

ABSTRACT

SOME GEOMETRIC PROPERTIES OF COMPRESSIBLE FLUID FLOWS AND CERTAIN CLASSES OF SUCH FLOWS OBTAINED BY INTRINSIC METHODS

by Lester B. Fuller

Steady nonviscous nonheat-conducting flow of a compressible fluid in the absence of external forces is discussed in this paper. In particular the geometry of such flows is studied, the dynamical equations which characterize these flows are reformulated, and some classes of flows are obtained.

In connection with the geometry, a condition is obtained which is necessary and sufficient for the existence of stream surfaces that contain the vortex lines. These are called Lamb surfaces. Then, two necessary and sufficient conditions are found; one for streamlines to be geodesics on Lamb surfaces and the other for streamlines to be asymptotics on stream surfaces. More generally, two necessary and sufficient conditions are obtained; one for the existence of stream surfaces on which streamlines are geodesics and the other for the existence of stream surfaces on which streamlines are asymptotics.

Some of these conditions mentioned involve the magnitude of the velocity vector, and so, relationships between it and the geometry are observed. Thus, the geometry and the dynamics are related.

Concerning the dynamical equations and their solutions, the following has been done. A system of equations equivalent to the dynamical equations is obtained using two families

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of stream surfaces and the family of constant pressure surfaces as coordinate surfaces. From this reformulated system of equations, two classes of plane flows are found, and a single ordinary differential equation is obtained whose solution leads to a class of three dimensional flows.

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CHAPTER I

INTRODUCTION

1.1 Preliminaries

As the title of this paper indicates, it is our purpose herein to make some observations concerning certain fluid flows, the equations which characterize them, and their geometry. Consequently, it seems appropriate that we begin with a discussion and explanation of some of these terms.

A fluid flow is a set of functions which satisfies a certain system of nonlinear partial differential equations, and so, we first consider these equations which are referred to as the dynamical equations throughout this paper. We write the system using the standard summation convention ($i = 1, 2, 3$) and, following it, explain the notation and discuss its derivation.

$$(1.11) \quad \nabla_i \rho v^i = 0$$

$$(1.12) \quad v^i \nabla_i v_j = - \frac{1}{\rho} \nabla_j p \quad (j = 1, 2, 3)$$

$$(1.13) \quad v^i \nabla_i \eta = 0$$

In these equations we are using the customary notation of tensor calculus, so v^i and v_j are the contravariant and covariant components respectively of a vector field called the velocity vector field, and ∇_i represents the

covariant derivative. The quantities p , ρ , and η are scalar point functions known as pressure, density, and entropy, respectively. Consequently, the terms $\nabla_i \eta$ and $\nabla_j p$ are gradients of scalar point functions.

The dynamical equations are derived from the basic principles of conservation of mass, momentum, and energy. These derivations are given in most standard fluid mechanics texts, such as in the first chapter of H. Lamb [8]. The form of equation (1.13) we are using is nicely developed in R. Courant and K. O. Friedrichs [4, p. 14-16].

These equations are the mathematical model for steady nonviscous nonheat-conducting flow of a fluid in the absence of external forces. By the term steady we mean that the quantities appearing in these equations depend on position only and are independent of time; by nonviscous we mean the force an element of fluid exerts on an adjacent element is normal to their common surface; by nonheat-conducting we mean there is no flow of heat from a hotter portion of the fluid to a cooler portion except that which takes place by convection, that is, by the motion of the fluid itself.

A set of functions consisting of a vector function $\bar{v}$, with components v^i or v_j , and three scalar point functions η , p , and ρ , which identically satisfy the dynamical equations in some region of three dimensional space, is called a flow. Briefly, then, a flow is a solution of equations (1.11) to (1.13). In the special case where the density, ρ , is constant, the flow is called incompressible.

Otherwise, it is known as compressible. If the vector function $\bar{v}$ is such that $\text{curl } \bar{v} = \bar{0}$ throughout a region of three dimensional space, the flow is irrotational in that region. Otherwise, it is called rotational. Throughout this paper we use $\bar{\omega}$ to denote $\text{curl } \bar{v}$ and call it the vorticity or vortex vector. If the vorticity and velocity vectors of a flow are parallel at each point in a region, that is $\bar{v} \times \bar{\omega} = \bar{0}$, then the flow is called Beltrami.

Each curve of the family of integral curves of the velocity vector field is called a streamline, and, similarly, a vortex line is a member of the family of integral curves of the vortex vector field. From the physical viewpoint, a streamline in a steady flow is the path of a fluid particle. Thus we see that certain families of curves may be associated with a fluid flow.

It is also possible to associate various families of surfaces with a flow, as the following remarks indicate. (We tacitly assume that any requirements such as continuity and differentiability of functions is satisfied.) The presence of the pressure gradient in (1.12) brings to mind the concept from vector calculus of level surfaces of a scalar point function. In this case, these are surfaces on which p is constant and which have the property that, at each point, a surface normal is collinear with the pressure gradient. The partial differential equation $v^i \nabla_i \theta = 0$ has two linearly independent solutions which we shall denote by $\theta_1(x^1, x^2, x^3)$ and $\theta_2(x^1, x^2, x^3)$, where x^1, x^2 , and

x^3 denote independent variables. If we let C_1 and C_2 be arbitrary constants, then $\theta_1(x^1, x^2, x^3) = C_1$ and $\theta_2(x^1, x^2, x^3) = C_2$ represent families of surfaces known as stream surfaces.

From the two preceding paragraphs, we notice that we may associate with a flow geometric objects such as families of curves and families of surfaces. Furthermore, for a given flow, these curves and surfaces may have special properties. For example, the streamlines may be straight, or the streamlines and vortex lines may coincide as in a Beltrami flow. Hence, we describe a flow in terms of these geometric quantities and their properties and speak of the geometry of a flow. In case the streamlines are plane curves, and there exist stream surfaces which are planes with the property that all quantities of the flow do not change in the direction normal to them, we say the flow is a plane flow.

1.2 Background Concerning Geometry

We would like to mention some of the previous research in the area of the geometry of fluid flows which, to some extent, motivates the geometric considerations undertaken in this paper, or has a direct bearing upon them.

D. Gilbarg [5] poses the following question: In what way is the flow pattern (i.e. the streamlines) related to the velocity, and to what extent does one depend on the other? Or, we might state the question this way: to what

extent does the geometry of a flow determine its dynamics? Some partial answers to this question have been obtained. In the case of steady, incompressible, plane flow, Gilbarg determines all incompressible flows having the same flow pattern as an arbitrary given flow of an incompressible fluid. In fact, he shows that, if the given flow does not have a constant velocity magnitude along each individual streamline, the only flows with the same streamline pattern are those having velocity fields that are proportional to that of the given flow. R. C. Prim [12] extends these findings of Gilbarg to the three dimensional case. R. Wasserman [21] shows that, if the velocity vector $\bar{v} = q\bar{e}$ ($|\bar{e}| = 1$), corresponding to each incompressible (compressible) flow with $\nabla_i t^i = 0$ there is a compressible (incompressible) flow having the same streamlines and constant pressure surfaces. He also proves the converse, namely, that if a compressible and incompressible flow have the same (non-straight) streamlines and constant pressure surfaces, then they both have $\nabla_i t^i = 0$. R. C. Prim [13] points out that among all flows having the same streamlines and constant pressure surfaces, there is a flow containing a special family of surfaces, which is not necessarily present in all flows. We shall elaborate more fully upon this in Chapter II, but, at present, simply point out that much of Chapters II and III are motivated by the articles mentioned in this section.

In other areas of flow geometry, contributions have been made by N. Coburn, E. R. Suryanarayan, and C. Truesdell. N. Coburn [3] points out certain properties of the vorticity and velocity vectors and demonstrates, the interesting fact, that the pressure gradient lies in the osculating plane of the streamlines. E. R. Suryanarayan [16] decomposes the dynamical equations in terms of the tangent, principal normal, and binormal of the vortex lines and considers a flow in which the vortex lines are right circular helices. In another paper [17] he discusses the geometry of flows (known as complex lamellar flows) containing surfaces orthogonal to the streamlines and derives conditions for the orthogonal intersection of certain surfaces existing in these flows. C. Truesdell [19] writes a comprehensive article in which he emphasizes the generality of intrinsic methods (to be discussed in Chapter II of this paper) in analyzing flow geometry.

1.3 Background Concerning Solutions of the Dynamical Equations

In attempting to solve the dynamical equations, numerous difficulties are encountered, and so it is natural to make some simplifying assumptions. One such assumption is that there are Beltrami flows ($\bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}} = \bar{\mathbf{0}}$) satisfying these equations. Such flows, however, would include irrotational flows as a special case, and the further assumption that $\bar{\boldsymbol{\omega}} = \mathbf{0}$ would render the dynamical equations still less

complex. Additional simplification could be brought about if flows that are both incompressible and irrotational are sought. So, the various types of flows one might seek, in order of increasing difficulty, are

- (a) irrotational and incompressible,
- (b) irrotational but compressible, and
- (c) Beltrami.

We now discuss some of the progress that has been made in the special cases mentioned above. Case (a) leads to Laplace's equation and classical potential theory. For solutions in this case see H. Lamb [8, Chap. 5]. Case (b) is analyzed by R. Courant and K. O. Friedrichs [4, Chap. 4], and in the case of supersonic plane flow, a class of solutions known as Prandtl-Meyer flows is presented. In case (c) Beltrami [1] sets forth, and Nemenyi and Prim [11] discuss flows in which the streamlines are coaxial helices. This is done for an incompressible flow.

In the special case where pressure is constant on streamlines, R. Wasserman [21,22] obtains flows in which the streamlines are helices and another class of three dimensional compressible fluid flows.

1.4 Objectives of this Paper

As we pointed out, there seems to be an increase in difficulty as one proceeds from irrotational, incompressible flows to irrotational flows which allow compressibility effects, and then to Beltrami flows. One might look upon

this progression as being one in which we go from a consideration of flows without vortex lines ($\bar{\omega} = 0$) to a consideration of flows in which streamlines and vortex lines coincide ($\bar{\mathbf{v}} \times \bar{\omega} = \bar{0}$). As the next stepping stone of difficulty in the advance from irrotational flow to general steady non-viscous nonheat-conducting flows, in Chapter II we shall discuss the geometry of flows in which streamlines and vortex lines form surfaces. As we shall see, this need not always be the case, even when $\bar{\mathbf{v}} \times \bar{\omega} \neq 0$. In Chapter III, the classification of flows already indicated in this section will be elaborated upon, and relations between the geometry and $|\bar{\mathbf{v}}|$ shown. In Chapter IV, we shall reformulate the dynamical equations (1.11) to (1.13) and then, in Chapter V, use this reformulation to obtain two classes of plane flows. In Chapter VI, we obtain a class of three dimensional flows, and in both Chapters V and VI, the geometry of the flows obtained is of the type discussed in Chapter II.

There is no attempt to solve any boundary value problems in this paper, but, rather, we confine our considerations to the dynamical equations only. It is also to be understood that, in our discussion of geometry, we are concerned with what is commonly referred to, in differential geometry texts, as geometry in the small. Also, the solutions we obtain are restricted to some limited region of three dimensional space.

CHAPTER II

THE GEOMETRY OF COMPRESSIBLE FLOWS

2.1 The Equation of State

Equations (1.11) to (1.13) are a set of five equations in six unknowns, but there is another relationship among the unknowns which allows us to consider these as a system of five equations in five unknowns. We shall now introduce this relationship.

The laws of thermodynamics inform us that for any given medium only two of the quantities, pressure, density, and entropy, may be independent [4, p. 4]. This fact is expressed in the equation of state, $\rho = f(p, \eta)$.

If the medium is a gas in which the internal energy is simply proportional to the temperature (with proportionality constant c_v), the gas is called polytropic, and the equation of state is of the form

$$(2.11) \quad \rho = p^{1/\gamma} S(\eta),$$

where γ is a constant having a value between 1 and $\frac{5}{3}$ for most media. $S(\eta)$ is given by

$$S(\eta) = [(\gamma - 1) \exp c_v^{-1}(\eta - \eta_0)]^{\frac{1}{\gamma}}$$

with η_0 an appropriate constant [4, p. 6,7]. The assumption that a gas is polytropic is made in most applications. For example, air at moderate temperature may be considered polytropic with $\gamma = 1.40$.

In this paper we shall frequently assume that a separable equation of state holds, namely one of the form

$$(2.12) \quad \rho = P(p) S(\eta),$$

with $P(p)$ and $S(\eta)$ assuming only positive values. We notice that the equation of state for a polytropic gas is a special case of (2.12).

2.2 Intrinsic Methods

As we have already observed, various families of curves and surfaces accompany a flow. Furthermore, these curves and surfaces have associated with them certain quantities of a geometric nature such as curvature, torsion, principal normals and binormals, and first and second fundamental forms. When these quantities, which are inherent to a flow, are employed in its description, we speak of using intrinsic methods or intrinsic quantities.

In the analysis of the geometry of flows, the use of intrinsic methods has been a great aid [19, 10, p. 105, 570]. Both as illustrative examples, and for future reference, we derive some well known formulas by an intrinsic approach. In particular, we avail ourselves of the moving trihedron associated with a streamline, which consists of the three orthonormal vectors $\bar{t}$, $\bar{n}$, and $\bar{b}$; $\bar{t}$ being tangent to the streamline, $\bar{n}$ the principal normal, and $\bar{b}$ the binormal.

