INERTIAL SUBALGEBRAS OF ALGEBRAS SEPARABLE OVER THEIR CENTERS

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ABSTRACT

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By

Nicholas Steven Ford

Let A be a finitely generated algebra over a commutative ring R. A separable R-subalgebra Γ of A having the property that $\Gamma + N = A$, where N is the Jacobson radical of A, is called an inertial R-subalgebra of A [E. C. Ingraham, Trans. Amer. Math. Soc. 124 (1966)].

Theorem: Suppose A is a finitely generated R-algebra which is separable over its center C. If A possesses an inertial R-subalgebra Γ, then

(1) $S = \Gamma \cap C$ is an inertial R-subalgebra of C.

(2) $A \simeq \Gamma \otimes_S C$.

The inertia subgroup, denoted I_A, of a group G of R-automorphisms of an R-algebra A is defined to be

$$\{\sigma \in G \mid \sigma(a) - a \in N = \text{rad } A \ \forall\, a \in A\}.$$

Theorem: Suppose A is a finitely generated, faithful R-algebra which is separable over its center C. Assume C has no idempotents other than 0 and 1, and that A possesses a finite group G of R-automorphisms which restricts faith-

fully to C in such a way that the G-fixed subring of C is R. If $I_A = I_C$ and $|I_A|$ is a unit in R, then the existence of an inertial R-subalgebra of C implies the existence of an inertial R-subalgebra of A.

The uniqueness statement is said to hold for an R-algebra A provided any two inertial R-subalgebras B and B* of A are isomorphic via an inner automorphism of A generated by $1 - n$, where $n \in N = \text{rad } A$. We denote the Brauer group of a commutative ring R by $\mathfrak{B}(R)$.

<u>Theorem</u>: Suppose R is a semi-local ring and C is a finitely generated, commutative R-algebra with inertial R-subalgebra S. Then the natural mapping $\mathfrak{B}(S) \to \mathfrak{B}(C)$ is a monomorphism if and only if the uniqueness statement holds for every R-algebra which is central separable over C.

The generalized quaternion algebra, denoted (C,x,y), over a commutative ring C is a free C-module having basis $\{1 \in C, \alpha, \beta, \alpha\beta\}$ with multiplication induced by $\alpha^2 = x \in C$, $\beta^2 = y \in C$, and $\beta\alpha = -\alpha\beta$. We denote the n-by-n matrices over a commutative ring R by $\mathfrak{M}_n(R)$, and the localization of the integers $\mathbb{Z}$ at the maximal ideal (p) by $\mathbb{Z}_p$.

<u>Example</u>: Let $R = \mathbb{Z}_5$ and $C = \mathbb{Z}_5 \oplus 5\mathbb{Z}_5 i$, where $i^2 = -1$. Then $(C,-1,-1)$ is an R-algebra which possesses two non-isomorphic inertial R-subalgebras, namely: $\mathfrak{M}_2(R)$ and $(R,-1,-1)$.

INERTIAL SUBALGEBRAS OF ALGEBRAS SEPARABLE OVER THEIR CENTERS

By

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To My Parents.

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TABLE OF CONTENTS

INTRODUCTION

Let A be a finitely generated algebra over a commutative ring R. A separable R-subalgebra B of A having the property that $B+N = A$, where N is the Jacobson radical of A, is said to be an inertial R-subalgebra of A.

The present work deals with inertial subalgebras of those finitely generated algebras which are separable over their centers. We investigate the relationship between inertial subalgebras of these algebras and inertial subalgebras of their centers. We also exhibit conditions under which the question of isomorphism of inertial subalgebras of such algebras can be formulated as a question involving Brauer groups (Definition 3.1).

Chapter I consists of definitions and basic results which are needed in the body of the thesis.

In Chapter II we show that the existence of an inertial R-subalgebra of a finitely generated R-algebra which is separable over its center always implies the existence of an inertial R-subalgebra of the center. The main result of this chapter is a partial converse. Suppose A is a faithful R-algebra possessing a finite group G of R-automorphisms. The inertia subgroup I_A of G is defined to be

$$\{\sigma \in G \mid \sigma(a)-a \in N = \text{rad } A \quad \forall\, a \in A\}.$$

Theorem: Let A be a finitely generated R-algebra which is separable over its center C. Suppose C has no idempotents except 0 and 1, and that A possesses a finite group G of R-automorphisms which restricts faithfully to C in such a way that the G-fixed subring of C is R. If $I_A = I_C$ and $|I_A|$ is a unit in R, then the existence of an inertial R-subalgebra S of C implies the existence of an inertial R-subalgebra Γ of A. Furthermore, $A \simeq \Gamma \otimes_S C$.

One notes that this result simultaneously extends a theorem of Ingraham ([12], theorem 2.10) and a theorem of DeMeyer ([7], theorem 1.5).

In Chapter III we investigate the uniqueness (up to isomorphism) of inertial R-subalgebras of finitely generated R-algebras which are separable over their centers. We say the uniqueness statement holds for a finitely generated R-algebra A provided any two inertial R-subalgebras B and B^* of A are isomorphic via an inner automorphism of A generated by $1 - n$ where $n \in N = \text{rad } A$. We denote the Brauer group (definition 3.1) of a commutative ring R by $\mathfrak{B}(R)$.

Theorem: Suppose R is a semi-local ring and C is a finitely generated, commutative R-algebra with inertial R-subalgebra S. Then the natural mapping $\mathfrak{B}(S) \rightarrow \mathfrak{B}(C)$ is a monomorphism if and only if the uniqueness statement holds for every R-algebra whieh is central separable over C.

Chapter III is concluded with the only known example of an R-algebra which possesses non-isomorphic inertial R-subalgebras. The example will be seen to be a finitely generated R-algebra which is separable over its center. It is also endowed with a finite group of automorphisms which satisfies all the conditions mentioned in the first theorem of this introduction.

Chapter I

PRELIMINARIES

In this chapter we introduce notation, establish conventions, and present the basic results needed in the remainder of the thesis. We supply proofs only for those results whose proofs are not clear in the literature. References are given for all other results. For clarity we divide the chapter into three sections: §1 - Basic Ring Theoretic Results, §2 - Separable Algebras and Related Properties, §3 - Background and Properties of Inertial Subalgebras.

All rings are assumed to be associative and to possess an identity element 1. A commutative ring is said to be _local_ if it possesses a unique maximal ideal, and _semi-local_ if it possesses only finitely many maximal ideals. It will be our convention that ring homomorphisms map the identity element to the identity element. By an algebra A over a commutative ring R we mean a ring A together with a ring homomorphism θ from R into the center of A. Many of the algebras we have occasion to deal with are such that θ is, in fact, a monomorphism. Such algebras are said to be R-faithful. When we say an algebra is _finitely generated_ or _projective_, we mean the algebra is finitely generated or projective as a module over its ground ring R.

§1 Basic Ring Theoretic Results

Let A be a ring or an R-algebra. We will denote the Jacobson radical of A by rad A and the center of A by $Z(A)$. When there is little chance for confusion, we will use the abbreviated notation $N = \text{rad } A$, $C = Z(A)$, and $\mathfrak{n} = \text{rad } C$.

<u>Lemma 1.1</u>: (Original Nakayama Lemma) Suppose A is a ring and M is a finitely generated A-module. Then $(\text{rad } A)\cdot M = M$ if and only if $M = (0)$.

<u>Lemma 1.2</u>: (Generalized Nakayama Lemma [20]) If R is a commutative ring and M is a finitely generated R-module, then an ideal $\mathfrak{A}$ of R has the property that $\mathfrak{A}\cdot M = M$ if and only if $\text{annih}_R(M) + \mathfrak{A} = R$, where $\text{annih}_R(M) = \{r \in R \mid r\cdot M = (0)\}$.

<u>Lemma 1.3</u>: (Ingraham [12]) Let A be a finitely generated R-algebra and let $\cap(\mathfrak{m}A)$ denote the intersection of $\mathfrak{m}A$ as $\mathfrak{m}$ runs over all maximal ideals of R.

(a) $(\text{rad } R)\cdot A \subseteq N$.

(b) There exists a positive integer n such that $N^n \subseteq \cap(\mathfrak{m}A)$.

(c) If A is R-projective, then $(\text{rad } R)\cdot A = \cap(\mathfrak{m}A)$.

(d) If A is R-separable*, then $N = \cap(\mathfrak{m}A)$.

(*) See definition 1.11

<u>Proposition 1.4</u>: An R-subalgebra B of a finitely generated R-algebra A has the property that $\text{rad } B = B \cap N$.

Proof: It follows from the corollary on page 126 of |2| that $B \cap N \subseteq \text{rad } B$. Suppose $(\text{rad } B)\cdot A \not\subseteq N$. Then there exists some maximal right ideal $\mathcal{M} \subseteq A$ such that $(\text{rad } B)\cdot A + \mathcal{M} = A$. Hence $(\text{rad } B)\cdot A/\mathcal{M} = A/\mathcal{M}$, and therefore $A/\mathcal{M} = (0)$ by lemma 1.1. This is a contradiction, and so $\text{rad } B \subseteq (\text{rad } B)\cdot A \subseteq N$.

Let R be a commutative ring, and let $S \subseteq R$ be a multiplicative system. The module of quotients ([8], page 25) of an R-module M with respect to the multiplicative system S is generally denoted M_S. However, when S is the complement of a maximal ideal $\mathfrak{m}$ of R, it is customary to denote the module of quotients of an R-module M with respect to the multiplicative system $R - \mathfrak{m}$ by $M_{\mathfrak{m}}$. The module of quotients $M_{\mathfrak{m}}$ is naturally isomorphic to $M \otimes_R R_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules where $R_{\mathfrak{m}}$ is the ring of quotients with respect to the multiplicative system $R - \mathfrak{m}$. The ring of quotients $R_{\mathfrak{m}}$ is a local ring with unique maximal ideal $\mathfrak{m}\cdot R_{\mathfrak{m}}$ such that $R_{\mathfrak{m}}/\mathfrak{m}\cdot R_{\mathfrak{m}} \simeq R/\mathfrak{m}$, and we refer to $R_{\mathfrak{m}}$ as the localization of the ring R at the maximal ideal $\mathfrak{m}$. When viewed as an R-module, the module of quotients $R_{\mathfrak{m}}$ is R-flat.

Proposition 1.5: If A is a finitely generated R-algebra, then $N_{\mathfrak{m}} \subseteq \text{rad } (A_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of R.

Proof: Since $R_{\mathfrak{m}}$ is R-flat, it is clear that $N_{\mathfrak{m}} \subseteq A_{\mathfrak{m}}$. One can show that the maximal right ideals of $A_{\mathfrak{m}}$ have the form $\mathcal{M}_{\mathfrak{m}}$ for certain of the maximal right ideals $\mathcal{M}$ of A. Since each $\mathcal{M}_{\mathfrak{m}} \supseteq N_{\mathfrak{m}}$, one sees that $N_{\mathfrak{m}} \subseteq \cap\, \mathcal{M}_{\mathfrak{m}} = \text{rad } (A_{\mathfrak{m}})$.

It is well-known (see, e.g., proposition I.4.1 of [8]) that a module over a local ring is finitely generated and projective if and only if it is free of finite rank. Thus if M is a finitely generated and projective R-module, then $M_{\mathfrak{m}}$ is free of finite rank over $R_{\mathfrak{m}}$. This rank is referred to as the $\mathfrak{m}$-rank of M and is denoted $\text{rank}_{\mathfrak{m}}(M)$. If there exists an integer n such that $\text{rank}_{\mathfrak{m}}(M) = n$ for all maximal ideals $\mathfrak{m}$ of R, we say the rank of M (denoted $\text{rank}_R(M)$) is well-defined and equal to n.

<u>Definition 1.6</u>: A commutative ring R is said to be <u>connected</u> if it possesses no idempotents except 0 and 1.

<u>Theorem 1.7</u>: ([8], page 32) If R is a (commutative) connected ring, then for every finitely generated and projective R-module M the rank of M is well-defined and finite.

<u>Proposition 1.8</u>: ([4], page 141) A finitely generated and projective module of well-defined rank over a semi-local ring R is free (of finite rank).

<u>Corollary 1.9</u>: A finitely generated and projective module over a connected, semi-local ring is free of finite rank.

