THE APPLICATION OF BAYESIAN STATISTICS TO AUDITING: DISCRETE VERSUS CONTINUOUS PRIOR DISTRIBUTIONS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

ALBERT KENNING FRANCISCO

1972

This is to certify that the

thesis entitled

THE APPLICATION OF BAYESIAN STATISTICS TO AUDITING:
DISCRETE VERSUS CONTINUOUS PRIOR DISTRIBUTIONS

presented by

Albert K. Francisco

has been accepted towards fulfillment
of the requirements for

Ph.D degree in Business - Accounting

[signature]
Major professor

Date May 5, 1972

O-7639

that

Stud

info

buti

have

such

the l

Bayes

with

uous

prior

which

metho

poor

rates

ABSTRACT

THE APPLICATION OF BAYESIAN STATISTICS TO
AUDITING: DISCRETE VERSUS CONTINUOUS
PRIOR DISTRIBUTIONS

By

Albert Kenning Francisco

It has been suggested in the auditing literature that auditors adopt Bayesian statistical techniques. Studies have shown that auditors are willing to provide information that can be used to construct prior distributions for this purpose. Practicing auditors, however, have not widely adopted Bayesian techniques, even though such techniques have received considerable exposure in the literature. The articles suggesting the use of Bayesian techniques have all demonstrated such techniques with discrete prior distributions.

Although less difficult to work with than continuous prior distributions in simple examples, discrete prior distributions have several practical disadvantages which may partly explain the lack of adoption of Bayesian methods by auditors. Discrete prior distributions are poor approximations of the continuous range of possible rates found in most audit attributes sampling situations.

Th

di

pr

It

a

po

pu

req

dis

usu

over

is c

lati

It i

Sketc

resea

prior

distr

mode)

addit

a giv

a bin

and s

The derivation of specific discrete prior distributions is difficult, because a number of points must be assigned probabilities, and these probabilities must total 1.00. It is difficult for an auditor to perceive the meaning of a discrete distribution because only a few of the possible points are assigned positive probabilities. Involved computations (many multiplication and division operations) are required for the solution of any practical problem using discrete distributions and Bayesian statistics. This would usually require a computer.

Bayesian methods using beta prior distributions can overcome most of these difficulties. Such a distribution is continuous, and fits the large number of possible population error rates better than does the discrete model. It is easier for an auditor to visualize continuous curves. Sketches of representative distributions developed in this research can allow an auditor to quickly pick the specific prior desired, and write down the numeric parameters of that distribution. Tables of statistics (mean, variance, and mode) of these representative distributions can provide additional information to the auditor seeking a prior for a given audit situation. The revision of a beta prior by a binomial sample result is not difficult. A few addition and subtraction operations are required. These can be per

sop

thi

a s

one

but

giv

able

quic

and

tion

which

compu

Table

moder

tions

Simps

can b

gener

poste

table

adopte

performed by an auditor with the aid of nothing more sophisticated than a piece of paper and a pencil. Once this is done, the auditor can refer again to his tables for a sketch of the posterior distribution (or a very similar one) and for statistics he wishes to know for that distribution. Another set of tables developed in this study can give confidence intervals. What would have taken considerable computer time with a discrete prior distribution is quickly taken care of with this method using a few tables and a quick addition and subtraction operation.

Although tables and sketches of the beta distribution are not currently available, techniques exist with which such information can be generated. The formulas for computing statistics of beta distributions are not complex. Tables of such statistics can be generated in seconds on modern computers. Similarly, sketches of such distributions can be created by plotters attached to computers. Simpson's rule for the evaluation of definite integrals can be used as the basis for computer programs which will generate confidence intervals based upon representative posterior beta distributions. Examples of sketches and tables are included in the study.

It is concluded that, although auditors have not adopted Bayesian statistical methods, such methods hold promise. There are more sophisticated statistical techniques available than those demonstrated in the articles which have proposed Bayesian methods for use in auditing. Such techniques, such as the beta prior distribution discussed here, may allow auditors to benefit from the advantages of Bayesian statistics that have been claimed in the literature.

THE APPLICATION OF BAYESIAN STATISTICS TO AUDITING: DISCRETE VERSUS CONTINUOUS PRIOR DISTRIBUTIONS

By

Albert Kenning Francisco

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Accounting and
Financial Administration

1972

com

Meac

and

expr

Jone

supp

to p

Denv

in w

prov

orga

tion

Nati

for

nati

for

ACKNOWLEDGMENTS

Thanks are due the members of my dissertation committee, Professors Alvin Arens (chairman), George Mead, and Robert Staudte for their review of my thesis and constructive suggestions. Appreciation is also expressed to Professors James Don Edwards and Gardner Jones who, as department chairmen, ensured financial support during my Ph.D. program. I am also indebted to Professor James E. Sorensen of the University of Denver for arousing my interest in the general area in which the dissertation was done.

Appreciation is due the organizations which provided financial support for my Ph.D. program. These organizations include the American Accounting Association, Ernst & Ernst, and the U. S. Government for its National Defense Education Act fellowship.

Finally, appreciation is expressed to my parents, for encouraging me to remain in school when other alternatives arose, and to my wife Barbara and daughter Eileen for help and encouragement throughout my graduate studies.

I w

of

I would like to thank Mrs. Donna Ford for an excellent job of typing the final manuscript.

AC

LIS

LIS

Cha

I

II

TABLE OF CONTENTS

Ch

VI

VII

A

SE

APPENDIX Page

LIST OF TABLES

LIST OF FIGURES

A.

tion

tion

orga

enti

comm

econ

abil

oppos

or ur

the e

quent

becom

value

repor

examp

tal b

CHAPTER I

INTRODUCTION

A. The Auditor in Society

The purpose of accounting is to provide information useful for decision making which affects the allocation of resources. Such decisions occur at all levels of organized society from the individual level to that of entire political entities.[1] The reliability of accounting communication is essential to the effective conduct of economic exchange and taxation. A high degree of acceptability of accounting communication is needed between the opposite parties in such transactions. Either intentional or unintentional errors could retard the functioning of the entire system.

The more separated the parties (or the more infrequent the contact between the parties) the more the reports become susceptible to inaccuracy and misunderstanding. The value of an independent third party reviewing the accounting report and vouching for its reliability is apparent. For example, investors are separated from the use of their capital by management. Thus, the financial statements of most

pub
cer
as
ado

in
aud
vid
inc
org
Off
bus
grou
and

howe
upon
this
opin
ments
regul
true
accor
assur

publicly owned corporations are audited by independent certified public accountants (CPAs). Audits are imposed as a matter of governmental regulations or can be voluntarily adopted by the parties concerned.[2]

Audits are performed by a variety of organizations in the United States. Almost everyone has heard of the tax audit, performed by Internal Revenue Service agents on individual or business records to substantiate claims made on income tax returns. The Congress of the United States has organized its own auditing agency, the General Accounting Office, to report on the spending of public monies. Many business organizations have their own internal auditing group which continually checks upon the accuracy of records and upon compliance with managerial policies.

The most common type of audit in the business world, however, is performed by an independent firm of accountants upon the financial statements of a business. The goal of this type of audit is for the auditing firm to render an opinion upon the fairness with which the financial statements issued to stockholders, creditors, various government regulatory agencies, and the public at large reflect the true condition of the business entity. This written opinion accompanies the statements when they are issued and provides assurance that statement users are entitled to rely upon the sta

the

B.

eve

rec

tod

aud

This

an a

not

incr

be f

a pa

ance

by a

item

and

majo

tor

sophi

refir

statements to a greater extent than if management had issued them with no outside check upon their accuracy.

B. Audit Sampling

At one time, an audit involved the examination of every transaction entered into by the firm and the resulting record in the accounts.[3] Because of the increased size of today's business entities and the number of transactions, auditors now restrict their tests to a sample of the records. This reduces the time required to perform an audit and gives an acceptable degree of reliability that major errors will not appear in the statements. Any sampling plan, however, increases the risk that a material error will occur and not be found by the auditor; since he selects and examines only a part of the available evidence. This risk must be balanced against the advantages of sampling.

Until rather recently, auditors in general sampled by a "seat of their pants" approach and took a sample of items from some population on the basis of past experience and intuition. This method worked reasonably well in the majority of cases, and the judgment of an experienced auditor is an indefinable quality which can not be replaced by sophisticated mathematics. However, it can be aided and refined by certain tools, as it has been recently with the

lin
too

aud
hap

ite
che
und
2,0
ite
seq
inde
usi
samp
been
With
cifi
even
on t

audi
the
etc.,

limited adoption of classical statistics as an auditing tool.

Several methods are used in practice to obtain audit samples. Some of these methods are the block, month, haphazard, random, and statistical methods of sampling.

A block sample is obtained by selecting a group of items listed together in a sequence. For example, out of checks numbered from 1 to 10,000 written during the year under examination, the auditor may examine those numbered 2,083 through 2,336. Because the selection of one sample item is more likely to occur if another close to it in the sequence is also drawn, the selections of items are not independent of one another and inferences can not be made using statistical theory on the basis of such a non-random sample. This is true even though a random method may have been used to select the particular block of items examined. With some filing systems it is so difficult to locate specific items that the use of block sampling is justified, even though statistically based inferences can not be made on the results.

A relatively common practice, especially on smaller audits, has been for the auditor to select one month out of the year and then to carefully examine entries, vouchers, etc., for that month. Similar items relating to the rest of the year are not examined. As was the case in block sampling, statistical inferences can not be made on the basis of such a procedure. The simplicity and convenience of this method make it useful where specific documents are filed by month instead of by serial number.

Another method used consists of the auditor selecting a few items here and a few there, then making his decisions on that basis. Appropriately, this method is known as the haphazard method for obtaining audit samples. As with the other methods described above, statistical inferences can not be made because the selection of one item is not statistically independent of the selection of other items.

All three methods described above are non-random selection methods, called judgmental methods. Any inferences made on the basis of these methods are based solely on the judgment of the individual auditor.

Statistically valid inferences are, in general, made only on the basis of a truly random sample. A random sample plan is any method of sampling in which the probability of a given item being selected is not affected by another item being selected. Samples can be either with replacement (a given item can be drawn more than once) or without replacement (a given item can not be drawn again once it has been chosen). Different statistical theories have been derived to deal with these two cases.

C. Classical Statistics

If a random sample is drawn, inferences can be made on the basis of statistical theory. There are many models for different situations in classical statistics, but all share one common trait--probabilities can be mathematically derived from a random sample result. These probabilities, given a specific population and sample therefrom, should be the same no matter which statistician computes them. Judgmental inference, on the other hand, provides no means for different auditors to come up with the same answers in complex situations. Not only are statistical methods more consistent from one user to the next, but a body of mathematical theory is available to support statistical results. No similar theory backs the user of judgmental methods.

In the last several years classical statistics has become popular as an auditing tool. The larger CPA firms engaged in auditing have statistical sampling worksheets, tables, etc., which use classical statistical methods to arrive at inferences based on audit samples. This increased popularity is probably due to several factors. Among them are the inherent applicability of classical statistics to many audit sampling situations, the better education in st

an

In

us

ma

th

in

D.

but

cur

tio

mat

Bay

som

sou

tic

so

for

of

able

statistics of recent college graduates entering the firms, and the leadership of various groups such as the American Institute of Certified Public Accountants in promoting the use of new methods.

Recent research suggests that Bayesian statistics may be even more applicable to certain audit situations than classical statistics. Bayesian methods are discussed in the following section.

D. Bayesian Statistics

Bayesian statistics is similar to classical methods, but uses a prior probability distribution as well as the current sample result. The combination of the two distributions is called a posterior distribution and is a sort of mathematical average of the two obtained with the use of Bayes theorem.

Bayesian methods recognize the subjective nature of some types of audit evidence. Instead of ignoring these sources as classical statistics models do, Bayesian statistics attempts to accept them as part of the analysis. In so doing, the auditor, based on his past experience, is forced to put down on paper what he expects of various parts of the audit. This can provide more evidence that a suitable audit will be conducted than do alternative methods

wh
th
as
do
ac

E.

ly
to
Cla
lim
pra
whi
and

amou
dete
meth
used
Beca
II i
vari

which ignore these sources of information altogether. With the increasing frequency of lawsuits naming auditing firms as defendents for not living up to expectations in the work done, adequate defense is more and more important to the accountant engaged in auditing.

E. Purpose of this Study

The purpose of this study is to explore more deeply the Bayesian method, which has been proposed and shown to be of at least limited applicability in auditing.[4] Classical statistics, as useful as it is, has certain limitations, which will be explored. In addition, certain practical difficulties inherent in the Bayesian methods which have been proposed in the literature will be discussed and an alternative proposed.

Any audit involves the gathering of evidence. The amount and kinds of evidence which are obtained should be determined by the audit variables. Whether judgmental methods, classical statistics, or Bayesian statistics is used, it is necessary that the audit variables be considered. Because the subject is so important to any audit, Chapter II is devoted to a discussion of audit evidence and audit variables.

ye
ha
ti
the
the
cla
a d
ed

sug
Exa
pri
sit
geor
tion
betw
Chap
and
dist
ence

stat

Auditors have used classical statistics for several years. It has proven to be a valuable audit tool. But it has certain limitations, which advocates of Bayesian statistics have claimed their method can overcome. In addition, the method which auditors call classical statistics is not the method used by statisticians who consider themselves classicists. A discussion of this distinction, as well as a description of "classical auditing" statistics is presented in Chapter III.

The accounting literature has included several suggestions that Bayesian statistics be used by auditors. Examples have shown how Bayesian statistics with discrete prior distributions can be applied to selected auditing situations. Although both the binomial and the hypergeometric distributions have been used as prior distributions in those examples, no explanation of the difference between the two or the advantages of either has appeared. Chapter IV of this study shows how the discrete binomial and hypergeometric distributions can be used as prior distributions in auditing situations and explains the difference between the two.

Although continuous distributions are used in many statistical applications, such distributions have not appeared in the literature as prior distributions in applications of Bayesian statistics to auditing. One continuous distribution which has potential in those applications is the beta. To be used in auditing, however, it is necessary to have tables and graphs of the beta. The use of the beta as a prior distribution in auditing, as well as the development of the necessary tables and graphs, is discussed in Chapter V.

The application of Bayesian statistics to auditing, with either discrete or continuous prior distributions, requires that the relationship between the audit variables and prior distributions be considered. Additionally, an auditor attempting to use Bayesian methods needs information about subjective probabilities and the criticism which the use of such probabilities has brought from statisticians who feel statistical methods should not be applied to situations which can not be repeated. Others have held that improved decision making capability is sufficient justification for the use of subjective probabilities. These topics are explored in Chapter VI of this study.

F. Limitations of the Study

In audit variables sampling, classical statistics gives almost the same results as those obtained through the use of Bayesian statistics. This is because the prior distribution provided by the auditor in variables sampling situations has a large variance. Internal control evaluations, work on other accounts, and prior audit experience gives little evidence to indicate whether the cash account should be $100,000 or $150,000, for example. In attributes sampling, on the other hand, these sources of information do give the auditor a good basis from which to predict likely error rates, so that the prior distribution provided by the auditor can contribute significant information. For this reason, the study is concerned only with attributes sampling.

A Bayesian method using a continuous prior distribution is proposed in this study. Illustrative tables and sketches, not complete enough for general use, are shown. Extended tables and sketches would require computer time and other resources beyond those available.

The general subject of Bayesian statistics in auditing includes many subsets which have not yet been researched. Although the exploration of most of these areas is beyond the scope of this study, some are mentioned in the last section of Chapter VII.

In
Wa

Pr

ca
19

CHAPTER I--FOOTNOTES

[1]The Study Group on Introductory Accounting, A New Introduction To Accounting (Seattle, Washington: Price Waterhouse Foundation, 1971), p. 11.

[2]Ibid., p. 18.

[3]Bookkeepers' Handy Guide, (New York: The Ronald Press Company, 1936), p. 518.

[4]For example, see John A. Tracy, "Bayesian Statistical Methods in Auditing, The Accounting Review, January, 1969, p. 90.

CHAPTER II

AUDIT EVIDENCE AND AUDIT VARIABLES

A. Audit Evidence

"The objective of the ordinary examination of financial statements by the independent auditor is the expression of an opinion on the fairness with which they present financial position and results of operations."[1] The auditor's opinion, however, must be based upon evidence obtained by the auditor in the course of his examination.

> Sufficient competent evidential matter is to be obtained through inspection, observation, inquires and confirmations to afford a reasonable basis for an opinion regarding the financial statements under examination.[2]

Audit evidence is a general term which includes all those factors in a given audit upon which the opinion is based. In addition to written material, such as reconciliations and confirmations, the term evidence covers such intangible factors as the competence of personnel in the client's accounting system.

The auditing profession provides few specific guidelines for auditors to use in their determination of what to require as evidence or how much evidence is necessary in a given situation.

> The amount and kinds of evidential matter required to support an informed opinion are matters for the auditor to determine in the exercise of his professional judgment after a careful study of the circumstances in a particular case.[3]

There are many kinds of audit evidence, some more important to the auditor than others. Mautz and Sharaf categorized audit evidence into nine general types in _The Philosophy of Auditing_:

1. Physical examination by the auditor of the thing represented in the accounts
2. Statements by independent third parties
   - written
   - oral
3. Authoritative documents
   - prepared outside the enterprise under examination
   - prepared inside the enterprise under examination
4. Statements by officers and employees of the company under examination
   - formal
   - informal
5. Calculations performed by the auditor
6. Satisfactory internal control procedures

7. Subsequent actions by the company under examination and others

8. Subsidiary or detailed records with no significant indications of irregularity

9. Interrelationships with other data[4]

The overall result of combining the individual items of evidence is used as the basis for the auditor's opinion. This involves an evaluation of the evidence gathered in light of the materiality of the several accounting propositions which the opinion is to cover. In this evaluation, several factors, or "audit variables", are considered by the auditor in determining whether sufficient evidence has been accumulated, as outlined in the following sections.

B. Audit Variables

An audit is an investigation, the extent of which depends upon specific circumstances encountered. Even under the most ideal conditions a certain minimum amount of evidence must be gathered. For any given audit there will be a minimum audit program which will be applied if all circumstances and the results of all tests meet the auditor's best expectations.[5]

> The variables of the audit are those factors or circumstances of a particular audit situation which should affect an auditor's decision on which procedures to apply and the extent of their application.[6]

Some audit variables determine the minimum audit program, while others result in procedures beyond the minimum audit program.