Letting the velocity $\bar{v} = q \bar{t}$, (1.12) becomes

$$qt^i \nabla_i qt_j = -1/\rho \nabla_j p, \quad j = 1, 2, 3.$$

Expanding the left side of this equation and using the fact that $\nabla_i (q^2/2) = q \nabla_i q$ yields

$$(2.21) \quad q^2 t^i \nabla_i t_j + [t^i \nabla_i (q^2/2)] t_j = -1/\rho \nabla_j p, \quad j = 1, 2, 3.$$

We substitute into the first term on the left from the Frenet formula, $t^i \nabla_i t_j = \kappa n_j$, where κ is the streamline curvature, and obtain

$$(2.22) \quad \nabla_j p = -\rho [t^i \nabla_i (q^2/2)] t_j - \rho q^2 \kappa n_j, \quad j = 1, 2, 3,$$

which says that the pressure gradient lies in the osculating plane of the streamline.

Before deriving a second important result by means of intrinsic methods, we define a function

$$h(p, \eta) = \int_{p_0}^p \frac{du}{\rho(u, \eta)},$$

where p_0 is a function of η only. This function $h(p, \eta)$ is known as the enthalpy, and from its definition, we see that

$$\nabla_j h = \frac{1}{\rho} \nabla_j p + \frac{\partial h}{\partial \eta} \nabla_j \eta.$$

Now we may write (1.12) in the form

$$qt^i \nabla_i qt_j = \frac{\partial h}{\partial \eta} \nabla_j \eta - \nabla_j h.$$

As in the derivation of (2.22), we obtain

$$[t^i \nabla_i (q^2/2)] t_j + q^2 \kappa n_j = \frac{\partial h}{\partial \eta} \nabla_j \eta - \nabla_j h, \quad j = 1, 2, 3.$$

Forming the scalar product of both sides of this equation with $\bar{t}$ and using (1.13) results in

$$t^i \nabla_i (q^2/2) = - t^i \nabla_i h.$$

Collecting terms, we have

$$t^i \nabla_i (q^2/2 + h) = 0.$$

We set

$$(2.23) \quad q^2/2 + h = B,$$

recognizing that B is constant along each streamline.

The fact that $q^2/2 + h$ is constant along each streamline is known as Bernoulli's law, and B is called Bernoulli's constant although it may vary from streamline to streamline. If B has the same value on each streamline, (2.23) expresses what is known as the strong form of Bernoulli's law.

A third example involving the moving trihedron of the streamline is an expression for the vorticity derived by N. Coburn [3, p. 118]. It is

$$(2.24) \quad \omega_j = q \left(b^k \frac{\partial t_k}{\partial n} - n^k \frac{\partial t_k}{\partial b} \right) t_j + \frac{\partial q}{\partial b} n_j + \left(q \kappa - \frac{\partial q}{\partial n} \right) b_j, \\ j = 1, 2, 3,$$

where partial derivatives with respect to n and b represent directional derivatives in the direction of the

principal normal and binormal respectively.

2.3 On the Existence and Properties of Certain Stream Surfaces

R. C. Prim [13, p. 436] has mentioned certain surfaces that exist in some flows that are intimately related with the substitution principle. Before generalizing a result of Prim's, we summarize this principle. It says that, for a separable equation of state, a solution of the dynamical equations is a member of an infinite family of flows having the same constant pressure surfaces and streamlines. Furthermore, the flows of this family are related by the equations

$$\begin{aligned}\bar{v}^* &= \bar{v}/m & p^* &= p \\ \rho^* &= m^2 \rho & S(\eta^*) &= m^2 S(\eta)\end{aligned}$$

where m is any scalar field such that $v^i \nabla_i m = 0$.

In these equations, the quantities without the stars (except for m) refer to one flow and the starred quantities to another flow. Prim points out that in such a family of flows there is one for which the strong Bernoulli law holds, and it has the special property of possessing a family of surfaces with unit normals $\bar{N}$ for which $\bar{N} \times (\bar{v} \times \nabla \times \bar{v}) = \bar{0}$. In general, however, given a vector field $\bar{v}$, the problem of finding a family of surfaces such that, for each member of the family, the direction cosines of its normal, at every point, are proportional to the components of the member of the vector field at the point, is not solvable

[24, p. 202]. Consequently, it is "natural" to seek a necessary and sufficient condition for the existence of such surfaces and to further investigate the geometry of flows containing them. We shall do this, but before proceeding further with this investigation, we introduce some terminology and state the problem in a slightly different fashion, which will be helpful to us.

We shall use the term $\bar{w}$ congruence of curves (or just $\bar{w}$ congruence) to mean the family of integral curves of a $\bar{w}$ vector field, or, more generally, by congruence of curves we shall mean a family of curves with the property that exactly one member of the family passes through each point of some region of space [5, p. 78]. A congruence of curves for which there exists a family of surfaces intersecting it orthogonally is called a normal congruence of curves. A necessary and sufficient condition for a congruence of curves to be normal is that any tangent vector field $\bar{w}$ associated with it satisfies the condition $\bar{w} \cdot \nabla \times \bar{w} = 0$ [24, p. 202].

As a consequence of these remarks, we may rephrase the property of the Prim substitution flows, relating to the existence of certain surfaces, by saying that in such flows the integral curves of the $\bar{v} \times (\nabla \times \bar{v})$ vector field form a normal congruence. So, we could view our inquiry as that of seeking a necessary and sufficient condition for the existence of such a normal congruence in a flow.

Theorem 2.31: The $\bar{v} \times \bar{\omega}$ congruence is normal if and only if at least one of the following holds:

- (a) entropy is constant on the vortex lines,
- (b) pressure is constant on the streamlines.

Proof: We write (1.12) in the form

$$(\bar{v} \cdot \nabla) \bar{v} = -(1/\rho) \nabla p$$

and apply the vector identity

$$(\bar{v} \cdot \nabla) \bar{v} = (1/2) \nabla (\bar{v} \cdot \bar{v}) - \bar{v} \times (\nabla \times \bar{v})$$

with $\bar{v} = q\bar{t}$. We get

$$(2.312) \quad \rho (\bar{v} \times \bar{\omega}) - \rho \nabla (q^2/2) = \nabla p.$$

We form the curl and obtain

$$\nabla \rho \times (\bar{v} \times \bar{\omega}) + \rho \nabla \times (\bar{v} \times \bar{\omega}) - \nabla \rho \times \nabla (q^2/2) = \bar{0}.$$

Forming the scalar product with $\bar{v} \times \bar{\omega}$ and transposing gives

$$\bar{v} \times \bar{\omega} \cdot \nabla \times (\bar{v} \times \bar{\omega}) = 1/\rho [(\bar{v} \times \bar{\omega}) \cdot \nabla \rho \times \nabla (q^2/2)].$$

From the equation of state, we see this may be written

$$\bar{v} \times \bar{\omega} \cdot \nabla \times (\bar{v} \times \bar{\omega}) = \frac{1}{\rho} \bar{v} \times \bar{\omega} \cdot \left[\left(\frac{\partial \rho}{\partial p} \nabla p + \frac{\partial \rho}{\partial \eta} \nabla \eta \right) \times \nabla \frac{q^2}{2} \right].$$

Equation (2.312) shows that $\nabla (q^2/2)$, ∇p , and $\bar{v} \times \bar{\omega}$ are coplanar, and hence, their scalar triple product is zero.

Therefore,

$$(2.315) \quad \bar{v} \times \bar{\omega} \cdot \nabla \times (\bar{v} \times \bar{\omega}) = \frac{1}{\rho} \frac{\partial \rho}{\partial \eta} \left[(\bar{v} \times \bar{\omega}) \cdot \nabla \eta \times \nabla \frac{q^2}{2} \right].$$

If, to the right side of this equation, we apply the identity of Lagrange, which says that

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c}),$$

and drop the term involving $\bar{v} \cdot \nabla \eta$ (by (1.13) this is zero), then (2.315) becomes

$$(\bar{v} \times \bar{\omega}) \cdot \nabla \times (\bar{v} \times \bar{\omega}) = - \frac{1}{\rho} \frac{\partial \rho}{\partial \eta} (\bar{\omega} \cdot \nabla \eta) (\bar{v} \cdot \nabla \frac{q^2}{2}).$$

It is to be inferred from Section 2.1 that ρ **depends on** η **in such a way** that $\frac{\partial \rho}{\partial \eta} \neq 0$. Thus $(\bar{v} \times \bar{\omega}) \cdot \nabla \times (\bar{v} \times \bar{\omega}) = 0$ if $\bar{\omega} \cdot \nabla \eta = 0$ or $\bar{v} \cdot \nabla (q^2/2) = 0$, and, conversely, if $(\bar{v} \times \bar{\omega}) \cdot \nabla \times (\bar{v} \times \bar{\omega}) = 0$, then either $\bar{\omega} \cdot \nabla \eta = 0$ or $\bar{v} \cdot \nabla (q^2/2) = 0$. Equation (2.312), however, implies that $\bar{v} \cdot \nabla (q^2/2) = 0$ if and only if $\bar{v} \cdot \nabla p = 0$, and the theorem is proved.

Hence, entropy constant on vortex lines or pressure constant on streamlines guarantees the existence of a family of surfaces orthogonal to the $\bar{v} \times \bar{\omega}$ congruence. In keeping with the terminology of Truesdell [18, p. 133], we call these surfaces Lamb surfaces.

Corollary 2.32: If entropy is constant on the vortex lines, then the entropy is a constant on each Lamb surface of the flow.

Proof: By Theorem 2.31, the Lamb surfaces exist in the flow, and $\bar{v} \times \bar{\omega}$ is a vector field normal to the Lamb

surfaces. But, by hypothesis, $\bar{\omega} \cdot \nabla \eta = 0$, and by (1.13), $\bar{v} \cdot \nabla \eta = 0$. Therefore,

$$\nabla \eta \times (\bar{v} \times \bar{\omega}) = (\bar{\omega} \cdot \nabla \eta) \bar{v} - (\bar{v} \cdot \nabla \eta) \bar{\omega} = \bar{0}$$

and entropy is constant on the Lamb surfaces.

Corollary 2.33: If entropy is constant on the vortex lines, the Bernoulli constant is the same for each streamline in a Lamb surface. That is, B is a constant on each Lamb surface.

Proof: Again, Theorem 2.31 implies the existence of the Lamb surfaces, and that $\bar{v} \times \bar{\omega}$ is a vector field normal to the Lamb surfaces. The Crocco-Vazsonyi equation [14, p. 186] states that

$$\bar{v} \times \bar{\omega} = \nabla B - T \nabla \eta,$$

where T represents absolute temperature. By Corollary 2.32, $\bar{v} \times \bar{\omega} = \alpha \nabla \eta$ for some scalar point function α . Hence,

$$\nabla B = (\alpha + T) \nabla \eta,$$

and the gradients of B and η are parallel, so their level surfaces coincide.

By introducing the term Bernoulli surface for a level surface of B , we may draw the following conclusions from Theorem 2.31 and its corollaries, concerning a flow in which pressure is not constant on streamlines. If Lamb surfaces exist, then Bernoulli surfaces and constant entropy surfaces

coincide and in fact are the same as the Lamb surfaces. Or, if the Bernoulli surfaces and constant entropy surfaces differ, then the flow cannot contain Lamb surfaces.

Two articles, one by R. C. Prim [13] and another by P. Smith [15], motivated our investigation. For this reason, we use the term Prim-Smith flow, throughout the remainder of this paper, to designate a flow with the following two properties:

- (a) it possesses Lamb surfaces,
- (b) its constant pressure surfaces are not stream surfaces.

2.4 A Special Congruence of Curves Lying on the Lamb Surfaces

Theorem 2.41: At each point of a Prim-Smith flow or a flow in which constant pressure surfaces are stream surfaces, the vectors $\nabla\eta$, $\nabla\rho$, ∇p and ∇q are coplanar.

Proof: Case I - Prim-Smith flow: The equation of motion in the form

$$(2.42) \quad \nabla q^2/2 - \bar{v} \times \bar{\omega} = -\frac{1}{\rho} \nabla p$$

implies that ∇q , $\bar{v} \times \bar{\omega}$ and ∇p are coplanar. In a Prim-Smith flow, however, $\bar{v} \times \bar{\omega}$ and $\nabla\eta$ are collinear. So ∇q lies in the plane of $\nabla\eta$ and ∇p . From the equation of state, it is clear that $\nabla\rho$ is also in the plane of ∇p and $\nabla\eta$, and hence, the four gradients, $\nabla\eta$, $\nabla\rho$, ∇p , and ∇q are coplanar.

Case II - Constant pressure surfaces are stream surfaces:

In this case $\bar{t} \cdot \nabla p = 0$. This, together with equation (2.42), implies that $\bar{t} \cdot \nabla q = 0$. Since $\bar{t} \cdot \nabla \eta = 0$ in the flows under consideration, the equation of state implies that $\bar{t} \cdot \nabla \rho = 0$. So at each point of the flow $\nabla \eta$, $\nabla \rho$, ∇p and ∇q lie in a plane normal to the streamline through the point.

We observe that in Case II, η , ρ , p , and q are constant on the streamlines and extend this property in the following corollary.

Corollary 2.43: In a flow with constant pressure surfaces and Lamb surfaces that are distinct, η , ρ , p , and q are constant on the integral curves of the unit vector field given by

$$(2.44) \quad \bar{x} = \frac{[\bar{t} \cdot \nabla(\frac{q^2}{2})]\bar{w} - [\bar{w} \cdot \nabla(\frac{q^2}{2})]\bar{t}}{\sqrt{[\bar{t} \cdot \nabla(\frac{q^2}{2})]^2 + [\bar{w} \cdot \nabla(\frac{q^2}{2})]^2 - 2[\bar{t} \cdot \nabla(\frac{q^2}{2})][\bar{w} \cdot \nabla(\frac{q^2}{2})]\bar{w} \cdot \bar{t}}},$$

where $\bar{\omega} = \omega \bar{w}$ with $|\bar{w}| = 1$.

