INVERSE LIMITS OF FINITE SPACES

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ABSTRACT

INVERSE LIMITS OF FINITE SPACES

By

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We make the following definitions. Let S be a topological space. Subset $X \subseteq S$ is <u>co-dense</u> in S iff for each $s \in S$, $\overline{\{s\}} \cap X \neq \emptyset$. Let $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}$ be the inverse limit space of the inverse system $\{X_\alpha, f_{\alpha\beta}\}$ over I, a directed indexing set. Let $\pi_\alpha : X_\infty \to X_\alpha$ be the usual projection for each $\alpha \in I$. Subset $X \subseteq X_\infty$ is <u>strongly dense</u> in X_∞ iff for each $\alpha \in I$ and $x_\alpha \in X_\alpha$, $\pi_\alpha^{-1}(x_\alpha) \cap X \neq \emptyset$. For T_0 - space X an inverse limit space X_∞ is a <u>finite resolution</u> of X iff each X_α is a finite T_0 - space and X can be imbedded as a strongly dense subspace in X_∞. We will usually write $X \subseteq X_\infty$ identifying X with its image via this imbedding. It is the purpose of the thesis to investigate the properties of these T_0 - compactifications of X.

In the following outline of principal results, X is a T_0 - space unless otherwise specified. The numbering of the results does not correspond to the numbering in the thesis.

<u>Theorem 1</u>: Let B be a basis for X. Then there exists a finite resolution of X, written $X_\infty(B)$, where I is the set of nonempty finite subsets of B. We say X_∞ is constructible from basis B.

<u>Theorem 2</u>: Each finite resolution of X is homeomorphic to a finite resolution constructible from some basis B.

<u>Theorem 3</u>: Let $\{X^i\}$ be any collection of T_0 - spaces with finite resolutions $\{X_\infty^i\}$. Then the product ΠX_∞^i is a finite resolution of the product ΠX^i.

<u>Theorem 4</u>: Let S be a compact T_0 - space and let $F \subseteq S$ be a subspace such that $\{x \in S: \overline{\{x\}} = \{x\}\} \subseteq F$. Then F is compact.

<u>Theorem 5</u>: Let X_∞ be a finite resolution of a Hausdorff space X. Then the following are equivalent.

(1) X is compact.

(2) X is co-dense in X_∞.

(3) $X = \{x \in X_\infty: \overline{\{x\}} = \{x\}\}$.

<u>Theorem 6:</u> Let X_∞ be a finite resolution of a compact Hausdorff space X. Then

(1) X has the fixed point property iff X_∞ has the fixed point property.

(2) X_∞ and X have the same Čech homology groups.

(3) X is a strong deformation retract of X_∞.

INVERSE LIMITS OF FINITE SPACES

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INTRODUCTION

Let $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}$ be the inverse limit space of finite T_0 - spaces, X_α. For each α, let $\pi_\alpha : X_\infty \to X_\alpha$ be the restriction to X_∞ of the natural projection from the product space, ΠX_α, to the factor space, X_α. Say that subspace $X \subseteq X_\infty$ is <u>strongly dense</u> in X_∞ iff $\pi_\alpha^{-1}(x_\alpha) \cap X \neq \emptyset$ for each α and $x_\alpha \in X_\alpha$. Further, X_∞ is a <u>finite resolution</u> of a T_0 - space, X, iff X may be imbedded as a strongly dense subspace of X_∞. It is the purpose of this thesis to investigate the properties of these T_0 - compactifications of the space, X. We make the one notational convenience that the closure of a singleton (i.e. $\overline{\{x\}}$) will usually be written without brackets.

The first chapter is a listing of general results and properties of inverse limit systems and spaces which are references throughout the thesis. In the second chapter an inverse system of finite T_0 - spaces is constructed from a general topological space, X, and any basis for X. This system is used in the third chapter to show the existence of finite resolutions. After establishing that this construction is naturally present for every finite resolution, some general properties of these spaces are discussed. In chapter four the much more rigid positioning of a compact Hausdorff space in its finite resolutions is investigated. Chapter five is a collection of illustrative examples for the previous chapters. The sixth chapter deals with some of the algebraic topological properties inherited from compact Hausdorff spaces by their finite resolutions.

CHAPTER ONE

PRELIMINARIES

The following definitions and theorems, with the exception of L and M, are generally known and are included for reference for the remainder of the thesis. The notation here differs slightly from that of [3] or [1] and [2], but the content is the same. Most proofs are not given and alphabetic labelling is used for distinction as reference theorems.

<u>Definition</u>: An indexing set, I, is said to be <u>directed</u> by a reflexive, transitive quasi-ordering, $\leq$, on I if for any pair $\alpha, \beta \in I$ there is some $\gamma \in I$ with $\alpha, \beta \leq \gamma$.

In the following, the letters I, J, K will refer to directed indexing sets which are directed by $\leq$ unless otherwise indicated.

<u>Definition</u>: Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of topological spaces. For each pair $\alpha \leq \beta$ let $f_{\alpha\beta}: X_\beta \to X_\alpha$ be a continuous map. Require $f_{\alpha\alpha}$ to be the identity map for each $\alpha \in I$ and for $\alpha \leq \beta \leq \gamma$ require $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$. The collection of spaces and maps $\{X_\alpha, f_{\alpha\beta}\}_I$ is called an <u>inverse system of spaces over I</u>.

<u>Definition:</u> Let $\{X_\alpha, f_{\alpha\beta}\}_I$ be an inverse system of spaces over I. In the product space, ΠX_α, the subspace

$$X_\infty = \{x \in \Pi X_\alpha : \alpha \le \beta \text{ implies } f_{\alpha\beta}(x_\beta) = x_\alpha\}$$

is called the *inverse limit space* of the inverse system $\{X_\alpha, f_{\alpha\beta}\}$. (By $x_\alpha \in X_\alpha$ we mean the α-th coordinate of $x \in \Pi X_\alpha$ for each $\alpha \in I$. That is, for $\pi_\alpha : \Pi X_\alpha \to X_\alpha$ the coordinate projection onto the factor space, we have $\pi_\alpha(x) = x_\alpha$.) We also use the notation $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}$. The X_α are called *factor spaces* of the limit space. The maps $f_{\alpha\beta}$ are called the *bonding maps* of the inverse system. This structure and notation may be referred to in these preliminaries and in later chapters without complete prefacing.

Definition: Let $\{X_\alpha, f_{\alpha\beta}\}$ and $\{Y_\gamma, g_{\gamma\delta}\}$ be inverse systems of spaces over I and J respectively. Let $\Psi: J \to I$ be any map. For each $\gamma \in J$, let $\varphi_\gamma : X_{\Psi(\gamma)} \to Y_\gamma$ be a continuous map. The collection of maps $\{\varphi_\gamma\}_{\gamma \in J}$ is an *inverse system of maps* iff whenever $\gamma \le \delta$ in J, the following diagram commutes.

$$\begin{tikzcd} X_{\Psi(\delta)} \arrow[r, "f_{\Psi(\gamma)\Psi(\delta)}"] \arrow[d, "\varphi_\delta"'] & X_{\Psi(\gamma)} \arrow[d, dashed, "\varphi_\delta"] \\ Y_\delta \arrow[r, dashed, "g_{\gamma\delta}"] & Y_\gamma \end{tikzcd}$$

Usually $I = J$ with Ψ the identity map. Now let X_∞ and Y_∞ be the inverse limit spaces of the above systems. Define $\Phi: X_\infty \to Y_\infty$ as follows. For each $x \in X_\infty$, let

$$\Phi(x) = (\varphi_\gamma(x_{\Psi(\gamma)}))_{\gamma \in J}$$

That is, if $\pi_\gamma : \prod Y_\gamma \to Y_\gamma$ is the coordinate projection, we indicate $\Phi(x)$ by giving all of its coordinates. Φ is called the induced or <u>limit map of the inverse system of maps</u>, $\{\varphi_\gamma\}$ and we write $\Phi = \varprojlim \{\varphi_\gamma\}$.

Henceforth π_α, for each $\alpha \in I$, will represent the restriction of the coordinate projection to the subspace, X_∞, unless otherwise specified.

<u>Theorem A</u>: (i) The composition of limit maps is the limit map of the corresponding inverse system of compositions.

(ii) The inverse of a limit map is the limit map of the corresponding inverse system of inverse maps.

<u>Theorem B</u>: Let $\Phi: X_\infty \to Y_\infty$ be a limit map as above. Then Φ is continuous and unique with respect to $\pi_\gamma \circ \Phi = \varphi_\gamma \circ \pi_{\Psi(\gamma)}$.

<u>Definition</u>: For directed set, I, $J \subseteq I$ is <u>cofinal</u> in I iff for each $\alpha \in I$ there is $\beta \in J$ such that $\alpha \leq \beta$.

<u>Theorem C</u>: Let Φ be given as above and let $\Psi(J)$ be cofinal in I. If each φ_γ, $\gamma \in J$, is an injection (bijection, homeomorphism) then Φ is an injection (bijection, homeomorphism).

<u>Theorem D</u>: For $J \subseteq I$, the system $\{X_\alpha, f_{\alpha\beta}\}_J$ obtained by restricting indexes to J is also an inverse system. Indi-

cating the limit spaces as X_∞^I and X_∞^J, the collection of identity maps $i_\alpha : X_\alpha \to X_\alpha$ for each $\alpha \in J$ induces a limit map $\Phi : X_\infty^I \to X_\infty^J$ called the natural projection into X_∞^J. Φ sends $(x_\alpha)_{\alpha \in I}$ to $(x_\alpha)_{\alpha \in J}$. If J is cofinal in I then the inverse limit spaces are homeomorphic by Theorem C.

Theorem E: The coordinate projections $\pi_\alpha : X_\infty \to X_\alpha$ are continuous and the following diagram commutes when $\alpha \leq \beta$ in I.

$$\begin{tikzcd} & X_\infty \arrow[dl, "\pi_\beta"'] \arrow[dr, "\pi_\alpha"] & \\ X_\beta \arrow[rr, "f_{\alpha\beta}"] & & X_\alpha \end{tikzcd}$$

Theorem F: For each $\alpha \in I$ let B_α be a basis for X_α. Then the collection $\{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in B_\alpha,\ \alpha \in I\}$ is a basis for X_∞.

We note that by Theorem D any cofinal $J \subseteq I$ may replace I in this construction.

Theorem G: If each X_α is Hausdorff then $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}$ is a closed subspace of ΠX_α.

Theorem H: Let each X_α be compact and Hausdorff (or finite) and nonvoid. Then X_∞ is compact and nonvoid (nonvoid).

Theorem I: Let $A \subseteq X_\infty$ and let $A_\alpha = \pi_\alpha(A)$. Then $\{A_\alpha\}$ and $\{\overline{A_\alpha}\}$ are inverse systems of subsets via the restricted bonding maps and we have

$$\overline{A} = \cap\, \pi_\alpha^{-1}\left(\overline{\pi_\alpha(A)}\right) = \cap\, \pi_\alpha^{-1}\left(\overline{A_\alpha}\right) = \varprojlim \{\overline{A_\alpha}\} .$$

<u>Theorem J</u>: <u>Examples</u>

(1) Let each X_α be a distinct copy of space, X, and let map $f_{\alpha\beta}$ be the identity whenever $\alpha \le \beta$. Then X_∞ is homeomorphic to X.

(2) If indexing set, I, has a maximum element, Ω, then X_∞ is homeomorphic to X_Ω.

The remaining results present the well known equivalent definition of an inverse limit space as a universal mapping space. Initially the idea of a Hausdorff inverse limit was hoped to be useful. Theorems L and M show that nothing is gained by its definition. See Lemma 5, page 29 and Example B, page 38.

<u>Definition</u>: Let $\{X_\alpha, f_{\alpha\beta}\}$ be a collection of spaces and continuous maps. Space Y with continuous maps $\{g_\alpha\}$ is a <u>left approximation</u> of $\{X_\alpha, f_{\alpha\beta}\}$ iff the diagram below commutes for $\alpha \le \beta$.