The variety of situations encountered by an auditor is quite large. To set forth here some of the more important variables which the auditor may encounter, let us summarize briefly two recent items from accounting literature. Extended discussions about audit variables appear in a 1970 article by Anderson, Giese, and Booker,[7] and in a 1970 thesis by Arens.[8]

Audit variables may be broken down into several catagories. Here three general categories will be considered: 1) the auditor's environment, 2) the client's environment, 3) internal factors peculiar to that client.

The auditor's environment includes several factors which affect the evidence required for an audit. The relative sizes of the auditing firm and the client are an important variable.[9] If the auditing fee paid by that one client is a substantial portion of the auditing firm's billings, questions may arise as to the auditor's appearance of independence. In such circumstances the auditor should be extremely cautious to avoid arousing suspicion about his work and therefore should take relatively large samples.

Another variable from the auditor's environment is the auditor's background. If he has had excellent experience with this client and similar clients, he may be willing to reduce sample sizes somewhat because his risk appears to be lower than with other clients. In addition, previous years' results of specific audit tests are important to the auditor, when the results are available. If tests in the area have produced few errors in past years, the auditor will have some confidence that the area will again produce few errors.

Another category of audit variables relates to the client's environment. The absolute size of the client's organization is of interest to the auditor.[10] The auditor of a small client which has little public exposure faces less risk than the auditor of a larger client. This is simply because there are fewer parties that can be damaged. In the audit of a smaller client the only directly interested party may be a banker, who knows nearly as much about the client's affairs as does the auditor. Consequently, the auditor is less likely to have his work attacked. Another variable of interest to the auditor is the client's industry.[11] Some industries are noted for steady growth and others for wide fluctuations in sales and earnings and resulting management difficulties. An auditor would require more evidence in the audit of a client in an unstable industry than for a similar client in a stable one.[12] Another variable is the type of legal form used by the client. Different legal obligations are placed upon the auditor depending on the client's legal form, e.g. a corporation or a partnership. If the client is listed on a major stock exchange, requirements are often imposed which may increase the risk assumed by the auditor. Similarly, if securities issued by the client are subject to the registration requirements of the Securities and Exchange Commission, special rules are imposed by law upon the auditor. The mode of financing chosen by the client is a variable of interest to the auditor, as is the profitability of the client.[13] Finally, general economic conditions can change the auditor's risk, and these should be considered. A sharp downturn in the economy can have adverse effects upon many organizations, and difficulties caused by such a slump can result in close scrutiny of the auditor's work.

The final category of audit variables is composed of items from within the client's organization. Internal auditing results are often an important variable to the outside auditor, as are other items related to internal control.[14] Another factor is the applicability of the client's records to auditing and the condition of those records. Each test made in the course of an audit has variables associated with it, e.g. the population size and the materiality of the item subject to the test.[15]

Many more variables could be listed. These should, however, be sufficient for the purposes at hand. In an audit the auditor weighs all these and more factors in his mind and then produces an audit program, complete with sample sizes and specific items to be examined. In the past much of this process has been based upon arbitrary judgments and previous practice. Audit variables should have an effect upon the audit, whether judgmental or statistical methods are used. This effect is discussed in the following section.

C. Effect of Variables on Audit

When the auditor discovers variables which indicate that more than the minimum audit program is necessary, there are several routes he can take. If he feels the tests are adequate for the circumstances, he can increase sample sizes, thus increasing the probability that the sample result will reflect the actual situation. In some audits he may wish to change the procedures performed in order to anticipate a greater variety of possible discrepancies. When the situation is highly abnormal, new personnel may be assigned to the audit to bring in skills especially applicable to the areas of concern. Finally, the timing of the tests may be changed.

These actions are, however, not without cost to the auditor. Increased sample sizes, additional procedures, timing variations, and more experienced personnel are all actions which increase the cost of an audit. Situations exist where all or a good portion of these costs can be passed on to the client. In other cases, the auditor himself must absorb them. This limits the extent to which the auditor can increase his costs without incurring losses.

Once the auditor understands the variables in a particular engagement, he must have an algorithm to determine the evidence required. Because this study is about statistical sampling in auditing, it is concerned only with the size of the samples taken. The specific procedures used in the audit, the timing of the tests, and the personnel are important items, but are not within the scope of this study. However, the audit variables do influence the selection of the population, the definition of the attribute to be tested, and the precision limits and confidence level deemed necessary by the auditor. In the following chapters the variables will be considered in the context of Bayesian and classical statistics.

CHAPTER II--FOOTNOTES

[1]Committee on Auditing Procedure of the American Institute of Certified Public Accountants, Auditing Standards and Procedures (New York: American Institute of Certified Public Accountants, 1963), p. 9.

[2]Ibid., p. 16.

[3]Ibid., p. 36.

[4]R. K. Mautz and Hussein A. Sharaf. The Philosophy of Auditing (American Accounting Association, 1961), p. 86.

[5]R. K. Mautz and Donald L. Mini, "Internal Control Evaluation and Audit Program Modification," The Accounting Review, April, 1966, p 284.

[6]Alvin A. Arens, The Adequacy of Audit Evidence Accumulation in Public Accounting (unpublished Ph.D. thesis, University of Minnesota, 1970), p. 17.

[7]H. M. Anderson, J. W. Giese and Jon Booker, "Some Propositions About Auditing," The Accounting Review, July, 1970, p. 524.

[8]Arens, p. 23-76.

[9]Anderson, Giese and Booker, p. 528.

[10]Ibid., p. 528.

[11]Arens, p. 41.

[12]Anderson, Giese and Booker, p. 529.

[13]Ibid., p. 529.

[14]Ibid., p. 528.

[15]Arens, p. 47.

A. In

bution

random

one or

ment.[1]

"A Ber

outcom

study

ing re

exampl

one au

bank c

cash a

passes

variabl

CHAPTER III

ATTRIBUTES SAMPLING AND CLASSICAL STATISTICS

A. Introduction

Statistics involves the use of probability distributions to make inferences about some population using random samples taken from that population. The taking of one or more samples from a population is called an experiment.[1] One execution of an experiment is called a trial. "A Bernoulli trial is an experiment which has two possible outcomes, generally called success and failure."[2] In this study a trial will either produce an "error" in the accounting records, or will not produce an error. There are many examples of Bernoulli trials in auditing: the result of one audit is either an unqualified opinion or it is not; a bank confirmation either reconciles satisfactorily with the cash account or fails to; an accounting student either passes a course or fails it.

When one or more trials of a Bernoulli random variable occur, the probability distribution of the number of errors found in those trials in called a binomial. The probability of any possible result occurring can be computed if two things are known

n the number of trials

p the probability of success on any one trial.

The computation is made with the following formula in which x is the number of errors for which the probability is being found:

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

where $\binom{n}{x} = \frac{n!}{(n-x)!x!}$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots\cdots (2) \cdot (1)$$

$$q = 1-p$$

Because there are tables and approximation methods available, the formula can be bypassed. It is significant with the binomial that the probability of a success must be constant from one trial to the next, implying either replacement of each item sampled in the population before another is taken or an infinitely large population.

The hypergeometric distribution is similar to the binomial except that it is drawn from a population of finite size, resulting in a changing probability of success *after* each trial. Computations are somewhat more involved than

Assum

which

follo

the s

For s

geome

proba

is re

more

to ap

stati

in th

B. C

audit

dence

some

a samp

than with the binomial because of this added variable. Assume the existance of a finite population of m items of which w are successes. If a sample of size n is drawn the following formula gives the probability of z successes in the sample:

$$p(z) = \frac{\binom{w}{k}\binom{m-w}{n-k}}{\binom{m}{n}}, \quad k = 0,1,2,\ldots\ldots n$$

........using the convention $\binom{b}{a}=0$ for $a>b$.[3]

For small samples taken from large populations the hypergeometric closely approximates the binomial because the probability of a success changes very little when one item is removed from the large population. Because tables are more readily available for the binomial, it is often used to approximate the hypergeometric for large populations.

Statisticians and auditors have used classical statistics in different manners. The distinction is covered in the following section.

B. Classical Statistics as Used by Auditors

One way in which classical statistics is used in auditing is the creation of confidence intervals. Confidence intervals are ranges within which the actual value of some variable is likely to be. For example, the result of a sample may allow an auditor to state that he is 80%

conf

subj

reser

fider

to st

0.05%

sari

fider

abou

resu

fixe

of i

an e

the

sampl

ical

mator

90% c

be cor

Class

lies i

confident that the rate of errors made on payroll checks subject to audit is less than 0.4%. Such a statement represents a one sided confidence interval. A two sided confidence interval would result if a sample allowed an auditor to state that the error rate on payroll checks was between 0.05% and 0.10% with 80% certainty.

Auditors using classical statistics do not necessarily understand the meaning of such statements about confidence intervals. Classical statistics makes statements about an unknown population parameter on the basis of sample results. The unknown population parameter is considered fixed.[4] Suppose an auditor takes a sample from a population of items. Each sample item taken is either an error or not an error. The auditor will want a confidence interval about the percentage of errors (p) in the population. If the sample consists of 100 items, 20 of which are errors, classical statistics would say that the maximum likelihood estimator of the population error rate is $\hat{p} = 0.20$. A two sided 90% confidence interval for the population error rate would be computed in the following manner:

$$\text{Confidence Limits} = \hat{p} \pm z_{5\%}\sqrt{\frac{pq}{n}} = 0.20 \pm 1.645(0.04)$$
$$= 0.20 \pm 0.0658.$$

Classical statistics would interpret this to say that p lies in the interval 0.1342 to 0.2658 with confidence coefficient 0.90.[5] This means that if a very large number of samples were taken from this same population and a confidence interval constructed from each in the above manner, 90% of those intervals would contain the actual population error rate p.

This is not the manner in which auditors have typically used classical statistics. They have gone through the computations shown above and then made probabilistic statements about the population parameter, as though it were the random variable instead of the sample result. This is shown by statements made about the population parameter, such as the following from Arkin.[6]

> ....if a random sample of 100 contains 50% black balls, the universe from which it was drawn probably (99% probability) would not have contained less than about 37.1% or more than 62.9% black balls....

This statement was made about a large binomial population, from which a sample was found to contain 50% black balls. It is clear that the person making this statement considers the population parameter p a variable, while the sample parameter $\hat{p}$ is fixed at 0.50. The following equation can be constructed from Arkin's statement:

$$\text{Probability } (.371 < p < .629 \mid \hat{p}\ 0.5) = 0.99.$$

The <u>only</u> variable in this equation is the population parameter p. Classical statistics would not interpret the sample result in this manner, because it implies that the population parameter p is not fixed, rather, that it was drawn from some large number of possible populations. These populations would have some distribution of p among themselves. This distribution would be the prior. Thus, the "classical auditing statistics" is not the classical stattics, but is some combination of it and Bayesian statistics. In this study the term "classical statistics" will refer to the way such methods have been applied to auditing.

The following example illustrates the application of classical statistics to auditing. Assume that an auditor takes a sample of 169 from a large population and observes 55 errors in the sample. He would compute

$$\sigma\% = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{(\frac{55}{169})(\frac{114}{169})}{169}} = 0.0361$$

A one-sided confidence interval would be computed from the sample mean ($\bar{X}$) of 55/169, or 32.6%, and the product of $\sigma\%$ and a factor taken from a table of the normal distribution. If the auditor desired a 97.8% confidence level, this normal factor would be 2.03.[7]

$$\begin{aligned} \text{Confidence Limit} &= \bar{X} + 2.03(\sigma\%) \\ &= 0.326 + 2.03(0.0361) \\ &\doteq 0.326 + 0.0735 \\ &= 0.400. \end{aligned}$$

Thus, the auditor using classical statistical methods would have a 97.8% confidence level that the error rate in the population would be less than 40%. If his maximum acceptable error rate is 40% or greater, he will accept the sample results and go no further. On the other hand, if his maximum acceptable error rate is smaller than 40%, he can either reject the entire population as containing such a large percentage of errors as to be useless, or expand his sample by taking additional sample units in an attempt to produce an acceptable sample result. This decision depends upon the audit variables. The relationship between classical statistics and audit variables is treated in the following section.

C. Effect of Audit Variables

As discussed in Chapter II, audit variables are the relationships which exist between circumstances encountered by the auditor and the appropriate sample sizes. If the circumstances are good (excellent internal control, few errors in the records, proper valuation methods, low risk of the client having difficulty in meeting the claims of its creditors, etc.), the auditor is able to form an unqualified opinion with relatively small sample sizes. This is due to few problem areas being encountered in the audit and the low risk of sanctions being applied to the auditor as a result of that engagement. In a less favorable situation, sample sizes must be increased in order that the auditor can enjoy a similar degree of risk, because more questionable items are likely to arise during the audit. These warn the auditor that he must be more prepared to defend an unqualified opinion rendered as a result of that engagement.

In the classical statistics model, the auditor can keep the same small sample size under worsening audit variables only by increasing the error rate he is willing to accept or by decreasing the confidence level he has that the actual error rate is within bounds acceptable to him. Classical statistics allows the auditor to consider the audit variables in calculating confidence intervals, but only indirectly. One method of incorporating the variables is to change the degree of confidence required. For example, assume a test in the audit of one client has produced no errors in each of the last six audits. The same test in another audit has produced high error rates in every one of the last six years. The auditor may require an 80% confidence level in the "errorless" engagement, and a 90% level in the other. As a result of a change in one of the audit variables, the auditor wishes

to have a greater level of certainty in the more risky engagement. If the auditor does not change the confidence level, but instead decides to allow an increased acceptable error rate, the risk of sanctions being imposed upon him increases.

One audit variable which the classical statistics model does explicitly consider is the size of the population from which the sample is taken. On page 29, an example shows how an auditor might construct a confidence interval using classical statistical methods. In that example, a large population (meaning that the population is considerably larger than the sample size) was assumed. If this had not been the case, it would have been necessary to correct the result with a "finite correction factor." The formula for this factor is

$$\sqrt{\frac{N - n}{N - 1}}$$

where the population size is referred to as "N" and the sample size as "n". It should be noted that this factor is very close to 1.00 when N is large relative to n, resulting in the correction factor being passed by, as in the example mentioned above. If n is a relatively large part of the population, this factor must be referred to, however. It is used by multiplying it by the product of the value obtained from the normal table and $\sigma_{\%}$ in the confidence limit computation.

In summary, classical statistics does not provide a way in which subjective information can be directly inserted into the model. Adjustments can be made to sample size by way of the acceptable error rate and the confidence level, but it is not a simple task to determine which of these to change and by how much it should be changed to reflect a given change in the audit variables. This difficulty is illustrated by the prevalence of 90%, 95%, and 99% confidence levels in academic studies of widely varying subjects, to the almost total exclusion of other confidence levels.[8]

In contrast, Bayesian statistics directly incorporates this subjective information, as set forth in the following chapters.

CHAPTER III--FOOTNOTES

[1]Harold J. Larson, Introduction to Probability Theory and Statistical Inference (John Wiley and Sons, 1969), p. 17.

[2]Ibid., p. 106.

[3]Ibid., p. 116.

[4]William Mendenhall, Introduction to Probability and Statistics (Wadsworth, 1967), p. 151.

[5]Ibid., p. 163, 4.

[6]Herbert Arkin, Handbook of Sampling For Auditing and Accounting, Volume I (McGraw-Hill, 1963), p. 74.

[7]See any introductory statistics text, under "normal probability distribution."

[8]William H. Kraft Jr., "Statistical Sampling for Auditors: A New Look", Journal of Accountancy, August, 1968, p. 49.

CHAPTER IV

ATTRIBUTES SAMPLING USING BAYESIAN STATISTICS: DISCRETE PRIOR DISTRIBUTIONS

A. Bayes Theorem

Bayes theorem is the basis of a method which allows inferences to be made about a population on the basis of more than one probability distribution derived from that population. Classical statistics, discussed in the previous chapter, does not allow more than one input distribution in an analysis. Bayes theorem is named for the Reverend Thomas Bayes, one of the early writers on probability theory. Recently his theorem has been applied to a wide variety of situations and is especially important in many branches of applied statistics.[1]

Bayes theorem itself is a method for combining two probability distributions. The first distribution is called a *prior*: the second is derived from the results of a *current sample*, and the result of the combination is known as the *posterior* distribution. Conceptually the method is not difficult, but in practice the decision as to which distributions to combine and the meaning of the result present difficulties.

In the discrete form, Bayes theorem takes the following structure. "Suppose that we are given k events $A_1, A_2, \ldots\ldots, A_k$ such that:

1. $A_1 \cup A_2 \cup \ldots\ldots \cup A_k = S$
2. $A_i \cap A_j = \emptyset$, for all $i \neq j$

(these events form a partition of S): then for any event $E \subset S$,

$$P(A_j|E) \quad \frac{P(A_j)P(E|A_j)}{\sum_{i=1}^{k} P(A_i)P(E|A_i)}, \quad j = 1,2,\ldots\ldots,k."^2$$

The notation above may be read $\cup$ (union), $\cap$ (intersection), $\neq$ (not equal), S (the entire possible sample space), $\emptyset$ (the empty set--no possible result), $|$ (given), $\sum$ (summation from the point under the symbol by intervals of one to the point above the symbol), and $\subset$ (an element of).

To use Bayesian statistics in an auditing attributes sampling situation, several steps must be completed. First, a suitable prior distribution must be provided which expresses the expectations of the auditor regarding a particular sample outcome. Secondly, an actual sample must be taken from the population and the results (sample size and number of errors found in the sample) recorded. That information is used with Bayes formula to revise the prior distribution and obtain the posterior distribution. Finally, it is necessary that the posterior distribution be used to create a confidence interval in a manner similar to that used with the normal distribution in classical statistics.

The prior distribution provided by the auditor can be either a discrete or a continuous distribution. The remainder of this chapter discusses the discrete case, while a continuous prior distribution is the subject of Chapter V.

B. Discrete Prior Distributions

The following examples show how Bayesian statistics might treat one auditing situation with the use of two discrete models that have been proposed in the literature. The auditor would have to provide information about the expected population error rate under consideration before the sample was taken. This is in contrast to the classical approach, which would base the entire analysis on the results of the sample.

Suppose the auditor felt, based on the results of other samples taken during the current audit and on previous years' results, that only an 80% population error rate (A) or a 20% rate (B) were possible and he felt that A was twice as likely to occur as B. Since the total probability (that of A and B) must equal 1.00, the auditor must believe that

$$P(A) = 2/3$$

$$P(B) = 1/3.$$

The Bayesian statistician could use the formula for the binomial distribution which was discussed in Chapter 3.