Proof: Case I - Prim-Smith flow: By the symmetry in (2.44), $\bar{x} \cdot \nabla q = 0$. Since $\bar{t} \cdot \nabla \eta = 0$, according to the energy equation, and $\bar{w} \cdot \nabla \eta = 0$ in a Prim-Smith flow, $\bar{x} \cdot \nabla \eta = 0$. According to Theorem 2.41, ∇p and $\nabla \rho$ lie in the plane spanned by $\nabla \eta$ and ∇q . Therefore, $\bar{x} \cdot \nabla p = \bar{x} \cdot \nabla \rho = 0$ and the corollary's conclusion holds for a Prim-Smith flow.

Case II - Constant pressure surfaces are stream surfaces:

As previously noted $\bar{t} \cdot \nabla p = 0$ if and only if $\bar{t} \cdot \nabla (q^2/2) = 0$. By hypothesis the constant pressure surfaces are distinct from the Lamb surfaces, so that $\bar{w} \cdot \nabla (q^2/2) \neq 0$. Therefore $\bar{x} = \pm \bar{t}$, and as observed immediately preceding this corollary, η , ρ , p , and q are constant on the streamlines, in this case. Thus the corollary is proved.

2.5 Geodesics and Asymptotics on Lamb Surfaces

We have already observed in Case II of the proof of Theorem 2.41 that, when streamlines lie on constant pressure surfaces, η , p , ρ , and q are constant on the streamlines. Since the equation of motion, (2.22), implies that $\bar{b} \cdot \nabla p$ is invariably zero, the additional property of p being constant on the streamlines ($\bar{t} \cdot \nabla p = 0$) implies that the streamlines are geodesics [24, p. 99] on the constant pressure surfaces. Conversely, if a streamline is (a geodesic) on a constant pressure surface, $\bar{t} \cdot \nabla p = 0$, and as we have shown in Case II of Theorem 2.41, η , ρ , p , and q are constant on the streamlines. We summarize these remarks in a theorem. Admittedly, the term geodesic could be omitted from the statement, but we insert it because we wish to draw attention to this geodesic property.

Theorem 2.51: The functions η , ρ , p , and q are constant on the streamlines of a flow if and only if the streamlines are (geodesics) on the surfaces of constant pressure.

In the case of constant pressure surfaces, then, the streamlines need merely to lie on the surfaces in order to be geodesics. In general, of course, this is not the case, and we now obtain a necessary and sufficient condition for a streamline to be a geodesic on a Lamb surface. Forming the cross product of (2.24) with $\bar{v}$, we obtain

$$(2.52) \quad \bar{v} \times \bar{\omega} = q \left(\frac{\partial q}{\partial n} - q\kappa \right) \bar{n} + q \frac{\partial q}{\partial b} \bar{b}.$$

If $\bar{v} \times \bar{\omega} \neq 0$, then (2.52) implies that $\partial q / \partial b = \bar{b} \cdot \nabla q = 0$ if and only if $\bar{v} \times \bar{\omega}$ is collinear with $\bar{n}$, the principal normal of the streamline. But, at each point of a Lamb surface, $\bar{v} \times \bar{\omega}$ is collinear with a surface normal. So, a streamline is a geodesic on a Lamb surface if and only if $\bar{b} \cdot \nabla q = 0$. That is, q constant on each curve of the $\bar{b}$ congruence is a necessary and sufficient condition for streamlines to be geodesics on Lamb surfaces. Theorem 2.41 and the fact that $\bar{b} \cdot \nabla p = 0$ imply that η , ρ , p , and q are constant on each curve of the $\bar{b}$ congruence whenever q is. Therefore, we can summarize our remarks as follows:

Theorem 2.53: A necessary and sufficient condition for streamlines to be geodesics on Lamb surfaces is that η , ρ , p , and q are constant on the $\bar{b}$ congruence.

Since q being constant on each curve of the $\bar{b}$ congruence (or $\bar{b}$ congruence) implies η , ρ , and p are also, and conversely, we may consider Theorems 2.51 and 2.53 as

relating dynamic properties of a flow to geometric properties. In fact, they say that $\bar{t} \cdot \nabla q = 0$ is a necessary and sufficient condition for streamlines to be geodesics on constant pressure surfaces, and $\bar{b} \cdot \nabla q = 0$ is a necessary and sufficient condition for streamlines to be geodesics on Lamb surfaces (in a flow containing such surfaces). As a consequence of these remarks, we have the following corollary.

Corollary 2.54: $\bar{t} \cdot \nabla q = \bar{b} \cdot \nabla q = 0$ if and only if the Lamb surfaces and constant pressure surfaces coincide.

A further consideration of Theorems 2.51 and 2.53 reveals that the $\bar{X}$ vector of Corollary 2.43 is collinear with the $\bar{t}$ vector when streamlines are geodesics on constant pressure surfaces, and collinear with the $\bar{b}$ vector when streamlines are geodesics on Lamb surfaces. This leads us to ask if, under certain conditions, $\bar{X}$ is collinear with $\bar{n}$. If it were, pressure would be constant on each curve of the $\bar{n}$ congruence. Equation (2.22) shows that $\bar{n} \cdot \nabla p = 0$ if and only if $\kappa = 0$, that is, the streamlines are straight. So, we may immediately conclude, that if the streamlines are not straight, the $\bar{X}$ and $\bar{n}$ vectors cannot be collinear.

Since a straight line that lies entirely in a surface is an asymptotic line of a surface [5, p. 237], the remarks of the last paragraph draw our attention to asymptotic curves on Lamb surfaces. Equation (2.52) implies that $\bar{v} \times \bar{\omega}$ and $\bar{b}$ are collinear if and only if

$$\kappa = \frac{1}{q} \frac{\partial q}{\partial n} = \bar{n} \cdot \nabla \ln q.$$

At each point of a flow containing Lamb surfaces, $\bar{v} \times \bar{\omega}$ is in the direction of a normal to the Lamb surface through the point. So, at each point of a flow, a unit normal to the Lamb surface and the $\bar{b}$ vector coincide if and only if $\kappa = \bar{n} \cdot \nabla \ln q$. Thus we have proved the following:

Theorem 2.55: In a flow containing Lamb surfaces, the streamlines are asymptotics on these surfaces if and only if $\kappa = \bar{n} \cdot \nabla \ln q$.

E. R. Suryanarayan [17] has shown that (excluding straight streamlines) $\kappa = \bar{n} \cdot \nabla \ln q$ is a necessary and sufficient condition for Lamb surfaces and constant pressure surfaces to intersect orthogonally, provided entropy is constant throughout the region of flow. His assumption of constant entropy is not necessary, however, as his conclusion follows if $\bar{n} \cdot \nabla \eta = \bar{t} \cdot \nabla \eta = 0$. So, a slight modification of his statement would be this: in any Prim-Smith flow with non-straight streamlines, $\kappa = \bar{n} \cdot \nabla \ln q$ is a necessary and sufficient condition for Lamb surfaces and constant pressure surfaces to intersect orthogonally, as well as for streamlines to be asymptotics on the Lamb surfaces.

We close this section by making a few remarks concerning some similarities between Beltrami flows and flows containing Lamb surfaces. R. C. Prim [13, p. 434] has pointed out that $\nabla \eta$, $\nabla \rho$, ∇p , and ∇q are collinear in a Beltrami flow. From (2.52), however, $\bar{v} \times \bar{\omega} = \bar{0}$ implies $\bar{b} \cdot \nabla q = 0$. Hence, for a Beltrami flow, η , ρ , p , and q are constant on the $\bar{b}$ congruence. From equation (2.52),

it is also clear that $\kappa = \bar{n} \cdot \nabla \ln q$ in a Beltrami flow. Consequently, the conditions mentioned in Theorem 2.53 concerning geodesics on Lamb surfaces, and the condition in Theorem 2.55 for asymptotics on Lamb surfaces, are both enjoyed by a Beltrami flow. Of course, we do not have Lamb surfaces in a Beltrami flow except, perhaps, in a degenerate sense.

2.6 Geodesics on Stream Surfaces

In the last section, we considered two cases in which streamlines were geodesics on stream surfaces. It is our purpose here to obtain a necessary and sufficient condition for the existence of stream surfaces on which streamlines are geodesics.

Equation (2.24) is not only valid for the vorticity vector of a fluid flow but, also, for $\text{curl } \bar{v}$ of an arbitrary vector field $\bar{v} = q\bar{t}$ where $|\bar{t}| = 1$. We choose $\bar{v} = \bar{t}$, an arbitrary unit vector, and substitute. This gives

$$\nabla \times \bar{t} = (\bar{b} \cdot \frac{\partial \bar{t}}{\partial n} - \bar{n} \cdot \frac{\partial \bar{t}}{\partial b}) \bar{t} + \kappa \bar{b}.$$

Thus $\bar{n} \cdot \nabla \times \bar{t} = 0$.

Theorem 2.61: If the pressure gradient is not tangent to the streamline throughout a flow, then a necessary and sufficient condition for the existence of surfaces on which streamlines are geodesics is that $\bar{t} \cdot \nabla p$ be constant along the $\bar{b}$ congruence.

Proof: For suitable scalar functions α and β , the equation of motion may be written as

$$\alpha \bar{t} + \beta \bar{n} = \nabla p.$$

Forming the curl yields

$$(2.62) \quad \nabla \alpha \times \bar{t} + \alpha \nabla \times \bar{t} + \nabla \beta \times \bar{n} + \beta \nabla \times \bar{n} = \bar{0}.$$

Dotting with $\bar{n}$ and using the fact that $\bar{n} \cdot \nabla \times \bar{t} = 0$ yields

$$\bar{b} \cdot \nabla \alpha + \beta \bar{n} \cdot \nabla \times \bar{n} = 0.$$

Therefore, $\beta \bar{n} \cdot \nabla \times \bar{n} = -\bar{b} \cdot \nabla \alpha$.

By assumption $\beta \neq 0$, and so $\bar{n} \cdot \nabla \times \bar{n} = 0$ if and only if $\alpha = \bar{t} \cdot \nabla p$ is constant along the $\bar{b}$ congruence. But $\bar{n} \cdot \nabla \times \bar{n} = 0$ means the $\bar{n}$ congruence is a normal congruence, so there is a family of surfaces cutting it orthogonally. On these surfaces the streamlines are geodesics.

2.7 Asymptotics on Stream Surfaces

In Section 2.5 we obtained a necessary and sufficient condition for streamlines to be asymptotics on Lamb surfaces. More generally, one might seek a necessary and sufficient condition for the existence of stream surfaces on which streamlines are asymptotics. We do this in this section. Before embarking on such an investigation, however, we note that the problem could be rephrased by saying we seek a necessary and sufficient condition for the $\bar{v} \times \nabla p$ congruence to be a normal congruence. Furthermore, since

∇p lies in the osculating plane of the streamlines, we are seeking a condition for which there exists a scalar point function ψ such that $\bar{t} \cdot \nabla \psi = 0$ and $\bar{n} \cdot \nabla \psi = 0$. Hence we must satisfy the integrability conditions [2, p. 186-187], namely

$$(t^j n^\ell - n^j t^\ell) \nabla_j b_\ell = 0.$$

Using the Frenet formula $t^j \nabla_j b_\ell = -\tau n_\ell$, where τ is the torsion of the streamlines, and the fact that $t^\ell n^j \nabla_j b_\ell = -b^\ell n^j \nabla_j t_\ell$, the integrability condition becomes

$$(2.71) \quad \tau = b^\ell n^j \nabla_j t_\ell.$$

To investigate (2.71) further, we make use of the vector identity $\nabla(\bar{u} \cdot \bar{v}) = (\bar{u} \cdot \nabla)\bar{v} + (\bar{v} \cdot \nabla)\bar{u} + \bar{u} \times (\nabla \times \bar{v}) + \bar{v} \times (\nabla \times \bar{u})$. Replacing $\bar{u}$ by $\bar{n}$ and $\bar{v}$ by $\bar{t}$, we have

$$(2.72) \quad 0 = (\bar{n} \cdot \nabla)\bar{t} + (\bar{t} \cdot \nabla)\bar{n} + \bar{n} \times (\nabla \times \bar{t}) + \bar{t} \times (\nabla \times \bar{n}).$$

Using the Frenet formula $(\bar{t} \cdot \nabla)\bar{n} = -\kappa \bar{t} + \tau \bar{b}$ and forming the scalar product of the right side of (2.72) with $\bar{b}$, we obtain

$$0 = \bar{b} \cdot (\bar{n} \cdot \nabla)\bar{t} + \tau - \bar{t} \cdot \nabla \times \bar{t} + \bar{n} \cdot \nabla \times \bar{n}.$$

Substituting from equation (2.71) yields

$$(2.73) \quad 2\tau = \bar{t} \cdot \nabla \times \bar{t} - \bar{n} \cdot \nabla \times \bar{n}.$$

From equation (2.62) and the fact that $\bar{n} \cdot \nabla \times \bar{t} = 0$, we

observe that

$$\bar{t} \cdot \nabla \times \bar{t} = \frac{1}{\alpha} (\bar{b} \cdot \nabla \beta - \beta \bar{t} \cdot \nabla \times \bar{n}),$$

and

$$\bar{n} \cdot \nabla \times \bar{n} = -\frac{1}{\beta} \bar{b} \cdot \nabla \alpha.$$

Substitution into (2.73) shows that

$$2\tau = \bar{b} \cdot \left(\frac{1}{\alpha} \nabla \beta + \frac{1}{\beta} \nabla \alpha \right) - \frac{\beta}{\alpha} \bar{t} \cdot \nabla \times \bar{n}.$$

By a well known vector identity,

$$\nabla \cdot \bar{b} = \nabla \cdot (\bar{t} \times \bar{n}) = \bar{n} \cdot \nabla \times \bar{t} - \bar{t} \cdot \nabla \times \bar{n} = -\bar{t} \cdot \nabla \times \bar{n},$$

and we have

$$2\tau = \bar{b} \cdot \frac{\nabla \frac{\beta^2}{2} + \nabla \frac{\alpha^2}{2}}{\alpha\beta} + \frac{\beta}{\alpha} \nabla \cdot \bar{b} = \frac{\bar{b} \cdot \nabla |\nabla p|^2}{2\alpha\beta} + \frac{\beta}{\alpha} \nabla \cdot \bar{b}.$$

Hence, we have proved the following theorem.

