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MIXED NORM GENERALIZATIONS OF WEIGHTED BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^N$

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# MIXED NORM GENERALIZATIONS OF WEIGHTED BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^N$

By

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# ABSTRACT

## MIXED NORM GENERALIZATIONS OF WEIGHTED BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^N$

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Let f be an analytic function in the unit ball B of $\mathbb{C}^N$ for which the "mixed norm"

$$\left(\int_0^1 \left(\int_{\partial B} |f(r\tau)|^p \, d\sigma(\tau)\right)^{q/p} \omega(r) r^{2N-1} \, dr\right)^{1/q}$$

is finite. Here $0 < p < \infty$, $0 < q < \infty$, $\omega$ is a suitable radial weight function, and $\sigma$ is normalized Lebesgue measure on $\partial B$. Note that when $p = q$, because of the "polar coordinates" formula, the space of all such functions is just the Bergman space with weight $\omega$. General mixed norm spaces were studied extensively by Benedek and Panzone.

We begin by generalizing a collection of results gotten by Luecking for the Bergman spaces or the Hardy spaces. Boundedness of certain Bergman projections is proven first, using vector-valued integration and some facts due to Forelli and Rudin. Representation of the dual space of our mixed norm spaces follows from this. Then a representation of functions in our mixed norm spaces is obtained (by using duality) and several equivalent norms are produced (by refining arguments of Luecking).

We also state a general "Carleson measure theorem" for our mixed norm spaces whose proof depends largely on geometry and the connection between the nonisotropic metric and the invariant Poisson kernel. Several consequences are noted, including a theorem originally due to Cima and Wogen. Other related methods and results are given, among them a Carleson measure theorem for mixed norm spaces in the polydisc, a generalization of a result of Hastings.

To my wife, parents, and family
for their patience and love

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## INTRODUCTION

Let f be an analytic function in the unit ball B of $\mathbb{C}^N$ for which the "mixed norm"

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is finite. Here $0 < p < \infty$, $0 < q < \infty$, $\omega$ is a suitable radial weight function (e.g., $\omega(r) = (1 - r^2)^\alpha$ for $\alpha > -1$), and $\sigma$ is normalized Lebesgue measure on $\partial B$. The space of all such functions is denoted by $A^{pq}_\omega$. Note that when p = q, because of the "polar coordinates" formula, this is just the weighted Bergman space $A^p_\omega$. General mixed norm spaces were studied extensively by Benedek and Panzone in [2].

The preliminaries are taken care of in Chapter 1. All notation and definitions are given, several metrics to be used later are discussed, and $A^{pq}_\omega$ is shown to be a Banach space for certain $\omega$.

Chapter 2 generalizes results gotten by Luecking in [11] for the Bergman spaces $A^p_\omega$ or the Hardy spaces $H^p$. Boundedness of certain Bergman projections is proven first, using vector-valued integration and some facts due to Forelli and Rudin in [7]. Representation of the dual space of $A^{pq}_\omega$ for certain p, q, and $\omega$ follows from this. Then by refining arguments of Luecking a representation of functions in $A^{pq}_\omega$ is obtained (duality is also used) and several equivalent norms are produced.

In Chapter 3 we state a general "Carleson measure theorem" for our mixed norm spaces whose proof depends largely on geometry and the connection between the nonisotropic metric and the invariant Poisson kernel. Several consequences are noted, including a theorem originally due to Cima and Wogen in [4]. Other related methods and results are given, among them a Carleson measure theorem for mixed norm spaces in the polydisc which generalizes a result of Hastings in [8].

## CHAPTER 1

In section one of this chapter, a mixed norm generalization of the weighted Bergman space in the unit ball of $\mathbb{C}^N$ is defined, and notation is set forth. Basic properties of our spaces of functions are given in section two, and the various metrics we will find useful are described in section three.

### §1 Notation and Definitions

The classical Bergman space $A^p$ on the unit ball $B = B_N$ in $\mathbb{C}^N$ $(0 < p < \infty)$ is the set of functions $f \in H(B_N)$ satisfying

$$\left(\int_B |f(z)|^p \, dm(z)\right)^{1/p} < \infty.$$

Here N is a positive integer, $\mathbb{C}^N$ is equipped with the usual inner product defined for $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in $\mathbb{C}^N$ by $\langle z, w\rangle = \sum_{i=1}^{N} z_i \overline{w_i}$ and with the associated norm $|z| = \langle z, z\rangle^{1/2}$, $H(B_N)$ is the set of holomorphic functions on $B_N$, and m is Lebesgue measure on $B_N$ normalized so that $m(B_N) = 1$. Using "polar cooordinates" (see [15], 1.4.3), this integral may be written as

$$\left(2N \int_I \left(\int_S |f(r\tau)|^p \, d\sigma(\tau)\right) r^{2N-1} \, dr\right)^{1/p}$$

where $I = [0, 1)$, $S = S_N = \partial B_N = \left\{z \in \mathbb{C}^N \mid |z| = 1\right\}$, and $\sigma = \sigma_N$ is the rotation

invariant positive Borel measure on $S_N$ with $\sigma(S_N) = 1$.

We shall study the following weighted "mixed norm" generalizations of the Bergman spaces. If $0 < p, q < \infty$ and if $\omega$ is a nonnegative weight function on $[0, 1)$ satisfying $\int_I \omega(r) r^{2N-1}\, dr < \infty$, define

$$A^{pq}_{\omega} = \left\{ f \in H(B) \mid \|f\|_{p,q,\omega} = \left(\int_I \left(\int_S |f(r\tau)|^p\, d\sigma(\tau)\right)^{q/p} \omega(r) r^{2N-1}\, dr\right)^{1/q} < \infty \right\}.$$

Note that when $\omega \equiv 1$ and $q = p$, this is precisely the Bergman space. Also note that

$$\|f\|_{p,q,\omega} = \left(\int_I \|f_r\|^q_{L^p(S)} \omega(r) r^{2N-1}\, dr\right)^{1/q},$$

where, for $0 \le r < 1$, $f_r$ is the function defined on S by $f_r(\tau) = f(r\tau)$. Using this notation, we also define

$$A^{\infty q}_{\omega} = \left\{ f \in H(B) \mid \|f\|_{\infty,q,\omega} = \left(\int_I \|f_r\|_{L^\infty(S)} \omega(r) r^{2N-1}\, dr\right)^{1/q} < \infty \right\},$$

$$A^{p\infty} = H^p = \left\{ f \in H(B) \mid \|f\|_{p,\infty} = \sup_{0<r<1} \|f_r\|_{L^p(S)} < \infty \right\},$$

$$\text{and } A^{\infty\infty} = H^\infty = \left\{ f \in H(B) \mid \|f\|_{\infty,\infty} = \sup_{0<r<1} \|f_r\|_{L^\infty(S)} < \infty \right\}.$$

The spaces $H^p$ and $H^\infty$ are the classical Hardy spaces. In the special case that $\omega(r) = (1 - r^2)^\alpha$ with $\alpha > -1$ $[\text{or } \omega \equiv 1]$, we write $A^{pq}_{\alpha}$ $[\text{or } A^{pq}]$ instead of $A^{pq}_{\omega}$, and $\|\ \|_{p,q,\alpha}$ $[\text{or } \|\ \|_{p,q}]$ instead of $\|\ \|_{p,q,\omega}$. We also write $dm_\alpha(z) = (1 - |z|^2)^\alpha\, dm(z)$. The set of (equivalence classes of) measurable functions satisfying the integrability condition defining $A^{pq}_{\omega}$ is denoted by $L^{pq}_{\omega}$. It is easy to check that

$$\|f + g\|^s_{p,q,\omega} \le \|f\|^s_{p,q,\omega} + \|g\|^s_{p,q,\omega}$$

where $s = \min\{p, q, 1\}$; thus $\left(A^{pq}_\omega, \|\ \|^s_{p,q,\omega}\right)$ and $\left(L^{pq}_\omega, \|\ \|^s_{p,q,\omega}\right)$ are metric spaces, and are normed linear spaces if $1 \le p, q \le \infty$. The basic reference for mixed norm spaces (including, but not limited to, our spaces $L^{pq}_\omega$) is Benedek and Panzone, [2].

We will follow the usual practice of writing c and C for positive constants ("small" and "large" respectively) that may vary from line to line. Dependence on some parameter(s), for example p and N, may be emphasized by writing $C = C(p, N)$. If two positive quantities $A(x)$ and $B(x)$ have ratio bounded above and below as x ranges over some index set X, we say $A(x)$ and $B(x)$ are equivalent, and we write $A(x) \sim B(x)$ for every $x \in X$. For example, $(1 - r^2) \sim (1 - r)$ for every $0 \le r \le 1$; this particular fact will be used repeatedly without mention.

Note that by two applications of Hölder's inequality, $L^{p_1q_1}_{\alpha_1} \subset L^{p_2q_2}_{\alpha_2}$ if either

$$p_2 \le p_1 \text{ and } q_2 < q_1 \text{ and } (\alpha_1 + 1)/q_1 < (\alpha_2 + 1)/q_2$$

or

$$p_2 \le p_1 \text{ and } q_2 = q_1 \text{ and } \alpha_1 \le \alpha_2.$$

In either case, the containment is proper (unless $p_1 = p_2$, $q_1 = q_2$, and $\alpha_1 = \alpha_2$), since then there is some s satisfying $N/p_1 + (\alpha_1 + 1)/q_1 < s < N/p_2 + (\alpha_2 + 1)/q_2$, and $f(z) = (1 - \langle z, \zeta\rangle)^{-s}$ (with $\zeta \in S$ fixed) defines a function in $A^{p_2q_2}_{\alpha_2}$ not in $A^{p_1q_1}_{\alpha_1}$:

$$\|f\|^q_{p,q,\alpha} = \int_I \left(\int_S |1 - \langle r\zeta, \tau\rangle|^{-sp}\, d\sigma(\tau)\right)^{q/p} (1 - r^2)^\alpha r^{2N-1}\, dr$$

$$\sim \int_I \left((1 - r)^{N-sp}\right)^{q/p}(1 - r)^{\alpha} r^{2N-1}\, dr$$

by Proposition 1.4.10 of [15], and this integral is finite if and only if $s < N/p + (\alpha + 1)/q$.

For $1 \leq p \leq \infty$, p' is the conjugate of p defined by $1/p + 1/p' = 1$. The dual space of a Banach space X is denoted by $X^*$. We will use z and w for typical elements of $B_N$, and we may write $z = r\tau$ and $w = \rho\eta$, where $r, \rho \in I$ and $\tau, \eta \in S_N$. When N = 1, $B_1$ and $S_1$ will also be written as $\mathbb{D}$ and $\mathbb{T}$ respectively, and $\mathbb{T}$ will often be identified with $[0, 2\pi)$ without warning.

## §2 Basic Properties of $A^{pq}_{\omega}$

The completeness of $A^{pq}_{\alpha}$ is a consequence of the following growth condition. Our statements and proofs here will resemble those in [16].

Proposition 1.2.1 : If $f \in A^{pq}_{\alpha}$ $(0 < p, q \leq \infty, -1 < \alpha)$, then

$$|f(z)| \leq C\|f\|_{p,q,\alpha}(1 - |z|)^{-(N/p + (\alpha+1)/q)} \text{ for every } z \in B$$

for some C independent of f.

