^3	0.69	7.07
	flexure	0.663×10^{-3}	0.164	1.86×10^3	0.97×10^3	0.51	6.30
(0,1,0)	dilatation	0.529×10^{-3}	0.13	5.03×10^3	2.16×10^3	0.91	8.40
	distortion	∞	∞	3.10×10^3	1.53×10^3	∞	∞
	flexure	0.529×10^{-3}	0.13	1.47×10^3	0.97×10^3	0.61	5.61
(0,0,1)	dilatation	0.529×10^{-3}	0.13	5.03×10^3	2.16×10^3	0.91	8.40
	distortion	0.529×10^{-3}	0.13	2.46×10^3	1.53×10^3	0.69	7.07
	flexure	2.64×10^{-3}	0.67	2.37×10^3	0.97×10^3	1.35	12.6

* $KH = 0.54\ Wb/m^2$

† $k = 1(cm)^{-1}$ and $a = \sqrt{3}$ cm

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