§2 Separable R-algebras and Related Properties

The concept of a separable R-algebra as it is presented here was first introduced by M. Auslander and O. Goldman in [1] which was published in 1960. Using this concept, the authors were able to extend the definition of the Brauer group of a field to that of any commutative ring.

Let A be an R-algebra, and denote its opposite algebra by A°. We refer to $A^e = A \underset{R}{\otimes} A^\circ$ as the enveloping algebra of A. The algebra A has the structure of a left A^e-module induced via $(a \otimes a')\cdot b = aba'$. There is a natural A^e-module map μ from A^e to A induced by $\mu(a \otimes a') = aa'$. Let us denote the kernel of μ by $\mathcal{J}$. We then have the following exact sequence of A^e-modules:

$$0 \to \mathcal{J} \to A^e \xrightarrow{\mu} A \to 0$$

<u>Proposition 1.10</u>: The following conditions on an R-algebra A are equivalent:

(a) A is projective as a left A^e-module under the above structure.

(b) $0 \to \mathcal{J} \to A^e \xrightarrow{\mu} A \to 0$ splits as a sequence of left A^e-modules.

(c) A^e contains an element ϵ (necessarily an idempotent) such that $\mathcal{J}\epsilon = 0$ and $\mu(\epsilon) = 1$.

<u>Definition 1.11</u>: An R-algebra A is called R-separable if it satisfies the equivalent conditions of prop. 1.10.

We now list those formal properties of separable R-algebras which we will have occasion to employ in the succeeding chapters. These results can also be found in Chapter II of the notes of DeMeyer and Ingraham [8].

<u>Proposition 1.12</u>: (Transitivity) Let S be a commutative, separable R-algebra and let A be a separable S-algebra. Then A is naturally an R-algebra and is R-separable. If, on the other hand, A is given to be a separable R-algebra and S is any R-subalgebra of the center of A, then A is separable over S.

<u>Proposition 1.13</u>: An R-algebra A is separable if and only if A is separable as an algebra over its center $Z(A)$ and $Z(A)$ is separable as an R-algebra.

<u>Proposition 1.14</u>: Let A be a separable R-algebra and let $\mathfrak{U}$ be a two-sided ideal of A. Then $A/\mathfrak{U}$ is a separable R-algebra. Furthermore, $Z(A/\mathfrak{U}) = [Z(A) + \mathfrak{U}]/\mathfrak{U}$.

<u>Proposition 1.15</u>: If A is a separable R-algebra and C is a commutative R-algebra, then $A \otimes_R C$ is a separable C-algebra with center $Z(A \otimes_R C) \cong Z(A) \otimes_R C$. Conversely, if C is a commutative R-algebra which possesses R as a direct summand and A is an R-algebra such that $A \otimes_R C$ is C-separable, then A is R-separable.

<u>Proposition 1.16</u>: If A is a separable R-algebra, then every A-module which is R-projective is A-projective.

A faithful R-algebra A is said to be <u>central separable over R</u> provided it is R-separable and $Z(A) = R$. In view of proposition 1.4, every separable R-algebra is central separable when considered as an algebra over its center. Central separable R-algebras are generalizations of central simple algebras over a field, and many of the structure theorems for central simple algebras have analogues in the central separable setting.

Proposition 1.17: If A is central separable over R, then A is R-projective.

Proposition 1.18: If A is central separable over R, then R is a direct summand of any R-subalgebra B of A.

Proposition 1.19: Suppose A is a central separable R-algebra. Then there is a one-one correspondence between ideals $\mathfrak{a}$ of R and two-sided ideals $\mathfrak{A}$ of A given by $\mathfrak{a} \to \mathfrak{a}A$ and $\mathfrak{A} \to \mathfrak{A} \cap R$.

Proposition 1.20: If A is a central separable R-algebra, then $N = \mathfrak{n}A$ where $N = \text{rad } A$ and $\mathfrak{n} = \text{rad } R$.

We conclude this section with two results on separability which are of a more advanced nature than those preceding.

The following theorem of S. Endo and Y. Watanabe provides an immensely useful characterization of separability over a commutative ring.

Theorem 1.21: (Endo-Watanabe [10]) The following statements concerning a finitely generated R-algebra are equivalent:

(a) A is a separable R-algebra .

(b) $A_{\mathfrak{m}}$ is a separable $R_{\mathfrak{m}}$-algebra for every maximal ideal $\mathfrak{m}$ of R.

(c) $A/\mathfrak{m}A$ is a separable $R/\mathfrak{m}$-algebra for every maximal ideal $\mathfrak{m}$ of R.

The final result of this section has been proved only recently by Dean Sanders, a fellow graduate student at Michigan State.

Theorem 1.22: (Sanders [16]) A separable R-subalgebra of a finitely generated R-algebra is itself finitely generated.

§3. Background and Properties of Inertial Subalgebras

The following theorem due to J. H. Maclagan Wedderburn appeared in a paper titled "On Hypercomplex Numbers" published in the London Mathematical Society Proceedings, Ser 2, vol. 6, Feb. 1908 - Jan. 1909.

The Original Wedderburn Principal Theorem: If A is an algebra in which every element, which has no inverse, is nilpotent, it can be expressed in the form $A = B + N$ where B is a primitive algebra and N is the maximal nilpotent invariant subalgebra.

Remark: In Wedderburn's terminology, an "algebra" is a finite dimensional algebra over a field, a "primitive algebra" is a division ring, and an "invariant subalgebra" is an ideal (so that N mentioned above is the nil radical = Jacobson radical of A).

The theorem was subsequently refined and generalized by Wedderburn and others, most notably A. Malcev who contributed a "uniqueness statement". Using Hochschild cohomology, C. W. Curtis and I. Reiner present in [5] a unified version which is usually referred to as:

The Wedderburn-Malcev Theorem: Let A be a finite dimensional algebra over a field F such that A/N is F-separable. Then A possesses a separable F-subalgebra B such that $A = B \oplus N$. Furthermore B is unique up to an inner automorphism of A generated by an element of the form $1 - n$ for some n in $N = \text{rad } A$.

The following definitions appear in a paper titled "Inertial Subalgebras of Algebras over a Commutative Ring" by E. C. Ingraham [12].

Definition 1.23: Let A be a finitely generated algebra over a commutative ring R. A separable R-subalgebra B of A such that $B + N = A$ (where $N = \text{rad } A$) is said to be an *inertial R-subalgebra of A*.

Definition 1.24: A commutative ring R is said to be an *inertial coefficient ring* (I.C.-ring) if every finitely generated R-algebra A such that A/N is R-separable possesses an inertial R-subalgebra.

Definition 1.25: The *uniqueness statement* is said to hold for a commutative ring R if for any two inertial R-subalgebras B and B' of a finitely generated R-algebra A there exists an element $n \in N$ such that $(1-n)B(1-n)^{-1} = B'$.

Remark: In view of Theorem 1.22, we see that every R-inertial subalgebra of a finitely generated R-algebra is itself finitely generated.

In the context of these definitions,Azumaya's main theorem [2] can be restated as:

Theorem: (Azumaya) A Hensel ring is an inertial coefficient ring for which the uniqueness statement holds.

We find the following property of inertial subalgebras quite useful.

Proposition 1.26: Suppose A is a finitely generated R-algebra with inertial R-subalgebra B. Then there exists a one-one correspondence between the maximal left (resp. right) ideals of A and the maximal left (resp. right) ideals of B given by $\mathcal{M} \to \mathcal{M} \cap B$ and $\mathfrak{m} \to \mathfrak{m} + N$ where $N = \operatorname{rad} A$.

Proof: Since $N = \cap \mathcal{M}$ where $\mathcal{M}$ runs over all the maximal left (resp. right) ideals of A, there exists a one-one correspondence between the maximal left (resp. right) ideals of A and those of A/N. Likewise there is a one-one correspondence between the maximal left (resp. right) ideals of B and those of $B/\operatorname{rad} B$. Now $\operatorname{rad} B = B \cap N$ by proposition 1.4, and thus $A/N = (B+N)/N \simeq B/B \cap N = B/\operatorname{rad} B$. Tracing through the mappings, we see the correspondence is as indicated.

It was in [12] that Ingraham noted that it was possible to shift the focus of attention from the coefficient ring to the algebra itself. Rather than attempting to classify inertial coefficient rings, one could ask for criteria for deciding whether or not a given algebra contains an inertial subalgebra. Since the major result in this paper has a direct bearing on the present work, we quote it in detail.

A faithful, commutative R-algebra C is said to be a (G,R)-algebra if it possesses a finite group G of R-automorphisms such that the G-fixed subring of C is R. For a given maximal ideal $\mathfrak{m}$ of C we denote

$$\{\sigma \in G \mid \sigma(c) - c \in \mathfrak{m} \ \forall\, c \in C\}$$

by $J_{\mathfrak{m}}$.

<u>Theorem 1.27</u>: (Ingraham [12]). Let C be a finitely generated, connected (G,R)-algebra. Then the following are equivalent:

(a) C contains an inertial R-subalgebra S.

(b) $C/\mathfrak{n}$ is R-separable and $J_{\mathfrak{m}} = J_{\mathfrak{m}'}$ for any two maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of C.

If (a) and (b) hold, denoting $\bigcap_{\mathfrak{m}} J_{\mathfrak{m}}$ by I, we have $S = C^I$ and S is a galois extension of R (in the sense of Chase, Harrison, and Rosenberg) with group G/I. Hence S is finitely generated, projective, and is the unique inertial R-subalgebra of C.

<u>Theorem 1.28:</u> (Ingraham [12]) Let C be a finitely generated (G,R)-algebra with R connected. Then the following are equivalent:

(a) C contains an inertial R-subalgebra S.

(b) $C/\mathfrak{n}$ is R-separable and $J_{\mathfrak{m}} = J_{\mathfrak{m}'}$ whenever $\mathfrak{m}$ and $\mathfrak{m}'$ are maximal ideals of C excluding the same primitive idempotent.

Chapter II

EXISTENCE OF INERTIAL SUBALGEBRAS OF ALGEBRAS SEPARABLE OVER THEIR CENTERS

Let A be a finitely generated R-algebra which is separable over its center C. In this chapter we investigate the relationship between inertial R-subalgebras of A and inertial R-subalgebras of C. We show that the existence of an inertial R-subalgebra of A always implies the existence of an inertial R-subalgebra of C. The main result of this chapter gives circumstances under which the converse is true. Viewed with respect to the main result, the chapter can be roughly subdivided as follows: preliminaries and necessary conditions, discussion of the setting, proof of the conjecture, applications, and examples.

<u>Proposition 2.1</u>: Suppose A is a finitely generated R-algebra which is separable over its center C. If A possesses an inertial R-subalgebra B, then $A = BC$. Moreover, $A \simeq B \otimes_S C$ under the correspondence $\sum b_i \otimes c_i \longleftrightarrow \sum b_i\, c_i$ where $S = Z(B)$.

<u>Proof</u>: In view of proposition 1.20 we have $N = \mathfrak{n}A$ where $N = \text{rad } A$ and $\mathfrak{n} = \text{rad } C$. Then

$$\mathfrak{n}\cdot(A/BC) = (\mathfrak{n}A + BC)/BC = (N + BC)/BC = A/BC,$$

since $B + N = A$ by hypothesis. Therefore $A = BC$ by the Original Nakayama Lemma (lemma 1.1).

Now since $A = BC$ it is clear that $S = Z(B) \subseteq Z(A) = C$. Thus it makes sense to speak of $B \underset{S}{\otimes} C$.

Let $\mu: B \underset{S}{\otimes} C \to A$ be the natural C-module mapping induced by $\mu(b \otimes c) = bc$. Now $\mu[(b \otimes c)(b' \otimes c')] = \mu(bb' \otimes cc') = bb'cc' = bc\ b'\ c' = \mu(b \otimes c)\mu(b' \otimes c')$, since C is the center of A. Hence μ is a C-algebra homomorphism.

It is clear that μ maps onto A since $\mu(B \underset{S}{\otimes} C) = BC = A$. Thus to prove μ is an isomorphism we need only show it is one-one. Since $\ker \mu$ is a two-sided ideal of $B \underset{S}{\otimes} C$, there exists an ideal $\mathfrak{A} \subseteq C$ such that $\mathfrak{A}(B \underset{S}{\otimes} C) = \ker \mu$ (proposition 1.19). Let $\alpha \in \mathfrak{A} = C \cap \ker \mu$. Then

$$0 = \mu(\alpha) = \mu(\alpha \cdot 1) = \alpha\mu(1) = \alpha \cdot 1 = \alpha.$$

Thus $\mathfrak{A} = (0)$, and therefore $\ker \mu = (0)$. This proves μ is one-one, and the assertion now follows.