Proof : First suppose $0 < p < \infty$. Since $|f|^p$ is subharmonic, for $0 < r < \rho < 1$ and $\tau \in S$ we have

$$|f(r\tau)|^p \leq \int_S |f(\rho\eta)|^p P((r/\rho)\tau, \eta)\, d\sigma(\eta) \leq 2^N(\rho-r)^{-N}\int_S |f(\rho\eta)|^p\, d\sigma(\eta),$$

i.e.,

(2.1) $$|f(r\tau)|(\rho - r)^{N/p} \le C\|f_\rho\|_{L^p(S)}.$$

Here P denotes the invariant Poisson kernel defined for $z \in B$ and $\zeta \in S$ by

$$P(z, \zeta) = \big((1 - |z|^2)/|1 - \langle z, \zeta\rangle|^2\big)^N.$$

For basic facts concerning the invariant Poisson kernel, see [15], section 3.3. If $q = \infty$, the result follows from (2.1) immediately. If $0 < q < \infty$, we then have

$$|f(r\tau)|^q \int_r^1 (\rho - r)^{Nq/p}(1 - \rho)^\alpha \rho^{2N-1}\, d\rho \le C\|f\|_{p,q,\alpha}.$$

Letting $x = (\rho - r)/(1 - r)$, for $1/2 \le r < 1$ it follows that

$$C\|f\|_{p,q,\alpha} \ge |f(r\tau)|^q (1 - r)^{(N/p + (\alpha+1)/q)q} \int_0^1 x^{Nq/p} (1 - x)^\alpha [(1 - r)x + r]^{2N-1}\, dx$$
$$\sim |f(r\tau)|^q (1 - r)^{(N/p + (\alpha+1)q)q} \int_0^1 x^{Nq/p} (1-x)^\alpha\, dx$$

so the result follows. If $0 \le r < 1/2$, the result follows from the maximum modulus theorem.

Now suppose $p = \infty$. Then $|f(r\tau)| \le \|f_\rho\|_{L^\infty(S)}$ for $0 \le r \le \rho < 1$, and the result for $0 < q \le \infty$ is proven by the same procedures. □

This growth estimate will also be a consequence of Theorem 3.3.1.

Corollary 1.2.2 : $A^{pq}_\alpha$ is a closed subspace of $L^{pq}_\alpha$ $(0 < p, q \le \infty, -1 < \alpha)$, and is hence complete. So $A^{pq}_\alpha$ is a Banach space if $1 \le p, q \le \infty$.

Proof : Suppose $f_n \to f$ in $L^{pq}_{\alpha}$ with $f_n \in A^{pq}_{\alpha}$. By Proposition 1.2.1 $f_n$ is uniformly Cauchy on compact subsets of B, so $f_n$ is uniformly convergent on compact subsets of B to some g since $L^{pq}_{\alpha}$ is complete (see [2], p. 304). But g is analytic ([15], 1.1.4) and f = g a.e., so $g \in A^{pq}_{\alpha}$ and $f_n \to g$ in $A^{pq}_{\alpha}$. $\square$

Proposition 1.2.3 : If $f \in A^{pq}_{\omega}$ $(0 < p, q < \infty)$, then $\lim_{r \to 1^-} \|f_r - f\|_{p,q,\omega} = 0$.

This follows immediately from the dominated convergence theorem. (For details, see [16], Proposition 3.3.) So the functions analytic in a neighborhood of B form a dense subset of $A^{pq}_{\omega}$.

§3 The Nonisotropic Metric and the Pseudohyperbolic "Metric"

There are several notions of "distance" in $B_N$ (or $\overline{B_N}$, or $S_N$) and each has its own advantages. We will have occasion to use three.

The isotropic metric will refer to the usual metric in $\mathbb{C}^N$, and B(z, r) will denote a corresponding ball, i.e., $B(z, r) = \{w \in \mathbb{C}^N \mid |z - w| < r\}$.

The nonisotropic metric d is defined on $\overline{B_N}$ by $d(z, w) = |1 - \langle z, w\rangle|^{1/2}$. It satisfies the triangle inequality on $\overline{B_N}$ and is a metric on $S_N$; see [15], 5.1.2. Define $\hat{B}(0, \delta) = S_N$, and for $\delta > 0$, $\rho \in (0, 1)$, and $\eta \in S_N$, define $\hat{B}(\rho\eta, \delta) = \{\tau \in S_N \mid (d(\tau, \eta))^2 < \delta(1 - \rho)\}$. Note that $\sigma\hat{B}(\rho\eta, \delta) \sim \delta^N(1 - \rho)^N$; see [15], 5.1.4.

The pseudohyperbolic "metric" $\rho$ is defined on $B_N$ by $\rho(z, w) = |\Phi_w(z)|$ where $\Phi_w$ is the automorphism of $B_N$ given for $w \neq 0$ by

$$\Phi_w(z) = \left(w - \langle z, w\rangle w/\langle w, w\rangle - (1 - |w|^2)^{1/2}(z - \langle z, w\rangle w/\langle w, w\rangle)\right)/(1 - \langle z, w\rangle)$$

and for $w = 0$ by $\Phi_0(z) = -z$. The corresponding "balls" are $E(w, \delta) = \Phi_w^{-1}(\delta B_N) = \{z \in B_N | \rho(z, w) < \delta\}$ for $w \in B_N$ and $0 < \delta < 1$. Note that $mE(w, \delta) \sim \delta^{2N}(1 - |w|)^{N+1}$; see [15], 2.2.7.

We will have need of the following.

Lemma 1.3.1 : Fix $0 < r < 1$ and $0 < \delta$ small. Then

$$(r - \delta)/(1 - r\delta) < |z| < (r + \delta)/(1 + r\delta) \text{ for every } z \in E(r, \delta).$$

(Here $E(r, \delta)$ means $E(w, \delta)$, with $w = (r, 0, \ldots, 0) \in B_N$.)

Proof : Write $z = (z_1, z_2, \ldots, z_N) = (z_1, z')$ and suppose $z \in E(r, \delta)$. Then

$$\delta^2|1 - rz_1|^2 > |r - z_1|^2 + (1 - r^2)|z'|^2,$$
$$\text{i.e., } 2r(1 - \delta^2)\text{Re } z_1 > (r^2 - \delta^2) + (1 - \delta^2 r^2)|z_1|^2 + (1 - r^2)|z'|.$$

Now, $4r\text{Re } z_1 + r^2|z'|^2 < r^2 + 4r|z_1| - r^2|z_1|^2 \leq 4r < 4$, so

$$\begin{aligned}&(r^2 - \delta^2) + (1 - \delta^2 r^2)|z|^2 \\ &= \left((r^2 - \delta^2) + (1 - \delta^2 r^2)|z_1|^2 + (1 - r^2)|z'|^2\right) + r^2(1 - \delta^2)|z'|^2 \\ &< 2r(1 - \delta^2)(\text{Re } z_1 + r|z'|^2/2) \leq 2r(1 - \delta^2)\left(|z_1|^2 + r|z'|^2\text{Re } z_1 + r^2|z'|^4/4\right)^{1/2} \\ &< 2r(1 - \delta^2)(|z_1|^2 + |z'|^2)^{1/2} = 2r(1 - \delta^2)|z|.\end{aligned}$$

Hence $(|z|, 0') \in E(r, \delta)$, and $(r - \delta)/(1 - r\delta) < |z| < (r + \delta)/(1 + r\delta)$. □

## CHAPTER 2

Most of the results of this chapter are generalizations of work of Luecking in [11]. Section one deals with boundedness of the Bergman projection, and this result is used to identify the dual space of our mixed norm spaces in section two. Sections three and four are concerned with representations of the mixed norms and of functions in the mixed norm spaces.

### §1 Boundedness of the Bergman Projection on $L^{pq}_{\alpha}$

Suppose s > -1. The Bergman kernel $K_s$ is defined by

$$K_s(z, w) = (1 - |w|^2)^s/(1 - \langle z, w\rangle)^{N+1+s}$$

for $z, w \in B_N$. Note that:

(a) for fixed $w \in B$, $K_s(\cdot, w) \in A^{pq}_{\omega}$

and (b) for fixed $z \in B$, $K_s(z, \cdot) \in L^{pq}_{\omega}$ if $\int_I (1 - r^2)^{sq}\omega(r)r^{2N-1}\, dr < \infty$, e.g., if $\omega(r) \sim (1 - r^2)^{\alpha}$ with $sq > -(\alpha + 1)$. (But $K_s(z, \cdot)$ is not conjugate holomorphic unless $s = 0$.)

Both observations follow because the respective denominators are bounded above and below in B.

The Bergman projection $T_s$ is defined by

$$T_s f(z) = \binom{N+s}{N} \int_B K_s(z, w) f(w)\, dm(w)$$

for $z \in B_N$ and f for which the integrands are in $L^1(dm)$. In general, the binomial coefficient $\binom{N+s}{N}$ is $\Gamma(N + s + 1)/\Gamma(N + 1)\Gamma(s + 1)$. It is clear that, for fixed s, $T_s f$ is holomorphic when defined.

In this section, a condition on s, p, q, and $\alpha$ will be found which ensures that $T_s$ is bounded on $L^{pq}_\alpha$; there will be no dependence on p other than $p \geq 1$. In [7], Forelli and Rudin showed that $T_s$ is bounded on $L^p(dm)$ $(1 \leq p < \infty)$ if and only if $(s + 1)p > 1$. Then Békollé ([1]) showed that $T_s$ is bounded on $L^p(dm_\alpha)$ $(1 < p < \infty, -1 < \alpha)$ if and only if $(s + 1)p > \alpha + 1$. (He actually showed this for more general weights satisfying a "$B_p$ condition", a condition analogous to Muckenhoupt's $A_p$ condition introduced in [13].) An important tool will be the following pair of facts due to Forelli and Rudin in [7], Proposition 2.7:

(1.1a) $\int_B |K_s(z, w)|(1 - |w|^2)^{-c}\, dm(w) \leq C(1 - |z|^2)^{-c}$ for every $z \in B$ if $0 < c < s + 1$;

(1.1b) $\int_B |K_s(z, w)|(1 - |z|^2)^{-c}\, dm(z) \leq C(1 - |w|^2)^{-c}$ for every $w \in B$ if $0 < c + s < s + 1$.

<u>Theorem 2.1.1</u> : $T_s$ maps $L^{pq}_\alpha$ $(1 \leq p \leq \infty, 1 < q < \infty, -1 < \alpha)$ boundedly into $A^{pq}_\alpha$ if $(s + 1)q > \alpha + 1$. Furthermore, $T_s f = f$ and $T_s \bar{f} \equiv \overline{f(0)}$ for every $f \in A^{pq}_\alpha$.

<u>Proof</u> : As noted in [15], Proposition 7.1.2, $T_s f = f$ and $T_s \bar{f} \equiv \overline{f(0)}$ is true for $f \in H^\infty(B)$, hence for $f \in A^{pq}_\alpha$ by density of $H^\infty(B)$ in $A^{pq}_\alpha$, once continuity is verified.

If $1 < p < \infty$, vector-valued integration and Hölder's inequality yield that

$$\|(T_s f)_r\|_{L^p(S)} = C \left\| \int_I \int_S K_s(r\,\cdot\,, \rho\eta) f_\rho(\eta)\, d\sigma(\eta) \rho^{2N-1}\, d\rho \right\|_{L^p(S)}$$

$$\leq C\int_I \left\|\int_S K_s(r\,\cdot\,, \rho\eta) f_\rho(\eta)\, d\sigma(\eta)\right\|_{L^p(S)} \rho^{2N-1}\, d\rho$$
$$= C\int_I \left(\int_S \left|\int_S K_s(r\tau, \rho\eta) f_\rho(\eta)\, d\sigma(\eta)\right|^p d\sigma(\tau)\right)^{1/p} \rho^{2N-1}\, d\rho$$
$$\leq C\int_I \Big[\int_S \Big(\int_S |K_s(r\tau, \rho\eta)||f_\rho(\eta)|^p\, d\sigma(\eta)\Big)$$
$$\times\Big(\int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)\Big)^{p/p'} d\sigma(\tau)\Big]^{1/p} \rho^{2N-1}\, d\rho.$$

But $\int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)$ is independent of $\tau$, so by Fubini's theorem, the expression above is less than

$$C\int_I \Big[\Big(\int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)\Big)^{p/p'+1}\Big(\int_S |f_\rho(\eta)|^p\, d\sigma(\eta)\Big)\Big]^{1/p} \rho^{2N-1}\, d\rho$$
$$= C\int_I \int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)\, \|f_\rho\|_{L^p(S)} \rho^{2N-1}\, d\rho.$$

This estimate can also be verified in a similar way if $p = 1$ or $\infty$.