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

If a sample of 3 were taken and no errors were found then

$$P(x=0|A) = \binom{3}{0}(0.8)^0(0.2)^3 = 0.008$$

$$P(x=0|B) = \binom{3}{0}(0.2)^0(0.8)^3 = 0.512$$

would be calculated. Using Bayes theorem, the following computations would be made

$$P(A|X=0) = \frac{P(X=0|A)P(A)}{P(X=0|A)P(A)+P(X=0|B)P(B)}$$

$$= \frac{.008(2/3)}{.008(2/3)+.512(1/3)} = 0.03$$

$$P(B|X=0) = \frac{P(X=0|B)P(B)}{P(X=0|A)P(A)+P(X=0|B)P(B)}$$

$$= \frac{.512(1/3)}{.008(2/3)+.512(1/3)} = 0.97$$

He would check his results, knowing that the total probability of A or B must equal 1.00.

$$P(A|X=0) + P(B|X=0) = 0.03 + 0.97 = 1.00$$

The auditor is 97% certain that the population error rate equals 0.20. This example is, of course, highly simplified, but techniques are discussed in the following sections which are capable of incorporating much more complex distributions that might better reflect reality

The Bayesian approach generally requires a smaller sample size to reach the same confidence level as the classical approach. This is not free information, however. The auditor must have some reason for the prior distribution he provides, whether it is the result of his internal control inquiries, other audit tests, or previous years' audit results. Bayesian statistics simply provides a method for formalizing the information available. Chapter VI discusses the problems inherent in deriving a realistic prior distribution.

In a 1968 Journal of Accountancy article,[3] William H. Kraft, Jr., proposed using Bayesian statistics to solve a problem similar to the preceding. He detailed a method where the prior distribution (with six possible error rates) can be specified from information the auditor has gained previously. The binomial formula, used in the preceding example and in Kraft's article, was meant for a situation in which the population size is infinite or in which each item sampled is replaced before another is taken. This implies a constant probability of obtaining a given result from one sample item selection to the next. More information can be obtained from the same size sample, however, if each item is not replaced as it is sampled, making the remaining population smaller with each successive item drawn, and thus increasing the probability that any one remaining item will be selected on one of the remaining draws. A more complicated formula is necessary to gain this advantage because the binomial does not consider the decrease in population size which occurs as sample items are removed.

The distribution which permits the use of this added information is known as the hypergeometric and was discussed with the binomial in Chapter III. In two 1969 articles, John A. Tracy[4,5] proposed the use of the hypergeometric distribution in auditing situations and gave examples which depicted results obtained using the hypergeometric with Bayesian statistics in a payroll check examination.

The following example depicts the same situation as that in the binomial example above, except that the population size is only 20. Using the hypergeometric formula from Chapter III,

$$P(X) = \frac{\binom{w}{k}\binom{m-w}{n-k}}{\binom{m}{n}} \quad \text{for } k = 0,1,2,\ldots.$$

the following results are obtained.

$$P(X=0|B) = \frac{\binom{4}{0}\binom{16}{3}}{\binom{20}{3}} = \frac{\frac{16!}{13!3!}}{\frac{20!}{17!3!}} = \frac{2\cdot 14}{3\cdot 19}$$

$$P(X=0|A) = \frac{\binom{16}{0}\binom{4}{3}}{\binom{20}{3}} = \frac{\frac{4!}{3!1!}}{\frac{20!}{17!3!}} = \frac{1}{15\cdot 19}$$

Using Bayes formula, the following results are obtained.

$$P(B|X=0) = \frac{P(X=0|B)\,P(B)}{P(X=0|A)\,P(A)+P(X=0|B)\,P(B)}$$

$$= \frac{\frac{2\cdot 14}{3\cdot 19}\left(\frac{2}{3}\right)}{\frac{2\cdot 14}{3\cdot 19}\left(\frac{2}{3}\right)+\frac{1}{15\cdot 19}\left(\frac{1}{3}\right)} = \frac{280}{281} = \underline{\underline{0.9964}}$$

$$P(A|X=0) = \frac{P(X=0|A)\,P(A)}{P(X=0|A)\,P(A)+P(X=0|B)\,P(B)}$$

$$= \frac{\frac{1}{15\cdot 19}\left(\frac{1}{3}\right)}{\frac{2\cdot 14}{3\cdot 19}\left(\frac{2}{3}\right)+\frac{1}{15\cdot 19}\left(\frac{1}{3}\right)} = \frac{1}{281} = \underline{\underline{0.0036}}$$

Since $P(A|X=0) + P(B|X=0) = 1.0000$, the results check.

It should be noted that the revised probability of B under the hypergeometric assumption and a population of 20 is 99.64%. This is considerably more than the similar probability of 97% under the binomial distribution and the infinite population size assumption. The greater probability obtained with the hypergeometric distribution is the

result of the additional information gained by sampling from a relatively small population without replacement of items selected for the sample. As the population size increases, less and less advantage is gained until finally the cost of using the more complex hypergeometric formula is greater than the value of the additional information gained. Thus, where the sample size is large relative to the population size and drawing is without replacement, the hypergeometric would be used. In situations where the population is much larger than the sample the binomial would be used whether or not sampling is with replacement.

In this section the use of Bayesian statistics and discrete prior distributions in auditing was discussed. Most of what has been written by others about the application of Bayesian statistics to auditing has relied upon discrete distributions. The exception is a thesis written at the University of Minnesota by Corless,[6] which is concerned with the ability of auditors to quantify expectations into both discrete and continuous prior distributions. However, it does not concern itself with the use that might be made of the distributions. Some implications of Corless' study are considered in Chapter V. Important practical difficulties found in the use of discrete prior distributions are discussed in the next section.

C. Difficulties Found in the Use of Discrete Prior Distributions

Discrete prior distributions have been the basis of Bayesian methods proposed to date for use in auditing situations. Several important difficulties are found, however, in practical applications of such methods. These difficulties include poor approximation of continuous phenomenon, inconvenience of deriving discrete prior distributions, and the complex computations which would require a computer in practical situations.

The phenomenon being represented by the prior distribution is continuous. Not only is the actual number of errors in the population unknown (or a sample would be unnecessary), but often the auditor does not know the precise number of items in the population being sampled. He is interested in rates instead of absolute numbers. In this situation, the actual error rate could take a vast number of specific values because both the numerator and denominator of the fraction which determines the rate are unknown. Any listing of a workable number of possible error rates making up a discrete distribution must necessarily be an approximation of the actual continuous distribution. Even a relatively large number of possible rates may be an inaccurate approximation because most of the probability could easily be concentrated between just two or three of the listed rates with probabilities close to zero assigned to the majority of the listed possible rates.

An auditor who wishes to use Bayesian statistics can approximate his feelings about a given attribute with a relatively large number of possible rates in such a fashion that the total of the probabilities is 1.00. This would give a discrete prior usable with Bayes theorem and the results of a current sample to produce a meaningful posterior distribution. Tracy used this approach in his <u>Accounting Review</u> article.[7] He took 100 possible error rates (0.002, 0.003, 0.004,....,0.099, 0.100, 0.101) and assigned a probability to each of these which reflected the auditor's prior thinking about that particular error rate. First he took an easy approach by assigning an equal probability to each of the 100 error rates so that the prior distribution included p(rate=0.002) = 0.01, p(rate= 0.003) = 0.01, p(rate=0.004) = 0.01, etc. Such a prior is called a uniform distribution. This uniform prior did not, however, contribute much information to the posterior distribution. The result was similar to a classical statistics analysis of the same problem. He then decreased the variance of the prior by lowering the probability associated with higher or lower error rates and by increasing the probabilities associated with error rates near 2.2% to 3.1%. As would be expected, this second prior had a greater effect upon the posterior than did the original. The difficulty with this approach is that listing 100 different probabilities is a considerable task. Furthermore, making certain they sum to 1.00 while simultaneously reflecting the information desired makes this method of deriving prior distributions so time consuming as to be impractical. It is important, also, to note that error rates other than the 100 listed are usually possible (anywhere between 0.02 and 0.03 for instance) but all of these possibilities are assigned a probability of zero with such a discrete prior distribution. This can only contribute error to the result.

To decrease the difficulty of listing 100 different probability figures, the number of points considered can be reduced. In the Kraft article[8] only six points were assigned probabilities. These represented error rates of 0.001, 0.01, 0.02, 0.03, 0.04, and 0.05. It would be a rare audit situation indeed which could be accurately represented by so few possible error rates, although the accuracy could be improved by increasing the number of points as Tracy did. By increasing the number of error rates assigned probabilities, it becomes increasingly obvious that this method of deriving prior distributions produces only an approximation to some continuous distribution which admits a range of possible rates instead of a finite number, in most practical situations. Statistical techniques exist which deal directly with continuous distributions instead of approximating them with various discrete points. One of these techniques is considered in the next chapter.

A final difficulty with the use of discrete distributions is that the revision of a discrete prior distribution by a discrete sample result requires a large number of multiplication and division operations, even for relatively simple problems. Most practical auditing problems would require a computer for solution. This is an important drawback, because computers and a means of translating information about the prior distribution and current sample result into a computer-readable form without error are not often available to the auditor in the field. Even with the availability of computers, discrete distributions would require significant input and processing costs.

The introduction of Bayesian statistics to auditors by the use of discrete prior distributions has been an excellent way to provide information about Bayesian statistics. Discrete distributions are easier to work with in simple examples when tables are not available than continuous distributions. But the practical limitations of discrete prior distributions in all but the most limited audit applications of Bayesian statistics are too important to ignore. Discrete prior distributions provide a poor approximation to continuous phenomenon, are inconvenient for auditors to derive, are difficult to perceive, and require involved computations when used with Bayes theorem. The use of Bayesian methods with discrete prior distributions which have been proposed in the auditing literature is not feasible in practice. An alternative, the use of Bayesian statistics with continuous prior distributions, is discussed in the following chapter.

CHAPTER IV--FOOTNOTES

[1]Harold J. Larson, _Introduction to Probability Theory and Statistical Inference_ (John Wiley & Sons, Inc., 1969), p. 46.

[2]Ibid., p. 47.

[3]William H. Kraft, Jr., "Statistical Sampling For Auditors: A New Look," _Journal of Accountancy_, August, 1968, p. 49.

[4]John A. Tracy, "Bayesian Statistical Methods in Auditing," _The Accounting Review_, January, 1969, p. 90.

[5]John A. Tracy, "Bayesian Statistical Confidence Intervals For Auditors," _Journal of Accountancy_, July, 1969, p. 41.

[6]John C. Corless, "The Assessment of Prior Distributions For Applying Bayesian Statistics in Auditing Situations," Unpublished Ph.D. Dissertation, University of Minnesota, 1971.

[7]Tracy, _The Accounting Review_, p. 90.

[8]Kraft, p. 49.

CHAPTER V

BAYESIAN STATISTICS AND THE BETA: A CONTINUOUS PRIOR DISTRIBUTION

A. The Beta Distribution

Continuous prior distributions, as well as the discrete ones discussed in the previous chapter, can be used with Bayesian statistics in auditing situations. In attempts to apply Bayesian statistics to auditing which have appeared in the literature to date, discrete prior distributions have been almost exclusively employed. However, many other fields have found continuous distributions or a mixture of the two more useful than discrete distributions alone. For example, the normal distribution is widely employed in natural science studies. The normal is not well suited to application as a prior distribution in auditing situations because the revision of a normal prior distribution by Bayes theorem can lead to widely varying posterior distributions which can be difficult to evaluate.

One continuous prior distribution which produces a more usable posterior distribution when revised by the results of an audit attributes sample is the beta. Mathematically, a beta function is one which relates two unknown variable terms by the use of three constants. The unknown variables will be referred to as "x" and as "p(x)". The value of p(x) is a function of the value of x. The three constants will be called "α" (alpha), "β" (beta), and "c". The relationship between these terms takes the form[1]

$$p(x) = cx^{\alpha-1}(1-x)^{\beta-1} \quad \text{(beta function)}.$$

A beta function can be graphed with x represented by the horizontal axis and p(x) represented by the vertical axis.

When a graph of a beta function is drawn in this study, it will be limited to the region between x=0 and x=1. Auditors are concerned with rates of occurrence in attributes sampling situations. In this discussion, these rates will be referred to as error rates to make visualization easier, although it is recognized that other rates may sometimes be of interest. Two facts about rates are of interest here. First, rates of occurrence such as those found in auditing situation are never negative. Secondly, such rates are never greater than 100%. The "x" from the discussion of the beta function above will represent an audit error rate. Therefore, it will be considered only

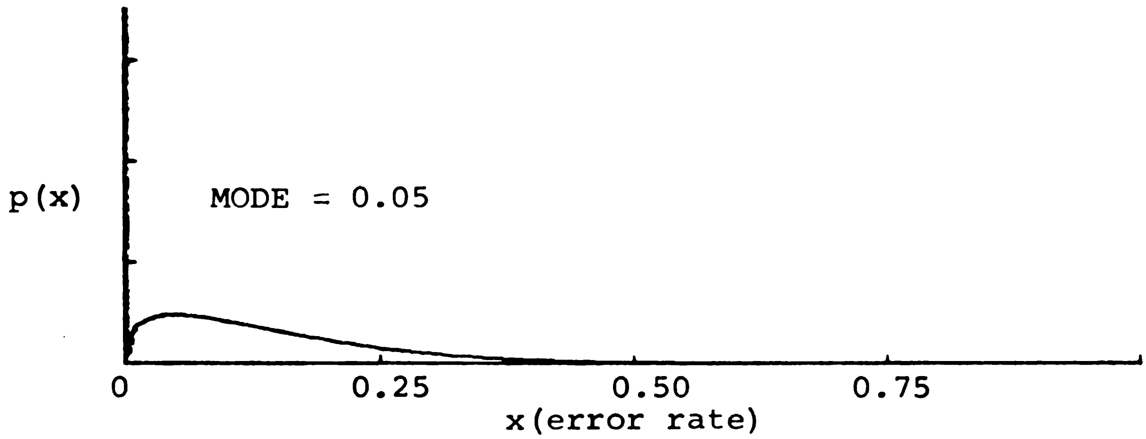

FIGURE 1. BETA DISTRIBUTION GRAPH $\alpha=1.40$ $\beta=8.60$

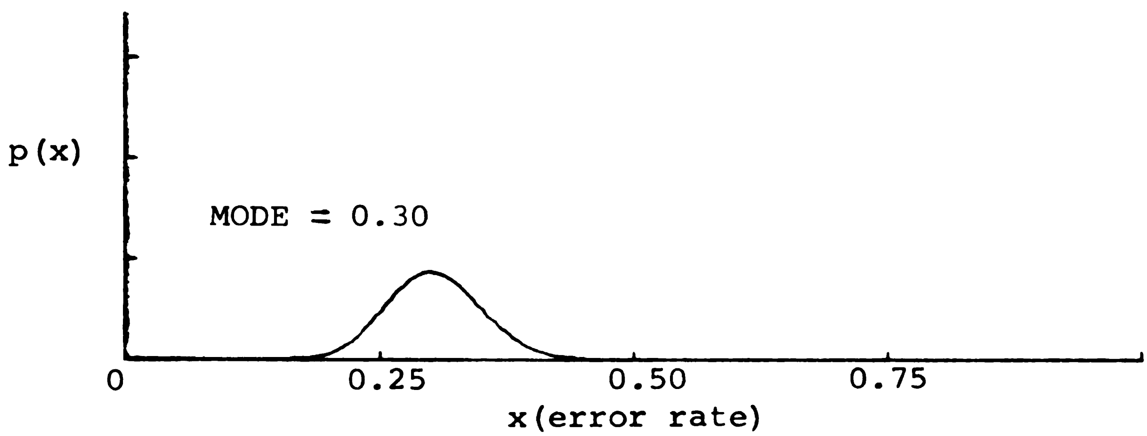

FIGURE 2. BETA DISTRIBUTION GRAPH $\alpha=30.40$ $\beta=69.60$

to the extent that its value lies between x=0 and x=1, inclusive.

The beta function is quite flexible, and by proper choice of the constants, α, β, and c, its graph can be made to assume a wide variety of shapes. The graphs of two beta distributions are shown in Figures 1 and 2.

Auditors will, in general, be interested only in beta distributions which have an "upside down bowl" shape such as those in Figures 1 and 2. It can be shown that any beta which exhibits such a shape will be the result of a function with constants α and β both greater than one.[2]

To make the beta function into a probability distribution function (such as a prior distribution for use by auditors), it is necessary to insure that the area between the curve and the x axis (see Figures 1 and 2) be made equal to 1.00. Once α and β have been chosen, this can be done by selecting the constant c in the beta function above so that this condition holds.

Under the conditions described, p(x) can be computed at any point x along the horizontal axis, and plotted on a graph. It will never have a negative value.

To illustrate the use of a beta function, an example will be considered with $\alpha=2$ and $\beta=8$. First, the constant c must be found so that the area between the curve and the x axis between x=0 and x=1 is equal to 1.00. The function takes the form

$$p(x) = cx^{2-1}(1-x)^{8-1} = cx(1-x)^{7}.$$

The area under a beta curve between x=0 and x=1 can be computed with gamma functions if both α and β are integers.

Since α and β are integers in this case, gamma functions will be used to compute the area under

$$p(x) = x(1-x)^7$$

and the reciprocal of the area used in place of c in the original formula to make the area between the curve and the x axis between x=0 and x=1 equal to 1.00. The value of a beta integral over x=0 to x=1 with α and β both integers can be computed from the following formula.[3]

$$\text{Area} = \frac{(\alpha-1)!\,(\beta-1)!}{(\alpha+\beta-1)!}$$

In the case at hand:

$$\text{Area} = \frac{(2-1)!\,(8-1)!}{(2+8-1)!} = \frac{1!\,7!}{9!} = \frac{1}{9.8} = \frac{1}{72}.$$

The reciprocal of this area will be substituted for c in the original formula. The value of c is therefore equal to 72 and the formula becomes:

$$p(x) = 72x(1-x)^7$$

A rough graph of that function can be drawn by computing several points from the formula, plotting them, and connecting the points. Such a graph for the function above is shown in Figure 3. The points used are shown below. Later in this chapter, it will be shown how the highest point on a beta curve can be computed. In this case that point occurs at $x = 1/8$. This information is useful in plotting the curve shown in Figure 3.