Theorem 2.74: If ∇p is not collinear with either $\bar{t}$ or $\bar{n}$, a necessary and sufficient condition for the existence of stream surfaces on which streamlines are asymptotics is that τ , the torsion of the streamlines, be given by the expression

$$\tau = \frac{1}{2} \left[\frac{\bar{b} \cdot \nabla |\nabla p|^2}{2\alpha\beta} + \frac{\beta}{\alpha} \nabla \cdot \bar{b} \right],$$

where

$$\alpha = \bar{t} \cdot \nabla p \quad \text{and} \quad \beta = \bar{n} \cdot \nabla p.$$

We conclude this chapter with a few remarks about Theorems 2.61 and 2.74. In both theorems, it is assumed that $\bar{n} \cdot \nabla p$ is not zero. This simply means that the special case of straight streamlines is not considered. It happens that a straight line is both a geodesic and an asymptotic on a surface containing it. Theorem 2.74 also fails to take into account the situation in which $\alpha = \bar{t} \cdot \nabla p = 0$. In this case, equation (2.62) implies that $\bar{n} \cdot \nabla \times \bar{n} = 0$, and from (2.73), we see that a necessary and sufficient condition for streamlines to be asymptotics on stream surfaces is that $2\tau = \bar{t} \cdot \nabla \times \bar{t}$.

CHAPTER III

RELATIONS BETWEEN $\bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}}$ AND THE MAGNITUDE OF THE VELOCITY VECTOR

3.1 Introductory Remarks

It has been proven by M. H. Martin [9, p. 470] that for plane flow, a necessary and sufficient condition for an irrotational flow is that q , the magnitude of the velocity vector, depends on pressure only. In this chapter, we would like to extend this remark by showing relationships between q and $\bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}}$.

We assume throughout this chapter that surfaces of constant pressure are not stream surfaces, and we denote by $\psi = \text{constant}$ and $\phi = \text{constant}$, where ψ and ϕ are scalar point functions, two distinct families of stream surfaces. Thus, we consider q to be a function of p , ϕ , and ψ .

3.2 Beltrami Flows

The equation of motion may be written in the form

$$\bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}} = \nabla(q^2/2) + 1/\rho \nabla p.$$

Expanding the term involving q yields

$$(3.21) \quad \bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}} = (q q_p + 1/\rho) \nabla p + q q_\phi \nabla \phi + q q_\psi \nabla \psi.$$

By assumption $\bar{\mathbf{t}} \cdot \nabla p \neq 0$, $\bar{\mathbf{t}} \cdot \nabla \phi = 0$, and $\bar{\mathbf{t}} \cdot \nabla \psi = 0$, so (3.21) implies that

$$(3.22) \quad q q_p + 1/\rho = 0,$$

and consequently,

$$(3.23) \quad \bar{v} \times \bar{\omega} = q(q_\phi \nabla \phi + q_\psi \nabla \psi).$$

It follows immediately from this equation that $\bar{v} \times \bar{\omega} = \bar{0}$ if q depends on pressure only, since then $q_\phi = q_\psi = 0$. On the other hand, suppose $\bar{v} \times \bar{\omega} = \bar{0}$ ($q \neq 0$) throughout a region of space. For the coordinate system under consideration, $\nabla \phi$ and $\nabla \psi$ are not zero, and so $\bar{v} \times \bar{\omega} = \bar{0}$ implies one of the following two possibilities:

$$(a) \quad \nabla \phi = - \frac{q_\psi}{q_\phi} \nabla \psi \quad \text{or} \quad (b) \quad q_\phi = q_\psi = 0.$$

Case (a) implies that the surfaces on which ϕ is constant, and the surfaces on which ψ is constant coincide, which contradicts the known independence of ϕ and ψ . Therefore, $q_\phi = q_\psi = 0$, and q depends on pressure only. In summary we state

Theorem 3.24: If constant pressure surfaces are not stream surfaces of a flow, the flow is Beltrami if and only if the magnitude of the velocity vector depends only on the pressure.

3.3 Prim-Smith Flows

Proceeding to flows in which streamlines and vortex lines do not coincide but form Lamb surfaces, we observe a special property of the velocity magnitudes, which we now state.

Theorem 3.31: If constant pressure surfaces are not stream surfaces, a compressible fluid flow contains Lamb surfaces if and only if q , the velocity magnitude, is a function of pressure and entropy only.

Proof: We recall that Lamb surfaces are stream surfaces on which η , the entropy, is constant provided $\bar{t} \cdot \nabla p \neq 0$. Therefore, we may replace ϕ by η in equation (3.23) and obtain

$$(3.32) \quad \bar{v} \times \bar{\omega} = q(q_\eta \nabla \eta + q_\psi \nabla \psi).$$

By Theorem 2.31, if pressure is not constant on streamlines, $\bar{\omega} \cdot \nabla \eta = 0$ whenever Lamb surfaces occur in a flow, and therefore, forming the scalar product of (3.32) with $\bar{\omega}$ yields

$$(3.33) \quad 0 = q q_\psi \bar{\omega} \cdot \nabla \psi.$$

Since $q \neq 0$, equation (3.33) implies either $q_\psi = 0$ or $\bar{\omega} \cdot \nabla \psi = 0$. The latter equation can not hold, however, since $\bar{\omega} \cdot \nabla \psi = 0$, $\bar{v} \cdot \nabla \eta = 0$, $\bar{\omega} \cdot \nabla \eta = 0$, and $\bar{v} \cdot \nabla \psi = 0$ imply that

$$(\bar{v} \times \bar{\omega}) \times \nabla \eta = (\bar{v} \times \bar{\omega}) \times \nabla \psi = \bar{0}.$$

Hence, $\nabla \eta$ and $\nabla \psi$ are collinear, which contradicts the independence of η and ψ . Therefore, $q_\psi = 0$, where $\psi = \text{constant}$ represents any family of stream surfaces distinct from the family of Lamb surfaces, and thus in a flow containing Lamb surfaces, $q = q(p, \eta)$.

Conversely, let us consider a flow in which $q = q(p, \eta)$, and the constant pressure surfaces are not stream surfaces. Then, by a method analogous to the one for deriving equation (3.23) we obtain

$$\bar{v} \times \bar{\omega} = q \, q_{\eta} \nabla \eta.$$

Hence, $\bar{\omega} \cdot \nabla \eta = 0$, and according to Theorem 2.31, the $\bar{v} \times \bar{\omega}$ congruence is normal. In other words, the flow is of Prim-Smith type.

3.4 A Classification of Steady Compressible Flows

In the table below, we have attempted to picture a classification of all flows of the type mentioned in our introduction for which $\bar{t} \cdot \nabla p \neq 0$. The left portion of the interior of the table represents flows for which $\bar{v} \times \bar{\omega} = \bar{0}$, the upper part for irrotational flows and the lower part for flows in which streamlines and vortex lines coincide. The right side of the table represents flows for which $\bar{v} \times \bar{\omega} \neq \bar{0}$, the upper portion for flows in which streamlines and vortex lines form surfaces and the lower part for all others. We have also indicated the restrictions on the magnitude of the velocity in these cases.

$\bar{v} \times \bar{\omega} = \bar{0}$	$\bar{v} \times \bar{\omega} \neq \bar{0}$
$\bar{\omega} = \bar{0}$ $q = q(p)$	streamlines and vortex lines form surfaces $q = q(p, \dot{S})$
streamlines and vortex lines coincide $q = q(p)$	others $q = q(p, \psi, S)$

$$(\bar{t} \cdot \nabla p \neq 0)$$

CHAPTER IV

A REFORMULATION OF THE DYNAMICAL EQUATIONS

4.1 Preliminaries

A transformation of variables is frequently helpful in putting a differential equation or system of equations into a more desirable form. In the area of fluid mechanics, the hodograph transformation is an example of this. We illustrate it in the case of compressible, irrotational, plane flow.

First, we introduce two functions, $\psi(x,y)$ and $\phi(x,y)$, where $\psi(x,y)$ is a constant on each streamline, and $\phi(x,y)$ is constant on each orthogonal trajectory of the streamlines. This can be done in the case of plane flow, and under the assumption of an irrotational flow, $\phi(x,y)$ may be considered as a potential function, so $\nabla\phi = \bar{v}$. A function, such as $\psi(x,y)$, which is constant on each streamline, is called a stream function. Next, we let θ be the angle measured counterclockwise from a positive x-axis to the velocity vector, and as usual, $q = |\bar{v}|$. The other quantities ρ and c , which appear below, are dependent on θ and q . The function ρ , again, represents density, and c is the sound speed defined by the equation $c^2 = \frac{\partial p}{\partial \rho}$ (it is a fundamental property of all actual media that, entropy remaining constant, the pressure increases

with increasing density). It can be shown [23, Chap. 4] that a compressible, irrotational, plane flow can then be characterized by the following pair of equations, where the intrinsic variables ψ and ϕ are used as independent variables rather than x and y .

$$\rho q \frac{\partial \theta}{\partial \psi} - \left(\frac{q^2}{c^2} - 1 \right) \frac{\partial q}{\partial \phi} = 0 \quad (4.11)$$

$$q \frac{\partial \theta}{\partial \phi} - \rho \frac{\partial q}{\partial \psi} = 0.$$

Interchanging the roles of the variables q, θ and ϕ, ψ by using

$$\frac{\partial q}{\partial \phi} = - \frac{1}{D} \frac{\partial \psi}{\partial \theta}$$

$$\frac{\partial q}{\partial \psi} = \frac{1}{D} \frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial \theta}{\partial \phi} = \frac{1}{D} \frac{\partial \psi}{\partial q}$$

$$\frac{\partial \theta}{\partial \psi} = - \frac{1}{D} \frac{\partial \phi}{\partial q}$$

where $D = \begin{vmatrix} \phi_q & \phi_\theta \\ \psi_q & \psi_\theta \end{vmatrix}$, we get from (4.11)

$$\rho q \frac{\partial \phi}{\partial q} - \left(\frac{q^2}{c^2} - 1 \right) \frac{\partial \psi}{\partial \theta} = 0 \quad (4.12)$$

$$\rho \frac{\partial \phi}{\partial \theta} - q \frac{\partial \psi}{\partial q} = 0$$

These are known as the hodograph equations. We note that equations (4.12) are linear while equations (4.11) are not. Furthermore, we can, if we wish, eliminate either ϕ or ψ and get a second order linear equation for a single variable. For applications of these equations see R. Courant and K. O. Friedrichs [4, p. 248-259].

A second type of transformation of variable occurring in the field of fluid mechanics is mentioned by R. von Mises [20, p. 433] and employed by M. H. Martin [9, p. 465-484] in the case of plane flow. This change of variable amounts to using the pressure, p , and a stream function (intrinsic quantities) as independent variables, and considering the other variables and the coordinates x, y as unknown functions of them. In other words, under the assumption that pressure is not constant on streamlines, the streamlines and isobars (curves on which pressure is constant) are taken as curvilinear coordinates rather than the streamlines and their orthogonal trajectories as in equations (4.11). Using this approach, M. H. Martin started with the system of differential equations

$$\rho \left(\frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v \right) + \frac{\partial p}{\partial x} = 0$$

$$\rho \left(\frac{\partial v}{\partial x} u + \frac{\partial v}{\partial y} v \right) + \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0,$$

introduced a stream function ψ , and considered ρ as a

known functions of p and ψ . He then reduced the problem of finding u , v , x and y as functions of p and ψ to the integration of one quasi-linear partial differential equation for a single unknown function.

It is our purpose in this chapter to extend this technique, which was fruitful for M. H. Martin in the case of plane flow, to a similar technique for three dimensional flows. As in Martin's approach, the streamlines will be coordinate curves. The other coordinate curves will consist of two distinct families of curves on which the pressure is constant. Such coordinate curves are realized by introducing a family of constant pressure surfaces which are not stream surfaces and two distinct families of stream surfaces as coordinate surfaces. Using this coordinate system, we shall reformulate the dynamical equations, (1.11) to (1.13), and obtain an equivalent system.

4.2 Reformulation - Step One

We shall assume, now and throughout the remainder of this paper, that a separable equation of state holds (see Section 2.1) and rewrite the dynamical equations replacing η with S in the last equation.

$$\left. \begin{aligned} \nabla_i \rho v^i &= 0 \\ v^i \nabla_i v_j &= -\frac{1}{\rho} \nabla_j p, \quad j = 1, 2, 3 \\ v^i \nabla_i S &= 0 \end{aligned} \right\} \quad (A)$$

The quantities v^i ($i = 1, 2, 3$), p , and S are considered as the five unknowns of this system, and it is assumed that $P(p)$ is known. Hence, ρ is determined from the equation of state. Since $v^i = qt^i$ and $|\bar{t}| = 1$, we may also consider the five dependent variables as q , p , S , and two components of $\bar{t}$.