Using this estimate and Hölder's inequality, we have

$$\|T_s f\|^q_{p,q,\alpha} = C\int_I \|(T_s f)_r\|^q_{L^p(S)} (1 - r^2)^\alpha r^{2N-1}\, dr$$
$$\leq C\int_I \Big[\int_I \int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)\, \|f_\rho\|_{L^p(S)} \rho^{2N-1}\, d\rho\Big]^q (1 - r^2)^\alpha r^{2N-1}\, dr$$
$$\leq C\int_I \Big(\int_B |K_s(r\tau, \rho\eta)|(1 - \rho^2)^{-\delta q'}\, dm(\rho\eta)\Big)^{q/q'}$$
$$\times\Big(\int_B |K_s(r\tau, \rho\eta)|(1 - \rho^2)^{\delta q}\|f_\rho\|^q_{L^p(S)}\, dm(\rho\eta)\Big)(1 - r^2)^\alpha r^{2N-1}\, dr,$$

where $\delta$ will be chosen later. But

$$\int_B |K_s(r\tau, \rho\eta)|(1 - \rho^2)^{-\delta q'}\, dm(\rho\eta) \leq C(1 - r^2)^{-\delta q'}$$

by (1.1a), as long as $0 < \delta q' < s + 1$. Using this and Fubini's theorem, we have

$$\|T_s f\|^q_{p,q,\alpha}$$

$$\leq C\int_I \left((1 - r^2)^{-\delta q'}\right)^{q/q'} \left(\int_B |K_s(r\tau, \rho\eta)|(1 - \rho^2)^{\delta q} \|f_\rho\|^q_{L^p(S)}\, dm(\rho\eta)\right)(1 - r^2)^{\alpha} r^{2N-1}\, dr$$
$$= C\int_I \|f_\rho\|^q_{L^p(S)} (1 - \rho^2)^{\delta q} \int_I \int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)\, (1 - r^2)^{\alpha - \delta q} r^{2N-1}\, dr\, \rho^{2N-1}\, d\rho.$$

But

$$\int_I \int_S |K_s(r\tau, \rho\eta)|\, d\sigma(\eta)\, (1 - r^2)^{\alpha-\delta q} r^{2N-1}\, dr$$
$$= \int_B |K_s(r\tau, \rho\eta)|(1 - r^2)^{-(-\alpha+\delta q)}\, dm(r\tau)$$
$$\leq C(1 - \rho^2)^{-(-\alpha+\delta q)} = C(1 - \rho^2)^{\alpha-\delta q}$$

by (1.1b), as long as $0 < -\alpha + \delta q + s < s + 1$. So

$$\|T_s f\|^q_{p,q,\alpha} \leq C\int_I \|f_\rho\|^q_{L^p(S)} (1 - \rho^2)^{\alpha} \rho^{2N-1}\, d\rho = C\|f\|^q_{p,q,\alpha}.$$

To choose suitable s, note that there exists $\delta$ satisfying

$$0 < \delta q' < s + 1 \text{ and } 0 < -\alpha + \delta q + s < s + 1$$

if and only if

$$(s + 1)q > \alpha + 1. \quad \square$$

As in [7], p. 594, we immediately get the following.

Corollary 2.1.2 : For $1 \leq p < \infty$, $1 < q < \infty$, and $-1 < \alpha$, $\|f\|_{p,q,\alpha} \leq C\|\mathrm{Re}\, f\|_{p,q,\alpha}$ for every $f \in H(B)$ with $f(0) = 0$.

Proof : Choose $s > (\alpha + 1)/q - 1$. Let $u = \text{Re } f$ and fix $0 < r, \rho < 1$. Then $f_\rho = T_s(f_\rho) = T_s(f_\rho + \overline{f_\rho}) = 2T_s(u_\rho)$, so

$$\int_S |f_\rho(r\tau)|^p \, d\sigma(\tau) = 2^p \int_S |T_s u_\rho(r\tau)|^p \, d\sigma(\tau).$$

Thus

$$\|f\|_{p,q,\alpha}^q = \int_I \left(\int_S |f_\rho(r\tau)|^p \, d\sigma(\tau)\right)^{q/p} (1 - r^2)^\alpha r^{2N-1} \, dr$$
$$= 2^q \int_I \left(\int_S |T_s u_\rho(r\tau)|^p \, d\sigma(\tau)\right)^{q/p} (1 - r^2)^\alpha r^{2N-1} \, dr$$
$$= 2^q \|T_s u_\rho\|_{p,q,\alpha}^q \le 2^q A^q \|u_\rho\|_{p,q,\alpha}^q,$$

and the result follows upon letting $\rho \to 1$. □

The "inner norm" $L^p(S)$ was not critical in Theorem 2.1.1. If $N = 1$, the Bergman kernel satisfies $K_s(re^{i\theta}, \rho e^{i\varphi}) = K_s(re^{i(\theta - \varphi)}, \rho)$ and a change of variables is possible, so $L^p(S_1)$ may be replaced by any Banach space $X$ on $S_1$ satisfying $\|F_\theta\|_X \le C\|F\|_X$ for all $\theta \in [0, 2\pi)$ and for all $F \in X$. (Here, and only here, we write $F_\theta$ for the function defined on $S_1$ by $F_\theta(e^{i\varphi}) = F(e^{i(\varphi - \theta)})$. No confusion with $f_r$ should result.) The beginning of the proof is then

$$\left\|\int_\Pi K_s(re^{i\cdot}, \rho e^{i\theta}) f_\rho(e^{i\theta}) \, d\theta\right\|_X = \left\|\int_\Pi K_s(r, \rho e^{i(\theta - \cdot)}) f_\rho(e^{i\theta}) \, d\theta\right\|_X$$
$$= \left\|\int_\Pi K_s(r, \rho e^{i\varphi}) (f_\rho)_{-\varphi}(e^{i\cdot}) \, d\varphi\right\|_X \le C \, \|f_\rho\|_X \int_\Pi |K_s(r, \rho e^{i\varphi})| \, d\varphi$$

and the rest of the proof goes through.

If $N > 1$, this can be imitated to some extent. Write $e_1 = (1, 0, \ldots, 0) \in S_N$. For $\tau \in S$, let $\mu_\tau$ be a unitary transformation on $S$ (i.e., $\langle \mu_\tau \eta_1, \mu_\tau \eta_2 \rangle = \langle \eta_1, \eta_2 \rangle$ for every $\eta_1, \eta_2 \in S$) with $\mu_\tau e_1 = \tau$. So

$$\int_S K_s(r\tau, \rho\gamma)f_\rho(\gamma)\, d\sigma(\gamma) = \int_S K_s(r\mu_\tau^{-1}\tau, \rho\mu_\tau^{-1}\gamma)f_\rho(\gamma)\, d\sigma(\gamma)$$
$$= \int_S K_s(re_1, \rho\eta)f_\rho(\mu_\tau\eta)\, d\sigma(\eta)$$

where $\eta = \mu_\tau^{-1}\gamma$. If X is a Banach space on $S_N$ satisfying

(1.2) $\|F(\mu.\eta)\|_X \leq C\|F(\cdot)\|_X$ for every $\eta \in S_N$ and every $F \in X$,

then $\|\int_S K_s(r\cdot, \rho\gamma)f_\rho(\gamma)\, d\sigma(\gamma)\|_X \leq C\, \|f_\rho\|_X \int_S |K_s(re_1, \rho\eta)|\, d\sigma(\eta)$. The inequality (1.2) is trivial when $X = C(S_N)$ for any N. When N = 2, we may take

$$\mu_\tau = \begin{bmatrix} \tau_1 & -\bar{\tau}_2 \\ \tau_2 & \bar{\tau}_1 \end{bmatrix}$$

A computation then shows that $|\tau_1 - \tau_2| = |\mu_{\tau_1}\eta - \mu_{\tau_2}\eta|$ for every $\tau_1, \tau_2, \eta \in S_2$, so that (1.2) holds when $X = \mathrm{Lip}_\alpha S_2$, $0 < \alpha \leq 1$. (Recall that $\mathrm{Lip}_\alpha S_2 = \{f \in C(S_2) \mid |f(\tau_1) - f(\tau_2)|/|\tau_1 - \tau_2|^\alpha < K(f) < \infty$ for all $\tau_1, \tau_2 \in S_2$ with $\tau_1 \neq \tau_2\}$.)

## §2 Representation of the Dual Space of $A^{pq}_\alpha$

Representation of the dual space of $A^{pq}_\alpha$ will follow from boundedness of the Bergman projection (Theorem 2.1.1). The case N = 1, $\alpha = 0$ was handled by Shapiro in [16], Corollary 3.6. In [11], Theorem 2.1, Luecking identified the dual space of $A^p_\alpha$ using the boundedness of the Bergman projection on $A^p_\alpha$ and $A^{p'}_{\alpha(1-p')}$.

Given $g \in A^{p'q'}_{\alpha(1-q')}$, define the linear functional $L_g$ on $A^{pq}_\alpha$ by $L_g f = \langle f, g\rangle = \int_B f\bar{g}\, dm$ for $f \in A^{pq}_\alpha$. By two applications of Hölder's inequality,

$$|\langle f, g\rangle| \leq \|f\|_{p,q,\alpha}\|g\|_{p',q',\alpha(1-q')}.$$

Theorem 2.2.1 : Suppose that $-1 < \alpha$, $1 < p < \infty$, and $\max\{1, \alpha + 1\} < q < \infty$. Then the map taking g to $L_g$ is a linear homeomorphism of $A^{p'q'}_{\alpha(1-q')}$ onto the dual space of $A^{pq}_{\alpha}$.

Proof : As noted above, any $g \in A^{p'q'}_{\alpha(1-q')}$ defines a bounded linear functional $L_g$ on $A^{pq}_{\alpha}$, with $\|L_g\| \leq \|g\|_{p',q',\alpha(1-q')}$.