<u>Definition</u>: A left approximation $\{X, f_\alpha\}$ of $\{X_\alpha, f_{\alpha\beta}\}$ is <u>universal</u> over a given category of left approximations $\{Y_i, g_{\alpha i}\}$ iff for each Y_i there is a unique continuous map $h_i\colon Y_i \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
X_\beta \arrow[rr, "f_{\alpha\beta}"] & & X_\alpha \\
& Y \arrow[ul, "g_\beta"] \arrow[ur, "g_\alpha"'] &
\end{tikzcd}$$

Left approximation

$$\begin{tikzcd}
& X \arrow[dl, "f_\beta"'] \arrow[dr, "f_\alpha"] & \\
X_\beta \arrow[rr, "f_{\alpha\beta}"] & & X_\alpha \\
& Y_i \arrow[ul, "g_{\beta i}"] \arrow[ur, "g_{\alpha i}"'] \arrow[uu, "h_i"] &
\end{tikzcd}$$

Universal approximation

<u>Theorem K</u>: $\{L, \pi_\alpha\}$ is the inverse limit space of inverse system $\{X_\alpha, f_{\alpha\beta}\}$ iff L is universal over the category of all left approximations.

<u>Definition</u>: Let X be Hausdorff. Left approximation $\{X, f_\alpha\}$ is a <u>Hausdorff inverse limit</u> of inverse system $\{X_\alpha, f_{\alpha\beta}\}$ iff $\{X, f_\alpha\}$ is universal over the category of all Hausdorff left approximations.

<u>Theorem L</u>: Let $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}$. Let subspace $X \subseteq X_\infty$ be Hausdorff. Then X is a Hausdorff inverse limit of $\{X_\alpha, f_{\alpha\beta}\}$ iff $X = X_\infty$.

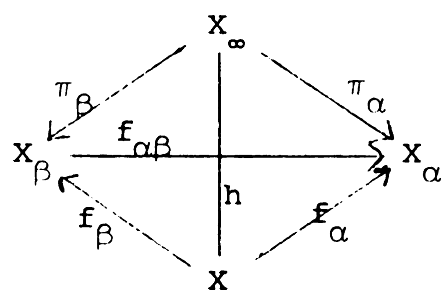
<u>Proof</u>: For each $x \in X_\infty$, $\{x\}$ is Hausdorff. Hence there is a unique h_x: $\{x\} \to X$ such that the diagram below commutes for $\alpha \leq \beta$. This implies that all coordinates of x and $h_x(x)$ are the same. Hence $x \in X$. The converse is trivial.

$$
\begin{tikzcd}
& X \arrow[dl, dashed, "\pi_\beta"'] \arrow[dr, "\pi_\alpha"] & \\
X_\beta \arrow[rr, "f_{\alpha\beta}"] & & X_\alpha \\
& \{x\} \arrow[uu, "h_x"] \arrow[ul, "\pi_\beta"] \arrow[ur, dashed, "\pi_\alpha"'] &
\end{tikzcd}
$$

<u>Theorem M</u>: Suppose $\{X, f_\alpha\}$ is a Hausdorff inverse limit of $\{X_\alpha, f_{\alpha\beta}\}$ and $\{X_\infty, \pi_\alpha\}$ is the inverse limit. Then X and X_∞ are homeomorphic.

<u>Proof</u>: By Theorem L we need only show that X is imbedded in X_∞. The universal mapping property of X_∞ implies that there is a unique continuous map h: $X \to X_\infty$ such that the diagram below commutes. Map h is one-to-one since

points are distinguished uniquely by their coordinates. If distinct points $x,y \in X$ or their images have the same coordinates then maps from the singletons into X as in the diagram of Theorem L are not unique. Now X_∞ and X are subspaces of the product ΠX_α and the usual projections p_α are such that $p_\alpha|_X = f_\alpha$ and $p_\alpha|_{X_\infty} = \pi_\alpha$. Then the map h^{-1}: $h(X) \to X$ is continuous into the product since for each α, $p_\alpha \circ h^{-1} = \pi_\alpha$ is continuous.



CHAPTER TWO

CONSTRUCTIONS

In this chapter we construct a particular inverse limit space whose factor spaces and bonding maps are defined in terms of finite collections of basic open sets from an arbitrary topological space and the partitions that these collections generate.

Definition: A collection $\mathfrak{u}$ of subsets of a set X is a partition of X iff the subsets of X in $\mathfrak{u}$ are nonvoid and pairwise disjoint and $\cup\mathfrak{u} = X$. Partition $\mathfrak{u}$ properly partitions a subset $F \subseteq X$ iff F is a union of subsets that are elements of $\mathfrak{u}$. Partition $\mathfrak{v}$ properly refines partition $\mathfrak{u}$ iff each subset in $\mathfrak{u}$ is properly partitioned by $\mathfrak{v}$.

In this chapter, X is a topological space and B is a basis for X. Let $I = \{\alpha \subseteq B: \alpha \neq \emptyset$ and finite$\}$. For each $\alpha \in I$ define the relation (α) on X by $x(\alpha)y$ iff

$$\{U \in \alpha: x \in U\} = \{U \in \alpha: y \in U\}$$

Lemma 1: (i) For each $\alpha \in I$, (α) is an equivalence relation.

(ii) Let X_α be the partition of X induced by (α) for each $\alpha \in I$. Direct I by inclusion writing $\alpha \leq \beta$ if α is contained in β. For $\alpha \leq \beta$ the partition X_β properly refines X_α.

(iii) We will consider X_α as a point set and will write $x_\alpha \subseteq X$ when considering $x_\alpha \in X_\alpha$ as a subset of X. Let $x_\alpha \in X_\alpha$. For each $x \in x_\alpha \subseteq X$ we have

$$x_\alpha = \cap \{U \in \alpha: x \in U\} - \cup \{U \in \alpha: x \notin U\}$$

Hence partition X_α properly partitions each $U \in \alpha$.

<u>Proof</u>: (i) Clear since relation (α) is based on the equality of subsets of $\alpha \in I$.

(ii) This follows directly from the definition of $\alpha \leq \beta$. The proof of part (iii) also helps clarify this point.

(iii) Now $y \in x_\alpha$ iff $x(\alpha)y$ iff $\{U \in \alpha: x \in U\} = \{U \in \alpha: y \in U\}$ iff $y \in \cap \{U \in \alpha: x \in U\} - \cup\{U \in \alpha: x \notin U\}$. Of course we have $x_\alpha = \{y \in X: x(\alpha)y\}$. Finally for $U_o \in \alpha$, $x \in U_o$ and x_α the equivalence class of x, we have shown that $x_\alpha \subseteq U_o$ by the above and so X_α properly partitions U_o.

For each $\alpha \in I$ we define $f_\alpha: X \to X_\alpha$ as follows. For each $x \in X$, $f_\alpha(x) = x_\alpha \in X_\alpha$ iff $x \in x_\alpha \subseteq X$. This natural projection is well defined and surjective by the definition of X_α as a partition. Now topologize X_α so that f_α is an identification map. The resulting space is sometimes called a decomposition space for X. (See [5], p. 244)

Whenever $\alpha \leq \beta$ define $f_{\alpha\beta}: X_\beta \to X_\alpha$ as follows. For each $x_\beta \in X_\beta$, $f_{\alpha\beta}(x_\beta) = x_\alpha$ iff $x_\beta \subseteq x_\alpha \subseteq X$. This map is well defined and surjective since partition X_β properly refines partition X_α. Given all of the previous constructions we prove the following.

<u>Theorem 1</u>: (i) $U_\alpha \subseteq X_\alpha$ is open in X_α iff $\bigcup_{x_\alpha \in U_\alpha} x_\alpha \subseteq X$ is open in X.

(ii) Each X_α is a finite T_o - space.

(iii) For $\alpha \leq \beta$, $f_{\alpha\beta}$ is an identification map.

(iv) The collection $\{X_\alpha, f_{\alpha\beta}\}_I$ is an inverse system of spaces.

<u>Proof</u>: (i) This is simply restating that the map f_α is an identification map.

(ii) X_α is finite since $\alpha \in I$ is a finite subset of B. Let $x_\alpha, y_\alpha \in X_\alpha$ be distinct elements. Let $x \in x_\alpha \subseteq X$ and $y \in y_\alpha \subseteq X$. By definition $f_\alpha(x) = x_\alpha$ and $f_\alpha(y) = y_\alpha$. Further, since x and y are in distinct equivalence classes, the definition of (α) implies that there is some $U \in \alpha$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. In either case by (i) above and (iii) of Lemma 1, $f_\alpha(U)$ is open in X and provides T_o - separation for x_α and y_α.

(iii) By our definitions we have $f_{\alpha\beta} \circ f_\beta = f_\alpha$ and so $f_{\alpha\beta}^{-1} = f_\beta \circ f_\alpha^{-1}$. Further the proper refinement of X_α given by X_β implies that for each open $U_\alpha \subseteq X_\alpha$,

$$f_\beta^{-1}(f_\beta(f_\alpha^{-1}(U_\alpha))) = f_\alpha^{-1}(U_\alpha).$$

Now since f_α and f_β are identifications, U_α is open in X_α iff $f_\alpha^{-1}(U_\alpha) = f_\beta^{-1}(f_\beta(f_\alpha^{-1}(U_\alpha))) = f_\beta^{-1}(f_{\alpha\beta}^{-1}(U_\alpha))$ is open in X iff $f_{\alpha\beta}^{-1}(U_\alpha)$ is open in X_β. And so $f_{\alpha\beta}$ is an identification.

(iv) For $\alpha \leq \beta \leq \gamma$, $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ from the definitions. Similarly, $f_{\alpha\alpha}$ is the identity on X_α for each $\alpha \in I$.

Let $X_\infty(B) = \varprojlim\{X_\alpha, f_{\alpha\beta}\}$ be the inverse limit space of the inverse system just constructed, where B indicates that such a limit space exists for each basis, B, of X. We note that by Theorem H, $X_\infty(B) \neq \emptyset$. We define a map $f:X \to X_\infty(B)$ as follows. We give each coordinate of images using the canonical projection $\pi_\alpha:X_\infty \to X_\alpha$ for each $\alpha \in I$. For each $x \in X$ and $\alpha \in I$ let $\pi_\alpha(f(x)) = f_\alpha(x)$. That the resulting map f is well defined and continuous follows from Theorem B and (1) of Theorem J. That is, f is the induced or limit map from $X = \varprojlim\{X, i_{\alpha\beta}\}_I$ into $X_\infty = \varprojlim\{X_\alpha, f_{\alpha\beta}\}_I$ where the inverse system of maps is $\{f_\alpha : \alpha \in I\}$, since $\pi_\alpha \circ f = f_\alpha$ and the limit map is unique with respect to this property.

<u>Theorem 2</u>: The map $f:X \to f(X)$ is a continuous open surjection.

<u>Proof</u>: Continuity is shown above and the surjective property is obvious. To show that f is open it is sufficient to show that for each $U \in B$, $f(U)$ is open in $f(X)$. Let $\alpha = \{U\} \subseteq B$. From the proof of (ii) in Theorem 1, $f_\alpha(U) = U_\alpha$ is open in X. Since π_α is continuous, $\pi_\alpha^{-1}(U_\alpha)$ is open in X_∞ and then $\pi_\alpha^{-1}(U_\alpha) \cap f(X)$ is open in $f(X)$. Claim that $f(U)$ is $\pi_\alpha^{-1}(U_\alpha) \cap f(X)$. This is quite clear when from (iii) Lemma 1 we see that U_α is

indeed a singleton in X_α and that $f_\alpha^{-1}(U_\alpha) = U$. Further, from the discussion preceding this theorem,

$$\pi_\alpha^{-1}(U_\alpha) \cap f(X) = f(f_\alpha^{-1}(U_\alpha)).$$

CHAPTER THREE

FINITE RESOLUTIONS: EXISTENCE AND GENERAL PROPERTIES

In this chapter we see how the constructions of the previous chapter motivate the definition of a finite resolution of a T_o - space. We further find that these constructions encompass all finite resolutions. First we find how T_o - separation of the given space affects the structure previously given.

<u>Theorem</u> <u>3</u>: The mapping $f:X \to X_\infty$ of Theorem 2 is an imbedding iff X is a T_o - space.

