THE ESSENTIAL P. L. CHARACTER OF CERTAIN COMPACT SETS IN TOPOLOGICAL MANIFOLDS

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ABSTRACT

THE ESSENTIAL P.L. CHARACTER OF CERTAIN COMPACT SETS IN TOPOLOGICAL MANIFOLDS

by David Emanuel Galewski

Chapters I and II of this thesis are a study of the neighborhoods of o-dimensional sets and arcs in topological manifolds. We prove that a compact o-dimensional set in the interior of an n-manifold has a connected neighborhood which embeds in euclidean n-space and that an arc in the interior of a topological n-manifold has a P.L. n-manifold neighborhood. These theorems enable one to extend results on the properties of embeddings in P.L. manifolds to topological n-manifolds.

A topological embedding $f : S^{n-1} \to S^n$ is said to have a <u>P.L. homotopy approximation</u> if given any neighborhood N of $f(S^{n-1})$ there exists a P.L. embedding $g : S^{n-1} \to S^n$ such that $g(S^{n-1}) \subset N$ and f is homotopic to g in N. In chapter III of this thesis we prove that every embedding $f : S^{n-1} \to S^n$ with $n \neq 4$ has a P.L. homotopy approximation.

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INTRODUCTION

P. H. Doyle in [7] asks whether there exist small connected neighborhoods of compact o-dimensional sets in topological n-manifolds which embed in R^n. In Chapter I of this thesis we answer this question affirmatively and in Chapter II we show that an arbitrary embedding of an arc in a topological n-manifold has small P.L. neighborhoods. These results enable one to extend known facts about embeddings of compact o-dimensional sets and arcs in P.L. manifolds to topological manifolds.

In Chapter III we change direction and consider the problem of approximating a topological embedding of S^{n-1} in S^n by one which is P.L. In particular we show that if $f : S^{n-1} \to S^n$ is a topological embedding, N a neighborhood of $f(S^{n-1})$ and $n \neq 4$, then there exists a P.L. embedding $g : S^{n-1} \to S^n$ such that $g(S^{n-1}) \subset N$ and f is homotopic to g in N.

Notation and Terminology

The following notation will be used for certain sets and topological spaces.

$$Z = \{n \mid n \text{ is a positive integer}\},$$

$$R^n = \{x \mid x = (x_1,\ldots,x_n) \text{ an n-tuple of real numbers}\}$$

where R^n is given the topology determined by the euclidian metric d_n. The subsets

$$D^n = \{x \in R^n \mid d_n(0,x) \leq 1\},$$
$$E^n = \{x \in R^n \mid d_n(0,x) < 1\} \text{ and}$$
$$S^{n-1} = \{x \in R^n \mid d_n(0,x) = 1\}$$

of R^n are given the subspace topology.

A homeomorphic image of D^n, E^n, or S^{n-1} is called a <u>closed n-cell</u>, <u>open n-cell</u>, or <u>(n-1)-sphere</u> respectively.

A <u>topological n-manifold M^n</u> is a separable metric space in which each point has a neighborhood whose closure is a closed n-cell. The <u>interior of M^n</u>, denoted by $\overset{\circ}{M}{}^n$, is the set of all points which have open n-cell neighborhoods; the <u>boundary of M^n</u>, denoted by $\dot{M}^n$, is defined to be $M^n - \overset{\circ}{M}{}^n$. The boundary, interior and closure of a subset A of a topological space X will be denoted by B_XA, Int_XA and Cl_XA respectively, and the subscript 'X' will be suppressed if there is no possibility of ambiguity.

The basic terminology for topological manifold theory such as locally flat, tame, bicollared, etc., may be found in [3] and [10] while the terminology, notation, and basic facts of piecewise linear, P.L., topology may be found in [11] and [17]. To simplify notation we sometimes use the same symbol for a polyhedron and its triangulation.

The end of a proof will be denoted by the symbol '$\square$'.

CHAPTER I

NEIGHBORHOODS OF o-DIMENSIONAL COMPACT SETS IN TOPOLOGICAL MANIFOLDS

P. H. Doyle has shown in [7] that every compact o-dimensional subset of a topological n-manifold has a neighborhood which embeds in R^n and asked whether there perhaps exist small connected neighborhoods which embed in R^n. In this chapter we answer this question affirmatively.

<u>Lemma 1.1</u>: Let σ^n be an n-simplex in R^n and $f : \sigma^n \to R^n$ be an embedding such that $f(\sigma^n)$ is an n-cell with bicollared boundary. Then there exists a homeomorphism $g : R^n \to R^n$ such that $g \circ f|_{\sigma^n} = 1_{\sigma^n}$.

