PROPERTIES OF L_p DERIVATIVES

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THESIS

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Major professor

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ABSTRACT

PROPERTIES OF L_p DERIVATIVES

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Suppose a real valued function f, defined on an interval I, possesses an ordinary derivative $f'(x)$ at each point of I. Then f' need not be a continuous function. However, A. Denjoy, J.A. Clarkson, and Z. Zahorski have shown that f' possesses the following four properties, defined here for an arbitrary function g.

1. A function g is said to be of Baire class one if g is the pointwise limit of a sequence of continuous functions.

2. A function g, defined on an interval I, is said to have the Darboux property if, on every subinterval $[a,b]$ of I, g takes on all intermediate values.

3. A function g is said to have the Denjoy property if for every open interval (a,b), $g^{-1}((a,b))$ either is empty or has positive measure.

4. A function g is said to have the Zahorski property if for every open interval (a,b), x in $g^{-1}((a,b))$, and $\{I_n\}$ a sequence of closed intervals converging to x with

$$|g^{-1}((a,b)) \cap I_n| = 0$$

for every n, then

$$\lim_{n\to\infty} \frac{|I_n|}{d(x,I_n)} = 0,$$

where $|E|$ denotes the Lebesgue measure of a set E, and

$$d(x,I_n) = \inf\{|x - y| : y \in I_n\}.$$

H.W. Oliver showed more generally that if a function f has a k^{th} Peano derivative $f_k(x)$ at each point of an interval, then f_k has properties 1, 2, and 3 listed above. C.E. Weil showed, furthermore, that f_k has property 4.

In Chapter I of this paper we look at the definition of a k^{th} L_p derivative for a function as given by A.P. Calderon and A. Zygmund. This type of differentiation is a generalization of the Peano type. In Chapter II it is shown that if a continuous function f possesses a k^{th} L_p derivative $f_k^{L_p}(x)$

at each point of an interval, then $f_k^{L_p}$ has the above listed properties 1, 2, 3, and 4. We also prove some other properties of $f_k^{L_p}$ dealing with its relationship to approximate and ordinary derivatives. In Chapter III we show that if we assume only that f is measurable and possesses a k^{th} L_p derivative at each point of an interval, then $f_k^{L_p}$ still has property 1.

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To Mother and Dad

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Chapter I. Introduction and Definitions

Suppose a real valued function f, defined on an interval I, possesses an ordinary derivative $f'(x)$ at each point of I. Although f' need not be continuous under these circumstances, it does possess certain properties which make it closely related to continuous functions. Consider the following definitions of these properties.

<u>Definition 1.1</u>. A function g is said to belong to Baire class one if g is the pointwise limit of a sequence of continuous functions.

<u>Definition 1.2</u>. A function g, defined on an interval I, is said to have the Darboux property if, on every subinterval $[a,b]$ of I, g takes on all intermediate values, i.e. if $g(a) \neq g(b)$, and if c lies between $g(a)$ and $g(b)$, there is a number d, $a < d < b$, for which $g(d) = c$.

<u>Definition 1.3</u>. A function g is said to have the Denjoy property if for every open interval (a,b), $g^{-1}((a,b))$ either is empty or has positive measure.

<u>Definition 1.4.</u> A sequence of closed intervals $\{I_n\}$ is said to converge to a point x if x is not in the union of the I_n and if every neighborhood of x contains all but a finite number of the intervals I_n.

<u>Definition 1.5</u>. A function g is said to have the Zahorski property if for every open interval (a,b), x in $g^{-1}((a,b))$ and $\{I_n\}$ a sequence of closed intervals converging to x with

$$|g^{-1}((a,b)) \cap I_n| = 0,$$

for every n, then

$$\lim_{n\to\infty} \frac{|I_n|}{d(x,I_n)} = 0,$$

where $|E|$ denotes the Lebesgue measure of a set E, and

$$d(x,I_n) = \inf\{|x-y| : y \in I_n\}.$$

If f' is the derivative of f at each point of an interval I, it is well known that f' is of Baire class one and that f' has the Darboux property. Although f' need not be continuous, Denjoy and Clarkson have shown that f' has the Denjoy property (see [6] and [3]). Zahorski refined this property in [19] and proved that f' possesses what we have named

the Zahorski property. These four properties do not classify derivatives. Indeed, they are possessed by more general types of derivatives. Consider the following definition.

<u>Definition 1.6</u>. A function f, defined on an interval I, is said to have a k^{th} Peano derivative at x_0, $k = 1,2,\ldots,$ if there exist numbers $f_1(x_0)$, $f_2(x_0),\ldots,$ $f_k(x_0)$ such that

$$f(x_0+h) - f(x_0) - hf_1(x_0) - \ldots - \frac{h^k}{k!} f_k(x_0) = o(h^k)$$

as $h \to 0$.

The numbers $f_1(x_0),\ldots,f_k(x_0)$ can easily be shown to be unique, and if f has a k^{th} Peano derivative $f_k(x_0)$ at x_0, then it also has an n^{th} Peano derivative $f_n(x_0)$, $n = 1,2,\ldots,k-1$. Notice that $f_1(x_0) = f'(x_0)$, the ordinary first derivative. If f has an ordinary k^{th} derivative, $f^{(k)}(x_0)$, at x_0, then Taylor's theorem shows that $f_k(x_0)$ exists and equals $f^{(k)}(x_0)$. However $f_k(x_0)$ may exist without $f^{(k)}(x_0)$ existing.

<u>Example 1.1</u> Let $f(x) = x^3 \sin \frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$. Then $f_1(0) = f'(0) = 0$, and since

$$h^3 \sin \frac{1}{h} = o(h^2),$$

as $h \to 0$, we have $f_2(0) = 0$. However, $f^{(2)}(0)$ does not exist.

Suppose now that a function f, defined on an interval I possesses a k^{th} Peano derivative at each point of I. Denjoy [5] and Oliver [13] have shown that under these circumstances f_k belongs to the first Baire class. Oliver also has shown that f_k has the Darboux property and the Denjoy property. Weil [18] has given another proof that f_k has the Denjoy property, as well as proving that f_k has the Zahorski property.

Calderon and Zygmund [2] have introduced a generalization of the k^{th} Peano derivative as follows:

<u>Definition 1.7</u>. A function f, defined on an interval I, is said to have a k^{th} L_p derivative at x_0, $1 \leq p < \infty$, $k = 1,2,\ldots,$ if there exist numbers $f_0^{L_p}(x_0)$, $f_1^{L_p}(x_0),\ldots,f_k^{L_p}(x_0)$ such that

$$\left\{ \frac{1}{h} \int_0^h \left| f(x_0+t) - f_0^{L_p}(x_0) - t f_1^{L_p}(x_0) - \cdots - \frac{t^k}{k!} f_k^{L_p}(x_0) \right|^p dt \right\}^{\frac{1}{p}}$$
$$= o(h^k),$$

as $h \to 0$.

As with the k^{th} Peano derivative, the numbers $f_0^{L_p}(x_0), f_1^{L_p}(x_0), \ldots, f_k^{L_p}(x_0)$ can be shown to be unique, and if f has a k^{th} L_p derivative $f_k^{L_p}(x_0)$ at x_0, then it also has an n^{th} L_p derivative $f_n^{L_p}(x_0)$ at x_0 for $n = 1,2,\ldots,k-1$. In this last definition it appears that the value of f at x_0 is irrelevant. However, if we know that f has a k^{th} L^p derivative everywhere on an interval I, then we can show that $f(x) = f_0^{L_p}(x)$ for almost every x in I by the following reasoning. Call x a Lebesgue point of f if

$$\lim_{h\to 0} \frac{1}{h} \int_0^h f(x+t)\,dt = f(x)$$

By Lebesgue's theorem we know that almost every point in I is a Lebesgue point of f (see [11] for a detailed discussion.) Using the fact that f has a k^{th} L_p derivative at each point x in I, it is not difficult to show that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h f(x+t)\,dt = f_0^{L_p}(x),$$

for each x in I. Hence $f_0^{L_p}(x) = f(x)$ for each Lebesgue point x of f.

Another useful fact that is easy to show is that if f has a k^{th} Peano derivative at a point x, then

f has a k^{th} L_p derivative at x for any $1 \leq p < \infty$, and $f_k(x) = f_k^{L_p}(x)$. Furthermore if $1 \leq q < p < \infty$, and f has a k^{th} L_p derivative at x, then f has a k^{th} L_q derivative at x, and $f_k^{L_p}(x) = f_k^{L_q}(x)$.

One of the first differences to be noticed between the Peano and L_p types of differentiation is that while the existence of the k^{th} Peano derivative of a function f at each point of an interval implies that f is continuous on that interval, the same is not true for the k^{th} L_p derivative as the following example shows. This example also shows that the k^{th} L_p derivative is a true generalization of the k^{th} Peano derivative, i.e. a k^{th} L_p derivative is not necessarily a k^{th} Peano derivative.

Example 1.2. Here we construct a function on the interval $I = [0,1]$ which has a first L_p derivative at each point of I (indeed it will have an ordinary derivative on $(0,1])$, but which is discontinuous at 0. For each positive integer n, let $I_n = [\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{8^{pn}}]$. On I_n we will choose f so that

1) f has an ordinary derivative at each point of I_n,

2) $f(\frac{1}{2^n}) = f(\frac{1}{2^n} + \frac{1}{8^{pn}}) = 0$,

3) $f'_+(\frac{1}{2^n}) = f'_-(\frac{1}{2^n} + \frac{1}{8^{pn}}) = 0$, where f'_+(or f'_-) denotes the right (or left) derivative at x,

4) $\max\limits_{x \in I_n} f(x) = 1$.

For x in $I - \bigcup\limits_{n=1}^{\infty} I_n$, we let $f(x) = 0$. Since $f(0) = 0$, and since every right neighborhood of 0 contains infinitely many of the intervals I_n, f is not continuous at 0. f clearly has an ordinary derivative at each point in $(0,1]$. Next we will show that $f_0^{L^p}(0) = f_1^{L^p}(0) = 0$, i.e. we will show that

$$\lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{h}\int_0^h |f(t)|^p dt\right\}^{\frac{1}{p}} = 0.$$

Let $0 < h < 1$, and choose the positive integer N so that $\frac{1}{2^{N+1}} < h \leq \frac{1}{2^N}$.

Then

$$\frac{1}{h}\left\{\frac{1}{h}\int_0^h |f(t)|^p dt\right\}^{\frac{1}{p}} \leq 2^{N+1}\left\{2^{N+1}\int_0^{\frac{1}{2^N}+\frac{1}{8^{pN}}} |f(t)|^p dt\right\}^{\frac{1}{p}}$$

$$= 2^{N+1}\left\{2^{N+1}\sum_{n=N}^{\infty}\int_{I_n} |f(t)|^p dt\right\}^{\frac{1}{p}}$$

$$\leq 2^{N+1}\left\{2^{N+1} \sum_{n=N}^{\infty} \int_{I_n} 1\ dt\right\}^{\frac{1}{p}}$$

$$= 2^{N+1}\left\{2^{N+1} \sum_{n=N}^{\infty} \frac{1}{8^{pn}}\right\}^{\frac{1}{p}}$$

$$= 2^{N+1}\left\{2^{N+1}\left(\frac{8^p}{8^{pN}(8^p-1)}\right)\right\}^{\frac{1}{p}}$$

$$= \frac{16\cdot 2^{1/p}}{(8^p-1)^{1/p}} \cdot \frac{1}{2^{2N-N/p}}$$

$$\leqq \frac{16\cdot 2^{1/p}}{(8^p-1)^{1/p}} \cdot \frac{1}{2^N} .$$

Now as $h \to 0$, $N \to \infty$, and $\frac{1}{2^N} \to 0$; so the example is established.

In Chapter II of this paper we will show that if, as in the case of the Peano derivative, we start with a continuous function f defined on an interval I, and assume the existence of the k^{th} L_p derivative of f at each point of I, then $f_k^{L_p}$ also possesses the four properties mentioned above as belonging to ordinary derivatives and Peano derivatives. In Chapter III we will show that if we dispense with the assumption that f be continuous on I, we can still show that $f_k^{L_p}$ belongs to Baire class one.

Before beginning the proofs of these results, we should look at yet another type of differentiation which will be quite useful in what follows. This is the notion of an approximate derivative which was introduced by Denjoy [4]. Consider the following definitions.

<u>Definition 1.8</u>. Let E be a set of real numbers, and let x be any real number. We say that x is a point of density one, or just a point of density, of E if

$$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{|E \cap I|}{|I|} = 1.$$

We say that x is a point of dispersion of E if

$$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{|E \cap I|}{|I|} = 0.$$

<u>Definition 1.9</u>. We say that a function f has A as an approximate limit at x_0, i.e.

$$\lim_{x \to x_0} {}_{ap} f(x) = A,$$

if there is a set E having x_0 as a point of density one such that

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = A;$$

or equivalently, if for each $\varepsilon > 0$, the set $F_\varepsilon = \{x : |F(x) - A| \geq \varepsilon\}$ has x_0 as a point of dispersion.

<u>Definition 1.10</u>. A function f is said to have an approximate derivative $f'_{ap}(x_0)$ at x_0 if

$$\lim_{x \to x_0} {}_{ap} \frac{f(x) - f(x_0)}{x - x_0} = f'_{ap}(x_0).$$

Suppose that f is a real valued function, defined on an interval I, possessing an approximate derivative at each point of I. Tolstoff [15] has shown that under these circumstances f'_{ap} is of Baire class one. Khintchine [8] has shown that Rolle's theorem holds for approximate derivatives. It then readily follows that f'_{ap} has the Darboux property. The proofs of Khintchine and Tolstoff are long and complicated. A shorter proof of the fact that f'_{ap} is of Baire class one and has the Darboux property has been given by Goffman and Neugebauer [7]. Marcus [10] has shown that f'_{ap} has the Denjoy property. Weil [18] gave another proof that f'_{ap} has the Denjoy property and, furthermore, showed that it has the Zahorski property.