We commence our reformulation of system (A) with its first equation (the continuity equation), by substituting qt^i for v^i and obtain

$$\nabla_i \rho q t^i = 0.$$

Expanding this yields

$$\rho q \nabla_i t^i + t^i \nabla_i \rho q = 0.$$

Since

$$\nabla_i \ln \rho q = \frac{1}{\rho q} \nabla_i \rho q,$$

dividing by ρq results in

$$\nabla_i t^i + t^i \nabla_i \ln \rho q = 0.$$

Substituting from the separable equation of state for ρ gives

$$\nabla_i t^i + t^i \nabla_i \ln PSq = 0.$$

Using this equation and the equation of motion in the form of (2.22), with ρ replaced by PS , system (A) becomes the equivalent system

$$\begin{aligned}
 (4.21) \quad & \nabla_i t^i + t^i \nabla_i \ln PSq = 0 \\
 (4.22) \quad & PSq^2 \kappa n_j + PS[t^i \nabla_i (q^2/2)] t_j = -\nabla_j p, \quad j = 1, 2, 3 \\
 (4.23) \quad & v^i \nabla_i S = 0.
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} (4.21) \\ (4.22) \\ (4.23) \end{aligned}} \right\} (B)$$

4.3 Reformulation - Step Two

We avail ourselves of the coordinate system mentioned in the last paragraph of Section 4.1, letting $X^1 = p$, the pressure, $X^2 = \psi$, and $X^3 = \phi$, where ψ is constant on each member of one family of stream surfaces and ϕ is constant on each member of another (distinct) family of stream surfaces. In this coordinate system, the streamlines are coordinate curves along which ψ and ϕ are constants and p varies. Furthermore, we let N^i denote a unit normal to the family of stream surfaces on which ϕ is constant and let $'n^i$ denote a vector cross product of t^i and N^i such that t^i , $'n^i$, and N^i form a right hand orthogonal system (see Figure 1).

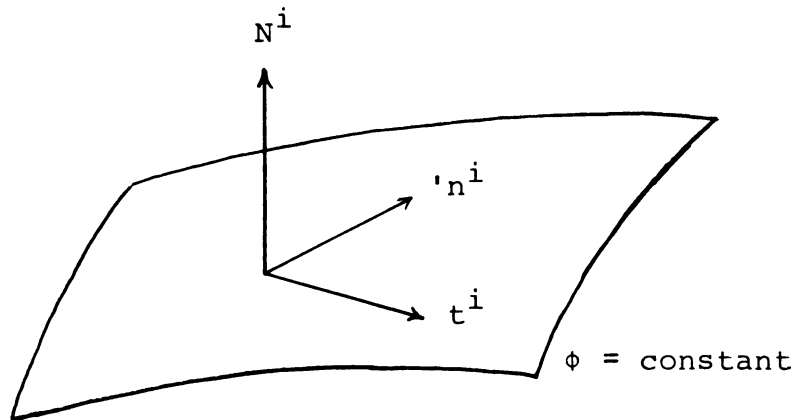


Figure 1.

From equation (4.22) we obtain the components of the pressure gradient in the directions of these three orthogonal vectors. Since $t^j n_j = 0$ and $t^j t_j = 1$, the dot product of (4.22) with t^j yields

$$(4.31) \quad t^j \nabla_j p = - P S t^i \nabla_i (q^2/2).$$

Making use of the Frenet formula $\kappa n_j = t^i \nabla_i t_j$ and the fact that $'n^j$ and N^j are orthogonal to t^j , the scalar product of (4.22) with $'n^j$ and N^j produces the following two equations.

$$'n^j \nabla_j p = - P S q^2 'n^j t^i \nabla_i t_j$$

$$N^j \nabla_j p = - P S q^2 N^j t^i \nabla_i t_j.$$

According to the energy equation, $t^i \nabla_i S = 0$. Therefore,

$$t^i \nabla_i (S q^2/2) = (q^2/2) t^i \nabla_i S + S t^i \nabla_i (q^2/2) = S t^i \nabla_i (q^2/2),$$

and (4.31) may be written as

$$t^j \nabla_j p = - P t^i \nabla_i (S q^2/2).$$

Consequently, system (B) may be written in the following form.

$$\begin{array}{lcl}
 (4.32) & \nabla_i t^i + t^i \nabla_i \ln pSq = 0 & \\
 (4.33) & t^j \nabla_j p = - p t^i \nabla_i (Sq^2/2) & \\
 (4.34) & 'n^j \nabla_j p = - pSq^2 'n^j t^i \nabla_i t_j & \\
 (4.35) & N^j \nabla_j p = - pSq^2 N^j t^i \nabla_i t_j & \\
 (4.36) & t^i \nabla_i S = 0 &
 \end{array} \quad \left. \vphantom{\begin{array}{l} (4.32) \\ (4.33) \\ (4.34) \\ (4.35) \\ (4.36) \end{array}} \right\} (C)$$

It is true that the terms $'n^j t^i \nabla_i t_j$ and $N^j t^i \nabla_i t_j$ are commonly called the geodesic and normal curvature, respectively, and given abbreviated notations. For future computational purposes, however, it is more convenient to leave them in their present form, so we do it.

4.4 Reformulation - Step Three

Upon examining system (C), we observe the term Sq^2 appearing in three of the five equations. The first equation of this system may also be written in terms of Sq^2 , and then (4.32) to (4.35) may be considered as a system of four equations with four dependent variables, p , Sq^2 , and t^i . We now rewrite (4.32).

Multiplying (4.32) by two and using a property of logarithms, we get

$$2 \nabla_i t^i + t^i \nabla_i \ln p^2 S^2 q^2 = 0.$$

Using another property of the logarithm function and the fact that $t^i \nabla_i \ln S = 0$, this equation may be written in

the form

$$2 \nabla_i t^i + t^i \nabla_i \ln p^2 S q^2 = 0.$$

Consequently, by replacing $S q^2$ with u , system (C) may be written as follows.

$$\left. \begin{aligned} 2 \nabla_j t^j + t^j \nabla_j \ln p^2 u &= 0 \\ t^j \nabla_j p &= - p t^i \nabla_i (u/2) \\ 'n^j \nabla_j p &= - p u 'n^j t^i \nabla_i t_j \\ N^j \nabla_j p &= - p u N^j t^i \nabla_i t_j \\ t^j \nabla_j S &= 0 \end{aligned} \right\} \quad (D)$$

4.5 Reformulation - Step Four

We now observe a few facts which allow us to simplify system (D). In the coordinate system we are using, $x^1 = p$, and p is independent of x^2 and x^3 . Therefore,

$$\nabla_j p = \begin{cases} 1 & j = 1 \\ 0 & j = 2, 3. \end{cases}$$

Consequently,

$$t^j \nabla_j p = t^1, \quad 'n^j \nabla_j p = 'n^1, \quad N^j \nabla_j p = N^1.$$

The unit vector t^i is tangent to the streamlines, along which only $x^1 = p$ varies, and hence $t^2 = t^3 = 0$. As a result, such expressions as $t^i \nabla_i t^j$ and $t^i \nabla_i S$, in

system (D), become $t^1 \nabla_1 t_j$ and $t^1 \nabla_1 S$. In the light of these remarks, the coordinate system we are considering permits us to write system (D) (and hence the original dynamical equations) in the form

$$\begin{array}{lcl}
 (4.51) & 2 \nabla_1 t^1 + t^1 \nabla_1 \ln p^2 u = 0 & \\
 (4.52) & t^1 = - P t^1 \nabla_1 (u/2) & \\
 (4.53) & 'n^1 = - P u 'n^j t^1 \nabla_1 t_j & \\
 (4.54) & N^1 = - P u N^j t^1 \nabla_1 t_j & \\
 (4.55) & t^1 \nabla_1 S = 0 &
 \end{array} \quad \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \end{array}} \right\} (E)$$

4.6 Reformulation - Step Five - Introduction of the Metric Coefficients and Final Form of (4.51) and (4.52)

Proceeding with our reformulation we let $\bar{e}_1, \bar{e}_2$, and $\bar{e}_3$ denote a set of base vectors for the x^1, x^2, x^3 coordinate system chosen such that $\bar{e}_i$ is tangent to the curve on which x^i varies. Then the vector $\bar{t}$ may be written in the form

$$\bar{t} = t^1 \bar{e}_1 + t^2 \bar{e}_2 + t^3 \bar{e}_3.$$

Since $t^2 = t^3 = 0$,

$$\bar{t} = t^1 \bar{e}_1.$$

But, $\bar{t}$ is a unit vector, and therefore,

$$\bar{t} \cdot \bar{t} = 1 = (t^1)^2 \bar{e}_1 \cdot \bar{e}_1.$$

Letting $\bar{e}_1 \cdot \bar{e}_1 = g_{11}$,

$$t^1 = 1/\sqrt{g_{11}} .$$

We introduce the reciprocal base vectors $\bar{e}^i$ ($i = 1, 2, 3$) defined by $\bar{e}^i \cdot \bar{e}_j = \delta_j^i$, where δ_j^i is the Kronecker delta. Since $\bar{N}$ is a unit normal vector to the family of surfaces $X^3 = \phi = \text{constant}$, $N_1 = N_2 = 0$ and,

$$\bar{N} \cdot \bar{N} = 1 = (N_3)^2 \bar{e}^3 \cdot \bar{e}^3 .$$

Denoting $\bar{e}^3 \cdot \bar{e}^3$ by g^{33} , we have

$$N_3 = 1/\sqrt{g^{33}} .$$

From differential geometry one knows that g_{11} , as we have defined it, is an element of the metric tensor and that g^{33} is the reduced cofactor of another member of the metric tensor, namely g_{33} . Hence, we shall denote by g_{ij} ($i, j = 1, 2, 3$) the elements of the space metric tensor and by g^{ij} the reduced cofactor of g_{ij} . Since $t_j = g_{ji} t^i$ and $t^2 = t^3 = 0$, we see that

$$t_j = g_{j1} t^1 = \frac{g_{j1}}{\sqrt{g_{11}}} .$$

Similarly $N^j = g^{ji} N_i$ implies that

$$N^j = \frac{g^{j3}}{\sqrt{g^{33}}} .$$

We have defined $'n^i$ as the vector product of N^i crossed with t^i , and consequently [2, p. 145],

$${}'_n{}^i = \frac{e^{ijk} N_j t_k}{\sqrt{g}} ,$$

where g is the determinant of the g_{ij} . Substituting the expressions for N_j and t_k in terms of the elements of the metric tensor we obtain

$${}'_n{}^1 = \frac{-g_{12}}{\sqrt{gg_{11}g^{33}}} , \quad {}'_n{}^2 = \frac{g_{11}}{\sqrt{gg_{11}g^{33}}} , \quad \text{and} \quad {}'_n{}^3 = 0.$$

Prior to substituting into system (D), we collect the formulas just derived in one place for ready reference and state a formula for the divergence of a vector v^i [2, p. 171].

$$(4.61) \quad \left\{ \begin{array}{l} t^j = \left(\frac{1}{\sqrt{g_{11}}} , 0, 0 \right) \\ N_j = \left(0, 0, \frac{1}{\sqrt{g^{33}}} \right) \\ {}'_n{}^j = \frac{1}{\sqrt{gg_{11}g^{33}}} (-g_{12}, g_{11}, 0) \end{array} \right. \quad \begin{array}{l} t_j = \frac{g_{j1}}{\sqrt{g_{11}}} \\ N^j = \frac{g^{j3}}{\sqrt{g^{33}}} \end{array}$$

$$(4.62) \quad \nabla_j v^j = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} v^j)}{\partial x^j}$$

Using (4.62), the components of t^j given in (4.61), and the fact that $\ln P^2 u$ is a scalar, equation (4.51) may be written as

$$\frac{2}{\sqrt{g}} \frac{\partial}{\partial x^1} \sqrt{\frac{g}{g_{11}}} + \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial x^1} \ln P^2 u = 0.$$

Multiplying by $\sqrt{g_{11}}$ and recalling that $x^1 = p$, this equation becomes

$$\frac{\partial}{\partial p} \ln \frac{p^2 u g}{g_{11}} = 0.$$

Substituting for the component t^1 in (4.52) and realizing that

$$\nabla_1 \left(\frac{u}{2} \right) = \frac{\partial}{\partial p} \left(\frac{u}{2} \right),$$

we obtain

$$\frac{1}{\sqrt{g_{11}}} = - \frac{p}{\sqrt{g_{11}}} \frac{\partial}{\partial p} \left(\frac{u}{2} \right).$$

Multiplying by $2\sqrt{g_{11}}$ yields

$$2 + p \frac{\partial u}{\partial p} = 0.$$

4.7 Reformulation - Step Six - Another Form of (4.53) and (4.54)

In equations (4.53) and (4.54), we have the expression $t^1 \nabla_1 t_j$ appearing. To expand this we use the formula

$$(4.71) \quad \nabla_k t_j = \frac{\partial t_j}{\partial x^k} - \Gamma_{jk}^\ell t_\ell,$$

where Γ_{jk}^ℓ is a Christoffel symbol of the second kind expressed by

$$(4.72) \quad \Gamma_{jk}^\ell = \frac{g^{\ell i}}{2} \left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right).$$

Substituting into (4.53) for n^j and t^1 as given in (4.61), we obtain

$$\frac{-g_{12}}{\sqrt{g g_{11} g^{33}}} = \frac{-Pu}{\sqrt{g g_{11} g^{33}}} \left(\frac{-g_{12}}{\sqrt{g_{11}}} \nabla_1 t_1 + \frac{g_{11}}{\sqrt{g_{11}}} \nabla_1 t_2 \right)$$

or

$$(4.73) \quad g_{12} = Pu \left(\sqrt{g_{11}} \nabla_1 t_2 - \frac{g_{12}}{\sqrt{g_{11}}} \nabla_1 t_1 \right) .$$

As we shall now show, however, $\nabla_1 t_1 = 0$, and by means of (4.71) we have

$$(4.74) \quad g_{12} = Pu \sqrt{g_{11}} \left(\frac{\partial t_2}{\partial x^1} - \Gamma_{21}^{\ell} t_{\ell} \right) .$$

To prove that $\nabla_1 t_1 = 0$, we observe that $t^i \nabla_i t^j t_j = 0$. This implies that $t^j t^i \nabla_i t_j = 0$. Substituting from (4.61) for the components of $\bar{t}$, we see that $(1/g_{11}) \nabla_1 t_1 = 0$. Hence, $\nabla_1 t_1 = 0$.