<u>Theorem 2.2</u>: Suppose A is a finitely generated R-algebra which is separable over its center C. If A contains an inertial R-subalgebra B, then C possesses a unique inertial R-subalgebra $Z(B) = B \cap C$.

<u>Proof</u>: It follows from proposition 2.1 that C includes $Z(B)$. Therefore $B \cap C \subseteq Z(B) \subseteq B \cap C$, or $B \cap C = Z(B)$.

Being an inertial R-subalgebra of A, B is separable over R. Therefore $Z(B)$ is R-separable by proposition 1.13. Thus we have only to show that $Z(B) + \mathfrak{n} = C$, where $\mathfrak{n} = \operatorname{rad} C$.

Now $\mathfrak{n} = C \cap N$ by proposition 1.4. Moreover, $Z(A/N) = [C + N]/N$ by proposition 1.14 since A is C-separable. Likewise $Z(B/B \cap N) = [Z(B) + B \cap N]/B \cap N$ since B is R-separable. With these facts in mind the following chain of isomorphisms is clear:

$$[Z(B) + \mathfrak{n}]/\mathfrak{n} \simeq Z(B)/Z(B) \cap \mathfrak{n} = Z(B)/Z(B) \cap B \cap N \simeq$$

$$[Z(B) + B \cap N]/B \cap N = Z(B/B \cap N) \simeq Z([B + N]/N) = Z(A/N)$$

$$= [C + N]/N \simeq C/C \cap N = C/\mathfrak{n}.$$

The composite of these isomorphisms is the identity map, whence $Z(B) + \mathfrak{n} = C$. Therefore $Z(B)$ is an inertial R-subalgebra of B.

Uniqueness follows easily. Since C is a direct summand of A (proposition 1.18) and so is finitely generated over R, inertial R-subalgebras of C will also be finitely generated over R (Theorem 1.22). That any two finitely generated inertial subalgebras of a finitely generated and commutative algebra coincide is a consequence of proposition 2.6 of [12].

Q.E.D.

Let us denote the n-by-n matrices over a ring R by $\mathcal{M}_n(R)$.

<u>Proposition 2.3</u>: Let C be a finitely generated, commutative R-algebra. If C contains an inertial R-subalgebra S, then $\mathcal{M}_n(S)$ is an inertial R-subalgebra of $\mathcal{M}_n(C)$ for each integer n.

Proof: When we refer to $\mathcal{M}_n(S)$ as a subalgebra of $\mathcal{M}_n(C)$, we actually mean that subalgebra $\mathcal{S} \simeq \mathcal{M}_n(S)$ of $\mathcal{M}_n(C)$ defined by $\mathcal{S} = \{(c_{ij}) | c_{ij} \in S\}$. It is well-known that $\mathcal{M}_n(S)$ is separable over S with separability idempotent

$$e = \sum_{i=1}^{n} e_{ir} \otimes e_{ri}$$

for any $r \leq n$. Thus $\mathcal{S}$ is separable over S and hence separable over R by transitivity (proposition 1.12). That rad $\mathcal{M}_n(C) = \{(c_{ij}) | c_{ij} \in \mathfrak{n} = \text{rad } C\}$ is theorem 6.15 of [15]. It is therefore clear that $\mathcal{S}$ + rad $\mathcal{M}_n(C) = \mathcal{M}_n(C)$. Thus $\mathcal{M}_n(S)$ is an inertial R-subalgebra of $\mathcal{M}_n(C)$.

Let us suppose that C is a finitely generated, commutative R-algebra possessing a finite group G of R-automorphisms of C. Then G can be extended to a group $\overline{G}$ of R-automorphisms of $\mathcal{M}_n(C)$ is an obvious way. If $\{e_{ij}\}$ is the set of n^2 matrix units of $\mathcal{M}_n(C)$, we define $\overline{\sigma} \in \overline{G}$ by $\overline{\sigma}\ (\sum c_{ij}\ e_{ij}) = \sum \sigma(c_{ij})e_{ij}$. That $\overline{\sigma}$ is indeed an R-automorphism of $\mathcal{M}_n(C)$ follows from the fact that the multiplication constants of the matrix units are in the G-fixed subring of C.

Definition 2.4: A finitely generated, faithful R-algebra A possessing a finite group G of R-automorphisms with G-fixed subring $A^G = \Lambda$ is said to be a (G,Λ,R)-algebra. If A is a commutative (G,R,R)-algebra, we refer to it as a (G,R)-algebra (see [12]).

Let A be a (G,Λ,R)-algebra. We denote by I_A the normal subgroup of G defined by $\{\sigma \in G | \sigma(a)-a \in N\ \forall\ a \in A\}$.

When there is no confusion as to which (G,Λ,R)-algebra is under discussion we will use the abbreviated notation I for I_A.

We now propose to abstract the setting of proposition 2.3. Let A be a finitely generated R-algebra which is separable over its center C. Suppose further that C is a connected (G,R)-algebra. Let us assume for the moment that A contains an inertial R-subalgebra B. Then $A \simeq B \underset{S}{\otimes} C$ where $S = Z(B)$ by proportion 2.2. It follows from theorem 1.27 that S can be characterized as being the I_C-fixed subring of C, i.e. $S = C^{I_C}$. Hence I_C can be extended to a group $\overline{I} \simeq I_C$ of S-automorphisms of A by defining $\overline{\sigma} \in \overline{I}$ as $\overline{\sigma}(b \otimes c) = b \otimes \sigma(c)$, where $\sigma \in I_C$. Let

$$I_A = \{\overline{\sigma} \in \overline{I} \mid \overline{\sigma}(a) - a \in N \ \forall \ a \in A\}.$$

For $\sigma \in I_C$ we see that

$$\begin{aligned}\overline{\sigma}(a) - a &= \overline{\sigma}\ (\sum_{j=1}^{r} b_j \otimes c_j) - \sum_{j=1}^{r} b_j \otimes c_j = \\ &= \sum_{j=1}^{r} b_j \otimes (\sigma(c_j) - c_j) \in nA = N.\end{aligned}$$

Therefore $\overline{\sigma} \in I_A$, implying that $I_C = \overline{I} = I_A$.

We are thus motivated to consider the following setting which we hereafter refer to as (*):

(*) A is a finitely generated R-algebra which is separable over its center C. Further, C is a (G,R)-algebra with G extendable to A in such a way that $I_A = J_C$. We denote this common subgroup of G by I. The situation is indicated by the following diagram:

$$\begin{tikzcd}
& \Gamma \arrow[d, equal] & \Lambda \arrow[d, equal] \\
A \arrow[r, phantom, "\supseteq"] \arrow[d, dash] & A^I \arrow[r, phantom, "\supseteq"] \arrow[d, dash] & A^G \arrow[d, dash] \\
C \arrow[r, phantom, "\supseteq"] & C^I \arrow[r, phantom, "\supseteq"] \arrow[d, equal] & C^G \arrow[d, equal] \\
& S & R
\end{tikzcd}$$

Remark 1: In the terminology of definition 2.4, A is seen to be a (G,Λ,R)-algebra.

Remark 2: If C is connected and possesses an inertial R-subalgebra T, then $T = S$ by theorem 1.27

Proposition 2.5: Let A be an R-algebra in the setting (*). Suppose C is connected and contains an inertial R-subalgebra S. Then $\Gamma \simeq \Lambda \underset{R}{\otimes} S$. Moreover, Γ is R-separable if and only if Λ is R-separable. When this is the case, $Z(\Gamma) = S$ if and only if $Z(\Lambda) = R$.

Proof: The technique employed is due to DeMeyer in [7], and this result is a sharper version of his Theorem 3. We first need to state a definition and to quote a short lemma.

Definition: A (not necessarily commutative) (G,Λ,R)-algebra A is said to be a galois extension of Λ provided there exists $\{x_i,y_i\} \subseteq A$, $1 \le i \le n$ such that $\sum_{i=1}^{n} x_i\sigma(y_i) = \delta_{1,\sigma}$ $\forall\ \sigma \in G$ where

$$\delta_{1,\sigma} = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1 \end{cases} .$$

<u>Lemma</u>: (DeMeyer [7]) Let A and B be R-algebras with common subalgebra C. Suppose G is a group of R-automorphisms of A and of B in such a way that both A and B are galois extensions of C. If $f: A \to B$ is a ring homomorphism which commutes with G and fixes C, then f is an isomorphism.

In the setting of proposition 2.5, there exists a natural ring homomorphism $h: \Lambda \otimes_R S \to \Gamma$ induced by $h(\lambda \otimes s) = \lambda s$. Now S is galois over R with group G/I by theorem 1.27, from which it follows that Γ is galois over Λ. Since R is a direct summand of S by corollary III.1.3 (2) of [8], $\Lambda \subseteq \Lambda \otimes_R S$. Let G/I act on $\Lambda \otimes_R S$ by $\bar{\sigma}(\lambda \otimes s) = \lambda \otimes \bar{\sigma}(s)$. Then $(\Lambda \otimes_R S)^{G/I} = \Lambda$ by corollary III.1.3 (3) of [8]. Now since it fixes Λ and commutes with G/I, h is an isomorphism by the above lemma. Therefore $\Gamma \simeq \Lambda \otimes_R S$.

Since R is a direct summand of S, we conclude from proposition 1.15 that Γ is separable over R if and only if Λ is separable over R.

Now if Λ is central separable over R, then Γ is central separable over S by proposition 1.15. Conversely, assume $S = Z(\Gamma)$. If $x \in Z(\Lambda)$, then $x \in Z(\Gamma) = S$ since $\Gamma = \Lambda S$. Therefore $x \in \Lambda \cap S = C^G = R$.

Q.E.D.

We will find the so-called "trace map" provides a convenient characterization of inertial subalgebras of algebras satisfying (*).

Definition 2.6: Let G be a finite group of automorphisms of an R-algebra A. The associated trace map τ_G of G, $\tau_G\colon A \to \Lambda = A^G$, is defined by $\tau_G(a) = \sum_{\sigma\in G} \sigma(a)$ $\forall\, a \in A$.

Remark: $\tau_G(A)$ is a two-sided ideal of $\Lambda = \Lambda^G$. Thus $\tau_G\colon A \to \Lambda$ is onto if and only if $1 \in \tau_G(A)$.

Proposition 2.7: Suppose A is an R-algebra satisfying (*). Then $\tau_I(A) = \Gamma$ if and only if $r = |I|$ is a unit in R.

Proof: Suppose $r = |I|$ is a unit in R. Then

$$\tau_I(\tfrac{1}{r}) = \sum_{\sigma\in I} \sigma(\tfrac{1}{r}) = \sum_{\sigma\in I} \frac{1}{r}\,\sigma(1) = \frac{1}{r}\sum_{\sigma\in I} 1 = \frac{1}{r}\,|I| = 1.$$

Thus $\tau_I(A) = A^I = \Gamma$.

Conversely, let $\tau(A) = \Gamma$ where we denote τ_I by τ. We know for $\sigma \in I$ that $\sigma(a) - a \in N = \operatorname{rad} A$ $\forall\, a \in A$. Thus $\tau(a) - ra \in N$ $\forall\, a \in A$. If r is not a unit, it lies in some maximal ideal $\mathfrak{m}$ of R. Since $\tau(A) = \Gamma$, there exists $x \in A$ such that $\tau(x) = 1$. Then $1 - rx = \tau(x) - rx \in N$, and so $1 \in N + \mathfrak{m}A$. This implies $\mathfrak{m}(A/N) = A/N$. By the Generalized Nakayama Lemma (lemma 1.2), this implies that $\mathfrak{m} + \operatorname{annih}_R(A/N) = R$. Since A is R-faithful, $\operatorname{annih}_R(A/N) = R \cap N$; and since A is a finitely generated R-algebra, $R \cap N = \operatorname{rad} R$ by proposition 1.4. But $\mathfrak{m} + \operatorname{rad} R = R$ is a contradiction, since $\operatorname{rad} R \subseteq \mathfrak{m}'$ for all maximal ideals $\mathfrak{m}'$ of R. Hence $r \notin \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R, and therefore is a unit in R.