<u>Proof</u>: Since each factor space, X_α, is a T_o - space, $f(X)$ is always a T_o - space. Hence if map f is an imbedding then X is a T_o - space.

Conversely, if X is a T_o - space then we need only show that the map is injective and the result follows by Theorem 2. Let $x,y \in X$ be distinct points. In basis, B, there is some open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. In either case, letting $\alpha = \{U\} \in I$ we find by the definition of $f:X \to X_\infty$ that $f(x) \neq f(y)$ since their α-th coordinates $f_\alpha(x)$ and $f_\alpha(y)$ are distinct. Hence the map f is an imbedding.

<u>Definition</u>: Let $X_\infty = \varprojlim\{X_\alpha, f_{\alpha\beta}\}_I$. Subset $F \subseteq X_\infty$ is <u>strongly</u> <u>dense</u> in X_∞ iff for each $\alpha \in I$ and $x_\alpha \in X_\alpha$,

$$\pi_\alpha^{-1}(x_\alpha) \cap F \neq \emptyset.$$

We note that strongly dense implies dense in the usual sense for a subspace because of Theorem F. It also implies that each π_α is surjective. Henceforth when X is a T_o-space and imbedded in an inverse limit space as in Theorem 3 we will write $X \subseteq X_\infty$, identifying X with its image. By the definitions in the constructions we have seen that $\pi_\alpha \circ f = f_\alpha$ in the above. Then considering $X \subseteq X_\infty$ we have $f_\alpha = \pi_\alpha|_X$ and where confusion is not likely, we will write $\pi_\alpha : X \to X_\alpha$.

<u>Definition</u>: Let $X_\infty = \varprojlim\{X_\alpha, f_{\alpha\beta}\}_I$ and let X be a T_o - space. Then X_∞ is a finite resolution of X iff

(i) Each X_α is a finite T_o - space

(ii) X can be imbedded as a strongly dense subspace of X_∞.

Of course the constructions motivate this definition and it is clear that the inverse limit space of Theorem 3 meets these requirements. The strong density follows from the nonvoid condition on the subsets of a partition. That is, each $x_\alpha \in X_\alpha$ corresponds to a nonvoid subset of X and so considering $X \subseteq X_\infty$ after imbedding, $\pi_\alpha^{-1}(x_\alpha) \cap X = x_\alpha \subseteq X$. We temporarily call such a finite resolution and any naturally equivalent to it a <u>basic finite resolution</u>, since it is constructed from a basis for X. However we now prove the following.

<u>Theorem</u> <u>4</u>: Every finite resolution of a T_o - space is basic.

<u>Proof</u>: Let $X_\infty = \underleftarrow{\lim}\{X_\alpha, f_{\alpha\beta}\}_I$ be a finite resolution of T_o - space X. By Theorem F we construct basis $B = \bigcup_{\alpha \in I} G_\alpha$ for X where $G_\alpha = \{\pi_\alpha^{-1}(U_\alpha) \cap X: U_\alpha \text{ open in } X_\alpha\}$ for each $\alpha \in I$.

We must now show the following.

(i) That X_α has the same number of points as the finite space generated by G_α via the relation (G_α) as described in Lemma 1.

(ii) That $\pi_{\alpha}|_X : X \to X_\alpha$ is an identification map.

(iii) That the collection $\{G_\alpha\}$ is cofinal in the set of all finite subsets of B, ordered by inclusion.

Then by Theorem D, X_∞ is naturally equivalent to the basic finite resolution constructed from B.

(i) Since X_α is a T_o - space we have for each $x_\alpha \in X_\alpha$ that

$$\{x_\alpha\} = \cap\{U_\alpha \in \pi_\alpha(G_\alpha): x_\alpha \in U_\alpha\} - \cup\{U_\alpha \in \pi_\alpha(G_\alpha): x_\alpha \notin U_\alpha\}$$

Since inverse maps commute with intersection, union and complementation, the characterization of points in the finite space generated by relation (G_α) given by (iii) of Lemma 1 shows that the partition associated with (G_α) is identical to that of $\pi_\alpha^{-1}(X_\alpha) \cap X$.

(ii) Since X is strongly dense in X_∞ the map $\pi_\alpha: X \to X_\alpha$ is surjective. The surjective restriction of an identification is again an identification. The map $\pi_\alpha: X \to X_\alpha$ is the restriction of $\pi_\alpha: \Pi X_\alpha \to X_\alpha$ which is an identification map

(in fact an open mapping).

(iii) Let $\mathfrak{U}$ be any finite subset of B. For each $U \in \mathfrak{U}$ choose some $\alpha \in I$ such that $U \in G_\alpha$. Label the resulting finite subset $I_0 \subseteq I$. Since I is a directed set there is some $\beta \in I$ so that $\alpha \leq \beta$ for each $\alpha \in I_0$. Hence it is sufficient to show that for $\alpha \leq \beta$, $G_\alpha \subseteq G_\beta$. Let U_α be open in X_α. Then $f_{\alpha\beta}^{-1}(U_\alpha) = V_\beta$ is open in X_β and $\pi_\beta^{-1}(V_\beta) \cap X \in G_\beta$. By Theorem E we get

$$\pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1} f_{\alpha\beta}^{-1}(U_\alpha) = \pi_\alpha^{-1}(U_\alpha)$$

and so
$$\pi_\beta^{-1}(V_\beta) \cap X = \pi_\alpha^{-1}(U_\alpha) \cap X \in G_\alpha.$$

Since U_α was arbitrary we have $G_\alpha \subseteq G_\beta$. So $\mathfrak{U} \subseteq G_\beta$ for β given as above and $\{G_\alpha\}$ is cofinal as required and X_∞ is basic.

When necessary for clarity we will write $X_\infty(B)$ to indicate that the basis B has generated the finite resolution. In general for any $\alpha \in I$ we write X_α is the finite space generated by α to indicate all of the construction that yields the factor space X_α.

We have already noted that if X_∞ is a finite resolution for X then strong density implies that $\pi_\alpha : X_\infty \to X_\alpha$ is surjective for each $\alpha \in I$. That this condition on the projections is sufficient for a given inverse limit space of finite T_0 - spaces to be a finite resolution of some T_0 - space, X, is also clear. For each $\alpha \in I$ and $x_\alpha \in X_\alpha$ choose $z(x_\alpha)$, an arbitrary element of the nonempty subset

$\pi_\alpha^{-1}(x_\alpha)$. Then X_∞ is clearly a finite resolution of

$$X = \{z(x_\alpha) : \alpha \in I \text{ and } x_\alpha \in X_\alpha\}.$$

Further note that a finite resolution of a T_0 - space, X, is also a finite resolution of any set or space between X and X_∞ itself. That is, if $X \subseteq Y \subseteq X_\infty$ then X_∞ is a finite resolution of Y since Y inherits the strong density of X.

If F is a subspace of T_0 - space X, then F is a T_0 - space and for basis B of X, $F \cap B = \{F \cap U: U \in B\}$ is a basis for space F. The expected relationship between the finite resolutions $F_\infty(F \cap B)$ and $X_\infty(B)$ is shown in the following.

<u>Theorem 5</u>: $F_\infty(F \cap B)$ is homeomorphic to the subspace

$$\{x \in X_\infty: \pi_\alpha^{-1}(\pi_\alpha(x)) \cap F \neq \emptyset \text{ for each } \alpha \in I\} \subseteq X_\infty.$$

<u>Proof</u>: Intersecting each subset of a partition of X with the subspace, F, yields a partition of F. Let

$$F \cap \alpha = \{F \cap U: U \in \alpha\} \text{ for each } \alpha \in I.$$

The partition $F_{F\cap\alpha}$ given by relation $(F \cap \alpha)$ is that of the intersection of partition X_α of X and the subspace F for each $\alpha \in I$. For convenience we relabel this space as F_α. As usual $I = \{\alpha \subseteq B: \alpha \neq \emptyset \text{ and finite}\}$ and for each $\alpha \in I$ we define $\varphi_\alpha: F_\alpha \to X_\alpha$ as follows. For each $z_\alpha \in F_\alpha$ let $\varphi_\alpha(z_\alpha) = x_\alpha$ iff $z_\alpha \subseteq x_\alpha \subseteq X$. By our previous constructions each φ_α is an imbedding. Also the collection $\{\varphi_\alpha\}$ is an inverse system of homeomorphisms into and so induces the map $\varprojlim \{\varphi_\alpha\} = \Phi : F_\infty(F \cap B) \to X_\infty(B)$ by Theorem B. Further, F_∞ is imbedded in X_∞ by Φ via Theorem C. Also recall the fact that Φ is unique with respect to $\pi_\alpha \circ \Phi = \varphi_\alpha \circ \pi_{F\cap\alpha}$ where $\pi_{F\cap\alpha}: F_\infty \to F_\alpha$ is the canonical projection.

Let $z \in F_\infty$ so that $\Phi(z) = x \in X_\infty$. Suppose there is some $\alpha \in I$ such that $\pi_\alpha^{-1}(\pi_\alpha(x) = x_\alpha) \cap F = \emptyset$. Then $x_\alpha \notin \varphi_\alpha(F_\alpha)$ by the definition of φ_α. But $\pi_\alpha(\Phi(z)) = x_\alpha$ which contradicts the uniqueness condition for map Φ given above. Hence $\pi_\alpha^{-1}(x_\alpha) \cap F \neq \emptyset$ for each $\alpha \in I$.

Conversely, let $x \in X_\infty$ such that for each $\alpha \in I$ $\pi_\alpha^{-1}(x_\alpha) \cap F \neq \emptyset$ and consider point $z \in F_\infty$ with coordinates $\pi_{F\cap\alpha}(z) = \pi_{F\cap\alpha}(\pi_\alpha^{-1}(x_\alpha) \cap F)$. Then for each $\alpha \in I$ the definition of φ_α implies that $\varphi_\alpha(\pi_{F\cap\alpha}(z)) = x_\alpha$ and so $\Phi(z) = x$ since $\varphi_\alpha \circ \pi_{F\cap\alpha} = \pi_\alpha \circ \Phi$. Hence $\Phi(F_\infty)$ is the subspace of the conclusion.

Products of finite resolutions also follow an expected pattern.

<u>Theorem 6</u>: Let X and Y be T_o - spaces with finite resolutions X_∞ and Y_∞ respectively. The product $X_\infty \times Y_\infty$ of the finite resolutions is a finite resolution of the product $X \times Y$.

<u>Proof</u>: Let $B(X)$ and $B(Y)$ be the bases which generate X_∞ and Y_∞ and we write;

$$X_\infty(B(X)) = \varprojlim \{X_\alpha, f_{\alpha\gamma}\} \text{ and } Y_\infty(B(Y)) = \varprojlim \{Y_\beta, g_{\beta\delta}\}.$$

We first prove that the limit space

$$Z_\infty = \lim_{\leftarrow} \{X_\alpha \times Y_\beta,\ h^{\gamma\delta}_{\alpha\beta} = f_{\alpha\gamma} \times g_{\beta\delta}\}$$

is a finite resolution of $Z = X \times Y$ generated by the basis $B(X) \times B(Y)$. The second part of the proof shows that Z_∞ and $X_\infty \times Y_\infty$ are homeomorphic.

Let $\alpha \times \beta = \{U \times V\colon U \in \alpha$ and $V \in \beta\}$. Now $\alpha \times \beta$ is a finite subset of the basis $B(X) \times B(Y)$ for Z. Let $Z_{\alpha\beta}$ be the finite space generated by $\alpha \times \beta$. Since X_α and Y_β represent partitions of X and Y and are topologized by identification, and since $\alpha \times \beta$ consists of all possible products from α and β, $Z_{\alpha\beta} = X_\alpha \times Y_\beta$. Clearly the collection $\{\alpha \times \beta\}$ is cofinal in

$$\{\xi\colon \xi \subseteq B(X) \times B(Y) \text{ and } \xi \text{ finite and nonvoid}\}$$

and $\{f_{\alpha\gamma} \times g_{\beta\delta}\}$ are the bonding maps necessary. Hence Z_∞ is a finite resolution of Z as required.