Expanding (4.54) in a manner analogous to that used in expanding (4.53) yields

$$(4.75) \quad g^{13} = \frac{-Pu}{\sqrt{g_{11}}} \left(g^{23} \nabla_1 t_2 + g^{33} \nabla_1 t_3 \right) .$$

Since $\nabla_1 t_1 = 0$, (4.73) implies that

$$\nabla_1 t_2 = \frac{g_{12}}{Pu \sqrt{g_{11}}} .$$

Using this expression for $\nabla_1 t_2$ in (4.75), we obtain

$$g^{13} = \frac{-g^{23} g_{12}}{g_{11}} - \frac{Pu g^{33}}{\sqrt{g_{11}}} \nabla_1 t_3 .$$

Multiplying by g_{11}/g^{33} and employing (4.71) gives

$$(4.76) \quad \frac{g_{11}g^{13} + g^{23}g_{12}}{g^{33}} = -Pu\sqrt{g_{11}} \left(\frac{\partial t_3}{\partial x^1} - \Gamma_{31}^{\ell} t_{\ell} \right).$$

Since g^{ij} is the reduced cofactor of g_{ij} ,

$$g^{13} = \frac{g_{21}g_{23} - g_{22}g_{31}}{g}, \quad g^{23} = \frac{g_{21}g_{31} - g_{11}g_{32}}{g}$$

$$g^{33} = \frac{g_{11}g_{22} - (g_{12})^2}{g}$$

and

$$\frac{g_{11}g^{13} + g^{23}g_{12}}{g^{33}} = \frac{g_{31}[(g_{21})^2 - g_{11}g_{22}]}{g_{11}g_{22} - (g_{12})^2} = -g_{31}.$$

Therefore, (4.76) may be written

$$(4.77) \quad g_{31} = Pu\sqrt{g_{11}} \left(\frac{\partial t_3}{\partial x^1} - \Gamma_{31}^{\ell} t_{\ell} \right).$$

4.8 Reformulation - Step Seven - Final Form of (4.53) and (4.54)

In the last section, (4.53) and (4.54) of system (E) became equations (4.74) and (4.77), respectively. We write these two equations in the form

$$g_{1j} = Pu\sqrt{g_{11}} \left(\frac{\partial t_j}{\partial x^1} - \Gamma_{j1}^{\ell} t_{\ell} \right), \quad j = 2, 3,$$

and put them entirely in terms of u , P , and the elements of the metric tensor. To accomplish this, we first substitute

the expressions for t_j (and t_ℓ) given in (4.61). This yields

$$g_{1j} = P_u \sqrt{g_{11}} \left[\frac{\partial}{\partial p} \left(\frac{g_{j1}}{\sqrt{g_{11}}} \right) - \Gamma_{j1}^\ell \frac{g_{\ell 1}}{\sqrt{g_{11}}} \right], \quad j = 2, 3.$$

Solving for the term involving the Christoffel symbols, we obtain

$$\Gamma_{j1}^\ell g_{\ell 1} = \sqrt{g_{11}} \frac{\partial}{\partial p} \left(\frac{g_{1j}}{\sqrt{g_{11}}} \right) - \frac{g_{1j}}{P_u}.$$

Since $\Gamma_{j1}^\ell g_{\ell 1} = \frac{1}{2} \frac{\partial g_{11}}{\partial x^j}$ [5, p. 98], we see that (4.53)

and (4.54) of system (E) may finally be written

$$\frac{\partial g_{11}}{\partial x^j} = 2 \left[\sqrt{g_{11}} \frac{\partial}{\partial p} \left(\frac{g_{1j}}{\sqrt{g_{11}}} \right) - \frac{g_{1j}}{P_u} \right], \quad j = 2, 3.$$

4.9 Reformulation - Final Step

Combining the results we have obtained following system of equations (E) and recognizing that (4.55) simply implies that S depends on x^2 and x^3 only, we see that from the dynamical equations we have derived the following system of equations.

$$\begin{array}{ll}
(4.91) & \frac{\partial}{\partial p} \left(\frac{p^2 u g}{g_{11}} \right) = 0 \\
(4.92) & 2 + p \frac{\partial u}{\partial p} = 0 \\
(4.93) & \frac{\partial g_{11}}{\partial x^2} = 2 \left[\sqrt{g_{11}} \frac{\partial}{\partial p} \left(\frac{g_{12}}{\sqrt{g_{11}}} \right) - \frac{g_{12}}{p u} \right] \\
(4.94) & \frac{\partial g_{11}}{\partial x^3} = 2 \left[\sqrt{g_{11}} \frac{\partial}{\partial p} \left(\frac{g_{13}}{\sqrt{g_{11}}} \right) - \frac{g_{13}}{p u} \right] \\
(4.95) & S = S(x^2, x^3)
\end{array} \quad \left. \vphantom{\begin{array}{l} (4.91) \\ (4.92) \\ (4.93) \\ (4.94) \\ (4.95) \end{array}} \right\} (F)$$

On the other hand, each step in deriving (F) from (A) is reversible. So, suppose we introduce into Euclidean three space two independent families of surfaces and let the curves in which they intersect be curves along which p varies. One of the families of surfaces can be taken as a family such that on each member x^2 is constant and the other as one such that on each member x^3 is constant. Then, if system (F) has a solution consisting of $u = f(p, x^2, x^3)$, $S = h(x^2, x^3)$ and x, y , and z as functions of p, x^2 , and x^3 , in a region where the Jacobian is not zero, we may obtain a solution of system (A) consisting of q, p, S and t^i as functions of x, y , and z .

It should, perhaps, be remarked that equations (4.93) and (4.94) of system (F) can be put in a more compact form provided g_{12} and g_{13} are not zero. Factor g_{12} and g_{13} from the brackets, replace $1/p$ by $-(1/2)u_p$ (using equation 4.92), and introduce logarithms. Then these

equations become

$$(4.96) \quad \frac{\partial g_{11}}{\partial x^2} = g_{12} \frac{\partial}{\partial p} \ln \left(\frac{ug_{12}^2}{g_{11}} \right)$$

$$(4.97) \quad \frac{\partial g_{11}}{\partial x^3} = g_{13} \frac{\partial}{\partial p} \ln \left(\frac{ug_{13}^2}{g_{11}} \right)$$

In the next two chapters, however, we analyze situations in which $g_{12} = 0$, and for this reason, leave (F) in its present form.

CHAPTER V

TWO CLASSES OF PLANE FLOWS

5.1 A Coordinate System for Plane Flows

In this chapter we obtain two classes of plane flows using system (F). So, we consider the streamlines as lying in planes parallel to an xy -coordinate plane.

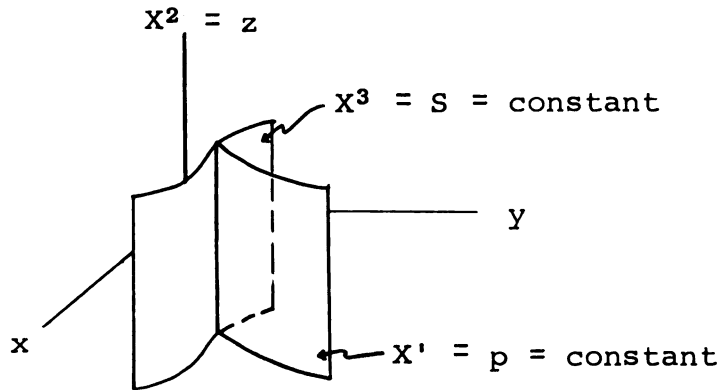


Figure 2

Under these conditions, we may choose the family of stream surfaces composed of these planes as a family of coordinate surfaces. We do this and let $x^2 = z$ be a constant on each of these coordinate planes. As a second family of coordinate surfaces, we choose the stream surfaces that are cylinders, each of which has a streamline for directrix and a line parallel to the z axis as generator (see Figure 2). These are surfaces on which x^3 is constant. But these cylindrical stream surfaces are Lamb surfaces

because the vortex vector is perpendicular to the xy -plane in flows such as we are considering, and consequently, the vortex lines are generators of these surfaces on which X^3 is constant. By Corollary 2.32, the function S that appears in the separable equation of state is constant on each of these Lamb surfaces. Hence, S may be taken as the X^3 variable. To complete our coordinate system, we take, as the third coordinate family, the family of cylinders each of which has an isobar in the xy -plane as directrix and a line parallel to the z axis as generator. This will complete our coordinate system as long as pressure is not constant on streamlines.

5.2 The Metric Tensor

The rectangular Cartesian coordinate system with x, y, z coordinates and the coordinate system of Section 5.1 are related by the transformation

$$x = x(p, S) \quad y = y(p, S) \quad z = X^2,$$

where we assume that the Jacobian

$$J = x_p y_S - x_S y_p \neq 0.$$

Since

$$g_{ij} = \frac{\partial x}{\partial x^i} \frac{\partial x}{\partial x^j} + \frac{\partial y}{\partial x^i} \frac{\partial y}{\partial x^j} + \frac{\partial z}{\partial x^i} \frac{\partial z}{\partial x^j},$$

the metric tensor is as follows:

$$(g_{ij}) = \begin{pmatrix} x_p^2 + y_p^2 & 0 & x_p x_S + y_p y_S \\ 0 & 1 & 0 \\ x_p x_S + y_p y_S & 0 & x_S^2 + y_S^2 \end{pmatrix}$$

Furthermore, the determinant of this matrix is

$$g = (x_p y_S - y_p x_S)^2 ,$$

and consequently,

$$g^{13} = \frac{-(x_p x_S + y_p y_S)}{(x_p y_S - y_p x_S)^2} \quad \text{and} \quad g^{33} = \frac{x_p^2 + y_p^2}{(x_p y_S - y_p x_S)^2} .$$

5.3 Equations (F) in the Case of Plane Flow

From the metric tensor of the previous section, we observe that $g_{21} = 0$ and g_{11} is independent of x^2 , so that equation (4.93) of equations (F) is identically satisfied.

We assume $g_{13} \neq 0$, replace (4.94) by (4.97) and solve for the expression $\frac{\partial}{\partial p} \ln \left(\frac{u}{g_{11}} \right)$ as follows:

$$\frac{1}{g_{13}} \frac{\partial g_{11}}{\partial S} = \frac{\partial}{\partial p} \ln \left(\frac{u}{g_{11}} \right) + \frac{\partial}{\partial p} \ln g_{13}^2$$

$$\frac{\partial}{\partial p} \ln \left(\frac{u}{g_{11}} \right) = \frac{1}{g_{13}} \left(\frac{\partial g_{11}}{\partial S} - 2 \frac{\partial g_{13}}{\partial p} \right)$$

Next, we substitute the metric tensor components and simplify.

This yields

$$\frac{\partial}{\partial p} \ln \left(\frac{u}{x_p^2 + y_p^2} \right) = \frac{1}{x_p x_S + y_p y_S} \left[\frac{\partial}{\partial S} (x_p^2 + y_p^2) - 2 \frac{\partial}{\partial p} (x_p x_S + y_p y_S) \right].$$

Expanding and simplifying the right side gives

$$\frac{\partial}{\partial p} \ln \left(\frac{u}{x_p^2 + y_p^2} \right) = \frac{-2(x_{pp}x_S + y_{pp}y_S)}{x_p x_S + y_p y_S}.$$

Substituting for g and g_{11} from the previous section, equation (4.91) becomes

$$\frac{\partial}{\partial p} \left[\frac{p^2 u (x_p y_S - y_p x_S)^2}{x_p^2 + y_p^2} \right] = 0$$

The problem of finding a plane flow satisfying (F) is now the problem of finding functions $x(p, S)$, $y(p, S)$ and $u(p, S)$ which satisfy

$$(5.31) \quad \frac{\partial}{\partial p} \ln \left(\frac{u}{x_p^2 + y_p^2} \right) = \frac{-2(x_{pp}x_S + y_{pp}y_S)}{x_p x_S + y_p y_S}$$

$$(5.32) \quad \frac{\partial}{\partial p} \left[\frac{p^2 u (x_p y_S - y_p x_S)^2}{x_p^2 + y_p^2} \right] = 0$$

$$(5.33) \quad p \frac{\partial u}{\partial p} = -2$$

where the last equation is a slight variation of (4.92).

Before proceeding to solve equations (5.31) to (5.33), we make a remark concerning the requirement that $g_{13} \neq 0$. This says that the isobars and streamlines are not perpendicular. Suppose, however, that they were. Then ∇p

and $\bar{t}$ would be parallel and $\bar{n} \cdot \nabla p$ would be zero. From equation (2.22) we find that $\bar{n} \cdot \nabla p = -\rho q^2 \kappa$, and it follows that $\kappa = 0$, which means the streamlines are straight. Conversely, for straight streamlines $\kappa = 0$, and we observe from (2.22) that $\bar{t}$ is parallel to ∇p . This, in turn, implies that the streamlines and isobars are perpendicular. Thus the requirement that $g_{13} \neq 0$ excludes plane flows in which the streamlines are straight.

5.4 A Class of Plane Flows Obtained by a Separation of Variable Technique

We now return to equations (5.31) to (5.33) and attempt to find functions $x(p, S)$, $y(p, S)$ and $u(p, S)$ satisfying these equations under the assumption that $x = \alpha(p)$ and $y = \beta(p)\gamma(S)$. With x and y of this form, $x_S = 0$, and equation (5.31) becomes

$$\frac{\partial}{\partial p} \ln \left(\frac{u}{x_p^2 + y_p^2} \right) = \frac{\partial}{\partial p} \ln y_p^{-2} .$$

Integration yields

$$(5.41) \quad u = \frac{f^2(S)(x_p^2 + y_p^2)}{y_p^2} ,$$

where $f^2(S)$ is an arbitrary positive function.

