FINITE ELEMENT METHODS FOR PERIODIC SOLUTIONS OF DYNAMICAL SYSTEMS

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ABSTRACT

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This dissertation deals with the problem of computing the normal modes of nonlinear Hamiltonian systems. The normal modes are assumed to exist and depend continuously on the total energy E for E in some range.

The finite element method, in which the time variable is discretized, is applied to such systems and transforms the mathematical problem that is governed by nonlinear differential equations into one governed by a set of nonlinear algebraic equations which is to be solved for various values of E.

The main contribution in this work is to establish the existence of solutions of the nonlinear algebraic finite element equations by a contraction mapping argument. This is done by relating the Jacobian of the discrete problem to that of the exact problem. Furthermore, if the exact problem has an isolated branch of solutions for some E in a neighborhood of E_O, then, correspondingly the finite element solution that exists for E_O is also isolated and can be uniquely continued, for small mesh size h, in a neighborhood of E_O. Algorithms for implementing the numerical work are discussed and some illustrative numerical results are also presented.

This Dissertation is Dedicated

to

Late El-Sayed El-Mandouh, my father which
words can never express my gratitude to him;

to

Rowhia Shendy Zakry, my mother who provided all
the love and care during my years of growing up
and for having the insight and
courage to send me to study abroad;

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that this work was completed;

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Chapter 1 Introduction

This dissertation concerns numerical solutions of a class of nonlinear dynamic systems known as Hamiltonian systems. Such systems of ordinary differential equations arise in celestial mechanics as well as in physics and many branches of engineering. In the language of engineering mechanics we may describe such systems as being composed of discrete masses that are interconnected by perfect, nonlinear elastic springs.

The study of periodic solutions of such dynamic systems has remained a central problem in nonlinear mechanics since the days of Lagrange, Hamilton, Jacobi, Hill and Poincare. A fairly comprehensive account of the early work, up to 1920, can be found in a monograph by Birkhoff [1].

There has in fact been a large body of literature published on the subject of periodic solutions of nonlinear ordinary differential equations. It would be futile to attempt here an exhaustive review of past work. We mention briefly in passing that for single, second order nonlinear ordinary differential equations, use of the phase plane along with functional-analytic and topological techniques have proven fruitful in leading to many explicit results on the necessary and sufficient conditions for the existence of periodic solutions [2,3]. For systems of nonlinear ordinary differential equations in which nonlinearities are small, perturbation methods have been used [4,5,6].

The study of periodic solutions of general systems of nonlinear ordinary differential equations is a difficult subject.

For the so-called Hamiltonian systems, however, some research has been done. In particular, when the system has m degrees of freedom and is linear with a positive definite potential $V(x)$, $x = (x_1, x_2, \ldots, x_m)$, it has exactly m periodic solutions known as the normal modes. When the potential $V(x)$ contains terms higher than the quadratic, the approximation of it by a linear system is justified when the total energy E of the system is small. As E increases and terms in $V(x)$ higher than the quadratic become important, it is natural to ask what become of the linear normal modes.

In 1948, Seifert [7] showed that the nonlinear system with a convex potential possesses for any $E > 0$ at least one periodic solution that joins two distinct points x on $V(x) = E$ in the "configuration space" x. Seifert's method is geometric in nature and relies on the fact that solutions of the system are geodesics in the x-space with time t being a parameter.

In a series of papers [8 - 12] Rosenberg considered a class of periodic solutions of such Hamiltonian systems that are generalizations of the normal modes of the linear systems and referred to them as nonlinear normal modes. Rosenberg also gave precise definitions for such normal modes in terms of solutions of certain nonlinear boundary value problems.

A question of theoretical interest is the existence of periodic solutions of nonlinear Hamiltonian systems. Using the variational approach, by which special periodic solutions can be formulated as critical points of convex functionals on manifolds defined by constant integrals of the potential or kinetic energy of the system, Berger [13 - 16] and Gordon [17] have proved the existence of periodic solutions that are even or odd in time. Under more stringent assumptions on the potential function $V(x)$, Weinstein [18] has proved the existence of at least m periodic solutions of a Hamiltonian system with m degrees of freedom. A related work is due to Moser [19]. More recently Rabinowitz [20,21] has also established the existence of periodic solutions, using the variational approach, for prescribed total energy or given period of the motion.

In this dissertation we are concerned with the normal modes of nonlinear Hamiltonian systems for given total energies E. The normal modes will be assumed to exist and depend continuously on E for E in some range and we shall develop numerical methods for their computation. We mention that Rosenberg was the first to recognize that for $m \geq 2$ a nonlinear Hamiltonian system may possess more than m normal modes at some E and this number may change as E changes. Such "superabundant normal modes" obviously cannot exist for systems that are linear or nearly linear for such systems are known to possess exactly m normal modes.

For $m = 2$, Yen [22,23] and Johnson and Rand [24] have both developed alternative characterizations of the normal modes and used such characterizations to establish the continuous dependence of the normal modes on E and shed light on the question of bifurcation of new normal modes. These works involve, essentially, embedding the normal modes into a family of solutions of initial problems, characterized by a parameter, α say. The solutions of the systems generate a continuous mapping $f(\alpha)$ that also depends continuously on E and the normal modes correspond to the zeros of $f(\alpha)$.

There do not seem to exist any quantitative methods for nonlinear Hamiltonian systems of the type discussed above. In this dissertation we shall apply the finite element method to such systems and compute their normal modes for fixed total energy E. Upon discretizing the time variable t with mesh size h, the mathematical problem is transformed into a nonlinear algebraic one which is to be solved for various values of E.

An important question that arises immediately is whether such discretization, for small h, by the finite element method, preserves the normal modes of the system. This question is especially intriguing in view of the fact that the normal modes correspond to critical points, that are saddle point in general, in the variational formulations and projection type methods such as the finite element method, do not preserve saddle points.

The main contribution in this dissertation is to answer the above question in the affirmative under suitable conditions. The existence of solutions of the nonlinear algebraic finite element equations is established by a contraction mapping [25] argument by relating the Jacobian of the discrete problem to that of the exact problem. Furthermore, by the implicit function theorem [25] we know that if the exact problem has an isolated solution for some $E = E_0$ say, then this solution can be uniquely continued in a neighborhood of E_0. We shall show that, correspondingly, the finite element solution that exists for E_0 is also isolated and can be uniquely continued, for small mesh size h, in a neighborhood of E_0. We shall also address the question of numerical algorithms and present a number of numerical results.

The organization of this dissertation is as follows. In Chapter 2 we present some background materials. We shall present there the mathematical problem, define special classes of periodic solutions such as the normal modes and discuss the general question of existence of the normal modes under given E and the continuous dependence of the normal modes on E. In Chapter 3 we apply the finite element method to such systems and present our results on the existence of solutions of the nonlinear algebraic problem and their convergence to the exact solutions as the mesh size h tends to zero. We shall also discuss there the continuation problem as E varies and the numerical algorithm. In Chapter 4 we present numerical results for example problems with $m = 1$ and $m = 2$. Chapter 5 contains the summary and conclusions.

Chapter 2 Periodic Solutions of Hamiltonian Systems

In this chapter we review some background materials on periodic solutions of nonlinear autonomous systems. In Section 2.1 we formulate the mathematical problem and discuss the several classes of periodic solutions whose numerical approximation are sought in Chapter 3. In Section 2.2 we discuss the general questions of existence of such periodic solutions for given total energy E of the system, their continuous dependence on E, and their bifurcations.

2.1. Normal Modes of Nonlinear Systems

We shall consider autonomous systems of ordinary differential equations of the form

$$\frac{d^2x}{dt^2} + \text{grad } V(x) = 0 \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))$ is an m vector-valued function of time t and $V(x) = V(x_1, x_2, \ldots, x_m)$ is a C^1 real-valued function of $x_1, \ldots, x_m$, known as the potential or potential energy function. Further assumptions on $V(x)$ are to be specified. Such systems are often referred to as Hamiltonian systems and arise naturally in many physical problems. An example is given below.

Consider the mechanical system as shown in Figure 2.1. It consists of m masses, not necessarily equal, that are interconnected by nonlinear springs. The first and the last of the masses, m_1 and m_m, are connected to the ground by the end springs S_1

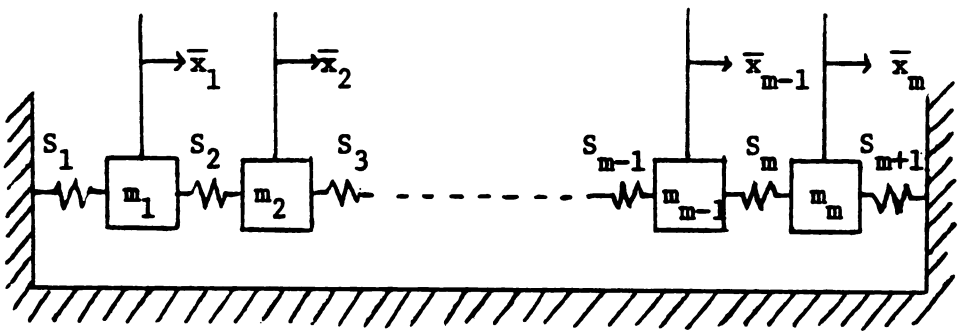

Figure 2.1 An m degree of freedom system

and S_{m+1}. We assume that the coordinates $\bar{x}_i$ are chosen such that $\bar{x}_i = 0$, $i = 1,2,\ldots,m$, correspond to an equilibrium configuration of the system. We also assume that each spring force S_i is an odd function of its deflection u_i from equilibrium. The equations of motion are then

$$m_i \frac{d^2\bar{x}_i}{dt^2} = S_i(\bar{x}_i - \bar{x}_{i-1}) - S_{i+1}(\bar{x}_i - \bar{x}_{i+1}) \quad i = 1,2,\ldots,m \tag{2.2}$$

with $\bar{x}_0 = \bar{x}_{m+1} = 0$. We assume further that the spring forces may be represented by their finite Taylor's expansion. It now follows from (2.2) and the oddness of each S_i in its argument that

$$m_i \frac{d^2\bar{x}_i}{dt^2} = \sum_{j=1,3,\ldots}^{r_i} a_{ij}(\bar{x}_{i-1} - \bar{x}_i)^j - \sum_{j=1,3,\ldots}^{r_{i+1}} a_{i+1j}(\bar{x}_i - \bar{x}_{i+1})^j$$

$$i = 1,2,\ldots,m \tag{2.3}$$

with $\bar{x}_0 = \bar{x}_{m+1} = 0$. By the change of variables

$$x_i = m_i^{1/2}\,\bar{x}_i, \quad i = 1,2,\ldots,m \tag{2.4}$$

and defining the potential energy function $V(x_1,x_2,\ldots,x_m)$ to be

$$V(x) = \sum_{i=1}^{m+1} \sum_{j=1,3,\ldots}^{r_i} \frac{a_{ij}}{j+1}\left(\frac{x_i - 1}{m_{i-1}^{1/2}} - \frac{x_i}{m_i^{1/2}}\right)^{j+1} \tag{2.5}$$

the system (2.3) becomes

$$\frac{d^2x_i}{dt^2} = -\frac{\partial V}{\partial x_i} \quad i = 1,2,\ldots,m$$

We shall assume, henceforth, that the potential function $V(x)$ in (2.1) is convex, positive definite and an even function of x with $V(0) = 0$ and $V(x)$ tending to infinity as $|x| \to \infty$. These assumptions ensure the existence of periodic solutions of the system (2.1), which we shall establish in the next two sections. The assumptions of convexity and evenness of $V(x)$ may also be relaxed in some instances, as we shall point out later.

Taking the inner product of (2.1) with $\frac{dx}{dt}$ and integrating with respect to t, we obtain

$$\frac{1}{2}\dot{x}^2 + V(x) = E \tag{2.6}$$

where E is a constant in time, the total energy of the system. The over dot in (2.6) denotes differentiation with respect to t. Equation (2.6), often referred to as the first integral of (2.1), expresses the conservation of energy.

For a given total energy E of the system and with the assumptions that we make on V, the solution $x(t)$ thus lies in the closed domain given by $V(x) \leq E$. It is well known that when the system (2.1) is linear with $V(x) = Q(x)$ where $Q(x)$ is a positive definite quadratic form, it possesses a set of m normal modes, each of which is a periodic motion with a period independent of E. When $V(x)$ is of the form $Q(x) + N(x)$ where $|N(x)| = O(|x|^3)$ for small x, the families of periodic solutions of (2.1) for small x may be considered as perturbations of the normal modes of the linear system. We now consider several special classes of periodic solutions of the system (2.1).

Among periodic solutions $x(t)$ of the system (2.1) with a least period τ, i.e. $x(t+\tau) = x(t)$ we wish to consider:

(1) Odd periodic solutions

$$x(0) = x(\tau/2) = 0, \; x(-t) = -x(t)$$

(2) Even periodic solutions

$$\dot{x}(0) = \dot{x}(\tau/2) = 0, \quad x(-t) = x(t)$$

(3) Normal modes

$$x(0) = \dot{x}(\tau/4) = 0, \quad x(-t) = -x(t), \quad x(t) = x(\tau/2 - t)$$

Any solution $x(t)$ of (2.1) gives rises to a curve, or a "trajectory", in the x-space, with t being a parameter. For a periodic solution, this trajectory is completely described by $x(t)$ for t in one period. As we indicate above, for a given E, the trajectory lies in the closed domain $V(x) \leq E$. It is clear that if $\dot{x} = 0$ at some t, then the point $x(t)$ is on the bounding surface $V(x) = E$. Thus a trajectory intersects the bounding surface $V(x) = E$ if $\dot{x}$ ever vanishes. It can also be established easily that if a trajectory intersects $V(x) = E$, it must do so orthogonally. The odd periodic solutions described above give rise to trajectories that pass through the origin of the x-space but need not intersect $V = E$. The even periodic solutions described above correspond to curves that interesect the bounding surface $V = E$ but need not pass through the origin. The normal modes described above correspond to curves that pass through the origin and intersect $V = E$ and are those studied by Rosenberg [8 - 12]. The normal modes are thus special cases of both odd and even (by a change of time origin) periodic solutions. We remark that while the existences of odd periodic solutions and normal modes depend on our assumption of evenness of $V(x)$ in x. The existence of even periodic solutions does not need this assumption. A well known result in this latter regard is due to Seifert [7].

In the case when $V(x) = Q(x)$, each normal mode is a straight line as the x_i's are proportional to one another. Now since $V(x) = E$ is an ellipsoid, the m principal axes are the normal modes.

In Chapter 3 we shall consider numerical methods for determining the special classes of periodic solutions of (2.1), with their existence being assumed. In the remainder of this chapter we shall discuss the existence of such periodic solutions and their continuous dependence on the total energy E of the system.

2.2. Existence of Periodic Solutions. Their Continuous Dependence on the Total Energy and Bifurcations

As we mentioned in the Introduction, the question of existence of periodic solutions of the system (2.1) has been studied by many authors. We shall consider here the periodic solutions of (2.1) for a given total energy E. We remark that the periods of the periodic solutions are functions of E and are to be determined along with the solutions. For the purposes of theoretical discussions of such periodic solutions and of their numerical computations, it is convenient to make the change of variable $t = \omega s$ in (2.1) where ω is some frequency parameter to be determined. The resulting system is

$$\frac{1}{\omega^2} x'' + \operatorname{grad} V(x) = 0 \tag{2.7}$$

where the prime denotes differentiation with respect to s. We shall then seek 2π-periodic solution $x(s)$ of (2.7) under the condition

$$\frac{1}{2\omega^2} x'^2 + V(x) = E \tag{2.8}$$

which follows from (2.6), along with the solution for ω.
It is obvious that such solutions of (2.7) and (2.8) correspond to $2\pi\omega$-periodic solutions $x(t)$ of (2.1) and (2.6).

Under the assumptions that we made on $V(x)$ in Section 2.1, the problem described by (2.7) and (2.8) is known to have at least one periodic solution for each E. We outline below one such existence proof based on the calculus of variations [26].

Let us consider the following variational problems:

(I) Determine the critical points of $\int_0^\pi [x'(s)]^2 ds$ over the admissible class of functions A_R defined by

$A_R = \{x(s) \mid x(s),\ x'(s) \in L_2[0,\pi]$ m vector-valued function such that

$$x(0) = x(\pi) = 0 \text{ and satisfies } \int_0^\pi V(x(s))\,ds = R\}$$

(II) Determine the critical points of $\int_0^\pi [x'(s)]^2 ds$ over the admissible class of functions A_R defined by

$A_R = \{x(s) \mid x(s), x'(s) \in L_2[0,\pi]$ m vector-valued functions such that

$$\int_0^\pi V(x(s))\,ds = R \text{ and } \int_0^\pi \operatorname{grad} V(x(s))\,ds = 0\}$$

(III) Determine the critical points of $\int_0^{\pi/2} [x'(s)]^2 ds$ over the admissible class of functions A_R defined by

$A_R = \{x(s) \mid x(s),\ x'(s) \in L_2[0,\pi/2]$ m vector-valued function such that

$$x(0) = 0 \quad \text{and} \quad \int_0^{\pi/2} V(x(s))\,ds = R\}$$

R in each problem above is a positive real number.

The Euler-Lagrange equations for each problem above can be shown to reduce to

$$x'' + \beta \text{ grad } V(x) = 0 \qquad (2.9)$$

where the Lagrange multiplier β is a positive number owing to the convexity assumption on $V(x)$ and is identified with ω^2. Also for Problem (II) above we have the natural boundary conditions

$$x'(0) = x'(\pi) = 0 \qquad (2.10)$$

and for Problem (III) above we have the natural boundary condition

$$x'(\pi/2) = 0 \qquad (2.11)$$

Thus the solutions for Problem (I) can be extended as odd 2π-periodic solutions, those for Problem (II) can be extended as even 2π-periodic solutions, and those for Problem (III) can be extended as 2π-periodic solutions that are normal modes.

The existence of solutions for Problems (I) and (II) has been established by Berger [14,15]. A similar existence proof, patterned after [14], can be made for Problem (III). The periodic solutions exist for all positive values of R and depend on R continuously.

The existence of periodic solutions for given E can now be made with the aid of the following lemma.

Lemma. In each of Problem (I), (II) and (III) above, the total energy E is a continuous function of R such that $E \to 0$ as $R \to 0$ and $E \to \infty$ as $R \to \infty$.

<u>Proof</u>. Let $x_R(s)$ and ω_R denote the solution of any of the three problems for a given R. By (2.8) we have

$$E = \frac{1}{2\omega_R^2} x_R'^2 + V(x_R)$$

Since x_R and ω_R depend continuously on R, so does the right hand side above. Thus $E = E(R)$ is continuous in R. Integrating the above from 0 to ℓ, where $\ell = \pi$ for Problems (I) and (II), and $\ell = \pi/2$ for Problem (III), we obtain

$$E(R) = \frac{1}{\ell}\left[\frac{1}{2\omega_R^2}\int_0^\ell [x_R'(s)]^2 ds + \int_0^\ell V(x_R(s))ds\right] \tag{2.12}$$

As $R \to 0$ we have $\int_0^\ell V(x_R(s))ds \to 0$. Since $V(x)$ is positive definite the above implies $x_R(s) = 0$ as $R \to 0$. By (2.7) we have

$$\frac{1}{\omega_R^2}\int_0^\ell [x_R'(s)]^2 ds = \int_0^\ell \operatorname{grad} V(x_R(s))\cdot x_R(s)\,ds \tag{2.13}$$

Thus the left-hand side tends to zero as $R \to 0$. It then follows from (2.12) that $E \to 0$ as $R \to 0$. On the other hand as $R \to \infty$, we have $\int_0^\ell V(x_R(s))ds \to \infty$. As the first term on the right of (2.12) is nonnegative, we thus have $E \to \infty$ as $R \to \infty$. The lemma is now proved.

The above lemma implies that the range of $E(R)$ for $R \geq 0$ is the entire half line $[0,\infty)$. For every $E > 0$ there is a corresponding R and hence there is a periodic solution. The solutions, however, need not depend continuously on E. We shall address the question of continuous dependence of the periodic solutions on E later in this section.

The existence of more than one family of periodic solutions can be established for small E when $V(x)$ is of the form $Q(x) + N(x)$

where $Q(x)$ in a positive definite quadratic form and $N(x)$ contains terms of degree higher than two in x. More specifically, we have the following theorem due to Liapunov [27].

Theorem 2.1. Suppose that the $V(x)$ above is real analytic and

$$\text{grad } V(x) = Ax + O(|x|^2) \tag{2.14}$$

where A is $m \times m$, real symmetric and nonsingular with positive eigenvalues $\lambda_1^2 \leq \lambda_2^2 \leq \ldots \leq \lambda_k^2$ $(1 \leq k \leq m)$. Then the system (2.1) possess k distinct periodic families of solutions near $x = 0$, provided that the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfy the irrationality conditions

$$\lambda_i \lambda_j^{-1} \neq \text{integers} \quad i,j = 1,2,\ldots,k, \quad i \neq j \tag{2.15}$$

The above theorem has been generalized by Berger [16].

Theorem 2.2. With the same hypotheses on $V(x)$ as in the above theorem, the system (2.1) possesses, for small $R > 0$, k families of periodic solutions $x^{(i)}(R)$, $i = 1,2,\ldots,k$, that are continuous in R, with the associated periods tending to $2\pi/\lambda_i$ as $R \to 0$. These families are distinct if $V(x)$ is even or if the numbers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfy the irrationality condition (2.15) of Liapunov.

Remark. The periodic solutions in Theorems 2.1 and 2.2 above can be made odd or even periodic solutions or normal modes if $V(x)$ is assumed to satisfy the evenness assumption.

<u>Remark</u>. The proof of Theorem 2.2 makes use of the variational problem discussed earlier. Using the lemma discussed earlier we can thus establish the existence of more than one branch of periodic solutions for small E.

The existence of at least m branches of periodic solutions for small E for systems with potential $V(x)$ of the form $Q(x) + N(x)$ was also established by Weinstein [18]. The existence of at least one periodic solution for a general given E was also proved by Gordon [17], Rabinowitz [20], in addition to the work of Seifert [7] cited earlier. It is to be noted that, in general, it is not clear whether such periodic solutions must depend on E continuously. Although the periodic solutions established by Berger depend on the parameter R continuously, as we pointed out above, and by the lemma in this section E also depends on R continuously, the inverse function $R = R(E)$ need not be continuous and the question of continuous dependence of the periodic solutions on E is not answered.

For $m = 2$, Yen [22, 23] and Johnson and Rand [24] established the continuous dependence of special classes of periodic solutions on E. More specifically, Yen's results apply to normal modes along which $V(x)$ is monotone and the results of Johnson and Rand apply to what they referred to as "minimal normal modes". In both these works the normal modes are embedded into one parameter families of solutions of initial value problems, characterized by a scalar parameter, α say. A certain mapping $f(\alpha)$ is generated by solutions of the initial value problems. The normal modes correspond to roots of $f(\alpha) = 0$. In particular, as

E changes, the simple roots of $f(\alpha) = 0$ continue to exist and depend continuously on E. Thus the normal modes that correspond to the simple roots of $f(\alpha) = 0$ are continuous functions of E.

Rosenberg [10] noted that a Hamiltonian system with m degrees of freedom may possess more than m distinct normal modes. In both the works of Yen and Johnson and Rand just cited, it was pointed out that such bifurcations occur at multiple root of $f(\alpha) = 0$. Similar results on the continuous dependence of periodic solutions of Hamiltonian systems on E and their bifurcations are to be expected for general $m > 2$, by generalizing the works mentioned above.

We conclude this section by reformulating the questions of continuous dependence of the periodic solutions on E and of bifurcation of the periodic solutions in a functional-analytic setting.

We substitute λ for $1/\omega^2$ and rewrite equations (2.7) and (2.8) as

$$f(x,\lambda) = 0 \tag{2.16}$$

$$T(x,\lambda,E) = 0 \tag{2.17}$$

The Jacobian matrix of (2.16) and (2.17) with respect to (x,λ), which we denote by $J_{(x,\lambda)}$, is equal to

$$J_{(x,\lambda)} = \begin{bmatrix} f_x & f_\lambda \\ T_x & T_\lambda \end{bmatrix} \tag{2.18}$$

We assume that for $E = E_0$, periodic solutions of (2.16) and (2.17) $x_0 = x(E_0)$ and $\lambda_0 = \lambda(E_0)$ exist. The Jacobian matrix in (2.18) at $E = E_0$ depends on x_0 and λ_0 and defines the linear eigenvalue problem

$$J_{(x_0,\lambda_0)} \begin{bmatrix} w \\ \nu \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.19}$$

where $w = w(s)$ lies in the same space as x and ν is a real number. It is well known that if $J_{(x_0,\lambda_0)}$ is nonsingular, i.e. equation (2.19) above has only the trivial solution $w = 0$, $\nu = 0$, then by the implicit function theorem, equations (2.16) and (2.17) have a unique solution $x(E)$ and $\lambda(E)$ for E near E_0, hence the periodic solution (x_0, λ_0) may be continued. On the other hand, if equation (2.19) has nontrivial solutions for (w,ν) bifurcations occur at $E = E_0$. The number of branches of periodic solutions near the bifurcation point E_0 depends on the dimension of the null space of $J_{(x_0,\lambda_0)}$, i.e. the number of linearly independent solutions (w,ν) of (2.19). We shall consider this question in more detail in the next chapter.

Chapter 3 Application of The Finite Element Method

In this chapter we apply the so-called finite element method to obtain approximate solutions for the class of problems defined in Chapter 2. We introduce the finite element method and develop the finite element equations in Section 3.1. We show in Section 3.2, upon assuming that such a problem has an exact solution, that the finite element equations for small mesh size also have solutions converging to the exact one as the mesh size tends to zero. In Section 3.3 we discuss the continuation problem as the total energy E is varied that yields isolated branches of solutions. Finally, in Section 3.4, we present a numerical algorithm which is used to obtain the approximate solutions. We also consider in Section 3.4 the problems of bifurcation and switching of solutions at some E where different branches of solutions meet.

3.1. The Finite Element Equations

The finite element method [28] is closely related to the well known classical Rayleigh-Ritz and Ritz-Galerkin methods. Let us first consider the variational problem of finding the critical points of some functional $I(u)$, say over some infinite dimensional space H. The Rayleigh-Ritz method consists in replacing H by a sequence of finite-dimensional subspaces $\{S_\ell\}$, $\ell = 1,2,\ldots$. Approximate solutions are then obtained by determining the critical points of the functional I over such finite dimensional subspaces.

In practice, some sequence of linearly independent "base functions"

$$\varphi_1, \varphi_2, \ldots$$

that is dense in H is chosen and the subspaces S_ℓ are taken as $S_\ell = \mathrm{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{n_\ell}\}$ for some integer n_ℓ, with $n_\ell \to \infty$ as $\ell \to \infty$. A trial function u_ℓ in S_ℓ is of the form

$$u_\ell = \sum_{i=1}^{n_\ell} q_i \varphi_i \tag{3.1}$$

with $q_1, q_2, \ldots, q_{n_\ell}$ being unknown coefficients. The critical points of $I(u_\ell)$ over S_ℓ are given by the set of q_i's $i = 1, 2, \ldots, n_\ell$, that are solutions of the equations

$$\frac{\partial I}{\partial q_i} = 0 \quad i = 1, 2, \ldots, n_\ell \tag{3.2}$$

Let us now consider the problem of solving the following equation

$$Lu = f \tag{3.3}$$

where L is an operator over an infinite dimensional subspace H and f is a given element in the range of L. If $\langle\ ,\ \rangle$ denotes some inner product, then (3.3) is equivalent to

$$\langle Lu, v\rangle = \langle f, v\rangle \quad \forall\ v \in H \tag{3.4}$$

Moreover, if L is a differential operator and $\langle\ ,\ \rangle$ is an integral, then integration by parts together with the boundary conditions can reduce the smoothness requirements on elements of H. A solution of the resulting system is referred to as a weak solution. The Ritz-Galerkin method consists in substituting the u_ℓ given in (3.1) above into (3.3) and then determining the q_i's $i = 1, 2, \ldots, n_\ell$, such that

$$\langle Lu_\ell, v_\ell\rangle = \langle f, v_\ell\rangle \quad \forall\ v_\ell \in S_\ell \tag{3.4}$$

or equivalently

$$\langle Lu_\ell, \varphi_i \rangle = \langle f, \varphi_i \rangle \quad \text{for all} \quad i = 1,2,\ldots,n_\ell \tag{3.5}$$

which is a system of n_ℓ equations in the q_i's $i = 1,2,\ldots,n_\ell$. Equation (3.5) implies that one must determine the q_i's $i = 1,2,\ldots,n_\ell$ such that the error

$$e_\ell = L\left(\sum_{i=1}^{n_\ell} q_i\varphi_i\right) - f \tag{3.6}$$

is orthogonal to each of the q_i's $i = 1,2,\ldots,n_\ell$.

We note that in general the Ritz-Galerkin method is applicable to a broader class of problems as solutions of variational problems satisfy their Euler-Lagrange equations which we take to be (3.3), while not every equation of the form (3.3) has a variational formulation with (3.3) being the associated Euler-Lagrange equations the two methods are often equivalent.

In the finite element method, the subspaces, which we denote by S^h are construced in a systematic way that simplifies the computations. Also, since h here is a "mesh size" the subspaces S^h become dense in $\bar{H}$ as $h \to 0$. As an illustration let us consider the space B of all scalar-valued functions on the interval $[0,\pi]$ subject to some smoothness requirements. The finite element subspaces are constructed as follows:

(a) The domain $[0,\pi]$ of $x(s)$, $x(s) \in B$, is divided into subintervals by means of points

$$s_i = s_{i-1} + h_i \quad i = 1,2,\ldots,n \quad \text{and} \quad s_0 = 0,\ s_n = \pi \tag{3.7}$$

which are called nodes or nodal points. We define the mesh size h by

$$h = \max_{1\leq i\leq n} h_i \tag{3.8}$$

The open subintervals $s_{i-1} < s < s_i$, $i = 1,2,\ldots,n$, are called finite elements. We suppose the subintervals are such that $h \to 0$ as $n \to \infty$.

(b) A "shape function" in the form of a polynomial of fixed degree $r \geq 1$ is introduced over each finite element $i = 1,2,\ldots,n$. Each shape function is thus characterized by $r + 1$ parameters.

(c) These shape functions are joined at the nodes by certain smoothness requirements and made to satisfy certain required boundary conditions to yield the so-called trial functions. The requirements mentioned above result in the elimination of some of the parameters.

(d) The number of the remaining free parameters characterizes the dimension of the finite element subspace which we call T^h. Let this number be ν say. The construction of T^h above implies the existence of a set of base functions ψ_j^h, $j = 1,2,\ldots,\nu$, with local support, such that

$$T^h = \text{span}\{\psi_1^h,\ldots,\psi_\nu^h\} \tag{3.8}$$

As $x = (x_1(s),\ldots,x_m(s))$ is a vector-valued function, in our problem, we use $T^h \times T^h \times\ldots\times T^h$ m-times as our finite element subspace which is denoted by S^h. Then any trial function $u^h \in S^h$ is of the form

$$u^h = (u_1^h,\ldots,u_m^h)$$

where

$$u_i^h = \sum_{j=1}^{\nu} q_{(i-1)\nu+j}\,\psi_j^h(s) \quad \text{for} \quad i = 1,2,\ldots,m \tag{3.9}$$

The weak formulation of our problem introduced in Chapter 2 is

$$\lambda\langle u'',v\rangle + \langle \text{grad } V(u),v\rangle = 0 \quad \forall\ v \in H$$

where $\langle\ ,\ \rangle$ denotes the $L_2[0,\pi]$ inner product for Problems I and II, and $L_2[0,\frac{\pi}{2}]$ for Problem III. Upon integrating the first term of the above equation by parts and using the boundary conditions we get

$$-\lambda\langle u',v'\rangle + \langle \text{grad } V(u),v\rangle = 0 \quad \forall\ v \in H \tag{3.10}$$

Hence the Ritz-Galerkin equations are

$$-\lambda^h\langle (u_i^h)',(\Psi_j^h)'\rangle + \langle (\text{grad } V(u^h))_i,\Psi_j^h\rangle = 0 \tag{3.11}$$
$$\text{for all } 1 \le j \le \nu \text{ and } 1 \le i \le m.$$

Now since λ^h is also undetermined, we add to our system of equations the averaged energy equation, i.e.,

$$\frac{\lambda^h}{2}\langle (u^h)',(u^h)'\rangle + \langle V(u^h),1\rangle - \pi E = 0 \tag{3.12}$$

We thus have in (3.11) and (3.12) a system of $\nu \times m + 1$ equations in the unknowns $q_{(i-1)\nu+j}$ and λ^h.

Our main task in the next section is to show that the above system possesses a solution (u^h,λ^h) that converges to the exact solution (u,λ) as the mesh size h tends to zero.

3.2. Existence of Approximate Solutions and Their Convergence

In this section we shall show that under suitable conditions, if the problem described in Chapter 2 has an exact solution, $(u(E),\lambda(E))$ say, for a range of E then the discrete problem also has a solution (u^h,λ^h) converging to (u,λ) as the mesh size h tends to zero. We assume that the exact solution u here is an even periodic solution.