Chapter II. Properties of k^{th} L_p Derivatives of Continuous Functions

The following theorem and its proof are due to Neugebauer [12].

<u>Theorem 2.1</u>. Let f be a measurable function defined on an interval I. Then at almost all points x at which f has a first L_p derivative, $f_1^{L_p}(x) = f'_{ap}(x)$.

Proof. Let x be a point at which f has a first L_p derivative, i.e. there exist numbers $f_0^{L_p}(x)$ and $f_1^{L_p}(x)$ such that

$$\left\{\frac{1}{h}\int_0^h |f(x+t) - f_0^{L_p}(x) - t\, f_1^{L_p}(x)|^p dt\right\}^{\frac{1}{p}} = o(h).$$

Let $\epsilon > 0$, and set

$E_\epsilon = \left\{t > 0: |f(x+t) - f_0^{L_p}(x) - t\, f_1^{L_p}(x)| \geq \epsilon t\right\}$ and

$F_\epsilon = \left\{t < 0: |f(x+t) - f_0^{L_p}(x) - t\, f_1^{L_p}(x)| \geq \epsilon |t|\right\}$. We need to show that E_ϵ and F_ϵ have 0 as a point of dispersion. Let $h > 0$ and set $E_h = E_\epsilon \cap [0,h]$. Then we have

$$\left\{\frac{1}{h}\int_0^h |f(x+t) - f_0^{L_p}(x) - t\, f_1^{L_p}(x)|^p dt\right\}^{\frac{1}{p}}$$

$$\geq \left\{ \frac{1}{h} \int_{E_h} \left| f(x+t) - f_0^{L_p}(x) - t\, f_1^{L_p}(x) \right|^p dt \right\}^{\frac{1}{p}}$$

$$\geq \left\{ \frac{1}{h} \int_{E_n} \epsilon^p t^p \, dt \right\}^{\frac{1}{p}}$$

$$\geq \left\{ \frac{1}{h} \int_0^{|E_h|} \epsilon^p t^p \, dt \right\}^{\frac{1}{p}}$$

$$= \left\{ \frac{\epsilon^p}{p+1} \cdot \frac{|E_h|}{h} \right\}^{\frac{1}{p}} |E_h|,$$

and this is $o(h)$, i.e.

$$\lim_{h \to 0} \left\{ \frac{\epsilon^p}{p+1} \cdot \frac{|E_h|}{h} \right\}^{\frac{1}{p}} \frac{|E_h|}{h} = 0.$$

Hence

$$\lim_{h \to 0} \frac{|E_h|}{h} = 0,$$

i.e. E_ϵ has 0 as a point of dispersion. Similarly F_ϵ has 0 as a point of dispersion. So

$$\lim_{t \to 0} {}_{ap} \frac{f(x+t) - f_0^{L_p}(x)}{t} = f_1^{L_p}(x).$$

If x also belongs to the Lebesgue set of f, then $f_0^{L_p}(x) = f(x)$, and hence $f_1^{L_p}(x) = f'_{ap}(x)$. Since almost every x at which f has a first L_p derivative belongs to the Lebesgue set of f, the theorem is proved.

<u>Remark</u>. Suppose that in the statement of the above theorem measurability is replaced by continuity. Then since every point of continuity of a function is a Lebesgue point of the function, we have $f(x) = f_0^{L_p}(x)$ for each point where f has a first L_p derivative. Hence at each such point $f_1^{L_p}(x) = f'_{ap}(x)$.

<u>Corollary 2.1</u>. Let f be a measurable function, defined on an interval I, possessing a first L_p derivative at each point of I. Then $f_1^{L_p}$

1) belongs to Baire class one,

2) has the Darboux property,

3) has the Denjoy property,

and 4) has the Zahorski property.

Proof. In the above theorem we saw that

$$\lim_{t\to 0} {}_{ap} \frac{f(x+t) - f_0^{L_p}(x)}{t} = f_1^{L_p}(x)$$

at each point x in I. Since $f(y) = f_0^{L_p}(y)$ for almost every y in I, we have

$$\lim_{t\to 0} {}_{ap} \frac{f_0^{L_p}(x+t) - f_0^{L_p}(x)}{t} = f_1^{L_p}(x),$$

i.e

$$\left(f_0^{L_p}\right)'_{ap}(x) = f_1^{L_p}(x)$$

for each x in I. As pointed out in Chapter I, an approximate derivative satisfies 1), 2), 3) and 4). So the proof is complete.

If we now consider higher order L_p derivatives, we obtain the following result.

<u>Theorem 2.2</u>. Let f be a measurable function possessing a k^{th} L_p derivative at each point of an interval I. Then $f_k^{L_p}(x)$ is the approximate derivative of $f_{k-1}^{L_p}(x)$ at almost every point x in I, i.e.

$$f_k^{L_p}(x) = \left(f_{k-1}^{L_p}\right)'_{ap}(x) \qquad \text{a.e. in I.}$$

Proof. Without loss of generality we may assume that $|I| < \infty$. Let $\varepsilon > 0$ be given. By the corollary to theorem 9 in [2] there exist a closed set $E \subset I$ with $|I - E| < \varepsilon$ and a decomposition $f = g + h$, where

1) $g \in C^k(I)$, i.e. g has k continuous derivatives on I,

2) $h(x) = 0$ for all $x \in E$,

and

3) $\left\{ \frac{1}{h} \int_0^h |h(x+t)|^p dt \right\}^{\frac{1}{p}} = o(h^k)$ as $h \to 0$

for each x in E.

Now let x be a point of density of E. If $x + u \in E$, then by 3)

$$h_{k-1}^{L_p}(x+u) = h_{k-1}^{L_p}(x) = 0.$$

So

$$\lim_{\substack{u \to 0 \\ x+u \in E}} \frac{h_{k-1}^{L_p}(x+u) - h_{k-1}^{L_p}(x)}{u} = 0,$$

i.e. $$\left(h_{k-1}^{L_p}\right)'_{ap}(x) = 0.$$

Hence

$$\begin{aligned} f_k^{L_p}(x) &= g_k^{L_p}(x) + h_k^{L_p}(x) \\ &= g^{(k)}(x) + 0 \\ &= \left(g^{(k-1)}\right)'_{ap}(x) + \left(h_{k-1}^{L_p}\right)'_{ap}(x) \\ &= \left(f_{k-1}^{L_p}\right)'_{ap}(x). \end{aligned}$$

So $f_k^{L_p}(x) = \left(f_{k-1}^{L_p}\right)'_{ap}(x)$ for almost every $x \in E$, and since this is true for each $\varepsilon > 0$, we have

$$f_k^{L_p}(x) = \left(f_{k-1}^{L_p}\right)'_{ap}(x)$$

for almost every $x \in I$.

The conclusion of this theorem cannot be strengthened to an "everywhere type" result, even if we assume that f is continuous, as the following example shows.

<u>Example 2.1.</u> Let $f(x) = \begin{cases} x^3 \sin \frac{1}{x}, & 0 < x \leq 1, \\ 0 & , x = 0. \end{cases}$

We will show that $f_2(x)$ (and hence $f_2^{L_p}(x)$) exists for each x, $0 \leq x \leq 1$, but $f_2(0) \neq (f_1)'_{ap}(0)$. We have

$$f'(x) = \begin{cases} -x \cos \frac{1}{x} + 3x^2 \sin \frac{1}{x}, & 0 < x \leq 1, \\ 0 & , x = 0 \end{cases} .$$

Also $f''(x)$ exists routinely for $0 < x \leq 1$, and since

$$f(h) = h^3 \sin \frac{1}{h} = o(h^2),$$

we have $f_2(0) = 0$.

Now if $(f_1)'_{ap}(0) = 0$, then for each $\varepsilon > 0$, the set $E_\varepsilon = \{h : |f'(h)| > \varepsilon h\}$ has 0 as a point of dispersion. We will show that this is not the case.

Let $F_\varepsilon = \{h : |\cos \frac{1}{h}| > \varepsilon\}$ for $\varepsilon > 0$. In particular consider

$$F_{\frac{1}{\sqrt{2}}} = \{h : |\cos \frac{1}{h}| > \frac{1}{\sqrt{2}}\}$$

$$= \bigcup_{n=1}^{\infty} \left(\frac{1}{n\pi + \frac{\pi}{4}} , \frac{1}{n\pi - \frac{\pi}{4}} \right).$$

For a fixed positive integer N,

$$\left| F_{\frac{1}{\sqrt{2}}} \cap \left[0, \frac{1}{N\pi - \frac{\pi}{2}}\right] \right| = \left| \bigcup_{n=N}^{\infty} \left(\frac{1}{n\pi + \frac{\pi}{4}}, \frac{1}{n\pi - \frac{\pi}{4}} \right) \right|$$

$$= \sum_{n=N}^{\infty} \left| \left(\frac{1}{n\pi + \frac{\pi}{4}}, \frac{1}{n\pi - \frac{\pi}{4}} \right) \right|$$

$$= \sum_{n=N}^{\infty} \left(\frac{1}{2n^2\pi - \frac{\pi}{8}} \right)$$

$$> \frac{1}{4} \sum_{n=N}^{\infty} \left(\frac{1}{n^2\pi - \frac{\pi}{4}} \right)$$

$$= \frac{1}{4} \sum_{n=n}^{\infty} \left| \left(\frac{1}{n\pi + \frac{\pi}{2}}, \frac{1}{n\pi - \frac{\pi}{2}} \right) \right|$$

$$= \frac{1}{4} \left| \left[0, \frac{1}{N\pi - \frac{\pi}{2}} \right] \right| .$$

So $F_{\frac{1}{\sqrt{2}}}$ does not have 0 as a point of dispersion.

Let $\epsilon > 0$, and consider

$$E_{\frac{1}{\sqrt{2}} - \epsilon} = \{h : |f'(h)| > \left(\frac{1}{\sqrt{2}} - \epsilon\right) h\}$$

$$= \{h : \left|\cos \frac{1}{h} + 3h \sin \frac{1}{h}\right| > \frac{1}{\sqrt{2}} - \epsilon\}$$

For a sufficiently large N,

$$E_{\frac{1}{\sqrt{2}}-\epsilon} \cap \left[0, \frac{1}{N\pi - \frac{\pi}{2}}\right] \supset F_{\frac{1}{\sqrt{2}}} \cap \left[0, \frac{1}{N\pi - \frac{\pi}{2}}\right].$$

So $E_{\frac{1}{\sqrt{2}}-\epsilon}$ does not have 0 as a point of dispersion.

Consequently, $f_2(0) \neq (f_1)'_{ap}(0)$.

Although a k^{th} L_p derivative of a continuous function need not be an approximate derivative, we will show that it has the four properties of derivatives mentioned in Chapter I. We will need a few preliminary results.

In [13] Oliver introduced certain types of mean value properties for k^{th} Peano derivatives. We will find it beneficial to exploit these properties, defined here in terms of k^{th} L_p derivatives.

<u>Definition 2.1</u>. If f has a k^{th} L_p derivative at each point of an interval I, we say f has properties M_j^k, $j = 0,1,\ldots,k-1$, if for each x and $x+h$ in I, there is an x' between x and $x+h$ such that:

$$\frac{f_j^{L_p}(x+h) - f_j^{L_p}(x) - \cdots - \frac{h^{k-j-1}}{(k-j-1)!} f_{k-1}^{L_p}(x)}{\frac{h^{k-j}}{(k-j)!}} = f_k^{L_p}(x').$$

The special case of property M_j^k when the left side of this inequality is equal to 0, we refer to as property R_j^k.

The proof of the following lemma is due to Oliver [13].

<u>Lemma 2.1</u>. Let an interval I and a positive integer k be fixed.

1) Let j be an integer from 0 to k-1. If every function which possesses a k^{th} L_p derivative at each point of I has property R_j^k, then every such function also has property M_j^k.

2) If a function f has property R_{k-1}^k and properties M_j^k, for each $j = 0,1,\ldots,k-2$, then f has property R_j^k for each $j = 0,1,\ldots,k-1$.

Proof. Statement 1) is proved in the usual manner by adding an appropriate polynomial to f. Suppose y and y+h belong to I. Set

$$g(x) = f(x) - \frac{f_j^{L_p}(y+h) - f_j^{L_p}(y) - \ldots - \frac{h^{k-j-1}}{(k-j-1)!} f_{k-1}^{L_p}(y)}{\frac{h^{k-j}}{(k-j)!}} \cdot \frac{(x-y)^k}{k!} .$$

Then

$$g_j^{L_p}(y+h) - g_j^{L_p}(y) - hg_{j+1}^{L_p}(y) - \ldots - \frac{h^{k-j-1}}{(k-j-1)!} g_{k-1}^{L_p}(y) = 0,$$

i.e. g satisfies the hypotheses of R_j^k ; and

$$g_k(x) = f_k(x) - \frac{f_j^{L_p}(y+h) - f_j^{L_p}(y) - \ldots - \frac{h^{k-j-1}}{(k-j-1)!} f_{k-1}^{L_p}(y)}{\frac{h^{k-j}}{(k-j)!}} .$$

Applying the conclusion of R_j^k to g, i.e., replacing x by x' and $g_k^{L_p}(x')$ by 0, the conclusion of M_j^k follows for f.