Now let $z \in Z_\infty$. Each coordinate $z_{\alpha\beta}$ of z has corresponding coordinates in X_α and Y_β. Call them $x_\alpha(z)$ and $y_\beta(z)$. They related in the partitionings as follows.

$$\pi_\alpha^{-1}[x_\alpha(z)] \cap X = p_x[\pi_{\alpha\beta}^{-1}(z_{\alpha\beta}) \cap Z]$$

$$\pi_\beta^{-1}[y_\beta(z)] \cap Y = p_y[\pi_{\alpha\beta}^{-1}(z_{\alpha\beta}) \cap Z]$$

where p_x and p_y are the first and second coordinate projections from the product space $Z = X \times Y$ and $\pi_{\alpha\beta}\colon Z_\infty \to Z_{\alpha\beta}$ is the usual coordinate projection from the inverse limit

space. Define the map $\Phi : Z_\infty \to X_\infty \times Y_\infty$ as follows. For each $z \in Z_\infty$ let $\Phi(z) = (x,y)$ iff for each α and β, $x_\alpha = x_\alpha(z)$ and $y_\beta = y_\beta(z)$. By this construction Φ is well defined and $(x,y) \in X_\infty \times Y_\infty$. Further, Φ is surjective for if $(x,y) \in X_\infty \times Y_\infty$ let $z \in \Pi Z_{\alpha\beta}$ such that for each α and β the coordinate

$$z_{\alpha\beta} = \pi_{\alpha\beta}[(\pi_\alpha^{-1}(x_\alpha) \cap X) \times (\pi_\beta^{-1}(y_\beta) \cap Y)] .$$

Now $z_{\alpha\beta}$ is well defined because of the constructed partitions and the resulting $z \in Z_\infty$ and $\Phi(z) = (x,y)$.

Φ is injective for if $w \neq z$ then for some $\alpha\beta$ we must have $w_{\alpha\beta} \neq z_{\alpha\beta}$ and hence either $x_\alpha(w) \neq x_\alpha(z)$ or $y_\beta(w) \neq y_\beta(z)$ and so $\Phi(w) \neq \Phi(z)$. Continuity and open map properties for Φ follow from the observation that from the construction $\Phi^{-1}(\pi_\alpha^{-1}(U_\alpha) \times \pi_\beta^{-1}(V_\beta)) = \pi_{\alpha\beta}^{-1}(U_\alpha \times V_\beta)$. That is, the inverse image of a basis for $X_\infty \times Y_\infty$ is a basis for Z_∞. This completes the proof.

For infinite products the theorem is simply a notational exercise and the result hinges on the fact that a finite collection of open sets in ΠX^i involves proper open subsets of X^i for only a finite number of the indexes, $i_1, i_2, \cdots, i_n$. Then

$$\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n = \{U^{i_1} \times U^{i_2} \times \cdots \times U^{i_n} \times \prod_{i \neq i_k} X^i : U^{i_k} \in \alpha_k, k=1, \cdots, n\}$$

is still a finite collection of open sets in ΠX^i. The finite spaces, mappings, etc. follow as before.

Continuous mappings onto T_o - spaces induce continuous mappings between certain of their finite resolutions in the following manner.

Theorem 7: Let X and Y be T_o - spaces with bases $B(X)$ and $B(Y)$. Write $X_\infty = \{X_\gamma, g_{\gamma\delta}\}_J$ and $Y_\infty = \{Y_\alpha, f_{\alpha\beta}\}_I$ to represent the finite resolutions generated by these bases. Let $\varphi: X \to Y$ be a continuous surjection such that

$$\varphi^{-1}(B(Y)) \subseteq B(X).$$

Then there exists a continuous map $\Phi: X_\infty \to Y_\infty$ such that $\Phi|_X = \varphi$.

Proof: By hypothesis the correspondence $\alpha \to \varphi^{-1}(\alpha)$ is a well defined order preserving map from I into J. Further, since φ is surjective, φ^{-1} preserves all necessary set operations so that by (iii) of Lemma 1

$$\varphi(X_{\varphi^{-1}(\alpha)}) = Y_\alpha$$

where equality means that the partition on Y is the same. In fact this shows that $X_{\varphi^{-1}(\alpha)}$ and Y_α are homeomorphic. Let the homeomorphism linking corresponding points (subsets) in the partitions be written as $\varphi_\alpha : X_{\varphi^{-1}(\alpha)} \to Y_\alpha$.

If $\pi_\alpha : Y_\infty \to Y_\alpha$ and $\pi_{\varphi^{-1}(\alpha)} : X_\infty \to X_{\varphi^{-1}(\alpha)}$ are the usual coordinate projections then for each $z \in X_{\varphi^{-1}(\alpha)}$ we have

$$\varphi_\alpha(z) = \pi_\alpha(\varphi(\pi^{-1}_{\varphi^{-1}(\alpha)}(z) \cap X)).$$

Also the construction preserves partitions in that

$$\pi_\alpha(\varphi(\pi^{-1}_{\varphi^{-1}(\alpha)}(\pi_{\varphi^{-1}(\alpha)}(x)) \cap X)) = \pi_\alpha(\varphi(x))$$

for each $x \in X$. By chasing subsets in partitions this shows that for $\alpha \leq \beta$ in I the diagram below commutes.

$$\begin{tikzcd} Y_\beta \arrow[r, "f"] & Y_\alpha \\ X_{\varphi^{-1}(\beta)} \arrow[u, "\varphi_\beta"] \arrow[r, "g"] & X_{\varphi^{-1}(\alpha)} \arrow[u, "\varphi_\alpha"'] \end{tikzcd}$$

Hence the collection $\{\varphi_\alpha : \alpha \in I\}$ is an inverse system of maps and we let $\Phi = \varprojlim \{\varphi_\alpha\}$ be the induced map of Theorem B. Then $\Phi: X_\infty \to Y_\infty$ and Φ is continuous and unique with respect to $\pi_\alpha \circ \Phi = \varphi_\alpha \circ \pi_{\varphi^{-1}(\alpha)}$. So for $x \in X$ we have for each $\alpha \in I$,

$$\pi_\alpha(\Phi(x)) = \varphi_\alpha(\pi_{\varphi^{-1}(\alpha)}(x)) = \pi_\alpha(\varphi(\pi^{-1}_{\varphi^{-1}(\alpha)}(\pi_{\varphi^{-1}(\alpha)}(x)) \cap X))$$

$$= \pi_\alpha(\varphi(x)).$$

Hence $\Phi(x) = \varphi(x)$ for each $x \in X$ and the proof is complete.

<u>Example 1</u>: We have constructed a finite resolution $X_\infty(B)$ of T_0 - space X, for each basis B of X. Order the bases by inclusion. For bases $B \subseteq B'$ the previous theorem implies the existence of a continuous map

$$\Phi_{BB'}: X_\infty(B') \to X_\infty(B)$$

such that $\Phi_{BB'}$ is the identity map on X. If we then write $X_\infty(B') = \varprojlim \{X_\alpha, f_{\alpha\beta}\}_I$, then $X_\infty(B)$ is homeomorphic to

$\lim\limits_{\leftarrow}\{X_\alpha, f_{\alpha\beta}\}_I$ where $I = \{\alpha \subseteq B \subseteq B' : \alpha \in I'\} \subseteq I'$. That is, $\Phi_{BB'}$ is just the natural projection given by Theorem D from $\lim\limits_{\leftarrow}\{X_\alpha, f_{\alpha\beta}\}_{I'}$ into $\lim\limits_{\leftarrow}\{X_\alpha, f_{\alpha\beta}\}_I$. Then $\{X_\infty(B), \Phi_{BB'}\}$ is an inverse system of spaces. By (2) of Theorem J the resulting inverse limit space is homeomorphic to $X(\tau)$ where τ is the basis of all open sets of X. Hence inverse limits of finite resolutions yield nothing more of interest. We call $X(\tau)$ the <u>total resolution</u> of X.

Homeomorphic finite resolutions occur for certain collections of bases as shown in the following.

<u>Lemma 2</u>: Let α be a finite nonvoid collection of open sets of space X. Let $\alpha+ = \{$intersections of sets from $\alpha\}$. Then the finite T_0 - spaces, X_α and $X_{\alpha+}$, generated by α and $\alpha+$ are homeomorphic.

<u>Proof</u>: It is sufficient to show that the partitions X_α and $X_{\alpha+}$ generated by relations (α) and $(\alpha+)$ are identical. Since $\alpha \subseteq \alpha+$, properly refines X_α and so for $x,y \in X$ $x(\alpha+)y$ implies $x(\alpha)y$. Conversely suppose that $x(\alpha)y$. Let $\mathfrak{U}_x = \{U \in \alpha+ : x \in U\}$ and $\mathfrak{U}_y = \{U \in \alpha+ : y \in U\}$.

We wish to show that $\mathfrak{U}_x = \mathfrak{U}_y$ which implies that $x(\alpha+)y$. Hence $x(\alpha)y$ iff $x(\alpha+)y$ and the partitions are identical.

Case 1: $\mathfrak{U}_x = \emptyset$. If $y \in U \in \alpha+$ then there is some $V \in \alpha$ with $y \in V$ as U is an intersection from α. Then $x(\alpha)y$ implies that $U \in \mathfrak{U}_x$. This contradicts $\mathfrak{U}_x = \emptyset$ and hence $\mathfrak{U}_y = \emptyset$.

Case 2: Let $x \in U \in \alpha+$. We may write $U = \cap U_i$ where each $U_i \in \alpha$. But $x \in U$ iff $x \in U_i \in \alpha$ for each i iff $y \in U_i \in \alpha$ for each i iff $y \in U$. Hence $\mathcal{U}_x = \mathcal{U}_y$.

Theorem 8: Given a basis B for a T_o - space X, construct basis $B+ = \{$finite intersections from $B\}$. Then finite resolutions $X_\infty(B)$ and $X_\infty(B+)$ are homeomorphic.

Proof: First note that if $I = \{\alpha \subseteq B: \alpha \neq \emptyset$ and finite$\}$ then $I+ = \{\alpha+ \subseteq B+ : \alpha \subseteq B\}$ is cofinal in $J = \{\gamma \subseteq B+ : \gamma \neq \emptyset$ and finite$\}$. Hence by Lemma 2 and Theorem C the finite resolutions above are homeomorphic. The homeomorphism intended here is $\varprojlim\{\varphi_\alpha\} = \Phi : X_\infty(B+) \to X_\infty(B)$ where $\varphi_\alpha : X_{\alpha+} \to X_\alpha$ is the homeomorphism of Lemma 2 that preserves subsets of the partitions.

Corollary: Let B' be any basis such that $B \subseteq B' \subseteq B+$. Then $X_\infty(B')$ and $X_\infty(B)$ are homeomorphic.

Proof: The induced homeomorphism of Theorem 8 is clearly the same as the projection of Example 1 where $B \subseteq B+$. Hence any B' in the position given yields a finite resolution homeomorphic to $X_\infty(B)$ as it is projected through this homeomorphism. More specifically, for $\alpha \subseteq B$, $\{\alpha+\}$ is cofinal in $\{\beta \subseteq B+ : \beta \neq \emptyset$ and finite$\}$ and this implies that $\{\alpha+ \cap B' : \alpha \subseteq B\}$ is cofinal in $\{\gamma \subseteq B' : \gamma \neq \emptyset$ and finite$\}$ and the generated finite spaces are still homeomorphic to the original X_α's.

CHAPTER FOUR

COMPACT HAUSDORFF SPACES

In this chapter we find that separation properties and then compactness for a given topological space force a more rigid positioning of this space in its finite resolutions. Several characterizations and properties are found in the compact Hausdorff case.

Definition: Let S be a topological space. Subspace $X \subseteq S$ is T_2 - separated in S iff for distinct $x, y \in X$ there are disjoint open sets U, V in S such that $x \in U$ and $y \in V$. Subspaces X and Y of S are T_2 - separated in S iff there are disjoint open sets U, V in S such that $X \subseteq U$ and $Y \subseteq V$.

Lemma 3: Let $X_\infty(B) = \varprojlim\{X_\alpha, f_{\alpha\beta}\}_I$ be a finite resolution of Hausdorff space X. Then X is T_2 - separated in X_∞. Further, if Y and Z are T_2 - separated in X then Y and Z are T_2 - separated in X_∞.