In order to establish the above we introduce the space $H = \{x(s) \mid x(s)$ is a vector-valued function such that $x(s), x'(s) \in L_2[0,\pi]\}$. H is a Hilbert space with the inner product

$$\langle x,y\rangle_H = \int_0^{\pi} [x(s)\cdot y(s) + x'(s)\cdot y'(s)]ds \tag{3.13}$$

and the corresponding norm

$$\|x\|^2 = \int_0^{\pi} [|x(s)|^2 + |x'(s)|^2]ds \tag{3.14}$$

If we denote by $\langle\ ,\ \rangle$ and $\|\cdot\|_2$ the $L_2[0,\pi]$ inner product and norm respectively, it immediately follows that

$$\max(\|x\|_2, \|x'\|_2) \leq \|x\| \quad \text{for all} \quad x \in H \tag{3.15}$$

Let $\tilde{H}$ denote the set of all bounded linear functionals on H which is a Banach space under the norm

$$\|f\| = \sup_{\substack{x\in H \\ \|x\|=1}} |f(x)| \tag{3.16}$$

where $|\cdot|$ denotes absolute value. For simplicity we shall not make distinction between norms on H and $\tilde{H}$. It can be easily checked that the product spaces $H \times \mathbb{R}$ and $\tilde{H} \times \mathbb{R}$ under the norms

$$\|(u,\lambda)\| = \|u\| + |\lambda| \quad \forall (u,\lambda) \in H \times \mathbb{R} \tag{3.17}$$

$$\|(f,\lambda)\| = \|f\| + |\lambda| \quad \forall (f,\lambda) \in \tilde{H} \times \mathbb{R} \tag{3.18}$$

are Banach spaces.

Let us consider now the finite element subspace S^h and define P to be the projection of H onto S^h, i.e.

$$\langle u,v^h\rangle_H = \langle Pu,v^h\rangle_H \quad \forall\ v^h \in S^h \tag{3.19}$$

let P_1 be a mapping from $H \times \mathbb{R}$ onto $S^h \times \mathbb{R}$ defined by

$$P_1(u,\lambda) = (Pu,\lambda) \quad \forall (u,\lambda) \in H \times \mathbb{R} \tag{3.20}$$

then P_1 is well defined since P is.

Let us also define ϕ to be the mapping from $H \times \mathbb{R}$ into $\tilde{H} \times \mathbb{R}$ by

$$\phi(u,\lambda) = (f(u,\lambda), T(u,\lambda)) \quad \forall (u,\lambda) \in H \times \mathbb{R} \tag{3.21}$$

where $f(u,\lambda)$ is defined implicitly by

$$f(u,\lambda)\cdot v = -\lambda \int_0^\pi u'\cdot v' ds + \int_0^\pi \text{grad } V(u)\cdot v ds \quad \forall v \in H \tag{3.22}$$

and $T(u,\lambda)$ is given by

$$T(u,\lambda) = \frac{\lambda}{2}\int_0^\pi |u'|^2 ds + \int_0^\pi V(u) ds - \pi E \tag{3.23}$$

we shall denote by $f^h(u,\lambda)$ the restriction of $f(u,\lambda)$ to S^h, that is

$$f^h(u,\lambda)\cdot v^h = -\lambda \int_0^\pi u'\cdot (v^h)' ds + \int_0^\pi \text{grad } V(u)\cdot v^h ds \; \forall v^h \in S^h \tag{3.24}$$

Finally, we let g be the mapping that restricts elements of $\tilde{H} \times \mathbb{R}$ to those of $\tilde{S}^h \times \mathbb{R}$ which is defined by

$$g(f,\lambda) = (f^h,\lambda) \quad \forall (f,\lambda) \in H \times \mathbb{R} \tag{3.25}$$

In what follows we shall assume that $V(x)$ is such that, for all $x,y \in H$ we have

$$|V(x) - V(y)| = O(\|x-y\|) \tag{3.26}$$

$$\left|\frac{\partial V(x)}{\partial x_i} - \frac{\partial V(y)}{\partial y_i}\right| = O(\|x-y\|) \quad 1 \le i \le m \tag{3.27}$$

and

$$\left|\frac{\partial^2 V(x)}{\partial x_i \partial x_j} - \frac{\partial^2 V(y)}{\partial y_i \partial y_j}\right| = O(\|x-y\|) \quad 1 \le i, j \le m \tag{3.28}$$

We now show that

$$\lim_{h\to 0} \|\phi^h(u,\lambda) - \phi^h(Pu,\lambda)\| = 0 \tag{3.29}$$

where it is understood that the norm is taken over $S^h \times \mathbb{R}$, not $\tilde{H} \times \mathbb{R}$. This proves the consistency of the diagram in Figure 3.1.

Theorem 3.1. For all $(u,\lambda) \in H \times \mathbb{R}$ we have

$$\|\phi(u,\lambda) - \phi(Pu,\lambda)\| \leq M(\lambda,u)\|u - Pu\| \tag{3.30}$$

where M is a constant that depends on λ and u.

Proof. Since $\|\phi(u,\lambda) - \phi(Pu,\lambda)\| = \|f(u,\lambda) - f(Pu,\lambda)\| + |T(u,\lambda) - T(Pu,\lambda)|$ it suffices to estimate each term in the right hand side of the above equation

$$\begin{aligned}
\|f(u,\lambda) - f(Pu,\lambda)\| &= \sup_{\substack{v\in H\\ \|v\|=1}} |(f(u,\lambda) - f(Pu,\lambda))\cdot v| \\
&= \sup_{\substack{v\in H\\ \|v\|=1}} \Big|-\lambda\int_0^\pi (u' - (Pu)')\cdot v' ds + \int_0^\pi (\text{grad } V(u) \\
&\qquad\qquad - \text{grad } V(Pu))\cdot v\, ds\Big| \\
&\leq \sup_{\substack{v\in H\\ \|v\|=1}} \{|\lambda|\,\|u' - (Pu)'\|_2\|v'\|_2 + \int_0^\pi O(\|u - Pu\|) \\
&\qquad\qquad |v|\,ds\} \\
&\leq \sup_{\substack{v\in H\\ \|v\|=1}} \{|\lambda|\,\|u' - (Pu)'\|_2\|v'\|_2 + O(\|u - Pu\|)\,\|v\|_2\} \\
&\leq \sup_{\substack{v\in H\\ \|v\|=1}} \{|\lambda|\,\|u - Pu\| + O(\|u - Pu\|)\}\,\|v\| \\
&\leq (|\lambda| + C(u,\lambda))\,\|u - Pu\|
\end{aligned}$$

$$
\begin{tikzcd}
\begin{array}{c} H \times \mathbb{R} \\ (u, \lambda) \end{array} \arrow[r, "\phi"] \arrow[d, "P_1"'] \arrow[dr, dashed, "g \circ \phi"] & \begin{array}{c} \tilde{H} \times \mathbb{R} \\ (f(u, \lambda), T(u, \lambda)) \end{array} \arrow[d, "g"] \\
\begin{array}{c} S^h \times \mathbb{R} \\ (Pu, \lambda) \end{array} \arrow[dr, "\phi^h = g \circ \phi"] & \begin{array}{c} \widetilde{S^h} \times \mathbb{R} \\ (f^h(u, \lambda), T(u, \lambda)) \end{array} \\
 & (f^h(Pu, \lambda), T(Pu, \lambda))
\end{tikzcd}
$$

Figure 3.1. Consistency diagram

where we have used the Schwartz inequality, (3.15) and (3.27). Thus we may write

$$\|f(u,\lambda) - f(Pu,\lambda)\| \leq C_1(u,\lambda)\,\|u - Pu\| \tag{3.31}$$

where C_1 is a constant that depends on u and λ. Similarly,

$$\begin{aligned}
|T(u,\lambda) - T(Pu,\lambda)| &= \left|\frac{\lambda}{2}\int_0^{\pi} \{(u')^2 - ((Pu)')^2\}ds + \int_0^{\pi} (V(u) - V(Pu))\,ds\right| \\
&\leq \frac{|\lambda|}{2}(\|u'\|_2^2 - \|(Pu)'\|_2^2) + O(\|u - Pu\|)\pi \\
&\leq \frac{|\lambda|}{2}(\|u'\|_2 + \|(Pu)'\|_2)(\|u'\|_2 - \|(Pu)'\|_2) \\
&\qquad + O(\|u - Pu\|) \\
&\leq \frac{|\lambda|}{2}(\|u'\|_2 + \|(Pu)'\|_2)\,\|u' - (Pu)'\|_2 + O(\|u - Pu\|) \\
&\leq [\frac{|\lambda|}{2}(\|u\| + \|Pu\|) + C_2(u,\lambda)]\,\|u - Pu\|
\end{aligned}$$

Thus we may write

$$|T(u,\lambda) - T(Pu,\lambda)| \leq C_3(u,\lambda) \quad \|u - Pu\| \tag{3.32}$$

where C_3 is a constant that depend on u and λ. Upon adding (3.31) and (3.32), we get

$$\|\phi(u,\lambda) - \phi(Pu,\lambda)\| \leq M(\lambda,u) \quad \|u - Pu\|.$$

If we assume that our finite element subspace S^h is consistent of order p with H, i.e.

$$\|u - Pu\| = O(h^p) \tag{3.33}$$

(For example if S^h is piecewise linear then $p = 2$ [28]). Using from (3.30) and (3.33), we now conclude that

$$\|\phi(u,\lambda) - \phi(Pu,\lambda)\| \leq M_1(u,\lambda)\,h^p \tag{3.34}$$

Next we establish the following corollary.

Corollary.

$$\|\phi^h(u,\lambda) - \phi^h(Pu,\lambda)\| \leq M_1(u,\lambda)h^p \tag{3.35}$$

Proof. This follows immediately from the fact that $g\circ\phi = \phi^h$ and

$$\|f^h(u,\lambda) - f^h(Pu,\lambda)\| = \sup_{\substack{v^h\in S^h \\ \|v^h\|=1}} |[f^h(u,\lambda) - f^h(Pu,\lambda)]\cdot v^h]$$

$$\leq \sup_{\substack{v\in H \\ \|v\|=1}} |(f(u,\lambda) - f(Pu,\lambda))\cdot v|$$

Definition. We say that $\{\phi^h(Pu,\lambda)\}$ is consistent of order p with $\phi^h(u,\lambda)$ if (3.35) holds.

Let us now consider the Jacobian matrix of $\phi(u,\lambda)$ with respect to (u,λ), which we refer to as $J_{(u,\lambda)}$. It is a bounded linear mapping from $H \times \mathbb{R}$ into itself defined as

$$J_{(u,\lambda)}\begin{bmatrix} w \\ \nu \end{bmatrix} = \begin{bmatrix} f_u & f_\lambda \\ T_u & T_\lambda \end{bmatrix}\begin{bmatrix} w \\ \nu \end{bmatrix} = \begin{bmatrix} f_u w + \nu f_\lambda \\ T_u w + T_\lambda \nu \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda\langle w', v'\rangle + \langle K(u)\cdot w, v\rangle - \nu\langle u', v'\rangle \\ \lambda\langle u', w'\rangle + \langle \text{grad } V(u)\cdot w, 1\rangle + \frac{\nu}{2}\|u'\|_2^2 \end{bmatrix} \quad \forall v \in H \tag{3.36}$$

where $(w,\nu) \in H \times \mathbb{R}$ and $K(u)$ is Hessian matrix of $V(u)$. We note that the Jacobian defined here differs, slightly in notations, from that introduced in (2.8).

Lemma 3.1.

$$\|J_{(u,\lambda)} - J_{(Pu,\lambda)}\| \leq N(u,\lambda)h^p \tag{3.37}$$

Proof. Let us consider for $(w,\nu) \in H \times \mathbb{R}$

$$(J_{(u,\lambda)} - J_{(Pu,\lambda)})\begin{bmatrix} w \\ \nu \end{bmatrix}$$

$$= \begin{bmatrix} <(K(u) - K(Pu))\cdot w, v> - \nu<u' - (Pu)', v'> \\ \lambda<u' - (Pu)', w'> + <(\text{grad } V(u) - \text{grad } V(Pu))\cdot w, 1> + \frac{\nu}{2}(\|u'\|_2^2 \\ - \|(Pu)'\|_2^2) \end{bmatrix}_{\forall v \in H}$$

It follows that

$$\|J_{(u,\lambda)} - J_{(Pu,\lambda)}\| = \sup_{\substack{(w,\nu)\in H\times\mathbb{R} \\ \|w\|+|\nu|=1}} \|(J_{(u,\lambda)} - J_{(Pu,\lambda)})\begin{bmatrix} w \\ \nu \end{bmatrix}\|$$

$$= \sup_{\substack{(w,\nu)\in H\times\mathbb{R} \\ \|w\|+|\nu|=1}} \Big\{ \sup_{\substack{v\in H \\ \|v\|=1}} |<(K(u) - K(Pu))\cdot w, v> - \nu<u' - (Pu)', v'>| + |\lambda<u' - (Pu)', w'> + <(\text{grad } V(u) - \text{grad } V(Pu))\cdot w, 1> + \frac{\nu}{2}(\|u'\|_2^2 - \|(Pu)'\|_2^2)| \Big\}$$

$$\leq \sup_{\substack{(w,\nu)\in H\times\mathbb{R} \\ \|w\|+|\nu|=1}} \Big\{ \sup_{\substack{v\in H \\ \|v\|=1}} (O(\|u - Pu\|)\|w\|_2\|v\|_2 + |\nu|\,\|u' - (Pu)'\|_2\|v'\|_2) + |\lambda|\,\|u' - (Pu)'\|_2\|w'\|_2 + O(\|u - Pu\|)\|w\| + \frac{|\nu|}{2}\|u' - (Pu)'\|_2 (\|u'\|_2 + \|(Pu)'\|_2) \Big\}$$

$$\leq \sup_{\substack{(w,\nu)\in H\times\mathbb{R} \\ \|w\|+|\nu|=1}} \{O(\|u - Pu\|)\|w\| + |\nu|\,\|u - Pu\| + |\lambda|\,\|u - Pu\|\,\|w\| + O(\|u - Pu\|)\|w\| + \frac{|\nu|}{2}(\|u\| + \|Pu\|)\|u - Pu\|)$$

$$\leq O(\|u - Pu\|) + (1 + |\lambda| + \|u\|)\,\|u - Pu\|$$

where we have used the Schwartz inequality, (3.15), (3.28) and the inequality $\|Pu\| \leq \|u\|$. We have thus shown

$$\|J_{(u,\lambda)} - J_{(Pu,\lambda)}\| \leq N_1(u,\lambda)\,\|u - Pu\| \tag{3.38}$$

Upon using (3.33), we now have

$$\|J_{(u,\lambda)} - J_{(Pu,\lambda)}\| \leq N(u,\lambda)\,h^p$$

Let us recall the Banach lemma [29]: Suppose that A and E are square matrices such that A^{-1} exists and $\|A^{-1}\|\|E\| < 1$, then $(A+E)^{-1}$ exists and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}$$

<u>Lemma 3.2</u>. If $J_{(u,\lambda)}$ is invertible, i.e., if $J^{-1}_{(u,\lambda)}$ exists then there exists an $h_1 > 0$ small enough so that $J^{-1}_{(Pu,\lambda)}$ exists for all $h \in (0, h_1]$ and is uniformly bounded.

<u>Proof</u>. Let us choose h_1 small enough such that

$$\|J^{-1}_{(u,\lambda)}\|N(u,\lambda)\,h_1^p < 1$$

then from (3.37)

$$\|J^{-1}_{(u,\lambda)}\|\|J_{(u,\lambda)} - J_{(Pu,\lambda)}\| < 1 \quad \text{for all} \quad h \in (0, h_1]$$

The Banach lemma with $A = J_{(u,\lambda)}$ and $E = J_{(Pu,\lambda)} - J_{(u,\lambda)}$ implies that $J^{-1}_{(Pu,\lambda)}$ exists and

$$\|J^{-1}_{(Pu,\lambda)}\| \leq \frac{\|J^{-1}_{(u,\lambda)}\|}{1 - \|J^{-1}_{(u,\lambda)}\|\|J_{(Pu,\lambda)} - J_{(u,\lambda)}\|} = \alpha(u,\lambda,h) \quad \text{for} \quad h \in (0, h_1] \tag{3.39}$$

Moreover since from (3.37) we have

$$\alpha(u,\lambda,h) \leq \alpha(u,\lambda,h_1) \quad \text{for all} \quad h \in (0,h_1] \tag{3.40}$$

it follows that

$$\|J^{-1}_{(Pu,\lambda)}\| \leq \alpha(u,\lambda,h_1) \quad \text{for all} \quad h \in (0,h_1] \tag{3.41}$$

which shows that $J_{(Pu,\lambda)}$ is uniformly bounded.

Definition. Let ρ be a small positive real number and let us define the following two spheres

$$S_\rho(u,\lambda) = \{(w,\nu) \in H \times \mathbb{R} \mid \|u-w\| + |\lambda-\nu| \leq \rho\}$$

$$S^h_\rho(Pu,\lambda) = \{(w^h,\nu^h) \in S^h \times \mathbb{R} \mid \|w^h - Pu\| + |\lambda-\nu^h| \leq \rho\}$$

Let us now consider the Jacobian matrix of $\phi^h(u^h,\lambda^h)$, which we will denote as $J^h_{(u^h,\lambda^h)}$, which is equal to $J_{(u^h,\lambda^h)}$ restricted to $S^h \times \mathbb{R}$, i.e. For all $(w^h,\nu^h) \in S^h \times \mathbb{R}$ we have

$$J^h_{(u^h,\lambda^h)} \begin{bmatrix} w^h \\ \nu^h \end{bmatrix} = J_{(u^h,\lambda^h)} \begin{bmatrix} w^h \\ \nu^h \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda^h\langle (w^h)', (v^h)'\rangle + \langle K(u^h)\cdot w^h, v^h\rangle - \nu^h\langle (u^h)', (v^h)'\rangle \\ \lambda^h\langle (u^h)', (w^h)'\rangle + \langle \text{grad } V(u^h)\cdot w^h, 1\rangle + \frac{\nu^h}{2}\|(u^h)'\|_2^2 \end{bmatrix} \forall v^h \in S^h \tag{3.42}$$

We now prove the following lemma.

Lemma 3.3. $J^h_{(x^h,\lambda^h)}$ is uniformly Lipschitz continuous on $S^h_\rho(Pu,\lambda)$, that is, for all (x^h,λ^h), $(y^h,\mu^h) \in S^h_\rho(Pu,\lambda)$ we have

$$\|J^h_{(x^h,\lambda^h)} - J^h_{(y^h,\mu^h)}\| \leq L(\rho,u,\lambda)\|(x^h,\lambda^h) - (y^h,\mu^h)\| \text{ for all } h \in (0,h_1] \tag{3.43}$$

where L is a constant depending on u,λ and ρ.

<u>Proof</u>. Let us consider

$$(J^h_{(x^h,\lambda^h)} - J^h_{(y^h,\mu^h)}) \begin{bmatrix} w^h \\ \nu^h \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda^h-\mu^h)<(w^h)',(v^h)'> + <(K(x^h)-K(y^h)).w^h,v^h> - \nu^h<(x^h)' \\ - (y^h)',(v^h)'> \\ \lambda^h<(x^h)',(w^h)'> - \mu^h<(y^h)',(w^h)'> + <(\text{grad } V(x^h) - \text{grad } V(y^h))\cdot w^h,1> \\ + \frac{\nu^h}{2}(\|(x^h)'\|_2^2 - \|(y^h)'\|_2^2) \end{bmatrix} \quad \forall v^h \in S^h$$

$$= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.44}$$

It can be easily shown, using a similar argument as in Lemma 3.1., that

$$\|b_1\| \leq (|\lambda^h-\mu^h| + (L_1(x^h,y^h)+1)\|x^h-y^h\|)(\|w^h\|+|\nu^h|)$$

or

$$\|b_1\| \leq L_2(x^h,y^h)(\|x^h-y^h\| + |\lambda^h-\mu^h|)(\|w^h\|+|\nu^h|) \tag{3.45}$$

and

$$|b_2| \leq \{|\lambda^h-\mu^h|\,\|x^h\| + |\mu^h|\,\|x^h-y^h\| + (L_3(x^h,y^h) + \frac{\|x^h\|+\|y^h\|}{2})$$
$$\|x^h-y^h\|\}(\|w^h\|+|\nu^h|)$$

or

$$|b_2| \leq L_4(x^h,y^h)(\|x^h-y^h\| + |\lambda^h-\mu^h|)(\|w^h\|+|\nu^h|) \tag{3.46}$$

From (3.44) through (3.46) we can conclude that

$$\|J^h_{(x^h,\lambda^h)} - J^h_{(y^h,\mu^h)}\| \leq L(\rho,u,\lambda)\,\|(x^h,\lambda^h)-(y^h,\mu^h)\|$$

where L depends on ρ, u and λ since

$$\max(\|y^h\|,\|x^h\|) \leq \|Pu\| + \rho \leq \|u\| + \rho \tag{3.47}$$

and

$$\max(|\lambda^h|, |\mu^h|) \leq |\lambda| + \rho \tag{3.48}$$

Remark. As a consequence of Lemma 3.3 we have, if (y^h, μ^h) is replaced by (Pu, λ),

$$\|J^h_{(x^h, \lambda^h)} - J_{(Pu, \lambda)}\| \leq L(\rho, u, \lambda)\rho \tag{3.49}$$

Lemma 3.4. There exist $\rho_0 > 0$ and $h_0 < h_1$ sufficiently small such that $\phi^h(u^h, \lambda^h)$ on $S^h_{\rho_0}(Pu, \lambda)$ satisfies

$$\|(u^h, \lambda^h) - (w^h, \nu^h)\| \leq B\|\phi^h(u^h, \lambda^h) - \phi^h(w^h, \nu^h)\| \quad \text{for all } h < h_0 \tag{3.50}$$

where (u^h, λ^h) and (w^h, ν^h) are in $S^h_{\rho_0}(Pu, \lambda)$ and B is a constant that depends on h_0, ρ_0, λ, u.

Proof. Consider $\phi^h(u^h, \lambda^h) - \phi^h(w^h, \nu^h)$. Upon applying the mean value theorem [29] we get

$$\phi^h(u^h, \lambda^h)^T - \phi^h(w^h, \nu^h)^T = \widetilde{J^h}\begin{bmatrix} u^h - w^h \\ \lambda^h - \nu^h \end{bmatrix} \tag{3.51}$$

where

$$\widetilde{J^h} = \int_0^1 J^h_{(tu^h + (1-t)w^h, t\lambda^h + (1-t)\nu^h)}\, dt \tag{3.52}$$

and T denotes matrix transposition. Let us consider the difference

$$\widetilde{J^h} - J_{(Pu, \lambda)} = \int_0^1 [J^h_{(tu^h + (1-t)w^h, t\lambda^h + (1-t)\nu^h)} - J_{(Pu, \lambda)}]dt$$

From (3.43) we have

$$\begin{aligned} \|\widetilde{J^h} - J_{(Pu, \lambda)}\| &\leq \int_0^1 \|J^h_{(tu^h + (1-t)w^h, t\lambda^h + (1-t)\nu^h)} - J_{(Pu, \lambda)}\|dt \\ &\leq L(\rho, u, \lambda)\int_0^1 \{t\|(u^h, \lambda^h) - (Pu, \lambda)\| + (1-t) \\ &\qquad \|(w^h, \nu^h) - (Pu, \lambda)\|\}dt \end{aligned}$$

$$\leq L(\rho,u,\lambda)\tfrac{1}{2}\{\|(u^h,\lambda^h) - (Pu,\lambda)\| + \|(w^h,\nu^h) - (Pu,\lambda)\|\}$$

Hence,

$$\|\widetilde{J^h} - J_{(Pu,\lambda)}\| \leq L(\rho,u,\lambda)\,\rho \tag{3.53}$$

Now from Lemma 3.2 we know that

$$\|J^{-1}_{(Pu,\lambda)}\| \leq \alpha(h,\lambda,u) \quad \text{for all} \quad h \in (0,h_1]$$

Thus if we choose $h_0 < h_1$ and $\rho_0 < \rho$ such that

$$L(\rho_0,u,\lambda)\,\alpha(h_0,u,\lambda)\,\rho_0 < 1 \tag{3.54}$$

the Banach lemma applies with $A = J_{(Pu,\lambda)}$ and $E = \widetilde{J^h} - J_{(Pu,\lambda)}$ and implies that $(\widetilde{J^h})^{-1}$ exists and satisfies

$$\|(\widetilde{J^h})^{-1}\| \leq \frac{\|J^{-1}_{(Pu,\lambda)}\|}{1 - \|J^{-1}_{(Pu,\lambda)}\|\,\|\widetilde{J^h} - J_{(Pu,\lambda)}\|}$$

$$\leq \frac{\alpha(u,\lambda,h_0)}{1 - \alpha(u,\lambda,h_0)\,L(\rho_0,u,\lambda)\,\rho_0} \quad \text{all} \quad h \in (0,h_0] \tag{3.55}$$

Hence from (3.51) we get

$$(u^h,\lambda^h) - (w^h,\nu^h) = (\widetilde{J^h})^{-1}(\phi^h(u^h,\lambda^h)^T - \phi^h(w^h,\nu^h)^T)$$

which implies, upon using (3.55), that

$$\|(u^h,\lambda^h) - (w^h,\nu^h)\| \leq \frac{\alpha(h_0,u,\lambda)}{1 - \alpha(h_0,u,\lambda)\;L(\rho_0,u,\lambda)\,\rho_0}$$

$$\|\phi^h(u^h,\lambda^h) - \phi(w^h,\nu^h)\| \tag{3.56}$$

Setting $B = \dfrac{\alpha(h_0,u,\lambda)}{1 - \alpha(h_0,\ u,\lambda)\,L(\rho_0,u,\lambda)\,\rho_0}$ we get (3.50).

We now make use of our assumption that the system $\phi(w,\nu) = 0$ has a solution $(u(E),\lambda(E))$ for some range of E and consider some fixed E.

<u>Definition</u>. We say that (u,λ) is an isolated solution if $J_{(u,\lambda)}$ is invertible.

We are ready to adjoin all the above lemmas to the fact that (u,λ) is an isolated solution of $\phi(w,\nu) = 0$ to show that the finite element approximate system i.e.

$$\phi^h(w^h,\nu^h) = 0$$

has a solution (u^h,λ^h) close to (u,λ).

<u>Theorem 3.2</u>. Let (u,λ) be an isolated solution of $\phi(u,\lambda) = 0$ for a given E and let $\{\phi^h(Pw,\nu)\}$ be consistent of order p with $\phi^h(w,\nu)$, that is (3.34) holds. Then for $\rho_0 < \rho$ and $h_0 < h_1$ sufficiently small, $\phi^h(x^h,\nu^h) = 0$ has a unique solution (u^h,λ^h) in $S^h_{\rho_0}(Pu,\lambda)$ for all $h \le h_0$. Moreover (u^h,λ^h) satisfies

$$\|(Pu,\lambda) - (u^h,\lambda^h)\| \le A(\rho_0,h,u,\lambda)h^p \tag{3.57}$$

<u>Proof</u>. Let us define $\psi^h(x^h,\nu^h)$ by

$$\psi^h(x^h,\nu^h) = (x^h,\nu^h)^T - J^{-1}_{(Pu,\lambda)}\phi^h(x^h,\nu^h)^T \tag{3.58}$$

then $\psi^h(x^h,\nu^h)$ maps $S^h_{\rho_0}(Pu,\lambda)$ into itself. To show this let $(x^h,\nu^h) \in S^h_{\rho_0}(Pu,\lambda)$ and consider

$$\begin{aligned}\psi^h(x^h,\nu^h) - (Pu,\lambda)^T &= (x^h,\nu^h)^T - (Pu,\lambda)^T - J^{-1}_{(Pu,\lambda)}\phi^h(x^h,\nu^h)^T \\ &= J^{-1}_{(Pu,\lambda)}[J_{(Pu,\lambda)}(x^h - Pu,\nu^h - \lambda)^T - \phi^h(x^h,\nu^h)^T] \\ &= J^{-1}_{(Pu,\lambda)}[J_{(Pu,\lambda)}(x^h - Pu,\nu^h - \lambda)^T + \phi^h(Pu,\lambda)^T \\ &\quad - \phi^h(x^h,\nu^h)^T + \phi^h(u,\lambda)^T - \phi^h(Pu,\lambda)^T]\end{aligned}$$

The mean value theorem implies that

$$\phi^h(Pu,\lambda)^T - \phi^h(x^h,\nu^h)^T = \widetilde{J}^h(x^h - Pu,\nu^h - \lambda)^T$$

where

$$\widetilde{J}^h = \int_0^1 J^h_{(t(x^h,\nu^h) + (1-t)(Pu,\lambda))}\,dt$$

Upon using (3.35) we get

$$\|\psi^h(x^h,\nu^h) - (Pu,\lambda)^T\| \le \|J^{-1}_{(Pu,\lambda)}\|\ \|J_{(Pu,\lambda)} - \widetilde{J}^h\|$$

$$[\|(x^h - Pu,\nu^h - \lambda)\| + M_1(u,\lambda)h^p]$$

Also since $\|J^{-1}_{(Pu,\lambda)}\| \le \alpha(h_0,\lambda,u)$ for all $h \le h_0$ and $\|J_{(Pu,\lambda)} - \widetilde{J}^h\| \le L(\rho_0,u,\lambda)\rho_0$ we get

$$\|\psi^h(x^h,\nu^h) - (Pu,\lambda)^T\| \le \alpha(h_0,\lambda,u)L(\rho_0,u,\lambda)\rho_0 + M_1(u,\lambda)\alpha(h_0,\lambda,u)h^p.$$

Upon choosing ρ_0 and h_0 sufficiently small to guarantee that

$$\alpha(h_0,\lambda,u)L(\rho_0,u,\lambda) < \frac{1}{2},\ \alpha(h_0,\lambda,u)M_1(u,\lambda)h_0^p < \frac{\rho_0}{2} \tag{3.59}$$

we get

$$\|\psi^h(x^h,\nu^h) - (Pu,\lambda)^T\| < \rho_0 \tag{3.60}$$

Hence

$$\psi^h(x^h,\nu^h) \in S^h_{\rho_0}(Pu,\lambda)$$

Next, we show that $\psi^h(x^h,\nu^h)$ is a contraction map on $S^h_{\rho_0}(Pu,\lambda)$, that is, if (x^h,ν^h) and (y^h,μ^h) are in $S^h_{\rho_0}(Pu,\lambda)$ we have

$$\|\psi^h(x^h,\nu^h) - \psi^h(y^h,\mu^h)\| \le K\|(x^h,\nu^h) - (y^h,\mu^h)\| \tag{3.61}$$

where K is a constant less than unity. To show this let us consider

$$\psi^h(x^h,\nu^h) - \psi^h(y^h,\mu^h) = (x^h,\nu^h)^T - (y^h,\mu^h)^T$$
$$- J^{-1}_{(Pu,\lambda)}\{\phi^h(x^h,\nu^h)^T - \phi^h(y^h,\mu^h)^T\}$$

$$= J^{-1}_{(Pu,\lambda)}\{J_{(Pu,\lambda)}[(x^h,\nu^h)^T - (y^h,\mu^h)^T] - \widetilde{J^h}[(x^h,\nu^h)^T - (y^h,\mu^h)^T]$$

$$= J^{-1}_{(Pu,\lambda)}(J_{(Pu,\lambda)} - \widetilde{J^h})[(x^h,\nu^h)^T - (y^h,\mu^h)^T]$$

Hence

$$\|\Psi^h(x^h,\nu^h) - \Psi^h(y^h,\mu^h)\| \leq \|J^{-1}_{(Pu,\lambda)}\|\|J_{(Pu,\lambda)} - \widetilde{J^h}\|\|(x^h,\nu^h)^T - (y^h,\mu^h)^T\|$$

$$\leq \alpha(h_O,\lambda,u)L(\rho_O,\lambda,u)\|(x^h,\nu^h)^T - (y^h,\mu^h)^T\| \tag{3.62}$$

If we set $K = \alpha(h_O,\lambda,u)L(\rho_O,\lambda,u)$, then (3.59) implies that $K < 1$. Thus $\Psi^h(x^h,\nu^h)$ is a contraction map on $S^h_{\rho_O}(Pu,\lambda)$.