Now take any $L \in (A^{pq}_{\alpha})^*$. Extend to $L \in (L^{pq}_{\alpha})^*$ by the Hahn-Banach theorem. Write $\omega(r) = (1 - r^2)^{\alpha}$, and note that $j \in L^{pq}$ if and only if $j/\omega^{1/q} \in L^{pq}_{\alpha}$. So define $\Lambda \in (L^{pq})^*$ by $\Lambda j = L(j/\omega^{1/q})$; then there exists some $k \in L^{p'q'}$ such that $\Lambda j = \int_B jk\, dm$. (See [2], Theorem 3.1 for the generalized representation theorem.) Let $h = \overline{k\omega^{1/q}}$. We have $h \in L^{p'q'}_{\alpha(1-q')}$ and $Lf = \langle f, h\rangle$ for all $f \in L^{pq}_{\alpha}$. Since $\overline{K_0(z, w)} = K_0(w, z)$, Fubini's theorem implies that

$$\langle T_0 f_1, f_2\rangle = \langle f_1, T_0 f_2\rangle \text{ for } f_1 \in L^{pq}_{\alpha} \cap L^2 \text{ and } f_2 \in L^{p'q'}_{\alpha(1-q')} \cap L^2. \tag{2.1}$$

(To justify the application of Fubini's theorem, note that $T_0f_1$ and $T_0f_2$ are in $L^2$ since $f_1$ and $f_2$ are, either by Theorem 2.1.1 or by Békollé's result.) Now, $q > \alpha + 1$ and $q' > \alpha(1 - q') + 1$, so by Theorem 2.1.1, $T_0$ is bounded on $L^{pq}_{\alpha}$ and $L^{p'q'}_{\alpha(1-q')}$. By continuity of $T_0$ and density of the respective spaces ([2], p. 308), (2.1) is also true for $f_1 \in L^{pq}_{\alpha}$ and $f_2 \in L^{p'q'}_{\alpha(1-q')}$. Let $g = T_0h$. So $g \in A^{p'q'}_{\alpha(1-q')}$, and for $f \in A^{pq}_{\alpha}$ we have $Lf = \langle f, h\rangle = \langle T_0f, h\rangle = \langle f, T_0h\rangle = \langle f, g\rangle$, i.e., $L = L_g$.

If $g \in A^{p'q'}_{\alpha(1-q')}$ defines the zero functional, then since $\overline{K_0(z, \cdot)} \in A^{pq}_{\alpha}$ for any fixed $z \in B$, we have $0 = \langle \overline{K_0(z, \cdot)}, g\rangle = \overline{T_0\, g(z)} = \overline{g(z)}$, i.e., $g \equiv 0$. So the map taking g to $L_g$ is a one-to-one, continuous, linear transformation of $A^{p'q'}_{\alpha(1-q')}$ onto

$(A^{pq}_\alpha)^*$. By the open mapping theorem, the map is actually a linear homeomorphism. □

One can use other (i.e., weighted) duality pairings (and other kernels) to get other representations of $(A^{pq}_\alpha)^*$.

## §3 Two Norm-Representation Theorems

In [11], Theorem 5.1, Luecking shows that

$$\|f\|_{H^p} \sim \Big(\sup_m \sum_{k=1}^{k_m} |f(a_{mk})|^p(1 - r_m)^N\Big)^{1/p} \text{ for all } f \in H^p$$

where $0 \le r_0 < r_1 < \ldots \to 1$ and $\{a_{mk}\}$ satisfies

(1) $|a_{mk}| = r_m$ for each $m = 0, 1, 2, \ldots$ and each $k = 1, 2, \ldots, k_m$,

(2) $r_m S_N \subset \bigcup_k E(a_{mk}, \delta)$ for each m for some $\delta = \delta(p)$ sufficiently small,

and (3) $E(a_{mk}, \varepsilon) \cap E(a_{mk'}, \varepsilon) = \emptyset$ for each m and each $k \neq k'$ for some $0 < \varepsilon < \delta$.

Such a set $\{a_{mk}\}$ will be called an $\varepsilon$-$\delta$ lattice; notice that this differs slightly from Luecking's use of the term, for he requires the condition $B_N = \bigcup_m \bigcup_k E(a_{mk}, \delta)$ instead of (2).

A close analysis of Luecking's proof yields the following.

<u>Theorem 2.3.1</u> : Fix $0 < p < \infty$, $0 < q < \infty$, and $-1 < \alpha$. Let $r_m = 1 - 2^{-m}$ for $m = 0, 1, 2, \ldots$ and suppose $\{a_{mk}\}$ is an $\varepsilon$-$\delta$ lattice for $\delta = \delta(p, q, \alpha)$ sufficiently small. Then

$$\|f\|_{p,q,\alpha} \sim \Big(\sum_{m=0}^{\infty} \big(\sum_{k=1}^{K_m} |f(a_{mk})|^p 2^{-mN}\big)^{q/p} 2^{-m(\alpha+1)}\Big)^{1/q} \text{ for every } f \in A_\alpha^{pq}.$$

Proof : Write $E_{mk} = E(a_{mk}, \varepsilon)$, $A_m = \bigcup_k E(a_{mk}, \varepsilon)$, $r_{m\varepsilon} = (r_m + \varepsilon)/(1 + r_m\varepsilon)$ and $I_{m\varepsilon} = \big[(r_m - \varepsilon)/(1 - r_m\varepsilon), (r_m + \varepsilon)/(1 + r_m\varepsilon)\big)$. Then by subharmonicity of $|f|^p$, the separation property (3), and Lemma 1.3.1,

$$\begin{aligned}\sum_k |f(a_{mk})|^p 2^{-mN} &\le 2^{-mN}\Big(\sum_k C\int_{E_{mk}} |f|^p\, dm / m(E_{mk})\Big)\\ &\le C2^{-mN}2^{m(N+1)}\sum_k \int_{E_{mk}} |f|^p\, dm = C2^m\int_{A_m} |f|^p\, dm\\ &= C2^m\int_{I_{m\varepsilon}} \Big(\int_S |f(r\tau)|^p\, d\sigma(\tau)\Big) r^{2N-1}\, dr\\ &\le C2^m\Big(\int_{I_{m\varepsilon}} r^{2N-1}\, dr\Big) \|f_{r_{m\varepsilon}}\|^p_{L^p(S)} \le C\|f_{r_{m\varepsilon}}\|^p_{L^p(S)}.\end{aligned}$$

Since we may assume that $\varepsilon < 1/3$ (at the cost of increasing the constant C), we have $r_{m\varepsilon} < (r_m + 1)/2 = r_{m+1}$, and thus

$$\begin{aligned}\sum_m \Big(\sum_k |f(a_{mk})|^p 2^{-mN}\Big)^{q/p} 2^{-m(\alpha+1)} &\le C\sum_m \|f_{r_{m+1}}\|^q_{L^p(S)} 2^{-m(\alpha+1)}\\ &\le C\int_I \|f_r\|^q_{L^p(S)} (1 - r)^\alpha r^{2N-1} dr = C\|f\|^q_{p,q,\alpha}.\end{aligned}$$

In the other direction, Luecking uses a change of variables, Fubini's theorem, and the "denseness" property (2) and actually shows that

$$c\|f_{r_m}\|^p_{L^p(S)} \le C\delta^p\|f_{r_{m\varepsilon}}\|^p_{L^p(S)} + \sum_k |f(a_{mk})|^p 2^{-mN}$$

so

$$\begin{aligned}c\|f\|^q_{p,q,\alpha} &\le \sum_m \Big(C\delta^p\|f_{r_{m+1}}\|^p_{L^p(S)} + \sum_k |f(a_{mk})|^p 2^{-mN}\Big)^{q/p} 2^{-m(\alpha+1)}\\ &\le K\sum_m \Big(C\delta^q\|f_{r_{m+1}}\|^q_{L^p(S)} + \big(\sum_k |f(a_{mk})|^p 2^{-mN}\big)^{q/p}\Big) 2^{-m(\alpha+1)}\end{aligned}$$

where $K = 1$ if $q \le p$ and $K = 2^{q/p-1}$ if $p < q$. Thus

$$c\|f\|^q_{p,q,\alpha} \le C\delta^q \sum_m \|f_{r_{m+1}}\|^q_{L^p(S)} 2^{-m(\alpha+1)} + C\sum_m \Big(\sum_k |f(a_{mk})|^p 2^{-mN}\Big)^{q/p} 2^{-m(\alpha+1)}$$
$$\le C\delta^q \|f\|^q_{p,q,\alpha} + C\sum_m \Big(\sum_k |f(a_{mk})|^p 2^{-mN}\Big)^{q/p} 2^{-m(\alpha+1)}$$

and the result follows for $\delta$ sufficiently small. □

We wish to note here that the same proof works if $(1-r)^\alpha$ is replaced by any radial weight $\omega$ satisfying $\omega(2r - 1) \le C\omega(r)$ for every $r \in [1/2, 1)$.

The fact that $\|f\|_{p,q,\alpha}$ dominates the weighted sum will also be a consequence of Theorem 3.3.1.

As a consequence of this theorem, no such $\varepsilon$-$\delta$ lattice can be a subset of the zero set of an $A^{pq}_\alpha$ function not identically zero.

The second theorem of this section concerns subsets G of B that are "large enough" so that $\|f\|_{p,q,\omega}$ and $\|f\chi_G\|_{p,q,\omega}$ are comparable for every $f \in A^{pq}_\omega$. (The constant of comparability may depend on G, as well as p, q, and $\omega$.) We make the following assumptions:

(1) $0 = r_0 < r_1 < \ldots \to 1$,

(2) $\{a_{mk}\}$ is an $\varepsilon$-$\delta$ lattice for some $\varepsilon$ and $\delta$ sufficiently small ($\delta = \delta(p, q, K)$ where K is as in (6) below),

(3) $(r_m + 1)/2 \le r_{m+M}$ for each $m = 0, 1, 2, \ldots$ for some integer M,

(4) $|(r_m - r_{m'})/(1 - r_m r_{m'})| > 3\gamma$ for each $m \ne m'$ for some $0 < \gamma < \varepsilon$,

(5) $\omega(r) \le C\omega(r_{m+1})$ for each m and for each $r \in (r_m, r_{m+1}]$,

(6) $\omega(r_{m+1})(r_{m+1} - r_m) \le K\omega(r_{m+1+M})(r_{m+1+M} - r_{m+M})$ for each m,

and (7) $\big(\omega(r_{m+1})(r_{m+1} - r_m)\big)^{p/q} \le C\omega(r)(1 - r_{m+1})$ for each m and for each r satisfying $|(r_{m+1} - r)/(1 - r_{m+1}r)| < \gamma$ for some p and q.

If $p = q$, these conditions are satisfied by $r_m = 1 - 2^{-m}$ and $\omega(r) = (1 - r^2)^\alpha$ with $\alpha > -1$. Conditions (3) and (4) say that $r_m \to 1$ "not too slowly", while condition (6) says that $r_m \to 1$ "not too quickly", at least for weights $\omega$ with suitably controlled growth.

<u>Theorem 2.3.2</u> : Suppose $0 < p \le q < \infty$, conditions (1) - (7) hold, and $\bigcup_m \bigcup_k E(a_{mk}, \delta) \subset G \subset B$. Then $\|f\|_{p,q,\omega} \sim \|f\chi_G\|_{p,q,\omega}$ for every $f \in A^{pq}_\omega$.

<u>Proof</u> : Since we may assume that $\gamma < 1/3$, condition (3) yields

$$r_{m-M} \le 2r_m - 1 \le (r_m - \gamma)/(1 - r_m\gamma) < (r_m + \gamma)/(1 + r_m\gamma) \le (r_m + 1)/2 \le r_{m+M}.$$

Then, as noted in the proof of Theorem 2.3.1, Luecking actually shows that

$$c\|f_{r_m}\|^p_{L^p(S)} \le C\delta^p\|f_{r_{m+M}}\|^p_{L^p(S)} + \sum_k |f(a_{mk})|^p(1 - r_m)^N$$

so

$$\begin{aligned} &c\sum_m \|f_{r_m}\|^q_{L^p(S)}\omega(r_m)r_m^{2N-1}(r_m - r_{m-1}) \\ &\le C\delta^q\sum_m \|f_{r_{m+M}}\|^q_{L^p(S)}\omega(r_m)r_m^{2N-1}(r_m - r_{m-1}) \\ &\qquad + \sum_m \Big(\sum_k |f(a_{mk})|^p(1 - r_m)^N\Big)^{q/p}\omega(r_m)r_m^{2N-1}(r_m - r_{m-1}) \\ &\le KC\delta^q\sum_m \|f_{r_m}\|^q_{L^p(S)}\omega(r_m)r_m^{2N-1}(r_m - r_{m-1}) \\ &\qquad + \sum_m \Big(\sum_k |f(a_{mk})|^p(1 - r_m)^N\Big)^{q/p}\omega(r_m)r_m^{2N-1}(r_m - r_{m-1}) \end{aligned}$$

by condition (6). Thus for $\delta$ small,

$$\sum_m \|f_{rm}\|^q_{L^p(S)}\omega(r_m)r_m{}^{2N-1}(r_m - r_{m-1})$$
$$\leq C\sum_m \Big(\sum_k |f(a_{mk})|^p(1 - r_m)^N\Big)^{q/p}\omega(r_m)r_m{}^{2N-1}(r_m - r_{m-1}).$$

(The reverse inequality is also true but is not needed here.)