Substituting from (5.41) into (5.32) and recalling that $x_S = 0$, we obtain

$$\frac{\partial}{\partial p} \left(\frac{Pf(S) x_p y_S}{y_p} \right)^2 = 0 ,$$

or

$$\frac{\partial}{\partial p} \left(\frac{p f(S) x_p y_S}{y_p} \right) = 0.$$

Integration gives

$$(5.42) \quad \frac{p(p) f(S) x_p y_S}{y_p} = h(S),$$

where $h(S)$ is an arbitrary function. Substituting for the derivatives of x and y in terms of $\alpha(p)$, $\beta(p)$, $\gamma(S)$ and their derivatives yields

$$\frac{p(p) f(S) \alpha'(p) \beta(p) \gamma'(S)}{\beta'(p) \gamma(S)} = h(S).$$

Separating the variables we have

$$\frac{p(p) \alpha'(p) \beta(p)}{\beta'(p)} = \frac{h(S) \gamma(S)}{f(S) \gamma'(S)} = k,$$

where k is the separation constant. From this we see that the following two equations hold.

$$(5.43) \quad \frac{\beta'(p)}{\beta(p)} = \frac{p(p) \alpha'(p)}{k}$$

$$(5.44) \quad \frac{\gamma'(S)}{\gamma(S)} = \frac{h(S)}{k f(S)}$$

In equation (5.33) we substitute for u from equation (5.41) and get

$$p f^2(S) \frac{\partial}{\partial p} \left(\frac{x_p^2 + y_p^2}{y_p^2} \right) = -2.$$

Therefore,

$$p \frac{\partial}{\partial p} \left(\frac{x_p^2 + y_p^2}{y_p^2} \right) = \frac{-2}{f^2(S)} .$$

But

$$\frac{\partial}{\partial p} \left(\frac{x_p^2 + y_p^2}{y_p^2} \right) = \frac{\partial}{\partial p} \left(\frac{x_p^2}{y_p^2} + 1 \right) = \frac{\partial}{\partial p} \left(\frac{x_p^2}{y_p^2} \right) ,$$

so that the previous equation becomes

$$p \frac{\partial}{\partial p} \left(\frac{x_p^2}{y_p^2} \right) = \frac{-2}{f^2(S)} .$$

In terms of the functions $\alpha(p)$, $\beta(p)$ and $\gamma(S)$, this is

$$p \frac{\partial}{\partial p} \frac{[\alpha'(p)]^2}{[\beta'(p)]^2 \gamma^2(S)} = \frac{-2}{f^2(S)} .$$

Separating variables we obtain

$$p \frac{d}{dp} \frac{[\alpha'(p)]^2}{[\beta'(p)]^2} = \frac{-2 \gamma^2(S)}{f^2(S)} = R$$

where R is a separation constant that must be negative.

From the last equation, we obtain two equations which must be satisfied by $\alpha(p)$, $\beta(p)$ and $\gamma(S)$, namely,

$$(5.45) \quad \frac{d}{dp} \left(\frac{\alpha'(p)}{\beta'(p)} \right)^2 = \frac{R}{p^2(p)} \quad (R < 0)$$

$$(5.46) \quad \gamma^2(S) = - \frac{R}{2} f^2(S)$$

Thus we have four equations, (5.43) to (5.46), imposing restrictions on $\alpha(p)$, $\beta(p)$, $\gamma(S)$, and the functions of integration, $f(S)$ and $h(S)$. We first determine $\alpha(p)$

and $\beta(p)$ from (5.43) and (5.45) and then, determine $\gamma(S)$ from (5.44) and (5.46). From (5.43) we see that

$$\frac{\alpha'(p)}{\beta'(p)} = \frac{k}{P(p) \beta(p)} .$$

Substituting this into (5.45) yields a differential equation for $\beta(p)$ alone, that is,

$$k^2 \frac{d}{dp} \left[\frac{1}{P(p) \beta(p)} \right]^2 = \frac{R}{P(p)} .$$

Integration yields

$$(5.47) \quad \beta^2(p) = \frac{k^2}{P^2(p) [R I(p) + C_1]} ,$$

where $I(p)$ is an integral of $1/P(p)$ and C_1 is a constant of integration chosen such that $RI(p) + C_1 > 0$ in the region of flow being considered.

We now proceed to find $\alpha(p)$. From (5.43), we see that

$$\alpha'(p) = \frac{k\beta'(p)}{P(p) \beta(p)} .$$

Logarithmic differentiation of (5.47) followed by substitution in this last equation for $\beta'(p)/\beta(p)$ gives

$$\alpha'(p) = \frac{-k}{2P^2(p)} \left[2P'(p) + \frac{R}{RI(p) + C_1} \right] ,$$

and integration yields

$$(5.48) \quad \alpha(p) = \frac{k}{P(p)} - \frac{kR}{2} \int \frac{dp}{P^2(p) [RI(p) + C_1]} + C_2 ,$$

where C_2 is a constant of integration.

Now, if we can choose the arbitrary functions $f(S)$ and $h(S)$ and the constants k , R such that equations (5.44) and (5.46) can both be satisfied, we will have a solution to the system of equations (5.31) to (5.33). Taking the logarithmic derivative of (5.46) yields

$$\frac{\gamma'(S)}{\gamma(S)} = \frac{f'(S)}{f(S)} .$$

Substituting from (5.44) and multiplying by $k f(S)$, we have

$$(5.49) \quad h(S) = k f'(S).$$

Consequently, choosing $f(S)$ and k determines $h(S)$.

In summary, we may obtain a plane flow satisfying equations (5.31) to (5.33) by proceeding according to the following steps.

- (1) Pick an arbitrary function $f(S)$ and two constants k and R ($R < 0$). Then $\gamma(S)$ is determined by (5.46). In order for the Jacobian (see Section 5.2) to be different from zero, $f(S)$ must not be a constant function.
- (2) Choose constants C_1 and C_2 , with C_1 such that $RI(p) + C_1 > 0$. Then, determine $\alpha(p)$ from (5.48) and $\beta(p)$ from (5.47). Care must be exercised in choosing C_1 and R (of first step) so that $\beta'(p)$ is not zero.
- (3) Finally, $u(p, S)$ is determined from (5.41).

It is of interest to note that in these flows a constant value of p implies a constant value of x . Hence, the surfaces of constant pressure are parallel planes, each of which is parallel to the yz -plane.

5.5 A Second Class of Plane Flows Obtained by a Separation of Variable Technique

Returning to equations (5.31) to (5.33) we attempt to find another solution, namely one of the form $x = \alpha(p)$, $y = \beta(p) + \gamma(S)$. The function u is again obtained from (5.31) and is given by

$$u = \frac{f^2(S) (x_p^2 + y_p^2)}{y_p^2},$$

where f is an arbitrary function of S . As in the previous section, we can start with equation (5.32) and obtain (5.42) which reads

$$\frac{P(p) f(S) x_p y_s}{y_p} = h(S),$$

where $h(S)$ is an arbitrary function. Under the present assumption that $y(p, S) = \beta(p) + \gamma(S)$, this equation leads to

$$(5.51) \quad \frac{P(p) f(S) \alpha'(p) \gamma'(S)}{\beta'(p)} = h(S).$$

Separating variables yields

$$\frac{P(p) \alpha'(p)}{\beta'(p)} = \frac{h(S)}{f(S) \gamma'(S)} = A,$$

where A is a separation constant. From this, we have the following two equations.

$$(5.52) \quad \beta'(p) = \frac{P(p) \alpha'(p)}{A}$$

$$(5.53) \quad \gamma'(S) = \frac{h(S)}{Af(S)}$$

In precisely the same way as in the previous section, we begin with (5.33) and find that

$$P(p) \frac{\partial}{\partial p} \left(\frac{x_p^2}{y_p^2} \right) = \frac{-2}{f^2(S)}.$$

But under the present assumptions, when we substitute for x_p and y_p , we obtain

$$P(p) \frac{d}{dp} \left[\frac{\alpha'(p)}{\beta'(p)} \right]^2 = \frac{-2}{f^2(S)} = D,$$

where D is a negative separation constant. This requires that

$$(5.54) \quad f^2(S) = -\frac{2}{D}$$

and

$$(5.55) \quad P(p) \frac{d}{dp} \left[\frac{\alpha'(p)}{\beta'(p)} \right]^2 = D.$$

Consequently, we see that to obtain a flow where $x(p,S)$ and $y(p,S)$ are of the assumed form, the various constants and functions involved must satisfy equations (5.52) to

(5.55) with u being given by the same equation, namely (5.41), as in the last section. Although $P(p)$ was assumed to be a given function, we notice that (5.52) and (5.55) place a restriction on $P(p)$ regardless of the form of $\alpha(p)$ and $\beta(p)$. We substitute from (5.52) into (5.55) and simplify as follows:

$$P(p) \frac{d}{dp} \left[\frac{A}{P(p)} \right]^2 = D$$

$$2A \frac{d}{dp} \left(\frac{A}{P(p)} \right) = D$$

Integration yields

$$\frac{2A^2}{P(p)} = Dp + E,$$

where E is a constant of integration. Therefore, $P(p)$ must be of the form

$$P(p) = \frac{2A^2}{Dp + E},$$

where A , D and E are constants with $D < 0$. This includes the equation of state used by Chaplygin, Karman, and Tsien [20, p. 278]. With this restriction on the form of $P(p)$, however, we can proceed to obtain a solution of the dynamical equations characterizing plane flow by the following procedure.

- (1) Pick constants A , D and E arbitrarily with $D < 0$.

- (2) Choose the functions $\alpha(p)$ and $h(S)$ arbitrarily (not constant).
- (3) Determine $f(S)$, to within a plus or minus sign from (5.54).
- (4) Determine $\beta(p)$ and $\gamma(S)$ by integration of (5.52) and (5.53), respectively.

CHAPTER VI

A CLASS OF THREE DIMENSIONAL FLOWS

6.1 Introductory Remarks

In Chapter V one of the families of coordinate surfaces that was introduced was a family of cylinders each member of which has a streamline as a directrix. This family of surfaces was a family of Lamb surfaces, and furthermore, each streamline was a geodesic on one of its members. We would now like to generalize what was done in the preceding chapter by seeking a three dimensional (rather than two dimensional) flow in which streamlines are geodesics on Lamb surfaces.

Since S is constant on the Lamb surfaces of a Prims-Smith flow (see Corollary 2.32), we may take it to be an independent variable, say X^3 . If we again let X^1 vary along streamlines and take $p = X^1$ and $\psi = X^2$, the assumption that streamlines are geodesics on Lamb surfaces implies that $g_{21} = 0$. This follows from the fact that the members of the $\bar{b}$ congruence of curves lie on the constant pressure surfaces (equation (2.22) implies $\bar{b} \cdot \nabla p = 0$) and also on the Lamb surfaces, if the streamlines are geodesics on these surfaces. The requirement that $g_{21} = 0$ allows us to omit (4.93) from equations (F).

6.2 The Metric Tensor

In the consideration of plane flows in Chapter V, there was a general relationship among the variables x , y and p , S , which we used (Section 5.2). In this chapter we assume a particular relationship among the variables x , y , z (rectangular Cartesian coordinates) and p , ψ , S , namely one of the form

$$(6.21) \quad x = x(p, \psi), \quad y = y(p, S), \quad z = z(p, \psi),$$

in which the Jacobian, J , is not zero. The metric tensor is then of the form

$$(g_{ij}) = \begin{pmatrix} x_p^2 + y_p^2 + z_p^2 & 0 & y_p y_S \\ 0 & x_\psi^2 + z_\psi^2 & 0 \\ y_p y_S & 0 & y_S^2 \end{pmatrix}$$

where zeros appear in the first row second column and second row first column due to the requirement that $g_{21} = g_{12} = 0$. Computing g_{12} and equating it to zero yields

$$(6.22) \quad x_p x_\psi + z_p z_\psi = 0.$$

We note that the determinant g is given by

$$(6.23) \quad g = y_S^2 (x_\psi^2 + z_\psi^2)(x_p^2 + z_p^2).$$

Before introducing the metric tensor into system of equations (F), we observe some of the geometric implications of (6.21). If ψ is fixed, from (6.21) we have $x = x(p)$

and $z = z(p)$, which may be considered as the parametric equations of a curve in the xz -plane. So, surfaces on which ψ is constant are cylinders with generators parallel to the y axis. Similarly, surfaces on which p is constant are cylinders with generators parallel to the y axis.

6.3 Introduction of the Metric Tensor into System of Equations (F)

As in Chapter V we seek a solution to system of equations (F) for which $g_{13} \neq 0$ and replace equation (4.94) with (4.97). We substitute the expressions for the terms of the metric tensor into (4.97) and get

$$\frac{\partial}{\partial S} (x_p^2 + y_p^2 + z_p^2) = y_p y_S \frac{\partial}{\partial p} \ln \left(\frac{u y_p^2 y_S^2}{x_p^2 + y_p^2 + z_p^2} \right) .$$

Taking the derivative with respect to S and dividing by $y_p y_S$ yields

$$\frac{2 y_{pS}}{y_S} = \frac{\partial}{\partial p} \ln \left(\frac{u y_p^2 y_S^2}{x_p^2 + y_p^2 + z_p^2} \right) .$$

The left side of this equation is

$$\frac{\partial}{\partial p} \ln y_S^2 ,$$

and therefore integrating with respect to p and solving for u we obtain

$$(6.31) \quad u = \frac{F^2(\psi, S)(x_p^2 + y_p^2 + z_p^2)}{y_p^2},$$

where $F^2(\psi, S)$ is an arbitrary positive valued function.