Proof: Since X is Hausdorff there are $U, V \in B$ with $U \cap V = \emptyset$, $x \in U$ and $y \in V$. Let $\alpha = \{U, V\} \in I$. Then by (iii) of Lemma 1 $\pi_\alpha(U) = U_\alpha$ and $\pi_\alpha(V) = V_\alpha$ are disjoint open sets in X_α. Continuity of π_α implies that $\pi_\alpha^{-1}(U_\alpha)$ and $\pi_\alpha^{-1}(V_\alpha)$ are disjoint open sets in X_∞ containing x and y respectively. Now let $\mathcal{U}$ and $\mathcal{V}$ be disjoint open sets of X with $Y \subseteq \mathcal{U}$ and $Z \subseteq \mathcal{V}$. Then $\mathcal{U} = \cup U$ and

$\mathfrak{V} = \cup V$ where U's and V's are basic open sets from B. For each U involved in $\cup U$ let $\alpha_U = \{U\} \in I$. Then the factor space X_{α_U} has two points x_{α_U} and y_{α_U} where $\pi_{\alpha_U}^{-1}(x_{\alpha_U}) \cap X = U$. Then clearly

$$[\cup \pi_{\alpha_U}^{-1}(x_{\alpha_U})] \cap X = \mathfrak{U}.$$

Similarly define α_V and x_{α_V} and get

$$[\cup \pi_{\alpha_V}^{-1}(x_{\alpha_V})] \cap X = \mathfrak{V}.$$

Since singletons x_{α_U} and x_{α_V} are open in X_{α_U} and X_{α_V} the sets $\mathfrak{U}' = \cup\pi_{\alpha_U}^{-1}(x_{\alpha_U})$ and $\mathfrak{V}' = \cup \pi_{\alpha_V}^{-1}(x_{\alpha_V})$ are open in X_∞ and they contain $\mathfrak{U}$ and $\mathfrak{V}$ and hence Y and Z respectively. If $\mathfrak{U}' \cap \mathfrak{V}' \neq \emptyset$ then since X is dense in X_∞, $\mathfrak{U}' \cap \mathfrak{V}' \cap X \neq \emptyset$. But this contradicts $\mathfrak{U} \cap \mathfrak{V} = \emptyset$ since by construction we have $\mathfrak{U}' \cap \mathfrak{V}' \cap X = \mathfrak{U} \cap \mathfrak{V}$. So $\mathfrak{U}' \cap \mathfrak{V}' = \emptyset$ and Y and Z are T_2 - separated subspaces in X_∞.

We note that a similar definition and result is easily proved for T_1 - spaces. Also, the condition of being a T_2-separated subspace is a strictly stronger condition than that of being a Hausdorff subspace. The importance of this is seen in the following.

<u>Definition</u>: Let S be a topological space. Subspace $X \subseteq S$ is <u>co-dense</u> in S iff for each $s \in S$, $\overline{s} \cap X \neq \emptyset$. The closure of course is taken in S.

<u>Example</u> <u>2</u>: Let $S = \{x,y,z\}$ with topology given by $\tau = \{\emptyset, S, \{x,y\}, \{y,z\}, \{y\}\}$. Then $X = \{x,z\}$ is co-dense in S.

Clearly, for T_1 - space S, X is co-dense in S iff $X = S$. However, Example 2 shows that X being co-dense in S does not imply that X is dense in S. Further, note that in Example 2 the subspace X is Hausdorff but not T_2-separated in S and $\overline{\{y\}} = S$. This shows that T_2-separation is necessary in the following.

<u>Lemma 4</u>: Let X be co-dense and T_2 - separated in S. Then $\bar{s} \cap X$ is a singleton for each $s \in S$.

<u>Proof</u>: Let $s \in S$ and let $x,y \in X$ be distinct points. By T_2 - separation there are disjoint U and V open in S with $x \in U$ and $y \in V$. Hence at most one of the elements x and y can be contained in $\overline{\{s\}}$.

We now prove a general result which is used in characterizing the position of a compact Hausdorff space in its finite resolutions.

<u>Lemma 5</u>: Let $X_\infty = \varprojlim\{X_\alpha, f_{\alpha\beta}\}_I$ be an inverse limit of finite spaces. Then X_∞ is compact.

<u>Proof</u>: Let X'_α be the same point set as X_α, but with the discrete topology. Then for each $\alpha \in I$ the correspondence $x'_\alpha \to x_\alpha$ defines a continuous bijection $\varphi_\alpha : X'_\alpha \to X_\alpha$. Then if $X'_\infty = \varprojlim\{X'_\alpha, f'_{\alpha\beta}\}_I$ where $f'_{\alpha\beta}(x'_\beta) = x'_\alpha$ iff $f_{\alpha\beta}(\varphi_\beta(x'_\beta)) = \varphi_\alpha(x'_\alpha)$, the collection $\{\varphi_\alpha\}$ is an inverse system of maps with limit map $\Phi : X'_\infty \to X_\infty$ also a continuous bijection. Since X'_α is compact and Hausdorff, X'_∞ is compact and Hausdorff by Theorems G & H. Then X_∞ is compact as the image of X'_∞ via Φ.

<u>Corollary</u>: The inverse limit space of an inverse sequence of finite spaces is the continuous image of the Cantor discontinuum.

<u>Proof</u>: This is discussed further in Example D of chapter five.

<u>Corollary</u>: Every finite resolution of a T_o - space is compact. Every finite resolution of a connected T_o - space is a T_o - continuum.

<u>Proof</u>: The first result follows directly from the previous lemma. The second follows from the additional observation that the space is dense in its finite resolution.

The following is a well known result on compact T_o - spaces.

<u>Lemma 6</u>: Let S be a compact T_o - space and let F be any nonvoid closed subset in S. Then F contains a closed singleton.

<u>Proof</u>: Let $\mathcal{J} = \{F_i \subseteq F : F_i \neq \emptyset$ and F_i closed in $F\}$. Since F is closed in S, each F_i is closed in S. Order $\mathcal{J}$ by inclusion and let $\mathcal{C}$ be any simple chain from $\mathcal{J}$. By the compactness of S, $\cap\,\mathcal{C} \neq \emptyset$ and so $\cap\,\mathcal{C} \in \mathcal{J}$. By Zorn's Lemma, $\mathcal{J}$ has a minimal element $M \neq \emptyset$. Let $x, y \in M$. The minimal property of M implies that $\overline{x} = \overline{y} = M$. Since S is a T_o - space this implies that $x = y$ and hence M is a singleton.

The next result is a direct consequence of Lemma 6 but to the best of our knowledge it is not well known.

Theorem 9: Let S be a compact T_o - space and let X be any subset of S such that every closed singleton of S is contained in X. Then X is compact.

Proof: Let $\mathfrak{U}$ be any open covering of X. Then $\mathfrak{U}$ also covers S. If not then $S - \cup\mathfrak{U}$ is nonvoid and closed and must contain a point in X. Contradiction! Now S is compact and so $\mathfrak{U}$ contains a finite subcovering of S and hence of $X \subseteq S$. Therefore X is compact.

Theorem 10: Let $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}_I$ be a finite resolution of Hausdorff space X. The following are equivalent.

(i) X is compact.

(ii) X is co-dense in X_∞.

(iii) $X = \{x \in X_\infty : \overline{\{x\}} = \{x\}\}$.

Proof: (i) implies (ii). By the strong density of X in X_∞, $\mathcal{J} = \{\pi_\alpha^{-1}(\overline{\pi_\alpha(z)}) \cap X : \alpha \in I\}$ is a collection of non-void closed subsets of X for each $z \in X$. Since I is directed, (ii) of Lemma 1 implies that $\mathcal{J}$ is a filterbase on X. X is compact and so $\cap\mathcal{J} \neq \emptyset$. By Theorem I, $\cap\mathcal{J} = \bar{z} \cap X$.

(ii) implies (iii). By Lemma 3, X is T_2 - separated in X_∞ and so by Lemma 4, $\{x \in X_\infty : \overline{\{x\}} = \{x\}\} \subseteq X$. Now let $x \in X$ and $y \in X$ with $y \neq x$. If $y \in X$ then Lemma 3 implies $y \notin \bar{x}$. If $y \notin X$ then X being co-dense implies that there exists $z \in X$ with $z \in \bar{y}$. Suppose $y \in \bar{x}$.

Then $z \in \overline{x}$ and so $z = x$. Hence $x \in \overline{y}$ and $y \in \overline{x}$ and this contradicts the T_0 - separation of X_∞. So $y \notin \overline{x}$. We have shown that for each $y \neq x$, $y \notin \overline{x}$. That is, $\overline{x} = x$ and $X \subseteq \{x \in X_\infty : \overline{\{x\}} = \{x\}\}$.

(iii) implies (i). Direct from Theorem 9.

We note that the equivalence of (ii) and (iii) is more general than given above. The structure of X_∞ is not necessary, but just that X is a T_2 - separated subspace of a compact T_0 - space. That is sufficient for (ii) implies (iii) and then (iii) implies (ii) by Lemma 6.

<u>Corollary</u>: Let X_∞ be a finite resolution of a compact Hausdorff space X. Then X is a maximal Hausdorff subspace of X_∞.

<u>Proof</u>: Let $X \subseteq Y \subseteq X_\infty$ such that Y is Hausdorff. If $y \in Y - X$ then there is some $x \in X$ such $x \in \overline{y}$, since X is co-dense in X_∞. Hence Y is not T_2 - separated in X_∞ but X_∞ is a finite resolution of $Y \supseteq X$. This contradicts Lemma 3 and so $Y - X = \emptyset$.

A final indication of the position of $X \subseteq X_\infty$ in the compact Hausdorff case is given in the following.

<u>Theorem 11</u>: Let $X_\infty(B) = \varprojlim \{X_\alpha, f_{\alpha\beta}\}_I$ be a finite resolution of a compact Hausdorff space X. Then X is a retract of X_∞.

<u>Proof</u>: Define $r: X_\infty \to X$ as follows. For each $z \in X_\infty$, let $r(z) = \overline{z} \cap X$. This map is well defined by Lemmas 3 & 4

and Theorem 10. That is, $\overline{z} \cap X$ is a singleton. Also, by Theorem 10, $r(x) = x$ for each $x \in X \subseteq X_\infty$.

To show that r is continuous it is sufficient to show that $r^{-1}(U)$ is open in X_∞ for each $U \in B$. Let $z \in X_\infty$ such that $r(z) \in U$. Since X is normal there is $V \in B$ with $r(z) \in V \subsetneq \overline{V} \subsetneq U$. Let $\alpha = \{V\} \in I$. We have seen that in X_α the set $\pi_\alpha(V) = V_\alpha$ is an open singleton. Then $\pi_\alpha^{-1}(V_\alpha)$ is open in X_∞. By this construction $\pi_\alpha(r(z)) = V_\alpha$. Hence $\pi_\alpha(z) = V_\alpha$ since if $\pi_\alpha(z) = X_\alpha - V_\alpha$ then

$$\pi_\alpha(r(z)) = \pi_\alpha(\overline{z} \cap X) \subseteq \pi_\alpha(\overline{z}) \subseteq \overline{\pi_\alpha(z)} = X_\alpha - V_\alpha$$

contradicting the construction. So $z \in \pi_\alpha^{-1}(V_\alpha)$. It remains to show that $\pi_\alpha^{-1}(V_\alpha) \subseteq r^{-1}(U)$. Suppose not and $y \in \pi_\alpha^{-1}(V_\alpha)$ such that $r(y) \notin U$. Then the normality of X implies the existence of $W \in B$ such that $W \cap \overline{V} = \emptyset$ and $r(y) \in W$.

Now let $\beta = \{V, W\} \in I$. In X_β, $\pi_\beta(V) = V_\beta$ and $\pi_\beta(W) = W_\beta$ are open points and V_α and V_β represent the same subset in the partitions of X. We have the following contradiction.

$$W_\beta = \pi_\beta(r(y)) = \pi_\beta(\overline{y} \cap X) \subseteq \pi_\beta(\overline{y}) \subseteq \overline{\pi_\beta(y)} = \overline{V}_\beta \subseteq X_\beta - W_\beta.$$

Hence $z \in \pi_\alpha^{-1}(V_\alpha) \subseteq r^{-1}(U)$ and so $r^{-1}(U)$ is open in X_∞.