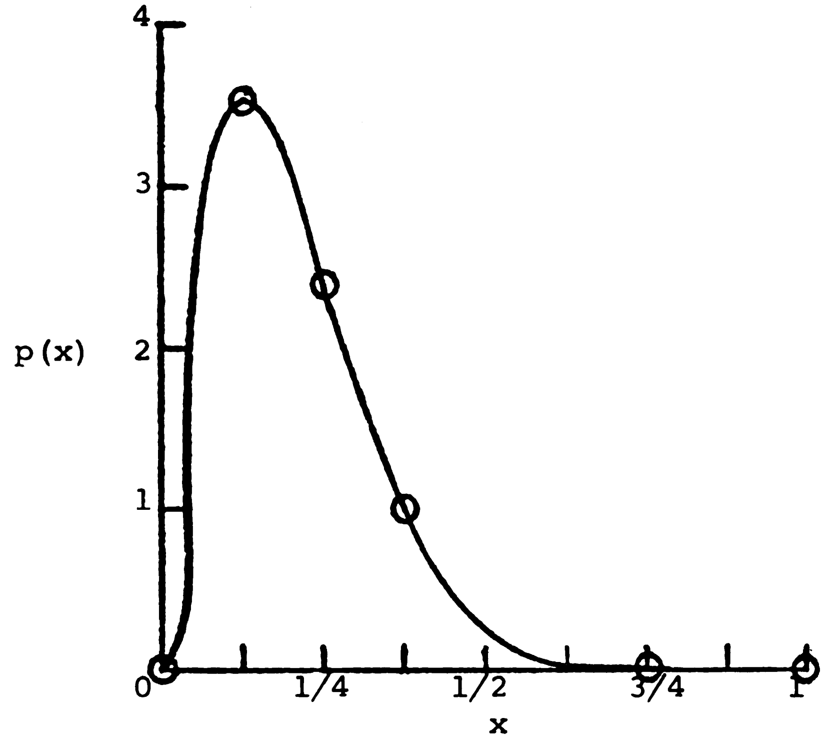

FIGURE 3. GRAPH OF BETA CURVE $\alpha = 2$ $\beta = 8$

x	p(x)
0	0
1/8	3.53
1/4	2.40
3/8	1.005
3/4	0.0033
1	0

The function used to find the area under the curve between x=0 and any point $x = x_1$ is the beta integral. For the example at hand, this integral takes the form

$$\text{Area} = c\int_0^{x_1} x(1-x)^7dx = 72\int_0^{x_1} x(1-x)^7dx.$$

This formula will be used to create confidence intervals based upon the beta distribution determined by $\alpha = 2$ and $\beta = 8$. It is necessary to integrate by parts to evaluate this function. The formula for integration by parts is

$$\int_0^{x_1} udv = uv\Big|_0^{x_1} - \int_0^{x_1} vdu.$$

In this instance u and v are defined as follows:

$$u = x \qquad\qquad dv = (1-x)^7dx$$

$$v = \int_0^{x_1}(1-x)^7dx = -\frac{1}{8}(1-x)^8\Big|_0^{x_1}$$

$$\int_0^{x_1} udv = x(-\frac{1}{8})(1-x)^8\Big|_0^{x_1} - \int_0^{x_1}(-)\frac{1}{8}(1-x)^8dx$$

$$= -\frac{1}{8}x(1-x)^8\Big|_0^{x_1} + \frac{1}{8}\int_0^{x_1}(1-x)^8dx$$

$$= -\frac{1}{8}x(1-x)^8\Big|_0^{x_1} - \frac{1}{72}(1-x)^9\Big|_0^{x_1}$$

To check this substitution the area under the curve from x=0 to x=1 will be evaluated.

$$\text{Area} = -\frac{1}{8}x(1-x)^8\Big|_0^1 - \frac{1}{72}(1-x)^9\Big|_0^1$$

$$= (0 - 0) - \frac{1}{72}(0) - (-)\ \frac{1}{72}(1-0)^9$$

$$= -(-)\frac{1}{72} = \frac{1}{72}\ .$$

This is the same answer as that obtained through the evaluation of the beta by gamma functions earlier. It should be noted that the constant c, equal to the reciprocal of the area computed above (c = 72), must be multiplied by the function to insure that the area under the curve between x=0 and x=1 equals 1.00. The formula for deriving confidence intervals from this distribution becomes

Area under curve from x=0 to $x=x_1$

$$= 72\left[-\frac{1}{8}x(1-x)^8\Big|_0^{x_1} - \frac{1}{72}(1-x)^9\Big|_0^{x_1}\right]$$

The area under the curve from x=0 to x=½ can be computed with the use of this formula.

Area under curve from x=0 to x=½

$$= 72\left[-\frac{1}{8}(\tfrac{1}{2})(\tfrac{1}{2})^8 - \frac{1}{8}(0) - \frac{1}{72}(\tfrac{1}{2})^9 + \frac{1}{72}\ (1-0)^9\right]$$

$$= 72\left[-\frac{1}{8}(\frac{1}{512}) - \frac{1}{72}\ (\frac{1}{512}) + \frac{1}{72}\right]$$

$$= 1 - \frac{9}{512} - \frac{1}{512} = \frac{512 - 10}{512} = \frac{502}{512} = 0.981.$$

When this value is compared to the graph of the function (Figure 3), the answer appears to be reasonable. In a similar fashion, the area under the beta curve with $\alpha = 2$ and $\beta = 8$ between x=0 and any other desired value of x can be computed. If the above computation were made for the purpose of obtaining a confidence interval, it would yield a 98.1% confidence coefficient that the error rate of the population did not exceed 50%. The confidence coefficient is the probability that the computed confidence interval contains the actual population error rate.

The methods demonstrated above are useful only when the constants α and β of the beta distribution under consideration are integers. If either constant is not an integer other computation techniques must be employed, one of which is discussed in section D of this chapter.

The Beta as a Prior Distribution. A prior distribution is required before Bayes theorem can be employed to produce a posterior. Prior distributions can be viewed as providing an "additional sample" from the population of interest to the auditor. The parameters α and β of the prior distribution can be looked upon as being representative of this additional sample. The sum $\alpha+\beta$ would roughly correspond to the size of the additional sample, while

α would roughly represent the number of errors found in that sample. Viewing the prior in this manner may aid an auditor who is attempting to determine the parameters α and β in a specific audit situation. Other information which the auditor may find useful in this task includes various statistics computed for possible prior distributions. Some of these statistics are the mean, variance, and mode. The formulas used to compute these statistics from the α and β parameters of a specific beta distribution will now be discussed.

The mean of any function is the expected value of that function. It can be shown that the mean of a beta distribution occurs at an error rate of [4]

$$\frac{\alpha}{\alpha+\beta}.$$

As an example, the mean of the beta distribution shown in Figure 1, page 51, will be computed. The constants α and β of that distribution are $\alpha = 1.40$ and $\beta = 8.60$.

$$\text{Mean} = \frac{\alpha}{\alpha+\beta} = \frac{1.40}{1.40 + 8.60} = \frac{1.40}{10.00} = 0.14$$

The variance of a distribution is an indication of the relative density with which the values would be expected to occur about the mean. The variance of a beta distribution is[5]

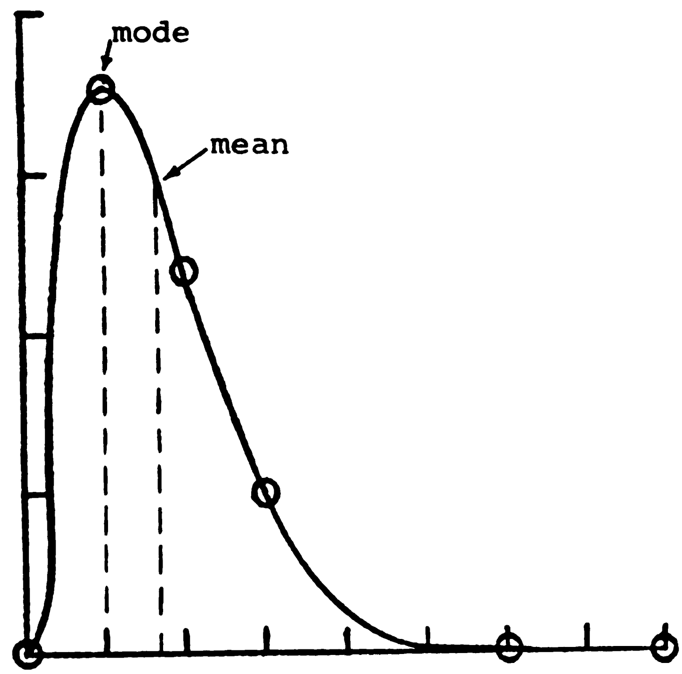

FIGURE 4. MEAN AND MODE OF BETA DISTRIBUTION

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} .$$

The variance of the beta distribution shown in Figure 1 will now be computed.

$$\text{Variance} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{(1.4)(8.6)}{(10.0)^2(10.0+1.0)}$$

$$= \frac{12.04}{1100.0} = 0.10945.......$$

Another measure which is sometimes used is the mode of a distribution. On a graph of a continuous function, the mode appears as the highest point on the curve, as shown in Figure 4.

The vertical distance of the beta curve from the x axis is computed from the formula:

$$p(x) = cx^{\alpha-1}(1-x)^{\beta-1}$$

To find the maximum value of p(x) the first derivative is found and set equal to zero.

$$\frac{dp(x)}{dx} = \frac{d(cx^{\alpha-1}(1-x)^{\beta-1})}{dx}$$

$$= c(\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} + (-1)cx^{\alpha-1}(\beta-1)(1-x)^{\beta-2}$$

$$= (\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} - x^{\alpha-1}(\beta-1)(1-x)^{\beta-2} = 0.$$

$$(\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} = (\beta-1)x^{\alpha-1}(1-x)^{\beta-2}$$

$$\frac{\alpha-1}{\beta-1} = x(1-x)^{-1}$$

$$\alpha-\alpha x-1+x = \beta x-x$$

$$\alpha-1 = (\beta-2+\alpha)x$$

The mode of the beta distribution occurs at

$$x = \frac{\alpha-1}{\alpha+\beta-2}.$$

To illustrate the computation of the mode of a beta distribution, the mode of the distribution shown in Figure 1 will be computed.

$$\text{Mode} = \frac{\alpha-1}{\alpha+\beta-2} = \frac{1.4 - 1}{1.4 + 8.6 - 2.0} = \frac{0.4}{8.0}$$

$$= 0.05$$

The mean, variance, and mode discussed here are useful to an auditor attempting to fit a prior beta distribution to evidence available to him. Once he has determined the specific constants (α and β) which determine the prior distribution, it is necessary to revise the prior by the results of a current audit sample. Determination of the prior distribution is discussed in Section B, while the revision of the prior by the results of a current audit sample is explained in Section C, and the procedure used to make inferences from the result is covered in Section D.

B. Selection of a Prior Distribution

Before Bayes formula or a posterior confidence interval can be used, a meaningful prior distribution must be selected by the auditor. This prior is based upon evidence actually available to the auditor: evidence which relates to the error rate of the population under consideration.

It is important that an auditor interested in the use of Bayesian statistics understand the meaning of the prior distribution he selects. To aid in this, various statistics can be computed about representative prior distributions. The mean, mode, and variance have been discussed in this study, and such statistics can be of value to an auditor who has had even limited statistical training. A short table of such statistics computed for selected beta distributions appears in Appendix A (Table 1).

A useful aid to the auditor faced with the problem of deriving a prior distribution might be a book of sketches of representative beta prior distributions. These might be arranged in a fashion similar to a police department's books of "mug shots" which include a number to identify each portrait. As the victim of a criminal act might thumb through the "mug shots" and note the identifying number of a picture which bears a resemblance to the offender, an auditor attempting to fit a prior distribution to specific circumstances might thumb through the sketches and note the appropriate α and β parameters for the distribution which best fits his situation. These sketches might be ordered in sections by mean or by mode to provide the auditor with a starting point. A short example of such a "mug book" of sketches of specific beta distributions appears in Appendix B.

The use of such sketches would save time. Instead of listing a number of probability points (the method used by those who have proposed the use of discrete prior distributions in auditing applications of Bayesian statistics), the auditor would merely look through the sketches until the appropriate prior was located. Marked on that sketch would be the parameters α and β which determine the function. After noting those parameters, the auditor would be ready to take the current sample and revise the prior with Bayes theorem.

Graphs of beta distributions, in the form of the sketches discussed above, can be produced by a plotter attached to a digital computer. The sketches shown in Appendix B were derived in that fashion. The computer programs which resulted in those sketches are shown in Appendix D. Also shown in that appendix are the programs used to produce the beta distribution tables which appear in Appendix A.

Several statistics can be used by the auditor in locating the appropriate prior distribution for his use. One mark with which he might begin is the mean of his prior expectations. This is not the single rate he feels is most likely to occur, based on his prior expectations, but rather is the expected value of the function. The mean is a useful measure and it is widely employed. However, when faced with the problem of specifying a distribution, the typical auditor might well have difficulty identifying the expected value of the result. He could specify the single error rate he feels is most likely to occur. This is the definition of the mode, another statistic discussed with reference to the beta distribution earlier in this chapter. Because it appears that the mode is less difficult for an auditor to use, it is used to identify the distributions shown in Appendix A and Appendix B of this study, although the mean is given as supplementary information.

One factor the user of Bayesian statitics would consider is the relative weights given to the prior distribution and the current sample in the derivation of the posterior. As the sum $\alpha + \beta$ is increased, with the mode of the prior held constant, the variance of the prior decreases, implying greater certainty. An auditor who has considerable evidence that an error rate is near 5% may use a prior distribution with a relatively large (when compared to the current sample size) $\alpha + \beta$. Another auditor with some smaller amount of evidence which also indicates an error rate near 5% is likely to select a prior in which the sum $\alpha + \beta$ is somewhat smaller than did the first auditor. If each draws the same current sample result, the prior selected by the first auditor will have a greater relative effect upon the posterior than that selected by the second auditor.

An auditor with no statistical training could not be expected to derive meaningful prior distributions for use with Bayes theorem. One study has shown that auditors are "willing to quantify their conclusions as probability distributions."[6] The same study[7] found that different auditors produce different prior distributions from the same data. This implies either a lack of agreement between the auditors as to the audit variables underlying the situation, or a difference in how each auditor subjectively interprets the audit variables. Either alternative suggests that more than a quick introductory lesson would be necessary before several auditors would reach similar independent conclusions from the same data.

Once an auditor has selected a prior distribution from a set of sketches or tables of statistics, he has the constants α and β which are required for the revision of a prior beta distribution by Bayes theorem. These constants are combined with the results of the current audit sample to find the posterior distribution upon which statistical inferences will be based. The operation of revising a beta prior distribution by a binomial sample result is discussed in the following section.

C. Revision of a Beta Prior by a Binomial Sample Result

After an auditor has selected a prior distribution and taken the current sample, only two steps remain in the use of Bayesian statistics with beta prior distributions in audit attributes sampling situations: The revision of the prior with Bayes formula and the creation of confidence intervals based upon the posterior beta distribution. This section discusses the revision of the prior distribution, while section D explains how a confidence interval can be obtained from a beta posterior distribution. An example illustrating the entire process appears in Section E of this chapter.

The revision of a beta prior distribution by a binomial sample result is unexpectedly simple. A beta distribution is determined by only two parameters, referred to as α and β. The result of an audit sample can be specified by n (the sample size) and x (the number of errors found in the sample). If the prior is a beta distribution with parameters α and β, and the sample a binomial of size n with x errors, the posterior will be a beta with parameters

$$\alpha' = \alpha + x - 1 \text{ and}$$

$$\beta' = \beta + n - x - 1.$$

The derivation of this result is shown in Appendix C. Here it is important only to note that revision of a beta prior by a binomial sample is a simple arithmetic manipulation requiring no higher mathematics than addition and subtraction, no computer, and no tables. The resulting posterior distribution is another beta, which may be further revised by additional binomial samples, in the same manner. The result will always be another beta distribution.

The Bayesian method discussed in the preceding paragraph would be relatively easy for an auditor in the field to use. If he had sketches of a number of possible beta prior distributions and tables of relevant statistics, he could select the prior which best represents the evidence available about a given error rate before a current sample is taken. He would then write down the appropriate constants α and β for that distribution. Once the current sample has been completed, its outcome can be summarized by two figures, n (the sample size), and x (the number of errors found in the sample). (Note that this x is not the same as the x used to represent the error rate in the beta function. Statisticians use the same term in two different ways.) The determination of the posterior distribution can be quickly accomplished with the aid of a pencil, a piece of paper, and the two formulas discussed previously in this section. The final step in the analysis involves drawing inferences from the posterior distribution thus obtained. This requires the derivation of confidence intervals from posterior beta distributions. The following section contains an example of the revision of a beta prior distribution by Bayes theorem and explains how confidence intervals can be obtained from beta posterior distributions.

D. Confidence Intervals from Beta Posterior Distributions

The procedure used to select a prior distribution, summarize the results of the current audit sample, revise the prior by those results, and obtain the posterior beta distribution has been discussed in the three preceding sections. Once this procedure has been followed in an actual audit situation, there remains only the problem of determining confidence intervals from the posterior distribution.

In the first section of this chapter, an example of the calculation of a confidence interval from a beta posterior distribution was shown. The method used in that example involved the evaluation of a beta integral through the calculus technique of integrating by parts. That method is not generally applicable to beta integrals in which the constants α and β are not integers. Since auditors would use beta distributions with non-integer constants, an alternative computation technique is required.

Mathematicians have developed approximation techniques for the evaluation of functions which can not be computed directly. One of these techniques is Simpson's rule for the approximation of definite integrals. It approximates the area under a given curve by summing the areas of a number of segments which approximate the area of portions of the desired area. A further discussion of Simpson's rule can be found in Appendix E.

Simpson's rule can be used to create tables of confidence intervals for the beta posterior distribution. If a set of representative posterior distributions were used as the basis for a group of confidence interval tables, the auditor with access to such tables could quickly determine the limits of the desired confidence interval, once he had completed the tasks leading to the posterior distribution. An abbreviated set of such tables, which were computed with the use of Simpson's rule, is shown in Appendix A (Table 2, 3, and 4). Those tables are shown as an example. They are not complete enough for use in practical situations. The computer programs which were used to obtain those tables appear in Appendix D.

The entire procedure followed by the auditor, from selecting a prior distribution through looking up the limits of the confidence interval in the tables, would take only a few minutes (aside from the time required to draw and examine sample documents, etc.). The auditor would not require the use of a computer, or even an adding machine.