Write $I_m = [r_m, r_{m+1})$, $E_{mk} = E(a_{mk}, \gamma)$, $A_m = \bigcup_k E(a_{mk}, \gamma)$, and $I_{m\gamma} = \Big[(r_m - \gamma)/(1 - r_m\gamma), (r_m + \gamma)/(1 + r_m\gamma)\Big)$. Then using condition (5) and the above estimate,

$$\|f\|^q_{p,q,\omega} = \sum_{m=0}^{\infty} \int_{Im} \|f_r\|^q_{L^p(S)}\omega(r)r^{2N-1}\,dr$$
$$\leq C\sum_{m=0}^{\infty} \|f_{rm+1}\|^q_{L^p(S)}\omega(r_{m+1})r_{m+1}{}^{2N-1}(r_{m+1} - r_m)$$
$$\leq C\sum_m \Big(\sum_k |f(a_{mk})|^p(1 - r_m)|^N\Big)^{q/p}\omega(r_m)r_m{}^{2N-1}(r_m - r_{m-1}).$$

By subharmonicity of $|f|^p$, disjointness of the $E_{mk}$'s, and condition (7), the above sum is less than

$$C\sum_m \Big(\sum_k (1 - r_m)^N\int_{Emk} |f|^p\,dm\,/m(E_{mk})\Big)^{q/p}\omega(r_m)r_m{}^{2N-1}(r_m - r_{m-1})$$
$$\leq C\sum_m \Big((1 - r_m)^{-1}\int_{Am} |f|^p\,dm\Big)^{q/p}\omega(r_m)r_m{}^{2N-1}(r_m - r_{m-1})$$
$$\leq C\sum_m \Big(\int_I \int_S |f(r\tau)|^p\,\chi_{Am}(r\tau)\,d\sigma(\tau)\,\omega(r)r^{2N-1}\,dr\Big)^{q/p}.$$

Finally, using Jensen's inequality (since $q/p \geq 1$) and disjointness of the $I_{m\gamma}$'s (condition (4)), this is dominated by

$$C\sum_m \int_I \Big(\int_S |f(r\tau)|^p\,\chi_{Am}(r\tau)\,d\sigma(\tau)\Big)^{q/p}\omega(r)r^{2N-1}\,dr$$
$$\leq C\sum_m \int_{Im\gamma}\Big(\int_S |f(r\tau)|^p\,\chi_G(r\tau)\,d\sigma(\tau)\Big)^{q/p}\omega(r)r^{2N-1}\,dr$$
$$\leq C\int_I \Big(\int_S |f(r\tau)|^p\,\chi_G(r\tau)\,d\sigma(\tau)\Big)^{q/p}\omega(r)r^{2N-1}\,dr$$
$$= C\|f\|^q_{p,q,\omega}. \quad \square$$

Luecking shows in [10] that $\|f\|_{p,\alpha} \sim \|f\chi_G\|_{p,\alpha}$ for every $f \in A^p_\alpha$ if and only if

(3.1) $m_\alpha(\mathbb{D} \cap B) \sim m_\alpha(G \cap B)$ for every isotropic ball B centered on $\mathbb{T}$

when N = 1, p = q, and $\omega(r) = (1 - r^2)^\alpha$ with $\alpha > -1$. (He also gives a more general result of the same nature in [11].) In this case we can also show directly that conditions (1) - (7) along with $\bigcup_m \bigcup_k E(a_{mk}, \delta) \subset G$ imply (3.1). The key is the geometric estimate

$$k_m \geq c(r_m^2 - \delta^2)/(1 - r_m^2) \quad (m \neq 0)$$

where $c = c(\delta, r_1)$.

## §4 Representation of $A^{pq}_\alpha$ Functions

The duality result (Theorem 2.2.1) and the equivalence of norms result (Theorem 2.3.1) can be used to obtain a representation of $A^{pq}_\alpha$ functions as sums of kernel functions. This generalizes Luecking's Corollary 4.4 in [11].

If $\upsilon$ is a weight function on {0, 1, 2, . . .}, we write $c = \{c_{mk}\}_{m,k} \in \ell^{pq}_\upsilon$ if $\left(\sum_{m=0}^{\infty} \left(\sum_{k=1}^{k_m} |c_{mk}|^p\right)^{q/p} \upsilon_m\right)^{1/q} = \|c\|_{p,q,\upsilon} < \infty$.

Theorem 2.4.1 : Suppose that $1 < p < \infty$, $1 < q < \infty$, and $-1 < \alpha$. Let $r_m = 1 - 2^{-m}$ and $\upsilon_m = (1 - r_m)^{1 + Nq'/p' + \alpha(1-q')} r_m^{2N-1}$ for m = 0, 1, 2, . . . , and suppose $\{a_{mk}\}$ is an $\varepsilon$-$\delta$ lattice for $\delta = \delta(p, q, \omega)$ sufficiently small. Then every $f \in A^{pq}_\alpha$ is of the form

$$f(z) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_m} c_{mk} \upsilon_m (1 - \langle z, a_{mk} \rangle)^{-N-1}$$

for some $c \in \ell^{pq}_{\upsilon}$, and any f of this form is in $A^{pq}_{\alpha}$.

Note that no claim of uniqueness of $c \in \ell^{pq}_{\upsilon}$ is being made.

Proof : As in the proof of Theorem 2.3.1,

$$\|g\|_{p',q',\alpha(1-q')} \sim \Big(\sum_m \big(\sum_k |g(a_{mk})|^{p'}\big)^{q'/p'} (1 - r_m)^{1 + Nq'/p' + \alpha(1-q')} r_m^{2N-1}\Big)^{1/q'}$$

$$\text{i.e., } \|g\|_{p',q',\alpha(1-q')} \sim \|g(a_{mk})\|_{p',q',\upsilon}.$$

Thus the map R: $A^{p'q'}_{\alpha(1-q')} \rightarrow \ell^{p'q'}_{\upsilon}$ defined by $(Rg)_{mk} = g(a_{mk})$ is a linear isomorphism. Hence R is one-to-one with closed range, and $R^*: \ell^{pq}_{\upsilon} \rightarrow A^{pq}_{\alpha}$ is onto. (Since $\max\{1, \alpha(1 - q') + 1\} < q' < \infty$, we have $(A^{p'q'}_{\alpha(1-q')})^* \sim A^{pq}_{\alpha}$ by Theorem 2.2.1, where the duality pairing does not involve a weight, and we have $(\ell^{p'q'}_{\upsilon})^* \sim \ell^{pq}_{\upsilon}$ by Theorem 3.1 of [2], where the duality pairing does involve the weight $\upsilon$.)

To identify $R^*$, take $g \in A^{p'q'}_{\alpha(1-q')}$ and $c \in \ell^{pq}_{\upsilon}$, supposing first that c has only finitely many nonzero terms. Then

$$\begin{aligned}\int_B (R^*c)g \, dm = \langle R^*c, g\rangle = \langle c, Rg\rangle &= \sum_m \Big(\sum_k c_{mk}\overline{g(a_{mk})}\Big)\upsilon_m \\ &= \sum_m \sum_k c_{mk}\upsilon_m \overline{\int_B g(z)(1 - \langle a_{mk}, z\rangle)^{-N-1} \, dm(z)} \\ &= \int_B \Big(\sum_m \sum_k c_{mk}\upsilon_m (1 - \langle z, a_{mk}\rangle)^{-N-1}\Big)\overline{g(z)} \, dm(z),\end{aligned}$$

so $R^*c(z) = \sum_m \sum_k c_{mk}\upsilon_m (1 - \langle z, a_{mk}\rangle)^{-N-1}$. To get the result for general $c \in \ell^{pq}_{\upsilon}$, use finite approximations to c (for which the result was just verified), the continuity of $R^*$, and the fact that convergence in $A^{pq}_{\alpha}$ implies pointwise convergence. □

## CHAPTER 3

This chapter is concerned with "Carleson measure theorems". Some of the classical results are described in the first section, some estimates (mostly geometric) are derived in the second section, and the third section is devoted to the statement and proof of the main theorem. Corollaries and related methods and results are given in sections four and five.

### §1 Some History

In the course of proving the corona theorem in [3], Carleson characterized those finite positive measures $\mu$ on $B_1$ such that $\left(\int_{B_1} |f|^p \, d\mu\right)^{1/p} \leq C\|f\|_{H^p}$ for every $f \in H^p$ $(0 < p < \infty)$. He showed that this holds if and only if $\mu S \leq C'(1 - \rho)$ for every set S of the form $S = S_{\rho\theta_0} = \left\{re^{i\theta} \mid \rho \leq r < 1 \text{ and } \theta_0 - \pi(1 - \rho) \leq \theta < \theta_0 + \pi(1 - \rho)\right\}$; such a measure $\mu$ is now often called a Carleson measure, and such sets S are called Carleson sets. The necessity of this geometric condition is easily shown by the proper choice of a function $f \in H^p$ which is suitably large on S; this procedure will be demonstrated in the proof of our Theorem 3.3.1. Carleson's proof of the sufficiency of the geometric condition used a complicated covering argument. Hormander ([9]) derived a version for more general regions in $\mathbb{C}^N$ using a maximal function, Marcinkiewicz interpolation, and a simpler covering argument. Using some of Hormander's ideas, Duren ([5]) generalized the theorem to the following:

$$\left(\int_B |f|^{p_2}\, d\mu\right)^{1/p_2} \le C\|f\|_{H^{p_1}} \text{ for every } f \in H^{p_1} \quad (0 < p_1 \le p_2 < \infty)$$

if and only if

$$\mu S \le C'(1 - \rho)^{p_2/p_1} \text{ for every Carleson set } S = S_{\rho\theta_0}.$$

The results mentioned so far have concerned the Hardy spaces; i.e., measures on B are compared to "measures on S". Other results have been gotten concerning the (weighted) Bergman spaces; i.e., measures on B are compared to measures on B. For example, there is a theorem due to Hastings ([8]) for the polydisc $\mathbb{D}^N$ (the product of N copies of $\mathbb{D} = B_1$):

$$\left(\int_{\mathbb{D}^N} |f|^{p_2}\, d\mu\right)^{1/p_2} \le C\left(\int_{\mathbb{D}^N} |f|^{p_1}\, dm\right)^{1/p_1} \text{ for every analytic function } f \text{ on } \mathbb{D}^N$$
$$(0 < p_1 \le p_2 < \infty)$$

if and only if

$$\mu S \le C'\left[(1 - \rho_1)\cdots(1 - \rho_N)\right]^{2p_2/p_1} \text{ for every set } S \text{ of the form } S = S_{\rho_1\theta_{01}} \times \cdots \times S_{\rho_N\theta_{0N}}.$$

The result of Luecking in [11], already discussed at the end of Section 3 of Chapter 2, can be viewed as a Carleson measure theorem for the measures $\chi_G dm_\alpha$ induced by "large enough" sets G in $B_1$. Cima and Wogen proved a Carleson measure theorem for weighted Bergman spaces in $B_N$ ([4]); the Cima-Wogen theorem is a consequence of Luecking's general technique in [12], and also of the proof of the main result here (see Theorem 3.4.3). The main result here, Theorem 3.3.1, is for weighted mixed norm generalizations of Bergman spaces in $\mathbb{C}^N$, and is thus of the second type mentioned.