To prove statement 2), let j be an integer, $0 \leq j \leq k-2$, and suppose x and $x+h$ are two points in I with

$$0 = \frac{f_j^{L_p}(x+h) - f_j^{L_p}(x) - \ldots - \frac{h^{k-j-1}}{(k-j-1)!} f_{k-1}^{L_p}(x)}{\frac{h^{k-j}}{(k-j)!}} .$$

So

$$0 = \frac{1}{\frac{h}{(k-j)}} \left[\frac{f_j^{L_p}(x+h) - f_j^{L_p}(x) - \ldots - \frac{h^{k-j-2}}{(k-j-2)!} f_{k-2}^{L_p}(x)}{} - f_{k-1}^{L_p}(x) \right].$$

Since f has property M_j^{k-1}, there is an x' between x and $x+h$ such that

$$\frac{f_j^{L_p}(x+h) - f_j^{L_p}(x) - \ldots - \frac{h^{k-j-2}}{(k-j-2)!} f_{k-2}^{L_p}}{\frac{h^{k-j-1}}{(k-j-1)!}} = f_{k-1}^{L_p}(x').$$

Consequently,

$$0 = \frac{f_{k-1}^{L_p}(x') - f_{k-1}^{L_p}(x)}{\frac{h}{k-j}} = \frac{f_{k-1}^{L_p}(x') - f_{k-1}^{L_p}(x)}{(x' - x)} \cdot \frac{(x'-x)}{\frac{h}{k-j}} .$$

Since f has property R_{k-1}^k, there exists an x'' between x' and x such that $f_k(x'') = 0$. Hence f has property R_j^k.

<u>Lemma 2.2</u>. Suppose f has a k^{th} L_p derivative at a point x. Then

$$\lim_{h\to 0} \frac{(k+1)!}{h^{k+1}} \int_0^h f(x+t) - \sum_{i=0}^{k-1} \frac{f_i^{L_p}(x)}{i!} t^i \, dt = f_k^{L_p}(x)$$

Proof. Let $\epsilon > 0$ be given. Since f has a k^{th} L_p derivative at x, there exists a $\delta > 0$ such that for $0 < |h| < \delta$

$$\left\{\frac{1}{|h|} \int_0^h \left| f(x+t) - \sum_{i=0}^{k} \frac{f_i^{L_p}(x)}{i!} t^i \, dt\right|^p\right\}^{\frac{1}{p}} < \frac{\epsilon|h|^k}{(k+1)!} .$$

Let h be such that $0 < |h| < \delta$. Then

$$\left|\frac{(k+1)!}{h} \int_0^h f(x+t) - \sum_{i=0}^{k-1} \frac{f_i^{L_p}(x)}{i!} t^i \, dt - h^k f_k^{L_p}(x)\right|$$

$$= \left|\frac{(k+1)!}{h} \int_0^h f(x+t) - \sum_{i=0}^{k} \frac{f_i^{L_p}(x)}{i!} t^i \, dt\right|$$

$$\leq \frac{(k+1)!}{|h|} \int_0^h |f(x+t) - \sum_{i=0}^{k} \frac{f_i^{L_p}(x)}{i!} t^i|dt$$

$$\leq (k+1)!\left\{ \frac{1}{|h|} \int_0^h |f(x+t) - \sum_{i=0}^{k} \frac{f_i^{L_p}(x)}{i!} t^i|^p dt\right\}^{\frac{1}{p}}$$

$$< \epsilon \, |h|^k,$$

where the next to the last inequality results from an application of Holder's inequality.

Lemma 2.3. Let f be continuous on an interval I and have the following properties:

1) f has property M_0^{k-1} on I,

2) f has a k^{th} L_p derivative at each point in I,

3) $f_k^{L_p}(x) > 0$ for each x in I (or $f_k^{L_p}(x) \geq 0$ for each x in I),

4) $f_{k-1}^{L_p}$ is upper semicontinuous on I.

Then $f_{k-1}^{L_p}$ is increasing on I (or nondecreasing on I).

Proof. As a first case suppose that $f_k^{L_p}(x) > 0$ for all x in I. Let a and b belong to I with $a < b$. We want to show that $f_k^{L_p}(a) < f_{k-1}^{L_p}(b)$. Since $f_{k-1}^{L_p}$ is upper semicontinuous on $[a,b]$, it attains a maximum on $[a,b]$. Let $x \in [a,b)$. Since $f_k^{L_p}(x) > 0$, and since by lemma 2.2

$$\lim_{h\to 0} \frac{(k+1)!}{h^{k+1}} \int_0^h f(x+t) - \sum_{i=0}^{k-1} \frac{f_i^{L_p}(x)}{i!} t^i \, dt = f_k^{L_p}(x) > 0$$

there exists an $0 < h' < b-x$ such that

$$f(x+h') - \sum_{i=0}^{k-1} \frac{f_i^{L_p}(x)}{i!} (h')^i > 0.$$

So

$$\frac{\left[f(x+h') - \sum_{i=0}^{k-2} \frac{f_i^{L_p}(x)}{i!} (h')^i\right]}{\frac{(h')^{k-1}}{(k-1)!}} - f_{k-1}^{L_p}(x) > 0.$$

Since f has property M_0^{k-1}, there is an x' such that $x < x' < x+h' < b$, and

$$\frac{\left[f(x+h') - \sum_{i=0}^{k-2} \frac{f_i^{L_p}(x)}{i!} (h')^i\right]}{\frac{(h')^{k-1}}{(k-1)!}} = f_{k-1}^{L_p}(x').$$

Hence

$$f_{k-1}^{L_p}(x') - f_{k-1}^{L_p}(x) > 0,$$

i.e.

$$f_{k-1}^{L_p}(x') > f_{k-1}^{L_p}(x).$$

So $f_{k-1}^{L_p}$ must attain its maximum at b, which implies that $f_{k-1}^{L_p}(a) < f_{k-1}^{L_p}(b)$, i.e. $f_{k-1}^{L_p}$ is increasing on I.

If $f_k^{L_p}(x) \geq 0$ on I, then for any $\varepsilon > 0$, set $g_\varepsilon(x) = f(x) + \varepsilon x^k$. Then $(g_\varepsilon)_k^{L_p}(x) = f_k^{L_p}(x) + k!\varepsilon > 0$ for all $x \in I$. Hence if a and b belong to I and $a < b$, then by the above argument,

$$(g_\varepsilon)_{k-1}^{L_p}(a) < (g_\varepsilon)_{k-1}^{L_p}(b),$$

i.e.

$$f_{k-1}^{L_p}(a) + k!\, a\varepsilon < f_{k-1}^{L_p}(b) + k!\, b\varepsilon,$$

or

$$f_{k-1}^{L_p}(a) < f_{k-1}^{L_p}(b) + k!(b-a)\varepsilon,$$

and since ε is arbitrary,

$$f_{k-1}^{L_p}(a) \leq f_{k-1}^{L_p}(b).$$

So $f_{k-1}^{L_p}$ is nondecreasing on I.

<u>Definition 2.2</u>. As in [9] we define differences $D_k(x,h;f)$ for a function f as follows:

$$D_1(x,h;f) = f(x+h) - f(x),$$

and for $k = 2,3,\ldots,$

$$D_k(x,h;f) = D_{k-1}(x,2h;f) - 2^{k-1}D_{k-1}(x,h;f).$$

When no confusion seems likely, we will write $D_k(x,h)$ for $D_k(x,h;f)$. The following lemma appears in [9] and is not difficult to prove by induction on k.

Lemma 2.4. For each $k = 1, 2, \ldots,$ there exist numbers $a_{k,j}$, $j = 0, 1, \ldots, k$ such that for any f,

1) $D_k(x,h;f) = a_{k,0}f(x) + \sum_{j=1}^{k} a_{k,j}\, f(x+2^{j-1}h)$,

2) $\sum_{j=0}^{k} a_{k,j} = 0$,

3) $\sum_{j=1}^{k} 2^{i(j-1)} a_{k,j} = 0$ for $i = 1, 2, \ldots, k-1$,

4) $\sum_{j=1}^{k} 2^{i(j-1)} a_{k,j} > 0$ for $i = k, k+1, \cdots$.

We set $\frac{1}{\lambda_k} = \sum_{j=1}^{k} 2^{k(j-1)} a_{k,j}$.

Lemma 2.5. Suppose f is continuous on an interval I and has a k^{th} L_p derivative at a point x in I. Then

$$\lim_{h\to 0} \frac{(k+1)!\lambda_k}{h^{k+1}} \int_0^h D_k(x_0,t)\, dt = f_k^{L_p}(x_0).$$

Proof. As a first case suppose $f(x_0) = 0$. Since f is continuous $f_0^{L_p}(x_0) = 0$ also. Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\left\{ \frac{1}{|h|} \int_0^h \left| f(x_0+2^{j-1}t) - \sum_{i=1}^{k} \frac{f_i^{L_p}(x_0)}{i!} (2^{j-1}t)^i \right|^p dt \right\}^{\frac{1}{p}}$$

$$< \frac{\varepsilon |h|^k}{(k+1)!k\lambda_k |a_{k,j}|} ,$$

for each $j = 1,2,\ldots,k$. An application of Holder's inequality yields

$$\frac{1}{|h|} \int_0^h \left| f(x_0+2^{j-1}t) - \sum_{i=1}^{k} \frac{f_i^{L_p}(x_0)}{i!} (2^{j-1}t)^i \right| dt$$

$$< \frac{\varepsilon |h|^k}{(k+1)!k|a_{k,j}|},$$

for each $j = 1,2,\ldots,k$.

Now let $0 < |h| < \delta$. Then

$$\left| \frac{(k+1)!\lambda_k}{h} \int_0^h D_k(x_0,t)\,dt - h^k f_k^{L_p}(x_0) \right|$$

$$= \left| \frac{(k+1)!\lambda_k}{h} \int_0^h D_k(x_0,t) - \frac{t^k}{k!\lambda_k} f_k^{L_p}(x_0)\ dt \right|$$

$$= (k+1)!\lambda_k \left| \frac{1}{h} \int_0^h \sum_{j=1}^{k} a_{k,j} f(x_0+2^{j-1}t) - \frac{t^k}{k!\lambda_k} f_k^{L_p}(x_0)\,dt \right|$$

$$= (k+1)!\lambda_k \Big\{ \Big| \frac{1}{h} \int_0^h \sum_{j=1}^{k} \Big[a_{k,j} \; f(x_0 + 2^{j-1} t)$$

$$- a_{k,j} \sum_{i=1}^{k} \frac{f_i^{L_p}(x_0)}{i!} (2^{j-1} t)^i \Big]$$

$$+ \sum_{j=1}^{k} \Big[a_{k,j} \sum_{i=1}^{k} \frac{f_i^{L_p}(x_0)}{i!} (2^{j-1} t)^i \Big] - \frac{t^k}{k!\lambda_k} f_k^{L_p}(x_0)\, dt \Big| \Big\}$$

$$\leq (k+1)!\lambda_k \Big\{ \sum_{j=1}^{k} \frac{1}{|h|} \int_0^h \Big| a_{k,j} \; f(x_0 + 2^{j-1} t)$$

$$- a_{k,j} \sum_{i=1}^{k} \frac{f_i^{L_p}(x_0)}{i!} (2^{j-1} t) \Big| dt$$

$$+ \frac{1}{|h|} \int_0^h \Big| \sum_{i=1}^{k} \frac{f_i^{L_p}(x_0)}{i!} \sum_{j=1}^{k} a_{k,j} (2^{j-1} t)^i$$

$$- \frac{t^k}{\lambda_k k!} f_k^{L_p}(x_0) \Big| dt \Big\}$$

$$< (k+1)!\lambda_k \Big\{ \sum_{j=1}^{k} \frac{\epsilon |h|^k}{(k+1)!\,k\lambda_k}$$

$$+ \frac{1}{|h|} \int_0^h \sum_{i=1}^{k-1} \Big| \frac{f_i^{L_p}(x_0)}{i!} t^i \sum_{j=1}^{k} a_{k,j} (2^{j-1})^i \Big| dt$$

$$+ \frac{1}{|h|} \int_0^h \Big| \frac{f_k^{L_p}(x_0)}{k!} t^k \sum_{j=1}^{k} a_{k,j} (2^{j-1})^k$$

$$- \frac{t^k}{\lambda_k k!} f_k^{L_p}(x_0) \Big| dt \Big\}$$

$$= (k+1)!\lambda_k\left\{\sum_{j=1}^{k} \frac{\varepsilon|h|^k}{(k+1)!k\lambda_k} + 0 + \frac{1}{|h|}\int_0^h \left|\frac{f_k^{L^p}(x_0)}{k!\lambda_k}\, t^k - \frac{f_k^{L^p}(x_0)}{k!\lambda_k}\, t^k\right| dt\right\}$$

$$= (k+1)!\lambda_k\left\{\sum_{j=1}^{k} \frac{\varepsilon|h|^k}{(k+1)!k\lambda_k}\right\}$$

$$= \varepsilon|h|^k.$$

Hence the lemma is proved for the case where $f(x_0) = 0$.