Now since $S^h_{\rho_O}(Pu,\lambda)$ is convex, Brower's fixed point theorem [25] implies that $\Psi^h(x^h,\nu^h)$ has a unique fixed point (u^h,λ^h), say, in $S^h_{\rho_O}(Pu,\lambda)$ for all $h \in (0,h_O]$. We thus have

$$\Psi^h(u^h,\lambda^h) = (u^h,\lambda^h)^T \tag{3.63}$$

which implies that

$$\phi^h(u^h,\lambda^h) = 0 \tag{3.64}$$

Moreover if we consider

$$(u^h,\lambda^h)^T - (Pu,\lambda)^T = \Psi^h(u^h,\lambda^h) - (Pu,\lambda)^T$$

$$= \Psi^h(u^h,\lambda^h) - \Psi^h(Pu,\lambda) + \Psi^h(Pu,\lambda) - (Pu,\lambda)^T$$

$$\|(u^h,\lambda^h) - (Pu,\lambda)\| \leq \|\Psi^h(u^h,\lambda^h) - \Psi^h(Pu,\lambda)\| + \|\Psi^h(Pu,\lambda) - (Pu,\lambda)^T\|$$

$$\leq \|J^{-1}_{(Pu,\lambda)}\|\|J_{(Pu,\lambda)} - \widetilde{J^h}\|\|(u^h,\lambda^h) - (Pu,\lambda)\|$$

$$+ \|\Psi^h(Pu,\lambda) - (Pu,\lambda)^T\|$$

where $\widetilde{J^h} = \int_0^1 J^h_{(t(Pu,\lambda) + (1-t)(u^h,\lambda^h))}\,dt$

Hence

$$\|(u^h,\lambda^h) - (Pu,\lambda)\| \leq \alpha(h_O,\lambda,u)L(\rho_O,u,\lambda)\|(u^h,\lambda^h) - (Pu,\lambda)\|$$

$$+ \|\Psi^h(Pu,\lambda) - (Pu,\lambda)^T\|$$

$$(1-\alpha(h_0,\lambda,u)L(\rho_0,u,\lambda))\|(u^h,\lambda^h)-(Pu,\lambda)\| \le \|\Psi^h(Pu,\lambda)-(Pu,\lambda)^T\| \tag{3.65}$$

Also since

$$\begin{aligned}\Psi^h(Pu,\lambda)-(Pu,\lambda)^T &= J^{-1}_{(Pu,\lambda)}\phi^h(Pu,\lambda)^T \\ &= J^{-1}_{(Pu,\lambda)}[\phi^h(Pu,\lambda)^T-\phi^h(u,\lambda)^T]\end{aligned}$$

we obtain upon using (3.35) that

$$\|\Psi^h(Pu,\lambda)-(Pu,\lambda)^T\| \le \|J^{-1}_{(Pu,\lambda)}\|\ M_1(u,\lambda)h^p \tag{3.66}$$

Combining (3.65) and (3.66), we get

$$\|(u^h,\lambda^h)-(Pu,\lambda)\| \le \frac{M_1(u,\lambda)\alpha(h_0,\lambda,u)}{1-\alpha(h_0,\lambda,u)L(\rho_0,u,\lambda)}h^p \quad \forall h \le h_0 \tag{3.67}$$

Taking $A(\rho_0,h_0,u,\lambda) = \dfrac{M_1(u,\lambda)\alpha(h_0,\lambda,u)}{1-\alpha(h_0,\lambda,u)L(\rho_0,u,\lambda)}$ (3.68)

then completes the proof of the theorem.

As a result of the proof above we also have the following.

<u>Corollary</u>. The discrete solution (u^h,λ^h) converges to the exact solution (u,λ) as the mesh size tends to zero.

<u>Proof</u>. We write

$$(u^h,\lambda^h)-(u,\lambda) = (u^h,\lambda^h)-(Pu,\lambda)+(Pu,\lambda)-(u,\lambda)$$

Thus

$$\begin{aligned}\|(u^h,\lambda^h)-(u,\lambda)\| &\le \|(u^h,\lambda^h)-(Pu,\lambda)\| + \|(Pu,\lambda)-(u,\lambda)\| \\ &\le \|(u^h,\lambda^h)-(Pu,\lambda)\| + \|Pu-u\|\end{aligned} \tag{3.69}$$

Combining (3.69) with (3.67), and from (3.33) we obtain

$$\|(u^h,\lambda^h)-(u,\lambda)\| \le D(u,\lambda,h_0,\rho_0)h^p \quad \forall h \in (0,h_0] \tag{3.70}$$

and the corollary is proved.

We conclude this section by remarking that the solutions considered here are even periodic solutions. These solutions become normal modes if they also satisfy the condition $x(\pi/2) = 0$. In fact, the proofs in this section are also valid, with minor modifications, for odd periodic solutions as well as for normal modes. In order to treat odd periodic solutions the natural boundary conditions $x'(0) = x'(\pi) = 0$ are to be replaced by the boundary conditions

$$x(0) = 0, \quad x(\pi) = 0 \tag{3.71}$$

Hence the space H is taken to be

$$H = \{x(s) \mid x(s) \text{ is vector-valued function such that } x(s),\ x'(s) \in L_2[0,\pi], \text{ with } x(0) = x(\pi) = 0\}$$

with the inner product

$$\langle x,y\rangle_H = \int_0^\pi x'(s)\cdot y'(s)\,ds \tag{3.72}$$

and the corresponding norm

$$\|x\|^2 = \int_0^\pi |x'(s)|^2 ds \tag{3.73}$$

Moreover, since for the above space

$$\|x\|_2 \leq \|x\| \tag{3.74}$$

(3.15) is valid and all the results of Section 3.2 hold. Similarly, to treat the normal modes the natural boundary conditions $x'(0) = x'(\pi) = 0$ are to be replaced by the boundary conditions

$$x(0) = 0, \quad x'(\pi/2) = 0 \tag{3.75}$$

and the space H is taken to be

$$H = \{x(s) \mid x(s) \text{ is vector-valued function such that}$$
$$x(s),\ x(s)' \in L_2[0,\pi/2] \text{ with } x(0) = 0\}$$

with the inner product and norm are as given by (3.71) and (3.72) respectively. Hence all results of Section 3.2 hold.

3.3. The Continuation of The Solution

We consider the continuation problem for the solution (u^h, λ^h) of $\phi^h(u^h, \lambda^h) = 0$ with E being the continuation parameter. We show such a solution (u^h, λ^h) can be continued if it is isolated. Also, we show that under certain conditions, a solution (u^h, λ^h) may be continued by skipping over bifurcation points where the Jacobian matrix is singular.

3.3.1. Continuation of an Isolated Solution

In Section 3.2 we proved that if the exact problem $\phi(w, \nu) = 0$ has an isolated solution (u_0, λ_0) for a given $E = E_0$, then the discrete problem $\phi^h(w^h, \nu^h) = 0$, for small mesh size h, correspondingly has an isolated solution (u_0^h, λ_0^h) for $E = E_0$. Moreover, if we assume that the exact problem has an isolated branch of solutions $(u(E), \lambda(E))$ in a neighborhood of E_0, say $|E - E_0| < \epsilon$, where ϵ is a positive real number, then the following theorem can be proved.

<u>Theorem 3.3</u>. The finite element solution (u_0^h, λ_0^h) can be uniquely continued, for small mesh size h, in the neighborhood $|E - E_0| < \epsilon$.

<u>Proof</u>. By Theorem 3.2, for each E in $|E - E_0| < \varepsilon$, the discrete problem has an isolated solution $(u^h(E), \lambda^h(E))$ with $u_0^h = u^h(E_0)$, $\lambda_0^h = \lambda^h(E_0)$. Moreover, by writting

$$(u^h(E), \lambda^h(E)) - (u^h(E_0), \lambda^h(E_0)) = (u^h(E), \lambda^h(E)) - (u(E), \lambda(E))$$

$$+ (u(E), \lambda(E)) - (u(E_0), \lambda(E_0)) + (u(E_0), \lambda(E_0)) - (u^h(E_0), \lambda^h(E_0))$$

and upon using (3.69) and the continuous dependence of $(u(E), \lambda(E))$ on E, we have

$$\| (u^h(E), \lambda(E)) - (u^h(E_0), \lambda^h(E_0)) \| \leq \delta(E_0, \varepsilon) \tag{3.76}$$

with $\delta \to 0$ as $\varepsilon \to 0$. Hence $(u^h(E), \lambda^h(E))$ depends continuously on E. Furthermore, for all E in $|E - E_0| < \varepsilon$ $\left(\frac{du^h(E)}{dE}, \frac{d\lambda^h(E)}{dE}\right)$ satisfies

$$J^h_{(u^h(E), \lambda^h(E))} \begin{bmatrix} \dfrac{du^h(E)}{dE} \\[2ex] \dfrac{d\lambda^h(E)}{dE} \end{bmatrix} = \phi_E^h(u^h(E), \lambda^h(E)) \tag{3.77}$$

Now since $J^h_{(u^h(E_0), \lambda^h(E_0))}$ is invertible and $\phi_E(u^h(E), \lambda^h(E))$ $= (0, -\pi)$, we have

$$\begin{bmatrix} \dfrac{du^h(E)}{dE} \\[2ex] \dfrac{d\lambda^h(E)}{dE} \end{bmatrix} = \left[J^h_{(u^h(E), \lambda^h(E))} \right]^{-1} \begin{bmatrix} 0 \\ -\pi \end{bmatrix} \tag{3.78}$$

Hence $\left(\frac{du^h(E)}{dE}, \frac{d\lambda^h(E)}{dE}\right)$ is uniquely defined for all E in $|E - E_0| < \varepsilon$. This then implies that the solution $(u^h(E_0), \lambda^h(E_0))$ can be uniquely continued with the continuation parameter E.

In the following theorem we show that Newton's method can be used to generate this approximate branch of solutions $(u^h(E), \lambda^h(E))$.

<u>Theorem 3.4</u>. Let $(u^h(E), \lambda^h(E))$ be the isolated branch of solutions of $\phi^h(w^h, \nu^h) = 0$ for E in $|E - E_0| < \varepsilon$ mentioned above. Suppose that there exist constants $L(\varepsilon, E_0)$ and $\rho = \rho(\varepsilon) > 0$ such that

$$\|J^h_{(u^h(E_0), \lambda^h(E_0))} - J^h_{(w^h, \nu^h)}\| \leq L(\varepsilon, E_0)$$

$$\|(u^h(E_0), \lambda^h(E_0)) - (w^h, \nu^h)\| \tag{3.79}$$

for $(w^h, \nu^h) \in S^h_\rho(u^h(E_0), \lambda^h(E_0))$. Then if ε is chosen small enough such that

$$\|(J^h_{(u^h(E_0), \lambda^h(E_0))})^{-1}\|^2 \; \varepsilon \; L(E_0, \varepsilon) < \frac{1}{2} \tag{3.80}$$

Newton iterates $\{(w^h_n, \nu^h_n)\}_{n=0}^{\infty}$ defined by

(a) $$(w^h_0, \nu^h_0) = (u^h(E_0), \lambda^h(E_0)) \tag{3.81}$$

(b) $$(w^h_{n+1}, \nu^h_{n+1})^T = (w^h_n, \nu^h_n)^T - [J^h_{(w^h_n, \nu^h_n)}]^{-1} \phi^h(w^h_n, \nu^h_n)$$

$$n = 0, 1, 2, \ldots \tag{3.82}$$

converges for each E in $|E - E_0| < \varepsilon$.

<u>Proof</u>. The existence of $\rho = \rho(\varepsilon)$ is known from Theorem 3.3. From Lemma 3.3 we know that $L(\varepsilon, E_0)$ exists.

Let us consider

$$(w^h_1, \nu^h_1)^T - (w^h_0, \nu^h_0)^T = -[J^h_{(w^h_0, \nu^h_0)}]^{-1} \phi^h(w^h_0, \nu^h_0)^T$$

Now since $J^h_{(w^h_0, \nu^h_0)} = J^h_{(u^h(E_0), \lambda^h(E_0))}$, hence non-singular, and $\phi^h(u^h(E_0), \lambda^h(E_0)) = 0$ for $E = E_0$ we get

$$\|(w_1^h,\nu_1^h) - (w_0^h,\nu_0^h)\| \le \|[J^h_{(u^h(E_0),\lambda^h(E_0))}]^{-1}\| \ \|\phi^h(u^h(E_0),\lambda^h(E_0)) - \phi^h(w_0^h,\lambda_0^h)\|$$

$$\le \|(J^h_{(u^h(E_0),\lambda^h(E_0))})^{-1}\| \ |E - E_0|$$

or

$$\|(w_1^h,\nu_1^h) - (w_0^h,\nu_0^h)\| < \|[J^h_{(u^h(E_0),\lambda^h(E_0))}]^{-1}\|\varepsilon \tag{3.84}$$

Moreover, since for the quadratic convergence of Newton's method [29] we require

$$\|[J^h_{(u^h(E_0),\lambda^h(E_0))}]^{-1}\| \ \|(w_1^h,\nu_1^h) - (w_0^h,\nu_0^h)\| L(\varepsilon,E_0) < \frac{1}{2} \tag{3.85}$$

Thus, by (3.84) condition (3.80) suffices for the quadratic convergence of Newton's iterates $\{(w_n^h,\nu_n^h)\}_{n=0}^{\infty}$ to the unique solution $(u^h(E),\lambda^h(E))$ of $\phi^h(w^h,\nu^h) = 0$ for each E in $|E - E_0| < \varepsilon$.

3.3.2. Continuation Past a Bifurcation Point

An energy $E = E_s$ where $J_{(u,\lambda)}$ becomes singular and different branches of solution meet is called a bifurcation point. For E in a neighborhood of E_s the number of branches of solution changes. The number of "bifurcated branches of solution" depends on the dimension of the null space of $J_{(u,\lambda)}$.

Let us consider the situation as depicted in Figure 3.2 where in a neighborhood of E_s the number of branches of solution changes from one to three. The corresponding situation for finite element equations $\phi^h(w^h,\nu^h) = 0$, with h being small but fixed, need not have a corresponding bifurcation point as illustrated in Figure 3.3. On the other hand the equation $\phi^h(w^h,\nu^h) = 0$ may have a corresponding bifurcation point as shown in Figure 3.4, i.e., there exists a bifurcation point E_s^h at which $J^h_{(u^h(E_s^h),\lambda^h(E_s^h)}$ is singular. The change of signs in the determinant of the Jacobian matrix $J^h_{(u^h(E_s^h),\lambda^h(E_s^h))}$ guarantees the existence of such bifurcation point. In this case we show in the following theorem that we can jump over such a bifurcation point in continuing the solution from E_1 to E_2 as shown in Figure 3.4.

<u>Theorem 3.5</u>. Let $(u^h(E),\lambda^h(E))$ be a smooth branch of solutions of $\phi^h(w^h,\nu^h) = 0$, on which $J^h_{(u^h(E),\lambda^h(E))}$ is non-singular for $E \in [E_1,E_2] - \{E_s^h\}$. Suppose that there exist constants $L(\rho,E)$ and $\rho > 0$ such that

$$\|J^h_{(u^h(E),\lambda^h(E))} - J^h_{(w^h,\nu^h)}\| \leq L(\rho,E)\,\|(u^h(E),\lambda^h(E)) - (w^h,\nu^h)\| \tag{3.86}$$

for $(w^h,\nu^h) \in S_\rho^h(u^h(E_1),\lambda^h(E_1))$. Then if E is such that

$$\{\|(J^h_{(u^h(E_1),\lambda^h(E_1))})^{-1}\|\}^2|E - E_1|\;L(\rho,E) < \frac{1}{2} \tag{3.87}$$

the Newton's iterates $\{(w_n^h,\nu_n^h)\}_{n=0}^{\infty}$ with initial guess $(w_0^h,\lambda_0^h) = (u^h(E_1),\lambda^h(E_1))$ converges for $E \neq E_s^h$.

<u>Proof</u>. Similar to that of Theorem 3.4.

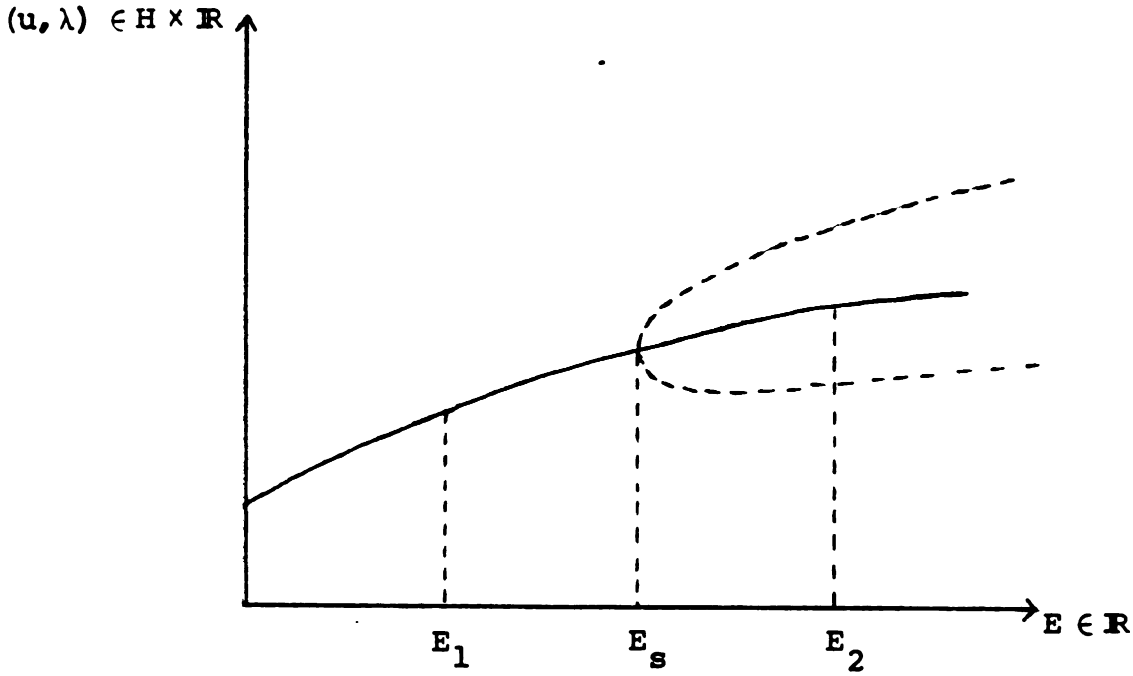

Figure 3.2. The problem $\phi(w, \nu) = 0$ has E_s as a bifurcation point

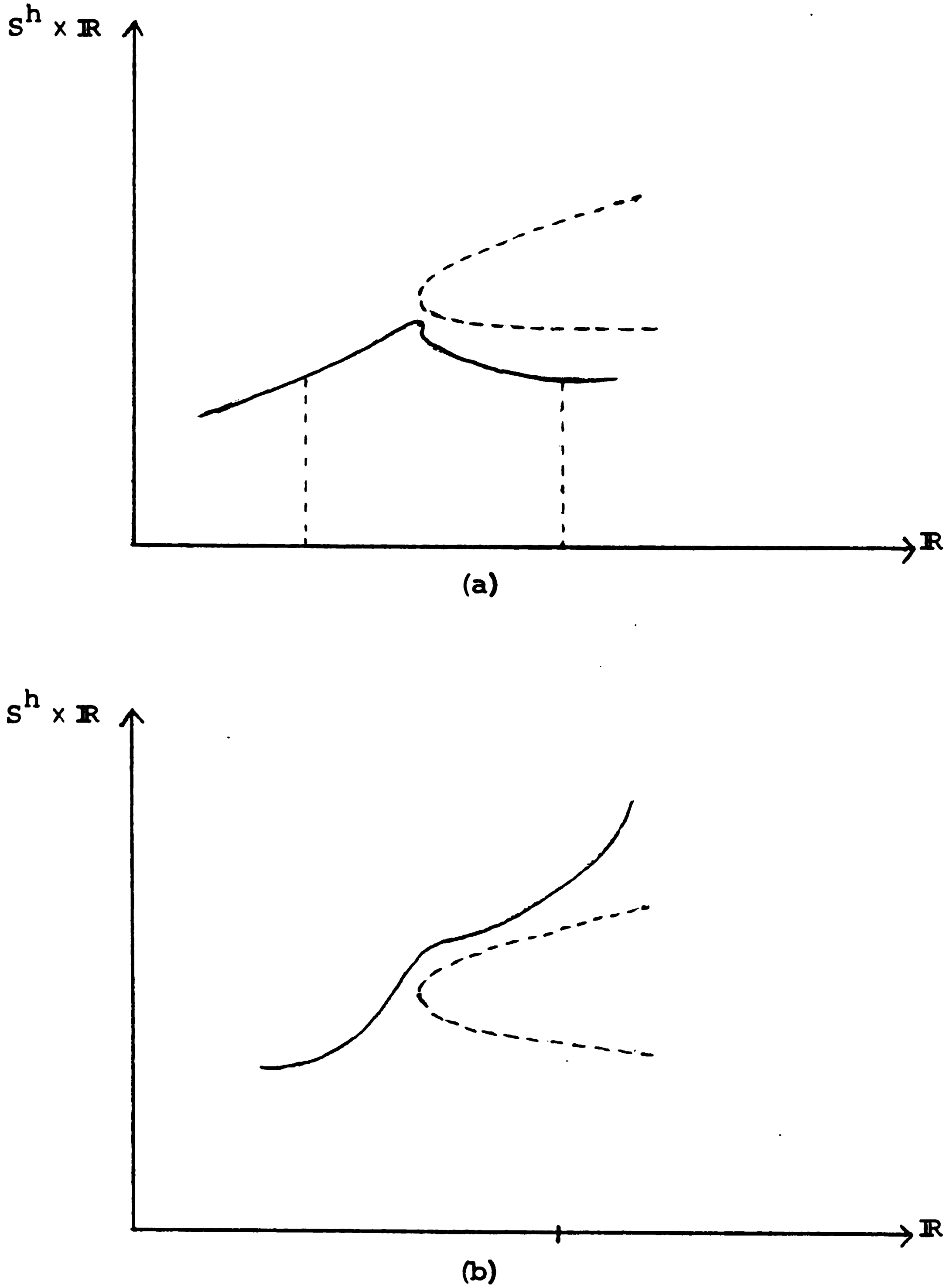

Figure 3.3. The problem $\phi^h(w^h, \nu^h) = 0$ does not have a corresponding bifurcation point.

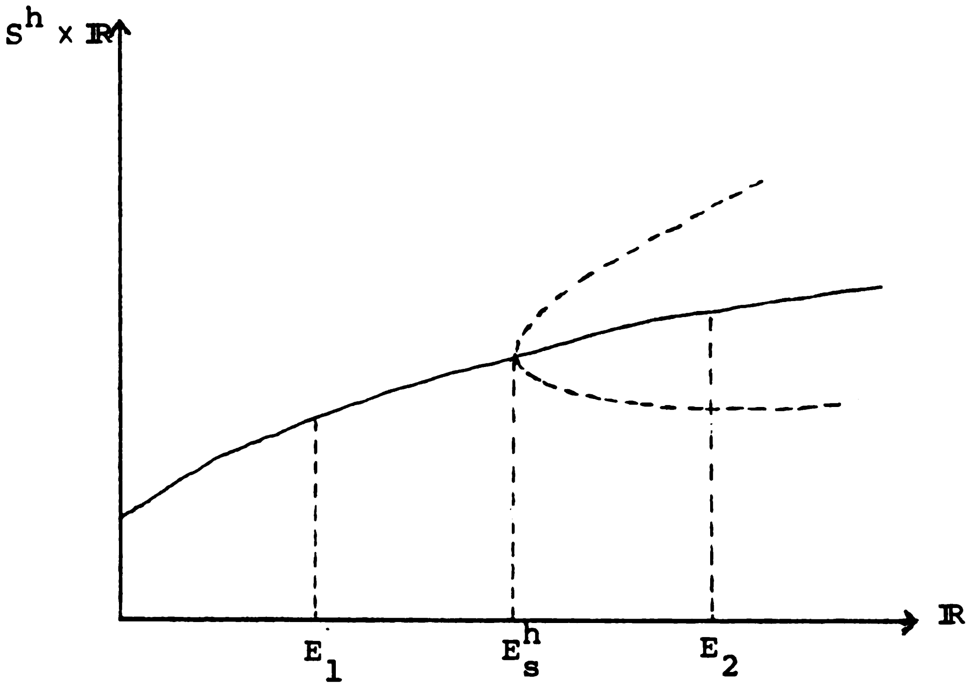

Figure 3.4. The problem $\phi^h(w^h, \nu^h) = 0$ has a corresponding bifurcation point E_s^h.

3.4. A Numerical Algorithm

In this Section we present an algorithm for determining an isolated solution branch as well as for treating bifurcated branches. The algorithm depends in part on the assumed form for the potential energy function, $V(x) = Q(x) + N(x)$, where $Q(x)$ is quadratic in x and $N(x)$ contains higher order terms. (Under slight modification it can be applied to a more general $V(x)$).

As $V(x) = Q(x) + N(x)$, for sufficiently small values of the energy E the nonlinear part of grad $V(x)$, i.e., grad $N(x)$, acts as a small perturbation to its linear part, i.e., grad $Q(x)$. We know that this linearized problem

$$-\lambda^h \langle (u^h)', (\Psi_i^h)' \rangle + \langle \text{grad } Q(u^h), \Psi_i^h \rangle = 0 \qquad (3.88)$$

$$i = 1, 2, \ldots, \nu$$

has exactly m-distinct linear normal modes each with frequency independent of E. We start our algorithm by obtaining these m-primary branches.

Our algorithm consists of the following steps:

<u>Step 1</u>. Solve the linearized problem (3.88) which can be written as

$$AQ = \lambda^h BQ \qquad (3.89)$$

which is a generalized eigenvalue problem. The matrices A and B above each contains $m \times m$ blocks with block size $\nu \times \nu$.

The eigenvalues and the corresponding eigenvectors of (3.89) are obtained using the "EISPACK" which is a collection of FORTRAN subroutines* which compute the eigenvalues and/or the eigenvectors of various classes of matrices [30].

By using the above routines we obtain $m \times \nu$ eigenvalues counting multiplicity and their corresponding eigenvectors. As we know that the linear problem has exactly m distinct normal modes it is necessary to omit the extraneous eigenvalues and eigenvectors introduced above.

We normalize each of the representative elements to the desired energy level by using the linear part of (3.12), that is

$$\frac{\lambda^h}{2} \langle (u^h)', (u^h)' \rangle + \langle Q(u^h), 1 \rangle = \pi E \tag{3.90}$$

which can be rewritten as

$$\frac{\lambda^h}{2} Q^T BQ + Q^T AQ - \pi E = 0 \tag{3.91}$$

We now let (u_L^h, λ_L^h) denote one of these representative normalized solutions, where $u_L^h(s) = ((u_L^h(s))_1, \ldots (u_L^h(s))_m)$ and

$$(u_L^h(s))_i = \sum_{j=1}^{\nu} q_{(i-1)\nu+j} \Psi_j^h(s) \; i = 1, 2, \ldots, m$$

<u>Step 2</u>. Apply Newton's method to (3.11) and (3.12) with initial guess (u_L^h, λ_L^h) to generate the sequence of iterates $\{(w_n^h, \nu_n^h)\}$ defined by

$$\text{(a)} \quad (w_0^h, \nu_0^h) = (u_L^h, \lambda_L^h) \tag{3.92}$$

$$\text{(b)} \quad (w_{n+1}^h, \nu_{n+1}^h)^T = (w_{n+1}^h, \nu_{n+1}^h)^T - (J^h_{(w_n^h, \nu_n^h)})^{-1} \phi^h (w_n^h, \nu_n^h)^T \tag{3.93}$$

*These subroutines are translations of the ALGOL procedures published in the handbook series of Springer-Verlag by Wilkinson and Reinch [31].

Solving (3.93) is equivalent to solving the linear system of equations

$$J^h_{(w^h_n, \nu^h_n)} \quad (\delta w^h_n, \delta\nu^h_n)^T = (w^h_n, \nu^h_n)^T \tag{3.94}$$

where

$$\delta w^h_n = w^h_{n+1} - w^h_n, \quad \delta\nu^h_n = \nu^h_{n+1} - \nu^h_n \tag{3.95}$$

To solve (3.94) we use Gaussian elimination procedure with partial pivoting to decompose $J^h_{(w^h_n, \nu^h_n)}$ into a product of a lower triangular matrix L and an upper triangular matrix U, i.e., we find a permutation matrix P such that

$$PJ^h_{(w^h_n, \nu^h_n)} = LU \tag{3.96}$$

Solving (3.94) is equivalent to solving the two linear systems

$$U(\delta u^h_n, \delta\lambda^h_u)^T = Y$$

and (3.97)

$$LY = P^{-1}(\delta u^h_n, \delta\lambda^h_n)^T$$

Iterative improvement and double precision can be used to increase the accuracy of the solution [32]. With the assumption that $J^h_{(u^h_L, \lambda^h_L)}$ is nonsingular, the above sequence of iterates converge to some limit, (u^h_f, λ^h_f) say.

<u>Step 3</u>. Increase the continuation parameter E by a small amount δE and repeat Step 2, with initial guess now set to be (u^h_f, λ^h_f). In doing this we may have skipped over singular points.

The steps 2 and 3 above generate an approximate solution branch $(u^h(E), \lambda^h(E))$.

Step 4. We return to the neighborhood of each singular point and locate it accurately (i.e. use false position or bisection to determine the zero E_s^h of $\det J^h_{(u^h,\lambda^h)}$). In particular, simple or odd order roots can be determined by the sign change in $\det J^h_{(u^h,\lambda^h)}$. (It is thus necessary to record the number of row changes in the LU-decomposition with partial pivoting.)

Step 5. To switch over to a bifurcated branch we must compute an approximation to a point on this bifurcated branch. To obtain such approximation we must construct several distinct tangent vectors $(\frac{du^h(E)}{dE}, \frac{d\lambda^h(E)}{dE})$, at the point $E = E_s^h$ best approximating the bifurcation value for E. We illustrate how this construction can be done in the following.

Let us recall that $(\frac{du^h}{dE}, \frac{d\lambda^h}{dE})$ is the solution of

$$J^h_{(u^h,\lambda^h)} \left(\frac{du^h}{dE}, \frac{d\lambda^h}{dE}\right)^T = (0, -\pi)^T$$

which implies

$$F^h_{u^h} \frac{du^h}{dE} + F^h_{\lambda^h} \frac{d\lambda^h}{dE} = 0 \tag{3.98}$$

and

$$T^h_{u^h} \frac{du^h}{dE} + T^h_{\lambda^h} \frac{d\lambda^h}{dE} = -\pi \tag{3.99}$$

For the solvability of (3.98) we must have

$$F^h_{\lambda^h} \in R(F^h_{u^h}) \tag{3.100}$$

where $R(F^h_{u^h})$ denotes the range of the operator $F^h_{u^h}$. Moreover, if $F^h_{u^h}$ is singular $J^h_{(u^h,\lambda^h)}$ will also be singular under mild conditions [33]. It suffices to study singular solutions at which (3.100) holdstogether with

$$\dim N(F^h_{u^h}) = \operatorname{codim} R(F^h_{u^h}) = k \geq 1 \tag{3.101}$$

From (3.101) we have the existence of elements $\gamma_j^h \in S^h$ and $\gamma_j^{h*} \in (S^h)^*$, where "*" refers to the adjoint space such that

$$N(F^h_{u^h}) = \operatorname{span}\{\gamma_1^h, \ldots, \gamma_k^h\} \tag{3.102}$$

$$N((F^h_{u^h})^*) = \operatorname{span}\{\gamma_1^{h*}, \ldots, \gamma_k^{h*}\} \tag{3.103}$$

with

$$\langle \gamma_j^{h*}, \gamma_i^h \rangle = \delta_{ij} \quad 1 \leq i,j \leq k$$

Now from (3.100) we can conclude that there exists a unique element $\gamma_0^h \in S^h$ such that

$$F^h_{\lambda^h} + F^h_{u^h}\gamma_0 = 0; \quad \langle \gamma_j^*, \gamma_0 \rangle = 0 \quad 1 \leq j \leq k \tag{3.104}$$

Hence from (3.98) and (3.104) we have

$$\frac{du^h}{dE} = \sum_{j=0}^{k} a_j^h \gamma_j^h \tag{3.105}$$

where

$$a_0^h = \frac{d\lambda^h}{dE} \quad \text{and} \quad a_j^h = \langle \gamma_j^*, \frac{du^h}{dE} \rangle \quad 1 \leq j \leq k \tag{3.106}$$

In order to construct several distinct tangents $\left(\frac{du_i^h}{dE}, \frac{d\lambda_i^h}{dE}\right)$ $i = 1,2,\ldots,$ we need to determine different sets of a_j^h $0 \leq j \leq k$. Techniques to obtain such a_j^h can be found in [33, 34]. We now take

$$u^h(E) = u^h(E_0) + (E - E_0)\frac{du^h(E_0)}{dE}$$
$$\lambda^h(E) = \lambda^h(E_0) + (E - E_0)\frac{d\lambda^h(E_0)}{dE} \tag{3.106}$$

as an initial guess to a solution on a bifurcated branch and return to Step 2, where E is taken far enough from E_0 to insure that the convergence of Newton's method is not to the original or primary branch of solution.

As an alternative algorithm that may be less costly, especially in the case when $m > 2$ we make the following modifications:

Let us consider the Jacobian matrix

$$J^h_{(u^h,\lambda^h)} = \begin{bmatrix} F^h_{u^h} & F^h_{\lambda^h} \\ T_{u^h} & T_{\lambda^h} \end{bmatrix}$$

where $F^h_{u^h}$ is a $(m \times \nu) \times (m \times \nu)$ matrix $F^h_{\lambda^h}$ and T_{u^h} are column and row vectors of order $m \times \nu$, and T_{λ^h} is a scalar.

To solve (3.94) it suffices to determine y^h and z^h satisfying

$$F^h_{u^h}\, y^h = F^h_{\lambda^h} \tag{3.107}$$

$$F^h_{u^h}\, z^h = u^h_n \tag{3.108}$$

and then

$$\delta\lambda^h_n = (T_{u^h_n}, z^h + \lambda^h_n)/T_{\lambda^h_n} - T^h_{u^h_n}\cdot y^h \tag{3.109}$$

$$\delta u^h_n = z^h - \delta\lambda^h_n y^h \tag{3.110}$$

We apply Gaussian elimination to $F^h_{u^h_n}$, instead of all $J^h_{(u^h_n,\lambda^h_n)}$, to obtain matrices P_1, L_1 and U_1 such that

$$P_1 F^h_{u^h_n} = L_1 U_1 \tag{3.111}$$

and to compute the null vectors of $F^h_{u^h_n}$ and $(F^h_{u^h_n})^*$ at $E = E^h_s$ we solve

$$F^h_{u^h} \gamma^h = \delta u^h \tag{3.112}$$

$$(F^h_{u^h})^* (\gamma^h)^* = \delta u^h \tag{3.113}$$

where δu^h denotes the last correction in Newton's scheme in Step 2.