To illustrate the use of a beta prior distribution and a binomial sample result in an application of Bayesian statistics to auditing, it will be assumed that an auditor has evidence regarding errors in some population which he feels is represented by a beta distribution with parameters α=1.4 and β=8.6. He takes a sample of 92 (n) items and finds 30 (x) errors. What are the parameters of the posterior beta distribution?

$$\alpha' = \alpha + x - 1 = 1.4 + 30.0 - 1.0 = \underline{\underline{30.4}}$$

$$\beta' = \beta + n - x - 1 = 8.6 + 92.0 - 30.0 - 1.0 = \underline{\underline{69.6}}$$

Illustrations of the shapes of both the prior and posterior distributions in this example can be found on page 51. It should be noted that the mode of the prior is 0.05, while the information contained in the sample results in the mode of the posterior equaling 0.30, certainly a significant shift for most auditors. The prior in this example has relatively less influence upon the posterior than does the sample result.

Confidence intervals can be constructed from the posterior beta distribution by referring to appropriate tables, such as those in Appendix A (Tables 2, 3, and 4). To use the confidence interval tables, the appropriate α and β parameters are located along the left side of the tables, while the confidence coefficient desired is located in the body of the table on that same line. The number at the top of the column in which the desired confidence coefficient appears is the upper limit of a one sided confidence interval. If the auditor in the example discussed in the preceding paragraph wished to construct a one sided confidence interval with a 97.8% confidence coefficient, the tables would show him that 97.8% of the area under a beta curve with parameters $\alpha = 30.4$ and $\beta = 69.6$ would lie below an error rate of

40%. In Chapter III, it was shown that the classical statistics approach resulted in a similar conclusion of a 97.8% confidence coefficient that the population error rate was below 40%. That result, however, was based upon a sample of 169, while the Bayesian example shown here requires a sample of only 92 to reach the same confidence interval. In each instance the sample error rate was 32.6%. Thus, the additional information contributed by the auditor's prior distribution reduced the sample size necessary to obtain the same assurance as that gained from the classical model.

The relative effect upon the posterior distribution by the prior and current sample result is determined by both the "gentleness" of the prior curve and the size of the sample taken. If a prior which is rather noncommittal (has a large variance) is revised by a large sample, the result will be largely determined by the sample. On the other hand, if a prior with a small variance is revised by a small sample, the result will be similar to the prior.[8] This matter is further considered in Chapter VI.

In summary, the use of a beta prior distribution by auditors in applications of Bayesian statistics to auditing requires several steps. In the first section of this chapter, beta distributions were discussed and it was demonstrated how computations can be made which yield information about specific beta curves. Several ways of aiding an auditor who is faced with the problem of determining a beta prior distribution for specific circumstances were discussed in the second section of this chapter. That was followed by an explanation of the process used to revise a beta prior by the results of a current audit attributes sample, while the confidence intervals necessary to the making of inferences based upon beta posterior distributions were the subject of this section. The last section of this chapter contains an example of the use of this method in a specific auditing situation.

E. An Example of the Use of a Continuous Prior Distribution

The preceding sections have explained the procedures and computations necessary to arrive at a posterior distribution from a beta prior distribution and a binomial audit sample result. The following example illustrates the application of a beta prior distribution to a common audit situation in which classical methods result in an illogical conclusion.

Assume an auditor has been doing the field work on a certain engagement for the past six years. Each year he has sampled 40 of the approximately 6000 sales invoices, and each year has found no errors. There have been no significant changes in the internal control system over sales or in the accounting staff. From the tables in Arkin[9], he determined that there was a 99% confidence coefficient that the actual error rate was less than 10.8%.

Bayesian methods allow the auditor to create a prior distribution which includes such information, when desired. In this situation, the auditor would be likely to use a prior with a very small mean and mode, probably in the neighborhood of 1% to 2%, because he feels strongly that the actual rate is at least that small, based upon past experience. Assume that this auditor is attempting to be very cautious in his prior and picks a beta prior with $\alpha = 2.98$ and $\beta = 59.02$. This means that he feels that "most likely" error rate is the mode of

$$\frac{\alpha - 1}{\alpha + \beta - 2} = \frac{2.98 - 1.00}{2.98 + 59.02 - 2.00} = 3.30\%,$$

and that the "average" expected value is the mean of

$$\frac{\alpha}{\alpha + \beta} = \frac{2.98}{2.98 + 59.02} = \frac{2.98}{60.00} = 4.83\%.$$

These values are both high (conservative), compared to the auditor's expectation of the actual rate.

As stated above, the actual sample yielded no errors (x=0) out of a sample of 40 sales invoices (n=40). To determine the posterior distribution, the following computations would be made:

$$\alpha' = \alpha + x - 1 = 2.98 + 0 - 1 = 1.98.$$

$$\beta' = \beta + n - x - 1 = 59.02 + 40 - 0 - 1 = 98.02.$$

A one-sided classical confidence limit was found above to yield a 99% probability that the actual rate was less than 10.8%. By referring to the beta table, a similar confidence interval can be constructed from the posterior Bayesian distribution (Table II in Appendix A). Such a confidence interval would give a 99% confidence coefficient that the actual error rate lies below 7.0%, instead of the 10.8% which the classical method resulted in. Thus, even a very "conservative" (in the sense of having mean and mode greater than the actual expectation of the auditor) prior distribution results in a more realistic posterior distribution and resulting confidence interval than did the classical methods. A prior which more accurately reflected the expectations of the auditor would result in an even more reasonable confidence interval.

It should be noted that an auditor using classical statistical methods in the example discussed above would probably approach the situation by reducing the confidence level he requires in the current year, knowing this will reduce the sample size. But he must make this decision by "feel", with no formal guide as to the effect of this change upon the analysis. The Bayesian method, on the other hand, allows the auditor to be as specific as he desires in making up a prior distribution based upon the results of previous years' audits.

The preceding situation is one to which Bayesian methods seem naturally adopted. Not all auditing situations are so constructed as to fit the Bayesian model in this manner, however. For example, the audit variables which affect the risk of the auditor being subject to sanctions because of the bankruptcy of the client can be considered. It would be difficult for an auditor, even one with considerable training and experience in statistical methods, to construct a prior distribution which meaningfully reflected this possibility. It appears that there is a sort of scale of audit variables, with those obviously applicable to Bayesian methods at one end, and others, such as the bankruptcy example mentioned here, at the opposite end.

Subjective probabilities, such as those used as prior beta distributions, have been the subject of some controversy. This is the subject of the following chapter.

CHAPTER V--FOOTNOTES

[1]Harold J. Larson, Introduction to Probability Theory and Statistical Inference (Belmont, California: John Wiley & Sons, Inc., 1969), p. 358.

[2]Irvin H. LaValle, An Introduction to Probability, Decision, and Inference (New York: Holt, Rinehart and Winston, 1970), p. 256.

[3]Larson, p. 358.

[4]Ibid., p. 305.

[5]Ibid., p. 305.

[6]John C. Corless, "The Assessment of Prior Distributions for Applying Bayesian Statistics in Auditing Situations," unpublished Ph.D. dissertation, University of Minnesota, 1971, p. 127.

[7]Ibid., p. 127.

[8]Robert Schlaifer, Introduction to Statistics for Business Decisions (New York: McGraw-Hill, 1961), p. 217.

[9]Herbert Arkin, Handbook of Sampling for Auditing and Accounting Volume I Methods (New York: McGraw-Hill, 1963), p. 506.

CHAPTER VI

SUBJECTIVE PROBABILITIES

A. Definition

An auditor who has decided to use Bayesian statistics must express in a prior distribution the evidence available to him. The mathematics associated with the beta distribution was discussed in Chapter V. But the auditor attempting to use Bayesian statistics needs more than formulas, or tables, or other such aids. He needs an understanding of what lies behind Bayes theorem. This chapter attempts to provide some of that information.

Probability distributions are popularly thought of as being based upon objective, verifiable evidence. Thus, in the usual sense of the term, a probability distribution would not vary even when derived by different auditors from the same sample. But a qualified auditor may have opinions as to the distributions underlying the sample taken, in addition to the sample result. Often these opinions are based upon facts just as real as the sample result, but not directly expressible as a probability distribution. These opinions may give rise to a "personalistic" or "subjectivist" probability distribution based upon the opinion of the qualified auditor instead of directly upon a sample. Since different auditors have varying perceptions and experiences, these subjective distributions are not necessarily constant. This makes the information contained in them no less real, however. It simply reflects the differing information contained in the opinions of the various auditors.

Information of this type is sometimes criticized because it is not based directly on solid, verifiable evidence. It is said that this type of information is so subjective as to be worthless. But if the auditor generating the information is qualified and has relevant experience, some part of the data will be ignored if this entire class of information is not used.[1]

If the Bayesian method were employed, the auditor would be forced to put down his expectations on paper. At the very least he would be forced to pay attention to this element, and if he were trained to use an input distribution such as the beta which lends itself to simple adjustments for various subjective distributions, it would be possible for him to approximate the sum total of his own experience which is relevant to the decisions being made.

Some who criticize the subjective school contend that unlimited repeatability is a prerequisite for an experiment to be associated with a probability distribution. They would limit probabilistic inference to games of chance, problems of social mass phenomena such as life insurance, and mechanical phenomena such as the movement of molecules in a gas.[2] A rebuttal against such criticisms is that the formal inclusion of subjective factors in the decision process is better than informal consideration outside the statistical sampling model. As a minimum, it provides written evidence of where information arose in the event of a lawsuit or inquiry into the audit after the fact. Another defense is that of usefulness. If one can obtain better results (in the sense of more information for the same cost or the same information for a lower cost) through the application of statistical principles to one time only events, then one is in a more desirable position using such tools than not using them.

When an auditor attempts to derive a prior distribution, it is imperative that he understand the variance statistic and use it or a similar measure to insure that his prior is not so clustered about a single point that other possibilities are excluded. In this the traditional conservatism of an auditor may be an asset because the use of a distribution with too small a variance has the result of putting more information in the prior than is justified by the supporting evidence. This may expose the auditor to sanctions for not being cautious enough in his work, while too large a variance in the prior allows a margin of safety for the auditor. As an auditor learns more about the use of statistics, he may approach a smaller variance with less fear when he has reason to justify the greater certainty implicit in such a distribution. The auditor just beginning to use Bayesian methods should be cautious to insure that any errors made will tend toward larger than necessary sample sizes.

Once it has been determined that Bayesian statistics is to be used, there comes the problem of driving a prior distribution that expresses the information available to the auditor before a current sample is taken. Several approaches have been suggested. In one the auditor lists error rates 0.00, 0.01, 0.02, 0.03,......, 0.98, 0.99, 1.00 and determines his "feelings" about the information available to him by associating a probability with each error rate in such a fashion that the sum of these 101 probabilities totals 1.00. Other approaches use more or fewer possible error rates, and still others utilize continuous distributions to reflect the prior information available. Both continuous and discrete prior distributions must be based upon available evidence. The relationship between the audit variables and the selection of priors is touched upon in the following section.

B. Audit Evidence and Prior Distributions

It has been shown[3] that auditors are willing to provide information about their feelings in an audit case which can be used to construct probability distributions adaptable to priors. One of these prior distributions can be combined with the results of an audit sample using Bayes theorem to produce a posterior probability distribution from which confidence intervals can be constructed or tests of hypotheses can be conducted. A problem that has plagued those considering the use of Bayesian statistics in auditing has been the difficulty of translating an auditor's feelings about the audit situation with which he is faced into a statistical distribution usable in the model.

The previously cited work with discrete distributions has required the precise listing of a number of sample points and the related probability of each. To avoid a sacrifice of accuracy in the approximation, the number of sample points must be large. This listing is not an easy task for an auditor to undertake, and he is further frustrated by the necessity that the sum of all the probabilities totals 1.00. Consistency is difficult to obtain with such a method.

If continuous distributions are also considered, however, the problem of creating a prior distribution from the auditor's feelings is simplified considerably. Because of the nature of continuous distributions, they are smooth curves which may be plotted so that the auditor can compare a specific curve with his feelings in a given situation. Once he has found a "picture" of the curve which best fits his perceptions, the mathematical specifications of that curve may be noted and used as input to a computer program or tables which have already solved the mathematics involved.

An auditor attempting to construct prior distributions must utilize information from his environment, from the client's environment, and from within the client's organization. Several audit variables from each of these three categories were discussed in Chapter II. They vary in applicability to Bayesian statistics.

Information from the auditor's environment is important in the derivation of meaningful prior distributions. The auditor's general experience with this and with similar clients aids in determining the variance associated with prior distributions. Specific sample results from other years' work papers provide an excellent source of information about the likely rate in the current year. Such information, it must be remembered, is only a supplement to current test results, even though the information contained therein is important.[4] There are numerous other audit variables from the auditor's environment, such as the relative sizes of the auditor and client, which provide some information useful in the determination of prior distributions.

The client's environment is important to the auditor. General economic conditions can determine whether the client succeeds or fails. If the national economy appears to be headed toward a depression, the auditor may be less likely to rely upon information from prior distributions derived in different economic circumstances. There are many other factors, such as the absolute size of the client, the industry, the legal form chosen by the client, listing on stock exchanges, registration with the Securities and Exchange Commission, and the mode of financing chosen by the client, which have some bearing upon the risk faced by the client's owners and creditors, and thus upon the risk that the audit results will be challenged. Such information can be used in determining the variance of prior distributions and the resulting weight contributed to the posterior by the prior distribution and current sample results.

Information from within the client's organization is especially important to the auditor attempting to derive a specific prior distribution. If the client has an internal auditing staff, this fact, as well as the results of tests by the internal audit staff, can be very helpful in determining rates which will probably be found in the samples taken, and the variability to be associated with the prior distributions. Other internal control factors also contribute information useful in the derivation of prior distributions. Finally, the auditor is interested in facts about a specific area when making inferences about that area. In deriving a prior distribution, he needs information about such things as the materiality of the items on the financial statements to which the specific sample relates. Many more items of information from the audit variables can be useful to the auditor in choosing prior distributions for the audit, but is is not the purpose of this study to explore such items in depth.

In addition to being reflected in the prior, the audit variables can be used in Bayesian statistics to determine the confidence coefficient and interval limits just as in classical statistics. This gives an additional input to the model for such information. In general, a smaller current sample size will be required with Bayesian statistics to produce a given confidence coefficient and interval limit as compared to classical statistics.

The ultimate decision as to whether Bayesian statistics or classical statistics will be used in auditing probably depends upon the ability of auditors to develop prior distributions. It was attempted, in this study, to develop more feasible approaches to Bayesian statistics in auditing than had before been proposed. Although the process of choosing a prior distribution in a specific audit situation is beyond the scope of this thesis, methods have been suggested which may aid in that process. This area is important to the development of Bayesian statistics in auditing, and requires study beyond that which has been devoted to the subject at this point in time.

CHAPTER VI--FOOTNOTES

[1]Sir Ronald Fisher, "The Nature of Probability," The Centennial Review, Summer, 1958, p. 272.

[2]Richard Von Ness, "Probability: An Objectivist View," reprinted in Edwin Mansfield, Elementary Statistics for Economics and Business (New York: Norton, 1970), p. 61.

[3]John C. Corless, "The Assessment of Prior Distributions for Applying Bayesian Statistics in Auditing Situations," unpublished Ph.D. disseration, University of Minnesota, 1971, p. 127.

[4]James E. Sorensen, "Bayesian Analysis in Auditing," The Accounting Review, July, 1969, p. 561.

CHAPTER VII

CONCLUSION

A. Summary

The objective of an audit by an independent accountant is to render an opinion on the fairness of financial statements prepared from the records being audited. The opinion must be based upon evidence obtained by the application of auditing procedures. Audit variables are the elements of an audit situation which should be considered in the decision about which procedures to apply and how extensively they should be applied. Some variables determine the minimum audit program and others govern modifications to that program.

The auditor's objective is to obtain sufficient evidence to justify an opinion at the least cost. Non-statistical methods, referred to as judgment sampling methods, have been used since the early 1900's when auditing changed from a check upon every transaction entered in the accounting records to a sampling process. More recently, classical statistics has also become commonly used in auditing to determine sample sizes and to construct confidence intervals about attributes, among other uses.

Classical statistics as used by auditors is not necessarily classical statistics. The difference lies in the interpretation of confidence intervals. Classical statistics considers population parameters, even though unknown, as fixed. Auditors, however, typically make statements about confidence intervals which refer to the population parameter as an unknown, implying that it is drawn from a group of possible values. This indicates that a distribution of population parameters must exist. Information about this distribution can be considered in the Bayesian formula as a prior distribution and can be used to make inferences about population parameters.

Bayesian statistics differs from classical methods in that information from two sources (the prior distribution and the current audit sample) is combined into a single distribution. Classical auditing statistics admits only information from the current sample into the model. The inferences made from classical auditing statistical models approximate those arising from Bayesian methods in which uniform prior distributions are used. Auditors typically have information from prior audits, similar engagements, other audit tests, internal control evaluations, and other sources which provide prior evidence about attributes tests. This prior information would seldom imply a uniform prior distribution, but instead would indicate certain rates that are more likely to occur than others. Auditors are taking larger samples than necessary to the extent that they rely upon classical auditing statistical methods when prior evidence indicates that some possible rates are more likely than others.

Several articles have shown how Bayesian statistical methods utilizing discrete distributions can be applied to auditing situations. Both the binomial and hypergeometric distributions have been discussed. Little has been said in the literature, however, about applications of continuous probability functions to Bayesian audit statistics.

Other fields of study, especially the natural sciences, have found the normal distribution to be of value in a wide variety of situations. When the normal distribution is revised with Bayes formula, the posterior distribution is sometimes very difficult to evaluate. Therefore, it is not well suited to applications of Bayesian statistics to auditing. A distribution which is adaptable to such uses is the beta. When a beta prior distribution (continuous) is revised by Bayes formula and a binomial sample (discrete), the result is always another beta distribution. This may be further revised by another binomial distribution, if desired. The combination of continuous and discrete distributions in this manner models the typical attributes auditing sample.

Revision of a beta prior distribution by a binomial sample result with Bayes theorem requires only a few addition and subtraction operations. No computer or tables are necessary. This makes the technique feasible for use by auditors who have had the necessary training, and who have tables and graphs of the prior and posterior distributions. Although these are not currently available to auditors, methods for producing the necessary tables and sketches do exist.