For the general case we set $g(x) = f(x) - f(x_0)$. Then

$$\begin{aligned} D(x_0,t;g) &= a_{k,0}g(x_0) + \sum_{j=1}^{k} a_{k,j}\, g(x_0+2^{j-1}t) \\ &= \sum_{j=1}^{k} a_{k,j}\, f(x_0+2^{j-1}t) - \sum_{j=1}^{k} a_{k,j}\, f(x_0) \\ &= \sum_{j=1}^{k} a_{k,j}\, f(x_0+2^{j-1}t) - \sum_{j=0}^{k} a_{k,j}\, f(x_0) \\ &\quad + a_{k,0}\, f(x_0) \\ &= D(x_0,t;f). \end{aligned}$$

By the previous considerations

$$\lim_{h\to 0} \frac{(k+1)!\lambda_k}{h^{k+1}} \int_0^h D_k(x_0,t;g)\,dt = g_k^{L^p}(x_0),$$

but clearly $g_k^{L_p}(x_0) = f_k^{L_p}(x_0)$, and hence

$$\lim_{h\to 0} \frac{(k+1)!\lambda_k}{h^{k+1}} \int_0^h D_k(x_0,t;f)\,dt = f_k^{L_p}(x_0).$$

<u>Theorem 2.3</u>. Let f be a continuous function, defined on an interval I, possessing a k^{th} L_p derivative at each point of I. Then $f_k^{L_p}$ is of Baire class one.

Proof. For each positive integer n let

$$f_n(x) = (k+1)!\lambda_k n^{k+1} \int_0^h D_k(x,t)\,dt.$$

By lemma 2.5 we have

$$\lim_{n\to\infty} f_n(x) = f_k^{L_p}(x)$$

for each x in I. Since f is continuous, we clearly have that each f_n is also, and the proof is complete.

<u>Definition 2.3</u>. As in [14] we call a sequence $\{f_n\}$ of functions, defined on an interval I, quasi-nonincreasing if for each x in I there exists a number $N(x)$ such that for all $n > N(x)$, $f_{n-1}(x) \geq f_n(x)$.

Lemma 2.6. Suppose f is continuous and possesses a k^{th} L_{p_L} derivative everywhere on an interval I, with $f_k^{L_p}(x) > 0$ at each point in I. Then there exists a quasi-nonincreasing sequence $\{f_n\}$ of continuous functions such that

$$\lim_{n\to\infty} f_n(x) = f_{k-1}^{L_p}(x) \quad \text{for each} \quad x \in I.$$

Proof. For each positive integer n define functions f_n and g_n by

$$f_n(x) = k!\ 2^{nk}\lambda_{k-1} \int_0^{\frac{1}{2^n}} D_{k-1}(x,t)\,dt,$$

$$g_n(x) = (k+1)!\ 2^{n(k+1)}\lambda_k \int_0^{\frac{1}{2^n}} D_k(x,t)\,dt.$$

Clearly f_n and g_n are continuous functions of x, and by lemma 2.5

$$\lim_{n\to\infty} f_n(x) = f_{k-1}^{L_p}(x) \quad \text{for each} \quad x \text{ in I},$$

and

$$\lim_{n\to\infty} g_n(x) = f_k^{L_p}(x) \quad \text{for each} \quad x \text{ in I}.$$

Let an x_0 in I be fixed. There exists a positive integer $N(x_0)$ such that for $n > N, g_n(x_0) > 0$. Now let $n > N$.

$$f_{n-1}(x_0) - f_n(x_0) = k!2^{(n-1)k}\lambda_{k-1}\int_0^{\frac{1}{2^{n-1}}} D_{k-1}(x_0,t)\,dt$$

$$- k!2^{nk}\lambda_{k-1}\int_0^{\frac{1}{2^n}} D_{k-1}(x_0,t)\,dt$$

$$= k!2^{nk-k+1}\lambda_{k-1}\int_0^{\frac{1}{2^n}} D_{k-1}(x_0,2t)\,dt$$

$$- k!2^{nk}\lambda_{k-1}\int_0^{\frac{1}{2^n}} D_{k-1}(x_0,t)\,dt$$

$$= k!2^{nk-k+1}\lambda_{k-1}\Big[\int_0^{\frac{1}{2^n}} D_{k-1}(x_0,2t)$$

$$- 2^{k-1} D_{k-1}(x_0,t)\,dt\Big]$$

$$= \frac{\lambda_{k-1}}{2^{n+k-1}(k+1)\lambda_k}\Big[(k+1)!2^{n(k+1)}\lambda_k\int_0^{\frac{1}{2^n}} D_k(x_0,t)\,dt\Big]$$

$$= \frac{\lambda_{k-1}}{2^{n+k-1}(k+1)\lambda_k}\, g_n(x_0)$$

$$> 0.$$

So $f_{n-1}(x_0) > f_n(x_0)$, and the proof is complete.

The following lemma and its proof are due to Saks [14].

<u>Lemma 2.7</u>. If $\{f_n\}$ is a quasi-nonincreasing sequence of continuous functions with $f_n(x)$ converging to $f(x)$ for each x in an interval I, then for each closed set P, there exists an interval (a,b) such that $(a,b) \cap P \neq \emptyset$, and f, restricted to P, is upper semicontinuous on $(a,b) \cap P$.

Proof. For each positive integer m, let $P_m = \{x \in P: f_n(x) \leq f_{n-1}(x)$ for all $n > m\}$. Since $\{f_n\}$ is quasi-nonincreasing, $P = \bigcup_{m=1}^{\infty} P_m$, and since each f_n is continuous, each P_m is closed. According to the Baire category theorem, there is an interval (a,b) and an integer m_0 such that $\emptyset \neq P \cap (a,b) \subset P_{m_0}$. On P_{m_0} $\{f_n : n > m_0\}$ is a nonincreasing sequence of continuous functions converging to f. So f is upper semicontinuous on P_{m_0} and hence on $(a,b) \cap P$.

The next lemma and proof are due to Oliver [13].

Lemma 2.8. Let f be defined on an interval $[a,b]$. Let P be a closed subset of $[a,b]$ and let f, restricted to P, be upper semicontinuous on P. For each component interval (c,d) of the complement of P, let f be nondecreasing and upper semicontinuous on the closure $[c,d]$. Then f is upper semicontinuous on $[a,b]$.

Proof. If x does not belong to P, then f is upper semicontinuous at x by hypothesis. If x belongs to P and is isolated on the right (or left) from P, then x is the left (or right) endpoint of a component interval of the complement of P, and so is upper semicontinuous on the right (or left). If x belongs to P and is an accumulation point from the right of P, choose $\delta > 0$ so that $f(y) < g(x) + \epsilon$, for all y in P such that $x \leq y \leq x + \delta$. Since x is an accumulation point from the right of P, δ may be chosen so that $x + \delta$ belongs to P. Let x' be any point such that $x < x' \leq x + \delta$. If x' is in P, $f(x') < f(x) + \epsilon$ by the choice of δ. If x' is in the complement of P, x' belongs to a component interval, say (c',d'), of the complement of P. Since $x + \delta$ and d' both belong to P, $d' \leq x + \delta$, so that $f(d') < f(x) + \varepsilon$. But f is nondecreasing on $[c',d']$, so that $g(x') \leq g(d') < g(x) + \varepsilon$. So f is upper semicontinuous on the right at each accumulation point from the right of P. Similarly f is upper semicontinuous on the left at each accumulation point from the left of P. Hence f is upper semicontinuous on $[a,b]$.

The next lemma has been proved for the k^{th} Peano derivative by Oliver [13]. The proof given here for the k^{th} L_p derivative is based on his proof. Like Oliver, we shall use without specific reference in this and subsequent proofs several well known results dealing with functions that are of Baire class one and/or have the Darboux property. We list these results here without proof (see [13] for a more complete list of such properties):

If f is monotone on (a,b), and has the Darboux property on $[a,b]$, then f is monotone on $[a,b]$.

If f is monotone and has the Darboux property on an interval, then it is continuous on the interval.

If f belongs to Baire class one on an interval, then the points of continuity of f are everywhere dense in the interval.

If f belongs to Baire class one, then f has the Darboux property if, and only if, for each number a, the sets $E_a = \{x : f(x) \leq a\}$, and $E^a = \{x : f(x) \geq a\}$ have closed connected components.

Lemma 2.9. Let f be continuous on an interval I and have the following properties:

1) f has property M_0^{k-1},
2) f has a k^{th} L_p derivative everywhere
3) $f_k^{L_p}(x) > 0$ for each x in I
   (or $f_k^{L_p}(x) \geq 0$ for each x in I)
4) $f_{k-1}^{L_p}$ has the Darboux property.

Then $f_{k-1}^{L_p}$ is increasing (or nondecreasing) and continuous on I.

Proof. Suppose first that $f_k^{L_p}(x) > 0$ everywhere. Let $G = \{x \in I\colon f_{k-1}^{L_p}$ is upper semicontinuous and increasing in some neighborhood of $x\}$. G is clearly an open set. Let $P = I - G$. By lemmas 2.6 and 2.7 there exists an interval (a,b) such that

$(a,b) \cap P \neq \emptyset$, and $f_{k-1}^{L_p}$ is upper semicontinuous on $(a,b) \cap P$. If $(c,d) \subset (a,b) - P$, then $f_{k-1}^{L_p}$ is upper semicontinuous and increasing on (c,d). Since it possesses the Darboux property, $f_{k-1}^{L_p}$ is actually increasing on $[c,d]$.

Now if we apply lemma 2.8, we see that $f_{k-1}^{L_p}$ is upper semicontinuous on (a,b). Therefore, by lemma 2.3 $f_{k-1}^{L_p}$ is increasing on (a,b), and so $(a,b) \subset G$. Hence $P = \emptyset$, and $f_{k-1}^{L_p}$ must be increasing and upper semicontinuous on I. Furthermore, since $f_{k-1}^{L_p}$ has the Darboux property, $f_{k-1}^{L_p}$ is actually continuous on I.

The case where $f_{k}^{L_p}(x) \geq 0$ on I can now be obtained from the above case as in lemma 2.3.

The following theorem was proved for the k^{th} Peano derivative by Oliver [13].

<u>Theorem 2.4</u>. Let f be continuous and possess a k^{th} L_p derivative at each point of an interval I. Then

1) $f_{k}^{L_p}$ has the Darboux property
2) f has properties M_j^k, $j = 0,1,\ldots,k-1$.

Proof. We prove this theorem by induction on k. For $k = 1$ we have from theorem 2.1 together with its following remark and corollary that $f^{L_p}(x) = f'_{ap}(x)$ for each x in I, and that $f_1^{L_p}$ has the Darboux property. Goffman and Neugebauer [7] have shown that if f has an approximate derivative everywhere on an interval, then f has property M_0^1. So the theorem is proved for the case $k = 1$.

Assume that the theorem is true for $k-1$. We want to show it is true for k. Since by theorem 2.3 $f_k^{L_p}$ is of Baire class one, in order to show that $f_k^{L_p}$ has the Darboux property we need only show that the connected components of $\{x : f_k^{L_p}(x) \geq a\}$ and $\{x : f_k^{L_p}(x) \leq a\}$ are closed for every number a. By considering the function $g(x) = f(x) - \frac{a}{k!} x^k$, we may reduce this to showing that the connected components of $\{x : f_k^{L_p}(x) \geq 0\}$ and $\{x : f_k^{L_p}(x) \leq 0\}$ are closed. So suppose $f_k^{L_p}(x) \geq 0$ for all x in the interior of an interval J. We must show that $f_k^{L_p}(x) \geq 0$ on the closure $\bar{J}$ of J. By lemma 2.9, if $f_k^{L_p}(x) \geq 0$ on the interior of J, then $f_{k-1}^{L_p}$ is increasing on the interior of J. Since by the inductive hypothesis $f_{k-1}^{L_p}$ has the Darboux property, $f_{k-1}^{L_p}$ is increasing on

$\bar{J}$. Let c be the left endpoint of $\bar{J}$, and suppose that $f_k^{L_p}(c) < 0$. Then by lemma 2.2 there is an $h > 0$ such that $c + h$ belongs to J and

$$f(c+h) - \sum_{i=0}^{k-1} \frac{f_i^{L_p}(c)}{i!} h^i < 0.$$

So $$\frac{f(c+h) - \sum_{i=0}^{k-2} \frac{f_i^{L_p}(c)}{i!} h^i}{\frac{h^{k-1}}{(k-1)!}} - f_{k-1}^{L_p}(c) < 0.$$

By the inductive hypothesis f has property M_0^{k-1}, and so there exists an x' with $c < x' < c + h$, such that

$$\frac{f(c+h) - \sum_{i=0}^{k-2} \frac{f_i^{L_p}(c)}{i!} h^i}{\frac{h^{k-1}}{(k-1)!}} = f_{k-1}^{L_p}(x').$$

Hence

$$f_{k-1}^{L_p}(x') - f_{k-1}^{L_p}(c) < 0,$$

but this contradicts the fact that $f_{k-1}^{L_p}$ is increasing on $\bar{J}$. So we must have $f_{k-1}^{L_p}(c) \geq 0$.

If d is the right endpoint of J, we can similarly show that $f_k^{L_p}(d) \geq 0$. So $\{x : f_k^{L_p}(x) \geq 0\}$ has closed connected components. Similarly we can show that the connected components of $\{x : f_k^{L_p}(x) \leq 0\}$ are closed, and hence $f_k^{L_p}$ has the Darboux property.

By lemma 2.1 we know that in order to show that f has properties M_j^k, $j = 0,1,\ldots,k-1$, it suffices to show that f has property R_{k-1}^k. So let x and $x+h$ be such that

$$f_{k-1}^{L_p}(x+h) - f_{k-1}^{L_p}(x) = 0.$$

Let $J = [x,x+h]$. If $f_k^{L_p}$ is identically zero on $\overset{\circ}{J}$, the interior of J, we are done. $f_k^{L_p}$ cannot be positive everywhere in $\overset{\circ}{J}$ because that would imply that $f_{k-1}^{L_p}$ is increasing on $\overset{\circ}{J}$ as seen in lemma 2.9, and since $f_{k-1}^{L_p}$ has the Darboux property, this would imply that $f_{k-1}^{L_p}(x+h) > f_{k-1}^{L_p}(x)$. Similarly $f_k^{L_p}$ cannot be negative everywhere in $\overset{\circ}{J}$. So $f_k^{L_p}$ must be either identically zero on $\overset{\circ}{J}$, or assume both positive and negative values. But the latter situation implies that there is a point x' in $\overset{\circ}{J}$ where $f_k^{L_p}(x') = 0$ since $f_k^{L_p}$ has the Darboux property.