The economy in this alternative algorithm comes from the fact that the LU-decomposition of $F^h_{u^h}$ has already been performed.

Chapter 4 Numerical Results

In this chapter we apply the numerical algorithm discussed in Chapter 3 to some special classes of problems introduced in Chapter 2 with $V(x) = Q(x) + N(x)$ where $Q(x)$ is quadratic in x. In Section 4.1 we review properties of the so-called piecewise linear shape functions that are used in the construction of the finite element subspace S^h. In Section 4.2 we consider free vibrations of nonlinear undamped single degree of freedom systems, with the special form of $V(x)$ mentioned above, for which exact solutions are known. The approximate finite element solutions are compared with the exact ones so as to assess the accuracy of the former. In Section 4.3 we consider free vibrations of non-linear systems with two degree of freedom and present examples in which bifurcated branches of solutions exist. In Section 4.3 we also compare the finite element solutions with approximate solutions obtained by a finite difference scheme.

4.1. The Linear Elements

In order to obtain approximate finite element solutions we choose for T^h the space of scalar-valued functions which are linear over each finite element and continuous at the nodes which we assume to be equally spaced, i.e.

$$s_i = s_{i-1} + h \quad i = 1,2,\dots,n \quad \text{and} \quad s_0 = 0,\ s_n = \pi \tag{4.1}$$

$$h = \pi/n \tag{4.2}$$

It can be easily verified that the functions defined by

$$\Psi_0^h(s) = -\frac{s}{h} + 1 \qquad 0 \le s \le h \tag{4.3}$$

$$\Psi_i^h(s) = \begin{cases} \frac{s}{h} - (i-1) & (i-1)h \le s \le ih \\ -\frac{s}{h} + (i+1) & ih \le s \le (i+1)h \end{cases} \tag{4.4}$$

$$\Psi_n^h(s) = \frac{s}{h} - (n-1) \tag{4.5}$$

constitute a basis for T^h [28] (Figure 4.1). Hence $\nu = n + 1$ and any trial function $u^h \in S^h$ is of the form $u^h = (u_1^h, \ldots, u_m^h)$ where

$$u_i^h = \sum_{j=0}^{n} q_{(i-1)n+j} \Psi_j^h(s), \quad 1 \le i \le m \tag{4.6}$$

Upon assuming that $V(x) = Q(x) + N(x)$ where $Q(x)$ is quadratic and $N(x)$ is quartic the finite element equations become

$$-\lambda^h BQ + AQ + N(Q) = 0 \tag{4.7}$$

$$\frac{\lambda^h}{2} Q^T BQ + \frac{1}{2} Q^T AQ + M(Q) - \pi E = 0 \tag{4.8}$$

where A and B are $m(n+1) x m(n+1)$ block matrices in which each block is an $(n+1) x (n+1)$ matrix whose entries are scalar multiples of the following tridiagonial positive definite matrices with entries

$$\int_0^{\pi} \Psi_i^h \Psi_j^h \, ds = \begin{cases} \frac{2h}{3} & i = j \ne 0 \text{ or } n \\ \frac{h}{3} & i = j = 0 \text{ or } n \\ \frac{h}{6} & |i - j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

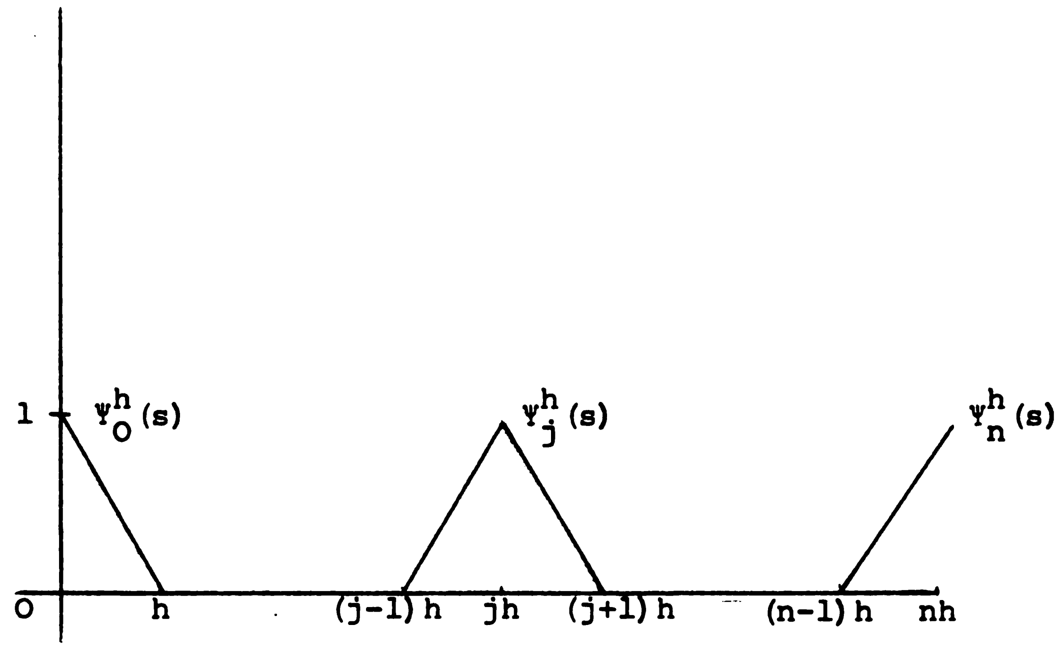

Figure 4.1 Piecewise linear basis functions

$$\int_0^{\pi} (\Psi_i^h)'(\Psi_j^h)'ds = \begin{cases} \frac{2}{h} & i = j \neq 0 \quad \text{or} \quad n \\ \frac{1}{h} & i = j = 0 \quad \text{or} \quad n \\ -\frac{1}{h} & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

and Q is $m(n+1)$ vector whose entries are the $q_{(i-1)(n+1)+j}$. $M(Q)$ is a scalar quartic function in $q_{(i-1)(n+1)+j}$ with coefficients involving the quantities

$$c_{ij\ell k} = \int_0^{\pi} \Psi_i^h \Psi_j^h \Psi_\ell^h \Psi_k^h \, ds$$

$$= \begin{cases} \frac{2h}{5} & i = j = k = \ell \neq 0 \quad \text{or} \quad n \\ \frac{h}{5} & i = j = k = \ell = 0 \quad \text{or} \quad n \\ \frac{h}{20} & i = j = \ell, |i-k| = 1 \\ \frac{h}{30} & i = j, \; \ell = k, \; |i-k| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

and $N(Q)$ is the gradient of $M(Q)$ with respect to Q.

We conclude this section by remarking that if the boundary conditions $x(0) = x(\pi) = 0$ are imposed on elements of T^h, then $\nu = n - 1$, $T^h = \text{span}\{\Psi_1^h, \ldots, \Psi_{n-1}^h\}$, and $q_{(i-1)(n+1)} = 0$ $= q_{(i-1)(n+1)+n}$ for $1 \leq i \leq m$. The resulting nonlinear algebraic system of equations as given by (4.7) and (4.8) is then modified by deleting the first and the last rows of each block of matrices A and B. Similarly if we are concerned with the approximate normal modes the mesh size h is equal to $\frac{\pi}{2n}$ instead of $\frac{\pi}{n}$ and the boundary condition $x(0) = 0$ is to be imposed on elements of T^h. Hence $\nu = n$, $T^h = \text{span}\{\Psi_1^h, \ldots, \Psi_n^h\}$ and $q_{(i-1)(n+1)} = 0$ for $1 \leq i \leq m$. The resulting nonlinear algebraic system of equations as given by (4.7) and (4.8) is then modified by deleting the first row of each block of matrices A and B.

4.2. Problems with a Single Degree of Freedom

The governing equation of motion for the free vibration of a single degree of freedom system in Figure 4.2 is

$$\ddot{x} + \frac{dV}{dx} = 0 \tag{4.12}$$

where without loss of generality the mass is normalized to unity. The first integral of the system (4.12) is

$$\frac{1}{2}\dot{x}^2 + V(x) = E \tag{4.13}$$

For a given total energy E equation (4.13) describes an equi-energy curve in the $(x, \dot{x})$ plane (known as the phase plane) with time being the parameter. It is known [35] that a solution $x(t)$ of (4.12) is periodic if and only if the corresponding equi-energy curve is closed.

Let us consider the special form of $V(x)$ we mentioned in Section 4.1, that is

$$V(x) = \frac{a_1}{2}x^2 + \frac{a_3}{4}x^4 \tag{4.14}$$

where a_1 and a_3 are constants with a_1 being positive. These constants are known as spring constants. Furthermore the spring is said to be "hard" if a_3 is positive, and is said to be "soft" if a_3 is negative. We have a "linear" spring if a_3 vanishes.

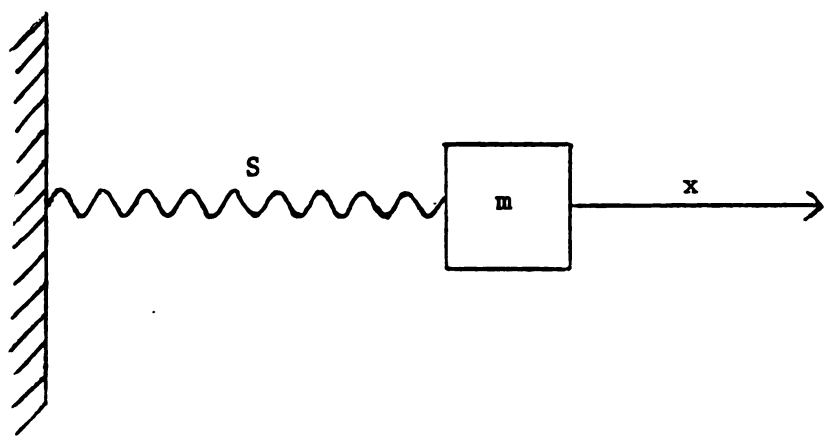

Figure 4.2 A single degree of freedom system

From equation (4.13) and the assumption of $a_1 > 0$ we may conclude that in a neighborhood of the origin $x = 0$, $\dot{x} = 0$ in the phase plane, the equi-energy curves are all closed and nearly ellipses. The maximum displacement A is readily shown to satisfy the relation

$$A^2 = \frac{-a_1 + \sqrt{a_1^2 + 4a_3E}}{a_3} \tag{4.15}$$

obtained by setting $\dot{x} = 0$ in (4.13) and solving for x^2. The above formula (4.15) holds for hard springs as well as soft springs and yields real and positive A^2 for small and positive E. On account of the symmetry of equi-energy curves in x and $\dot{x}$ the period τ of the motion can be expressed in terms of elliptic integrals [36]. For the case of a hard spring

$$\tau = \frac{4}{\sqrt{a_1 + a_3A^2}} K(k_1, \tfrac{\pi}{2}) \tag{4.16}$$

where $k_1^2 = \dfrac{a_3A^2}{2(a_1 + a_3A^2)}$ and $K(k_1, \frac{\pi}{2})$ is the complete elliptic integral of the first kind, i.e. $\displaystyle\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2\varphi}}$ and for the case of a soft spring

$$\tau = \frac{4\sqrt{2}}{\sqrt{2a_1 + a_3A^2}} K(k_2, \tfrac{\pi}{2}) \tag{4.17}$$

where $k_2^2 = \dfrac{-a_3A^2}{2a_1 + a_3A^2}$. We note, however, that (4.17) yields unbounded solutions as $\dfrac{-a_3A^2}{a_1} \to 1$ since $k_2 \to 1$ and hence $K \to \infty$. Thus (4.17) applies only for the range of energy levels or amplitude where (4.17) yields bounded solutions.

It is also known that, unlike the linear case $(a_3 = 0)$, the period of oscillation in general depends on the amplitude A. Figure 4.3 depicts the relationships between the amplitude A and the "circular frequency" $\omega = \frac{2\pi}{\tau}$ in the cases of linear, hard, and soft springs.

In Figure 4.4 the equi-energy curves are shown for hard springs and they are all closed curves. The arrows on the curves indicate the direction in which the point $(x(t),\dot{x}(t))$ moves with increasing t.

For soft springs the situation are somewhat more complicated where closed curves occur only for a range of small energy E, beyond which the equi-energy curves become open curves and periodic solutions are no longer possible see Figure 4.5.

In what follows we present several examples for the case of soft and hard springs. Approximate solutions are obtained by the finite element equations derived above and are treated as a continuation problem with the continuation parameter being E. In these examples we consider $V(x)$ as given by (4.14). The amplitude of vibration is given by (4.15) and the frequency of vibration is as given by (4.16) for hard springs and by (4.17) for soft springs.

In Figures 4.6 through 4.19, we present both the exact solution and the finite element solution for one degree of freedom systems with soft springs. Different mesh sizes are used, and the relationships between the frequency of vibration and the average energy as well as the relationships between the frequency and the amplitude of vibration, which for the finite element solution $u^h(s)$

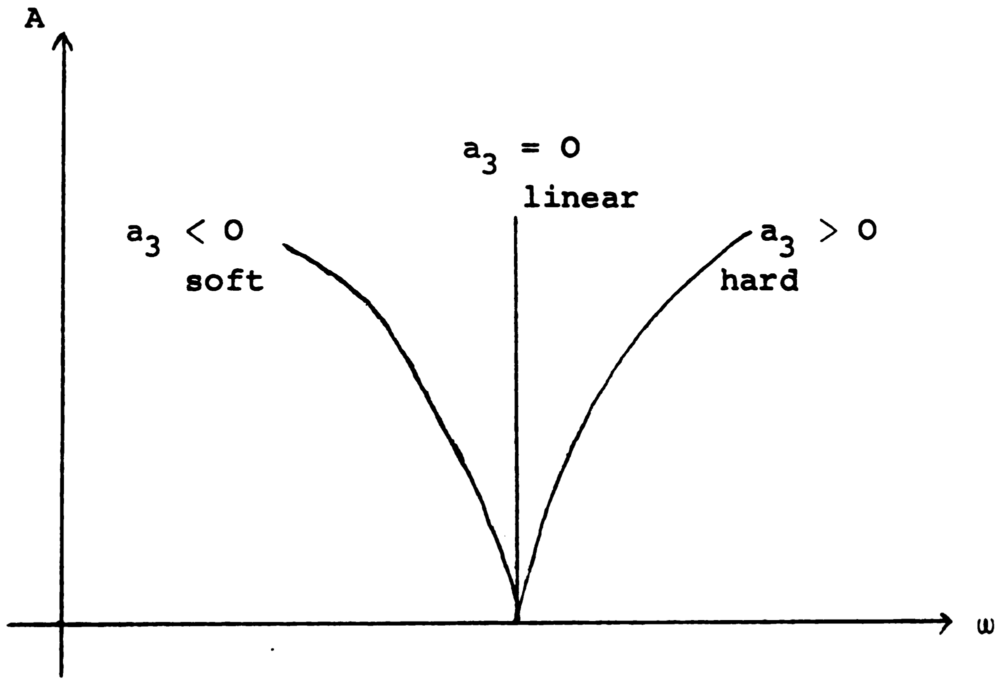

Figure 4.3 Dependence of amplitude on frequency with different types of spring characteristics

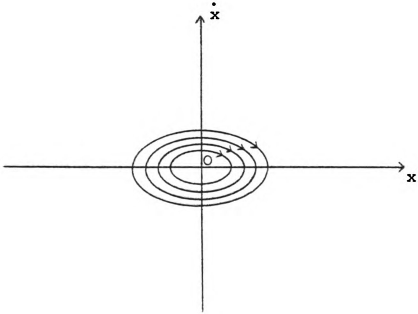

Figure 4.4 The equi-energy curves. Hard springs.

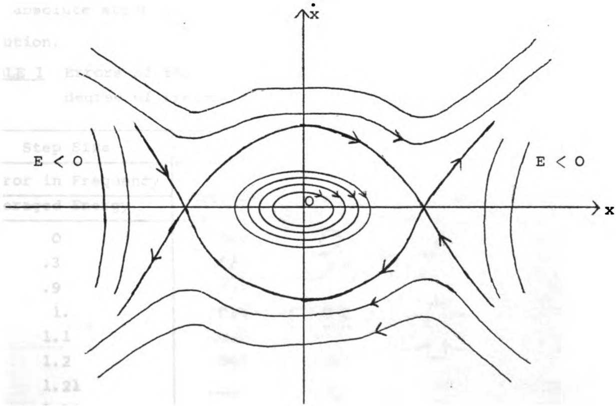

Figure 4.5 The equi-energy curves. Soft springs.

is taken as $\max_{0\leq i\leq n} |u^h(ih)|$, as shown. Since exact periodic solutions exist only for a finite range of energies, after which solutions become unbounded, finite element solutions likewise exist only for a finite range of energies. These two ranges are close to each other as the mesh size h becomes smaller. Our continuation technique as described in Section 3.4 fails to converge beyond this finite range of energies as the Jacobian matrix, which depends on the solution, is unbounded.

In Tables 1 through 4 below we recast some of the numerical results shown in Figures 4.6 through 4.13. These tables serve to exhibit the absolute and relative errors in the computed finite element solutions. The absolute error in each case is the absolute value of the difference between the exact solution and the computed finite element solution. The relative error is the absolute error divided by the absolute value of the exact solution.

TABLE 1 Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = -.16$.

Step Size	$\pi/5$		$\pi/9$	
Error in Frequency / Averaged Energy	Absolute	Relative	Absolute	Relative
0	.015	.016	.005	.005
.3	.015	.016	.005	.005
.9	.015	.019	.005	.006
1.	.016	.022	.005	.007
1.1	.020	.030	.006	.009
1.2	.061	.106	.013	.023
1.21	____	____	.016	.029
1.22	____	____	.022	.040
1.23	____	____	.034	.064

TABLE 2 Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = -.16$.

Step Size	π/5		π/9	
Error in Amplitude / Average Energy	Absolute	Relative	Absolute	Relative
0	0	0	0	0
.3	.013	.15	.003	.005
.9	.013	.16	.003	.005
1.0	.05	.030	.005	.006
1.1	.08	.040	.005	.006
1.2	.085	.043	.017	.008
1.21	———	———	.022	.010
1.22	———	———	.028	.013
1.23	———	———	.041	.019

TABLE 3 Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = -1.6$.

Step Size	π/6		π/12	
Error in Frequency / Average Energy	Absolute	Relative	Absolute	Relative
0	.011	.011	.003	.003
.06	.012	.012	.003	.003
.09	.013	.013	.003	.003
.11	.014	.020	.004	.006
.121	.045	.080	.009	.016
.122	.077	.141	.011	.021
.123	———	———	.016	.032
.124	———	———	.030	.062

TABLE 4 Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = -1.6$.

Step Size	π/6		π/12	
Error in Amplitude / Average Energy	Absolute	Relative	Absolute	Relative
0	0	0	0	0
.06	.009	.024	.002	.006
.09	.013	.025	.003	.006
.11	.017	.028	.004	.007
.12	.026	.040	.006	.009
.121	.032	.045	.007	.010
.122	.040	.058	.008	.011
.123	———	———	.009	.013
.124	———	———	.013	.018

The results in Tables 1 through 4 indicate that for any nonlinear soft spring constant the absolute and the relative errors in both the frequency and the amplitude depend on the mesh size used as well as the averaged energy. As the averaged total energy increases the absolute and relative errors increase slightly. As the averaged energy approaches the critical value where periodic solutions fails to exist the change in both the relative and absolute errors become rather significant. Furthermore, from Figures 4.6 through 4.13 we can conclude that the computed frequency always underestimates the exact frequency and the computed amplitude always overestimates the exact one. The range of errors encountered in both absolute and relative errors varies from less than 1% to about 7%, for the mesh sizes,

$\pi/9$ and $\pi/12$, hence our finite element method provides reasonably accurate approximations.

Figures 4.14 through 4.19 show similar results for systems with hard springs. Since exact periodic solutions exists for all energy level, a solution branch can be followed as far as we wish. The finite element equations also have the same property and the finite element solutions can be continued as far as we wish.

TABLE 5 Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = .6$.

Step Size	$\pi/5$		$\pi/9$	
Error in Frequency / Average Energy	Absolute	Relative	Absolute	Relative
0	.015	.016	.005	.005
.04	.016	.016	.005	.005
.12	.016	.016	.005	.005
.2	.017	.017	.005	.005
.3	.018	.017	.006	.006
.54	.019	.017	.006	.006
.78	.020	.017	.006	.006
.9	.030	.018	.007	.007

<u>TABLE 6</u> Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = .6$.

Step Size	$\pi/5$		$\pi/9$	
Error in Amplitude / Average Energy	Absolute	Relative	Absolute	Relative
0	0	0	0	0
.04	.005	.018	.002	.005
.12	.009	.02	.003	.006
.2	.012	.02	.004	.006
.3	.015	.021	.005	.007
.54	.021	.022	.007	.007
.78	.026	.023	.008	.007
.9	.028	.024	.009	.008

<u>TABLE 7</u> Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = 1.6$.

Step Size	$\pi/6$		$\pi/12$	
Error in Amplitude / Average Energy	Absolute	Relative	Absolute	Relative
0	.011	.011	.003	.028
.4	.015	.012	.004	.003
.8	.017	.012	.004	.003
1.2	.019	.012	.005	.003
1.6	.020	.012	.005	.003
2.0	.021	.012	.005	.003
2.8	.023	.012	.006	.003
3.6	.024	.012	.006	.003
4.4	.025	.012	.006	.003
5	.026	.012	.007	.003

TABLE 8 Errors of the finite element solutions for a single degree of freedom system. $a_1 = .896$, $a_3 = 1.6$.

Step Size	$\pi/6$		$\pi/12$	
Error in Amplitude / Average Energy	Absolute	Relative	Absolute	Relative
0	0	0	0	0
.4	.018	.023	.004	.006
.8	.023	.024	.006	.006
1.2	.027	.024	.007	.006
1.6	.029	.024	.007	.006
2.0	.032	.024	.008	.006
2.8	.035	.024	.008	.006
3.6	.038	.024	.009	.006
4.4	.040	.024	.01	.006
5	.042	.024	.01	.006

In Tables 5 through 8 the relative and absolute errors in the frequency and the amplitude for the case of hard springs are represented. As the averaged energy increases the changes in the relative and absolute errors are not significant. The errors in these tables do not exceed 4% in the range of E considered. Figures 4.14 through 4.19 show that both the computed frequency and the computed amplitude underestimate the corresponding exact ones.

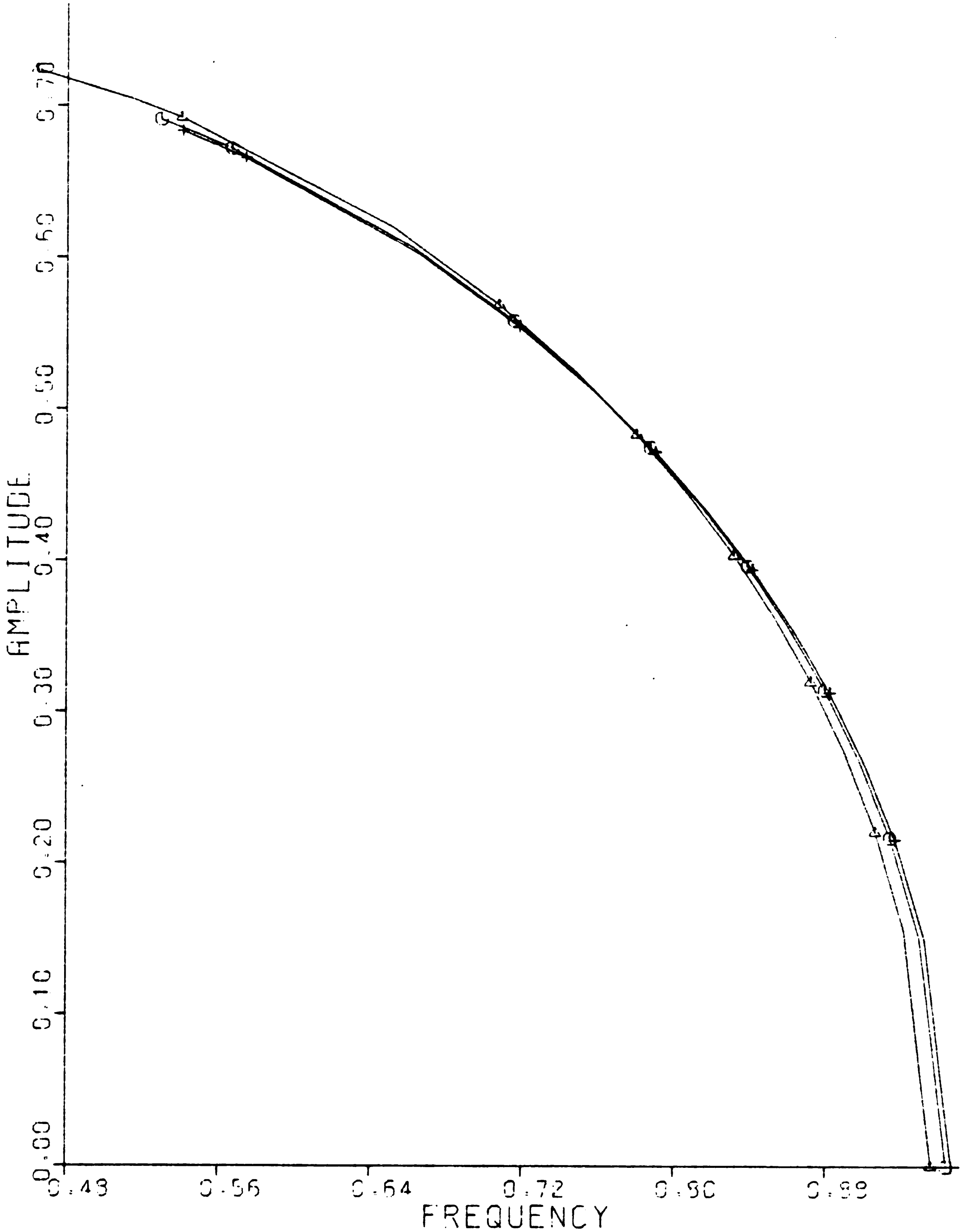

Figure 4.6 Amplitude - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: a = .896 and a_3 = -1.6. —+— Exact solution, —⊕— finite element solution h = π/12, and —△— finite element solution h = π/6.

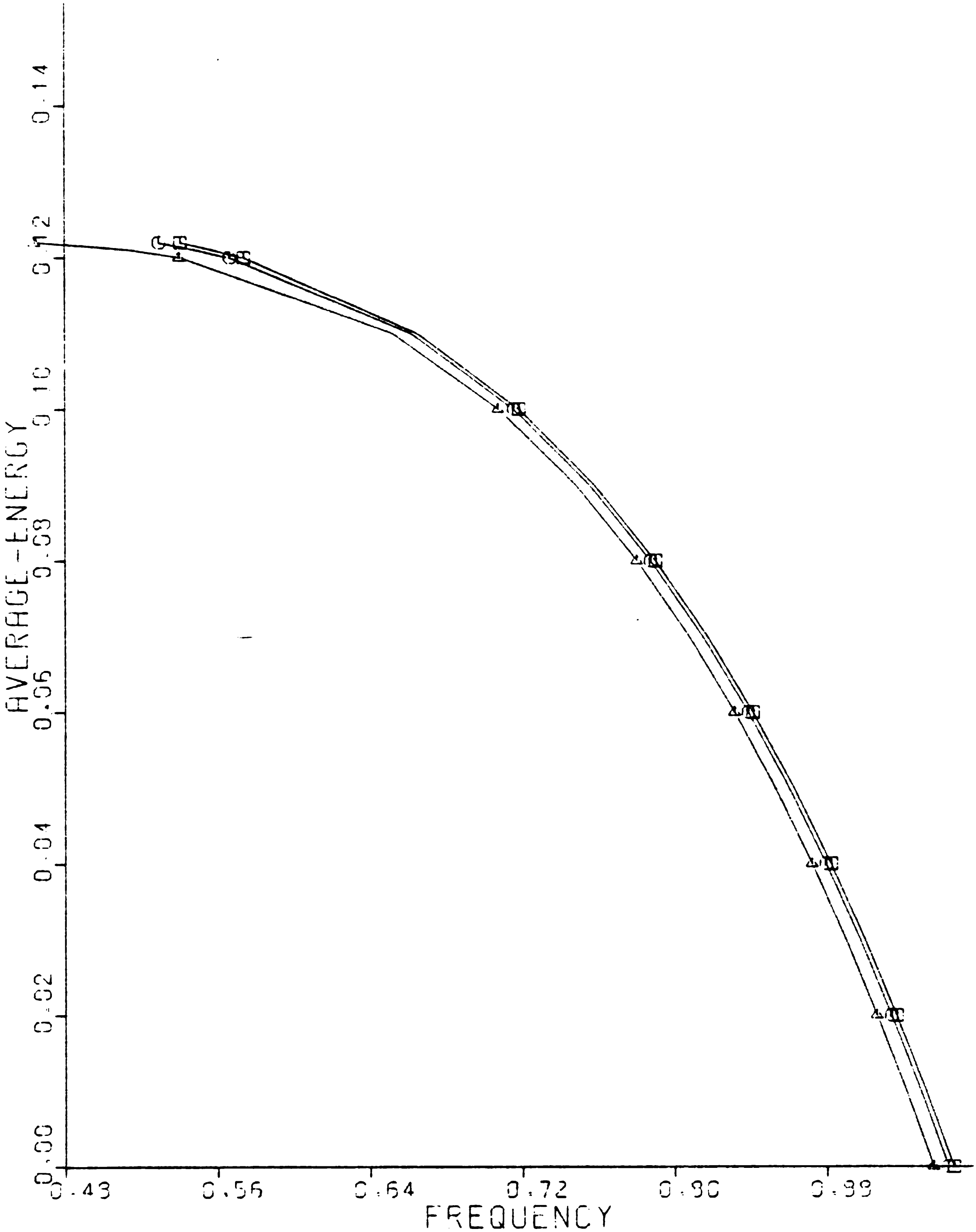

Figure 4.7 Average energy - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .896$ and $a_3 = -1.6$. —Δ— finite element solution $h = \pi/6$, —⊕— finite element solution $h = \pi/12$ and —+— exact solution.

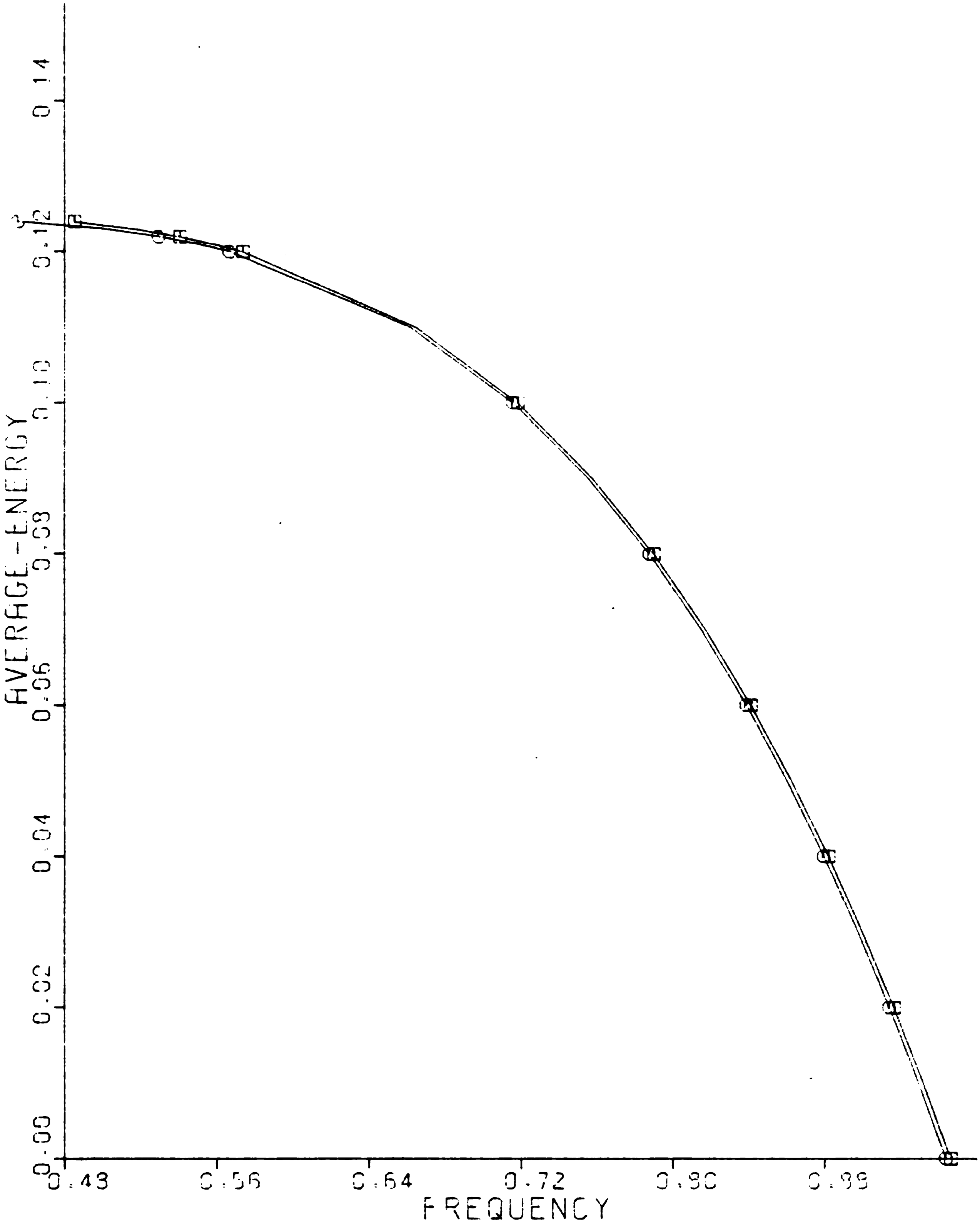

Figure 4.8 Average energy - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .896$ and $a_3 = -1.6$. exact solution and finite element solution $h = \pi/12$.