To produce the tables required for the derivation of confidence intervals from posterior beta distributions, it is necessary to evaluate beta integrals. Such integrals cannot be evaluated with the methods of calculus. They can, however, be approximated to any desired degree of accuracy with Simpson's rule for the approximation of definite integrals. Even though this approximation is not practical if done by hand, modern digital computers can perform the necessary calculations in seconds, at a very low cost.

Some applications of Bayesian statistics have been criticized because subjective probabilities were used. It is argued that statistics should be applied only to situations that can be repeated a large number of times. Sufficient justification for the use of such methods in the analysis of one time only events exists, however, if the application of these methods provides a basis for better decisions at a reasonable cost. In addition, the formal consideration of subjective factors in a statistical model is preferable to informal consideration.

B. Relative Advantages of Various Distributions In Applications of Bayesian Statistics to Auditing

If Bayesian statistics is to be used by auditors, a decision must be reached as to which of the many statistical distributions available best fits the auditing situation and is most efficient in actual use. Discrete distributions have been found less difficult for those not trained in higher mathematics to use than continuous distributions. But this study has shown that the actual revision of a prior distribution by a sample result is less complex if a continuous prior distribution, the beta, is used. Continuous distributions are also more exact representations of the prior information available to auditors.

In situations where discrete prior distributions are used, it was discovered that the hypergeometric provides more information from samples which are large relative to the size of the population than the binomial distribution. But this advantage was found to be outweighed if the sample was small compared to the population size because of the additional complexity of computations involving the hypergeometric.

The combination of a beta (continuous) prior distribution and a binomial (discrete) sample result with Bayes formula was found to closely fit the situation found in attributes audit sampling. Once the prior has been determined by the auditor and the sample taken, the actual operation of obtaining the posterior beta distribution is easily accomplished in a few minutes by simple addition and subtraction. This was not found to be the case with situations in which discrete prior distributions are used. If discrete priors are used, lengthy computations, generally involving a computer, are required to obtain the posterior in practical auditing situations. Evaluation of the beta prior and posterior distributions is the only difficult part of a Bayesian analysis using beta prior and posterior distributions and a binomial sample result. Graphs and tables of representative distributions can be prepared in advance, which would make the use of this technique feasible in actual auditing situations, even if no computer were available.

C. Conclusions

Judgment sampling was once relied upon almost exclusively in auditing, but has been supplemented by, and in some cases, replaced by, classical statistical methods. Statistical methods provide more consistent answers and are more readily defended in the event of a challenge to the results of the audit than judgmental methods. Some audit situations do not lend themselves to statistical methods, due to the small size of the audit, or the organization of the records, and in such cases judgmental methods should continue to be used.

It has been suggested that Bayesian statistics may be used in some auditing situations. Discrete distributions (the binomial and hypergeometric) have been used in examples where Bayesian statistics was suggested for use in attributes audit sampling. Discrete distributions are, however, unusable in practical auditing situations. They are poor approximations of the continuous range of possible values found in most audit attributes samples. The derivation of specific discrete prior distributions is difficult, because a number of points must be assigned probabilities, and these probabilities must total 1.00. Involved computations (many multiplication and division operations) are required for the solution of any practical problem using discrete prior distributions and Bayesian statistics. This would require a computer and a means for getting information about the prior and current sample result into the computer without error. These are usually not available in the field. At this time auditors have not adopted these methods, even though such methods have received rather wide exposure in the literature.

Bayesian methods using a beta prior distribution can overcome most of these difficulties. Such a distribution is continuous, and therefore fits the large number of possible population error rates better than does the discrete model. Visualization of the priors is easier because continuous curves can be drawn (plotted on a

computer). Sketches of representative distributions can allow an auditor to quickly pick the specific prior he wants and write down the numeric parameters of that distribution. Tables of statistics (mean, mode, variance, etc.) can be prepared for these representative distributions.

The revision of a beta prior by a binomial sample result is not difficult. Once this has been done, the auditor can refer again to his tables for a sketch of the posterior distribution (or a very similar one) and for statistics he wishes to know for that distribution. Another set of tables can give confidence intervals from that distribution (one sided or two sided) at several confidence levels. What would have taken considerable computer time with a discrete prior distribution is quickly taken care of with this method by using a few tables in a book and a quick addition and subtraction operation.

This study has attempted to help convert an infeasible technique with a great deal of potential into a method feasible for use by auditors. It has been limited in several respects, however. No attempt was made to discuss variables sampling in auditing. A proposal was made suggesting the use of a continuous prior distribution in auditing applications of Bayesian statistics. For this it was necessary that tables and sketches of representative beta distributions be developed. These were restricted to illustrative models. Additional resources would be necessary to the development of complete sets of these tables and sketches.

D. Suggestions for Further Research

More questions arose during this study than could possibly be answered by this project. One area yet open for investigation is the application of Bayesian statistical methods to variables sampling. In addition, there are numerous statistical distributions and combinations of distributions which may prove applicable to auditing applications and have not yet been investigated.

Some studies have been done which used Bayesian statistics with discrete prior distributions in "real world" audit situations. Much can be learned by more studies of this type, especially where applications of classical statistics and Bayesian statistics utilizing various distributions are compared.

The derivation of prior distributions from evidence available to the auditor is an area about which little is known. Studies drawing on the behavioral sciences are needed to make prior distributions in Bayesian statistical applications more meaningful to auditors. Comparisons can also be made between the various types of discrete and continuous distributions used as priors to determine the relative efficiency of each in arriving at correct decisions from the information available to auditors.

Finally, it would be helpful to have information as to how some of the many audit variables fit into the Bayesian model. This could provide insight as to the types of audit situations in which judgmental sampling, classical statistics, and Bayesian statistics are most applicable.

APPENDICES

APPENDIX A

BETA DISTRIBUTION TABLES

Table 1 of this Appendix gives statistics of selected beta distributions. The mode, mean, and variance of each distribution are shown. If explanation is necessary, any introductory statistics discusses these measures.

Also shown are the parameters α and β, and their sum. The sum $\alpha+\beta$ roughly corresponds to the sample size that would be required if the beta distribution were derived from a sample, while the value of α roughly corresponds to the number of errors found in that sample.

Tables 1, 2 and 3 of this appendix give approximations of the value of selected beta integrals evaluated from 0 to the point shown along the top of the table. These were found using Simpson's rule for the approximation of definite integrals. The computer programs are shown in Appendix D, while Simpson's rule is discussed in Appendix E.

TABLE 1

STATISTICS OF SELECTED BETA DISTRIBUTIONS

MODE	A+B	A	B	MEAN	VARIANCE
.01	10.00	1.08	8.92	.10800	.00876
.01	100.00	1.98	98.02	.01980	.00019
.02	10.00	1.16	8.84	.11600	.00932
.02	100.00	2.96	97.04	.02960	.00028
.03	10.00	1.24	8.76	.12400	.00937
.03	100.00	3.94	96.06	.03940	.00037
.04	10.00	1.32	8.68	.13200	.01042
.04	100.00	4.92	95.08	.04920	.00046
.05	10.00	1.40	8.60	.14000	.01095
.05	100.00	5.90	94.10	.05900	.00055
.10	10.00	1.80	8.20	.18000	.01342
.10	100.00	10.80	89.20	.10800	.00095
.15	10.00	2.20	7.80	.22000	.01560
.15	100.00	15.70	84.30	.15700	.00131
.20	10.00	2.60	7.40	.26000	.01749
.20	100.00	20.60	79.40	.20600	.00162
.25	10.00	3.00	7.00	.30000	.01909
.25	100.00	25.50	74.50	.25500	.00188
.30	10.00	3.40	6.60	.34000	.02040
.30	100.00	30.40	69.60	.30400	.00209
.40	10.00	4.20	5.80	.42000	.02215
.40	100.00	40.20	59.80	.40200	.00238
.50	10.00	5.00	5.00	.50000	.02273
.50	100.00	50.00	50.00	.50000	.00248

TABLE 2

ONE SIDED BETA CONFIDENCE COEFFICIENTS

		ERROR RATE, UPPER LIMIT OF CONFIDENCE INTERVAL									
A	B	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
1.08000	8.92000	.06838	.13877	.20646	.27055	.33076	.38704	.43945	.48813	.53323	.57492
1.98000	98.02000	.26612	.59722	.80487	.91214	.96239	.98451	.99382	.99760	.99909	.99966
1.16000	8.84000	.05384	.11538	.17714	.23729	.29499	.34986	.40170	.45045	.49612	.53875
2.96000	97.04000	.08183	.32726	.58303	.76914	.88207	.94333	.97406	.98860	.99516	.99801
1.24000	8.76000	.04221	.09553	.15136	.20727	.26206	.31505	.36584	.41420	.45999	.50313
3.94000	96.06000	.01954	.14534	.35827	.57504	.74498	.85840	.92617	.96347	.98271	.99213
1.32000	8.68000	.03295	.07876	.12881	.18034	.23192	.28266	.33198	.37953	.42506	.4684[illegible]
4.92000	95.08000	.00379	.05362	.18791	.37796	.56993	.72738	.83932	.91096	.95321	.9765[illegible]
1.40000	8.60000	.02562	.06468	.10918	.15631	.20448	.25267	.30020	.34659	.39149	.4346[illegible]
5.90000	94.10000	.00362	.01680	.08508	.21863	.39137	.56643	.71392	.82362	.89753	.943[illegible]
1.80000	8.20000	.00686	.02282	.04520	.07244	.10334	.13699	.17262	.20960	.24741	.28553
10.80000	89.20000	.00000	.00001	.00026	.00261	.01290	.04118	.09733	.18526	.29995	.42974
2.20000	7.80000	.00168	.00738	.01719	.03088	.04812	.06855	.09177	.11739	.14505	.17439
15.70000	84.30000	.00000	.00000	.00000	.00000	.00005	.00038	.00190	.00685	.01925	.04443
2.60000	7.40000	.00038	.00221	.00605	.01219	.02078	.03184	.04535	.06119	.07925	.0993[illegible]
20.60000	79.40000	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00005	.00026	.00102
3.00000	7.00000	.00008	.00061	.00198	.00448	.00836	.01380	.02091	.02979	.04048	.05297
25.50000	74.50000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00001
3.40000	6.60000	.00002	.00016	.00061	.00154	.00314	.00559	.00903	.01359	.01938	.02643
30.40000	69.60000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
4.20000	5.80000	.00000	.00001	.00005	.00015	.00037	.00076	.00139	.00234	.00368	.0054[illegible]
40.20000	59.80000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
5.00000	5.00000	.00000	.00000	.00000	.00001	.00003	.00008	.00017	.00031	.00055	.00083
50.00000	50.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE 3

ONE SIDED BETA CONFIDENCE COEFFICIENTS

ERROR RATE, UPPER LIMIT OF CONFIDENCE INTERVAL

A	B	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
1.08000	8.92000	.61339	.64883	.68142	.71134	.73877	.76387	.78682	.80775	.82683	.84418
1.98000	98.02000	.99987	.99995	.99997	.99997	.99997	.99996	.99995	.99995	.99993	.99992
1.16000	8.84000	.57846	.61534	.64952	.68113	.71031	.73719	.76191	.78461	.80541	.82444
2.96000	97.04000	.99920	.99969	.99989	.99996	.99999	1.00000	1.00000	1.00000	1.00000	1.00000
1.24000	8.76000	.54376	.58176	.61726	.65032	.68105	.70955	.73591	.76026	.78270	.80[illegible]34
3.94000	96.06000	.99654	.99853	.99939	.99975	.99990	.99996	.99998	.99999	.99999	.9[illegible]99
1.32000	8.68000	.50952	.54834	.58438	.61916	.65123	.68116	.70902	.73490	.75889	.78[illegible]07
4.92000	95.08000	.98870	.99476	.99765	.99898	.99957	.99982	.99993	.99997	.99999	1.00[illegible]00
1.40000	8.60000	.47597	.51531	.55260	.58785	.62105	.65223	.68143	.70871	.73413	.7[illegible]76
5.90000	94.10000	.97028	.98501	.99272	.99659	.99845	.99932	.99971	.99988	.99995	.99[illegible]98
1.80000	8.20000	.32377	.36165	.39896	.43550	.47107	.50555	.53882	.57080	.60142	.63[illegible]64
10.80000	89.20000	.55893	.67641	.77420	.84977	.90446	.94178	.96594	.98082	.98959	.99[illegible]55
2.20000	7.80000	.20507	.23676	.26919	.30209	.33520	.36832	.40124	.43379	.46582	.49[illegible]19
15.70000	84.30000	.08775	.15217	.23716	.33820	.44783	.55755	.65965	.74861	.82161	.87[illegible]27
2.60000	7.40000	.12134	.14500	.17014	.19656	.22406	.25243	.28150	.31105	.34094	.37[illegible]98
20.60000	79.40000	.00324	.00866	.01996	.04058	.07407	.12315	.18877	.26953	.36173	.46000
3.00000	7.00000	.06725	.08326	.10093	.12016	.14085	.16289	.18614	.21046	.23573	.26180
25.50000	74.50000	.00003	.00013	.00047	.00141	.00370	.00860	.01801	.03434	.06028	.09[illegible]20
3.40000	6.60000	.03498	.04490	.05627	.06909	.08336	.09903	.11606	.13439	.15396	.17468
30.40000	69.60000	.00000	.00000	.00000	.00002	.00006	.00020	.00057	.00149	.00351	.00757
4.20000	5.80000	.00786	.01087	.01458	.01907	.02441	.03066	.03786	.04607	.05532	.06563
40.20000	59.80000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
5.00000	5.00000	.00138	.00206	.00296	.00414	.00563	.00748	.00976	.01250	.01575	.01958
50.00000	50.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000.	.00000	.00000

TABLE 4

ONE SIDED BETA CONFIDENCE COEFFICIENTS

		ERROR RATE, UPPER LIMIT OF CONFIDENCE INTERVAL							
A	B	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.08000	8.92000	.94816	.98163	.98936	.98932	.98744	.98525	.98301	1.00000
1.98000	98.02000	.99973	.99933	.99862	.99744	.99566	.99312	.98966	1.00000
1.16000	8.84000	.94133	.98114	.99149	.99258	.99142	.98985	.98822	1.00000
2.36000	97.64000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.24000	8.76000	.93319	.97961	.99266	.99482	.99424	.99315	.99198	1.00000
3.94000	96.06000	.99996	.99989	.99976	.99958	.99936	.99918	.99910	1.00000
1.32000	8.68000	.92384	.97722	.99311	.99630	.99619	.99546	.99465	1.00000
4.92000	95.08000	1.00000	.99999	.99995	.99988	.99973	.99946	.99905	1.00000
1.40000	8.60000	.91338	.97406	.99298	.99723	.99752	.99706	.99651	1.00000
5.90000	94.10000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	.99995	1.00000
1.80000	8.20000	.84611	.94836	.98661	.99749	.99961	.99981	.99977	1.00000
10.80000	89.20000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.20000	7.80000	.75698	.90626	.97219	.99419	.99930	.99999	1.00000	1.00000
15.70000	84.30000	.99962	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.60000	7.40000	.65140	.84623	.94775	.98737	.99816	.99990	1.00000	1.00000
20.60000	79.40000	.98494	.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.00000	7.00000	.53717	.76821	.91016	.97497	.99571	.99969	1.00000	1.00000
25.50000	74.50000	.84898	.99895	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.40000	6.60000	.42324	.67478	.85692	.95420	.99079	.99917	.99999	1.00000
30.40000	69.60000	.47668	.97813	.99997	1.00000	1.00000	1.00000	1.00000	1.00000
4.20000	5.80000	.22702	.46377	.70160	.87543	.96575	.99541	.99989	1.00000
40.20000	59.80000	.01586	.48908	.97585	.99997	1.00000	1.00000	1.00000	1.00000
5.00000	5.00000	.09981	.26657	.50000	.73343	.90119	.98042	.99911	1.00000
50.00000	50.00000	.00002	.02193	.50000	.97807	.99998	1.00000	1.00000	1.00000

APPENDIX B

SKETCHES OF BETA DISTRIBUTIONS

Plots of selected beta distributions are shown in this Appendix. Certain statistics related to each of the curves shown here were given in Appendix A.

Two curves, with the same mode, are shown on each plate. The error rate corresponding to the highest point on each of the two curves is the same. indicating they share the same mode.

These plots were done by the CDC 3600 computer in the Michigan State University Computer Center. The actual program, written in the FORTRAN IV language, is shown in Appendix D.