The following proposition and its corollary illustrate the usefulness of the trace map.

<u>Proposition 2.8</u>: Suppose A is an R-algebra satisfying (*). If $r = |I|$ is a unit, then $\Gamma + N = A$.

<u>Proof</u>: By definition of I, $\sigma(a) - a \in N\ \forall\ a \in A$ and every $\sigma \in I$. Denoting τ_I by τ, we have

$$\tau(a) - ra \in N\ \forall\ a \in A.$$

Hence $\frac{1}{r}\tau(a) - a \in N$. But $\frac{1}{r}\tau(a) = \tau(\frac{1}{r}a)$. Thus

$$a - \tau(\tfrac{1}{r}a) \in N\ \forall\ a \in A,$$

or $\tau(A) + N = A$. Since $\tau(A) \subseteq \Gamma$ in general, this says $\Gamma + N = A$.

<u>Corollary 2.9</u>: Suppose A is an R-algebra satisfying (*). If $|I|$ is a unit, then $A = C\Gamma$.

<u>Proof</u>: Let $\mathfrak{n} = \text{rad } C$. Then $\mathfrak{n}\cdot(A/C\Gamma) =[C\Gamma + \mathfrak{n}A]/C\Gamma$. Since A is separable over C, $\mathfrak{n}A = N$ by proposition 1.20. Thus $[C\Gamma + \mathfrak{n}A]/C\Gamma = A/C\Gamma$ by proposition 2.8, and therefore $C\Gamma = A$ by the Original Nakayama Lemma (lemma 1.1)

<u>Proposition 2.10</u>: Let C be a commutative ring with $\text{rad } C = \mathfrak{n}$. Suppose M is a finitely generated and projective C-module. If $M/\mathfrak{n}M$ is free over $C/\mathfrak{n}$ with basis $\{x_i + \mathfrak{n}M\}_{i=1}^{n}$, then $\{x_i\}_{i=1}^{n}$ is a free basis for M over C.

<u>Proof</u>: (see [10], page 43). Let us denote the free C-module of rank n by $C^{(n)}$. There exists a natural C-module homomorphism φ: $C^{(n)} \to M$ defined by

$$\varphi[(c_1,\cdots,c_n)] = \sum_{i=1}^{n} c_i x_i.$$

Since $\{x_i + \mathfrak{n}M\}_{i=1}^{n}$ span $M/\mathfrak{n}M$, we have $\varphi(C^{(n)}) + \mathfrak{n}M = M$ or $\mathfrak{n}\cdot M/\varphi(C^{(n)}) = M/\varphi(C^{(n)})$. Hence $M = \varphi(C^{(n)})$ by Nakayama's Lemma.

Thus the sequence $0 \to \ker \varphi \to C^{(n)} \xrightarrow{\varphi} M \to 0$ is exact. Since M is C-projective the sequence splits, and $C^{(n)} = \ker \varphi \oplus L$ for some submodule L of $C^{(n)}$. Let $(c_1,\cdots,c_n) \in \ker \varphi$. Then $\sum_{i=1}^{n} c_i x_i = 0$ so that $\sum_{i=1}^{n} c_i x_i + \mathfrak{n}M = \overline{0}$, whence the coefficients in $C/\mathfrak{n}$ determined by the c_i are zero (i.e. $c_i \in \mathfrak{n}\ \forall\ i \le n$). Therefore $\ker \varphi \subseteq \mathfrak{n}M = \mathfrak{n} \ker \varphi \oplus \mathfrak{n} L$, whence $\ker \varphi = \mathfrak{n} \ker \varphi$. Since $\ker \varphi$ is a direct summand of M it is finitely generated over C. Thus $\ker \varphi = (0)$ by Nakayama's Lemma.

Corollary 2.11: Suppose A is a finitely generated R-algebra in the setting (*), and that $r = |I|$ is a unit in R. If C is (i) a principal ideal domain or (ii) semi-local and connected, then A possesses an I-invariant basis.

Proof: Since $r = |I|$ is a unit, $\Gamma + N = A$ by proposition 2.8. As we have previously noted, A being separable over C implies that $N = \mathfrak{n}A$. Therefore any generating set (resp. basis) for A over C is congruent modulo $\mathfrak{n}A$ to an I-invariant generating set (resp. basis) for A over C. Thus we need only show that $A/\mathfrak{n}A$ possesses a finite basis over $C/\mathfrak{n}$. This will follow from that fact that under either condition (i) or condition (ii) A possesses a finite basis over C.

(i) Suppose C is a principal ideal domain. Then it is well-known that any finitely generated C-module can be represented as a direct sum of a free C-module of finite rank and its torsion submodule: $A \simeq C^{(n)} \oplus t(A)$. However, since A is projective over C, $t(A) = (0)$.

(ii) Suppose C is connected and semi-local. It follows from corollary 1.9 that A is free of finite rank over C.

We now prove a preliminary version of our main result.

<u>Lemma 2.12</u>: Suppose R is a semi-local ring and A is a finitely generated R-algebra in the setting (*). Assume that $r = |I|$ is a unit in R. If C contains an inertial R-subalgebra S and $C/\mathfrak{n}$ is R-projective, then Γ is an inertial R-subalgebra of A and $\Gamma \simeq A/N$.

<u>Proof</u>: Since $r = |I|$ is a unit, $\Gamma + N = A$ by proposition 2.8. C is finitely generated over R since it is a direct summand of A (proposition 1.18), and thus is seen to be semi-local. Therefore A possesses an I-invariant basis $\{x_i\}_{i=1}^{n}$ over C by corollary 2.11. This in turn implies $\Gamma \cap N = (0)$. To see this we suppose $\gamma \in \Gamma \cap N$. Then, since $N = \mathfrak{n}A$, γ has a representation as $\gamma = \sum_{i=1}^{n} \mu_i x_i$ where $\mu_i \in \mathfrak{n}$. Moreover, since $\gamma \in \Gamma$, $\sigma(\gamma) = \gamma \ \forall\ \sigma \in I$. Thus $\sum_{i=1}^{n} \sigma(\mu_i)x_i = \sum_{i=1}^{n} \mu_i x_i$. Hence $\mu_i = \sigma(\mu_i) \ \forall\ \sigma \in I$. Therefore $\mu_i \in \mathfrak{n} \cap C^I = \mathfrak{n} \cap S$. However, $\mathfrak{n} \cap S = (0)$ by Lemma 2.3 [12], since $C/\mathfrak{n}$ is R-projective. Thus $\mu_i = 0 \ \forall_i$, whence $\Gamma \cap N = (0)$.

Now $C/\mathfrak{n}$ is necessarily R-separable since it is a homomorphic image of the separable R-algebra S:

$$C/\mathfrak{n} = (S+\mathfrak{n})/\mathfrak{n} \simeq S/S \cap \mathfrak{n}.$$

That A/N is R-separable follows from the transitivity of separability (proposition 1.12) since $A/N = A/\mathfrak{n}A$ is central-separable over $C/\mathfrak{n}$. Hence

$$\Gamma \simeq \Gamma/(0) = \Gamma/\Gamma \cap N \simeq [\Gamma + N]/N = A/N$$

is R-separable and so is an inertial R-subalgebra of A.

We note that in this instance we have a "classical" Wedderburn decomposition of A in the sense that $A = \Gamma \oplus N$, just as in the case where A is an algebra over a field.

We are now in a position to prove the main result of this chapter.

<u>Theorem 2.13</u>: Let A be a finitely generated (G,Λ,R)-algebra in the (*) setting. Suppose A is separable over its center C which is assumed to be connected. If $r = |I|$ is a unit and C contains an inertial R-subalgebra S, then $\Gamma = A^I$ is an inertial R-subalgebra of A. Furthermore $A \simeq \Gamma \underset{S}{\otimes} C \simeq \Lambda \underset{R}{\otimes} C$.

<u>Proof</u>: $A = \Gamma + N$ by proposition 2.8, since $r = |I|$ is a unit. However, to show Γ is R-separable is more challenging.

<u>Idea of the proof</u>: Let m be a maximal ideal of S. We will show $A/\mathfrak{m}A$ is an $S/\mathfrak{m}$-algebra which fulfills the hypotheses of the previous lemma, and so possesses an inertial $S/\mathfrak{m}$-subalgebra. We will further demonstrate that this inertial $S/\mathfrak{m}$-subalgebra is in fact $\Gamma/\mathfrak{m}\Gamma$. By the theorem of Endo-Watanabe (theorem 1.21), we conclude Γ is separable over S. Since S is separable over R, Γ is separable over R by transitivity (proposition 1.12).

As a first step we show that $A/\mathfrak{m}A$ is an $S/\mathfrak{m}$-algebra satisfying the setting (*).

(1) $A/\mathfrak{m}A$ is a central separable $C/\mathfrak{m}C$-algebra.

Since A is separable over C, $A/\mathfrak{m}A$ is separable over $Z(A/\mathfrak{m}A) \simeq [C + \mathfrak{m}A]/\mathfrak{m}A \simeq C/C \cap \mathfrak{m}A$ by proposition 1.14. Now $\mathfrak{m}C$ is a two-sided ideal of C, and so $C \cap \mathfrak{m}A = C \cap (\mathfrak{m}C)A = \mathfrak{m}C$ by proposition 1.19.

(2) $A/\mathfrak{m}A$ possesses a group $\overline{I}$ of $S/\mathfrak{m}$-automorphisms which restricts faithfully to $C/\mathfrak{m}C$ (i.e., the restriction of $\overline{I}$ to $C/\mathfrak{m}C$ is a group of $S/\mathfrak{m}$-automorphisms of $C/\mathfrak{m}C$ which is isomorphic to $\overline{I}$).

Since $\mathfrak{m} \subseteq S \subseteq C^I$, it follows that $\sigma(\mathfrak{m}A) = \mathfrak{m}A$ for every $\sigma \in I$. Thus each $\sigma \in I$ induces a well-defined mapping on $A/\mathfrak{m}A$ by $\sigma(a + \mathfrak{m}A) = \sigma(a) + \mathfrak{m}A$. It is clear that this mapping is an $S/\mathfrak{m}$-algebra automorphism. Let

$$J_{\mathfrak{m}A} = \{\sigma \in I \mid \sigma(a) - a \in \mathfrak{m}A \ \forall a \in A\}.$$

One easily sees that $J_{\mathfrak{m}A} \trianglelefteq I$ and that $I/J_{\mathfrak{m}A}$ acts faithfully on $A/\mathfrak{m}A$.

Now by hypothesis I also acts as a group of S-automorphisms on C. Therefore $I/J_{\mathfrak{m}C}$ acts as a faithful group of $S/\mathfrak{m}$-automorphisms on $C/\mathfrak{m}C$ where

$$J_{\mathfrak{m}C} = \{\sigma \in I \mid \sigma(c) - c \in \mathfrak{m}C \ \forall\ c \in C\}.$$

We now assert that $J_{\mathfrak{m}C} = J_{\mathfrak{m}A}$, one consequence of which is that $I/J_{\mathfrak{m}A}$ acts on $A/\mathfrak{m}A$ and restricts faithfully to $C/\mathfrak{m}C$. Since $\mathfrak{m}C = C \cap \mathfrak{m}A$, it is immediate that $J_{\mathfrak{m}A} \subseteq J_{\mathfrak{m}C}$. Let $\sigma \in J_{\mathfrak{m}C}$. By corollary 2.9 each $a \in A$ has a representation as $a = \sum_{i=1}^{n} c_i x_i$ where $c_i \in C$ and $x_i \in \Gamma$. Thus

$$\sigma(a)-a = \sigma(\sum_{i=1}^{n} c_i x_i) - \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} (\sigma(c_i)-c_i)x_i \in mC\Gamma = mA,$$

showing $\sigma \in J_{mA}$, from which we conclude $J_{mA} = J_{mC}$. We denote $I/J_{mA} = I/J_{mC}$ by $\bar{I}$.

(3) $(A/mA)^{\bar{I}} = [\Gamma+mA]/mA$ and $(C/mC)^{\bar{I}} = [S+mC]/mC$.