## §2 Some Estimates

Recall from section 3 of Chapter 1 that for $\rho \in (0, 1)$, $\eta \in S_N$, and $\delta > 0$ we defined $\hat{B}(\rho\eta, \delta) = \{\tau \in S_N \mid (d(\tau, \eta))^2 < \delta(1 - \rho)\}$, where d is the nonisotropic metric on $S_N$ given by $d(\tau, \eta) = |1 - \langle\tau, \eta\rangle|^{1/2}$; we also defined $\hat{B}(0, \delta) = S_N$. Also recall that $\sigma\hat{B}(\rho\eta, \delta) \sim (1 - \rho)^N$. (Now the constants of comparability involve $\delta$.) Our first lemma is the construction of a "nice" covering of S by sets $\hat{B}$.

<u>Lemma 3.2.1</u> : Suppose $\rho \in (0, 1)$ and $\delta > 0$ are fixed. Then $\hat{B}(\rho\eta_1, \delta), \ldots, \hat{B}(\rho\eta_{k_\rho}, \delta)$ can be chosen inductively so that

(1) $S \subset \bigcup_k \hat{B}(\rho\eta_k, \delta)$,

(2) $\hat{B}(\rho\eta_1, \delta/2), \ldots, \hat{B}(\rho\eta_{k_\rho}, \delta/2)$ are pairwise disjoint,

and (3) no point of S is in more than L of the sets $\hat{B}(\rho\eta_k, \delta)$, where $L = L(N, \delta) < \infty$ is independent of $\rho$.

<u>Proof</u> : In the induction scheme, if there exists some $\eta \in S \setminus \bigcup_k \hat{B}(\rho\eta_k, \delta)$, then add $\hat{B}(\rho\eta, \delta)$ to the collection; condition (2) will continue to hold because d is a metric. The scheme will terminate in finitely many steps because $\sigma\hat{B}(\rho\eta_k, \delta/2) \geq c(1 - \rho)^N$, $\sigma S = 1$, and (2) holds. To demonstrate condition (3), take any point $\eta \in S$; then $\hat{B}(\rho\eta, 2\delta)$ contains all of the sets $\hat{B}(\rho\eta_k, \delta)$ which contain $\eta$ because d is a metric; but (2) holds, $\sigma\hat{B}(\rho\eta_k, \delta/2) \geq c(1 - \rho)^N$, and $C(1 - \rho)^N \geq \sigma\hat{B}(\rho\eta, 2\delta)$. $\square$

The geometric facts above do not depend strongly on the particular metric being used. However, the nonisotropic metric is being used here because of its close connection with (the denominator of) the invariant Poisson kernel $P(z, \zeta) = \left((1 - |z|^2)/|1 - \langle z, \zeta\rangle|^2\right)^N$. In particular, it allows the following argument.

Fix $f \in H(B_N)$ and $r_0 \in (0, 1)$. Choose a positive integer M so large that $\delta < 1 - 2^{-M}$ and then let $R = 1 - (1 - r_0)2^{-M}$. Note that $0 < r_0 < R < 1$, so there is no problem with the existence of $f_R$ on S. Define u on $RB = \{Rz \mid z \in B\}$ by

$$u(r\tau) = \int_S |f_R(\zeta)|^p \left((R^2 - r^2)/|R - \langle r\tau, \zeta\rangle|^2\right)^N d\sigma(\zeta),$$

i.e., $u(r\tau) = \int_S |f_R(\zeta)|^p P((r/R)\tau, \zeta)\, d\sigma(\zeta) = P[|f_R|^p]\,((r/R)\tau)$. Then $|f|^p \le u$ on RB (see the proof of Theorem 5.6.2 in [15]) and $\int_S u_r\, d\sigma = u(0) = \int_S |f_R|^p\, d\sigma$ for $0 \le r < R$ (use the definition of u, Fubini's theorem, and the fact that $\int_S P((r/R)\tau, \zeta)\, d\sigma(\tau) = 1$). Most importantly, we have the following.

<u>Lemma 3.2.2</u> : With the setup above, if $z_0 = r_0\eta$, $z' = r_0\tau$, and $z = r\tau$, where $r \in [r_0, (r_0 + 1)/2]$ and $\tau \in \hat{B}(z_0, \delta)$, then $u(z_0) \sim u(z') \sim u(z)$.

<u>Proof</u> : Write $r_0' = r_0/R$ and $r' = r/R$. Fix $\zeta \in S$. Then by the triangle inequality, the fact that $\tau \in \hat{B}(z_0, \delta)$, and the choice of M, we have

$$\begin{aligned}
&|1 - \langle r_0'\tau, \zeta\rangle|^{1/2} \\
&\le |1 - \langle r_0'\tau, \tau\rangle|^{1/2} + |1 - \langle \tau, \eta\rangle|^{1/2} + |1 - \langle \eta, r_0'\eta\rangle|^{1/2} + |1 - \langle r_0'\eta, \zeta\rangle|^{1/2} \\
&\le (1 - r_0')^{1/2} + \delta^{1/2}(1 - r_0)^{1/2} + (1 - r_0')^{1/2} + |1 - \langle r_0'\eta, \zeta\rangle|^{1/2} \\
&\le (1 - r_0')^{1/2} + (1 - 2^{-M})^{1/2}(1 - r_0)^{1/2} + (1 - r_0')^{1/2} + |1 - \langle r_0'\eta, \zeta\rangle|^{1/2}
\end{aligned}$$

$$\leq 4|1 - \langle r_0'\eta, \zeta\rangle|^{1/2},$$

and $|1 - \langle r_0'\eta, \zeta\rangle|^{1/2} \leq 4|1 - \langle r_0'\tau, \zeta\rangle|^{1/2}$ similarly, so $P(r_0'\eta, \zeta) \sim P(r_0'\tau, \zeta)$, and $u(z_0) \sim u(z')$ follows by integrating over S with respect to $d\sigma(\zeta)$. To show that $u(z') \sim u(z)$, first note that

$$1 - r^2 \leq 1 - r_0'^2 < 2\big((2^M - 1)/(2^{M-1} - 1)\big)(1 - r') \leq C(1 - r'^2).$$

Secondly, for $\zeta \in S$,

$$|1 - \langle r'\tau, \zeta\rangle|^{1/2} \leq |1 - \langle r'\tau, r_0'\tau\rangle|^{1/2} + |1 - \langle r_0'\tau, \zeta\rangle|^{1/2}$$
$$\leq C(1 - r_0')^{1/2} + |1 - \langle r_0'\tau, \zeta\rangle|^{1/2} \leq C|1 - \langle r_0'\tau, \zeta\rangle|^{1/2}.$$

Finally, for $\zeta \in S$,

$$|1 - \langle r_0'\tau, \zeta\rangle|^2 = (1 - r_0'|\langle\tau, \zeta\rangle|)^2 + 2r_0'(|\langle\tau, \zeta\rangle| - \mathrm{Re}\langle\tau, \zeta\rangle)$$
$$\leq \big((2^M - 1)/(2^{M-1} - 1)\big)^2(1 - r'|\langle\tau, \zeta\rangle|)^2 + 2r'(|\langle\tau, \zeta\rangle| - \mathrm{Re}\langle\tau, \zeta\rangle)$$
$$\leq C|1 - \langle r'\tau, \zeta\rangle|^2.$$

So $u(z') \sim u(z)$ follows as before by integrating.

The claims made above also hold if $r_0 = 0$; some of the estimates become simpler. □

We will also need the following.

Lemma 3.2.3 : Fix a, b, c $\in \mathbb{R}$ satisfying $c > -1$, $b < -1 - c$, and $a < -1 - b - c$. Then for $0 \le \rho \le 1$, $\int_1^\infty ((1-\rho)x + \rho)^a x^b (x-1)^c \, dx$ is finite and bounded above by a constant $C = C(a, b, c)$ independent of $\rho$.

Proof : Use the fact that $1 \le (1-\rho)x + \rho \le x$, consider the cases $a < 0$ and $a \ge 0$ separately, and write $\int_1^\infty = \int_1^2 + \int_2^\infty$. □

§3 A Carleson Measure Theorem

With the preliminaries disposed of, we proceed to the statement and the proof of a Carleson measure theorem for mixed norm generalizations of Bergman spaces in the unit ball of $\mathbb{C}^N$.

Theorem 3.1 : Suppose $0 < p < \infty$, $0 < q_1 \le q_2 < \infty$, $-1 < \alpha$, and $0 < \delta < 1$, and suppose $\mu$ and $\nu$ are finite positive measures on S and [0, 1) respectively. Then the following are equivalent:

(3.1) $\left(\int_I \left(\int_S |f_r(\tau)|^p \, d\mu(\tau)\right)^{q_2/p} d\nu(r)\right)^{1/q_2} \le C \|f\|_{p,q_1,\alpha}$ for every $f \in A_\alpha^{pq_1}$.

(3.2) $\left(\mu(\hat{B}(\rho\eta, \delta))\right)^{1/p} \left(\nu([\rho, 1))\right)^{1/q_2} \le C'(1-\rho)^{N/p + (\alpha+1)/q_1}$ for every $\eta \in S$ and every $\rho \in [0, 1)$.

Proof : To show that (3.1) implies (3.2), fix $\eta \in S$ and $\rho \in [1/2, 1)$. (For $\rho$ near 0, (3.2) is true with large enough constant C' since $\mu$ and $\nu$ are finite measures.) Let $w = \rho\eta$ and define $f(z) = (1 - \langle z, w \rangle)^{-s}$ where $s > N/p + (\alpha+1)/q_1$. Take $z = r\tau$ where

$\rho < r < 1$ and $\tau \in \hat{B}(w, \delta)$ and let $z' = \rho\tau$. Then

$$|1 - \langle z, w\rangle|^{1/2} \leq |1 - \langle z, z'\rangle|^{1/2} + |1 - \langle z', w\rangle|^{1/2}$$
$$\leq |1 - \langle z, z'\rangle|^{1/2} + \big(|1 - \langle \eta, \tau\rangle| + |\langle \eta, \tau\rangle - \langle \rho\eta, \rho\tau\rangle|\big)^{1/2}$$
$$\leq (1 - r\rho)^{1/2} + \big(\delta(1 - \rho) + (1 - \rho^2)\big)^{1/2} \leq C(1 - \rho)^{1/2},$$

i.e., $|f(z)| > c(1 - \rho)^{-s}$ for such z. So the left-hand side of (3.1) is greater than

$$\Big(\int_\rho^1 \Big(\int_{\hat{B}(\rho\eta,\,\delta)} c(1 - \rho)^{-sp}\, d\mu(\tau)\Big)^{q_2/p} d\nu(r)\Big)^{1/q_2}$$
$$= c(1 - \rho)^{-s}\big(\mu(\hat{B}(\rho\eta, \delta))\big)^{1/p}\big(\nu([\rho, 1))\big)^{1/q_2}.$$

On the other hand,

$$\|f\|_{p,q_1,\alpha} = \Big(\int_I \Big(\int_S |1 - \langle r\tau, w\rangle|^{-sp}\, d\sigma(\tau)\Big)^{q_1/p}(1 - r^2)^\alpha r^{2N-1}\, dr\Big)^{1/q_1}$$
$$\sim \Big(\int_I \big((1 - r\rho)^{N-sp}\big)^{q_1/p}(1 - r)^\alpha r^{2N-1}\, dr\Big)^{1/q_1}$$

by Proposition 1.4.10 of [15], and under the change of variables $x = (1/r - \rho)/(1 - \rho)$, this becomes

$$C(1 - \rho)^{N/p + (\alpha+1)/q_1 - s}\Big(\int_1^\infty ((1-\rho)x + \rho)^{-(2N + 1 + \alpha + Nq_1/p - sq_1)}\, x^{Nq_1/p - sq_1}(x - 1)^\alpha\, dx\Big)^{1/q_1},$$

which is bounded above by $C(1 - \rho)^{N/p + (\alpha+1)/q_1 - s}$, with C independent of $\rho$, by Lemma 3.2.3. Combining these two estimates, (3.1) yields

$$\big(\mu(\hat{B}(\rho\eta, \delta))\big)^{1/p}\big(\nu([\rho, 1))\big)^{1/q_2}$$

$$\leq C(1-\rho)^s(1-\rho)^{N/p+(\alpha+1)/q_1-s} = C(1-\rho)^{N/p+(\alpha+1)/q_1},$$

which is (3.2).