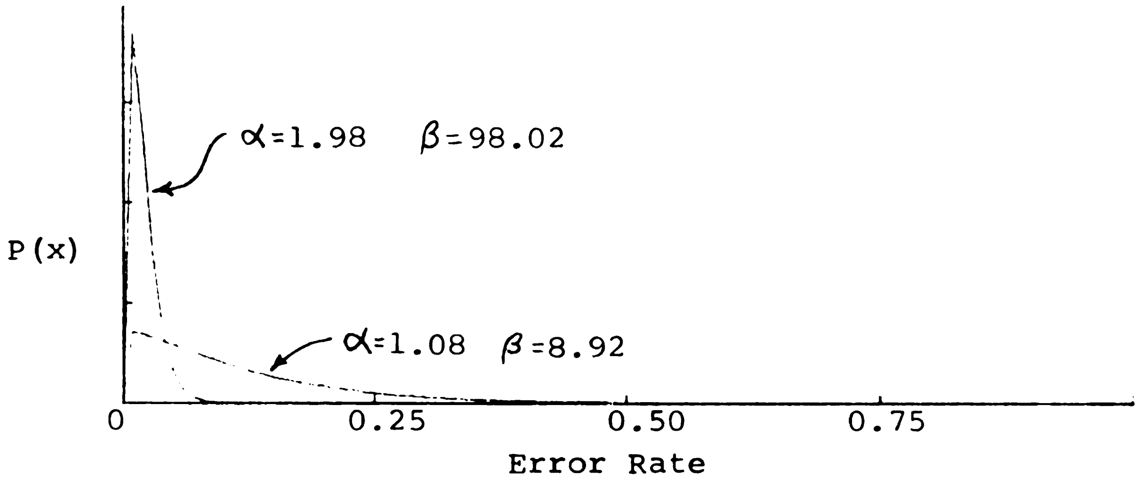

FIGURE 5. TWO BETA CURVES WITH MODE OF 0.01

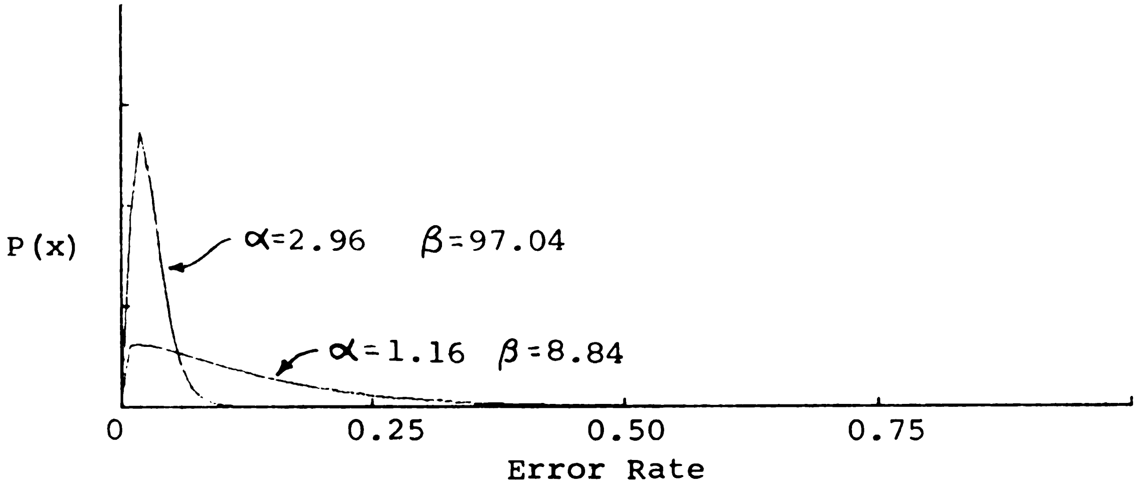

FIGURE 6. TWO BETA CURVES WITH MODE OF 0.02

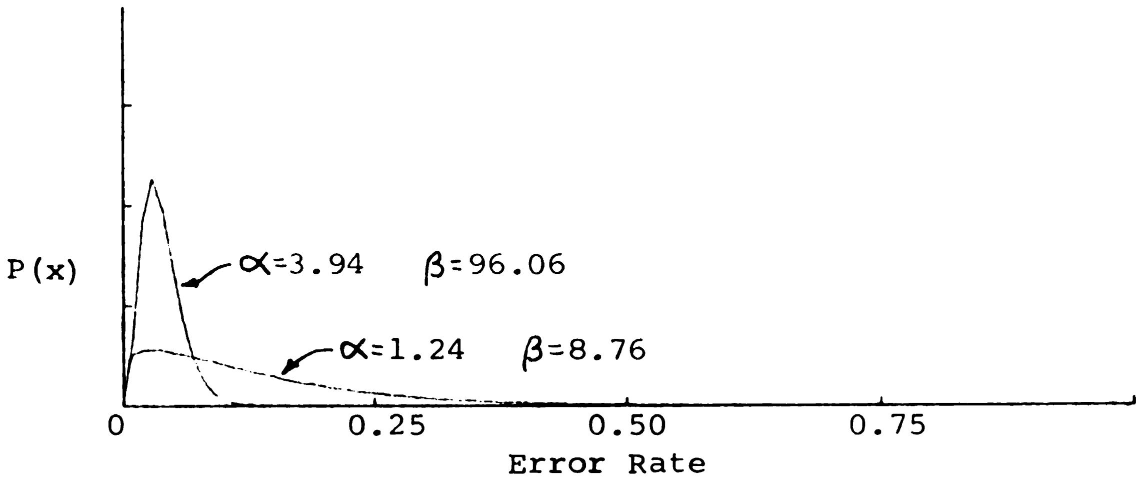

FIGURE 7. TWO BETA CURVES WITH MODE OF 0.03

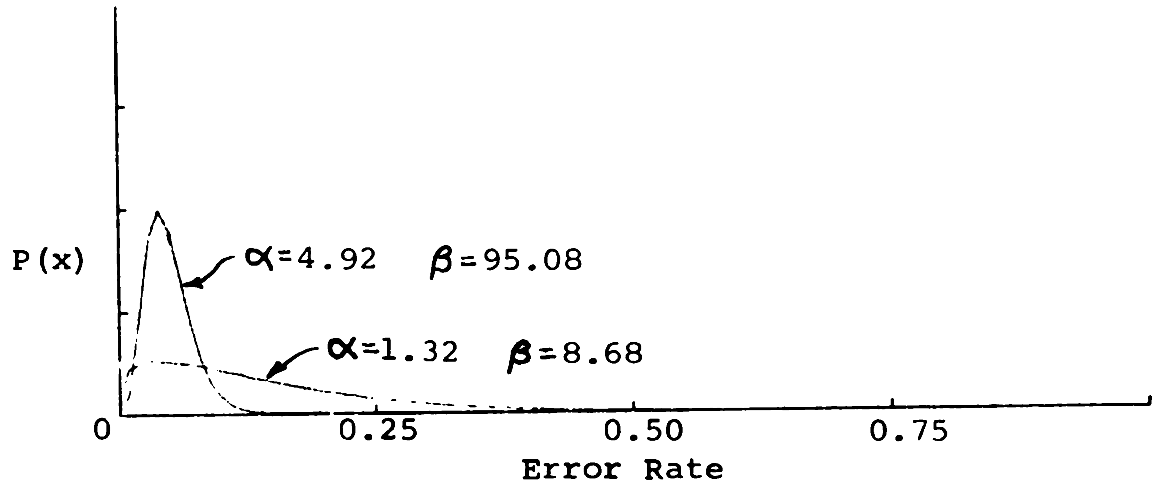

FIGURE 8. TWO BETA CURVES WITH MODE OF 0.04

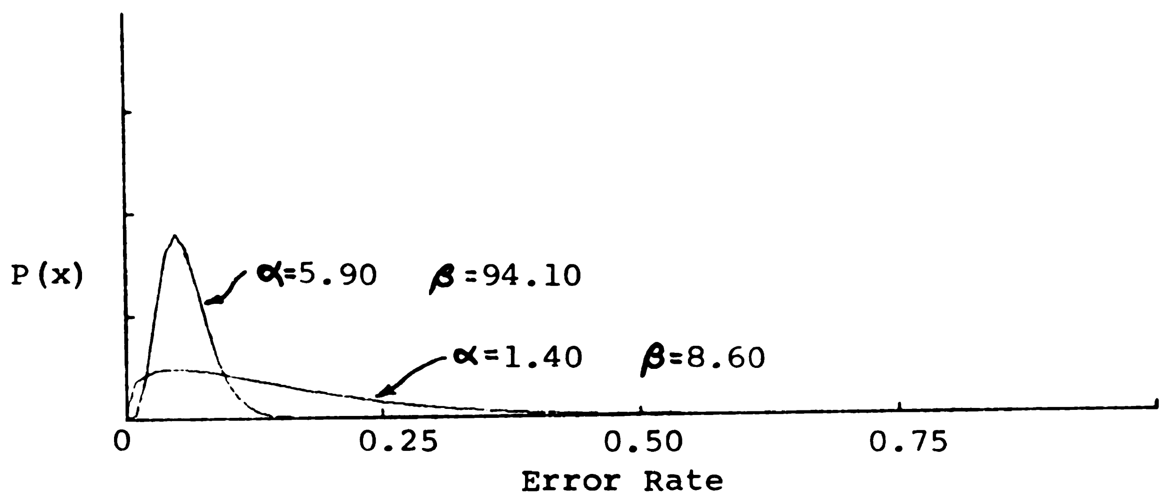

FIGURE 9. TWO BETA CURVES WITH MODE OF 0.05

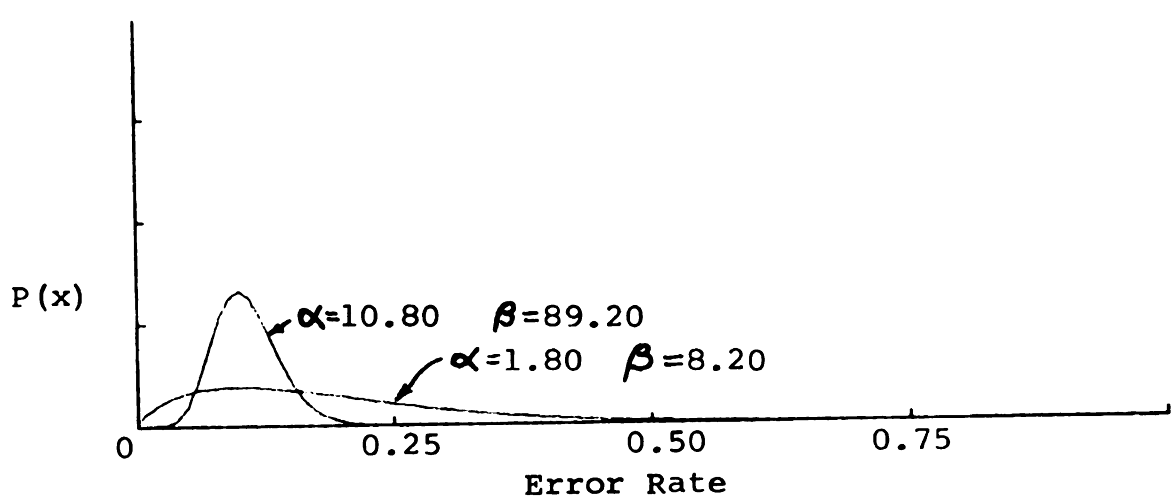

FIGURE 10. TWO BETA CURVES WITH MODE OF 0.10

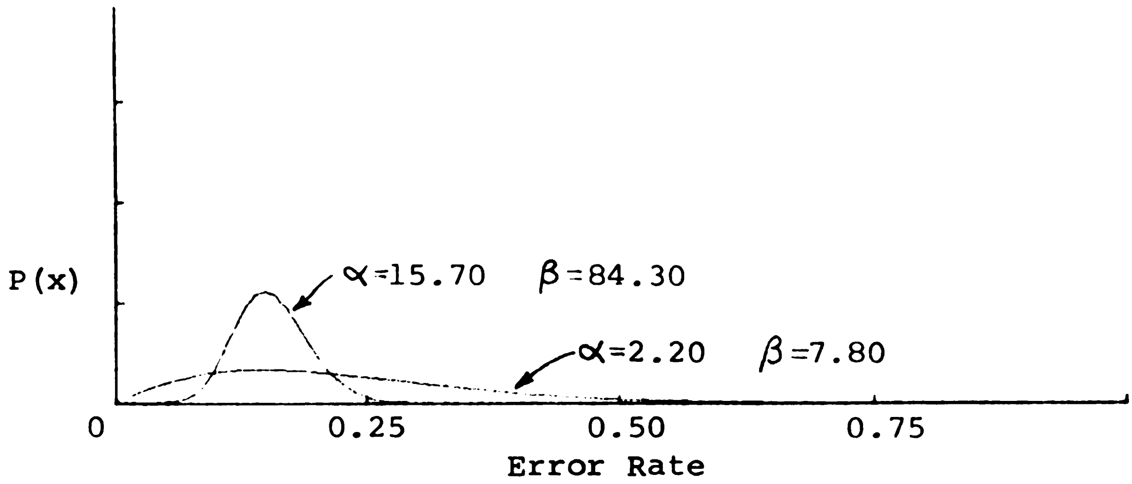

FIGURE 11. TWO BETA CURVES WITH MODE OF 0.15

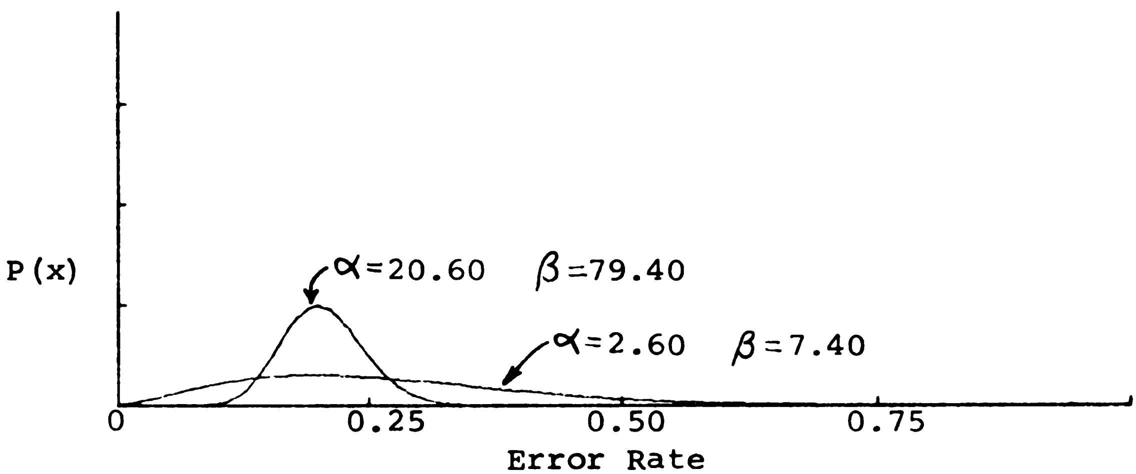

FIGURE 12. TWO BETA CURVES WITH MODE OF 0.20

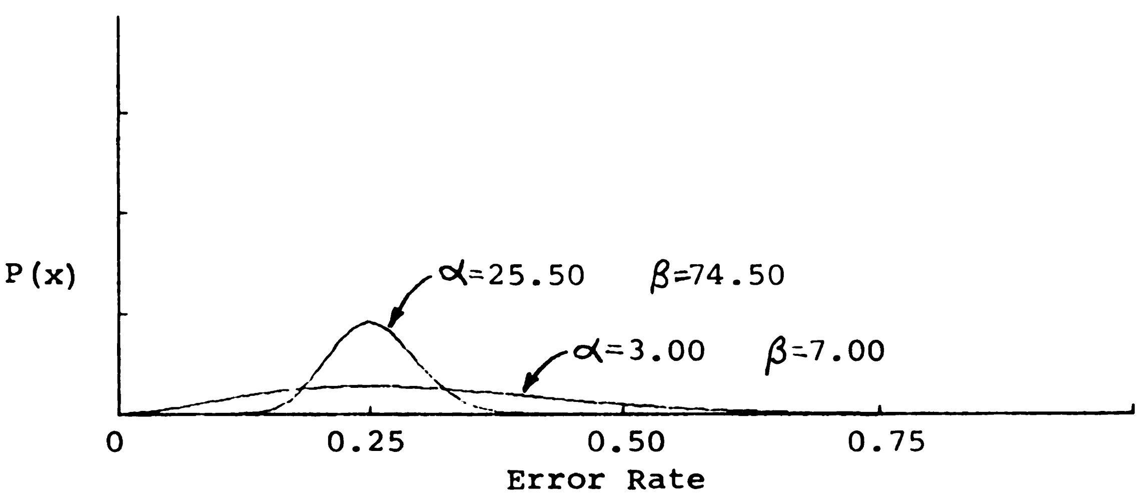

FIGURE 13. TWO BETA CURVES WITH MODE OF 0.25

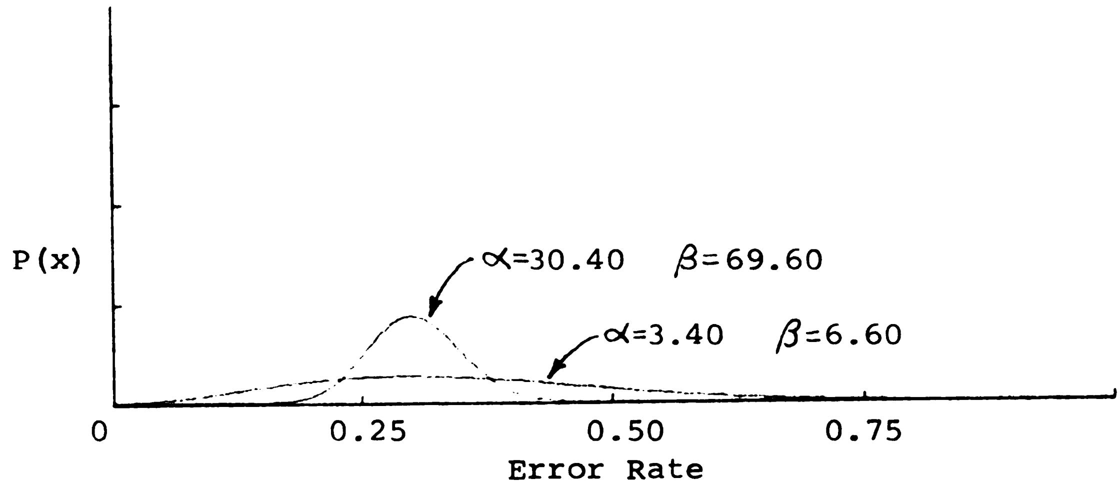

FIGURE 14. TWO BETA CURVES WITH MODE OF 0.30

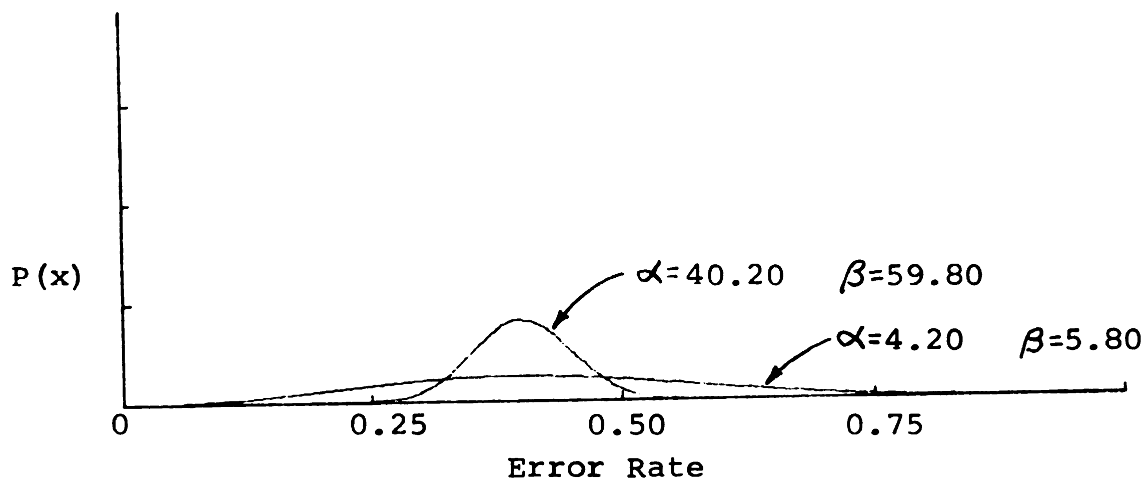

FIGURE 15. TWO BETA CURVES WITH MODE OF 0.40

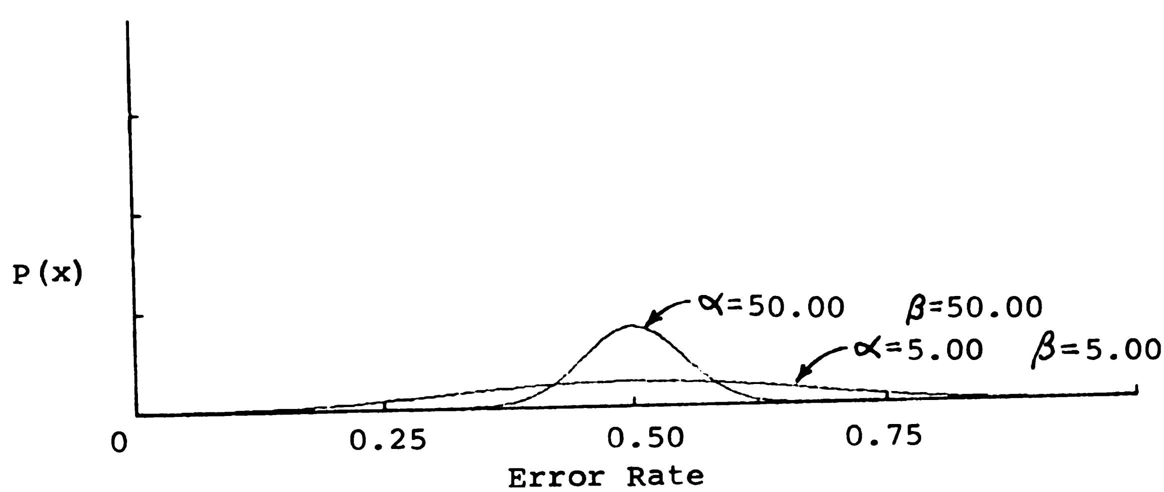

FIGURE 16. TWO BETA CURVES WITH MODE OF 0.50

APPENDIX C

REVISION OF A BETA PRIOR DISTRIBUTION BY A BINOMIAL SAMPLE RESULT

Given the prior distribution with mean, mode, variance, etc. chosen to reflect a belief about the underlying population parameter θ, and expressed as

$$f_\theta(\theta) = C_1 \theta^{\alpha-1}(1-\theta)^{\beta-1} \qquad \begin{matrix}\alpha>1\\ \beta>1\end{matrix}$$

where C_1 is chosen so that

$$C_1 \int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1} d\theta = 1$$

then the prior distribution can be revised using the results of a sample of n items, x of which were errors.