Clearly $[\Gamma+mA]/mA \subseteq (A/mA)^{\bar{I}}$. To prove the opposite inclusion, let $\bar{a} \in (A/mA)^{\bar{I}}$. Since $|\bar{I}|$ is a unit, the argument given in proposition 2.7 shows $\exists\ \bar{x} \in A/mA$ such that

$$\bar{a} = \tau_{\bar{I}}(\bar{x}) = \sum_{\bar{\sigma}\in\bar{I}} \bar{\sigma}(\bar{x}).$$

Now let $\{\sigma_i\}_{i=1}^{p}$ be a full set of coset representatives for J_{mA} in I. (i.e. $I = (\text{disjoint})\ J_{mA} \cup J_{mA}\sigma_2 \cup \cdots \cup J_{mA}\sigma_p$). Thus $\bar{a} = \sum_{i=1}^{p} \sigma_i(x) + mA$. Let $n = |J_{mA}|$ and $b = \sum_{i=1}^{p} \sigma_i(x)$ so that $\bar{b} = \bar{a}$. Claim $nb - \tau_I(x) \in mA$. For

$$nb - \tau_I(x) = n\sum_{i=1}^{p} \sigma_i(x) - \sum_{\sigma\in I} \sigma(x) = n\sum_{i=1}^{p} \sigma_i(x) -$$

$$- \sum_{\xi\in J_{mA}} (\sum_{i=1}^{p} \xi\sigma_i(x)) = n\sum_{i=1}^{p} \sigma_i(x) - \sum_{i=1}^{p} (\sum_{\xi\in J_{mA}} \xi\sigma_i(x)) =$$

$$= \sum_{\xi\in J_{mA}} [\sum_{i=1}^{p} (\sigma_i(x) - \xi\sigma_i(x))] \in mA$$

since

$$\sigma_i(x) - \xi\sigma_i(x) \in mA$$

by the definition of J_{mA}. Thus $n\bar{a} = \overline{\tau_I(x)}$, or

$$\bar{a} = \frac{1}{n}\overline{\tau_I(x)} = \overline{\tau_I(\frac{1}{n}x)} \in [\Gamma+mA]/mA.$$

An identical argument shows $(C/\mathfrak{m}C)^{\overline{I}} = [S+\mathfrak{m}C]/\mathfrak{m}C$.

(4) $C/\mathfrak{m}C$ is a connected $(\overline{I}, S/\mathfrak{m})$-algebra.

That $C/\mathfrak{m}C$ is connected follows from proposition 1.26 which states that there is a one-one correspondence between maximal ideals $\mathfrak{m} \subseteq S$ and maximal ideals $\mathcal{M} \subseteq C$. Since this correspondence is given by $\mathfrak{m} \to \mathfrak{m} + \mathfrak{n} = \mathcal{M} \to \mathcal{M} \cap S = \mathfrak{m}$ where $\mathfrak{n} = \text{rad } C$, it follows that there is exactly one maximal ideal lying over $\mathfrak{m}C$: namely $\mathfrak{m} + \mathfrak{n}$. Thus $C/\mathfrak{m}C$ is local, and hence connected. Now $(C/\mathfrak{m}C)^{\overline{I}} = [S+\mathfrak{m}C]/\mathfrak{m}C \simeq S/S \cap \mathfrak{m}C$ by (3), and we assert that $S \cap \mathfrak{m}C = \mathfrak{m}$. Since $S \cap \mathfrak{m}C$ is a two-sided ideal of S which includes $\mathfrak{m}$, it must equal either $\mathfrak{m}$ or S. Now if $S \cap \mathfrak{m}C = S$, then $1 \in \mathfrak{m}C$ so that $\mathfrak{m}C = C$. This implies that $S = \mathfrak{m} + \text{annih}_S(C)$ by the Generalized Nakayama Lemma. However, C is faithful over S and this leads to the contradiction that $\mathfrak{m} = S$. Therefore $S \cap \mathfrak{m}C = \mathfrak{m}$, and so $(C/\mathfrak{m}C)^{\overline{I}} = S/\mathfrak{m}$.

(5) $S/\mathfrak{m}$ is an $S/\mathfrak{m}$-inertial subalgebra of $C/\mathfrak{m}C$. Rad $(C/\mathfrak{m}C) = \cap\, \mathcal{M}/\mathfrak{m}C$ where the intersection is taken over all maximal ideals $\mathcal{M} \supseteq \mathfrak{m}C$. Clearly then, $\text{rad}(C/\mathfrak{m}C) \supseteq (\mathfrak{n}+\mathfrak{m}C)/\mathfrak{m}C$. Moreover, $C \xrightarrow{\eta} C/\mathfrak{m}C$ is an epimorphism and so $(\mathfrak{n}+\mathfrak{m}C)/\mathfrak{m}C \subseteq \text{rad}(C/\mathfrak{m}C)$. Hence

$$S/\mathfrak{m} + \text{rad}(C/\mathfrak{m}C) = (S+\mathfrak{m}C)/\mathfrak{m}C + (\mathfrak{n}+\mathfrak{m}C)/\mathfrak{m}C = (S+\mathfrak{n}+\mathfrak{m}C)/\mathfrak{m}C = C/\mathfrak{m}C,$$

proving the assertion.

(6) $A/\mathfrak{m}A$ is a finitely generated $S/\mathfrak{m}$-algebra in the setting (*). An argument identical to (5) shows $\text{rad}(A/\mathfrak{m}A) = (N+\mathfrak{m}A)/\mathfrak{m}A$. Hence $I_{A/\mathfrak{m}A} = \{\overline{\sigma} \in \overline{I} \mid \overline{\sigma}(\overline{a}) - \overline{a} \in \text{rad}(A/\mathfrak{m}A)\} = \overline{I}$.

Similarly, $I_{C/\mathfrak{m}C} = \bar{I}$, whence $I_{A/\mathfrak{m}A} = I_{C/\mathfrak{m}C} = \bar{I}$. In light of what we have previously shown, $A/\mathfrak{m}A$ is in the setting (*).

$$\begin{tikzcd}
A/\mathfrak{m}A \arrow[r, phantom, "\supseteq"] \arrow[d, dash] & (A/\mathfrak{m}A)^{\bar{I}} \arrow[d, dash] & \\
C/\mathfrak{m}C \arrow[r, phantom, "\supseteq"] & (C/\mathfrak{m}C)^{\bar{I}} \arrow[r, phantom, "="] & S/\mathfrak{m}
\end{tikzcd}$$

(7) $[\Gamma+\mathfrak{m}A]/\mathfrak{m}A \simeq \Gamma/\Gamma \cap \mathfrak{m}A$ is an $S/\mathfrak{m}$-inertial subalgebra of $A/\mathfrak{m}A$. $S/\mathfrak{m}$ is a field and therefore semi-local. Trivially, $(C/\mathfrak{m}C)/\mathrm{rad}(C/\mathfrak{m}C)$ is $S/\mathfrak{m}$-projective since all modules over a field are free. Therefore the assertion follows from Lemma 2.12 and (3).

(8) $\Gamma \cap \mathfrak{n}A = (\mathrm{rad}\ S)\Gamma$

Since $A = C\Gamma$ and $\mathfrak{n} = \mathrm{rad}\ C$, it follows that $\mathfrak{n}A = \mathfrak{n}\Gamma$. Therefore each $x \in \Gamma \cap \mathfrak{n}A$ can be represented as $x = \sum_{j=1}^{q} n_j x_j$ where $n_j \in \mathfrak{n}$ and $x_j \in \Gamma$. Thus

$$\tau_I(x) = \sum_{\sigma\in I} \sigma(x) = \sum_{\sigma\in I} \sigma(\sum_{j=1}^{q} n_j x_j) = \sum_{j=1}^{q} (\sum_{\sigma\in I} \sigma(n_j)x_j) =$$

$$= \sum_{j=1}^{q} \tau_I(n_j)x_j \in (S \cap \mathfrak{n})\Gamma = (\mathrm{rad}\ S)\Gamma.$$

Moreover since $x \in \Gamma$, $\tau_I(x) = \sum_{\sigma\in I} \sigma(x) = \sum_{\sigma\in I} x = rx$ where $r = |I|$. Thus $x = \frac{1}{r}\tau_I(x) \in (\mathrm{rad}\ S)\Gamma$, and therefore $\Gamma \cap \mathfrak{n}A \subseteq (\mathrm{rad}\ S)\Gamma$. The opposite inclusion holds since $\mathrm{rad}\ S = S \cap \mathfrak{n}$, and so equality is attained.

(9) $\Gamma \cap \mathfrak{m}A = \mathfrak{m}\Gamma$

Arguing as above, $\mathfrak{m}A = \mathfrak{m}(\Gamma C) = \mathfrak{m}\ (\Gamma(\mathfrak{n}+S)) = \mathfrak{m}\Gamma + \mathfrak{m}\Gamma\mathfrak{n} \subseteq \mathfrak{m}\Gamma + \mathfrak{n}\Gamma$ (Recall $\mathfrak{m} \subseteq S \subseteq \Gamma$). Therefore each $y \in \Gamma \cap \mathfrak{m}A$

can be represented as $y = \sum_{i=1}^{k} u_i x_i + \alpha$ where $u_i \in \mathfrak{m}$, $x_i \in \Gamma$ and $\alpha \in \mathfrak{n}\Gamma$. Inasmuch as $\mathfrak{m}\Gamma \subseteq \Gamma$, we conclude that

$$y - \sum_{i=1}^{k} u_i x_i \in \Gamma .$$

Therefore $\alpha \in \Gamma \cap \mathfrak{n}\Gamma = (\text{rad } S)\Gamma$ by (8). Since $\text{rad } S \subseteq \mathfrak{m}$ it follows that α, and therefore y, is an element of $\mathfrak{m}\Gamma$. The opposite inclusion is immediate, and so $\Gamma \cap \mathfrak{m}A = \mathfrak{m}\Gamma$.

With these technicalities behind us, it is straightforward to verify that Γ is an inertial R-subalgebra of A. Combining (7) and (9), we conclude that $\Gamma/\mathfrak{m}\Gamma$ is a separable $S/\mathfrak{m}$-algebra. Furthermore, since A is finitely generated over S, it follows that $\Gamma = \tau_I(A)$ is finitely generated over S. Therefore, by the well-known result of Endo and Watanabe (Theorem 1.21), it follows that Γ is a separable S-algebra. Now since S is a separable R-algebra, Γ is also a separable R-algebra by the transitivity of separability (theorem 1.12). Inasmuch as we have previously shown that $\Gamma + N = A$, we conclude that Γ is an inertial R-subalgebra of A.

The isomorphism $A \simeq \Gamma \otimes_S C$ is a consequence of proposition 2.1. Moreover, that $\Gamma \simeq \Lambda \otimes_R S$ follows from proposition 2.5. Therefore, since S is clearly an R-S bimodule, we have $A \simeq \Gamma \otimes_S C \simeq (\Lambda \otimes_R S) \otimes_S C \simeq \Lambda \otimes_R (S \otimes_S C) \simeq \Lambda \otimes_R C$.

Q.E.D.

In the case where $r = |I|$ is a unit in R, we have the following extension of Ingraham's theorem 1.27.

Corollary 2.14: Suppose A is an R-algebra satisfying (*). Assume that $I_A = I_C$ and that $r = |I|$ is a unit in R. Then A possesses an inertial R-subalgebra Γ if and only if A/N is R-separable and $J_{\mathfrak{m}} = J_{\mathfrak{m}'}$, for any two maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of C.

Remark 2.15: In the proof of theorem 2.13 the hypothesis of A being central separable over C was used only to conclude that $\operatorname{rad} A = (\operatorname{rad} C)A$. This "lifting of the radical" property is true in the more general setting where A is separable and projective over $C \subseteq Z(A)$ (see lemma 1.3 (c) and (d)). Thus we may relax (*) to this new setting and still obtain the same conclusions from theorem 2.13.

In the above context, theorem 2.13 is seen to be an extension of the following theorem due to F.R. DeMeyer.

Theorem 2.16: (DeMeyer [7]) Let A be a finitely generated, separable, and projective algebra over a commutative ring K. Assume that K is connected. Suppose G is a finite group of ring automorphisms of A which restricts faithfully to a group of ring automorphisms of K. Let $R = K^G$ and assume that K is a finitely generated, separable, and projective R-algebra. Then $B = A^G$ is R-separable and $A \simeq B \underset{R}{\otimes} K$. Moreover, if A is central over K, then B is central over R.