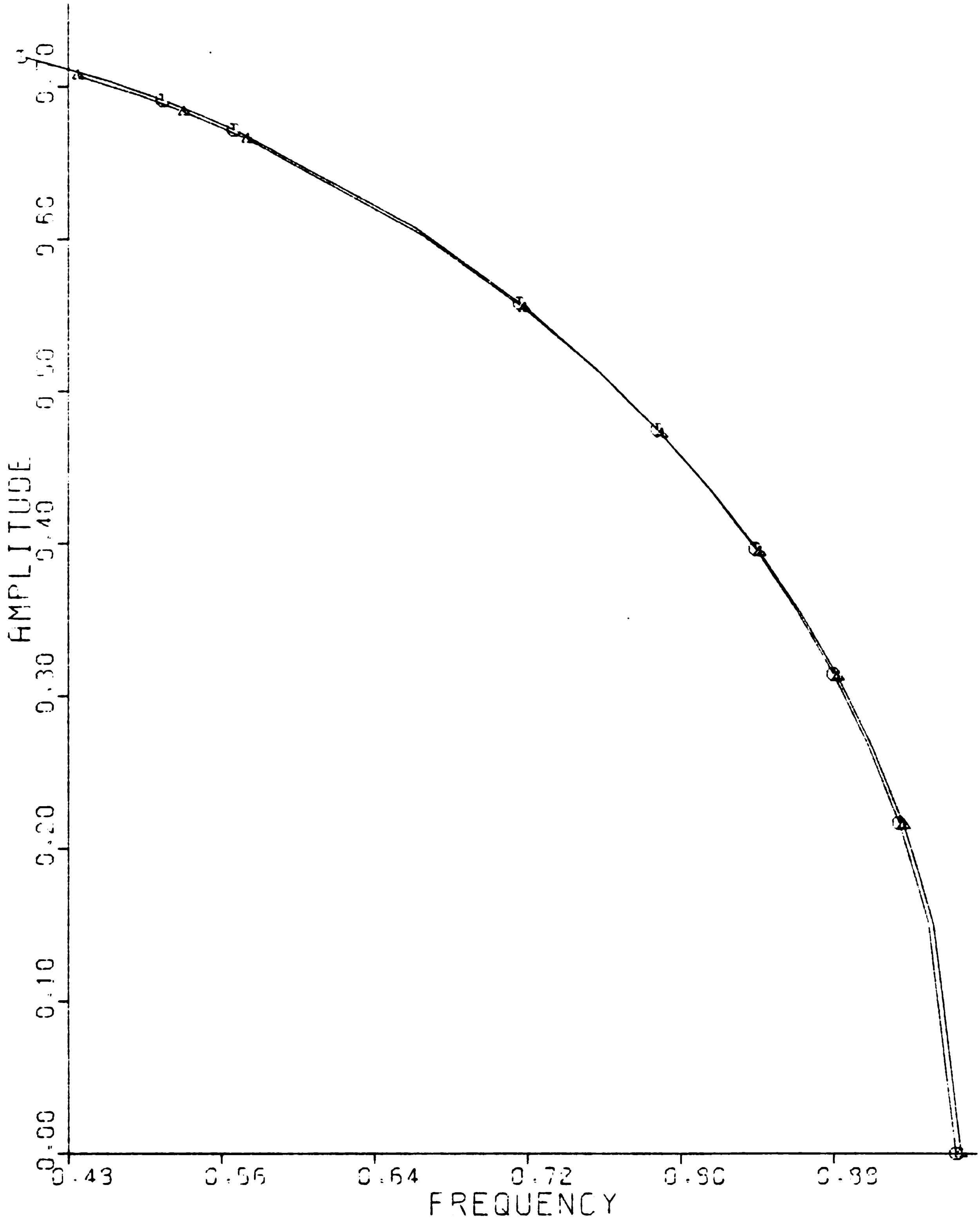

Figure 4.9 Amplitude-Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .896$ and $a_3 = -1.6$. —△— exact solution and —⊕— finite element solution $h = \pi/12$.

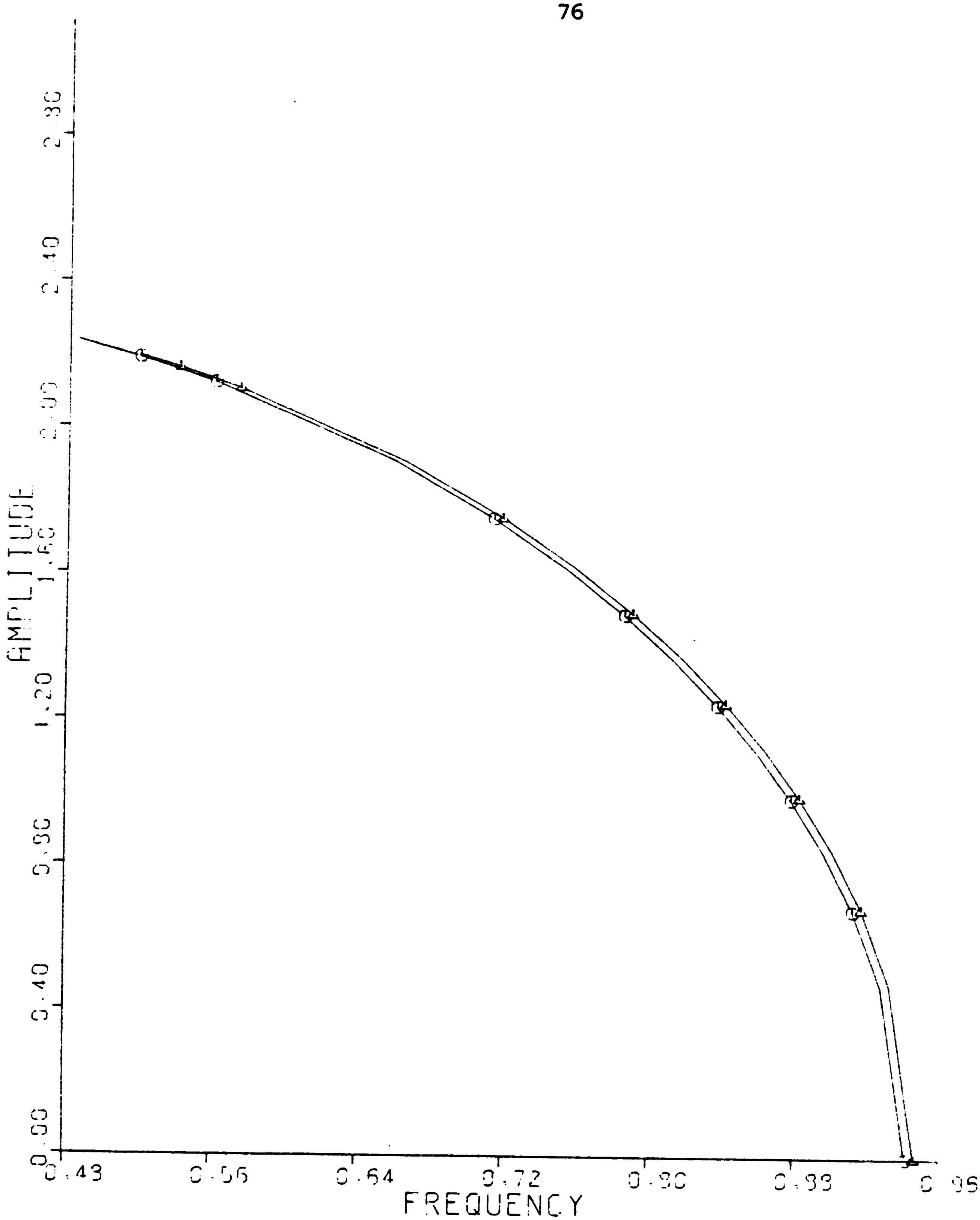

Figure 4.10 Amplitude - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .896$ and $a_3 = -.16$. —⊕— finite element solution $h = \pi/9$ and —△— exact solution.

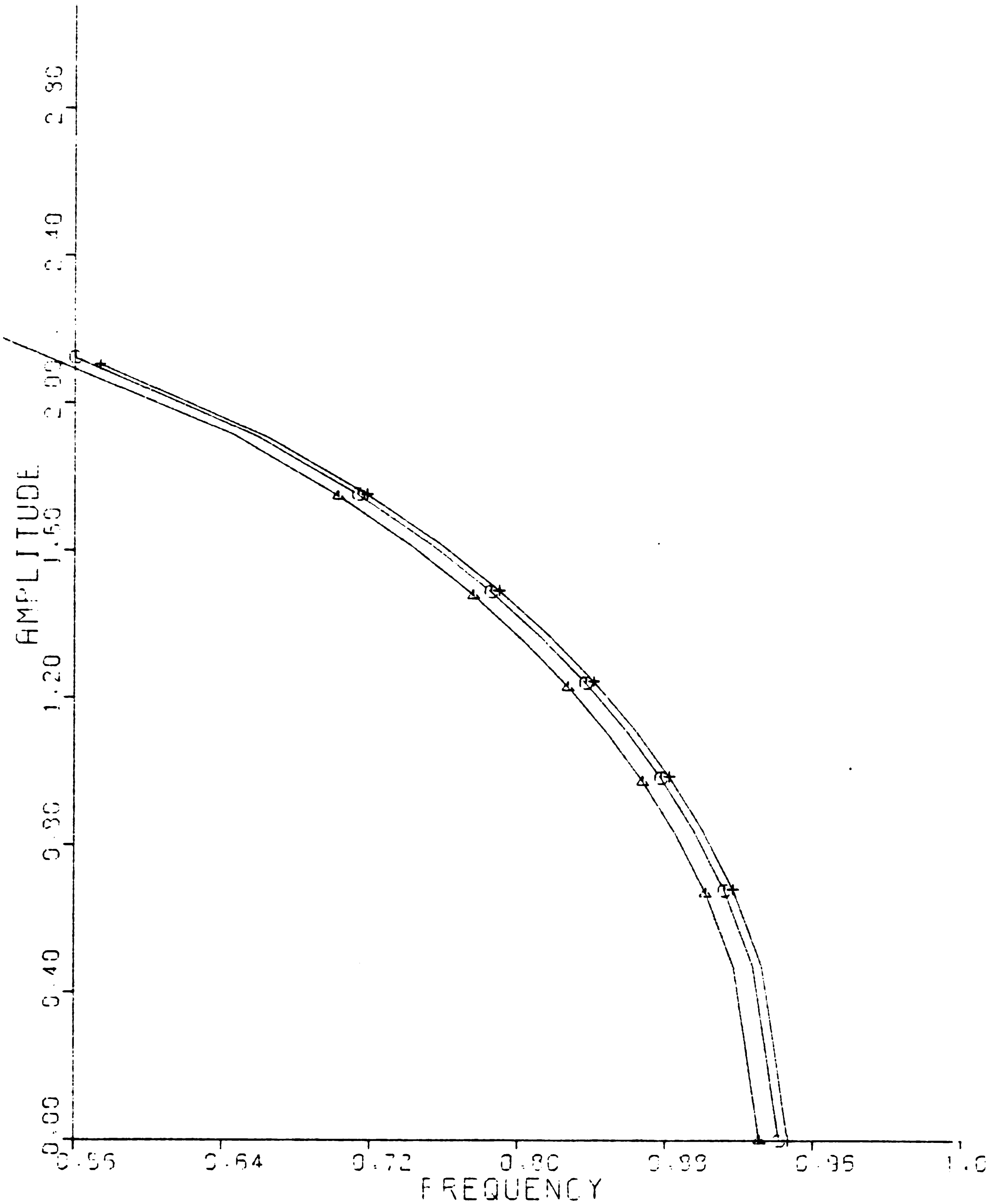

Figure 4.11 Amplitude - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .896$ and $a_3 = -.16$. —△— finite element solution $h = \pi/6$, —⊕— finite element solution $h = \pi/9$ and —+— exact solution.

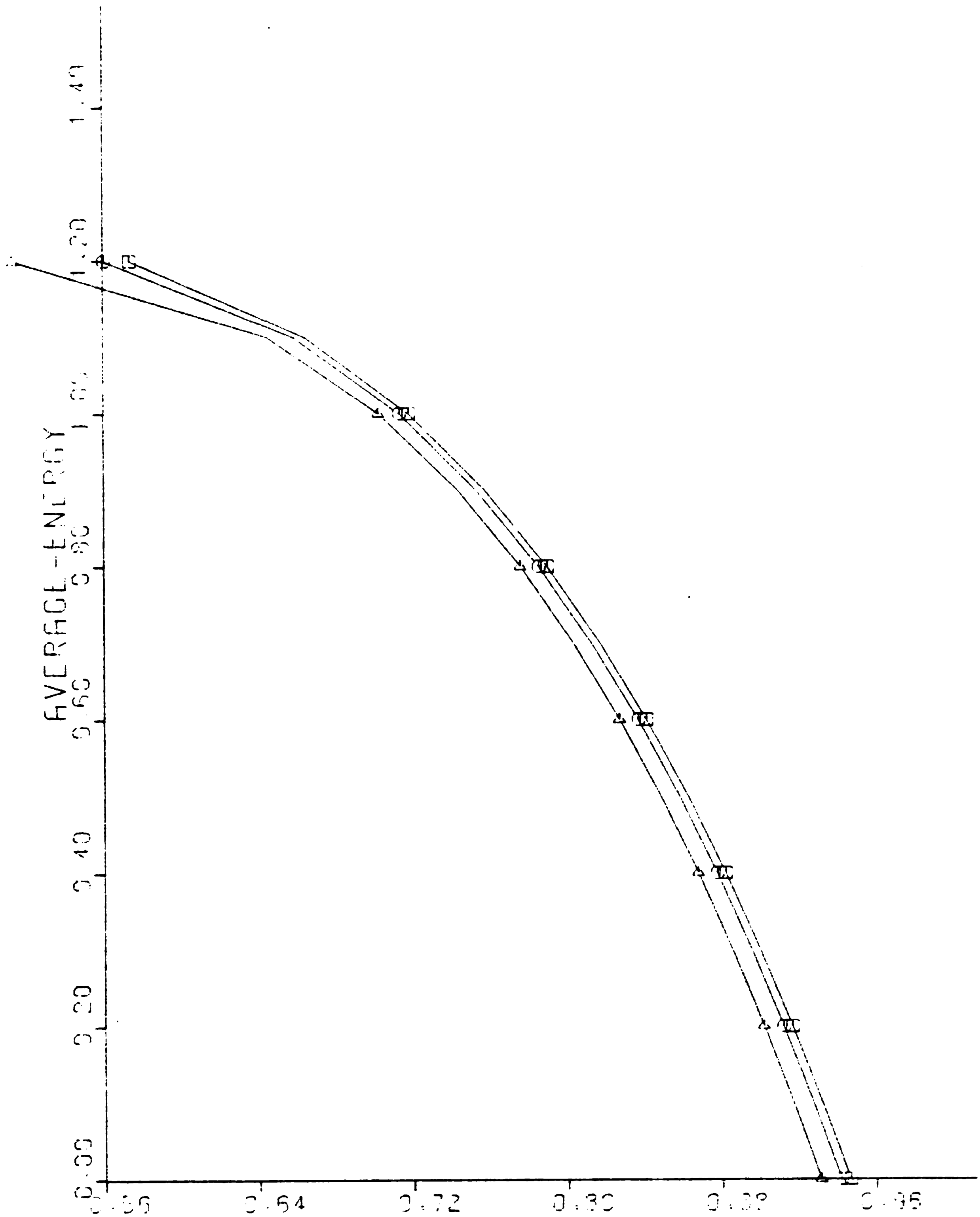

Figure 4.12 Average energy - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .869$ and $a_3 = -.16$. —△— finite element solution $h = \pi/5$, —⊕— finite element solution $h = \pi/9$ and —⊞— exact solution.

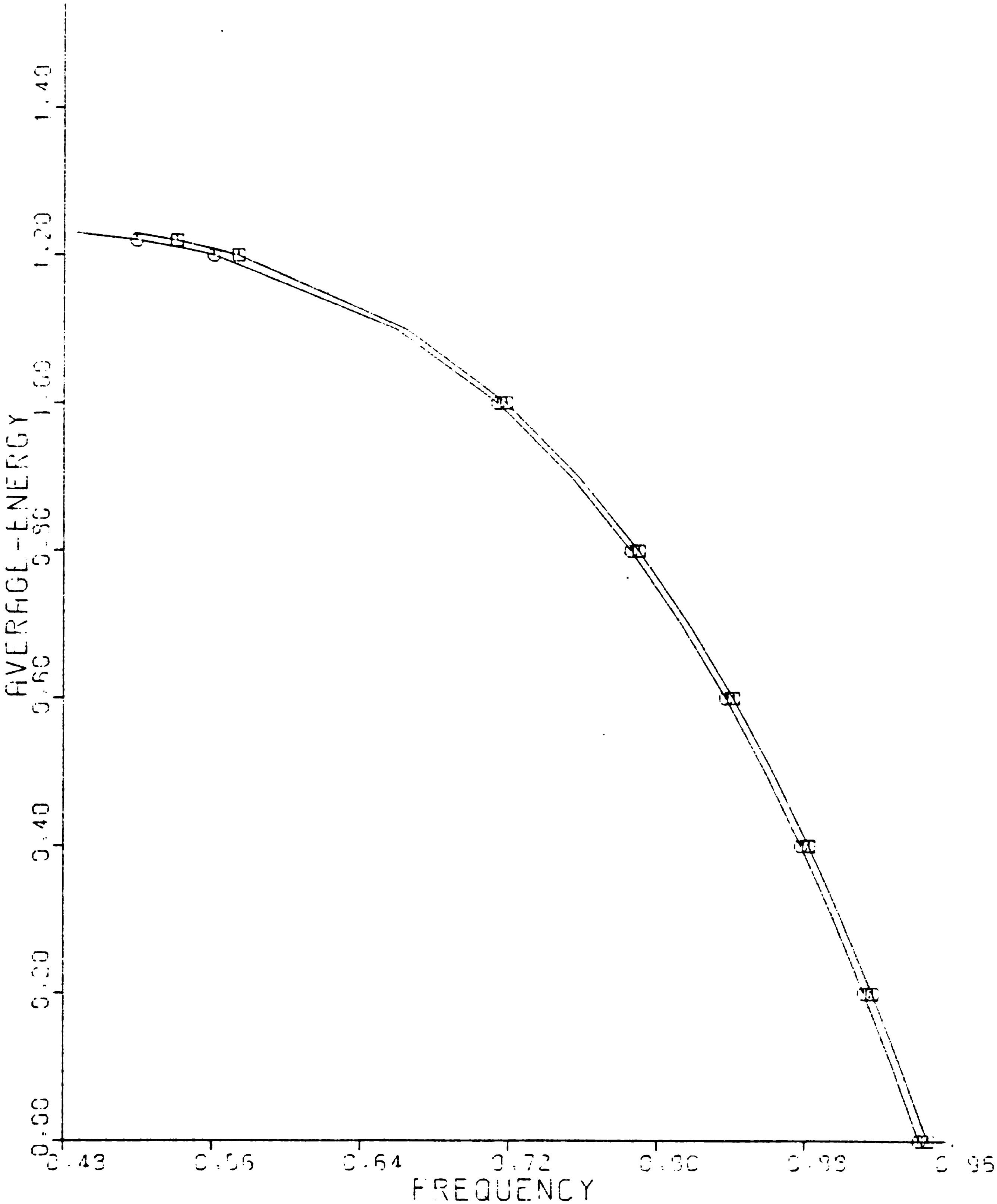

Figure 4.13 Average energy - Frequency relationships. One degree of freedom system. Soft spring. The spring constants: $a_1 = .896$ and $a_3 = -.16$. —⊕— finite element solution $h = \pi/9$ and —⊞— exact solution.

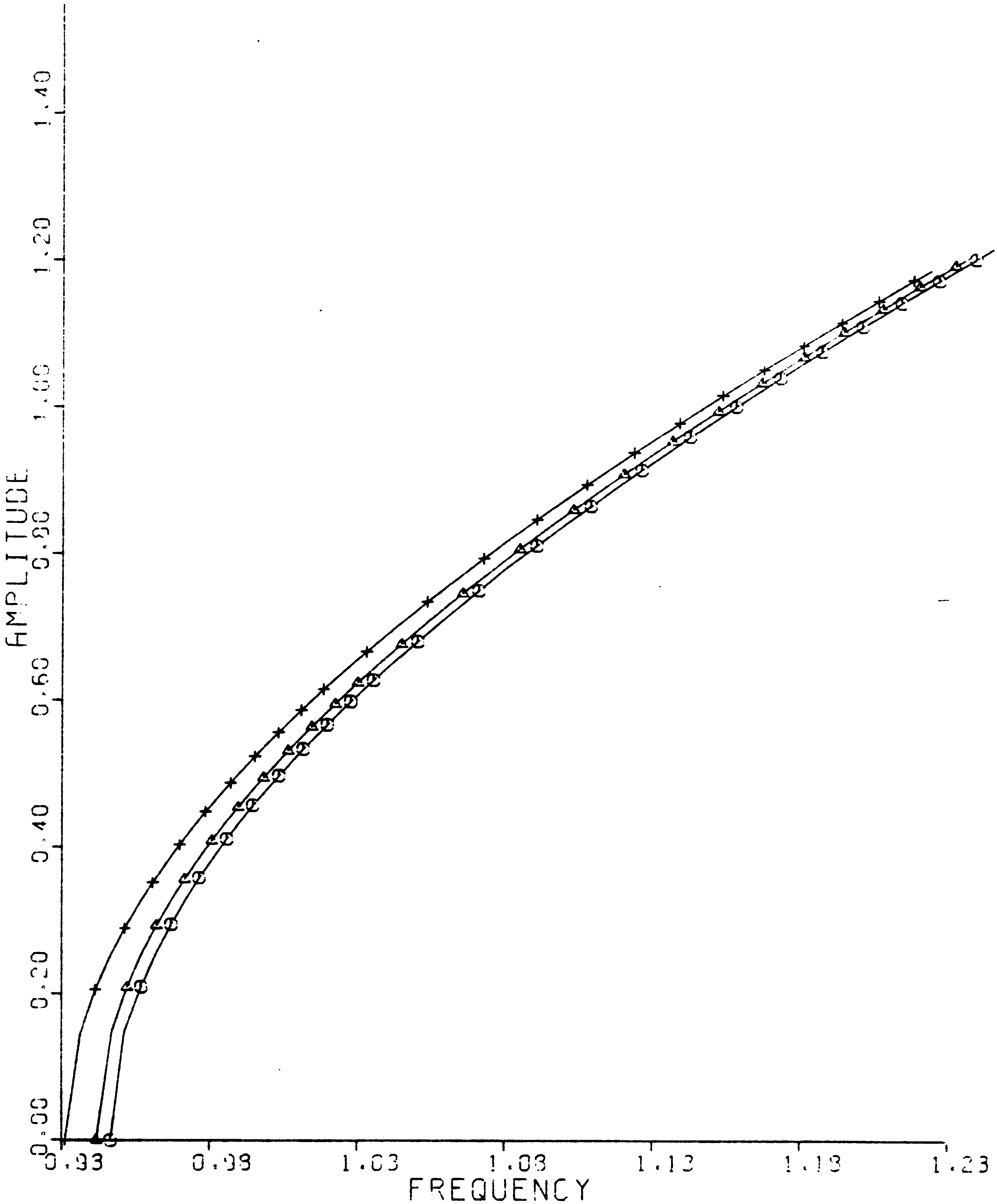

Figure 4.14 Amplitude - Frequency relationships. One degree of freedom system. Hard spring. The spring constants: $a_1 = .896$ and $a_3 = .6$. —+— finite element solution $h = \pi/5$, —Δ— finite element solution $h = \pi/9$ and —○— exact solution.

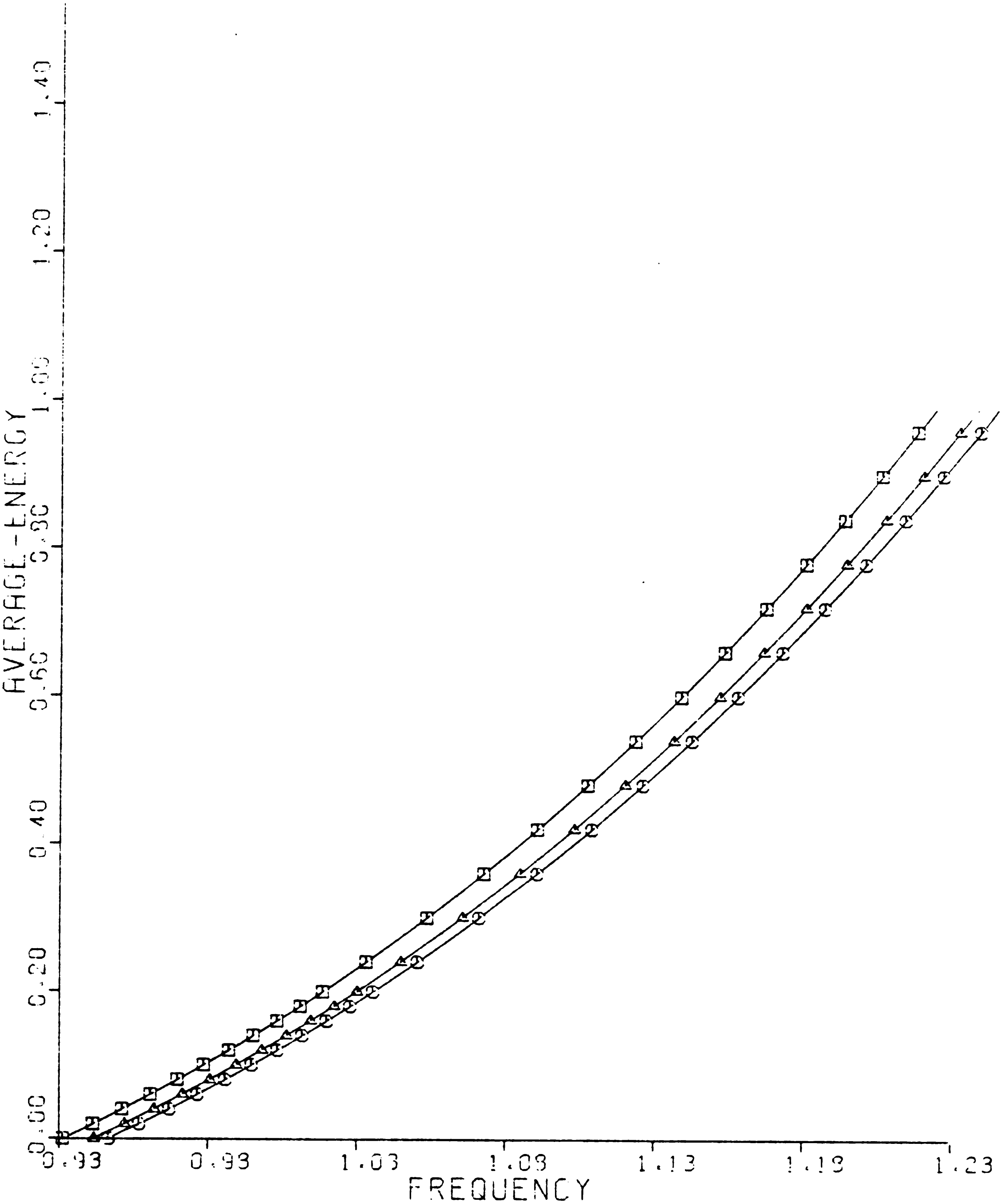

Figure 4.15 Average energy - Frequency relationships. One degree of freedom system. Hard spring. The spring constants: $a_1 = .896$ and $a_3 = .6$. —□— finite element solution $h = \pi/5$, —△— finite element solution $h = \pi/9$ and —○— exact solution.

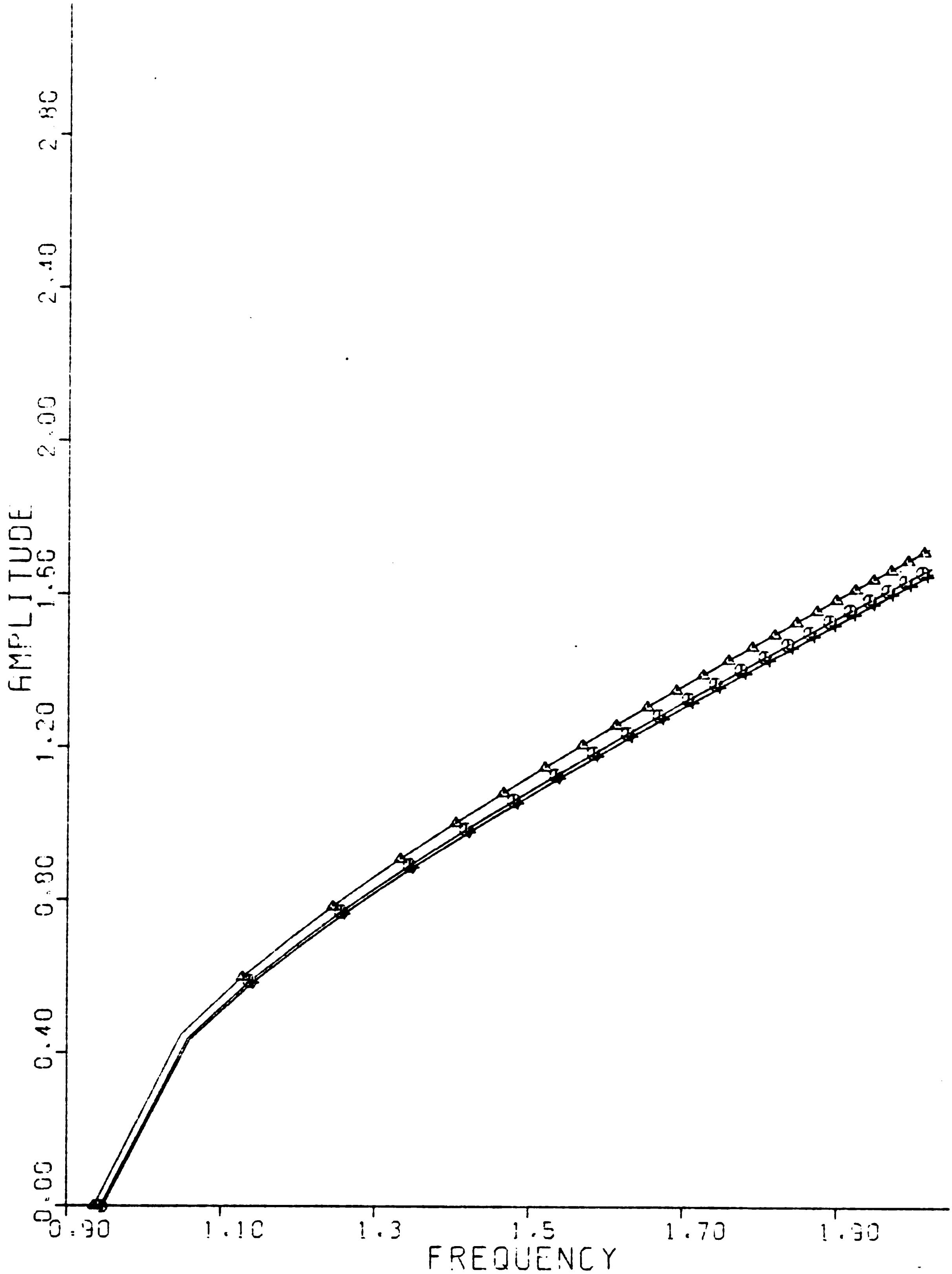

Figure 4.16 Amplitude - Frequency relationships. One degree of freedom system. Hard spring. The spring constants: $a_1 = .896$ and $a_3 = 1.6$. —△— finite element solution $h = \pi/6$, —⊕— finite element solution $h = \pi/12$ and —+— exact solution.