The hard part of the theorem is showing that (3.2) implies (3.1). Fix $f \in A_\alpha^{pq_1}$. As in the previous section, choose a positive integer M so large that $\delta < 1 - 2^{-M}$; let $r_m = 1 - 2^{-m}$, $R_m = 1 - 2^{-m-M}$, and $u_m = P[|f_{R_m}|^p]$ on $R_m B$ for $m = 0, 1, 2, \ldots$; and by Lemma 3.2.1 for $m = 1, 2, 3, \ldots$ we can choose $\hat{B}(r_m\eta_1, \delta), \ldots, \hat{B}(r_m\eta_{K_m}, \delta)$ so that

(1) $S \subset \bigcup_k \hat{B}(r_m\eta_k, \delta)$,

(2) $\hat{B}(r_m\eta_1, \delta/2), \ldots, \hat{B}(r_m\eta_{K_m}, \delta/2)$ are pairwise disjoint,

and (3) no point of S is in more than L of the sets $\hat{B}(r_m\eta_k, \delta)$, where L is independent of m.

For convenience, write $I_m = [r_m, r_{m+1})$, $z_{mk} = r_m\eta_k$, $\hat{B}_{mk} = \hat{B}(r_m\eta_k, \delta)$, and $\hat{B}_{0k} = \hat{B}_0 = S_N$. Then

$$\int_I \left(\int_S |f(r\tau)|^p \, d\mu(\tau)\right)^{q_2/p} d\nu(r) \leq \sum_{m=0}^{\infty} \int_{I_m} \left(\sum_{k=1}^{K_m} \int_{\hat{B}_{mk}} u_m(r\tau)\, d\mu(\tau)\right)^{q_2/p} d\nu(r)$$
$$\leq C\sum_m \left[\sum_k u_m(z_{mk})\left((\mu(\hat{B}_{mk}))^{1/p}(\nu(I_m))^{1/q_2}\right)^p\right]^{q_2/p}$$

by condition (1) and since $u_m(r\tau) \leq Cu_m(z_{mk})$ for $r \in I_m$ and $\tau \in \hat{B}_{mk}$ (Lemma 3.2.2). By assumption (3.2) and because $q_1 \leq q_2$, this is less than

$$\leq C\left\{\sum_m \left[\sum_k u_m(z_{mk})\left(2^{-m(N/p+(\alpha+1)/q_1)}\right)^p\right]^{q_1/p}\right\}^{q_2/q_1}$$
$$\leq C\left\{\sum_m \int_{I_m} \left(\sum_k \int_{\hat{B}_{mk}} u_m(r\tau)\, d\sigma(\tau)\right)^{q_1/p}(1-r)^\alpha r^{2N-1}\, dr\right\}^{q_2/q_1}$$

where we have used the facts that $\int_{I_m} (1-r)^\alpha r^{2N-1}\, dr \sim 2^{-m(\alpha+1)}$, $\sigma\hat{B}_{mk} \sim 2^{-mN}$, and

$u_m(z_{mk}) \le Cu_m(r\tau)$ for $r \in I_m$ and $\tau \in \hat{B}_{mk}$ (Lemma 3.2.2). Since condition (3) says the sets $\hat{B}_{mk}$ have "controlled overlap", we have

$$\sum_k \int_{\hat{B}mk} u_m(r\tau)\, d\sigma(\tau) \le C\int_S u_m(r\tau)\, d\sigma(\tau)$$
$$= Cu_m(0) = C\int_S |f_{Rm}(\tau)|^p\, d\sigma(\tau) \le C\int_S |f_{r'}(\tau)|^p\, d\sigma(\tau)$$

where $r' = (r + 2^M - 1)2^{-M}$, since $R_m \le r'$ for $r \in I_m$. (Note that this is the only place where analyticity of f is used, and the full strength of analyticity is not needed.) Inserting this estimate and using a change of variables, we have

$$\int_I \left(\int_S |f(r\tau)|^p\, d\mu(\tau)\right)^{q_2/p} dv(r)$$
$$\le C\left\{\sum_m \int_{Im} \left(\int_S |f_{r'}(\tau)|^p\, d\sigma(\tau)\right)^{q_1/p}(1 - r)^{\alpha} r^{2N-1}\, dr\right\}^{q_2/q_1}$$
$$\le C\left\{\int_I \left(\int_S |f_{r'}(\tau)|^p\, d\sigma(\tau)\right)^{q_1/p}(1 - r)^{\alpha} r^{2N-1}\, dr\right\}^{q_2/q_1}$$
$$\le C\left\{\int_I \left(\int_S |f_{r'}(\tau)|^p\, d\sigma(\tau)\right)^{q_1/p}(1 - r')^{\alpha} r'^{2N-1}\, dr'\right\}^{q_2/q_1} = C\|f\|^{q_2}_{p,q_1,\alpha}. \quad \square$$

The same proof actually yields the following.

<u>Theorem 3.3.1'</u>: Suppose $0 < p < \infty$, $0 < q_1 \le q_2 < \infty$, $-1 < \alpha$, and $0 < \delta < 1$, suppose $v$ is a finite positive measure on $[0, 1)$, and suppose for each $r \in [0, 1)$, $\mu_r$ is a finite positive measure on S satisfying $\mu_r(\hat{B}(\rho\eta, \delta)) \sim \mu_\rho(\hat{B}(\rho\eta, \delta))$ for $r \in [\rho, (\rho + 1)/2)$, $\rho \in [0, 1)$, and $\eta \in S$. Then the following are equivalent:

(3.1') $\left(\int_I \left(\int_S |f_r(\tau)|^p\, d\mu_r(\tau)\right)^{q_2/p} dv(r)\right)^{1/q_2} \le C\|f\|_{p,q_1,\alpha}$ for every $f \in A^{pq_1}_{\alpha}$,

(3.2') $\left(\mu_\rho\, (\hat{B}(\rho\eta, \delta))\right)^{1/p}\left(v([\rho, (\rho + 1)/2))\right)^{1/q_2} \le C'(1 - \rho)^{N/p + (\alpha+1)/q_1}$ for every

$\eta \in S$ and every $\rho \in [0, 1)$.

For example, $d\mu_r(\tau)$ could be $g(r, \tau)\, d\mu(\tau)$ for a suitable nonnegative function $g$ on $[0, 1) \times S_N$.

In light of the classical theorems (e.g., Duren's and Hastings'), it is natural to wonder whether Theorem 3.3.1 is true if $p$ is replaced by $p_2$ on the left and by $p_1$ on the right, with $0 < p_1 \le p_2 < \infty$. Our method of proof sheds no light on this question.

## §4 Some Corollaries

Condition (3.2) is easily verified for certain choices of $\mu$ and $\nu$, and thus we get the following corollaries.

Corollary 3.4.1 : $A^{p_1 q_1}_{\alpha_1} \subset A^{p_2 q_2}_{\alpha_2}$ if $0 < p_2 \le p_1 < \infty$, $0 < q_2 \le q_1 < \infty$, $-1 < \alpha_1$, $-1 < \alpha_2$, and $(\alpha_1 + 1)/q_1 \le (\alpha_2 + 1)/q_2$.

Proof : Take $\mu = \sigma$ and $d\nu(r) = (1 - r)^{\alpha_2}\, dr$, so that $\|f\|_{p_1, q_2, \alpha_2} \le C \|f\|_{p_1, q_1, \alpha_1}$ by Theorem 3.3.1, hence $A^{p_1 q_1}_{\alpha_1} \subset A^{p_1 q_2}_{\alpha_2}$. The slightly more general statement given then follows from Hölder's inequality as noted in Section 1 of Chapter 1. (However, Hölder's inequality alone is not enough to give this result.) $\square$

Corollary 3.4.2 : $\left(\int_I |f(r\tau_0)|^{q_2} (1 - r)^{q_2(N/p + (\alpha+1)/q_1) - 1}\, dr\right)^{1/q_2} \le C \|f\|_{p, q_1, \alpha}$ $(0 < p < \infty,\ 0 < q_1 \le q_2 < \infty,\ -1 < \alpha)$ for every $f \in A^{pq_1}_{\alpha}$ and every $\tau_0 \in S$.

Proof : Take $\mu = \delta_{\tau_0}$ ( = point mass at $\tau_0$) and $dv(r) = (1 - r)^{q_2(N/p + (\alpha+1)/q_1) - 1}\, dr$. □

Theorem 3.3.1 also yields Proposition 1.2.1, which was the growth condition used to show that $A^{pq}_\alpha$ is a Banach space: fix $z \in B$, take $\mu = (1 - |z|)^N \delta_{z/|z|}$ and $\nu = (1 - |z|)^{\alpha+1}\delta_{|z|}$.

We can also get the easier direction of Theorem 2.3.1, namely that

$$\Big(\sum_{m=0}^{\infty}\Big(\sum_{k=1}^{K_m}|f(a_{mk})|^p 2^{-mN}\Big)^{q/p} 2^{-m(\alpha+1)}\Big)^{1/q} \le C\|f\|_{p,q,\alpha} \text{ for every } f \in A^{pq}_\alpha$$

(where $\{a_{mk}\}$ is an $\varepsilon$-$\delta$ lattice): for $r \in [1 - 2^{-m}, 1 - 2^{-m-1})$, take $\mu_r = \sum_{k=1}^{K_m}\delta_{a_{mk}/|a_{mk}|}$ and $dv(r) = 2^{-m(Nq/p + \alpha)}\, dr$; Theorem 3.3.1' applies since $\sigma\hat{B}_{mk} \sim 2^{-mN} \sim \sigma\{\tau \in S_N \mid \rho(a_{mk}, |a_{mk}|\tau) < \delta\}$ (recall that $\rho$ is the pseudohyperbolic metric) so $\mu_r\hat{B}_{mk}$ = number of $a_{mk}$'s in $(1 - 2^{-m})\hat{B}_{mk} \sim C$.