We proceed to show that this theorem is indeed a special case theorem 2.13. Suppose A is any K-algebra satisfying the hypothesis of theorem 2.16. Then K is clearly a connected (G,R)-algebra, and is an inertial R-subalgebra of itself. Therefore $K = K^{I_C}$ by theorem 1.27, and so $I_C = \langle 1 \rangle$. This trivially implies that $I_C = I_A$ and that $|I|$ is a unit in R. Therefore A also satisfies the hypotheses of theorem 2.13.

We should point out that we use DeMeyer's techniques in proposition 2.5 in order to obtain the isomorphism $\Gamma \simeq \Lambda \underset{R}{\otimes} S$.

Using the techniques introduced [12] it is possible to relax the connectedness hypothesis of theorem 2.13 at the expense of a somewhat weakened conclusion.

Suppose R is a connected ring and C is a (G,R)-algebra. It is shown in lemma 2.14 of [12] that C possesses a finite set $\{e_i\}_{i=1}^{n}$ of primitive idempotents such that $C = \oplus \sum_{i=1}^{n} Ce_i$. Let e be one of the primitive idempotents and define $H_e = \{\sigma \in I | \sigma(e) = e\}$. It follows from the proof of theorem 2.15 of [12] that Ce is a connected (H_e,Se)-algebra.

<u>Theorem 2.17</u>: Let R be a connected ring and suppose A is a finitely generated (G,Λ,R)-algebra which is in the (*) setting. Assume further that A is separable over its center C and that $r = |I|$ is a unit. If C possesses an inertial R-subalgebra, then A possesses an inertial S-subalgebra where $S = C^I$.

<u>Proof</u>: That Ae is central separable over Ce follows easily from the fact that A is central separable over C and the fact that the projection mapping $\pi: A \to Ae$ is a ring homomorphism.

We now show that H_e restricts faithfully from Ae to Ce. It is a consequence of corollary 2.9 that $A = \Gamma C$ where $\Gamma = A^I$. Thus each $a \in A$ can be represented as $a = \sum_{i=1}^{n} c_i x_i$ where $c_i \in C$ and $x_i \in \Gamma$. Now suppose $\sigma \in H_e$ restricts to the identity map on Ce. Then $\sigma(ae) = \sigma[(\sum_{i=1}^{n} c_i x_i)e] = \sum_{i=1}^{n} \sigma(c_i)x_i e = \sum_{i=1}^{n} c_i x_i e = ae$ for all $ae \in Ae$, whence σ is the identity map on Ae.

It is immediate that $I_{Ae} = \{\sigma \in H_e | (\sigma(a)-a)e \in \text{rad}(Ae) = Ne \ \forall \ a \in A\} = \{\sigma \in H_e | (\sigma(c)-c)e \in \text{rad}(Ce) = \eta e \ \forall \ c \in C\} = I_{Ce}$

Finally, we show that Se is an inertial Se-subalgebra of Ce. Now it is a further consequence of theorem 2.15 of [12] that the inertial Re-subalgebra of Ce can be characterized as $(Ce)^{F_e}$ where $F_e = \{\sigma \in G | \sigma(e) = e$ and $\sigma(ce)-ce \in \eta e \ \forall c \in C\}$. Clearly $H_e = \{\sigma \in G | \sigma(e) = e$ and $\sigma(c)-c \in \eta \ \forall \ c \in C\} \subseteq F_e$, so that $Se = (Ce)^{H_e} \supseteq (Ce)^{F_e}$. Therefore it follows that $Se + \eta e = Ce$, whence Se is an inertial Se-algebra of Ce.

Having verified that Ae fulfills all the hypotheses of theorem 2.13, we can conclude that $(Ae)^{H_e}$ is an inertial Se-subalgebra of Ae. It follows that $B = \oplus \sum_{i=1}^{n} (Ae_i)^{H_{e_i}}$ is an inertial S-subalgebra of A.

Q.E.D.

<u>Remark</u>: In general one can conclude only that S contains the inertial R-subalgebra of C. In the event equality is obtained, B will be R-separable by transitivity and therefore will be an inertial R-subalgebra of A. Such equality is, of course, attained when C is connected, in which case the setting reverts to that of theorem 2.13.

We now present two examples which provide additional insight into the implications and limitations of the hypotheses of theorem 2.13.

Suppose A is a finitely generated R-algebra which is separable over its center C. If C is a connected (G,R)-algebra and A contains an inertial R-subalgebra B, then it is always possible to extend I_C to R-automorphisms of A in such a way that $I_C = I_A$ (page 20). As the following example points out, it is by no means true that every extension of I_C to R-automorphisms of A satisfies $I_C = I_A$. We first need to state a definition.

<u>Definition 2.18</u>: An algebra A over a commutative ring C is said to be a <u>generalized quaternion algebra</u> provided A is a free C-module having basis $\{1 \in C, \alpha, \beta, \alpha\beta\}$ with multiplication induced by $\alpha^2 = x \in C$, $\beta^2 = y \in C$, and $\beta\alpha = -\alpha\beta$. We will denote this algebra by (C,x,y).

<u>Example 2.19</u>: Let $A = (C,-1,-1)$ where $C = \mathbb{R}[x]/(x^2)$ and $\mathbb{R}$ is the field of real numbers. Let $G = \langle\sigma\rangle$ be the cyclic group of $\mathbb{R}$-automorphisms of C of order two with generator σ defined by $\sigma(r + s\bar{x}) = r - s\bar{x}$. Then A is separable over C, C is a connected $(G,\mathbb{R})$-algebra, and A possesses an inertial $\mathbb{R}$-subalgebra. Furthermore, there exists an extension $\tilde{G}$ of G to A in such a way that $I_C \ngeq I_A$.

<u>Discussion</u>: C is local with unique maximal ideal $\mathfrak{m} = (x)/(x^2)$. One sees $A/\mathfrak{m}A \simeq (\mathbb{R},-1,-1)$, which is the classical quaternion algebra. $(\mathbb{R},-1,-1)$ is well known to be

(central) separable over $R \simeq C/\mathfrak{m}$. Therefore A is separable over C by Endo-Watanabe (Theorem 1.21). One easily checks that A is, in fact, central separable over C.

Now $\mathbb{R}$ is clearly an $\mathbb{R}$-inertial subalgebra of C, and it is immediate that C is a $(G,\mathbb{R})$-algebra. Since rad $A = \mathfrak{m}A$, it follows easily that $(\mathbb{R},-1,-1)$ is an inertial R-subalgebra of A.

We define $\tilde{\sigma}\colon A \to A$ by $\tilde{\sigma}(c_1\cdot 1 + c_2\alpha + c_3\beta + c_4\alpha\beta) = \sigma(c_1)\cdot 1 + \sigma(c_3)\alpha + \sigma(c_2)\beta - \sigma(c_4)\alpha\beta$. It is a straightforward verification to show $\tilde{\sigma}$ is an R-automorphism of A which has order two. Thus $\tilde{G} = \langle\tilde{\sigma}\rangle$ is an extension of G to A. Now since A is central separable over C, it follows that rad $A = \mathfrak{m}A = \mathfrak{m}\cdot 1 \oplus \mathfrak{m}\cdot\alpha \oplus \mathfrak{m}\beta \oplus \mathfrak{m}\alpha\beta$. Since $\sigma(r+s\bar{x}) - (r+s\bar{x}) = 2s\bar{x} \in \mathfrak{m}$, we see that $\tilde{G} = I_C$. However, $\tilde{\sigma}(a)-a = \tilde{\sigma}(c_1\cdot 1 + c_2\alpha + c_3\beta + c_4\alpha\beta) - (c_1\cdot 1 + c_2\alpha + c_3\beta + c_4\alpha\beta) = (\sigma(c_1)-c_1)\cdot 1 + (\sigma(c_3)-c_2)\alpha + (\sigma(c_2)-c_3)\beta - (\sigma(c_4)+c_4)\alpha\beta$. Thus for an element $a \in A$ such that $\sigma(c_2)-c_3 \notin \mathfrak{m}$ (e.g. $c_2 = \frac{1}{2}$ and $c_3 = -\frac{1}{2}$), $\sigma(a)-a \notin$ rad A. Therefore $I_A = \langle 1\rangle$, and we have demonstrated that $\tilde{G} = I_C \supsetneq I_A$.

We remark that $\bar{G} = \langle\bar{\sigma}\rangle$, where $\bar{\sigma}(c_1\cdot 1 + c_2\alpha + c_3\beta + c_4\alpha\beta) = \sigma(c_1)\cdot 1 + \sigma(c_2)\alpha + \sigma(c_3)\beta + \sigma(c_4)\alpha\beta$, is an extension of G to $\mathbb{R}$-automorphisms of A in such a way that $I_A = I_C$.

As shown by the following example, the hypothesis that $r = |I|$ is a unit is not necessary for the existence of an inertial R-subalgebra of an algebra A in the setting (*).

<u>Example 2.20</u>: Let $A = C = Z[x]/(x^2)$, and let G be the cyclic group of Z-automorphisms of C of order 2 with

generator σ where $\sigma(a + b\overline{x}) = a - b\overline{x}$. Then C is a connected (G,Z)-algebra with inertial Z-subalgebra $S = Z$. However $|I|$ is not a unit in Z.

<u>Discussion</u>: Although C is not local, a straight forward calculation shows it is connected. As in example 2.19, $\mathfrak{n} = \text{rad } C = (x)/(x^2)$. From this it is clear that Z adds with rad C and so is a Z-inertial subalgebra of C. One easily checks that $I = \{\sigma \in G | \sigma(c)-c \in \mathfrak{n} \ \forall \ c \in C\} = G$. Therefore the order of I is 2 which is not a unit in Z.

Settings where the order of a group of automorphisms is a unit have precedent in the literature. For example, Y. Takeuchi shows in [18] that if a ring Γ is a galois extension of its center C with group G, then $n = |G|$ is a unit in C. A related result was obtained by T. Kanzaki in [13] where he proved that if Γ is galois over R with group G and $H = \{\sigma \in G| \ \sigma |_C = 1\}$, then $r = |H|$ is a unit in R.

Chapter III

UNIQUENESS OF INERTIAL SUBALGEBRAS OF ALGEBRAS SEPARABLE OVER THEIR CENTERS

The main objective of this chapter is to investigate the question of uniqueness (up to isomorphism) of inertial R-subalgebras of those finitely generated R-algebras which are separable over their centers. Suppose C is a finitely generated, commutative R-algebra with inertial R-subalgebra S. Let us denote the Brauer groups (definition 3.1) of S and of C by $\mathfrak{B}(S)$ and $\mathfrak{B}(C)$ respectively. We show that under certain conditions the inertial R-subalgebras of any central separable C-algebra are inner automorphic if and only if the induced mapping $\mathfrak{B}(S) \rightarrow \mathfrak{B}(C)$ is a monomorphism. We conclude by presenting the only known example of an R-algebra which possesses non-isomorphic inertial R-subalgebras. The example will be seen to be a member of the class of algebras studied in the previous chapter.

In 1960 Auslander and Goldman introduced the concept of the Brauer group of a commutative ring R. The Brauer group of R is an abelian group which reflects the variety of central separable R-algebras. It is also an important invariant of the ring R.

Let $\sigma(R)$ denote the set of isomorphism classes of central separable R-algebras. Let $\sigma^{\circ}(R)$ denote the subset of $\sigma(R)$ consisting of those classes each of whose elements is isomorphic to the endomorphism ring of a finitely generated, projective, and faithful R-module. Elements $A, B \in \sigma(R)$ are said to be equivalent $(A \sim B)$ provided there exists elements $X, Y \in \sigma^{\circ}(R)$ such that $A \underset{R}{\otimes} X \simeq B \underset{R}{\otimes} Y$.

Definition 3.1: Let $\mathfrak{B}(R)$ denote the set of equivalence classes of $\sigma(R)$ under the equivalence relation $\sim$. For arbitrary $[A], [B] \in \mathfrak{B}(R)$, the binary operation given by $[A][B] = [A \underset{R}{\otimes} B]$ is well-defined and makes $\mathfrak{B}(R)$ into an abelian group called the Brauer group of R ([8], page 60). The identity element of $\mathfrak{B}(R)$ is the class $[R]$ containing the ground ring. The inverse of any class $[A]$ is the class $[A^{\circ}]$ containing the opposite algebra A° of A.