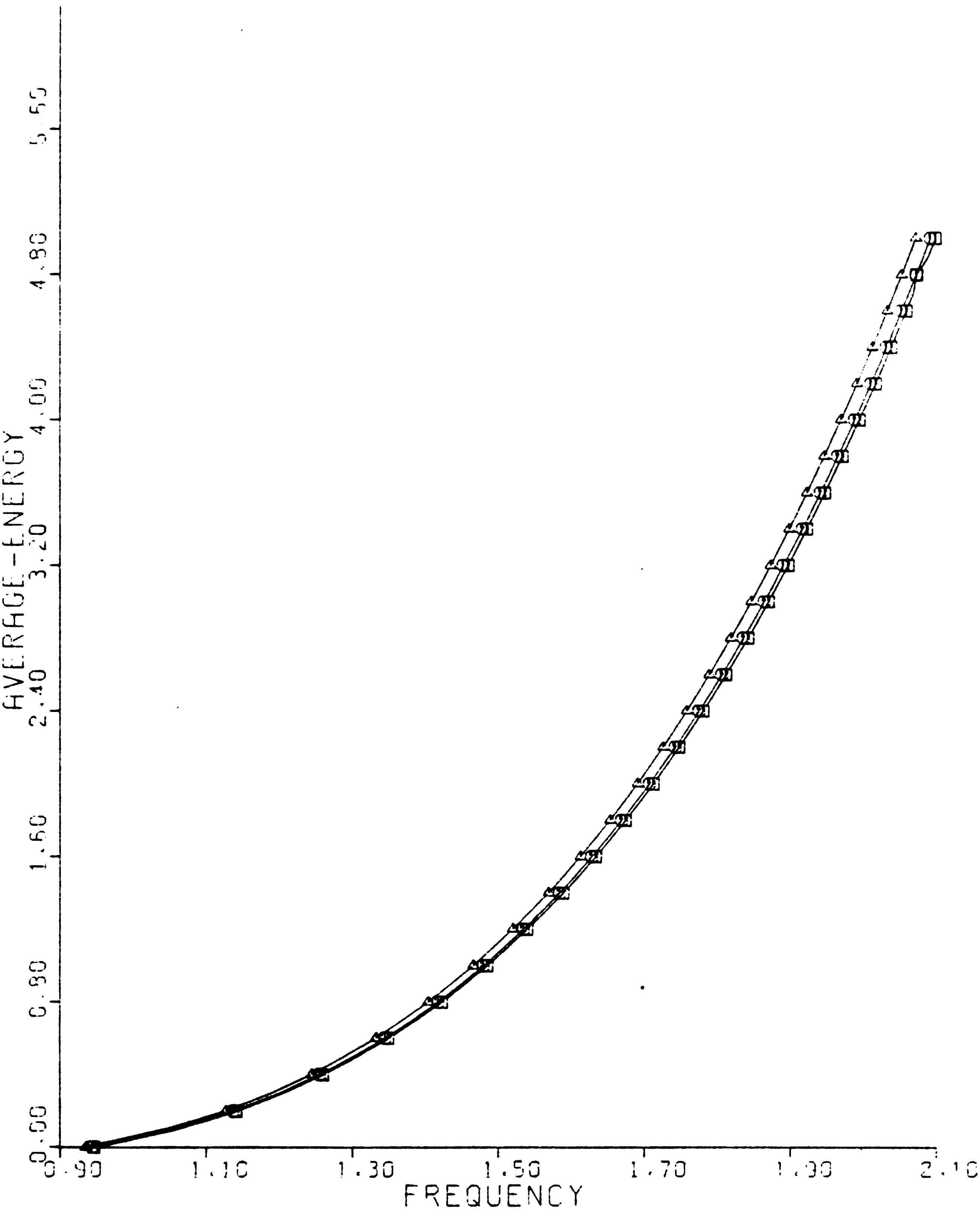

Figure 4.17 Average energy - Frequency relationships. One degree of freedom system. Hard spring constants: $a_1 = .896$ and $a_3 = 1.6$. —△— finite element solution $h = \pi/6$, —○— finite element solution $h = \pi/12$ and —□— exact solution.

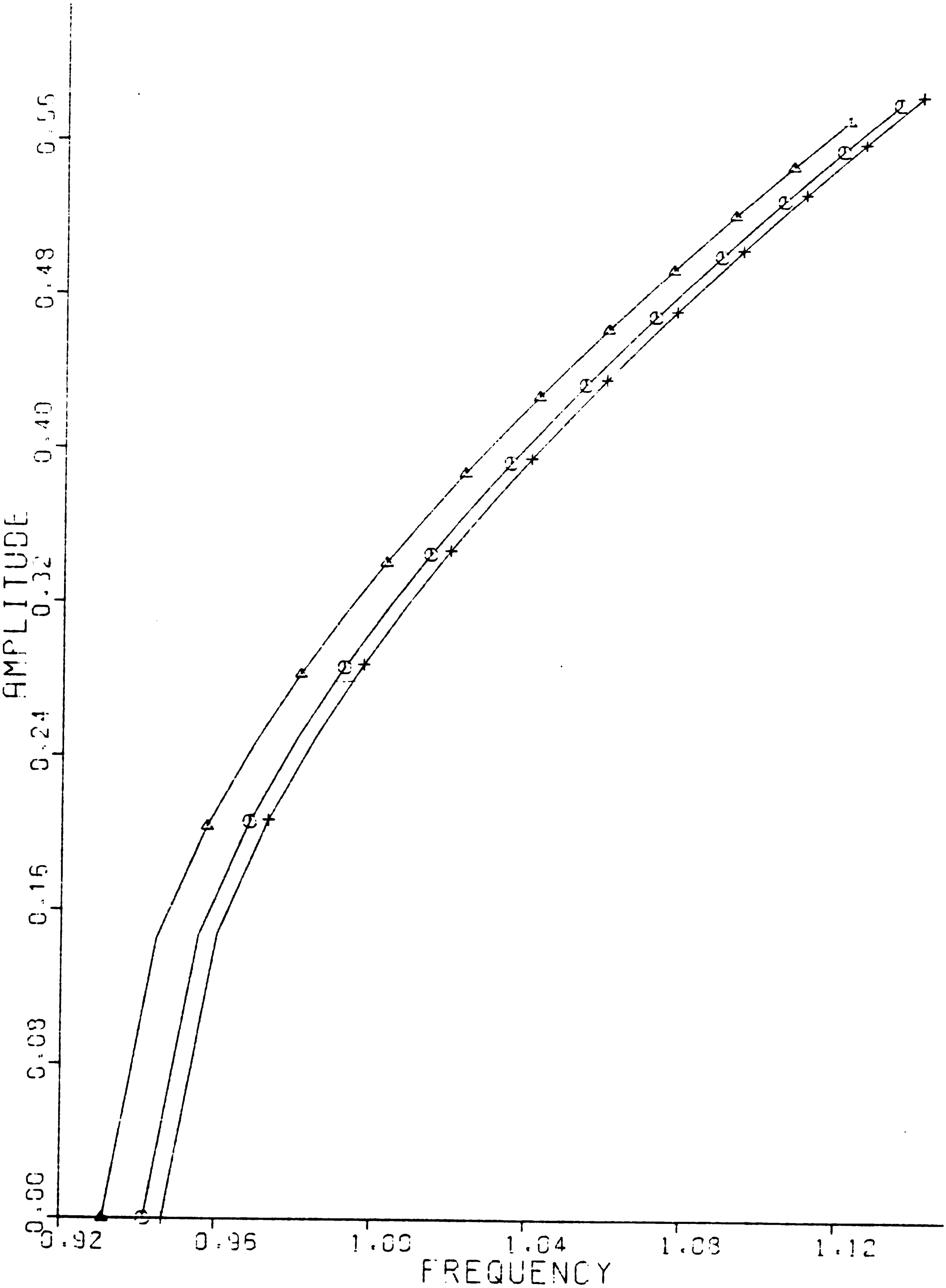

Figure 4.18 Amplitude - Frequency relationships. One degree of freedom system. Hard spring. The spring constants: $a_1 = .896$ and $a_3 = 1.6$. —△— finite element solution $h = \pi/5$, —⊕— finite element solution $h = \pi/9$ and —+— exact solution.

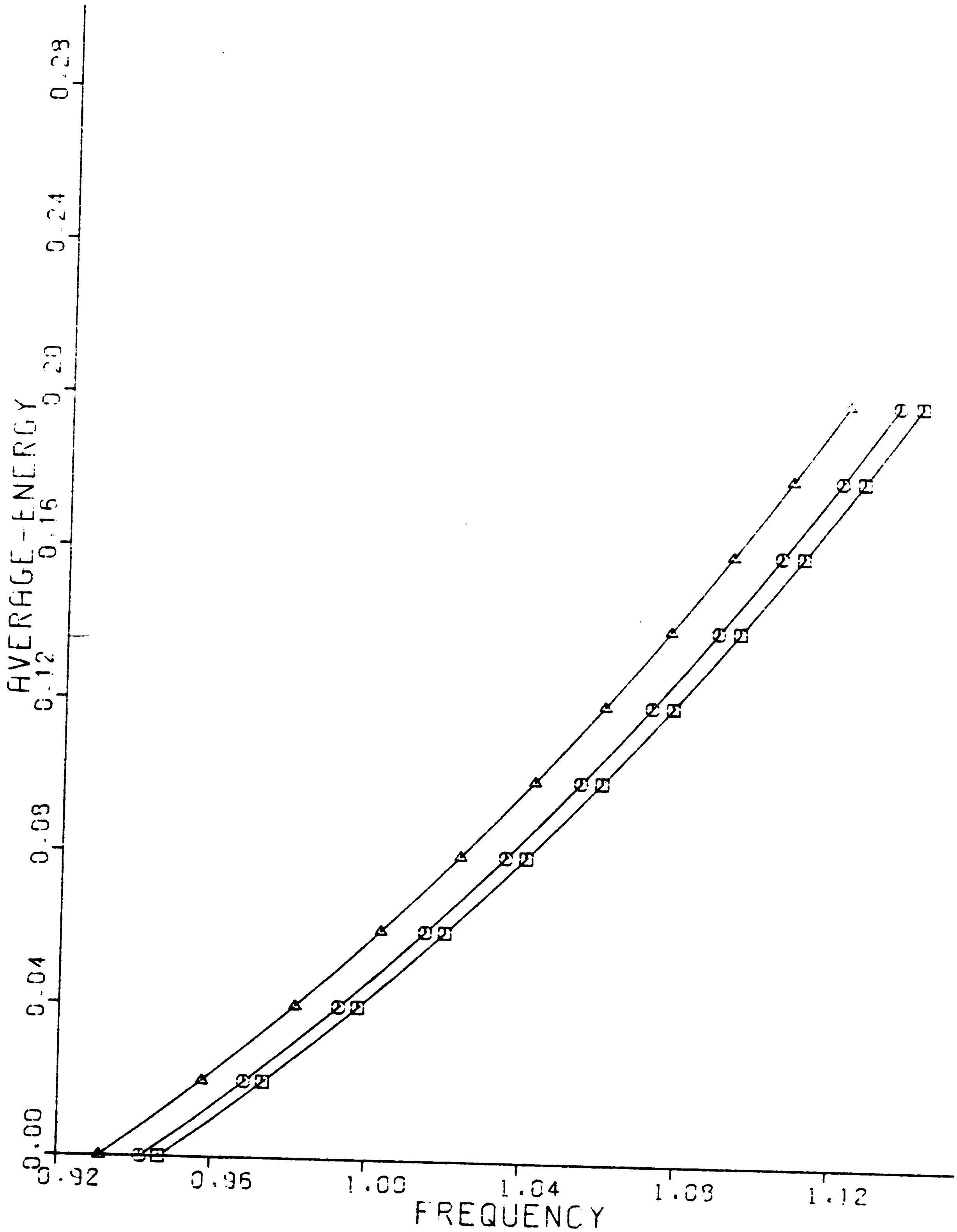

Figure 4.19 One degree of freedom system. Hard spring. The spring constants: $a_1 = .896$ and $a_3 = 1.6$. —△— finite element solution $h = \pi/5$, —○— finite element solution $h = \pi/9$ and —□— exact solution.

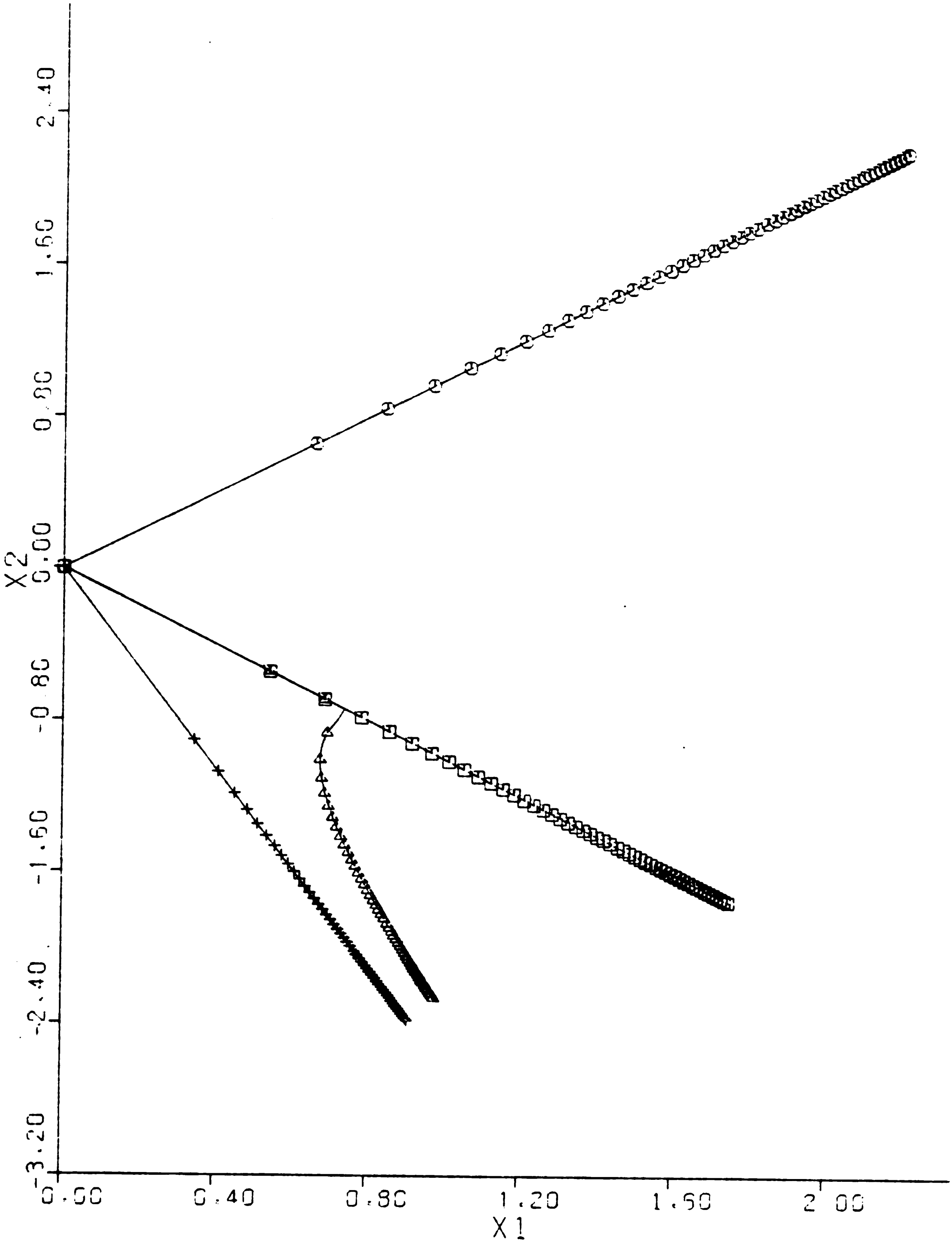

Figure 4.24 Loci of points on $V(x_1,x_2) = E$ corresponding to normal modes. Two degrees of freedom symmetric system. Finite element solutions $(h = \pi/3)$. $(a_1 = b_1 = .896,\ A_1 = .1536,$ $a_3 = b_3 = 1.6$ and $A_3 = .32)$. Symmetric mode, Anti-symmetric mode, Superabundant mode, Normal mode of the homogeneous system.

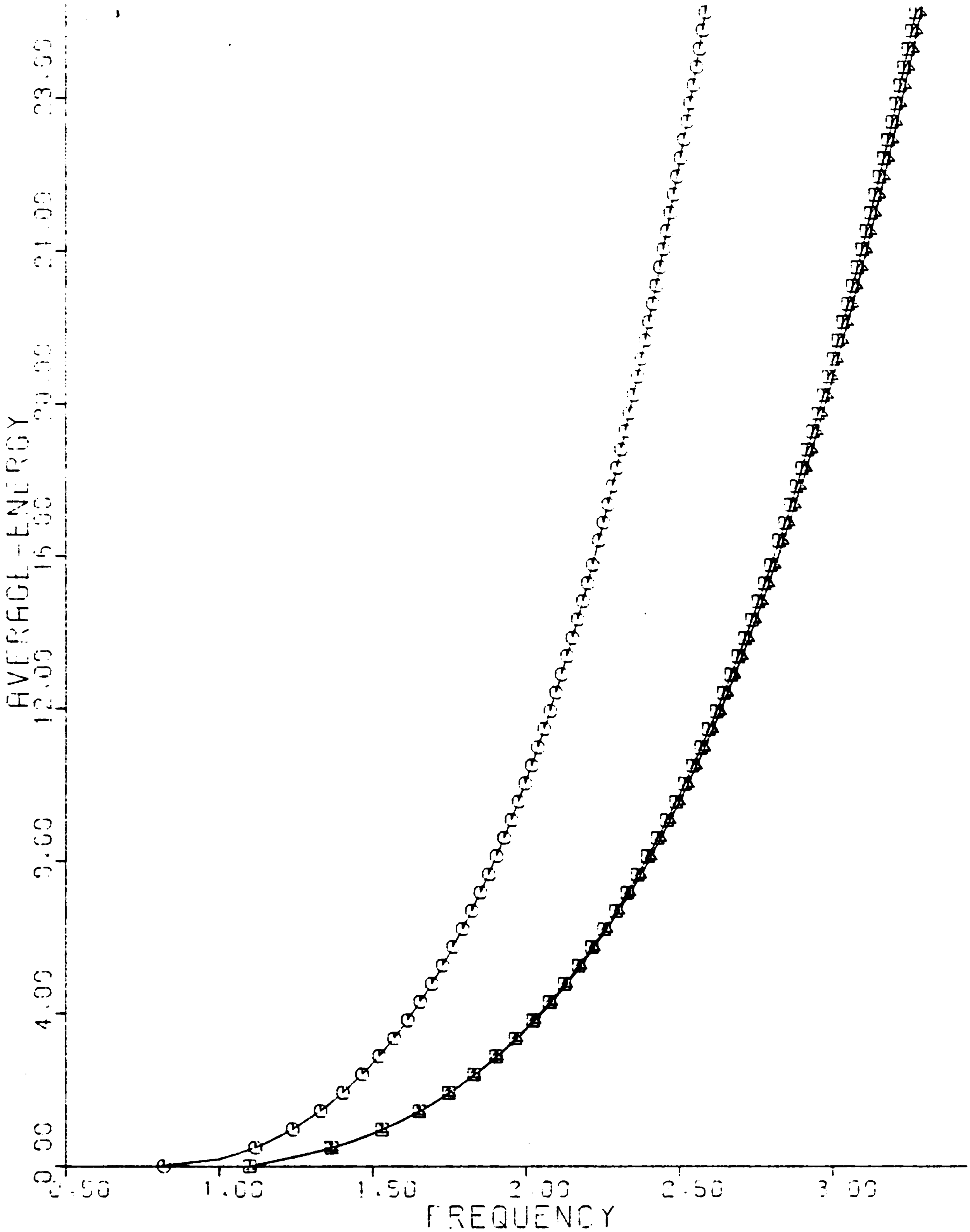

Figure 4.25 Average energy - Frequency relationships. Two degrees of freedom symmetric system. Finite element solutions ($h = \pi/3$). ($a_1 = b_1 = .896$, $A_1 = .1536$, $a_3 = b_3 = 1.6$ and $A_3 = 32$). Symmetric mode, Antisymmetric mode and Superabundant mode.

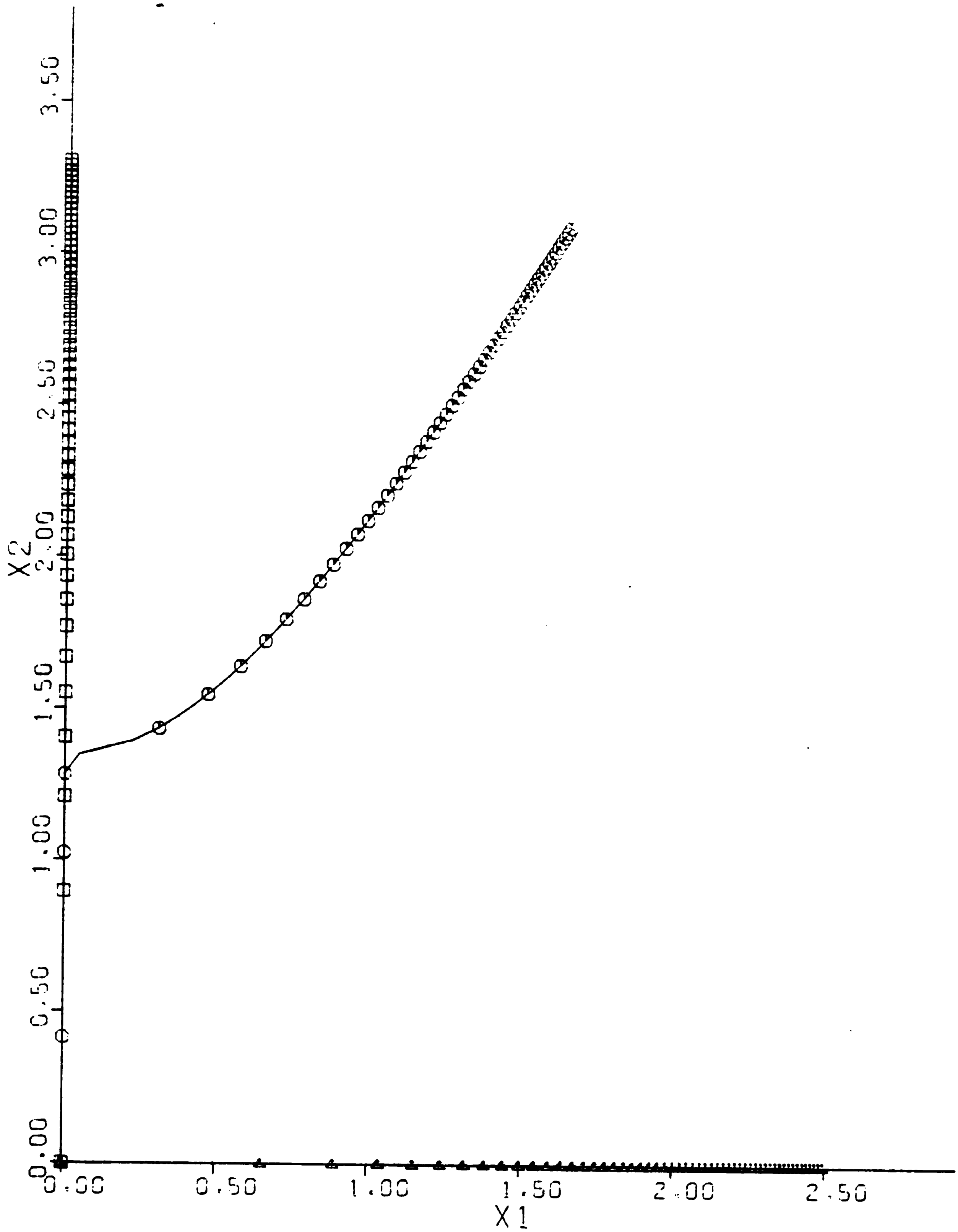

Figure 4.26 Loci of points on $V(x_1, x_2) = E$ that correspond to normal modes. Two degrees of freedom system. Finite element solutions ($h = \pi/5$). ($a_1 = .8$, $b_1 = .4$, $A_1 = 0$, $a_3 = .3$, $b_3 = .1$ and $A_3 = 0$). $x_1 = 0$, $x_2 = 0$ and superabundant mode.

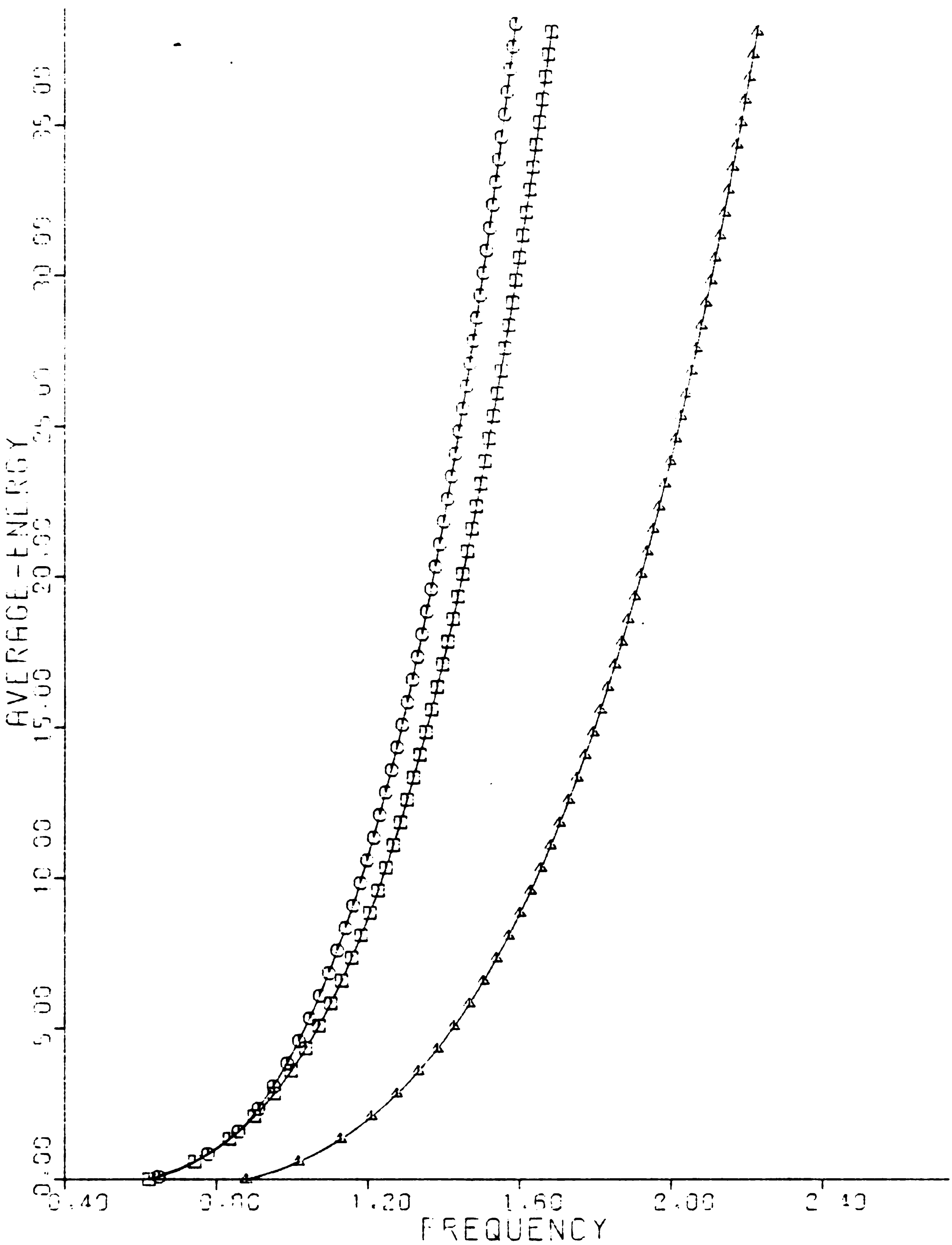

Figure 4.27 Average energy - Frequency relationships. Two degrees of freedom system. Finite element solutions ($h = \pi/5$). ($a_1 = .8$, $b_1 = .4$, $A_1 = 0$, $a_3 = .3$, $b_3 = .1$ and $A_3 = 0$). ⊞ $x_1 = 0$, Δ $x_2 = 0$ and ⊖ superabundant mode.

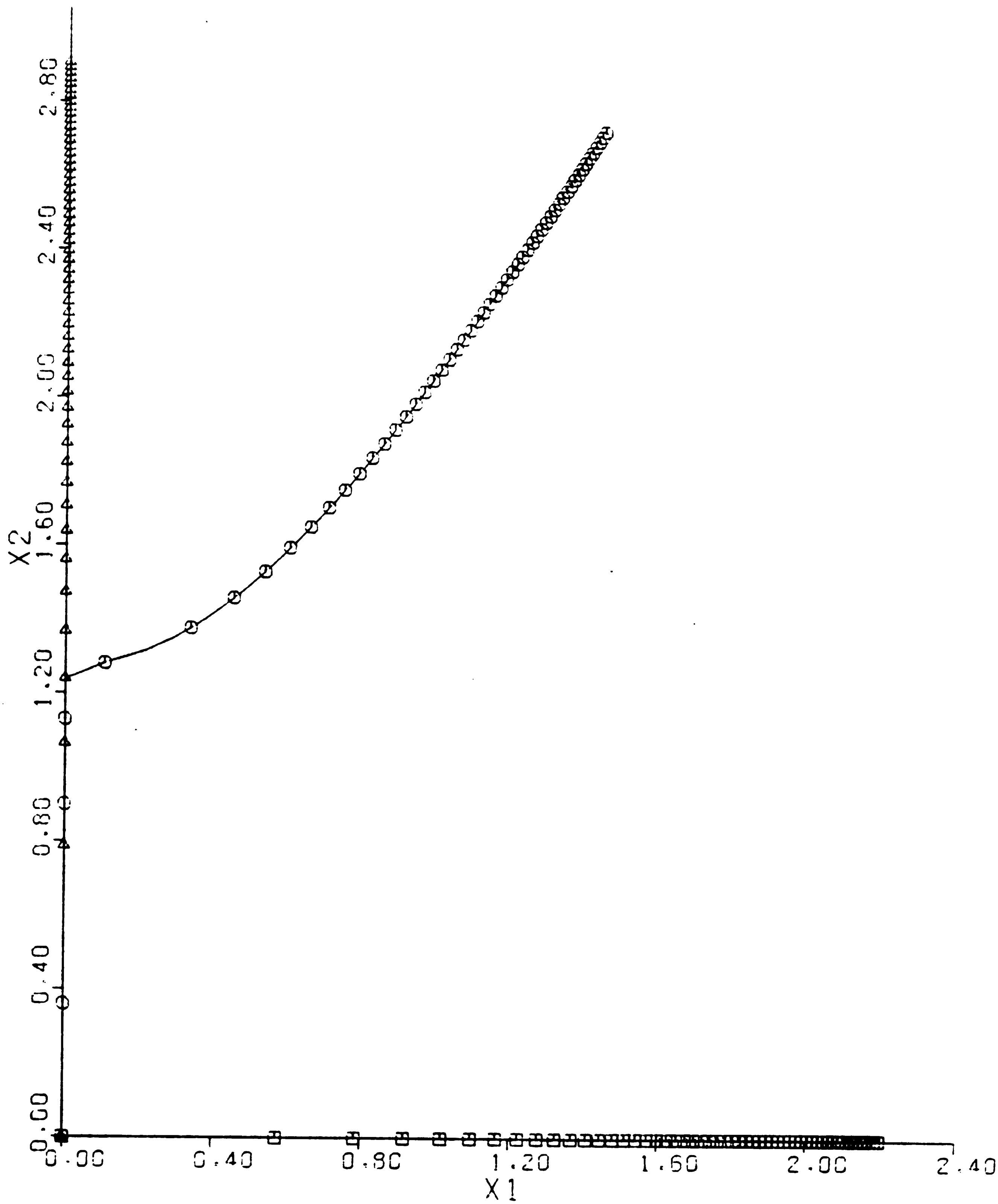

Figure 4.28 Loci of points on $V(x_1, x_2) = E$ that correspond to normal modes. Two degrees of freedom system. Finite difference solutions $(h = \pi/5)$. ($a_1 = .8$, $b_1 = .4$, $A_1 = 0$, $a_3 = .3$, $b_3 = .1$ and $A_3 = 0$). △ $x_1 = 0$, ⊞ $x_2 = 0$ and ⊕ superabundant mode.