<u>Corollary</u>: If X is a compact metric space then X_∞ is a semimetric space.

<u>Proof</u>: Let $d: X \times X \to R$ be a metric on X. For $y, z \in X_\infty$ let $d_\infty(y,z) = d(r(y), r(z))$. Then d_∞ is clearly a semi-metric on X_∞ and $d_\infty(y,z) = 0$ iff $r(y) = r(z)$.

The next general result shows that the retraction of Theorem 11 is the only one possible.

<u>Lemma 7</u>: Let X be a Hausdorff co-dense retract of space S. Then the retract $r: S \to X$ is unique and for each $s \in S$ $r(s) = \bar{s} \cap X$.

<u>Proof</u>: A Hausdorff retract of a space is clearly T_2 - separated in that space. Hence, by Lemma 4, the given mapping is well defined. Now let $q: S \to X$ be a retraction and suppose for some $s \in S$ that $q(s) \neq r(s)$. Since X is Hausdorff there are disjoint open sets U and V containing $q(s)$ and $r(s)$. Then $q^{-1}(U)$ and $q^{-1}(V)$ are disjoint open sets in S. But $r(s) \in \bar{s}$ and $r(s) \in q^{-1}(V)$ imply that $s \in q^{-1}(V)$ and by the construction $s \in q^{-1}(U)$. This contradiction implies that $q(s) = r(s)$ for each $s \in S$.

<u>Definition</u>: Let X be a Hausdorff (T_o -) space. Compact space, $\hat{X}$, is a <u>Hausdorff</u> (T_o -) <u>compactification</u> of X iff $\hat{X}$ is Hausdorff (T_o) and X is imbedded as a dense subspace of $\hat{X}$.

By Lemma 5, every finite resolution X_∞ of a T_o - space, X, is a T_o - compactification of X. In fact, if

$$A = \{\text{closed singletons in } X_\infty - X\}$$

and $X \cup A \subseteq Y \subseteq X_\infty$, then Y is a T_0 - compactification of X. If X is Hausdorff the situation is not as vague for Hausdorff compactifications.

Lemma 8: Let X_∞ be a finite resolution of Hausdorff space X. Let $X \subseteq \hat{X} \subseteq X_\infty$ such that $\hat{X}$ is a Hausdorff compactification of X. Then $\hat{X} = X \cup A$.

Proof: $\hat{X}$ inherits X_∞ as a finite resolution. By Theorem 10 each $x \in X$ is a closed singleton in X_∞ so $\hat{X} = X \cup A$.

CHAPTER FIVE

EXAMPLES

In most of these examples a reference is made to the previous chapters. We strive for a reasonable graphic representation. For $X_\infty = \varprojlim \{X_\alpha, f_{\alpha\beta}\}_I$, a finite resolution of X, we call each point $z \in X_\infty$ a <u>thread</u> in X_∞. Each $\alpha \in I$ will be called a <u>level</u> in X_∞. For most of the following examples, the number of levels will be countably infinite and the indexing set ordered as the nonnegative integers.

Then X_∞ may be easily represented by a graph which is in fact a <u>tree</u> with finite levels, where the k-th level has as many points as the space X_k, for each $k \in I$. Point x_k in level k is connected by an edge to point x_{k+1} in level $k+1$ iff $f_{k,k+1}(x_{k+1}) = x_k$. Say that x_{k+1} is a <u>predecessor</u> of x_k. Clearly, we may require this tree to be rooted since adding space X as a basic open set and as an additional element for each $k \in I$, to redefine I, does not affect this resolution up to homeomorphism and $0 = \{X\}$ is then minimum in the new indexing set with resulting finite space a singleton and the root of the tree.

<u>Example A</u>: Let $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ have the tower topology $\tau = \{\emptyset, \{x_1\}, \{x_1, x_2\}, \cdots, \{x_1, x_2, \cdots, x_n\}, \cdots, X\}$. In this case we will use basis $B = \tau$ and let $I = \{0, 1, \cdots\}$ where for each $k \geq 1$, $k = \{X, \{x_1\}, \cdots, \{x_1, \cdots, x_k\}\}$ and let

$O = \{X\}$. Then the sets of partition X_k are

$$\{\{x_1\},\{x_2\},\cdots,\{x_k\},\{x_{k+1},\cdots\}\}$$

for $k \geq 1$. That is, level k has $k+1$ points and we label them as $x_{k1}, x_{k2}, \cdots, x_{k,k+1}$. Each X_k also has the tower topology. The resulting bonding maps are

$$f_{k,k+1}(x_{k+1,j}) = x_{kj} \qquad j \leq k+1$$
$$= x_{k,k+1} \qquad j = k+2$$

The resulting finite resolution is represented as the tree in Diagram A.

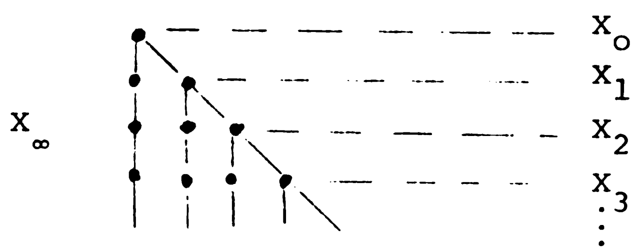


Diagram A

Only the topologies on the X_k are not indicated. They are necessary to indicate those collections of threads which are to be open sets in X_∞. By Theorem F, a basis for the topology of X_∞ is easily represented in the tree. Simply choose an open set in any level. The set of all threads radiating from this set is open in X_∞. The collection of all such open sets is a basis for X_∞. In the present example, the finite resolution is clearly a one point T_o-compactification of X, where the new point is the thread

$$x_\infty = (x_{01}, x_{12}, \cdots, x_{k,k+1}, \cdots)$$

This example is also an easy illustration for Lemma 1 and Theorems 1, 2 & 3.

Example B: Let $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ be the countable discrete space. Let B be the basis of singletons and for $k \geq 1$ let $k = \{\{x_1\}, \cdots \{x_k\}, X\} \in I$. Then X_k has $k+1$ points and the tree of Diagram A represents the resulting finite resolution X_∞, but in this case each level is a discrete space. X_∞ is clearly totally disconnected and is the one point Hausdorff compactification of the integers.

In general, if an inverse limit of finite spaces is Hausdorff then the topologies on these finite spaces may be taken as the discrete topology, since the bijection of Lemma 5 is then a homeomorphism. Hence a finite resolution, X_∞ of X, being Hausdorff implies that X_∞, and hence X, is totally disconnected. For example, the unit interval cannot be the inverse limit of finite spaces.

Example C: Let $X = [0,1]$ be the unit interval with basis B = {open intervals with dyadic rational end points}. It is understood that 0 and 1 are always included in the relative open interval when possible. Again, I is the non-negative integers where for each $k \in I$,

$$k = \{\text{intervals with endpoints in } n/2^k,\ 0 \leq n \leq 2^k\}.$$

Then $\{k \in I\}$ is cofinal in {finite subsets of B}. The resulting finite space, X_k, has $2^{k+1}-1$ points,

$$x_{k1},\dots,x_{k(2^{k+1}-1)}$$

where these points of the partition of $[0,1]$ represent the following subsets.

$$x_{k1} = [0, \frac{1}{2^k})$$
$$x_{k2} = \{\frac{1}{2^k}\}$$
$$x_{k3} = (\frac{1}{2^k}, \frac{2}{2^k})$$
$$x_{k4} = \{\frac{2}{2^k}\}$$
$$\vdots$$
$$x_{k(2^{k+1}-1)} = (\frac{2^k-1}{2^k}, 1]$$

The resulting tree appears in Diagram C.

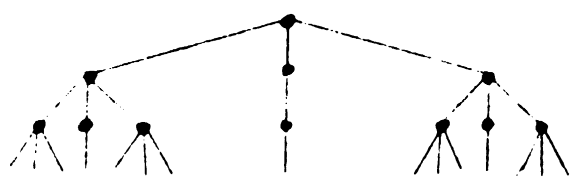


Diagram C

The points which have coordinates with just one predecessor are the dyadic rationals in $(0,1)$. Other points in each level have visibly a <u>left</u>, <u>right</u> and <u>dyadic</u> predecessor. Say that a thread is <u>eventually left</u> (<u>right</u>) <u>directed</u> iff there is some $k \in I$ such that the left (right) predecessor is chosen for all $n \geq k$ to define this thread. The point

$0 \in [0,1] \subseteq X_\infty$ is an eventually left directed thread and 1 is an eventually right directed thread. All other points of $[0,1] \subseteq X_\infty$ are represented by the complement of the set of points which are eventually left or right directed. Further, if $z \in X_\infty - X$ then $\bar{z} \cap X$ is a dyadic rational (See Theorem 10.) which in the diagram is that dyadic approached by the eventual left or right branching. We also note that $(0,1) \subseteq X_\infty$ is strongly dense in X_∞ and so this space is also a finite resolution of $(0,1)$ via the basis of dyadic intervals by Theorem 5. In fact, the set of dyadic rationals in $(0,1)$ is still strongly dense in X_∞. Of course $[0,1]$ is the unique Hausdorff compactification present in each case. See Lemma 8.

This example can be used to show the well known <u>Urysohn's Lemma</u>: Let Y be a normal space with $A,B \subseteq Y$ disjoint closed and nonvoid subsets. Then there is a continuous map $\varphi\colon Y \to [0,1] = X$ such that $\varphi(A) = 0$ and $\varphi(B) = 1$.

<u>Proof</u>: Since Y is normal there is an open set $U_{11} \subseteq Y$ with $A \subseteq U_{11}$ and $B \subseteq \bar{U}_{11}$. Define $\varphi_1\colon Y \to X_1$ by

$$U_{11} \to x_{11}$$

$$\bar{U}_{11} - U_{11} \to x_{12}$$

$$Y - \bar{U}_{11} \to x_{13}$$

Since $(Y - U_{11}) \cap A = \emptyset$ and $\bar{U}_{11} \cap B = \emptyset$ the normality of Y again implies the existence of open sets U_{21} and U_{22} such that

$$A \subseteq U_{21} \subseteq \overline{U}_{21} \subseteq U_{11} \subseteq \overline{U}_{11} \subseteq U_{22} \quad \text{and} \quad B \subseteq (Y - \overline{U}_{22}).$$

Define $\varphi_2 : Y \to X_2$ by

$$\begin{aligned} &U_{21} \to x_{21} \\ &\overline{U}_{21} - U_{21} \to x_{22} \\ &\vdots \\ &Y - \overline{U}_{22} \to x_{27} \end{aligned}$$

The construction of open sets U_{ij} continues in the usual manner for this lemma and similarly we define this system $\{\varphi_i\}$ of clearly continuous maps which also have the property, via the nested constructions,

$$f_{n(n+1)} \circ \varphi_{n+1} = \varphi_n \qquad n > 1$$

where the f_{ij} are the induced bonding maps. Hence $\{\varphi_i\}$ is an inverse system of maps with continuous limit map

$$\Phi : Y \to X_\infty = \varprojlim \{X_i, f_{ij}\}$$

by (1) of Theorem J and Theorem A.

Now let $r : X_\infty \to [0,1]$ be the retraction of Theorem 11. Then let $\varphi = r \circ \Phi$. By the definition of r and Theorem I, if $y \in A$ then

$$\begin{aligned} \varphi(y) = r(\Phi(y)) &= \bigcap_{k=1}^{\infty} [\pi_k^{-1} \overline{(\pi_k(\Phi(y))} \cap X] \\ &= \bigcap_{k=1}^{\infty} [\pi_k^{-1}(\{x_{k1}, x_{k2}\}) \cap X] \\ &= \bigcap_{k=1}^{\infty} [0, \tfrac{1}{2^k}] = 0 \end{aligned}$$

Similarly, $\varphi(y) = 1$ for each $y \in B$.