Suppose $f: R \to S$ is any ring homomorphism of R to the commutative ring S. Then f endows S with the structure of an R-algebra, and $\mathfrak{B}(f): \mathfrak{B}(R) \to \mathfrak{B}(S)$ defined by $\mathfrak{B}(f)\ [A] = [A \underset{R}{\otimes} S]$ is a homomorphism of the Brauer groups. Indeed, one sees that $\mathfrak{B}(\ )$ is a covariant functor from the category of commutative rings (and ring homomorphisms) to the category of abelian groups (and group homomorphisms).

Definition 3.2: The Brauer group $\mathfrak{B}(R)$ of a connected ring R is said to have the unique representation property provided each class $[A] \in \mathfrak{B}(R)$ has a representative D possessing no idempotents other than 0 and 1 and having the property that to each $B \in [A]$ there corresponds an integer n such that $B \simeq \mathcal{M}_n(D)$.

Remark 3.2.1: The representative D of the class $[A]$ is unique up to isomorphism. Suppose D^* is another representative of [A] having the same properties as D. Then there is an integer r such that $\mathfrak{M}_r(D^*) \simeq D$. Since D has no idempotents except 0 and 1, it follows that $r = 1$, whence $D^* \simeq D$.

Remark 3.2.1: The integer n associated with each $B \in [A]$ is also unique. Suppose $\mathfrak{M}_n(D) \simeq \mathfrak{M}_r(D)$. Then $\mathfrak{M}_n(D) \underset{R}{\otimes} R_{\mathfrak{m}} \simeq \mathfrak{M}_r(D) \underset{R}{\otimes} R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, or equivalently $\mathfrak{M}_n(D_{\mathfrak{m}}) \simeq \mathfrak{M}_r(D_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of R. Since $D_{\mathfrak{m}}$ is free of finite rank over $R_{\mathfrak{m}}$ by proposition 1.8, it follows that $n^2[D_{\mathfrak{m}}:R_{\mathfrak{m}}] = r^2[D_{\mathfrak{m}}:R_{\mathfrak{m}}]$ for every maximal ideal $\mathfrak{m}$ of R. Therefore $n = r$.

It is well-known that if F is a field then $\mathfrak{B}(F)$ has the unique representation property. In this instance the representative D of each class $[A] \in \mathfrak{B}(F)$ is seen to be a division ring. F. R. DeMeyer provides us with the following generalization of this result ([6], corollary 1):

Theorem 3.3: If K is a connected, semi-local ring, then $\mathfrak{B}(K)$ has the unique representation property.

R. Hoobler in his dissertation [11] extends the definition of the Brauer group from fields to commutative rings by introducing a stronger equivalence relation $\sim'$ on $\sigma(R)$ than did Auslander and Goldman. Central separable R-algebras $A, B \in \sigma(R)$ are said to be $\sim'$-equivalent $(A \sim' B)$ provided there exists integers u and t such that $\mathfrak{M}_u(A) \simeq \mathfrak{M}_t(B)$. Let $\mathcal{J}(R)$ denote the set of isomorphism classes of finitely generated, faithful, and projective R-modules.

$\sigma(R)/\sim'$ forms an abelian group,which is denoted $\overline{\mathfrak{B}}(R)$, under the operation $[A]\,[B] = [A \otimes_R B]$. This is a consequence of a result by Bass [3] which states that to each $P \in \mathcal{J}(R)$ there corresponds a $Q \in \mathcal{J}(R)$ and an integer n such that $P \otimes_R Q \simeq \oplus \sum_{i=1}^{n} R$. One immediately sees that there is a natural epimorphism from $\overline{\mathfrak{B}}(R)$ onto $\mathfrak{B}(R)$.

Definition 3.4: The uniqueness statement is said to hold for a finitely generated R-algebra A provided any two inertial R-subalgebras B and B' of A are isomorphic via an inner automorphism of A generated by an element of the form $1 - n$, for some $n \in N = \text{rad } A$.

Lemma 3.5: Let C be a finitely generated, commutative R-algebra with inertial R-subalgebra S. Suppose W is a central separable S-algebra. Then W is an inertial R-subalgebra of $W \otimes_S C$.

Proof: Since W is projective over S (proposition 1.17), W can be considered an R-subalgebra of $W \otimes_S C$ by identifying it with $W \otimes_S S$. That W is R-separable follows from the transitivity of separability (proposition 1.12). Now $\text{rad } (W \otimes_S C) = \mathfrak{n} \cdot (W \otimes_S C)$ by proposition 1.20, and so we have

$$W + \text{rad}(W \otimes_S C) = W \otimes_S S + W \otimes_S \mathfrak{n} = W \otimes_S (S+\mathfrak{n}) = W \otimes_S C.$$

Therefore W is an inertial R-subalgebra of $W \otimes_S C$.

Theorem 3.6: Let C be a finitely generated, commutative R-algebra possessing an inertial R-subalgebra S.

(a) Suppose $\mathfrak{B}(C) = \overline{\mathfrak{B}}(C)$. If the uniqueness statement holds for every central separable C-algebra then $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism.

(b) Suppose C is connected, and both $\mathfrak{B}(S)$ and $\mathfrak{B}(C)$ have the unique representation property. If $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism, then the inertial R-subalgebras of any central separable C-algebra are isomorphic as S-algebras.

Proof: (a) Suppose $[B] \in \ker \eta$ so that $[B \otimes_S C] = [C]$ in $\mathfrak{B}(C)$. Since $\mathfrak{B}(C) = \overline{\mathfrak{B}}(C)$, there exist integers n and m such that $(B \otimes_S C) \otimes_C \mathcal{M}_n(C) \simeq \mathcal{M}_m(C)$. Now $(B \otimes_S C) \otimes_C \mathcal{M}_n(C) \simeq B \otimes_S (C \otimes_C \mathcal{M}_n(C)) \simeq B \otimes_S (\mathcal{M}_n(C)) \simeq B \otimes_S (\mathcal{M}_n(S) \otimes_S C) \simeq \mathcal{M}_n(B) \otimes_S C$. Since both B and $\mathcal{M}_n(S)$ are central separable over S, $\mathcal{M}_n(B) \simeq B \otimes_S \mathcal{M}_n(S)$ is central separable over S also. Hence $\mathcal{M}_n(B)$ is an inertial R-subalgebra of $\mathcal{M}_m(C)$ by lemma 3.5. However, $\mathcal{M}_m(S)$ is also an inertial R-subalgebra of $\mathcal{M}_m(C)$ by proposition 2.4. Thus $B \otimes_S \mathcal{M}_n(S) \simeq \mathcal{M}_m(S)$ by hypothesis, whence $B \sim S$ in $\mathfrak{B}(S)$. Therefore $\ker \eta = (0)$, or, equivalently, $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism.

(b) Let A be any central separable C-algebra and suppose that B and B' are inertial R-subalgebras of A. Then $A \simeq B \otimes_S C \simeq B' \otimes_S C$ by proposition 2.2. Hence $[B \otimes_S C] = [B' \otimes_S C]$ in $\mathfrak{B}(C)$. Now $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism by hypothesis, so that $[B] = [B']$ in $\mathfrak{B}(S)$. Since $\mathfrak{B}(S)$ has the unique representation property, there exists integers r and t and a representative $D \in [B]$ possessing no idempotents except 0 and 1 such that $B \simeq \mathcal{M}_r(D)$ and $B' \simeq \mathcal{M}_t(D)$. It follows that $\mathcal{M}_r(D) \otimes_S C \simeq \mathcal{M}_t(D) \otimes_S C$, whence $\mathcal{M}_r(D \otimes_S C) \simeq \mathcal{M}_t(D \otimes_S C)$. Since $D \otimes_S C$ is a central separable C-algebra, and since $\mathfrak{B}(C)$ has the unique representation property, there exists an integer u and a representative

$\mathcal{B} \in [D \underset{S}{\otimes} C]$ containing no idempotents except 0 and 1 and such that $D \underset{S}{\otimes} C \simeq \mathcal{M}_u(\mathcal{B})$. Thus $\mathcal{M}_r(\mathcal{M}_u(\mathcal{B})) \simeq \mathcal{M}_t(\mathcal{M}_u(\mathcal{B}))$ or, equivalently, $\mathcal{M}_{ru}(\mathcal{B}) \simeq \mathcal{M}_{tu}(\mathcal{B})$. The dimension of such matrices is uniquely determined (remark 3.2.2), and so $r = t$. Therefore $B \simeq B'$ as S-algebras.

Q.E.D.

Proposition 3.7: Suppose A is a finitely generated R-algebra containing an inertial R-subalgebra B. Then every central idempotent of A is contained in B.

Proof: Any central idempotent $e \in A$ induces a ring direct sum decomposition of A in the usual way: $A = Ae \oplus A(1-e)$. The projection map $\pi: A \rightarrow Ae$ defined by $\pi(a) = ae$ is thus seen to be a ring epimorphism. Therefore $\pi(B) = Be$ is R-separable by proposition 1.14. Likewise, $B(1-e)$ is R-separable. Therefore the R-subalgebra B' of A defined by $B' = Be \oplus B(1-e)$ is R-separable, since it is the ring direct sum of two separable R-algebras. It is clear that $B' \supseteq B$, whence $B' + N = A$. Thus B' is an inertial R-subalgebra of A. Now both B and B' are finitely generated by theorem 1.22. Since any two nested and finitely generated inertial R-subalgebras coincide ([12], lemma 2.5), it follows that $B = B'$. Therefore e is an element of B.

Remark: Non-central idempotents of a finitely generated R-algebra need not be contained in a particular inertial R-subalgebra. This fact is nicely illustrated in example 3.9 where $A \simeq \mathcal{M}_2(C)$ possesses many non-central idempotents, none of which are in the inertial R-subalgebra $B = (S,-1,-1)$.

<u>Theorem 3.8</u>: Suppose R is a semi-local ring and C is a finitely generated, commutative R-algebra with inertial R-subalgebra S. Then $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism if and only if the uniqueness statement holds for every R-algebra which is central separable over C.

<u>Proof</u>: Since both C and S are finitely generated over R, they also are semi-local. Any semi-local ring can be decomposed into a direct sum of connected semi-local rings: $C = \oplus \sum_{i=1}^{n} Ce_i$ where $\{e_i\}_{i=1}^{n}$ is a set of primitive orthogonal idempotents such that $1 = \sum_{i=1}^{n} e_i$. Since $\{e_i\} \subseteq S$ by proposition 3.7, S can also be decomposed as a direct sum of its ideals generated by the same primitive idempotents: $S = \oplus \sum_{i=1}^{n} Se_i$. For the sake of convenience, let us denote Ce_i by C_i and Se_i by S_i. It is seen that S_i is an inertial R-subalgebra of C_i. As indicated in the proof of theorem 2.17, a decomposition of C induces a decomposition on any C-algebra A in such a way that A is (central) separable over C if and only if A_i is (central) separable over C_i for all $i \leq n$.

With this is mind, we reduce the theorem to the case where C is connected.

(a) The uniqueness statement holds for every central separable C-algebra if and only if it holds for every central separable C_i-algebra for every $i \leq n$.

Suppose the uniqueness statement holds for every central separable C_i-algebra for every $i \leq n$. Let B and B' be inertial R-subalgebras of an arbitrary central separable C-

algebra A. Then B_i and B_i' are inertial R-subalgebras of the central separable C_i-algebra A_i. Therefore there exists an inner automorphism θ_i of A_i such that $\theta_i(B_i) = B_i'$, where $\theta_i(a_i) = (e_i - n_i^*)a_i(e_i - n_i)$ and $n_i, n_i^* \in \operatorname{rad}(A_i) = N_i$. Then $\theta\colon A \to A$ defined by $\theta(a) = (1 - \sum_{i=1}^{n} n_i^*)a(1 - \sum_{i=1}^{n} n_i)$ is an inner automorphism of A generated by an element in $N = \operatorname{rad} A$, and is such that $\theta(B) = B'$.