We now proceed to substitute for the metric coefficients and u in (4.91) getting

$$\frac{\partial}{\partial p} \left[\frac{p^2(p) F^2(\psi, S) y_S^2 (x_\psi^2 + z_\psi^2) (x_p^2 + z_p^2)}{y_p^2} \right] = 0.$$

Simplification yields

$$(6.32) \quad \frac{\partial}{\partial p} \left[\frac{p^2(p) y_S^2 (x_\psi^2 + z_\psi^2) (x_p^2 + z_p^2)}{y_p^2} \right] = 0.$$

Shifting our attention to (4.92) and substituting for u from (6.31), we obtain

$$(6.33) \quad p(p) \frac{\partial}{\partial p} \left(\frac{x_p^2 + z_p^2}{y_p^2} \right) = \frac{-2}{F^2(\psi, S)}.$$

Thus, to find a three dimensional flow (of the form indicated by equations (6.21)) with the property that streamlines are geodesics on Lamb surfaces is reduced to finding x, y, z and u in terms of p, ψ , and S such that (6.22) and (6.31) to (6.33) are satisfied.

6.4 An Application of a Separation of Variable Technique

In accordance with a standard separation of variable technique, we assume there is a solution to the equations

just mentioned of the following form.

$$\begin{aligned}
 x &= \alpha_1(p) \beta_1(\psi) \\
 (6.41) \quad y &= \alpha_2(p) \gamma(S) \\
 z &= \alpha_1(p) \beta_2(\psi)
 \end{aligned}$$

Substitution into (6.22) results in

$$\begin{aligned}
 \alpha_1(p) \alpha_1'(p) [\beta_1(\psi) \beta_1'(\psi) + \beta_2(\psi) \beta_2'(\psi)] &= \\
 \frac{\alpha_1(p) \alpha_1'(p)}{2} \frac{d}{d\psi} [\beta_1^2(\psi) + \beta_2^2(\psi)] &= 0.
 \end{aligned}$$

Therefore, we must have (for $\alpha_1(p)$ not constant)

$$(6.42) \quad \beta_1^2(\psi) + \beta_2^2(\psi) = A,$$

where A is an arbitrary positive constant.

Substituting from (6.41) into (6.32) we find that

$$\frac{\partial}{\partial p} \left[\frac{P^2(p) \alpha_2^2(p) [\gamma'(S)]^2 \alpha_1^2(p) [\alpha_1'(p)]^2 [(\beta_1'(\psi))^2 + (\beta_2'(\psi))^2] [\beta_1^2(\psi) + \beta_2^2(\psi)]}{[\alpha_2'(p)]^2 \gamma^2(S)} \right] = 0$$

which means that

$$(6.43) \quad \frac{d}{dp} \left(\frac{P(p) \alpha_2(p) \alpha_1(p) \alpha_1'(p)}{\alpha_2'(p)} \right) = 0,$$

provided the various functions appearing are not constant.

Finally, we must substitute into equation (6.33) from (6.41). This gives

$$P(p) \frac{\partial}{\partial p} \frac{[\alpha_1'(p)]^2 [\beta_1^2(\psi) + \beta_2^2(\psi)]}{[\alpha_2'(p)]^2 \gamma^2(S)} = \frac{-2}{F^2(\psi, S)}.$$

Using (6.42) and isolating the terms involving only p gives

$$P(p) \frac{d}{dp} \left(\frac{\alpha_1'(p)}{\alpha_2'(p)} \right)^2 = \frac{-2\gamma^2(S)}{A F^2(\psi, S)} = C$$

where C is a negative separation constant. We are thus led to two conditions.

$$(6.44) \quad \frac{d}{dp} \left[\frac{\alpha_1'(p)}{\alpha_2'(p)} \right]^2 = \frac{C}{P(p)} \quad (C < 0)$$

$$(6.45) \quad \gamma^2(S) = -\frac{AC}{2} F^2(\psi, S)$$

Notice that (6.45) implies that F is independent of ψ .

For purposes of ready reference and summarization, we gather up those requirements which must be met in order to have a solution to system (F) of Chapter IV, of the type we are considering here. These requirements are

- (1) $J = y_S(x_\psi z_p - x_p z_\psi) \neq 0$. From (6.41) we see that this is equivalent to

$$\beta_1'(\psi)\beta_2(\psi) - \beta_1(\psi)\beta_2'(\psi) \neq 0.$$

- (2) $\beta_1^2(\psi) + \beta_2^2(\psi) = A$, where A is a constant.

$$(3) \quad \frac{P(p)\alpha_1(p)\alpha_1'(p)\alpha_2(p)}{\alpha_2'(p)} = D,$$

where D is a constant. This is just a restatement of (6.43).

$$(4) \quad \frac{d}{dp} \left(\frac{\alpha_1'(p)}{\alpha_2'(p)} \right)^2 = \frac{C}{P(p)} \quad (C < 0)$$

$$(5) \quad \gamma^2(S) = -\frac{AC}{2} F^2(S).$$

To obtain a flow, we could proceed according to the following steps:

- (I) Pick a positive constant A , a negative constant C , and an arbitrary function $F(S)$.
- (II) Determine $\gamma(S)$ from the equation in requirement (5).
- (III) Pick $\beta_1(\psi)$ such that $\beta_1^2(\psi) < A$, and then determine $\beta_2(\psi)$ from $\beta_2^2(\psi) = A - \beta_1^2(\psi)$. This can be done, for example, by choosing $A = 1$ and $\beta_1(\psi) = \cos \psi$. Then, $\beta_2(\psi) = \sin \psi$. Also, requirement (1) is satisfied as long as $|\beta_1(\psi)|$ is not equal to a multiple of $|\beta_2(\psi)|$.
- (IV) Satisfy requirements (3) and (4) as indicated below.

6.5 Determination of $\alpha_1(p)$ and $\alpha_2(p)$

Requirement (4) implies that $\alpha_1'(p)/\alpha_2'(p)$ is a known function of p , namely

$$\frac{\alpha_1'(p)}{\alpha_2'(p)} = \pm \sqrt{CI(p) + C_1},$$

where C_1 is an arbitrary constant chosen such that $CI(p) + C_1 > 0$ in the region of flow, and the function $I(p)$ is again an integral of $1/P(p)$. Let us call this

function $k(p)$. With this notation, from requirement (3) we see that

$$\alpha_1(p)\alpha_2(p) = \frac{D}{P(p)k(p)} ,$$

and hence, $\alpha_1(p)\alpha_2(p)$ is a known function of p . Let us denote it by $h(p)$. So, we have

$$(6.51) \quad \frac{\alpha_1'(p)}{\alpha_2'(p)} = k(p)$$

and

$$(6.52) \quad \alpha_1(p)\alpha_2(p) = h(p) .$$

From these two equations we obtain a single differential equation with dependent variable α_2 . If this differential equation can be solved, then $\alpha_1(p)$ may be obtained from (6.52), and we have a class of three dimensional flows. We proceed as follows.

Solving (6.52) for $\alpha_1(p)$ we get

$$\alpha_1(p) = \frac{h(p)}{\alpha_2(p)} .$$

Differentiation yields

$$\alpha_1'(p) = \frac{h'(p)\alpha_2(p) - h(p)\alpha_2'(p)}{\alpha_2^2(p)} .$$

Substituting this expression for $\alpha_1'(p)$ in (6.51) and multiplying by $\alpha_2^2(p)\alpha_2'(p)$ yields

$$h'(p)\alpha_2(p) - h(p)\alpha_2'(p) = k(p)\alpha_2^2(p)\alpha_2'(p) .$$

Collecting terms we find that

$$[h(p) + k(p)\alpha_2^2(p)]\alpha_2'(p) - h'(p)\alpha_2(p) = 0.$$

If we multiply by $2\alpha_2(p)$, this equation becomes

$$[h(p) + k(p)\alpha_2^2(p)] \frac{d}{dp}[\alpha_2^2(p)] - 2h'(p)\alpha_2^2(p) = 0.$$

Letting $Y(p) = \alpha_2^2(p)$ and solving for dY/dp results in

$$(6.53) \quad \frac{dY}{dp} = \frac{2h'(p)Y}{h(p) + k(p)Y}.$$

In order to put this in a standard form discussed by E.

Kamke [7, p. 24], we make the substitution $1/r =$

$h(p) + k(p)Y$. Differentiation of this yields

$$-\frac{1}{r^2} \frac{dr}{dp} = h'(p) + k'(p)Y + k(p)Y'(p).$$

Hence,

$$Y'(p) = -\frac{1}{k(p)} \left[h'(p) + k'(p)Y + \frac{1}{r^2} \frac{dr}{dp} \right].$$

We substitute into (6.53), change the dependent variable

from Y to r , and get

$$-\frac{1}{k(p)} \left[h'(p) + \frac{k'(p)}{k(p)} \left(\frac{1}{r} - h(p) \right) + \frac{1}{r^2} \frac{dr}{dp} \right] = \frac{2h'(p)r}{k(p)} \left(\frac{1}{r} - h(p) \right).$$

Solving for r'/r^2 we obtain

$$\frac{1}{r^2} \frac{dr}{dp} = -\frac{k'(p)}{k(p)} \frac{1}{r} + \left[\frac{h(p)k'(p)}{k(p)} - 3h'(p) \right] + 2h(p)h'(p)r.$$

Multiplying by r^2 we have

$$(6.54) \quad \frac{dr}{dp} = f_1(p)r + f_2(p)r^2 + f_3(p)r^3,$$

where $f_1(p)$, $f_2(p)$ and $f_3(p)$ are known functions of p with

$$f_1(p) = -\frac{k'(p)}{k(p)}$$

$$f_2(p) = \frac{h(p)k'(p)}{k(p)} - 3h'(p)$$

$$f_3(p) = 2h(p)h'(p).$$

Methods for solving (6.54) for various forms of the coefficients f_i ($i = 1, 2, 3$) are given on page 25 of E. Kamke.

In conclusion, the problem of finding a solution to system (F) where x , y , and z are of the form (6.41), and for which streamlines are geodesics on Lamb surfaces has been reduced to the problem of solving a single first order differential equation.

6.6 Some Remarks Concerning the Geometry of the Flows of This Chapter

If the requirement (6.42) is satisfied by choosing $\beta_1(\psi) = \cos \psi$, $\beta_2(\psi) = \sin \psi$ and $A = 1$, as suggested in Step IV in the procedure for obtaining a flow, equations (6.41) become

$$(6.61) \quad \begin{aligned} x &= \alpha_1(p) \cos \psi \\ y &= \alpha_2(p) \gamma(S) \\ z &= \alpha_1(p) \sin \psi. \end{aligned}$$

Let us consider the following three types of surfaces: (a) Lamb surfaces on which S is constant, (b) constant pressure surfaces, and (c) surfaces on which $\psi = \text{constant}$.

(a) If S is constant, (6.61) is of the form

$$x = \alpha_1 \cos \psi$$

$$z = \alpha_1 \sin \psi$$

$$y = f(\alpha_1)$$

where $f(\alpha_1)$ is obtained as follows. Equations (6.43) and (6.44) yield a functional relation between α_1 and α_2 from which we can determine α_2 in terms of α_1 . Substituting this result in the equation $y = \alpha_2(p) \gamma(S)$ and holding S fixed yields $y = f(\alpha_1)$. Consequently, the surfaces on which S is constant (these are Lamb surfaces) are surfaces of revolution [5, p. 49].

(b) When p is held fixed, $\alpha_1(p)$ is constant, say C_1 . Consequently, from (6.41) we obtain

$$x = C_1 \cos \psi$$

$$z = C_1 \sin \psi$$

$$y = C_2 \gamma(S)$$

where C_2 is the value of α_2 for some fixed p . Then $x^2 + z^2 = C_1^2$ and the surfaces of constant pressure are coaxial right circular cylinders [5, p. 47].

(c) When ψ is constant, (6.61) implies that $z = Kx$ [while $y = \alpha_2(p) \gamma(S)$]. As a consequence, $\psi = \text{constant}$ surfaces are planes. (See Figure 3.)

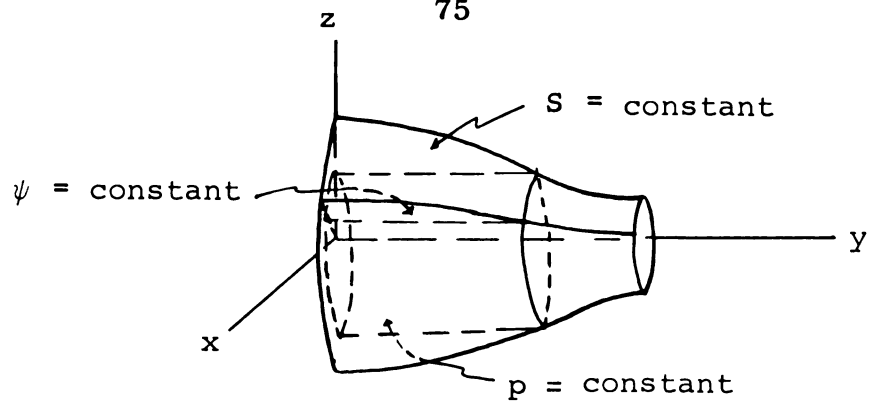


Figure 3.

From (a) and (c) we observe that the streamlines ($\psi = \text{constant}$ and $S = \text{constant}$) are meridian curves of surfaces of revolution, in fact, the Lamb surfaces. From (a) and (b) we conclude that the parallels of these Lamb surfaces are isobars. Furthermore, we note that the $\bar{\mathbf{x}}$ vector of Corollary 2.43 is tangent to these parallels and is collinear at each point with the binormal vector of the streamline through this point.

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