$$f_{X|\theta}(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

The joint (unconditional) distribution of X and θ is simply the product

$$f_{X,\theta}(x,\theta) = f_{X|\theta}(x|\theta)\, f_\theta(\theta)$$

$$f_{X,\theta}(x,\theta) = C_1\theta^{\alpha-1}(1-\theta)^{\beta-1}\binom{n}{x}\theta^x(1-\theta)^{n-x}$$

$$= \binom{n}{x} C_1 \theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}$$

The margional density of X can be found by integrating the joint density $f_{X,\theta}(x,\theta)$ over the range of θ (0 to 1).

$$f_X(x) = \int_{\substack{\text{Range} \\ \text{of } \theta}} f_{X,\theta}(x,\theta)\, d\theta$$

$$= \int_0^1 C_1 \binom{n}{x} \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} d\theta$$

$$= C_1 \binom{n}{x} \int_0^1 \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} d\theta$$

The posterior function $f_{\theta|X}(\theta|x)$ is found by dividing the joint distribution by the margional distribution.

$$f_{\theta|X}(\theta|x) = \frac{f_{X,\theta}(x,\theta)}{f_X(x)}$$

$$= \frac{C_1 \binom{n}{x} \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1}}{C_1 \binom{n}{x} \int_0^1 \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} d\theta}$$

$$= \frac{\theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1}}{\int_0^1 \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} d\theta}$$

However, the denominator is a definite integral, equal to a constant, C_2.

$$C_2 = \int_0^1 \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} d\theta$$

$$= \frac{\Gamma(\alpha+x)\, \Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)}$$

Since we wish the integral (cumulative distribution function) of the revised distribution to be equal to 1 over the range (0,1)

$$\int_0^1 \frac{1}{C_2} \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} d\theta = 1$$

then C_3 may be set equal to $\frac{1}{C_2}$ and

$$f_{\theta|X}(\theta|x) = C_3\theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}$$

which is another beta distribution, with parameters

$$\alpha' = \alpha + x - 1$$

$$\beta' = \beta + n - x - 1.$$

APPENDIX D

COMPUTER PROGRAMS

This appendix contains the computer programs used to produce the beta distribution tables in Appendix A and the plots in Appendix B. All programs were written in FORTRAN IV. The plotting program was run on the Michigan State University CDC 3600 machine, while the other routines were run on the CDC 6500. These programs are given here for reference only since certain software features used are not commonly available on other machines and changes would be necessary before they could be run on most other computers.

This set of programs was written as a group. Program TABLE computes statistics for a beta distribution with various modes and with A+B ($\alpha+\beta$) of either 10 or 100. It punches a deck of cards, one card for each mode and A+B combination, which gives A, B, and the constant c for the basic beta distribution formula discussed in Chapter V. In addition, TABLE prints a table of statistics giving the mean, mode, and variance of each distribution. The main program determines A, B, and the mode. Subroutine WORK computes the mean and variance of that function and prints the table. Subroutine SIMPSON uses the Simpson rule for the approximation of definite integrals to find the constant of integration, c, which will make the value of the integral over the range X=0 to X=1 equal to 1.00. Subroutine ONE finds the Y value of the beta function at each specific X value called for by SIMPSON.

Program TABLE and its subroutines follow on the next three pages.

TABLE 5. COMPUTER PROGRAM "TABLE"

```
      PROGRAM TABLE(INPUT,OUTPUT,PUNCH)
      REAL MODE
      PRINT 99
      PRINT 100
      DO 50 M=1,5
      Z=M
      MODE=Z/100.
      TOT=10.
   40 CONTINUE
      CALL WORK(TOT,MODE)
      TOT=TOT*10.
      IF (TOT.LT.105) GO TO 40
   50 CONTINUE
      DO 70 MA=10,30,5
      ZA=MA
      MODE=ZA/100.
      TOT=10.
   60 CALL WORK(TOT,MODE)
      TOT=TOT*10.
      IF (TOT.LT.105) GO TO 60
   70 CONTINUE
      DO 90 MB=40,50,10
      ZB=MB
      MODE=ZB/100.
      TOT=10.
   80 CALL WORK(TOT,MODE)
      TOT=TOT*10.
      IF (TOT.LT.105) GO TO 80
```

TABLE 5 (cont'd)

```
   90 CONTINUE
   95 FORMAT (*1*//////////)
  100 FORMAT (17X,*     MODE              A+B                A                B
     C                      MEAN               VARIANCE*)
      END
      SUBROUTINE WORK(TOT,MODE)
      REAL MEAN, MODE
      A=MODE*(TOT-2.0)+1.0
      B=TOT-A
      MEAN=A/(A+B)
      VAR=(A*B)/((A+B)**2*(A+B+1))
      PRINT 110, MODE, TOT, A, B, MEAN, VAR
      CALL SIMPSON(A,B)
  110 FORMAT (10X,4F15.2,2F20.5)
      RETURN
      END
      SUBROUTINE SIMPSON(A,B)

      REAL C
   35 AREA=0.0
      C=1.0
      XU=1.0
      XL=0.0
      H=(XU-XL)/1000.
      X=XL
      CALL ONE(C,X,Y,A,B)
      AREA=H*Y/3.
      DO 50 I=1,499
      X=X+H
```

TABLE 5 (cont'd.)

```
      CALL ONE(C,X,Y,A,B)
      AREA=AREA+4.*Y*H/3.
      X=X+H
      CALL ONE(C,X,Y,A,B)
      AREA=AREA+2.*Y*H/3.
   60 CONTINUE
      X=XU-H
      CALL ONE(C,X,Y,A,B)
      AREA = AREA +Y*4.*H/3.
      X=XU
      CALL ONE(C,X,Y,A,B)
      AREA=AREA+Y*H/3.
      C=ALOG10(1.0) - ALOG10(AREA)
   80 CONTINUE
      PUNCH 200,A,B,C
  200 FORMAT (2F10.4,E40.20)
      RETURN
      END
      SUBROUTINE ONE (C,X,Y,A,B)
      IF (X.EQ.1.0) GO TO 500
      IF(X.EQ.0.0) GO TO 500
      Y=C*(X**(A-1.0))*((1.0-X)**(B-1.0))
      GO TO 600
  500 Y=0.0
  600 CONTINUE
      RETURN
      END
```

Program PLOTIT uses the cards punched by TABLE as input data and directs a routine on file in the Michigan State University Computer Program Library to plot the curves in Appendix B. One plot is produced for each run of the program and several curves may be plotted on the same set of X and Y axes by running the appropriate input data cards together in one run. PLOTIT first initializes the plotter pointer, then directs the drawing of the X and Y axes, and finally computes and plots the appropriate X and Y values for the beta curve determined by the A, B, and C values found on the input card being read. The last call to PLOT prepares the plotter mechanism for the call of the next user.

Program PLOTIT appears on the following page.

TABLE 6. COMPUTER PROGRAM "PLOTIT"

```
      PROGRAM PLOTIT
   25 FORMAT (2F10.5,F40.20)
   32 READ 25,A,B,C
      IF (EOF,60) 180,35
   35 C=10.0**C
      CALL PLOT(0.0,0.0,0,10.0,1000.0)
      CALL PLOT(0.0,-1.0,2)
      CALL PLOT(0.0,0.0,0)
      X=0.0
      DO 140 IA=1,4
      XA=IA
      X=XA/4.0
      CALL PLOT(0.0,X,1)
      CALL PLOT(0.8,X,1)
  140 CALL PLOT(0.0,X,1)
      CALL PLOT(0.0,0.0,1)
      Y=0.0
      DO 145 IB=1,10
      XB=IB
      Y=XB*10.0
      CALL PLOT(Y,0.0,1)
      CALL PLOT(Y,0.01,1)
  145 CALL PLOT(Y,0.0,1)
      CALL PLOT(0.0,0.0,1)
      DO 150 I=1,100
      XI=I
      X=XI/100.0
      Y=C*(X**(A-1.0))*((1.0-X)**(B-1.0))
  150 CALL PLOT(Y,X,1)
      CALL PLOT(Y,X,2)
      CALL PLOT(0.0,0.0,-1)
      GO TO 32
  180 CONTINUE
      END
```

Program VALUE computes an approximation of the definite integral of the beta function discussed in Chapter V. The value of A, B, and C are gathered from punched cards previously prepared by program TABLE. The main program in VALUE reads this data, converts C from its logarithmic value and prints the values determined by the subroutines. Subroutine SIMPSON uses the Simpson rule for the approximation of definite integrals to find the integral value. Subroutine ONE computes the Y value for a given X value for the parameters A, B, and C read in by the main program. The approximations are not perfect. They could be made better by dividng the X axis into more than the 100 parts which SIMPSON uses. This would, however, require additional computer time which should be weighed against the added accuracy. The effect of this approximation may be noted as x_1 approaches 1.0 under the smaller modes in the last table in Appendix A. It can be seen that the approximation to the value of the integral reaches a peak and then decreases slightly before x_1=1.0. If two place accuracy were sufficient for the purposes at hand the accuracy here obtained should suffice; however, more computer time would be required for greater accuracy.

The following two pages contain program VALUE.

TABLE 7. COMPUTER PROGRAM "VALUE"

```
      PROGRAM VALUE(INPUT,OUTPUT,TAPE60=INPUT)
C
C*************************************************************************
C           PROGRAM VALUE COMPUTES AND PRINTS A TABLE
C           OF BETA FUNCTION INTEGRALS
C*************************************************************************
C
      DIMENSION AR(25)
  100 FORMAT (2F10.4,F40.20)
  150 FORMAT (*1*)
  175 FORMAT (/////////)
  200 FORMAT (*       A          B          0.01         0.02         0.03         0.
     C04         0.05         0.06         0.07         0.08         0.09         0.10*)
  250 FORMAT (12F10.5)
      PRINT 150
      PRINT 175
      PRINT 200
  275 READ 100,A,B,C
      IF (EOF(60)) 999,300,999
  300 C=10.0**C
      DO 800 I=1,10
      CALL SIMPSON(A,B,C,I,AREA)
  800 AR(I)=AREA
      PRINT 250,A,B,AR(1),AR(2),AR(3),AR(4),AR(5),AR(6),AR(7),AR(8),AR(
     C9),AR(10)
      GO TO 275
  999 CONTINUE
      END
      SUBROUTINE SIMPSON(A,B,C,I,AREA)
```

TABLE 7 (cont'd.)

```
   35 AREA=0.0
      XI=I
      XU=XI/100.0
      XL=0.0
      H=(XU-XL)/100.
      X=XL
      CALL ONE(C,X,Y,A,B)
      AREA=H*Y/3.
      DO 50 J=1,49

      X=X+H
      CALL ONE(C,X,Y,A,B)
      AREA=AREA+4.*Y*H/3.
      X=X+H
      CALL ONE(C,X,Y,A,B)
      AREA=AREA+2.*Y*H/3.
   50 CONTINUE
      X=XU-H
      CALL ONE(C,X,Y,A,B)
      AREA = AREA +Y*4.*H/3.
      X=XU
      CALL ONE(C,X,Y,A,B)
      AREA=AREA+Y*H/3.
      IF (AREA.LT.1.0) GO TO 80
      AREA = 1.0
   80 CONTINUE
      RETURN
      END
      SUBROUTINE ONE (C,X,Y,A,B)
      Y=C*(X**(A-1.0))*((1.0-X)**(B-1.0))
      RETURN
      END
```

APPENDIX E

EVALUATION OF INTEGRALS NOT SOLVABLE BY NORMAL MEANS

In order that statements may be made about the probability of the result of an experiment represented by a beta distribution it is necessary to evaluate definite integrals of the form

$$p(d<x<f) = \int_d^f cx^{\alpha-1}(1-x)^{\beta-1}dx.$$

Unfortunately, the calculus does not provide a general method for handling such functions. However, any definite integral can be considered as a series of areas, and the value of the integral approximated by any method which approximates the area. Simpson's rule for the approximation of definite integrals takes advantage of this fact and approximates the value of the integral by summing the areas of many small slices which themselves approximate small parts of the integral. It divides the width of the integral into n equal parts, each of width $h = \frac{1}{n}$. The area is bounded on the left by x_o and on the right by x_n, with the "inside" broken up by x_1, x_2, ...x_{n-2}, x_{n-1}. At each x point the appropriate vertical distance from the x axis to the curve is computed; y_o, y_1, ..., y_{n-1}, y_n. The approximate value of the integral is found by the formula

$$A_s = \frac{h}{3}\left[y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots\ldots + 2y_{n-2} + 4y_{n-1} + y_n\right].$$ [1]

This formula can be used to advantage in finding the approximate value of beta integrals which can not be directly computed by the usual techniques. In general, the larger the number of sections into which the area is divided, the better the approximation will be. For smooth curves such as the beta which do not have sharp peaks the approximation is quite good with a rather small number of divisions (say 20) and approaches the actual value rapidly as the number of divisions gets larger. With the digital computer, division of the integral width into 100 or 1000 parts for computation is easily accomplished and provides accuracy far beyond the needs of general audit use.

---

[1]George B. Thomas, Jr., Calculus and Analytic Geometry (Addison-Wesley, Reading, Massachusetts, 1968), p. 309.

SELECTED BIBLIOGRAPHY

SELECTED BIBLIOGRAPHY

Reference Works

Arens, Alvin A. "The Adequacy of Audit Evidence Accumulation in Public Accounting." Unpublished Ph.D. dissertation, Graduate School of Business, University of Minnesota, 1970.

Arkin, Herbert. Handbook of Sampling for Auditing and Accounting, Volume I, Methods. New York: McGraw-Hill Book Company, Inc., 1963.

Bookkeepers' Handy Guide. New York: The Ronald Press Company, 1936.

Committee on Auditing Procedure of the American Institute of Certified Public Accountants. Auditing Standards and Procedures. New York: AICPA, 1963.

Corless, John C. "The Assessment of Prior Distributions for Applying Bayesian Statistics in Auditing Situations." Unpublished Ph.D. dissertation, Graduate School of Business, University of Minnesota.

Larson, Harold J. Introduction to Probability Theory and Statistical Inference. Monterey, California: John Wiley and Sons, 1969.

LaValle, Irving H. An Introduction to Probability, Decision, and Inference. New York: Holt, Rinehart and Winston, 1970.

Mautz, R. K. and Sharaf, Hussein A. The Philosophy of Auditing. Evanston, Illinois: American Accounting Association, 1961.

Mendenhall, William. Introduction to Probability and Statistics. Belmont, California; Wadsworth Publishing Company, Inc., 1967.

Schlaifer, Robert. Introduction to Statistics for Business Decisions. New York: McGraw-Hill Book Company, Inc., 1961.

The Study Group On Introductory Accounting. A New Introduction To Accounting. Seattle, Washington: Price Waterhouse Foundation, 1971.

Thomas, Jr., George B. Calculus and Analytic Geometry. Reading, Massachusetts: Addison-Wesley Publishing Company, 1968.

Von Ness, Richard. "Probability: An Objectivist View," Elementary Statistics for Economics and Business. Edited by Edwin Mansfield. New York: Norton, 1970.

Periodical Literature

Anderson, H. M., Giese, J. W., Booker, Jon. "Some Propositions About Auditing", The Accounting Review, July, 1970.

Fisher, Sir Ronald. "The Nature of Probability," The Centennial Review, Summer, 1958.

Kraft, William H. "Statistical Sampling For Auditors: A New Look," Journal of Accountancy, August, 1968.

Mautz, R. K., and Mini, Donald L., "Internal Control Evaluation and Audit Program Modification," The Accounting Review, April, 1966.

Sorensen, James E. "Bayesian Analysis In Auditing", The Accounting Review, July 1969.

Tracy, John A. "Bayesian Statistical Confidence Intervals For Auditors," Journal of Accountancy, July, 1969.

Tracy, John A. "Bayesian Statistical Methods in Auditing," The Accounting Review, January, 1969.