One final consequence of Theorem 3.3.1 is actually a corollary of its proof. First, we need some terminology. For $\eta \in S$ and $0 < t < 1$, let $\Delta(\eta, t) = \{z \in B \mid |1 - \langle z, \eta\rangle|^{1/2} < t^{1/2}\}$. It is easy to check that the sets $\Delta$ are comparable to our Carleson sets (which were also defined in terms of the nonisotropic metric) in the following sense:

$$\begin{aligned}(4.1)\qquad &\{r\tau \in B \mid (1 - t/4) \le r < 1 \text{ and } \tau \in \hat{B}((1 - t/4)\eta, \delta)\}\\ &\subset \Delta(\eta, t)\\ &\subset \{r\tau \in B \mid (1 - 4t/\delta^2) \le r < 1 \text{ and } \tau \in \hat{B}((1 - 4t/\delta^2)\eta, \delta)\}.\end{aligned}$$

Also observe that

$$(4.2) \qquad m_\alpha(\Delta(\eta, t)) \sim t^{N+\alpha+1} \sim m_\alpha\big(\hat{B}((1 - ct)\eta, \delta)\big).$$

Now, we offer another proof of this result of Cima and Wogen ([4]):

Theorem 3.4.3 (Cima-Wogen) : $\int_B |f|^2\, d\lambda \le C\int_B |f|^2\, dm_\alpha$ for every $f \in H(B_N)$ if and only if $\lambda(\Delta(\eta, t)) \le C'm_\alpha(\Delta(\eta, t))$ for every $\eta \in S_N$ and $0 < t < 1$.

Proof : Take $p = q_1 = q_2 = 2$ in the proof of Theorem 3.3.1. Then having a "product measure" was not important there. The necessity of the condition $\lambda(\Delta(\eta, t)) \le C'm_\alpha(\Delta(\eta, t))$ follows from the second containment in (4.1), condition (3.2), and (4.2). Its sufficiency follows from the first containment in (4.1), (4.2), and condition (3.1). □

## §5 Related Methods and Results

For $\delta > 0$, $\rho \in (0, 1)$, and $\eta \in S_N$, define $\tilde{B}(\rho\eta, \delta) = \big\{\tau \in S_N \,\big|\, |\tau - \eta| < \delta(1 - \rho)\big\}$. Note that $\sigma(\tilde{B}(\rho\eta, \delta)) \sim \delta^{2N-1}(1 - \rho)^{2N-1}$. Consider the two conditions

(5.1) $\Big(\int_I \Big(\int_S |f_r(\tau)|^p\, d\mu(\tau)\Big)^{q_2/p} d\nu(r)\Big)^{1/q_2} \le C\|f\|_{p,q_1,\alpha}$ for every $f \in A_\alpha^{pq_1}$,

(5.2) $\big(\mu(\tilde{B}(\rho\eta, \delta))\big)^{1/p}\big(\nu([\rho, 1))\big)^{1/q_2} \le C'(1 - \rho)^{(2N-1)/p + (\alpha+1)/q_1}$ for every $\eta \in S$ and every $\rho \in [0, 1)$.

As usual, we suppose $0 < p < \infty$, $0 < q_1 \le q_2 < \infty$, $-1 < \alpha$, $0 < \delta < 1$, and $\mu$ and $\nu$ are finite positive measures on $S$ and $[0, 1)$ respectively. Then (5.2) implies (5.1) (but any attempt at a proof of (5.2) seems to require the exponent involving N, not 2N-1). Since sets $\tilde{B}(\rho\eta_1, \delta), \ldots, \tilde{B}(\rho\eta_{k_p}, \delta)$ can be chosen to cover S with "controlled overlap" as in Lemma 3.2.1, the key to the proof of this implication is to show that $u(z) \sim u(z_0)$ for z near $z_0$ where u is some suitable majorant of $|f|^p$ on some large enough ball in $B_N$. This can be done by a purely geometric argument which we describe here.

Fix $\gamma \in (1, 2)$ and choose a positive integer M so large that $\gamma^{M-1} > 1/(2 - \gamma)$. For m = 0, 1, 2, . . . let $r_m = 1 - \gamma^m$ and $R_m = 1 - \gamma^{m-M}$, and let $u_m$ be the least harmonic majorant of $|f|^p$ on $R_m B$. Fix $z_m = r_m\eta$ and $z = r\tau$ with $r_m \leq r < r_{m+1}$ and $\tau \in \tilde{B}(z_m, \delta)$. Note that

$$|z - z_m| \leq |r\tau - r_m\tau| + |r_m\eta - r_m\tau| \leq (1 + \delta)\gamma^m - \gamma^{m-1} - \delta\gamma^{2m}.$$

Let $a_m = \gamma^m - \gamma^{m-2}$ and $b_m = a_m - |z - z_m|$. Recall that B(z, r) is an isotropic ball; i.e., $B(z, r) = \{w \in \mathbb{C}^N \mid |z - w| < r\}$. Then, since we may assume $\delta < (\gamma - 1)/\gamma^2$, we have

$$1 > b_m/a_m \geq (\gamma - 1 - \delta\gamma^2 + \delta\gamma^{2-m})/(\gamma^2 - 1) > (\gamma - 1 - \delta\gamma^2)/(\gamma^2 - 1) > 0,$$

so $m(B(z, b_m)) \sim m(B(z_m, a_m))$, independently of m, and

$$u_m(z) = \Big(\int_{B(z, bm)} u_m \, dm\Big)/m(B(z, b_m))$$
$$\leq \Big(m(B(z_m, a_m))/m(B(z, b_m))\Big)\Big(\int_{B(zm, am)} u_m \, dm\Big)/m(B(z_m, a_m)) \leq Cu_m(z_m)$$

by the ordinary mean value property (which is not true for M-harmonic functions, e.g., those gotten by integrating with the invariant Poisson kernel as in the proof of

Theorem 3.3.1). To get the other direction, let $c_m = \gamma^{m-1} - \gamma^{m-M}$ and $d_m = c_m - |z - z_m|$. Then, since $\gamma^{M-1} > 1/(2-\gamma)$ and since we may assume $\delta < (\gamma^{M-1}(2-\gamma) - 1)/\gamma^M$, we have

$$1 > d_m/c_m \geq (2\gamma^{M-1} - \gamma^M - 1 - \delta\gamma^M + \delta\gamma^{M-m})/(\gamma^{M-1} - 1)$$
$$> (\gamma^{M-1}(2-\gamma) - 1 - \delta\gamma^M)/(\gamma^{M-1} - 1) > 0,$$

so $m(B(z_m, d_m)) \sim m(B(z, c_m))$ and $u_m(z_m) \leq Cu_m(z)$ as above.

We remark that if $1 \leq p < \infty$ and $1 \leq q_1 \leq q_2 < \infty$, "$f \in A_\alpha^{pq_1}$" could be replaced by "f positive subharmonic in $B_N$" in the argument above.

This method doesn't seem to work with nonisotropic balls $\hat{B}$ because the nonisotropic metric fails to give adequate control in all directions. This manifests itself in the argument above in "radii" $b_m$ and $d_m$ that may fail to be positive.

We close with a generalization of Hastings' result in [8] for the polydisc.

<u>Theorem 3.5.1</u> : Suppose $0 < p < \infty$, $0 < q_1 \leq q_2 < \infty$, and $-1 < \alpha_1, \ldots, \alpha_N$, suppose $\mu_1, \ldots, \mu_N$ are finite positive measures on $\mathbb{T} = S_1$, and suppose $\nu_1, \ldots, \nu_N$ are finite positive measures on $[0, 1)$. Then the following two conditions are equivalent:

$$(5.3)\quad \Big(\int_I \cdots \int_I \Big(\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} |f(r_1 e^{i\theta_1}, \ldots, r_N e^{i\theta_N})|^p \, d\mu_1(\theta_1) \cdots d\mu_N(\theta_N)\Big)^{q_2/p}$$
$$d\nu_1(r_1) \cdots d\nu_N(r_N)\Big)^{1/q_2}$$
$$\leq C\Big(\int_I \cdots \int_I \Big(\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} |f(r_1 e^{i\theta_1}, \ldots, r_N e^{i\theta_N})|^p \, d\theta_1 \cdots d\theta_N\Big)^{q_1/p}$$
$$(1 - r_1)^{\alpha_1} r_1 \, dr_1 \cdots (1 - r_N)^{\alpha_N} r_N \, dr_N\Big)^{1/q_1}$$

for every function f analytic on $\mathbb{D}^N$,

(5.4) $\left(\mu_1([\theta_{01} - \pi\delta_1, \theta_{01} + \pi\delta_1))\right)^{1/p} \times \cdots \times \left(\mu_N([\theta_{0N} - \pi\delta_N, \theta_{0N} + \pi\delta_N))\right)^{1/p}$
$\times \left(\nu_1([1 - \delta_1, 1))\right)^{1/q_2} \times \cdots \times \left(\nu_N([1 - \delta_N, 1))\right)^{1/q_2}$
$\leq C\delta_1^{(1/p + (\alpha_1+1)/q_1)} \times \cdots \times \delta_N^{(1/p + (\alpha_N+1)/q_1)}$
for every $\theta_{01}, \ldots, \theta_{0N} \in \mathbb{T}$ and every $\delta_1, \ldots, \delta_N \in (0, 1]$.

Proof : First fix $\theta_{01}, \ldots, \theta_{0N} \in \mathbb{T}$ and $\delta_1, \ldots, \delta_N \in (0, 1]$. Let $w_j = (1 - \delta_j)e^{i\theta_{0j}}$ for $j = 1, \ldots, N$ and define

$$f(z_1, \ldots, z_N) = (1 - z_1\overline{w_1})^{-s} \cdots (1 - z_N\overline{w_N})^{-s}$$

where $s > 1/p + (\alpha_j+1)/q_1$ for each j. Then $|f(z_1, \ldots, z_N)| > c\delta_1^{-s} \cdots \delta_N^{-s}$ if $z_j = r_je^{i\theta_j}$ satisfies $\theta_j \in [\theta_{0j} - \pi\delta_j, \theta_{0j} + \pi\delta_j)$ and $r_j \in [1 - \delta_j, 1)$ for each j. (See [6], p. 157.) So (5.3) implies (5.4) as before.

For the proof that (5.4) implies (5.3), fix $\gamma \in (1, 2)$, choose M so large that $\gamma^{M-1} > 1/(2 - \gamma)$, write $r_{mj} = 1 - \gamma^{m_j}$ and $R_{mj} = 1 - \gamma^{M-m_j}$ for $j = 1, \ldots, N$, and write $m = (m_1, \ldots, m_N) \in \{0, 1, 2, \ldots\}^N$. For each N-tuple m, let $u_m$ be the least n-harmonic (i.e., harmonic in each variable separately) majorant of $|f|^p$ on $R_{m1}\mathbb{D} \times \cdots \times R_{mN}\mathbb{D}$ (see [14], p. 52). If $\theta_j \in [\theta_{0j} - \pi(1 - r_{mj}), \theta_{0j} + \pi(1 - r_{mj}))$ and $r_j \in [r_{mj}, r_{mj+1})$ for each j, then

$$u_m(r_1e^{i\theta_1}, \ldots, r_Ne^{i\theta_N}) \sim u_m(r_{m1}e^{i\theta_{01}}, \ldots, r_{mN}e^{i\theta_{0N}})$$

by applying the geometric argument with isotropic balls given at the beginning of this section. (The argument is applied N times, one time on each "slice".) One other needed observation (see [14], p. 52) is that

$$u_m(0) = \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} |f(R_{m1}e^{i\theta_1}, \ldots, R_{mN}e^{i\theta_N})|^p \, d\theta_1 \cdots d\theta_N.$$

Then the proof is completed in the manner of the proof of Theorem 3.3.1. □

When $1 \leq p < \infty$ and $1 \leq q_1 \leq q_2 < \infty$ the theorem is also true with "f analytic on $\mathbb{D}^N$" replaced by "f positive N-subharmonic in $\mathbb{D}^N$".

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