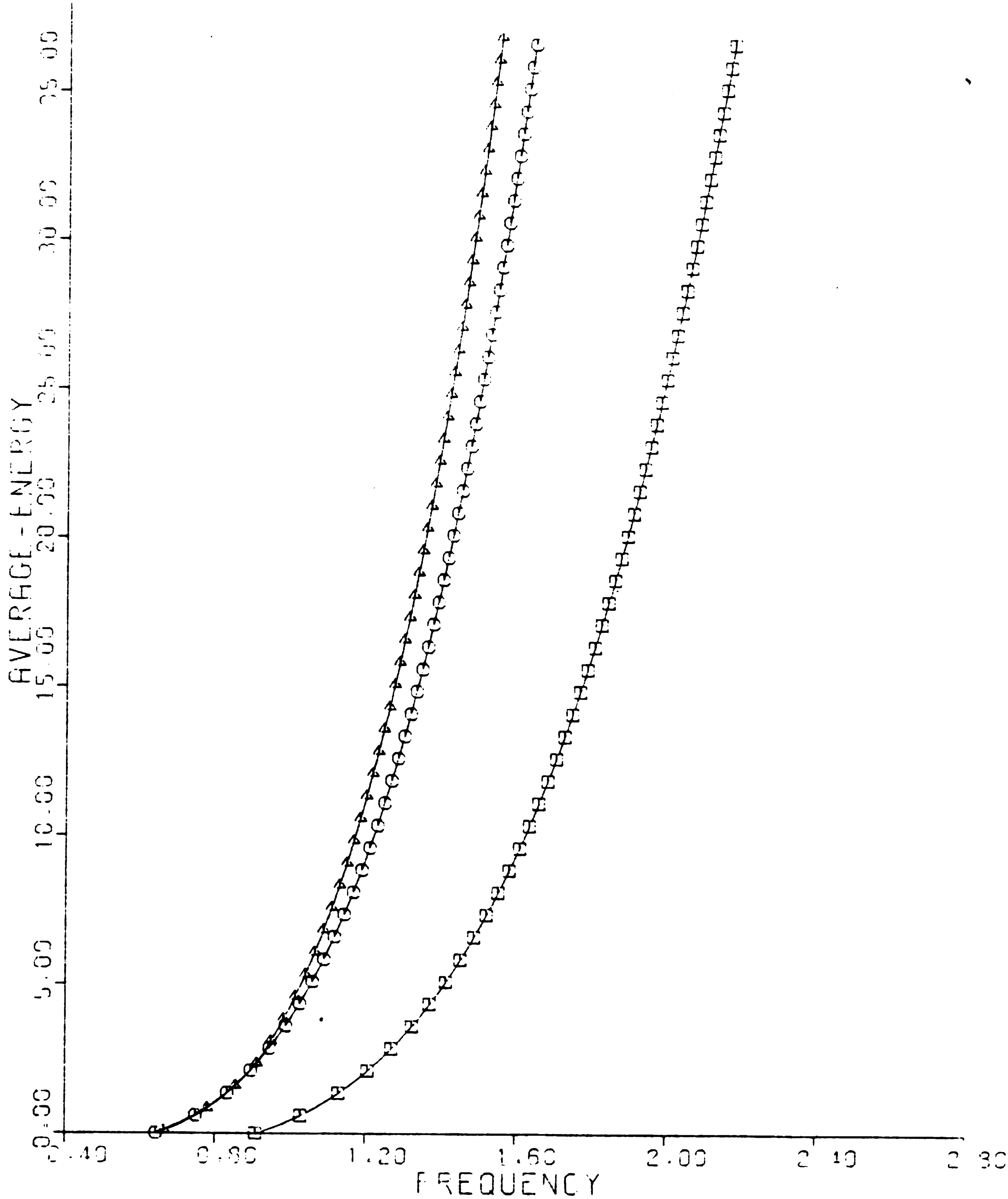

Figure 4.29 Average energy - Frequency relationships. Two degrees of freedom system. Finite difference solution ($h = \pi/5$). ($a_1 = .8$, $b_1 = .4$, $A_1 = 0$, $a_3 = .3$, $b_3 = .1$ and $A_3 = 0$). $x_2 = 0$, superabundant mode and $x_1 = 0$.

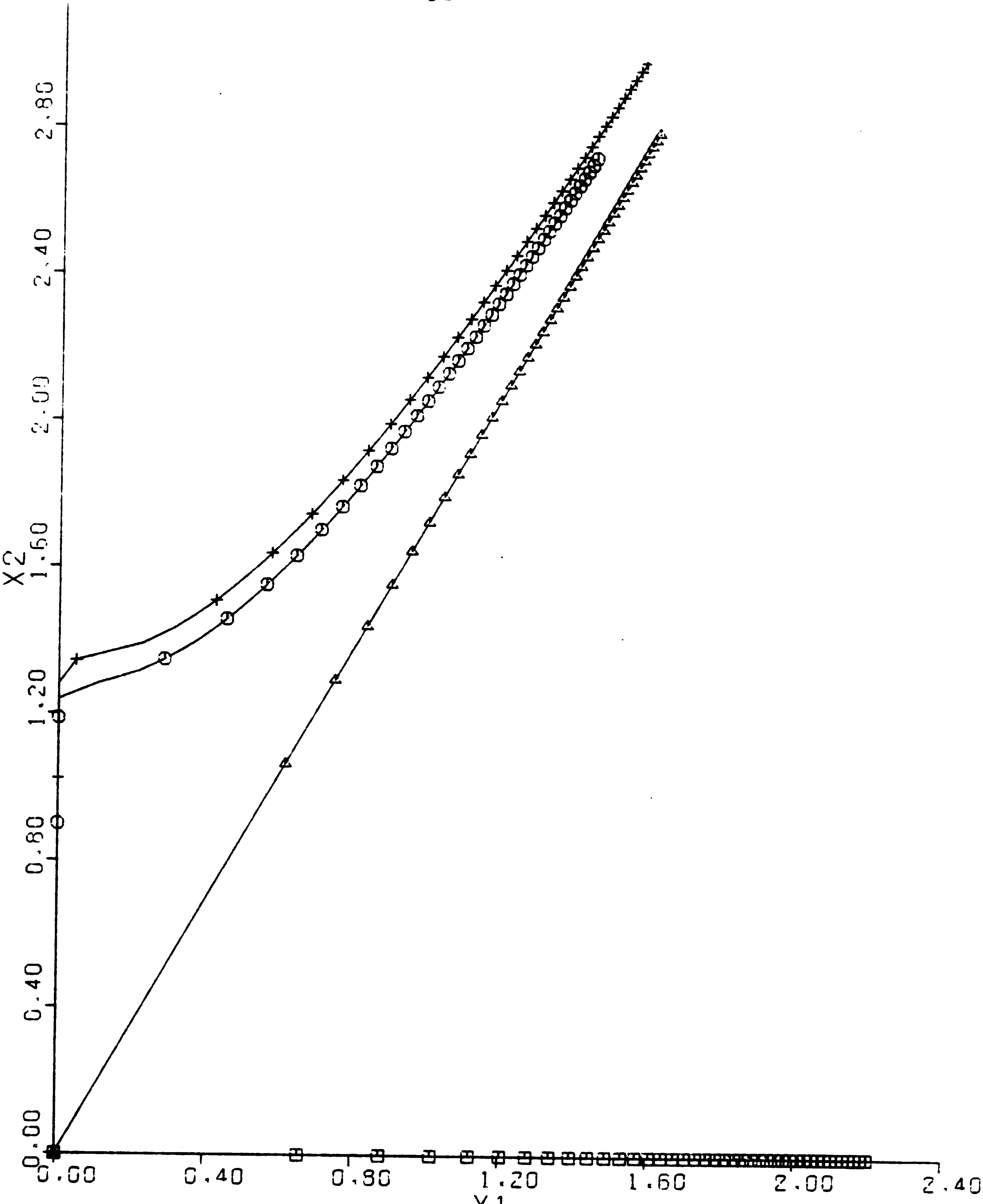

Figure 4.30 Loci of points on $V(x_1, x_2) = E$ that correspond to normal modes. Two degrees of freedom system. Finite difference and finite element solutions ($h = \pi/5$). ($a_1 = .8$, $b_1 = .4$, $A_1 = 0$, $a_3 = .3$, $b_3 = .1$ and $A_3 = 0$).
—⊡— $x_2 = 0$, —⊕— Finite difference superabundant mode, —+— finite element superabundant mode and —△—normal mode of the homogeneous system.

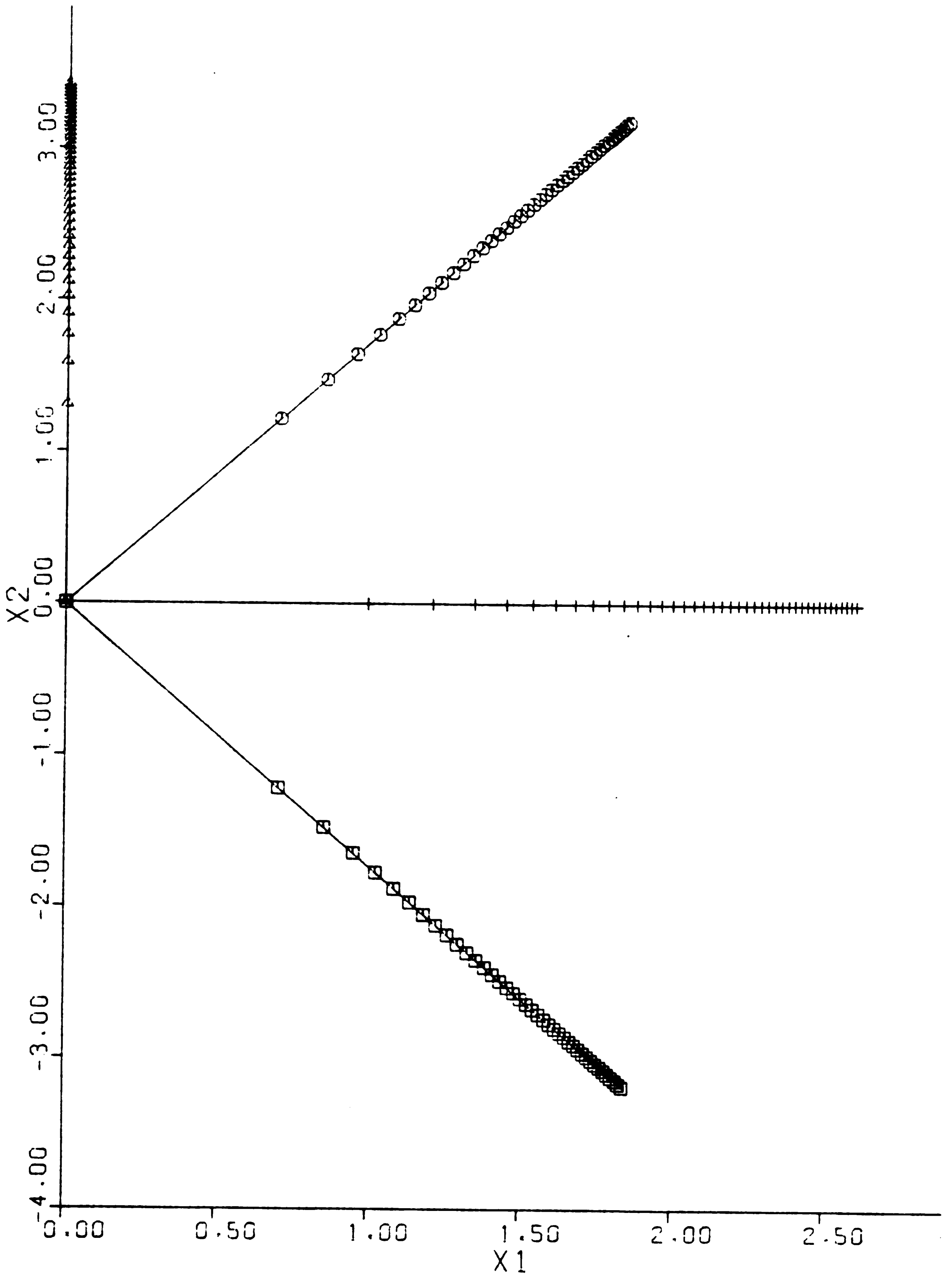

Figure 4.31 Loci of points on $V(x_1, x_2) = E$ that correspond to normal modes. Two degrees of freedom homogeneous system. Finite element and finite difference solutions $(h = \pi/5)$. $(a_1 = b_1 = A_1 = A_3 = 0,\ a_3 = .3$ and $b_3 = .1)$. —△— $x_1 = 0$, —+— $x_2 = 0$, —○— $x_2 = \sqrt{3}\, x_1$ and —□— $x_2 = -\sqrt{3}\, x_1$.

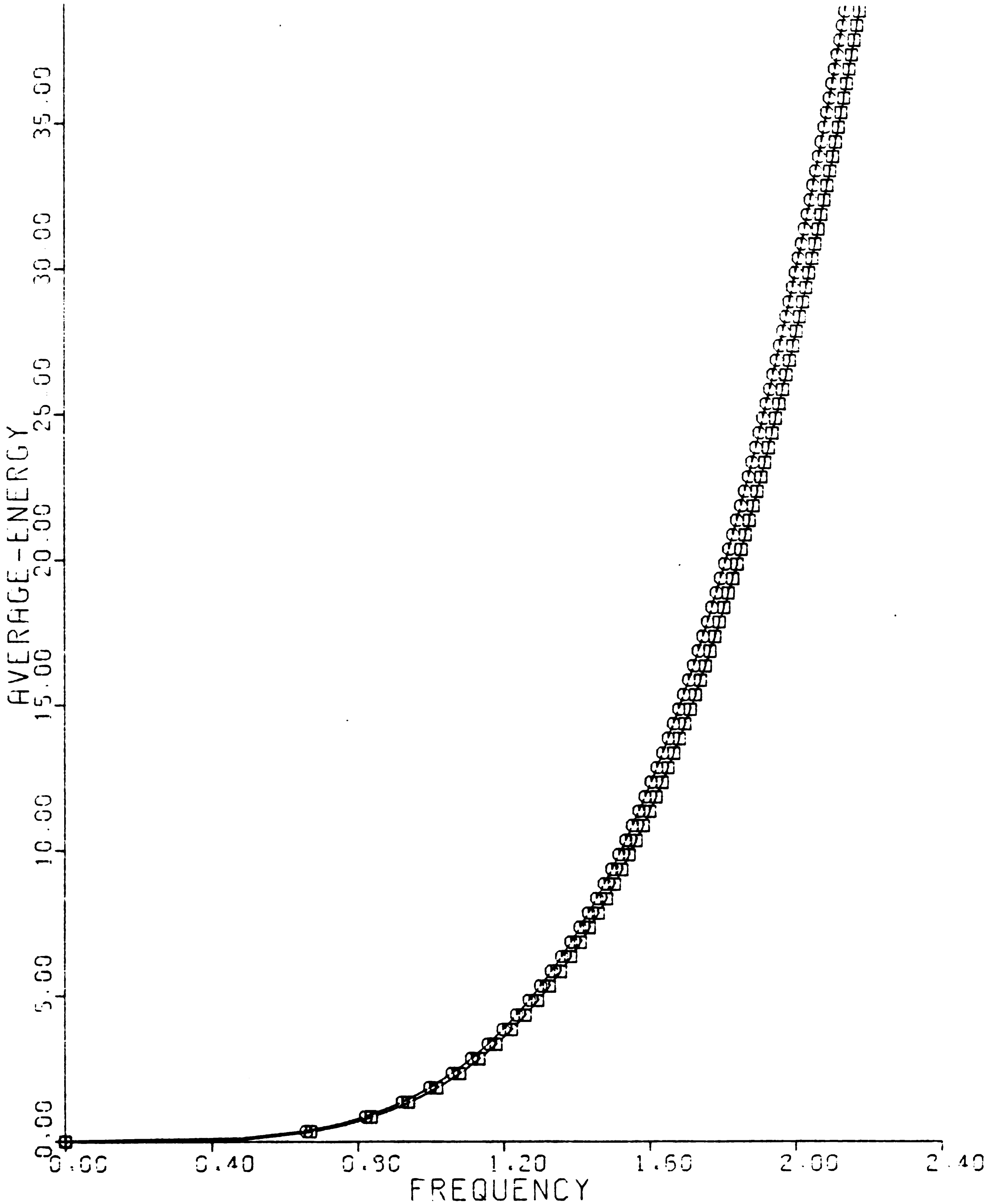

Figure 4.32 Average energy - Frequency relationships. Two degrees of freedom homogeneous system. Finite element and finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = A_1 = A_3 = 0$, $a_3 = .3$ and $b_3 = .1$). —⊕— $x_2 = 0$ finite difference and —⊞— $x_2 = 0$ finite element.

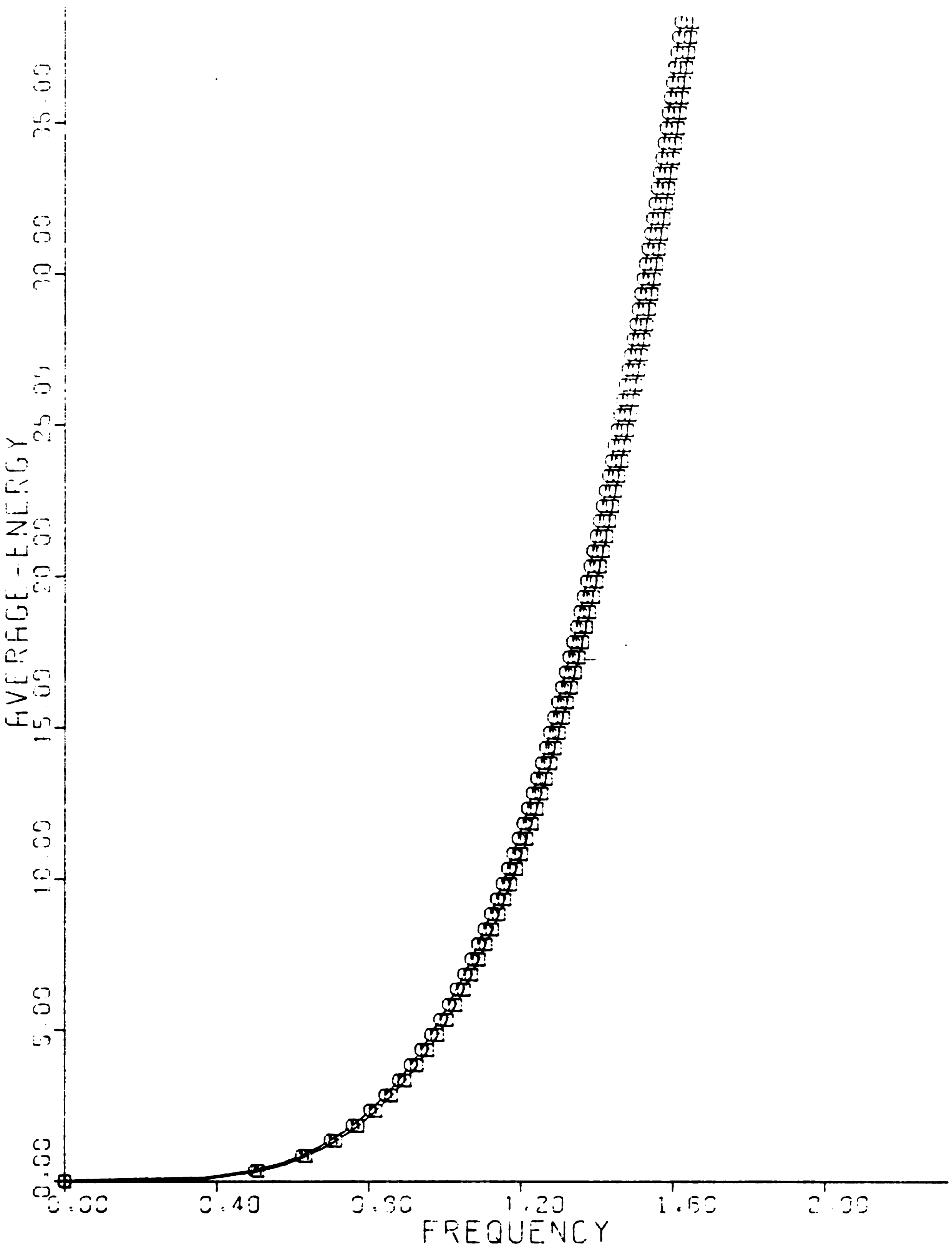

Figure 4.33 Average energy - Frequency relationships. Two degrees of freedom homogeneous system. Finite element and finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = A_1 = A_3 = 0$, $a_3 = .3$ and $b_3 = .1$). —⊕— $x_1 = 0$ finite difference and —⊞— $x_1 = 0$ finite element.

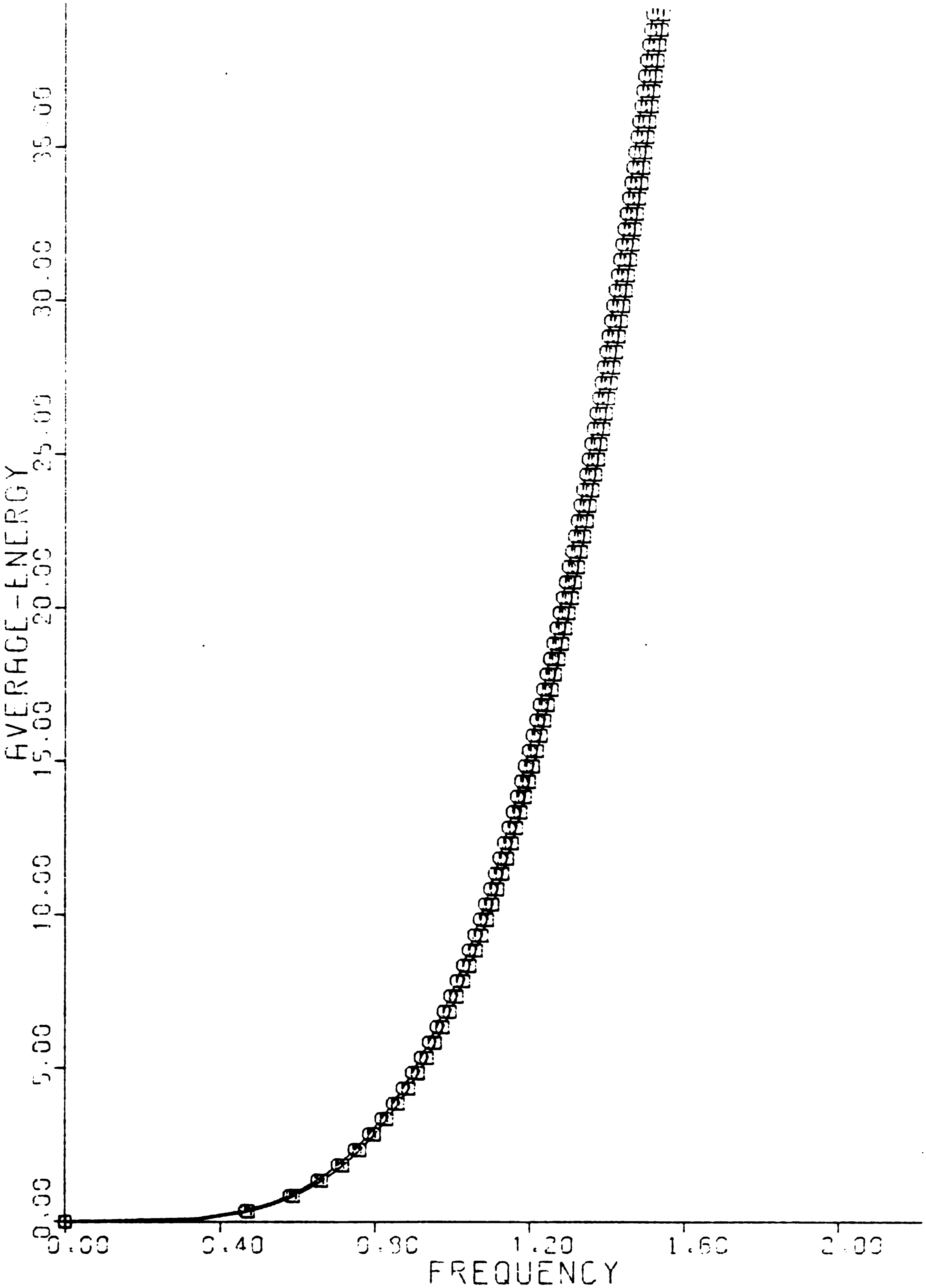

Figure 4.34 Two degrees of freedom homogeneous system. Finite element and finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = A_1 = A_3 = 0$, $a_3 = .3$ and $b_3 = .1$). $x_2 = \sqrt{3}\ x_1$ Finite difference and $x_2 = \sqrt{3}\ x_1$ finite element.

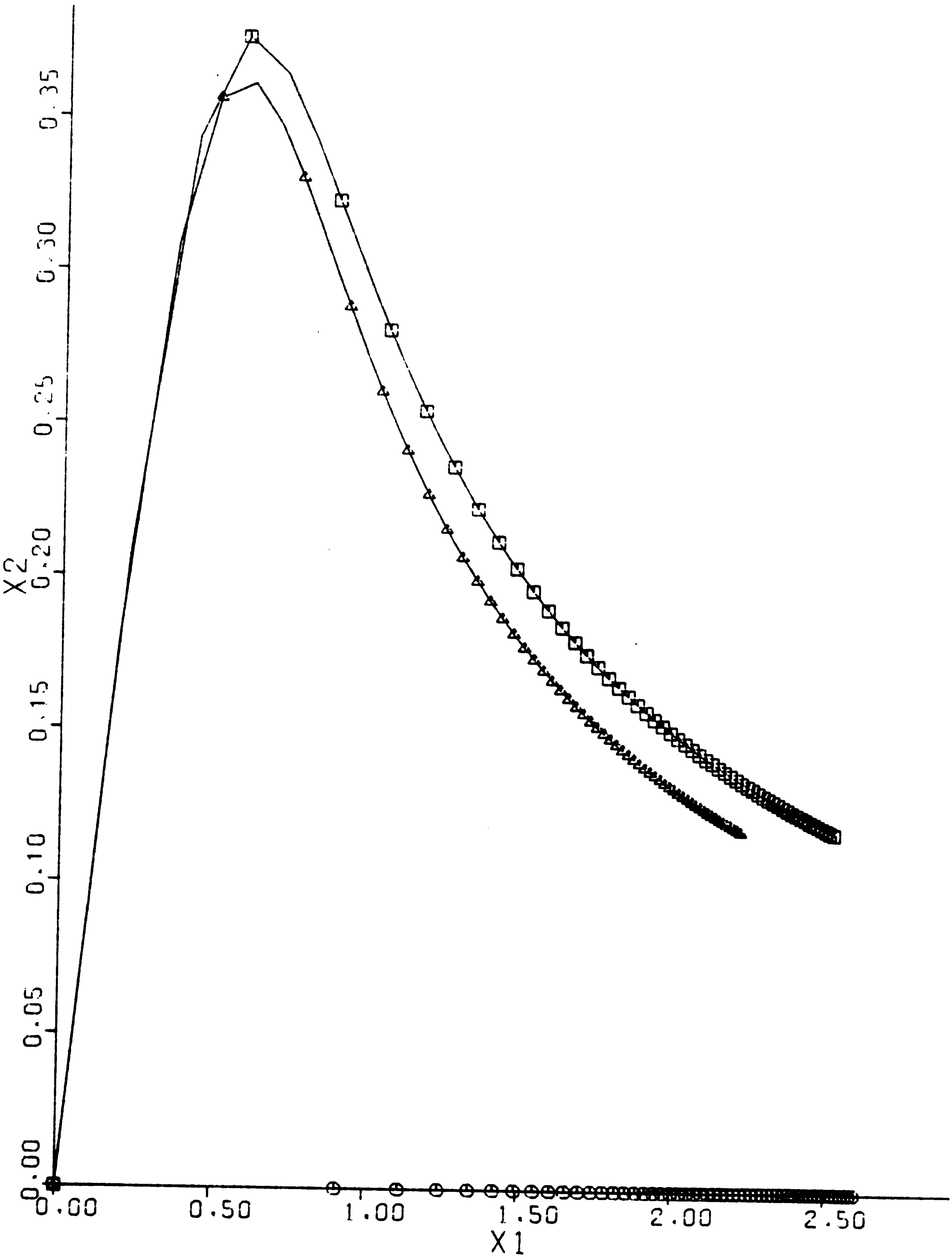

Figure 4.35 Loci of points on $V(x_1,x_2) = E$ that correspond to normal modes. Two degrees of freedom system. Finite element and finite difference solutions $(h=\pi/5)$. $(a_1 = b_1 = .8,\ A_1 = -.2,\ a_3 = .3,\ b_1 = .1$ and $A_3 = 0)$. —⊕— Finite difference. —⊞— Finite elements and —△— $x_2 = 0$.

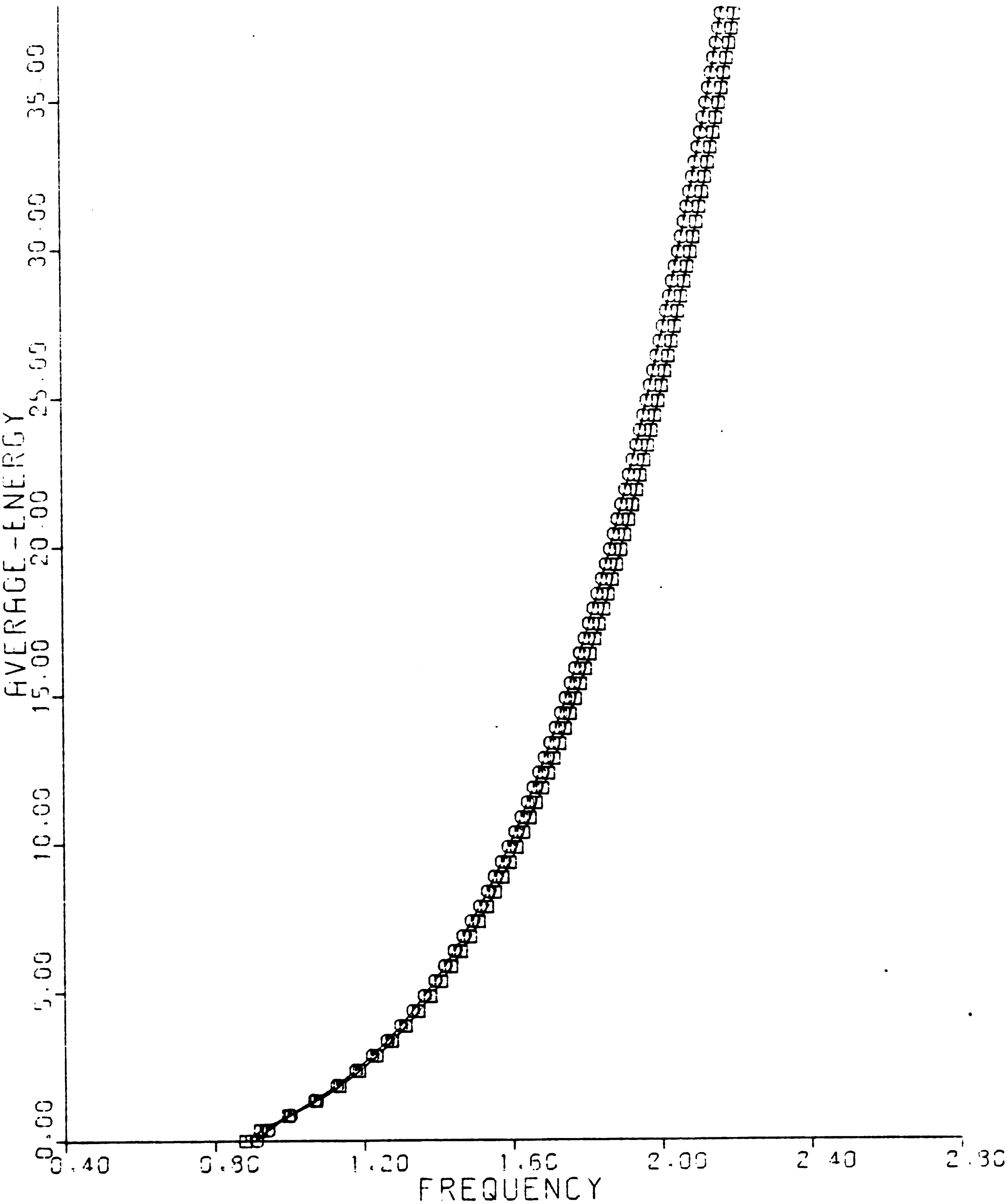

Figure 4.36 Average energy - Frequency relationships. Two degrees of freedom system. Finite element and finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = .8$, $A_1 = -.2$, $a_3 = .3$, $b_1 = .1$ and $A_3 = 0$). Finite difference and finite element.

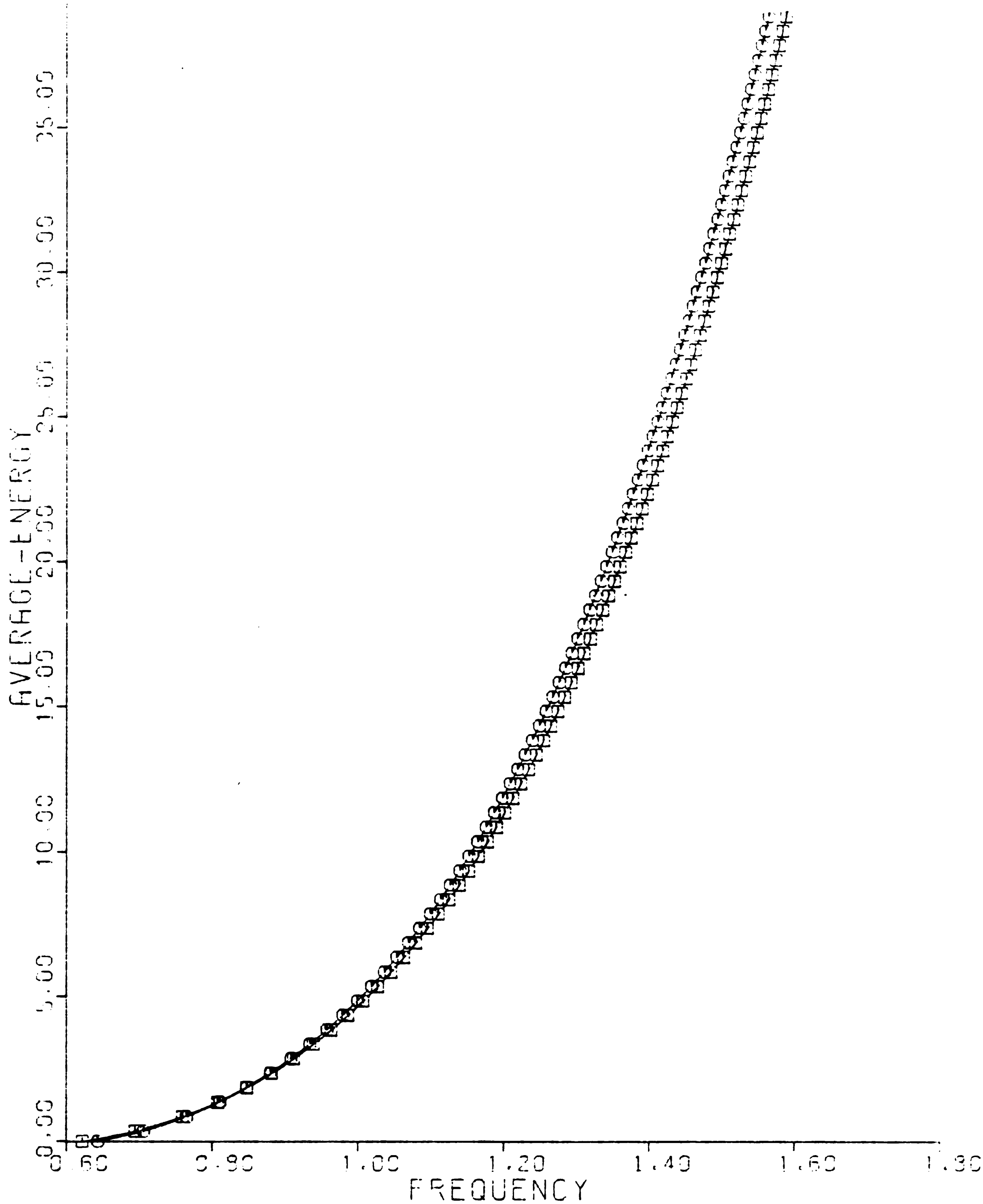

Figure 4.37 Average energy-Frequency relationships. Two degrees of freedom system. Finite element and finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = .8$, $A_1 = -.2$, $a_3 = .3$, $b_1 = .1$ and $A_3 = 0$). —⊞— Finite difference solution and —⊞— finite element solution.