Conversely, assume the uniqueness statement holds for all central separable C-algebras. Let i be any integer $\le n$, and let B_i and B_i' be inertial R-subalgebras of an arbitrary central separable C_i-algebra A_i. One sees that $B = S_1 \oplus \cdots \oplus S_{i-1} \oplus B_i \oplus S_{i+1} \oplus \cdots \oplus S_n$ and $B' = S_1 \oplus \cdots \oplus S_{i-1} \oplus B_i' \oplus S_{i+1} \oplus \cdots \oplus S_n$ are inertial R-subalgebras of the central separable C-algebra $A = C_1 \oplus \cdots \oplus C_{i-1} \oplus A_i \oplus C_{i+1} \oplus \cdots \oplus C_n$. Therefore there exists an inner automorphism θ of A such that $\theta(B) = B'$ where $\theta(a) = (1-n^*)a(1-n)$ and $n, n^* \in N = \operatorname{rad} A$. Then $\theta_i\colon A_i \to A_i$ defined by $\theta_i(a_i) = (e_i - n_i^*)a_i (e_i - n_i)$ is an inner automorphism of A_i generated by an element in $N_i = \operatorname{rad}(A_i)$, and is such that $\theta_i(B_i) = B_i'$.

(b) $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism if and only if each $\mathfrak{B}(S_i) \xrightarrow{\eta_i} \mathfrak{B}(C_i)$ is a monomorphism for every $i \le n$.

Every central separable S-algebra B can be decomposed as $B = \oplus \sum_{i=1}^{n} B_i$ where $B_i = Be_i$. Thus $B \otimes_S C = (\oplus \sum_{i=1}^{n} B_i) \otimes_S (\oplus \sum_{j=1}^{n} C_j) \simeq \oplus \sum_{(i,j)}^{(n,n)} B_i \otimes_S C_j$. Now $B_i \otimes_S C_j = (0)$ for $i \ne j$, since the set of idempotents $\{e_i\}$ is orthogonal. Also $B_i \otimes_S C_i = B_i \otimes_{S_i} C_i$, since $\oplus \sum_{i \ne j} S_j \subseteq \operatorname{annih}_S(C_i)$. Therefore

$B \otimes_S C \simeq \oplus \sum_{i=1}^{n} B_i \otimes_{S_i} C_i$.

It is well-known that the Brauer group distributes over finite direct sums. Hence we may identify $\mathfrak{B}(\oplus \sum_{i=1}^{n} C_i)$ with $\oplus \sum_{i=1}^{n} \mathfrak{B}(C_i)$, and $\mathfrak{B}(\oplus \sum_{i=1}^{n} S_i)$ with $\oplus \sum_{i=1}^{n} \mathfrak{B}(S_i)$. Thus $\eta[B] = [B \underset{S}{\otimes} C] = [\oplus \sum_{i=1}^{n} B_i \underset{S_i}{\otimes} C_i] = \oplus \sum_{i=1}^{n} [B_i \underset{S_i}{\otimes} C_i] = \oplus \sum_{i=1}^{n} \eta_i[B_i]$.

Now suppose that η_i is a monomorphism for every $i \leq n$, and let $\eta[B] = [C]$. Then $\eta_i[B_i] = [C_i]$ for every $i \leq n$, whence $[B_i] = [S_i]$ for every $i \leq n$ by assumption. This implies that $[B] = [S]$, and therefore that η is a monomorphism.

Conversely, assume η is a monomorphism. Suppose $\eta_i[B_i] = [C_i]$ for any $i \leq n$. It is clear that

$$B' = S_1 \oplus \cdots \oplus S_{i-1} \oplus B_i \oplus S_{i+1} \oplus \cdots \oplus S_n$$

is a central separable S-algebra, and that $\eta[B] = [B \underset{S}{\otimes} C] = \oplus \sum_{j=1}^{n} \eta_j[B'_j] = [C]$. Then $[B] = [S]$ by assumption, whence $[B_i] = [S_i]$. Therefore η_i is a monomorphism for every $i \leq n$.

In view of (a) and (b), it suffices to prove the theorem in the case where C is connected and semi-local. In this setting if M is a finitely generated projective C-module, then M is free of finite rank over C (corollary 1.9). This implies that $\overline{\mathfrak{B}}(C) = \mathfrak{B}(C)$. Therefore the uniqueness statement for all central separable C-algebras implies $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism by theorem 3.6 (a).

Conversely, assume $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is a monomorphism. Since S and C are connected, semi-local rings, both $\mathfrak{B}(S)$ and $\mathfrak{B}(C)$ have the unique representation property (theorem

3.3). Therefore the inertial R-subalgebras of every central separable C-algebra A are isomorphic as S-algebras by theorem 3.6 (b). We are done if we can show that, in fact, they are isomorphic via an inner automorphism of A generated by $1 - n$ where $n \in N = \text{rad } A$. First we need the following generalization of the Skolem-Noether Theorem which appears in [17].

Theorem (Sridharan): Let R be a semi-local ring. Suppose A and B are central separable R-algebras with R-algebra monomorphisms $f,g\colon B \to A$. Then there exists an inner automorphism θ of A such that $g = \theta \circ f$.

Suppose that $f\colon B \to B'$ is an S-isomorphism of two inertial S-algebras B and B' of a central separable C-algebra A. It then follows that the mapping $f \otimes 1\colon B \otimes_S C \to B' \otimes_S C$ induced by $f \otimes 1(b\otimes c) = f(b)\otimes c$ is a C-isomorphism. In view of proposition 2.1, the multiplication maps $\mu\colon B \otimes_S C \to A$ and $\mu'\colon B' \otimes_S C \to A$ are also C-isomorphisms. Then μ and $\mu' \circ f\otimes 1$ are C-algebra isomorphisms which map $B \otimes_S C$ onto A. Therefore there exists an inner automorphism θ of A such that $\mu = \theta \circ [\mu' \circ f\otimes 1]$ by Sridharan's theorem. This implies that $B = \mu(B \otimes_S S) = \theta(\mu'(f\otimes 1(B \otimes_S S))) = \theta(\mu'(B' \otimes_S S)) = \theta(B')$.

The following argument, due to Azumaya [2], shows that, moreover, B is conjugate to B' via an inner automorphism generated by $1 - n$, where $n \in N = \text{rad } A$. Suppose $\theta(x) = w\, x\, w^{-1}$. Since $A = B + N$, we can represent w as $w = v + n$ where $v \in B$ and $n \in N$. The element v is a unit modulo

N, and is therefore outside every maximal left and every maximal right ideal of A. It is thus both left and right invertible and therefore is a unit. Hence $wv^{-1} \equiv 1 \bmod N$, and $(wv^{-1})B(wv^{-1})^{-1} = w(v^{-1}Bv)w^{-1} = wBw^{-1} = \theta(B) = B'$. Therefore the uniqueness statement holds for A.

Q.E.D.

We conclude with an example of an R-algebra which has non-isomorphic inertial R-subalgebras. This example will be seen to be a member of the class of algebras discussed in Chapter II. Let us denote the localization of the integers Z at the maximal ideal (P) by Z_p.

<u>Example 3.9</u>: Let $R = S = Z_5$ and $C = Z_5 \oplus 5Z_5\, i \subseteq Z_5[i]$, where $i^2 = -1$. Let $A = (C,-1,-1)$ be the generalized quaternion algebra with basis $\{1,\alpha,\beta,\alpha\beta\}$. Then $B = (S,-1,-1)$ and $B' = \mathfrak{m}_2(S)$ are two non-isomorphic R-inertial subalgebras of A.

<u>Discussion</u>: (1) $A \simeq \mathfrak{m}_2(C)$ (and so is central separable over C). C is local with unique maximal ideal $\mathfrak{m} = 5Z_5 \oplus 5Z_5\, i$. It is not hard to see that $A/\mathfrak{m}A \simeq (C/\mathfrak{m},-1,-1) \simeq (Z/(5),-1,-1)$. Since Z/(5) possesses a root of unity (namely 2), it follows from page 18 of [9] that

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \begin{bmatrix} 1/2(1-i\alpha) & 1/2(\beta-i\alpha\beta) \\ 1/2(-\beta-i\alpha\beta) & 1/2(1+i\alpha) \end{bmatrix}$$

forms a matrix basis for $A/\mathfrak{m}A$ over $C/\mathfrak{m}$. Thus $A/\mathfrak{m}A \simeq \mathfrak{m}_2(C/\mathfrak{m})$

and so is separable over $C/\mathfrak{m}$. Therefore A is separable over C by Theorem 1.21 (Endo-Watanabe).

It is a straightforward computation to show $Z(A) = C$. Let $\gamma = x + y\alpha + z\beta + w\alpha\beta \in Z(A)$. Then $\gamma\alpha = \alpha\gamma$ or $x\alpha - y - z\alpha\beta + w\beta = x\alpha - y + z\alpha\beta - w\beta$. Hence $2z = 2w = 0$, or $z = w = 0$ since 2 is a unit in C. Similarly since $\gamma\beta = \beta\gamma$, we obtain $2y = 2w = 0$ or $y = w = 0$. Therefore $\gamma \in C$, showing $Z(A) = C$.

C being local, $\mathfrak{B}(C)$ has the unique matrix representation property (Theorem 3.3). Therefore there exists an integer n and a unique central separable C-algebra D which has no idempotents except 0 and 1 such that $A \simeq \mathcal{M}_n(D)$. Now D is free over C since D is projective over C (proposition 1.17) and C is local. Therefore $4 = [A:C] = [A:D][D:C] = n^2[D:C]$. There are only two possibilities:

$$\begin{cases} n=1 \Rightarrow [D:C] = 4 \Rightarrow A = D \Rightarrow A \text{ has no non-trivial idempotents} \\ n=2 \Rightarrow [D:C] = 1 \Rightarrow A \simeq \mathcal{M}_2(C) \ . \end{cases}$$

One checks that the element $e = \frac{1}{2} - \frac{3}{8}\beta - \frac{5}{8}i\ \alpha\beta$ is a non-trivial idempotent of A. Therefore the case $n = 1$ cannot hold, and we conclude that $A \simeq \mathcal{M}_2(C)$.

(2) $B = (S,-1,-1)$ is central separable over S. We see $S = Z_5$ is also local with unique maximal ideal $\mathfrak{m} = 5Z_5$. Hence $B/\mathfrak{m}B \simeq (Z_5/5Z_5,-1,-1) \simeq (Z/(5),-1,-1)$. We have seen in (1) that this algebra is separable over $Z/(5)$, and therefore B is separable over S by Endo-Watanabe (Theorem 1.21). A computation identical to that in (1) will show that $Z(B) = S$, since 2 is also a unit in S. Thus B is central separable over S.

(3) B is an inertial S-subalgebra of A. Since A is central separable over C, rad $A = \mathfrak{n}A = (\mathfrak{n},-1,-1)$ by proposition 1.20. It is clear that $S = Z_5$ adds with $\mathfrak{n} = 5Z_5 \oplus 5Z_5 i$. Therefore $(S,-1,-1) + (\mathfrak{n},-1,-1) = (C,-1,-1)$. That B is an inertial S-subalgebra now follows from (2).

(4) $\mathcal{M}_2(S)$ is an inertial S-subalgebra of A, but B is not isomorphic to $\mathcal{M}_2(S)$.

Now $A \simeq \mathcal{M}_2(C)$ by (1), and so $\mathcal{M}_2(S)$ is an inertial S-subalgebra of A by proposition 2.4. Since $Z_5 \subseteq \mathbb{R}$ (the field of real numbers) it follows that $B = (S,-1,-1) \subseteq (\mathbb{R},-1,-1)$. Now $(\mathbb{R},-1,-1)$ is the classical quaternion algebra which is well-known to be a division ring. Hence B has no non-trivial idempotents, and therefore cannot be isomorphic to $\mathcal{M}_2(S)$.

Remark 3.9.1: In view of Theorem 3.9, S and C are examples of finitely generated commutative R-algebras where S is an inertial R-subalgebra of C, but such that $\mathfrak{B}(S) \xrightarrow{\eta} \mathfrak{B}(C)$ is not a monomorphism.

Remark 3.9.2: Define $\sigma: C \to C$ by $\sigma(x + 5yi) = x-5yi$. Then σ is an R-automorphism of C of order 2. If we let $G = \langle\sigma\rangle$, then C is a connected (G,S)-algebra such that $2 = |G|$ is a unit in S. By the discussion following proposition 2.4, we see that G can be extended to A in such a way that $I_A = I_C$ (in this case $G = I_A = I_C$). Thus A is an R-algebra which satisfies the hypotheses of Theorem 2.16. Therefore even such "well-behaved"algebras as those discussed in Chapter II may possess inertial R-subalgebras which are not isomorphic.

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