Lemma 2.10. Let f be measurable and have a k^{th} L_p derivative at 0 with $f_i^{L_p}(0) = 0$, $i = 0,1,\ldots,k-1$, $f_k^{L_p}(0) = A$. Then

$$\lim_{h\to 0} {}_{ap} \frac{f(h)}{h^k} = \frac{A}{k!}$$

Proof. We will show here that

$$\lim_{h\to 0^+} {}_{ap} \frac{f(h)}{h^k} = \frac{A}{k!} \quad .$$

The proof that

$$\lim_{h\to 0^-} {}_{ap} \frac{f(h)}{h^k} = \frac{A}{k!}$$

is analogous.

Let $\epsilon > 0$ be given, and let $E_\epsilon = \{t > 0\colon |f(t) - A\frac{t^k}{k!}| > \epsilon\, t^k\}$. We must show that E_ϵ has 0 as a point of dispersion from the right. For any positive h let $E_h = E_\epsilon \cap [0,h]$. Then we have

$$\left\{ \frac{1}{h} \int_0^h |f(t) - A\frac{t^k}{k!}|^p \, dt \right\}^{\frac{1}{p}} \geq \left\{ \frac{1}{h} \int_{E_h} |f(t) - A\frac{t^k}{k!}|^p \, dt \right\}^{\frac{1}{p}}$$

$$\geq \left\{ \frac{1}{h} \int_{E_h} \epsilon^p \, t^{kp} \, dt \right\}^{\frac{1}{p}}$$

$$\geq \left\{ \frac{1}{h} \int_0^{|E_h|} \epsilon^p t^{kp} \, dt \right\}^{\frac{1}{p}}$$

$$= \left\{ \frac{\epsilon^p}{kp+1} \cdot \frac{|E_h|^{kp+1}}{h} \right\}^{\frac{1}{p}} ,$$

and as $h \to 0^+$, this must be $o(h^k)$, i.e.

$$\frac{\epsilon}{(kp+1)^{1/p}} \qquad \lim_{h \to 0^+} \left(\frac{|E_h|}{h} \right)^{k+\frac{1}{p}} = 0.$$

Hence $\lim\limits_{h \to 0^+} \dfrac{|E_h|}{h} = 0$, and so

$$\lim_{h \to 0^+}{}_{ap} \frac{f(h)}{h^k} = \frac{A}{k!} .$$

Besides the use that is made of the above lemma in this chapter, it has some other interesting consequences, which will be explored in Chapter III.

<u>Lemma 2.11</u>. Let 0 be a point of density of a set E. Then there are sequences $\{a_n\} \subset E$ and $\{b_n\} \subset E$ such that

1) $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} b_n = 0,$

2) $a_{n+1} < a_n$ and $b_{n+1} > b_n$ for each $n,$

and 3) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1.$

Proof. Here we construct the sequence $\{a_n\}$. The sequence $\{b_n\}$ can be constructed in a similar fashion. First we define a sequence $\{\delta_m\}$. Choose $\delta_1 > 0$, so that if $0 < h \leq \delta_1$, then

$$\frac{|[0,h] \cap E|}{h} > \frac{2}{3} .$$

In general, having chosen δ_m, choose δ_{m+1} so that $0 < \delta_{m+1} < \frac{m}{m+2} \delta_m$, and so that if $0 < h \leq \delta_{m+1}$, then

$$\frac{|[0,h] \cap E|}{h} > \frac{m+1}{m+2} .$$

Now pick $a_1 \in \left[\frac{2}{3} \delta_1, \delta_1 \right] \cap E$. Such an a_1 exists since

$$|[0,\delta_1] \cap E| > \frac{2}{3} \delta_1 .$$

Pick $a_2 \in \left[\frac{2}{4} a_1, \frac{3}{4} a_1 \right] \cap E$. Such an a_2 exists since

$$\left|\left[0,\frac{3}{4} a_1\right] \cap E\right| > \left(\frac{3}{4} a_1 \right) \frac{2}{3} = \frac{2}{4} a_1 .$$

Keep this process up, choosing $a_{n+1} \in \left[\frac{2}{4} a_n, \frac{3}{4} a_n \right] \cap E$, until we come to the smallest integer k_1 such that $a_{k_1} \leq \delta_2$. Notice that

$$a_{k_1} \geq \frac{2}{4} a_{k_1} - 1$$

$$> \frac{2}{4} \delta_2 \quad \text{(by the choice of } k_1\text{)}$$

$$> \delta_3 \quad \text{(by the choice of } \delta_3\text{)}.$$

Pick $a_{k_1+1} \in \left[\frac{3}{5} a_{k_1}, \frac{4}{5} a_{k_1} \right] \cap E$. Such a number exists since

$$\left| \left[0, \frac{4}{5} a_{k_1} \right] \cap E \right| > \left(\frac{4}{5} a_{k_1} \right) \frac{3}{4} = \frac{3}{5} a_{k_1} .$$

The process is as follows:

Having chosen a_n, choose $a_{n+1} \in \left[\frac{3}{5} a_n, \frac{4}{5} a_n \right] \cap E$,

and keep this up until arriving at the smallest integer k_2 such that $a_{k_2} \leq \delta_3$. Again we easily see that $a_{k_2} > \delta_4$.

In general, having found $\delta_{m+2} < a_{k_m} \leq \delta_{m+1}$ by the above process we choose

$$a_{k_m+1} \in \left[\frac{m+2}{m+4} a_{k_m}, \frac{m+3}{m+4} a_{k_m} \right] \cap E,$$

which we can do since

$$\left| \left[0, \frac{m+3}{m+4} a_{k_m} \right] \cap E \right| > \left(\frac{m+3}{m+4} \right) a_{k_m} \left(\frac{m+2}{m+3} \right) = \frac{m+2}{m+4} a_{k_m} .$$

Keep choosing $a_{n+1} \in \left[\frac{m+2}{m+4} a_n, \frac{m+3}{m+4} a_n \right] \cap E$ until we come to the smallest integer k_{m+1} such that $a_{k_{m+1}} \leq \delta_{m+2}$, and so on.

In this way we arrive at a sequence $\{a_n\}$. We clearly have $a_{n+1} < a_n$ for each n, and $\lim_{n\to\infty} a_n = 0$. Now fix an n, and choose the smallest of the integers in the sequence $\{k_m\}_{m=1}^{\infty}$ such that $a_n \leq a_{k_m}$. If $a_{k_{m+1}} \leq a_{n+1}$, then

$$\frac{m+2}{m+4} \leq \frac{a_{n+1}}{a_n} \leq \frac{m+3}{m+4} ,$$

and on the other hand, if $a_{k_{m+1}} > a_{n+1}$, then

$$\frac{m+3}{m+5} = \frac{(m+1)+2}{(m+1)+4} \leq \frac{a_{n+1}}{a_n} \leq \frac{(m+1)+3}{(m+1)+4} = \frac{m+4}{m+5} .$$

Now as $n \to \infty$, $m \to \infty$, and as $m \to \infty$

$$\frac{m+2}{m+4} \to 1,$$

$$\frac{m+3}{m+4} \to 1,$$

and $$\frac{m+4}{m+5} \to 1.$$

Hence $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$

<u>Lemma 2.12</u>. Let f be measurable and have a k^{th} L_p derivative at 0 with $f_i^{L_p}(0) = 0$, $i = 0,1,\ldots,k-1$, $f_k^{L_p}(0) = A$. If f is monotone on a right neighborhood of 0 and monotone on a left neighborhood of 0, then $A = f_k(0)$, the k^{th} Peano derivative of f at 0.

Proof. From lemma 2.10 we have

$$\lim_{\substack{h\to 0 \\ h\in E}} \frac{f(h)}{h^k} = \frac{A}{k!} ,$$

where E is a set of density 1 at 0.

As a first case suppose that f is monotone nondecreasing on a right neighborhood of 0. Let $\{a_n\} \subset E$ be a sequence as described in lemma 2.11. Let h be in the given right neighborhood of 0. Choose n so that $a_{n+1} \le h < a_n$. Then

$$\frac{f(a_{n+1})}{(a_n)^k} \le \frac{f(h)}{h^k} \le \frac{f(a_n)}{(a_{n+1})^k} ;$$

so

$$\left(\frac{a_{n+1}}{a_n}\right)^k \cdot \frac{f(a_{n+1})}{(a_{n+1})^k} \le \frac{f(h)}{h^k} \le \frac{f(a_n)}{(a_n)^k} \cdot \left(\frac{a_n}{a_{n+1}}\right)^k .$$

Now as $h \to 0^+$, $n \to \infty$ and the two outside members of the above inequality tend to $\frac{A}{k!} \cdot 1 = \frac{A}{k!}$. Hence

$$\lim_{h \to 0^+} \frac{f(h)}{h^k} = \frac{A}{k!} .$$

All other cases are treated analogously, and we conclude that

$$\lim_{h \to 0} \frac{f(h)}{h^k} = \frac{A}{k!} ,$$

i.e.

$$f_k(0) = A.$$

We are now in a position to prove a theorem which relates k^{th} L_p derivatives to k^{th} ordinary derivatives. This result will then be used to establish the Denjoy property and the Zahorski property for k^{th} L_p derivatives of continuous functions. The theorem is patterned after a result by Oliver [13] in which he shows that if a k^{th} Peano derivative f_k is bounded either above or below on an interval, then $f_k(x) = f^{(k)}(x)$, the ordinary k^{th} derivative of f at x, for each x in the interval.

<u>Theorem 2.5</u>. Let f be continuous and possess a k^{th} L_p derivative at each point of an interval I. If $f_k^{L_p}$ is bounded either above or below, then $f_k^{L_p}(x) = f^{(k)}(x)$, the ordinary k^{th} derivative of f at x, for each x in I.

Proof. Assume that $f_k^{L_p}$ is bounded below on I; say $f_k^{L_p}(x) > M$ for all x in I. (The proof for the case where $f_k^{L_p}$ is bounded above is analogous). Let x_0 be any point in I. We will show that the k^{th} Peano derivative of f at x_0 exists and that $f_k^{L_p}(x_0) = f_k(x_0)$. Having done this for all x_0 in I, we will use Oliver's result mentioned above to conclude that $f^{(k)}(x_0)$ exists for each x_0 in I and $f_k^{L_p}(x_0) = f^{(k)}(x_0)$.

For a fixed x_0 in I, let

$$g(x) = f(x+x_0) - \Big(f(x_0) + xf_1^{L_p}(x_0) + \frac{x^2}{2!} f_2(x_0) + \dots$$

$$+ \frac{x^{k-1}}{(k-1)!} f_{k-1}^{L_p}(x\ ) + \frac{x^k}{k!} M\Big) .$$

Then

$$g(0) = g_1^{L_p}(0) = \dots = g_{k-1}^{L_p}(0) = 0,$$

and if we let J represent the interval obtained by translating I in such a way that x_0 is moved to the origin, then

$$g_k^{L_p}(x) = f_k^{L_p}(x) - M > 0 \quad \text{for all} \quad x \in J.$$

In particular set $A = f_k^{L_p}(x_0) - M = g_k^{L_p}(0)$.

In order to show that $f_k^{L_p}(x_0) = f_k(x_0)$, it suffices to show that $g_k^{L_p}(0) = g_k(0)$, i.e. it suffices to show that

$$\lim_{h\to 0} \frac{g(h)}{h^k} = \frac{A}{k!} .$$

Since $g_k^{L_p}$ is positive on J, lemma 2.9 together with theorem 2.4 imply that $g_{k-1}^{L_p}$ is non-decreasing on J. Suppose $J = [a,b]$, and let

$$y_{k-1}^0 = \sup \{y \in J\colon g_{k-1}^{L_p}(y) \leq 0\}.$$

Let $J_{k-1}^0 = [a, y_{k-1}^0]$, $J_{k-1}^1 = [y_{k-1}^0, b]$.

Now $g_{k-1}^{L_p}(x) \geq 0$ on J_{k-1}^1, and so by the same argument used directly above, $g_{k-2}^{L_p}$ is nondecreasing on J_{k-1}^1, and similarly $g_{k-2}^{L_p}$ is nonincreasing on J_{k-1}^0. So let

$$y_{k-2}^{0} = \sup\{y \in J_{k-1}^{0} : g_{k-2}^{L_p} \geq 0\}$$

$$y_{k-2}^{1} = \sup\{y \in J_{k-1}^{1} : g_{k-2}^{L_p} \leq 0\},$$

and

$$J_{k-2}^{0} = [a, y_{k-2}^{0}],$$

$$J_{k-2}^{1} = [y_{k-2}^{0}, y_{k-1}^{0}]$$

$$J_{k-2}^{2} = [y_{k-1}^{0}, y_{k-2}^{1}]$$

$$J_{k-2}^{3} = [y_{k-2}^{1}, b].$$

Then

$$g_{k-3} \text{ is } \begin{cases} \text{nondecreasing on } J_{k-2}^{0}. \\ \text{nonincreasing on } J_{k-2}^{1}. \\ \text{nondecreasing on } J_{k-2}^{2}. \\ \text{nonincreasing on } J_{k-2}^{3}. \end{cases}$$

Continuing this process for a total of $k-1$ times, we arrive at 2^{k-1} intervals J_1^i, $i = 0,1,\ldots,2^{k-1} - 1$, with $J = \bigcup_{i=0}^{2^{k-1}-1} J_1^i$. On each of these intervals g is monotone and continuous. The point 0 can belong to at most two of these intervals (with the possible exception of degenerate intervals). Hence g is monotone on a right neighborhood of 0 and monotone on a left neighborhood of 0. So by lemma 2.12

$$\lim_{h\to 0} \frac{g(h)}{h^k} = \frac{A}{k!}$$

which is what we wanted to show.