In general a finite space X with an odd number of points, $x_1, x_2, \cdots, x_{2n+1}$, will be called a $2n+1$ - train or simply a train iff X is homeomorphic to the given set of points with subbasis $\{\{x_{2k-1}, x_{2k}, x_{2k+1}\}: k = 0, 1, \cdots, n+1\}$.

Example D: Let $X = [0,1]$ with the upper limit topology. Let B be the dyadic intervals of Example C with the right end points included. Now the partitioning of X occurs as a collection of open sets and so X_k has 2^k points and is discrete. Further since each point $x_k \in X_k$ has exactly two predecessors in level $k+1$, the resulting limit space X_∞ is homeomorphic to the Cantor discontinuum. See [4] pp. 97-100. In this case threads which are eventually right directed represent the dyadic rationals with the exception of $0 \in X$ which is eventually left directed in X_∞. All other eventually left directed threads make up $X_\infty - X$. We note that by Lemma 5 the Cantor discontinuum is a Hausdorff compactification of $[0,1]$ with the upper limit topology.

Example E: Let $X = [0,1]$ with basis B consisting of the dyadic intervals of Example C and the sets of dyadic rationals contained in each of these intervals. Each level space X_k is generated by the dyadic intervals with endpoints $n/2^k$ for $n = 0, 1, \cdots, 2^k$ and those sets of dyadic rationals contained in these intervals. From Example C, X_k has $3 \cdot 2^k - 1$ points. For example, X_1 has the following subsets of X as its points.

$$x_{11} = \{\text{dyadics in } [0,1/2)\}$$

$$x_{12} = [0,1/2) - x_{11}$$

$$x_{13} = \{1/2\}$$

$$x_{14} = \{\text{dyadics in } (1/2,1]\}$$

$$x_{15} = (1/2,1] - x_{14}$$

Basic open sets in X_1 are $\{x_{11}\}, \{x_{14}\}, \{x_{11}, x_{12}\}, \{x_{14}, x_{15}\}$. Clearly, x_{12} has two predecessors in X_2 and each of these has two predecessors in X_3, etc. These points represent partitioning of the nondyadic reals in $[0,1/2)$ into those in $[0,1/4)$ and $(1/4,1/2)$, etc. In each level these points are closed singletons by our construction. Hence for

$$a \in \pi_1^{-1}(x_{12}),$$

$\{a\}$ is a closed singleton by Theorem I. And since for each such $a \in X_\infty$ that is eventually right (left) directed,

$$\cap[\pi_k^{-1}(a_k) \cap X] = \emptyset,$$

we must have $a \in A$ where A is the subset of X_∞ defined in Lemma 8.

Now X is Hausdorff but not regular and so $X \cup A$, being compact, cannot be Hausdorff since every subspace, including X, must then be completely regular. What has occurred in this case is that some point of X is not a closed singleton in X_∞. Such a point is $0 \in X$. In coordinate form we write $0 = (x_{k1})$ and see that $(x_{k2}) \in \overline{\{0\}}$ since $x_{k2} \in \overline{\{x_{k1}\}}$ for each $k = 1,2,\cdots$. See Theorem I.

<u>Example</u> <u>F</u>: Consider the following plane continuum, $X = \bigcup_{n=0}^{\infty} I_n$, homeomorphic to the topological sine curve.

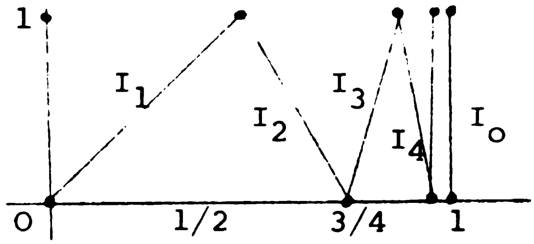


Diagram F

At step k, each interval I_n is partitioned as $[0,1]$ in Example C. To insure finiteness, X_k is generated by the usual open sets in I_n for $n = 1,2,\cdots,k$ with the exception that the only included endpoint occurs at $(0,0)$ and the open sets formed by "slicing" $\bigcup_{n=k+1}^{\infty} I_n - \{(1-\frac{1}{2^k}, 1)\}$ with horizontal lines through y-coordinates $j/2^k$ for $j = 0,1,\cdots,2^k$. The collection of all these open sets through steps $k = 1,2,\cdots$ is a basis for X and the collections at each step form a cofinal sequence for the set of finite subsets from this basis. At each step k, X_k has $(k+1)2^{k+1}-1$ points and is a <u>train</u> as described after Example C. In fact X_{2^p-1} in this example is a train of the same length as X_{2^p+p-1} in Example C. This means that the unit interval and the topological sine curve have resolutions involving the same sequence of finite spaces. Of course the bonding maps are considerably different as in the present example a "folding" of the X_{k+1} train onto the X_k train occurs.

Example G: Let X be the unit circle. We continue to form dyadic type nested partitions generated by nested collections of open sets. In this case let X have its usual position in the plane with center at the origin. The open sets generating X_k are all open arcs with endpoints on the radial lines $\theta = 2n\pi/2^k$ radians for $n = 1,2,\cdots,2^k$. For $k \geq 2$, finite space X_k is essentially a $2^{k+1} + 1$ train with its tails identified. So X_k has 2^{k+1} points and we call it a 2^{k+1} - block. A tree representing the resulting X_∞ appears in Diagram G. The three branches from the root represent the upper and lower open semicircles on the left and right with a dyadic subdivision as in Example C. The central branch contains only the two threads representing $(1,0)$ and $(-1,0)$ on the unit circle.

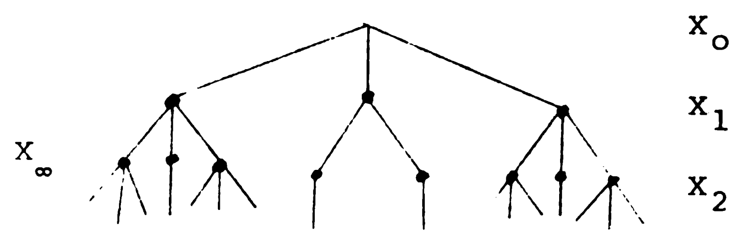


Diagram G

Example H: Let X be an annulus. By Theorem 6, the inverse limit of spaces X_k which are products of $2^{k+1} - 1$ trains and 2^{k+2} blocks is a finite resolution of the annulus, which is the product of an interval and a circle. Then X_k has $2^{k+2}(2^{k+1} - 1)$ points for $k = 1,2,\cdots$. We illustrate the subdivision X_1 and a schematic of this space as three nested 8-blocks in Diagram H. The partitioning of X continues both radially as in Example F and concentrically in X_2, etc. We note that the resulting inverse limit space is also a finite resolution for the open annulus as this subspace is strongly dense. Again by Lemma 8, X is the only compactification of the open annulus present in X_∞.

Diagram H

CHAPTER SIX

ALGEBRAIC STRUCTURES

In this chapter, X is always a compact Hausdorff space. We find then that some of the common algebraic structures relating to X are inherited by the finite resolutions of X.

Theorem 12: Let X_∞ be a finite resolution of X and let $\Phi:X_\infty \to X_\infty$ be a homeomorphism. Then $\Phi/_X$ is a homeomorphism from X onto X.

Proof: By (iii) of Theorem 10, X is exactly the subset of closed singletons in X_∞. Φ and Φ^{-1} are closed maps so

$$\Phi(X) \subseteq X \quad \text{and} \quad \Phi^{-1}(X) \subseteq X$$

Applying Φ to the latter inclusion shows that $X \subseteq \Phi(X)$ and hence $\Phi(X) = X$.

Corollary: X has the fixed point property iff X_∞ has the fixed point property.

Corollary: Let $H(X_\infty)$ and $H(X)$ be the homeomorphism groups of X_∞ and X respectively. Then the map $\mu:H(X_\infty) \to H(X)$ given by $\mu(\Phi) = \Phi/_X$ is a homomorphism.

Lemma 9: Let $\varphi:Y \to Y'$ and $\Phi:Z \to Z'$ be homeomorphisms and let $p:Z \to Y$ and $q:Z' \to Y'$ be continuous surjections such that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[r, "\Phi"] \arrow[d, "p"'] & Z' \arrow[d, "q"] \\ Y \arrow[r, "\varphi"] & Y' \end{tikzcd}$$

Then for each $y \in Y$ and $y' \in Y'$, $p^{-1}(y)$ and $q^{-1}(\varphi(y))$ are homeomorphic and $q^{-1}(y')$ and $p^{-1}(\varphi^{-1}(y'))$ are homeomorphic.

<u>Proof</u>: Simply by the commutativity. The homeomorphisms readily available are the restrictions of Φ and Φ^{-1} to $p^{-1}(y)$ and $q^{-1}(y')$ respectively.

<u>Lemma 10</u>: Let X_∞ be a finite resolution of X and let mapping $r:X_\infty \to X$ be the retraction of Theorem 11. Let $\varphi:X \to X$ be any homeomorphism that has an extension $\Phi:X_\infty \to X_\infty$. Then $r^{-1}(\varphi(x)) = \Phi(r^{-1}(x))$ for all $x \in X$.

<u>Proof</u>: With r taking the place of p and q in Lemma 9, we have the required commuting diagram since the closure operator commutes with homeomorphisms. For example, if $z \in X_\infty$

$$\begin{aligned} \varphi^{-1}(r(\Phi(z))) &= \varphi^{-1}(\overline{\Phi(z)} \cap X) \\ &= \varphi^{-1}(\Phi(\bar{z}) \cap X) \\ &= \Phi^{-1}(\Phi(\bar{z}) \cap X) \\ &= \bar{z} \cap X = r(z) \end{aligned}$$

Then $r^{-1}(x)$ and $r^{-1}(\varphi(x))$ are homeomorphic via Φ, which in this case means $\Phi(r^{-1}(x)) = r^{-1}(\varphi(x))$.

We note that by Theorem 12 we have also shown that every homeomorphism $\Phi: X_\infty \to X_\infty$ must send fiber $r^{-1}(x)$ into fiber $r^{-1}(\Phi(x))$ for each $x \in X$. If $z \in r^{-1}(x)$ and $z \neq x$ then say that z is a neighbor of $x \in X$.

Theorem 13: Let $X_\infty(B)$ be a finite resolution of X. Let $\varphi: X \to X$ be any homeomorphism such that $\varphi(B) = B$. Then there exists a homeomorphism $\Phi: X_\infty \to X_\infty$ extending φ.

Proof: This theorem is really a special case corollary to Theorem 7, which guarantees the existence of Φ as continuous. That Φ is a homeomorphism follows from the construction of homeomorphisms φ_α in Theorem 7, $\varphi(B) = B$ and Theorem C.

The necessity of $\varphi(B) = B$ is easily seen in light of Lemma 10 as $x \in X$ and its image may have fibers of differing cardinality in case $\varphi(B) \neq B$ and so no extension is possible. As an example, let $X = [0,1]$ and let B_1 be the usual open interval basis and B_2 is the dyadic interval basis of Example C. Define a new basis B as the union of the collections

$$[0,3/4) \cap B_1 \quad \text{and} \quad (1/4,1] \cap B_2$$

Consider the homeomorphism $\varphi(x) = 1-x$. Now let $x < 1/4$ be other than a dyadic rational point. Since the collection $\{(x,1/n) : n = 2,3,\cdots\} \subseteq B$, the construction of points in X_∞ implies that $r^{-1}(x)$ is not a singleton. Since $\varphi(x) > 3/4$ and not dyadic, the results in Example C imply $r^{-1}(\varphi(x)) = \varphi(x)$.

Corollary: Let $X_\infty(\tau)$ be the total resolution of X. Then $H(X)$ is a quotient of $H(X_\infty)$.

Proof: For every $\varphi \in H(X)$ we have $\varphi(\tau) = \tau$, since φ and φ^{-1} are open mappings. By Theorem 13 every $\varphi \in H(X)$ extends to some $\Phi \in H(X_\infty)$. Hence the homomorphism of the second corollary to Theorem 12 is surjective.

Of course there are examples where a basis courser than τ will suffice in the above. For instance, let $X = [0,1]$ and let $B = \{\text{open intervals}\}$ or more generally, let X be locally connected and let $B = \{\text{locally connected open sets}\}$. The following results discuss the case of arbitrary basis and examine some groups of homeomorphisms as subgroups of $H(X)$ and $H(X_\infty)$.