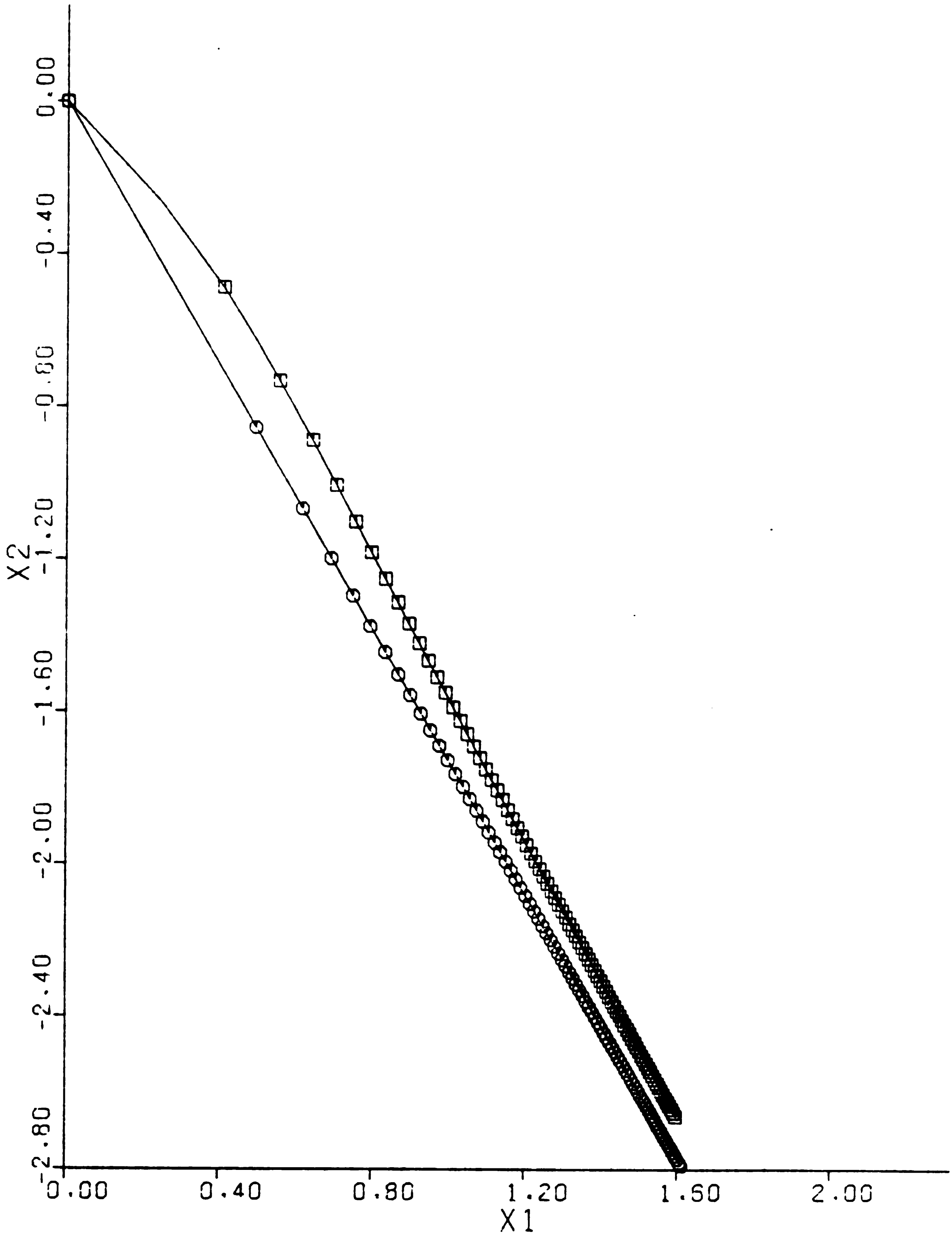

Figure 4.38 Loci of points on $V(x_1, x_2) = E$ that correspond to normal modes. Two degrees of freedom system. Finite element solution $(h = \pi/5)$. $(a_1 = b_1 = .8,\ A_1 = -.2,\ a_3 = .3,\ b_1 = .1$ and $A_3 = 0)$. —(I)— Normal mode of the homogeneous system and —[II]— normal mode.

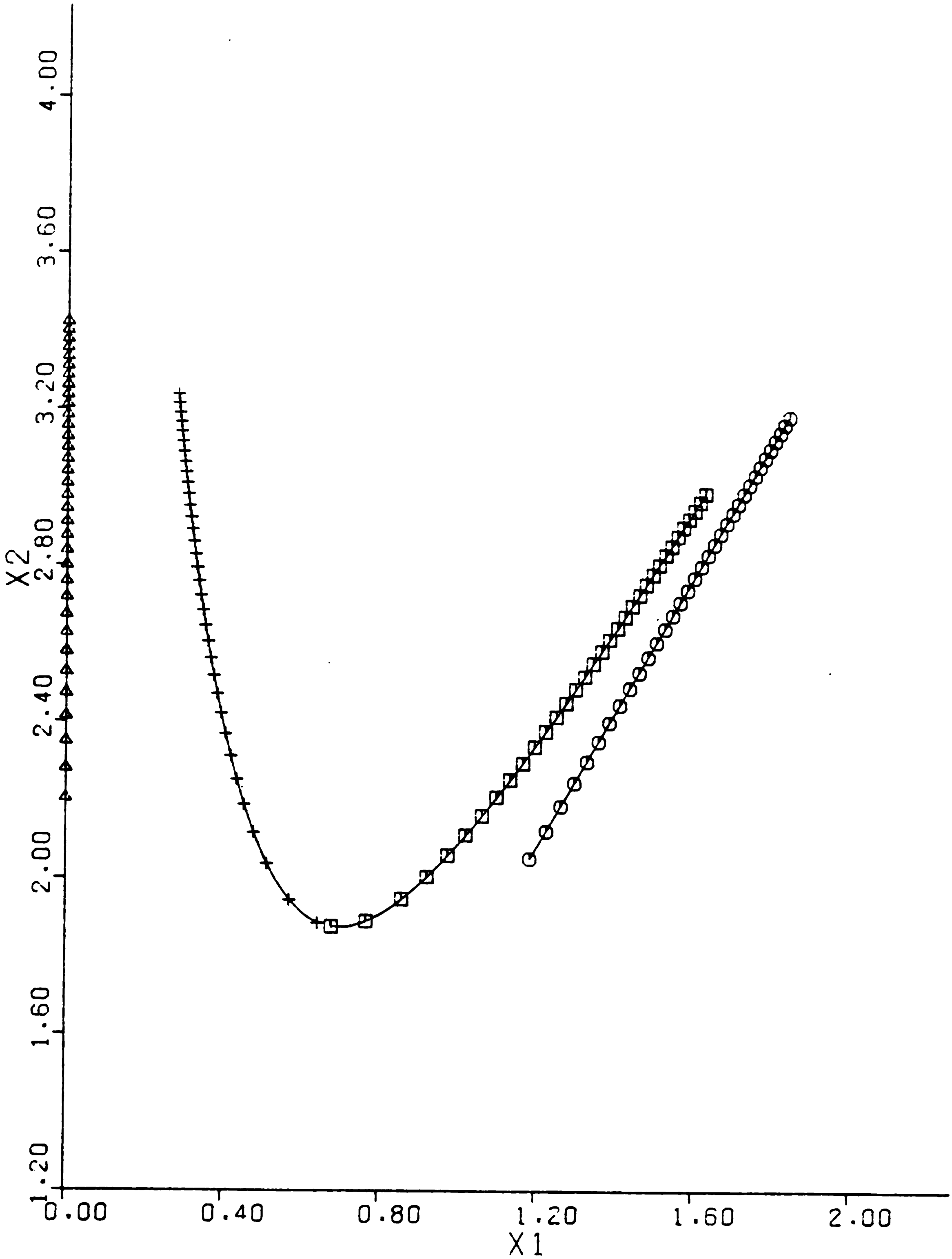

Figure 4.39 Loci of points on $V(x_1,x_2) = E$ that correspond to normal mode. Two degrees of freedom system. Finite element solutions $(h = \pi/5)$. ($a_1 = b_1 = .8$, $A_1 = -.2$, $a_3 = .3$, $b_1 = .1$ and $A_3 = 0$). △ $x_1 = 0$, □ Superabundant mode, + Superabundant mode and ○ $x_2 = \sqrt{3}\ x_2$.

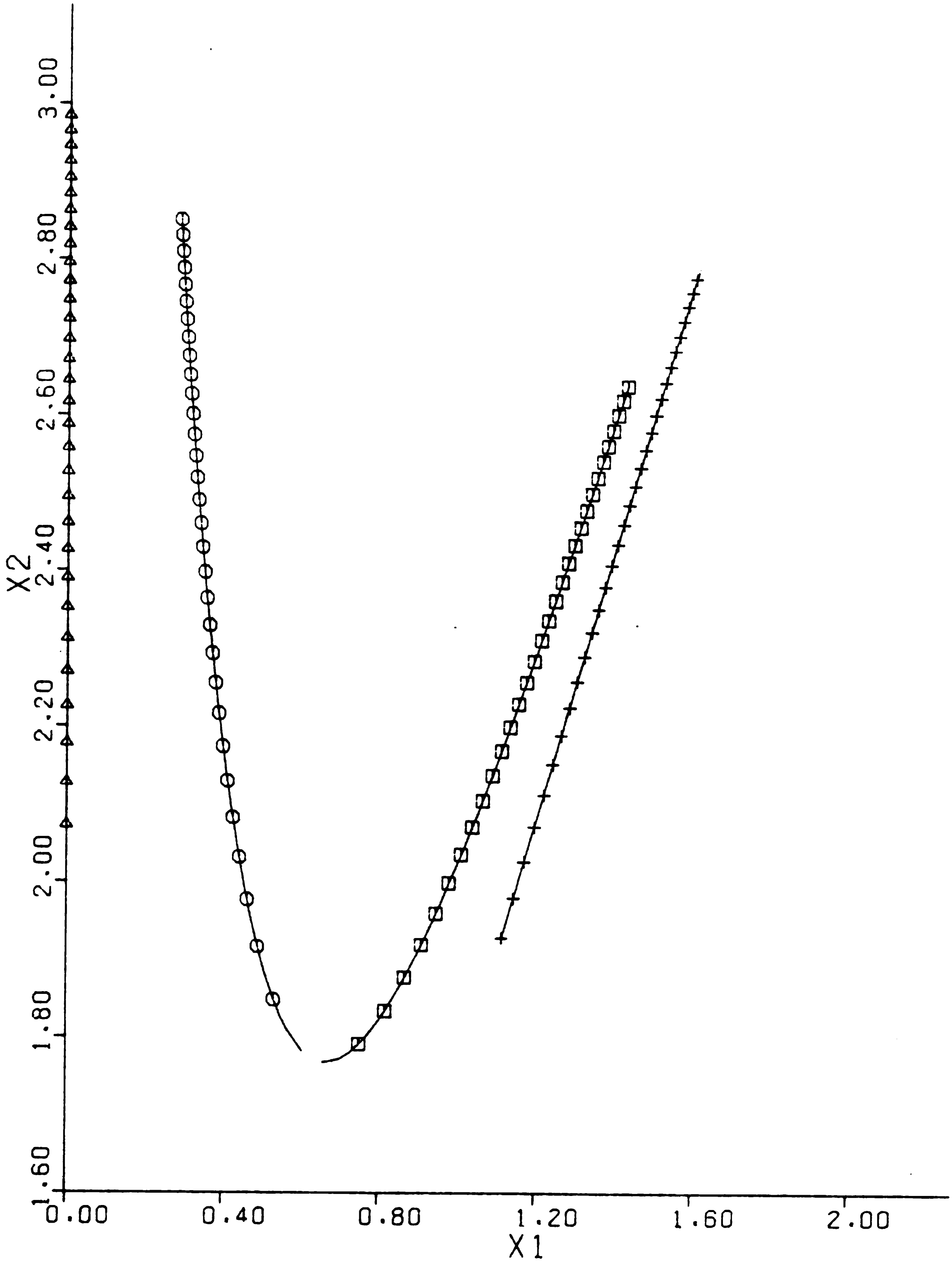

Figure 4.40 Loci of points on $V(x_1,x_2) = E$ that corresponds to normal modes. Two degrees of freedom system. Finite difference solutions $(h = \pi/5)$. $(a_1 = b_1 = .8,\ A_1 = -.2,\ a_3 = .3,\ b_1 = .1$ and $A_3 = 0)$. —△— $x_1 = 0$, —○— Superabundant mode, —□— Superabundant mode and —+— $x_2 = \sqrt{3}\, x_1$.

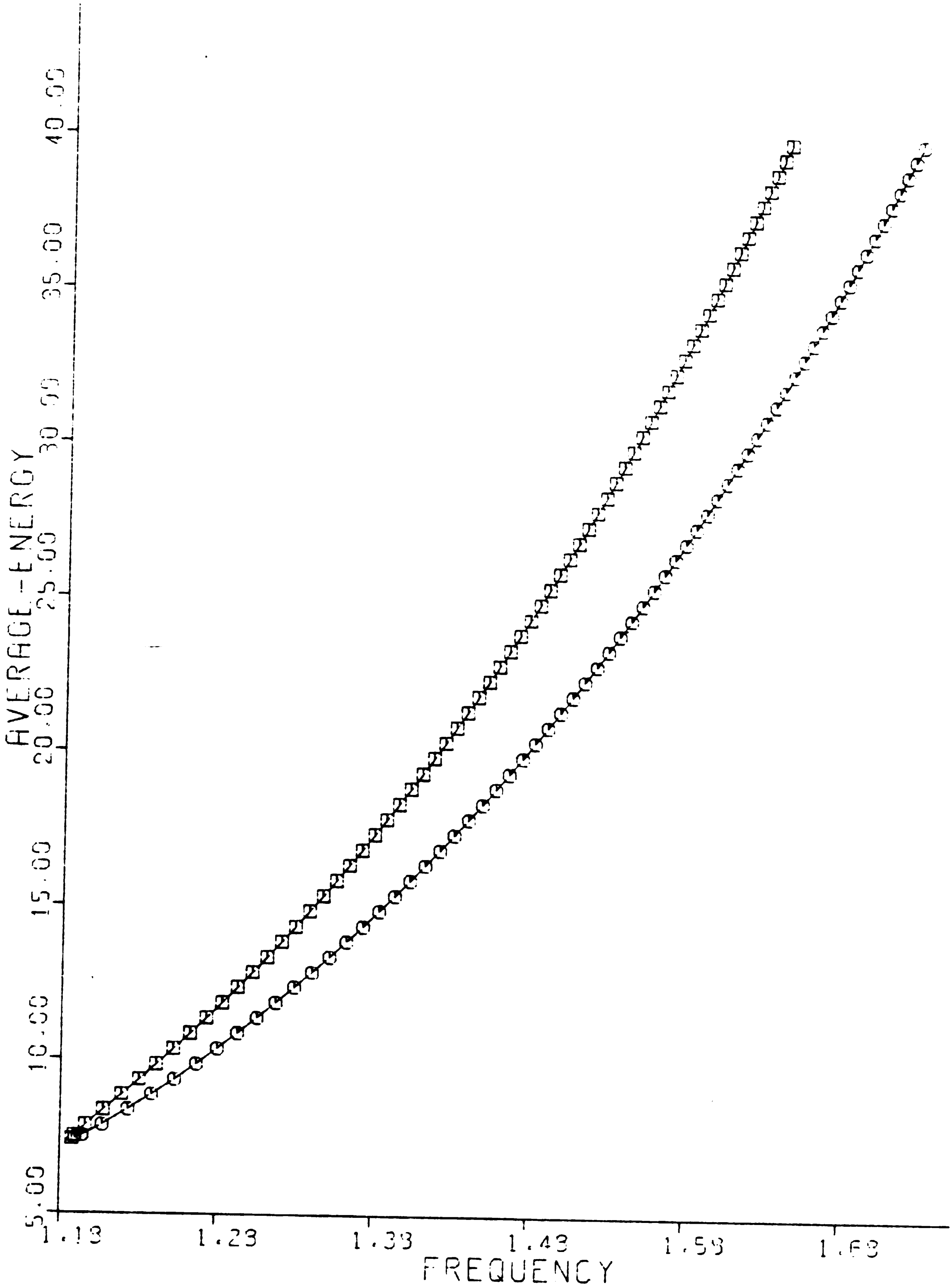

Figure 4.41 Average energy - Frequency relationships. Two degrees of freedom system. Finite element solutions $(h = \pi/5)$. $(a_1 = b_1 = .8,\ A_1 = -.2,\ a_3 = .3,\ b_1 = .1$ and $A_3 = 0)$. ⊞ , ⊕ Superabundant modes.

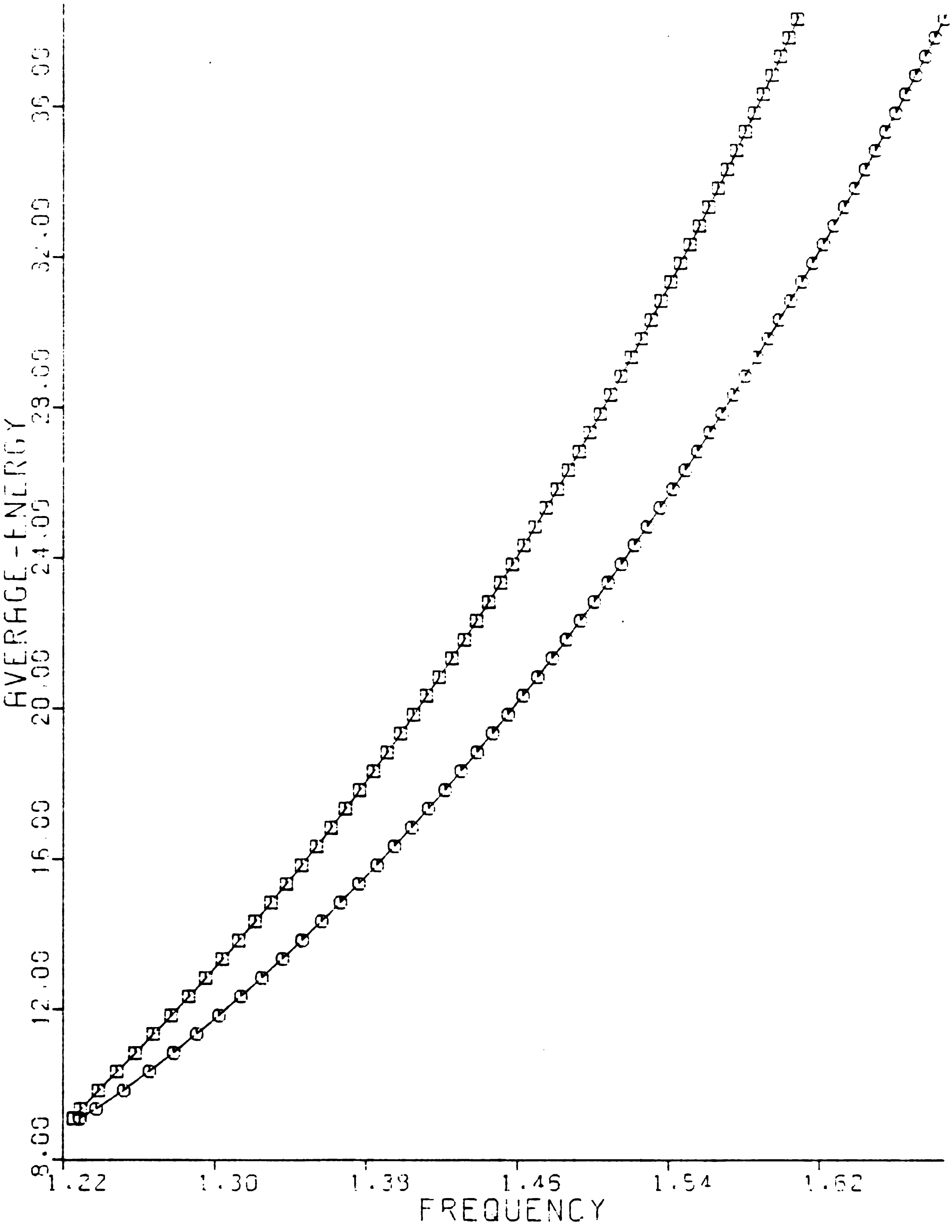

Figure 4.42 Average energy Frequency relationships. Two degrees of freedom system. Finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = .8$, $A_1 = -.2$, $a_3 = .3$, $b_1 = .1$ and $A_3 = 0$).

—⊞— , —⊕— Superabundant modes.

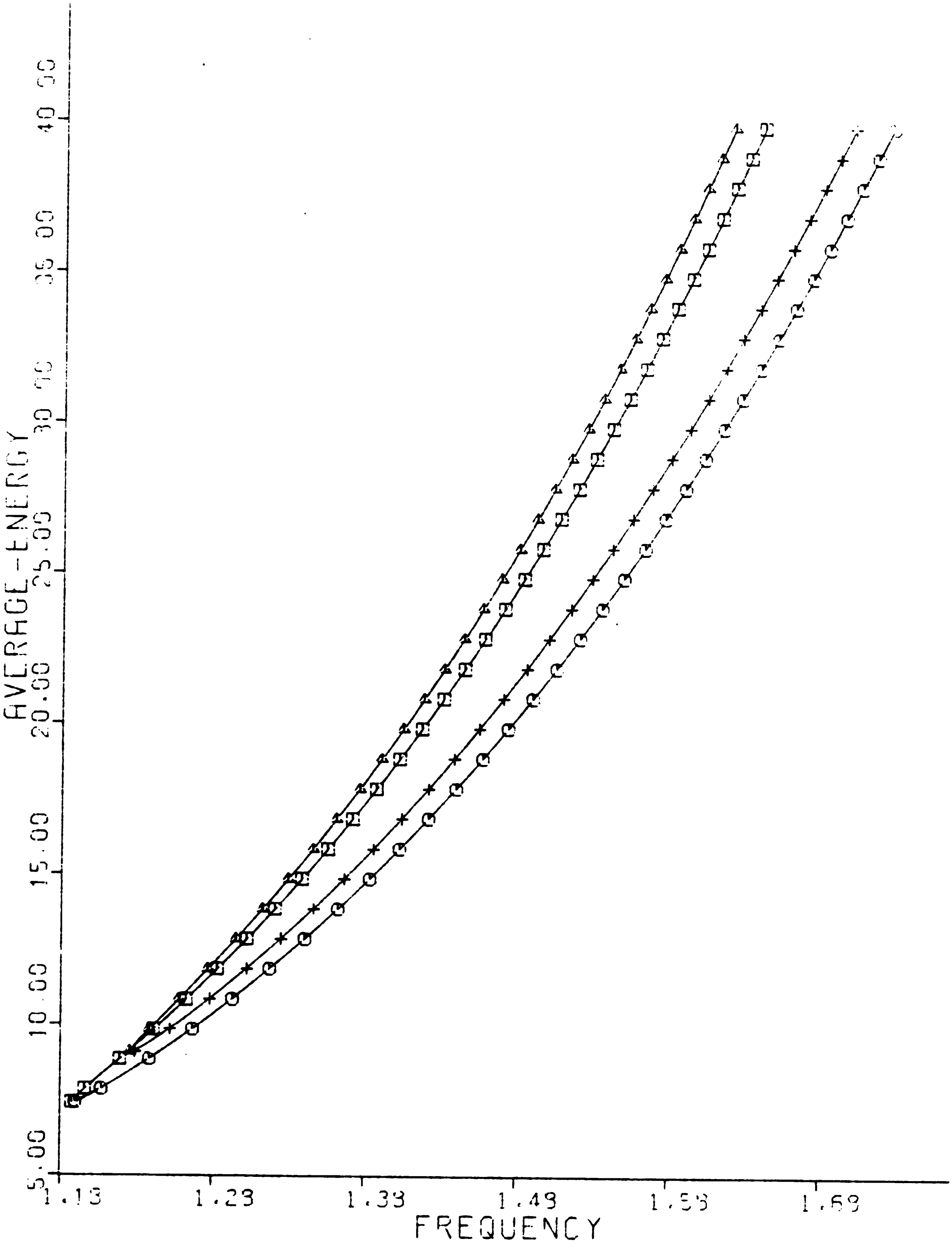

Figure 4.43 Average energy - Frequency relationships. Two degrees of freedom system. Finite element and finite difference solutions ($h = \pi/5$). ($a_1 = b_1 = .8$, $A_1 = -.2$, $a_3 = .3$, $b_1 = .1$ and $A_3 = 0$). , Finite element superabundant modes and, finite difference superabundant modes.

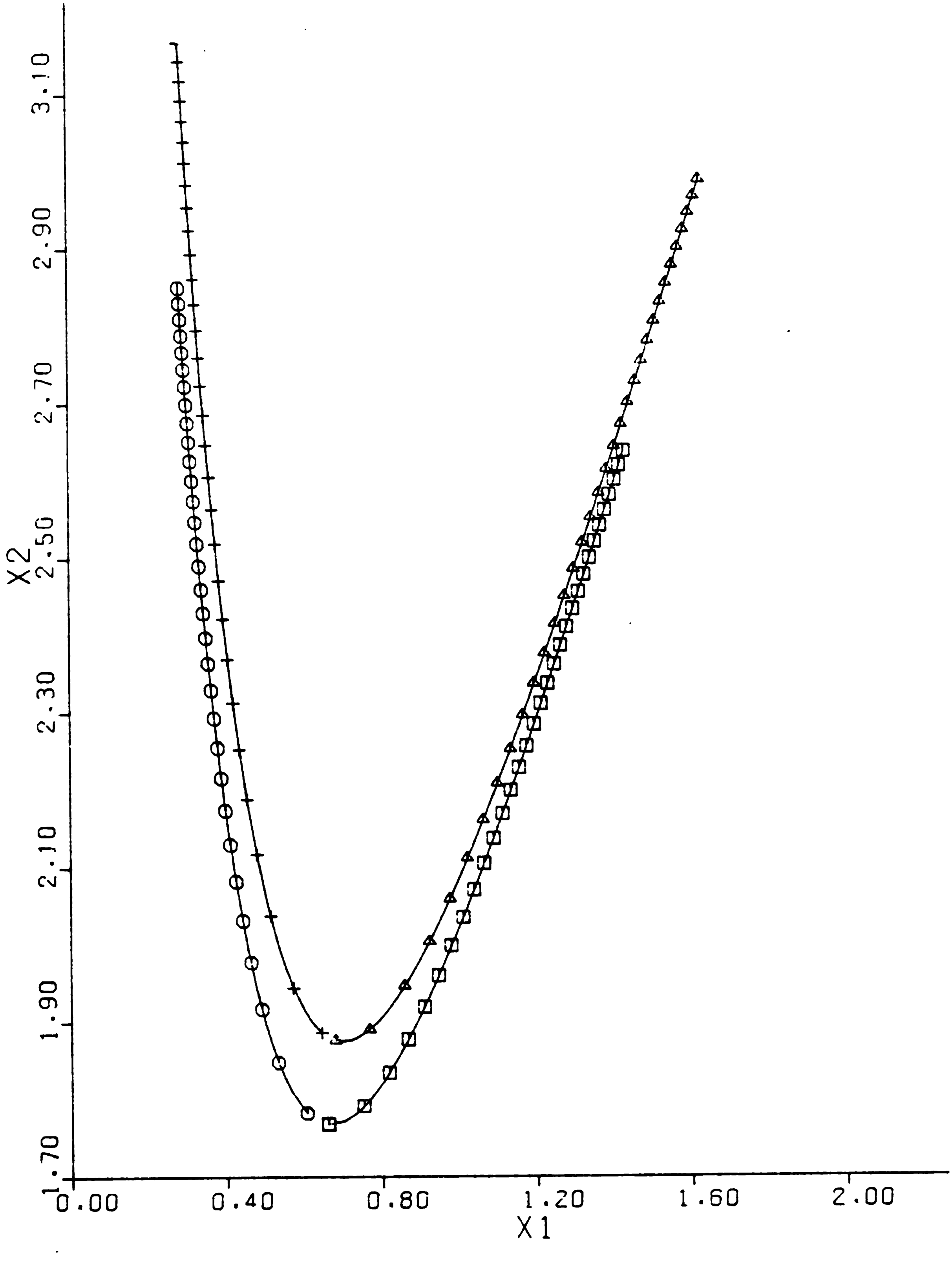

Figure 4.44 Loci of points on $V(x_1,x_2) = E$ that correspond to normal modes. Two degrees of freedom system. Finite element and finite difference solutions $(h = \pi/5)$. $(a_1 = b_1 = .8,\ A_1 = -.2,\ a_3 = .3,\ b_1 = .1$ and $A_3 = 0)$. —⊡—, —⊖— Finite element superabundant modes and —+—, —△— finite difference superabundant modes.

Chapter 5 Summary and Conclusions

We have been concerned with the computation of families of periodic solutions of nonlinear Hamiltonian systems. The periodic solutions here are generalizations of the normal modes of linear systems, for which the potential energy function is quadratic and positive definite. When the potential energy $V(x)$ contains terms higher than quadratic, the approximation of it by a linear system is justified only when the total energy E of the system is small. As E increases, terms of $V(x)$ higher than quadratic become important. It is natural to ask what become of the linear normal modes.

In Chapter 2 we formulated the mathematical problem and identified periodic solutions of the above mentioned system with critical points of some convex functional on manifolds defined by constant integrals of the potential energy of the system. We showed that for a given total energy E the variational problem had at least one periodic solution. Continuous dependence of such periodic solutions on the total energy E, however, was assumed.

In Chapter 3 the so called finite element method, in which the time variable was discretized, was applied to obtain approximate solutions for the class of nonlinear autonomous differential equations mentioned above. We showed, upon assuming that such a problem had an isolated solution for a given energy level E_0, that the finite element equations, for small mesh size h, also had a corresponding isolated solution. This was established by a contraction mapping argument by relating the Jacobian of the discrete problem to that of the exact problem. Furthermore, if the exact problem had an isolated branch of solution for some E in a neighborhood of E_0, then, correspondingly, the finite element solution at E_0 was also isolated and could be uniquely continued. Also in Chapter 3 we showed that under certain conditions we could continue a solution beyond a bifurcation point. A numerical algorithm which was presented that had been used to obtain approximate solutions. Techniques for switching solution branches at a bifurcation point by constructing several distinct tangent vectors at such a point were also included in Chapter 3.

In Chapter 4 the numerical algorithm discussed in Chapter 3 was applied to some special class of problems where the potential energy function contained quadratic terms plus higher even order terms. We first considered a single degree of freedom nonlinear system, with the special form of potential energy function mentioned above, for which exact solutions were known. The approximate finite element solutions obtained using the so called piecewise linear functions were compared with the exact ones so as to assess the accuracy of the former. We also considered in Chapter 4 two degrees of freedom nonlinear systems and presented examples in which bifurcated branches of solution existed. Finally, we compared the finite element solutions with approximate solutions obtained by a finite difference scheme.

In conclusion, isolated solutions are preserved by finite element method and can be uniquely continued. Bifurcation point need not be preserved by the finite element method. However, if a bifurcation point is preserved, then a solution can be continued past such a point, and bifurcated branches can also be obtained numerically.

One question that may be raised at this point is whether other discrete consistent schemes, for example, schemes that are derived from the finite difference method, also preserve isolated periodic solutions. Another direction of future research is to develope more efficient numerical algorithms that avoid explicit computations of the tangent vectors at bifurcation points along the lines of work in [33, 34, 37].

Finally, we point out that the methods developed here can perhaps be extended to forced vibration problems governed by non-autonomous systems of ordinary differential equations with periodic driving forces and possibly damping.

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