In [18] Weil proved that a function g of Baire class one has the Denjoy property on an interval I if, for every subinterval J of I on which g is bounded either above or below, g restricted to J has the Denjoy property. Using this result along with the fact that an ordinary k^{th} derivative has the Denjoy property, we have the following corollary to the last theorem.

<u>Corollary 2.5</u>. If f is a continuous function having a k^{th} L_p derivative at each point of an interval I, then $f_k^{L_p}$ has the Denjoy property on I.

<u>Theorem 2.6</u>. Let f be continuous and possess a k^{th} L_p derivative at each point of an interval I. Then $f_k^{L_p}$ has the Zahorski property.

Proof. Following along the lines of the proof in [18] we first consider a special case where

$$f_0^{L_p}(0) = f_1^{L_p}(0) = \ldots = f_k^{L_p}(0) = 0,$$

and assume that $\{I_n = [a_n, b_n]\}$ is a sequence of closed intervals, with positive endpoints, converging to 0 in such a way that, for each n, x in I_n implies $f_k^{L_p}(x) \geq c$, where c is a fixed positive number. Let n be a positive integer. Then the same inductive proof as that given in [18] will show that, for each positive integer j with $1 \leq j \leq k$, there is a partition

$$a_n = t_{j,0} < \cdots < t_{j,m(j)} = b_n$$

of I_n such that $m(j) \leq 2^j$ and, for each $i = 1,\ldots,m(j)$, one of the following holds for every x in $[t_{j,i-1}, t_{j,i}]$:

$$1(j):\ f_{k-j}^{L_p}(x) - f_{k-j}^{L_p}(t_{j,i-1}) \geq \left(\frac{c}{j!}\right)(x-t_{j,i-1})^j,$$

$$\text{and}\quad f_{k-j}^{L_p}(t_{j,i-1}) \geq 0$$

$$2(j):\ f_{k-j}^{L_p}(x) - f_{k-j}^{L_p}(t_{j,i}) \leq -\left(\frac{c}{j!}\right)(t_{j,i}-x)^j,$$

$$\text{and}\quad f_{k-j}^{L_p}(t_{j,i}) \leq 0$$

$$3(j):\ f_{k-j}^{L_p}(x) - f_{k-j}^{L_p}(t_{j,i-1}) \leq -\left(\frac{c}{j!}\right)(x-t_{j,i-1})^j,$$

$$\text{and}\quad f_{k-j}^{L_p}(t_{j,i-1}) \leq 0$$

$$4(j) \; : \; f_{k-j}^{L_p}(x) - f_{k-j}^{L_p}(t_{j,i}) \geq (\tfrac{c}{j!})(t_{j,i}-x)^j,$$

$$\text{and} \quad f_{k-j}^{L_p}(t_{j,i}) \geq 0.$$

In particular if we examine the situation when $j = k$ and recall that $f_0^{L_p} = f$, then we can say that, for each n, there is a partition

$$a_n = t_0 < \dots < t_m = b_n$$

of I_n such that $m \leq 2^k$, and for each $i = 1,\dots,m$, one of the following holds for every x in $[t_{i-1}, t_i]$:

(1) $f(x) \geq (\frac{c}{k!})(x-t_{i-1})^k$

(2) $f(x) \leq -(\frac{c}{k!})(t_i-x)^k$

(3) $f(x) \leq -(\frac{c}{k!})(x-t_{i-1})^k$

(4) $f(x) \geq (\frac{c}{k!})(t_i-x)^k$.

In case (1) or (3) holds on $[t_{i-1},t_i]$ we have

$$\left\{\int_{t_{i-1}}^{t_i} |f(x)|^p dx\right\}^{\frac{1}{p}} \geq (\tfrac{c}{k!}) \left\{\int_{t_{i-1}}^{t_i} (x-t_{i-1})^{kp} dx\right\}^{\frac{1}{p}}$$

$$= \left(\frac{c}{k!\,(kp+1)^{1/p}}\right) (t_i-t_{i-1})^{\frac{kp+1}{p}};$$

and in case (2) or (4) holds on $[t_{i-1}, t_i]$

$$\left\{\int_{t_{i-1}}^{t_i} |f(x)|^p dx\right\}^{\frac{1}{p}} \geq \left(\frac{c}{k!}\right) \left\{\int_{t_{i-1}}^{t_i} (t_i-x)^{kp} dx\right\}^{\frac{1}{p}}$$

$$= \left(\frac{c}{k!\,(kp+1)^{1/p}}\right) (t_i-t_{i-1})^{\frac{kp+1}{p}} .$$

Hence

$$\left(\frac{c}{k!\,(kp+1)^{1/p}}\right) (t_i-t_{i-1})^{\frac{kp+1}{p}} \leq \left\{\int_0^{t_i} |f(x)|^p dx\right\}^{\frac{1}{p}}$$

$$= t_i^{\frac{kp+1}{p}} \quad \varepsilon\,(t_i),$$

where $\varepsilon(t_i)$ approaches 0 as t_i approaches 0, because $f(0) = f_1^{L^p}(0) = \ldots = f_k^{L^p}(0) = 0$. With this estimate we can make the following approximation:

$$\frac{(b_n-a_n)}{b_n} = \sum_{i=1}^{m} \frac{(t_i-t_{i-1})}{b_n}$$

$$< \sum_{i=1}^{m} \frac{(t_i-t_{i-1})}{t_i}$$

$$\leq m \left(\frac{k!\,(kp+1)^{1/p}}{c}\right)^{\frac{p}{kp+1}} \left(\max_{i=1,\ldots,m} \varepsilon(t_i)\right)^{\frac{p}{kp+1}}$$

$$\leq 2^k \left(\frac{k!\,(kp+1)^{1/p}}{c}\right)^{\frac{p}{kp+1}} \left(\max_{i=1,\ldots,m} \varepsilon(t_i)\right)^{\frac{p}{kp+1}} .$$

Since I_n converges to 0, and since $\lim_{t_i \to 0} \varepsilon(t_i) = 0$,

$$\lim_{n\to\infty} \max_{t_i \in I_n} \varepsilon(t_i) = 0.$$

Hence

$$\lim_{n\to\infty} \frac{(b_n - a_n)}{b_n} = 0.$$

But $\frac{(b_n - a_n)}{b_n} = 1 - \frac{a_n}{b_n}$. So

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1,$$

and hence

$$\lim_{n\to\infty} \frac{(b_n - a_n)}{a_n} = \lim_{n\to\infty} \left(\frac{b_n}{a_n}\right) - 1$$

$$= 1 - 1 = 0.$$

This completes the proof of the special case.

The procedure for reducing the general case to the special one is the same as that employed by Weil in [18] and will not be repeated here.

We have thus shown that k^{th} L_p derivatives of continuous functions possess the four basic properties mentioned in Chapter I as being shared by ordinary derivatives, k^{th} Peano derivatives, and approximate derivatives. We now show that k^{th} L_p derivatives share a couple of other properties with k^{th} Peano derivatives and approximate derivatives. These properties deal with the relationship between these generalized derivatives and ordinary derivatives.

In [17] Weil showed that if f has a k^{th} Peano derivative (or an approximate derivative) at each point of an interval I, and if, for some interval (a,b), $f_k^{-1}((a,b))$ is not empty (or $(f'_{ap})^{-1}((a,b))$ is not empty), then

$$|\{x : f^{(k)}(x) \text{ exists and lies in } (a,b)\}| > 0$$

$$(\text{or } |\{x : f'(x) \text{ exists and lies in } (a,b)\}| > 0).$$

In his proof of these results Weil used only those properties of f_k (or f'_{ap}) which we have also shown to be possessed by $f_k^{L_p}$. So without repeating Weil's proof we state the following theorem.

<u>Theorem 2.7</u>. If f is a continuous function possessing a k^{th} L_p derivative at each point of an interval I, and if, for some interval (a,b), $(f_k^{L_p})^{-1}((a,b))$ is not empty, then

$$|\{x : f^{(k)}(x) \text{ exists and lies in } (a,b)\}| > 0.$$

The following theorem is already known to hold for k^{th} Peano derivatives and approximate derivatives (see [5] or [13], and [7] or [16]).

<u>Theorem 2.8</u>. Let f be continuous and possess a k^{th} L_p derivative at each point of an interval I. Then there is an everywhere dense, open set $G \subset I$ at each point of which the ordinary k^{th} derivative exists.

Proof. Let $U = \{x \in I$: for every right neighborhood of x or for every left neighborhood of x, $f_k^{L_p}$ is unbounded both above and below$\}$. Let G be the complement of U. Then $G = \{x \in I$: there is a right neighborhood of x and a left neighborhood of x in each of which $f_k^{L_p}$ is bounded above or below$\}$. G is open in I, and by theorem 2.5 $f^{(k)}(x)$ exists at each point $x \in G$. Since $f_k^{L_p}$ is in Baire class one, the set of points of continuity of $f_k^{L_p}$ is everywhere dense. But each point in U is a point of discontinuity of $f_k^{L_p}$; and so U can contain no interval. Hence G is everywhere dense.

Next we show that in a sense this result is the best possible.

Theorem 2.9. If an interval I, an everywhere dense, open set $G \subset I$, and an integer $k \geq 1$ are given, there is a continuous function f which has a k^{th} L_p derivative at every point of I and for which the set F of points where the ordinary k^{th} derivative exists contains G, and furthermore, if S is an open set with $G \subset S \subset F$, then $G = S$.

Proof. For $k \geq 2$ Oliver constructs a function f which has a k^{th} Peano derivative at each point of I, and hence a k^{th} L_p derivative at each point of I, and which satisfies the theorem. Here we will treat the remaining case, namely $k = 1$.

Denote by P the closed, nowhere dense set $I - G$. Let $\{(a_n,b_n)\}_{n=1}^{\infty}$ denote the sequence of intervals contiguous to P. Fix a particular integer n. Let $c_n = a_n + \frac{b_n-a_n}{4}$, and $d_n = a_n + \frac{3(b_n-a_n)}{4}$. Choose an integer $m_n \geq 2$ so that

$$\frac{1}{2^{m_n p}-1} < (c_n - a_n)^{3p}, \tag{2.9.1}$$

and choose an integer j_n so that

$$a_n + \frac{1}{2^{j_n}} + \frac{1}{2^{m_n p j_n}} < c_n,$$

and

$$b_n - \frac{1}{2^{j_n}} - \frac{1}{2^{m_n p j_n}} > d_n .$$

We shall first define f on (a_n, c_n) as follows:

For every integer $j \geq j_n$, let $I_{n,j} =$ $\left[a_n + \frac{1}{2^j}, a_n + \frac{1}{2^j} + \frac{1}{2^{m_n pj}}\right]$. On each $I_{n,j}$ we define a function g so that

1) g has an ordinary derivative at each point of $I_{n,j}$,

2) $g(a_n + \frac{1}{2^j}) = g(a_n + \frac{1}{2^j} + \frac{1}{2^{m_n pj}}) = 0,$

3) $g'_+(a_n + \frac{1}{2^j}) = g'_-(a_n + \frac{1}{2^j} + \frac{1}{2^{m_n pj}}) = 0,$

4) $\max_{x \in I_{n,j}} g(x) = 1.$

For $x \in (a_n, c_n) - \bigcup_{j=j_n}^{\infty} I_j$, we set $g(x) = 0$. Then for any $x \in (a_n, c_n)$, we set $f(x) = (x-a_n)g(x)$. We set $f(x) = 0$ for $x \in [c_n, d_n]$. Next we will similarly define f on (d_n, b_n). For every integer $j \geq j_n$, let $J_{n,j} = \left[b_n - \frac{1}{2^j} - \frac{1}{2^{m_n pj}}, b_n - \frac{1}{2^j}\right]$. On each $J_{n,j}$ we define a function g so that

1) g has an ordinary derivative at each point of $J_{n,j}$,

2) $g(b_n - \frac{1}{2^j}) = g(b_n - \frac{1}{2^j} - \frac{1}{2^{m_n pj}}) = 0,$

3) $g'_-(b_n - \frac{1}{2^j}) = g'_+(b_n - \frac{1}{2^j} - \frac{1}{2^{m_n pj}}) = 0$

4) $\max\limits_{x \in J_{n,j}} g(x) = 1.$

For $x \in (d_n, b_n) - \bigcup\limits_{j=j_n}^{\infty} J_j$, we set $g(x) = 0$. Then for $x \in (d_n, b_n)$, we set $f(x) = (b_n - x)g(x)$. Now $f(x)$ has been defined for each $x \in (a_n, b_n)$. We do this for each n and set $f(x) = 0$ for $x \in P$ to arrive at a function f defined on all of I.