Definition: Let $H(X)$ and $H(X_\infty)$ be the homeomorphism groups of X and $X_\infty(B)$ respectively. We then write

$$H(X,B) = \{\varphi \in H(X) : \varphi(B) = B\}$$

$$H(X_\infty,B) = \{\Phi \in H(X_\infty) : \Phi/_X \in H(X,B)\}$$

$$H_0(X_\infty,B) = \{\Phi \in H(X_\infty,B) : \Phi \text{ is the extension of Theorems 7 \& 13}\}$$

We note that $H_0(X_\infty,B)$ is well defined by Theorem B and is a subgroup of $H(X_\infty,B)$ by Theorem A. Simple examination then proves the following.

Lemma 11: The following inclusions are as subgroups.

(i) $H(X,B) \subseteq H(X)$

(ii) $H(X_\infty,B) \subseteq H(X_\infty)$

(iii) $H_o(X_\infty,B) \subseteq H(X_\infty,B)$

(iv) $H(X,B_1) \cap H(X,B_2) \subseteq H(X,B_1 \cup B_2)$

<u>Theorem 14</u>: $H_o(X_\infty,B)$ and $H(X,B)$ are isomorphic.

<u>Proof</u>: By Theorem B the correspondence $\varprojlim \varphi_\alpha \longleftrightarrow \varphi$ is one-to-one, where the φ_α are as in Theorem 7. Hence the correspondence is an isomorphism by Theorem A.

In the following, $H_p(X)$ and $H_p(X_\infty)$ are the p-th Čech homology groups of X and X_∞ respectively. The notation is essentially the same as in [4]. The underlying necessary group structure is not included in the notation. We let $\Sigma(X)$ be the collection of all finite open coverings of X. For $\alpha,\beta \in \Sigma(X)$ write $\alpha < \beta$ if β refines α. For finite $\alpha,\beta \subseteq B$, a given basis for X, write $\alpha \le \beta$ if β contains α. Let $I = \{\text{finite } \alpha \subseteq B\}$. The compactness of X implies that the collection, J, of basic finite open coverings from B is cofinal for both $(\Sigma(X),<)$ and $(I,\le)$. Unfortunately these orderings are not compactible and so we proceed further by defining for each $\alpha \in J$ and $x \in X$, the open set

$$U_\alpha(x) = \cap\ \{U \in \alpha\colon x \in U\}.$$

The finiteness of α also implies that the covering

$$\mathcal{U}_\alpha = \{U_\alpha(x)\colon x \in X\}$$

is finite for each $\alpha \in J$. By the construction, $\alpha < \mathcal{U}_\alpha$ for each $\alpha \in J$ and so $\{\mathcal{U}_\alpha\colon \alpha \in J\}$ is cofinal in $(\Sigma(X), <)$.

Further $\alpha < \beta$ implies that $\mathcal{U}_\alpha < \mathcal{U}_\beta$ since β contains α and $x \in X$ means that $U_\beta(x) = \cap \{U \in \beta : x \in U\} \subseteq \cap\{U \in \alpha : x \in U\}$.

Now let $H_p(\mathcal{U})$ be the p-th Cech homology group of the nerve of $\mathcal{U} \in \Sigma(X)$ considered as a simplicial complex. For any $\mathcal{V} > \mathcal{U}$ let $\pi_{\mathcal{U}\mathcal{V}}: \mathcal{V} \to \mathcal{U}$ be the usual projection and denote the resulting induced homomorphism by $_*\pi_{\mathcal{U}\mathcal{V}}$ from $H_p(\mathcal{V})$ to $H_p(\mathcal{U})$. The p-th Čech homology group of X is the inverse limit group

$$H_p(X) = \varprojlim \{H_p(\mathcal{U}), {}_*\pi_{\mathcal{U}\mathcal{V}}\}.$$

Let $X_\infty(B) = \varprojlim \{X_\alpha, f_{\alpha\beta}\}$ be a finite resolution of X. Let $H_p(X_\alpha)$ be the p-th Čech homology group of the factor space X_α. Each bonding map induces a homomorphism written ${}_*f_{\alpha\beta}: H_p(X_\beta) \to H_p(X_\alpha)$ and the groups $\varprojlim\{H_p(X_\alpha), {}_*f_{\alpha\beta}\}$ and $H_p(X_\infty)$ are isomorphic. Using all of the preceding constructions we prove the following.

<u>Theorem 15</u>: Let $X_\infty(B)$ be a finite resolution of X. Then $H_p(X)$ and $H_p(X_\infty)$ are isomorphic.

<u>Proof</u>: By our construction and (iii) of Lemma 1, $\pi_\alpha(\mathcal{U}_\alpha)$ refines every open covering of X_α and so by (2) of Theorem J, $H_p(X_\alpha)$ and $H_p(\mathcal{U}_\alpha)$ are isomorphic. Denote this isomorphism by ${}_*\varphi_\alpha$. Then the diagram below commutes via our construction whenever $\alpha \leq \beta$.

$$\begin{tikzcd} H_p(\mathcal{U}_\beta) \arrow[r, "{}_*\pi_{\mathcal{U}_\alpha \mathcal{U}_\beta}"'] \arrow[d, "{}_*\varphi_\beta"'] & H_p(\mathcal{U}_\alpha) \arrow[d, "{}_*\varphi_\alpha"] \\ H_p(X_\beta) \arrow[r, "{}_*f_{\alpha\beta}"'] & H_p(X_\alpha) \end{tikzcd}$$

By the group theoretic equivalent of Theorem C (See [3], p. 218) the induced map ${}_*\Phi = \varprojlim {}_*\varphi_\alpha$ is an isomorphism between the inverse limit groups $H_p(X)$ and $H_p(X_\infty)$.

The remaining results show that finite resolutions of compact Hausdorff spaces inherit path connectedness and then homotopy type.

Theorem 16: Let X be path connected and let X_∞ be a finite resolution of X. Then X_∞ is path connected.

Proof: Let $x \in X$ and let $z \in r^{-1}(x) - \{x\}$ where the map $r:X_\infty \to X$ is the unique retraction. We have $x \in \overline{z}$ and so $z \notin \overline{x}$ since X_∞ is a T_o - space. Then $\{x,z\}$ is always a Sierpinski subspace of X_∞ and z is the open singleton in $\{x,z\}$. Then the standard mapping $h:[0,1]\to X_\infty$ given by

$$h(t) = z \qquad t \in [0,1)$$
$$h(1) = x$$

is continuous and hence is a path from z to x. That is, the neighbors of $x \in X$ are in the path component of $x \in X_\infty$. The compactness of X implies that each $z \in X_\infty - X$ is a neighbor of some $x \in X$ and so X being path connected implies that X_∞ is path connected.

<u>Theorem 17</u>: Let X_∞ be a finite resolution of X. Then the unique retraction $r:X_\infty \to X$ is a strong deformation retraction.

<u>Proof</u>: Define $h:X_\infty \times I \to X_\infty$ as follows.

$$\begin{aligned} h(z,t) &= z \qquad && z \in X_\infty \text{ and } t \in [0,1) \\ h(z,1) &= r(z) && z \in X_\infty \end{aligned}$$

We have that $h(z,0)$ is the identity map on X_∞ and for each $x \in X$ and $t \in [0,1]$, $h(x,t) = x$. For the continuity of h we consider two cases.

Case I. $(z,t) \in X_\infty \times [0,1)$. Let U be an open neighborhood of z in X_∞. Then $(z,t) \in U \times [0,1)$ which is an open set in $X_\infty \times I$ and $h(U \times [0,1)) = U$. Hence h is continuous at (z,t).

Case II. $(z,1) \in X_\infty \times \{1\}$. Let U be an open neighborhood of $r(z)$ in X_∞. Then $U \cap X$ contains $r(z)$ and is an open set in X. Hence $(z,1) \in r^{-1}(U \cap X) \times I$, which is open in $X_\infty \times I$. Further, $h(r^{-1}(U \cap X) \times I) \subseteq U$ and so h is continuous at $(z,1)$. This completes the proof.

<u>Corollary</u>: X_∞ and X are of the same homotopy type and, in particular, the fundamental groups $\pi_1(X_\infty,z)$ and $\pi_1(X,r(z))$ are isomorphic.

APPENDIX

UNRESOLVED CONJECTURES

The following consists of several conjectures which we suspect from the structures investigated and the results already given. Partial results are included, although in each case the general approach to a proof or a counterexample is not yet visible.

If X_∞ is a finite resolution of a completely regular space X then only one of the possible Hausdorff compactifications of X can be contained between X and X_∞. If present, the structure is given by Lemma 8 as $X \cup A$ where A is the set of closed singletons in $X_\infty - X$.

<u>Conjecture A</u>: For each finite resolution X_∞ of a completely regular space X, $\hat{X} = X \cup A \subseteq X_\infty$ is a Hausdorff compactification of X.

<u>Conjecture B</u>: Given a Hausdorff compactification $\hat{X}$ of a space X, there exists a basis B for X such that after the usual imbeddings we have $\hat{X} = X \cup A \subseteq X_\infty(B)$.

We note that for each example in chapter five with completely regular X, the compactification corresponding to $X \cup A$ is easily described. The problem with more complex examples is the inability to fully appreciate the imagery of the construction. Further, by Theorem 9 and Lemma 8, to prove Conjecture A it is sufficient to show that the complete regularity of X implies that $X \cup A$ is Hausdorff. A partial step in this direction is given by the following two results which are directly implied by Lemma 3.

<u>Corollary</u>: Let $y, z \in X_\infty$ such that for some $\alpha \in I$, the subsets $Y = \pi_\alpha^{-1}(y_\alpha) \cap X$ and $Z = \pi_\alpha^{-1}(z_\alpha) \cap X$ are T_2 - separated in X. Then y and z are T_2 - separated in X_∞.

<u>Proof</u>: The sets $\mathfrak{U}'$ and $\mathfrak{V}'$ of Lemma 3 will suffice.

<u>Lemma</u>: Let X_∞ be a finite resolution of a regular space X. Let $x \in X$ and $a \in A$. Then x and a are T_2 - separated in X_∞.

<u>Proof</u>: Since $\bar{a} = a$, we have by Theorem I that there is some $\beta \in I$ such that $x \notin \pi_\beta^{-1}(\overline{a_\beta})$. Since X is regular, x and $\pi_\beta^{-1}(\overline{a_\beta}) \cap X$ are T_2 - separated in X. The corollary implies that x and a are T_2 - separated in X_∞.

It remains to show that distinct $a, b \in A$ are always T_2 - separated when X is completely regular. Any method similar to that above seems to fail as a proof.

We further remark that Conjecture B is equivalent to the following since strong density is all that is required to make a subspace of the compactification $\hat{X}$ inherit $\hat{X}_\infty$ as a finite resolution.

<u>Conjecture B′</u>: Let $\hat{X}$ be a Hausdorff compactification of space X. Then there exists a basis $\hat{B}$ for $\hat{X}$ such that from finite resolution $\hat{X}_\infty(\hat{B})$, the collection

$$\{\hat{\pi}_\alpha^{-1}(\hat{x}_\alpha) \cap \hat{X} : \hat{x}_\alpha \in \hat{X}_\alpha \text{ and } \alpha \in \hat{I}\}$$

contains no subset $\hat{F} \subseteq \hat{X} - X$.

The collection $\{\hat{\pi}_\alpha^{-1}(\hat{x}_\alpha) \cap \hat{X}\}$ given above may be replaced by the smallest algebra of subsets of $\hat{X}$ containing $\hat{B}$.

The remaining conjecture refers to Theorem 13. Recall here that X is compact and Hausdorff and φ is a homeomorphism on X such that $\varphi(B) = B$ for given basis B. By Theorem B, the extension Φ of Theorem 13 is unique such that for each $\alpha \in I$, $\pi_{\varphi(\alpha)} \circ \Phi = \varphi_\alpha \circ \pi_\alpha$.

<u>Conjecture</u> <u>C</u>: The extension of Theorem 13 is unique and hence $H(X)$ and $H(X_\infty(\tau))$ are isomorphic.

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