We first show that f is continuous on I. f is clearly continuous on each interval (a_n, b_n). Now suppose a_n is a left endpoint of an interval (a_n, b_n) contiguous to P. If $h > 0$ and sufficiently small, then

$$|f(a_n+h) - f(a_n)| = f(a_n+h),$$

and if $a_n + h \in (a_n, c_n) - \bigcup_{j=j_n}^{\infty} I_{n,j}$, then $f(a_n+h) = 0$; whereas if $a_n + h \in I_{n,j}$, for some $j \geq j_n$, then $f(a_n+h) = h\,g\,(a_n+h) \leq h$, by condition 4) on g. So f is continuous from the right at a_n, and similarly f is continuous from the left at each b_n. Next suppose that x is an accumulation point of P from the right, and let $h > 0$. If $x + h \in P$,

$$|f(x+h) - f(x)| = |0 - 0| = 0;$$

and if $x + h \in G$, then $x + h \in (a_n, b_n)$ for some n, and so $x + h = a_n + h'$ for some $0 < h' < (b_n - a_n)$; so

$$|f(x+h) - f(x)| = f(x+h) = f(a_n+h') \leq h' < (b_n-a_n).$$

As $h \to 0^+$, $(b_n - a_n) \to 0$ since x is an accumulation point of P. So f is continuous from the right at x. Similarly f is continuous from the left at each point x which is an accumulation point of P from the left. Hence f is continuous on I.

By the way f was defined, it clearly has an ordinary first derivative at each point in G. However, if S is another open set such that $G \subsetneq S$, then S must contain at least one of the endpoints a_n or b_n

for some n. We now show that f is not differentiable in the ordinary sense at any endpoint a_n or b_n. Fix an a_n. Choose a sequence $\{h_k\}$ such that $h_k \to 0$ as $k \to \infty$, and $a_n + h_k \in (a_n, c_n) - \bigcup_{j=j_n}^{\infty} I_{n,j}$ for each k. Then

$$\lim_{k\to\infty} \frac{f(a_n+h_k) - f(a_n)}{h_k} = \lim_{k\to\infty} \frac{0}{h_k} = 0.$$

Now according to condition 4) which we placed on the function g defined on $I_{n,j}$, we may find a number $h_j > 0$ such that $g(a_n+h_j) = 1$. Doing this for each $j \geq j_n$, we have a sequence $\{h_j\}$ with $h_j \to 0$ as $j \to \infty$, and

$$\begin{aligned}\lim_{j\to\infty} \frac{f(a_n+h_j) - f(a_n)}{h_j} &= \lim_{j\to\infty} \frac{f(a_n + h_j)}{h_j} \\ &= \lim_{j\to\infty} \frac{h_j g(a_n + h_j)}{h_j} \\ &= \lim_{j\to\infty} 1 \\ &= 1.\end{aligned}$$

So f does not have an ordinary derivative at a_n. Similary f does not have an ordinary derivative at any b_n.

In order to complete the proof, we must now show that f has a first L_p derivative at each point in I. As noted above, f has an ordinary derivative at each point $x \in G$, and so it has a first L_p derivative at each point $x \in G$. The following calculation shows that at each left endpoint a_n of an interval contiguous to P, the right-hand first L_p derivative $f_1^{L_p}(a_n) = 0$. Let a_n be such an endpoint, and let $0 < h < \frac{1}{2^{j_n}} + \frac{1}{2^{m_n p j_n}}$.

Choose an integer J so that $\frac{1}{2^J} < h \le \frac{1}{2^{J-1}}$. Then

$$(2.9.2) \quad \frac{1}{h}\left\{\frac{1}{h}\int_0^h |f(a_n+t)|^p dt\right\}^{\frac{1}{p}}$$

$$\le 2^J\left\{2^J \int_0^{\frac{1}{2^{J-1}}} (f(a_n+t))^p dt\right\}^{\frac{1}{p}}$$

$$\le 2^J\left\{2^J \sum_{j\ge J} \int_{I_{n,j}} t \, dt\right\}^{\frac{1}{p}}$$

$$\le 2^J\left\{\frac{2^J}{2^{J-1}} \sum_{j\ge J} |I_{n,j}|\right\}^{\frac{1}{p}}$$

$$= 2^J\left\{\frac{2^{m_n p+1}}{2^{m_n pJ}(2^{m_n p}-1)}\right\}^{\frac{1}{p}}$$

$$= \left(\frac{2^{m_n p+1}}{2^{m_n p}-1}\right)^{\frac{1}{p}} \frac{2^J}{2^{m_n J}}$$

$$\leq \left(\frac{2^{2p+1}}{2^{2p}-1}\right)^{\frac{1}{p}} \frac{2^J}{2^{2J}} \quad (\text{since} \quad m_n \geq 2)$$

$$= K \cdot \frac{1}{2^J} \quad \left(\text{where} \quad K = \left(\frac{2^{2p+1}}{2^{2p}-1}\right)^{\frac{1}{p}}\right)$$

$$< Kh .$$

So we have

$$\left\{ \frac{1}{h} \int_0^h |f(a_n+t)|^p dt \right\}^{\frac{1}{p}} = o(h),$$

as $h \to 0^+$. Similarly it can be shown that at each right endpoint b_n of an interval contiguous to P, the left-hand first L_p derivative $f_1^{L_p}(b_n) = 0$.

Next suppose that $x \in P$ is an accumulation point of P from the right. Let $h > 0$, and let $\mathcal{J}(h) = \{n : x < a_n, \text{ and } b_n < x + h\}$. There are two possible situations to consider here, namely $x + h \in P$ and $x + h \in G$. If $x + h \in P$, then

$$(2.9.3) \quad \frac{1}{h}\left\{\frac{1}{h} \int_0^h |f(x+t)|^p dt\right\}^{\frac{1}{p}} = \frac{1}{h}\left\{\frac{1}{h} \int_x^{x+h} (f(t))^p dt\right\}^{\frac{1}{p}}$$

$$= \frac{1}{h}\left\{\frac{1}{h} \sum_{n\in\mathcal{J}(h)} \int_{a_n}^{b_n} (f(t))^p dt\right\}^{\frac{1}{p}}$$

$$= \frac{1}{h}\left\{\frac{1}{h}\sum_{n\in\mathcal{S}(h)}\ \sum_{j=j_n}\left[\int_{I_{n,j}}(f(t))^p dt + \int_{J_{n,j}}(f(t))^p dt\right]\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{h}\left\{\frac{1}{h}\sum_{n\in\mathcal{S}(h)}\ \sum_{j=j_n}^{\infty}\left[|I_{n,j}| + |J_{n,j}|\right]\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{h}\left\{\frac{2}{h}\sum_{n\in\mathcal{S}(h)}\ \sum_{j=1}^{\infty}\frac{1}{2^{m_n pj}}\right\}^{\frac{1}{p}}$$

$$= \frac{1}{h}\left\{\frac{2}{h}\sum_{n\in\mathcal{S}(h)}\frac{1}{2^{m_n p}-1}\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{h}\left\{\frac{2}{h}\sum_{n\in\mathcal{S}(h)}(c_n-a_n)^{3p}\right\}^{\frac{1}{p}} \quad \text{(by (2.9.1))}$$

$$< \frac{1}{h}\left\{\frac{2}{h}\cdot h^{3p}\right\}^{\frac{1}{p}}$$

$$= 2^{1/p}\, h^{2-1/p}$$

$$\leq 2^{1/p} h.$$

On the other hand, if $x + h \in G$, then there is an integer $N(h) = N$ such that $x + h \in (a_N, b_N)$. So

$$(2.9.4)\quad \frac{1}{h}\left\{\frac{1}{h}\int_0^h |f(x+t)|^p dt\right\}^{\frac{1}{p}} = \frac{1}{h}\left\{\frac{1}{h}\int_x^{x+h}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{h}\left\{\frac{1}{h}\sum_{n\in\mathcal{S}(h)}\int_{a_n}^{b_n}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$+\frac{1}{h}\left\{\frac{1}{h}\int_{a_N}^{x+h}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$\leq 2^{1/p}h + \frac{1}{h}\left\{\frac{1}{h}\int_{a_N}^{x+h}(f(t))^p dt\right\}^{\frac{1}{p}},$$

according to calculation (2.9.3). We now must estimate the second term on the right hand side of the last inequality. We consider three cases:

Case 1. If $h \geq (c_N-a_N)$, then proceeding as in calculation (2.9.3), we have

$$\frac{1}{h}\left\{\frac{1}{h}\int_{a_N}^{x+h}(f(t))^p dt\right\}^{\frac{1}{p}} \leq \frac{1}{h}\left\{\frac{1}{h}\int_{a_N}^{b_N}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{h}\left\{\frac{2}{h}(c_N-a_N)^{3p}\right\}^{\frac{1}{p}}$$

$$\leq 2^{1/p}h.$$

Case 2. Assume that $h < c_N - a_N$, and that $h' < \frac{1}{2^{j_N}} + \frac{1}{2^{m_n p j_N}}$, where $h' = x + h - a_N$. Choose $J(h') = J$ as in calculation (2.9.2) to obtain

$$\frac{1}{h}\left\{\frac{1}{h}\int_{a_N}^{x+h}(f(t))^p dt\right\}^{\frac{1}{p}} \le \frac{1}{h'}\left\{\frac{1}{h'}\int_{a_N}^{a_N+h'}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$\le K\,h'$$

$$\le Kh.$$

Case 3. Again here assume that $h < c_N - a_N$, but now that $h' > \frac{1}{2^{j_N}} + \frac{1}{2^{m_N p j_N}}$. Then

$$\frac{1}{h}\left\{\frac{1}{h}\int_{a_N}^{x+h}(f(t))^p dt\right\}^{\frac{1}{p}} \le \frac{1}{h'}\left\{\frac{1}{h'}\int_{a_N}^{a_N+h'}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$= \frac{1}{h'}\left\{\frac{1}{h'}\int_{a_N}^{a_N+\frac{1}{2^{j_N-1}}}(f(t))^p dt\right.$$

$$\left. + \frac{1}{h'}\int_{a_N+\frac{1}{2^{j_N-1}}}^{a_N+h'}(f(t))^p dt\right\}^{\frac{1}{p}}$$

$$= \frac{1}{h'}\left\{\frac{1}{h'}\int_{a_N}^{a_N+\frac{1}{2^{j_N-1}}}(f(t))^p dt + 0\right\}^{\frac{1}{p}}$$

$$\le K\frac{1}{2^{j_N}} \quad \text{(as in calculation (2.9.2))}$$

$$\le Kh'$$

$$\le Kh.$$

Now if we let $h \to 0^+$, with $x + h \in G$, then

$$\frac{1}{h}\left\{\frac{1}{h}\int_0^h |f(x+t)|^p\right\}^{\frac{1}{p}} dt \leq \left[2^{1/p} + \max(2^{1/p}, K)\right] \quad h \to 0$$

So the right-hand first L_p derivative of f at x is 0. Similarly it can be shown that if x is an accumulation point of P from the left, then the left-hand first L_p derivative of f at x is 0.

Combining these results we have that $f_1^{L_p}(x)$ exists everywhere on I, and the theorem is proved.

Chapter III. A Property of k^{th} L_p and k^{th} Approximate Peano Derivatives of Measurable Functions

In this chapter we consider some further consequences of lemma 2.10. If we combine the notions of a k^{th} Peano derivative and an approximate derivative we have the following concept.

Definition 3.1. We will say that a function f has a k^{th} approximate Peano derivative at a point x_0, $k = 1,2,\ldots,$ if there exist numbers $f_1^{ap}(x_0)$, $f_2^{ap}(x_0),\ldots,f_k^{ap}(x_0)$ such that

$$\lim_{h\to 0}{}_{ap} \frac{1}{h^k}\left\{f(x_0+h) - f(x_0) - hf_1^{ap}(x_0) - \ldots - \frac{h^k}{k!} f_k^{ap}(x_0)\right\}$$
$$= 0.$$

Theorem 3.1. Let f be a measurable function possessing a k^{th} L_p derivative at a point x_0. Then $f_k^{L_p}(x_0)$ is a k^{th} approximate Peano derivative at (x_0); specifically,

$$f_k^{L_p}(x_0) = \left(f_0^{L_p}\right)_k^{ap}(x_0).$$

Proof.

Set

$$g(x) = f_0^{L_p}(x-x_0) - \left(f_0^{L_p}(x_0) + xf_1^{L_p}(x_0) + \ldots + \frac{x^k}{k!} f_k^{L_p}(x_0)\right).$$

In order to prove the theorem, it suffices to show that

$$g_k^{L_p}(0) = g_k^{ap}(0).$$

But since

$$g(0) = g_1^{L_p}(0) = \ldots = g_k^{L_p}(0) = 0,$$

it will suffice to show that

$$\lim_{h\to 0} {}_{ap} \frac{g(x+h)}{h^k} = 0,$$

and this follows immediately from lemma 2.10.

We want to show that the k^{th} approximate Peano derivative of a measurable function is of Baire class one if it exists on an interval, thus generalizing theorem 2.3. We need a couple of preliminary results.

Definition 3.2. As in [9] we define differences $\Delta_k(x,h;f)$ for a function f, $k = 1,2,\ldots$ by

$$\Delta_k(x,h;f) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh - \tfrac{1}{2} kh).$$

The following lemma is not difficult to prove using induction on k (see [1] or [9]).

Lemma 3.1. Let λ be any real number. Then

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\lambda + j - \tfrac{1}{2}k)^{i} = \begin{cases} 0, & i = 0,1,\ldots,k-1 \\ k!, & i = k. \end{cases}$$

Lemma 3.2. If f has a k^{th} approximate Peano derivative at a point x, then for any fixed number λ there is a set $F(\lambda,x)$ of density 1 at 0 such that

$$\lim_{\substack{h\to 0 \\ h\in F(\lambda,x)}} \frac{\Delta_k(x+\lambda h, h; f)}{h^k} = f_k^{ap}(x).$$

Proof. Let E be a set of density 1 at 0 such that

$$f(x+h) - \sum_{i=0}^{k} \frac{f_i^{ap}(x)}{i!} h^i = o(h^k)$$

as $h \to 0$, $h \in E$. Let λ be given, and define

$$F(\lambda,x) = \{h : \lambda h + jh - \tfrac{1}{2}kh \in E \text{ for each } j = 0,1,\ldots,k\}.$$

Then $F(\lambda,x)$ is of density one at 0. Let $\epsilon > 0$ be given. There exists a $\delta > 0$ such that if $h \in E$ and $|h| < \delta$, then

$$\left| f(x+h) - \sum_{i=0}^{k} \frac{f_i^{ap}(x)}{i!} h^i \right| < \frac{\epsilon}{k2^{k-1}} |h|^k ,$$

and furthermore if I is an interval containing 0 and $|I| < \delta$, then

$$\frac{|E \cap I|}{|I|} > 1 - \epsilon .$$

Consequently we can find a $0 < \delta' \leq \delta$, such that if $h \in F(\lambda,x)$ and $|h| < \delta'$, then

$$(3.2.1) \quad \left| f(x+\lambda h+jh-\tfrac{1}{2}kh) - \sum_{i=0}^{k} \frac{f_i^{ap}(x)}{i!}(\lambda+j-\tfrac{1}{2}k)^i h^i \right|$$

$$< \frac{\epsilon}{k2^{k-1}} |\lambda+j-\tfrac{1}{2}k|^k |h|^k$$

for each $j = 0,1,\ldots,k$, and if I is an interval containing 0 with $|I| < \delta'$, then

$$\frac{|F(\lambda,x) \cap I|}{|I|} > 1 - \epsilon .$$

If we now consider the right hand side of (3.2.1), we have

$$\frac{\epsilon}{k2^{k-1}} |\lambda+j-\tfrac{1}{2}k|^k |h|^k \leq \frac{\epsilon|h|^k}{k2^{k-1}} \sum_{n=0}^{k} \binom{k}{n} |\lambda|^n |j-\tfrac{1}{2}k|^{k-n}$$

$$< \frac{\epsilon|h|^k}{2^k} \sum_{n=0}^{k} \binom{k}{n} |\lambda|^n$$

$$< \begin{cases} \epsilon |h|^k |\lambda|^k & \text{if} \quad |\lambda| > 1 \\ \epsilon |h|^k & \text{if} \quad |\lambda| \le 1. \end{cases}$$

Let us first suppose that $|\lambda| > 1$ and that $h \in F(\lambda, x)$ with $|h| < \delta'$. Then

$$\left| \frac{\Delta_k(x+\lambda h, h; f)}{h^k} - f_k^{ap}(x) \right| = \left| \frac{1}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+\lambda h+jh - \tfrac{1}{2}kh) \right.$$

$$\left. - f_k^{ap}(x) \right|$$

$$\le \left| \frac{1}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left[f(x+\lambda h+jh - \tfrac{1}{2}kh) \right. \right.$$

$$\left. \left. - \sum_{i=0}^{k} \frac{f_i^{ap}(x)}{i!} (\lambda+j - \tfrac{1}{2}k)^i h^i \right] \right|$$

$$+ \left| \frac{1}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{i=0}^{k} \frac{f_i^{ap}(x)}{i!} (\lambda+j - \tfrac{1}{2}k)^i h^i \right.$$

$$\left. - f_k^{ap}(x) \right|$$

$$\le \sum_{j=0}^{k} \binom{k}{j} \epsilon |\lambda|^k$$

$$+ \left| \frac{1}{h^k} \sum_{i=0}^{k} \frac{f_i^{ap}(x)}{i!} h^i \sum_{j=0}^{k} (-1)^{k-j} (\lambda+j - \tfrac{1}{2}k)^i \right.$$

$$\left. - f_k^{ap}(x) \right|$$

$$\leq 2^k|\lambda|^k\varepsilon + |f_k^{ap}(x) - f_k^{ap}(x)| \quad \text{(by lemma 3.1)}$$

$$= 2^k|\lambda|^k\varepsilon .$$

In considering the case where $|\lambda| \leq 1$ we similarly obtain

$$\left| \frac{\Delta_k(x+\lambda h,h;f)}{h^k} - f_k^{ap}(x) \right| < 2^k\varepsilon ,$$

if $h \in F(\lambda,x)$ and $|h| < \delta'$.

In either case we have

$$\lim_{\substack{h\to 0\\ h\in F(\lambda,x)}} \frac{\Delta_k(x+\lambda h,h;f)}{h^k} = f_k^{ap}(x) .$$

<u>Theorem 3.2</u>. If f is measurable and has a k^{th} approximate Peano derivative at each point of an interval I, then f_k^{ap} belongs to Baire class one.

Proof. For each positive integer n and each integer p set

$$I_{n,p} = \left[\frac{p - \frac{3}{2}}{2^n} , \frac{p + \frac{3}{2}}{2^n} \right] ,$$

and

$$I_n = \left[-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right].$$

For each point $\frac{p}{2^n} \in I$, define

$$f_n\left(\frac{p}{2^n}\right) = \sup\left\{\alpha : \left|\left\{\tfrac{1}{2}kh \in I_n : \left|\left\{x \in I_{n,p} : \frac{\Delta_k(x,h;f)}{h^k}\right.\right.\right.\right.$$

$$\left.\left.\left.\left. > \alpha\right\}\right| > \tfrac{1}{2}|I_{n,p}|\right\}\right| > \tfrac{1}{2}|I_n|\right\}.$$

For each fixed n extend f_n linearly to arrive at a continuous function f_n on I.

Let $x_0 \in I$. We want to show that $f_n(x_0) \to f_k^{ap}(x_0)$. From lemma 3.2 we know that there is a set $F(0,x_0)$ of density 1 at 0 such that

$$\lim_{\substack{h\to 0 \\ h\in F(0,x_0)}} \frac{\Delta_k(x_0,h;f)}{h^k} = f_k^{ap}(x_0).$$

Set $G = \{\frac{1}{2}kh : h \in F(0,x_0)\}$. G clearly is of density 1 at 0.

Let $\varepsilon > 0$ be given. We shall find it convenient later to suppose that ε is so small that

$$1 - (k+1)\varepsilon > \frac{3}{4}$$

and

$$1 - 2\varepsilon^{\frac{1}{2k}} - \varepsilon > \frac{1}{2}.$$

Then there exists a $\delta > 0$ such that if we let

$$E_\epsilon = \left\{h : |f(x+h) - \sum_{i=0}^{k} \frac{f_i^{ap}}{i!} h^i| < \frac{\epsilon}{k2^{k-1}} |h|^k\right\},$$

$$G_\epsilon = \left\{ \frac{1}{2} k h : \left|\frac{\Delta_k(x_0, h; f)}{h^k} - f_k^{ap}(x_0)\right| < 2^k \epsilon \right\},$$

and $F_\epsilon = \left\{ \frac{1}{2} k h : j h - \frac{1}{2} k h \in E_\epsilon \text{ for each } j = 0, 1, \ldots, k\right\}$,

where we have seen that $F_\epsilon \subset G_\epsilon$ in lemma 3.2, then

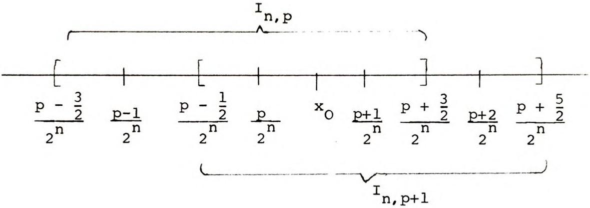
$$\frac{|F_\epsilon \cap I|}{|I|} > 1 - \epsilon, \quad \text{and} \quad \frac{|E_\epsilon \cap I|}{|I|} > 1 - \epsilon$$

for any interval I containing 0 with $|I| < \delta$.

Now choose a positive integer N so large that $\frac{1}{2^N} < \frac{\delta}{4}$. Let $n > N$, and find the unique integer p so that

$$\frac{p}{2^n} < x_0 \leq \frac{p+1}{2^n}$$

The situation is shown in the following diagram



Let $$\frac{1}{2}kh \in \left\{\left[-\frac{1}{2^{n+1}}, \frac{-\epsilon^{\frac{1}{2k}}}{2^{n+1}}\right] \cup \left[\frac{\epsilon^{\frac{1}{2k}}}{2^{n+1}}, \frac{1}{2^{n+1}}\right]\right\} \cap F_\epsilon,$$

and hold it fixed. For each $j = 0,1,\ldots,k$ we have

$$jh - \frac{1}{2}kh \in \left[-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right] \cap E_\epsilon .$$

Let $B_j = \{y - jh + \frac{1}{2}kh : y \in E_\epsilon\}$. Then for each $j = 0,1,\ldots,k$ we have

$$\frac{\left|B_j \cap \left[\frac{-3}{2^{n+1}}, \frac{1}{2^{n+1}}\right]\right|}{\frac{1}{2^{n+1}}} > 1 - \epsilon .$$

If we set $B = \bigcap_{j=0}^{k} B_j$, then

$$\frac{\left|B \cap \left[-\frac{3}{2^{n+1}}, \frac{1}{2^{n+1}}\right]\right|}{\frac{1}{2^{n-1}}} > 1 - (k+1)\varepsilon.$$

Furthermore, if $\lambda h \in B \cap \left[-\frac{3}{2^{n+1}}, \frac{1}{2^{n+1}}\right]$, then $x_0 + \lambda h \in I_{n,p}$, and $\lambda h + jh - \frac{1}{2}kh \in E_\varepsilon$ for each $j = 0,1,\ldots,k$. So performing calculations as in lemma 3.2 we have the following:

1) If $|\lambda| \leq 1$,

$$\left|\frac{\Delta_k(x_0 + \lambda h, h; f)}{h^k} - f_k^{ap}(x_0)\right| < 2^k \varepsilon$$

2) If $|\lambda| > 1$,

$$\begin{aligned}\left|\frac{\Delta_k(x_0 + \lambda h, h; f)}{h^k} - f_k^{ap}(x_0)\right| &< 2^k|\lambda|^k \varepsilon \\ &< 2^k\left(\frac{3}{2^{n+1}|h|}\right)^k \varepsilon \\ &< 2^k\left(\frac{3 \cdot 2^{n+1}k}{2^{n+2}\varepsilon^{\frac{1}{2k}}}\right)^k \varepsilon \\ &= (3k)^k \varepsilon^{1/2}\end{aligned}$$

If we set $c = (3k)^k$, then regardless of the absolute value of λ we have

$$\left|\frac{\Delta_k(x_0 + \lambda h,\ h;\ f)}{h^k} - f_k^{ap}(x_0)\right| < c\,\varepsilon^{1/2}.$$

We have shown so far that for a fixed

$$\frac{1}{k}\,k\,h \in \left\{ \left[-\frac{1}{2^{n+1}},\ \frac{-\varepsilon^{\frac{1}{2k}}}{2^{n+1}}\right] \cup \left[\frac{\varepsilon^{\frac{1}{2k}}}{2^{n+1}},\ \frac{1}{2^{n+1}}\right]\right\} \cap F_\varepsilon$$

we have

$$\left|\left\{x_0 + \lambda h \in I_{n,p} : \left|\frac{\Delta_k(x_0 + \lambda h,\ h;\ f)}{h^k} - f_k^{ap}(x_0)\right|\right.\right.$$

$$\left.\left. < c\,\varepsilon^{1/2}\right\}\right| > \frac{1 - (k+1)\,\varepsilon}{2^{n-1}}\ .$$

So

$$\left|\left\{\tfrac{1}{2}\,k\,h \in I_n : \left|\left\{ x \in I_{n,p}: \left|\frac{\Delta_k(x,h;f)}{h^k} - f_k^{ap}(x_0)\right| < c\,\varepsilon^{1/2}\right\}\right|\right.\right.$$

$$\left.\left. > \frac{1-(k+1)\,\varepsilon}{2^{n-1}}\right\}\right| > \frac{1-2\,\varepsilon^{\frac{1}{2k}} - \varepsilon}{2^n}\ .$$

In the beginning of this proof we specified that ε be so small that

$$1 - (k+1)\,\varepsilon > \frac{3}{4} \qquad \text{and} \qquad 1 - 2\,\varepsilon^{\frac{1}{2k}} - \varepsilon > \frac{1}{2}\ .$$

So we have the following:

$$\left|\left\{ \tfrac{1}{2} k h \in I_n : \left|\left\{ x \in I_{n,p} : \left|\frac{\Delta_k(x,h;f)}{h^k} - f_k^{ap}(x_0)\right|\right.\right.\right.$$

$$\left.\left.\left. < c\,\varepsilon^{1/2}\right\}\right| > \tfrac{1}{2}\,|I_{n,p}|\right\}\right| > \tfrac{1}{2}\,|I_n| .$$

This then implies that

$$f_k^{ap}(x_0) - c\,\varepsilon^{1/2} < f_n\left(\frac{p}{2^n}\right) < f_k^{ap}(x_0) + c\,\varepsilon^{1/2} .$$

In a similar manner we can find an N' such that for $n > N'$ and p such that $\frac{p}{2^n} < x_0 \leq \frac{p+1}{2^n}$ we have

$$f_k^{ap}(x_0) - c\,\varepsilon^{1/2} < f_n\left(\frac{p+1}{2^n}\right) < f_k^{ap}(x_0) + c\,\varepsilon^{1/2} .$$

We then let $N_0 = \max(N, N')$ and have that for $n > N_0$,

$$|f_n(x_0) - f_k^{ap}(x_0)| < c\,\varepsilon^{1/2} .$$

Hence $f_n(x_0) \to f_k^{ap}(x_0)$, and the theorem is proved.

<u>Corollary 3.1</u>. If f is measurable and has a k^{th} L_p derivative at each point of an interval I, then $f_k^{L_p}$ belongs to Baire class one.

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