<u>Proof</u>: By M. Brown [2] there exists a homeomorphism $g_1 : R^n \to R^n$ such that $g_1 \circ f(\sigma^n)$ is an n-simplex τ^n. Let $g_2 : R^n \to R^n$ be a linear homeomorphism determined by any bijection between the vertices of τ^n and σ^n; then $g_2 \circ g_1 \circ f(\sigma^n) = \sigma^n$ and $g_2 \circ g_1 \circ f(\dot{\sigma}^n) = \dot{\sigma}^n$. If r_x denotes the ray from the barycenter of σ^n meeting $\dot{\sigma}^n$ at x then the homeomorphism $g_3 : R^n \to R^n$ determined by $r_x \xrightarrow{\text{linear}} r_{(g_2 \circ g_1 \circ f)^{-1}(x)}$ is such that $g_3 \circ g_2 \circ g_1 \circ f(\sigma^n) = \sigma^n$ and $g_3 \circ g_2 \circ g_1 \circ f|_{\dot{\sigma}^n} = 1_{\dot{\sigma}^n}$. Finally let $g_4 : R^n \to R^n$ be the homeomorphism determined by $g_4|_{\sigma^n} = (g_3 \circ g_2 \circ g_1 \circ f|_{\sigma^n})^{-1}$ and $g_4|_{R^n - \mathring{\sigma}^n} = 1_{R^n - \mathring{\sigma}^n}$. Then the required homeomorphism is $g = g_4 \circ g_3 \circ g_2 \circ g_1$. □

Definition 1.2: The pair (D,f) is an n-euclidean domain in a topological n-manifold M^n if $D \subset M^n$ and $f : D \to R^n$ is an embedding such that $f(D)$ is open and connected in R^n.

Lemma 1.3: Let M^n be an n-manifold such that

(i) (D,f) is an n-euclidean domain in M^n,

(ii) $B^n \subset \mathring{M}^n$ is an n-cell with bicollared boundary,

and (iii) C is a subset of D such that $C \cap B^n = C \cap \dot{B}^n = q$. Then there exists an $(n-1)$ cell $B^{n-1} \subset \dot{B}^n \cap D$ with bicollared boundary in $\dot{B}^n$ and a homeomorphism $g : R^n \to R^n$ such that

(iv) $g \circ f|_{B^{n-1}}(B^{n-1}) = \sigma^{n-1}$ is an $(n-1)$-simplex in $R^{n-1} \times 0$

and (v) $v * \sigma^{n-1} \cap g \circ f(C) = g \circ f(q) \in \mathring{\sigma}^{n-1}$ for some $v \in R^n$.

Proof: Since $\dot{B}^n$ is bicollared there exists a neighborhood N of q in D and a homeomorphism $h : R^{n-1} \times (-1,2) \to N$ such that $h(0,0) = q$, $h(R^{n-1} \times 0) \subset \dot{B}^n$ and $h(R^n \times [0,2)) \subset B^n$. Let σ^{n-1} be an $(n-1)$-simplex in $R^{n-1} \times 0$ with $(0,0) \in \mathring{\sigma}^{n-1}$ and $v = (0,1) \in R^{n-1} \times (-1,2)$. Then $f \circ h(v * \sigma^{n-1})$ is an n-cell with bicollared boundary in R^n hence by Lemma 1.1 there exists a homeomorphism $g : R^n \to R^n$ such that $g \circ f \circ h|_{v * \sigma^{n-1}} = 1_{\sigma^{n-1}}$. Choose $B^{n-1} = h(\sigma^{n-1})$. □

Lemma 1.4: Let $\tau^n = v * \sigma^{n-1}$ be an n-simplex and C a compact subset of R^n with $n \geq 4$ such that

(i) $C \cap \tau^n = v$

(ii) $\tau^n - v$ is contained in the unbounded component of $R^n - C$.

Then given $\varepsilon > 0$ there exists a homeomorphism $k : R^n \to R^n$ such that

(iii) $k|_{\tau^n} = 1_{\tau^n}$

and (iv) $k(C) \subset N_\varepsilon(v)$ where $N_\varepsilon(v)$ is an ε-neighborhood of v in R^n.

<u>Proof</u>: Let $\bar{\tau}^n = \bar{v} * \bar{\sigma}^n$ be an n-simplex in R^n concentric to τ^n such that $\bar{\sigma}^{n-1}$ is parallel to σ^{n-1} and $C \cup \tau^n \subset \mathring{\bar{\tau}}^n$. Also let $X^n = \bar{\tau}^n - \mathring{\tau}^n$ be a P.L. manifold with triangulation T such that $|T^{(1)}|$ contains

(i) a P.L. 1-ball $B_1^1 = \check{\bar{\sigma}}^n * \check{\sigma}^n$ (where '$\check{}$' denotes barycenter),

(ii) a P.L. 1-ball B_2^1 such that $B_2^1 \cap B_1^1 = \dot{B}_2^1 \cap \dot{B}_1^1 = \dot{B}_1^1$ and $B_2^1 \cap C = \emptyset$ (since $\tau^n - v$ and $\dot{\bar{\tau}}^n$ are in the unbounded component of $R^n - C$),

and (iii) a P.L. 1-ball $B_3^1 = (\bar{\sigma}^{n-2} * v)^{\vee} * (\sigma^{n-2} * v)^{\vee}$ such that $B_3^1 \cap B_2^1 = \emptyset$ where σ^{n-2} is an (n-1)-face of σ^{n-1} and $\bar{\sigma}^{n-2}$ is the corresponding face of $\bar{\sigma}^{n-1}$. Next we choose a fine subdivision T' of T such that $N(B_3^1, T') \cap B_2^1 = \emptyset$ where $N(B_3^1, T')$ is a regular neighborhood of B_3^1 in T'. Then $Y^n = Cl(X^n - \mathring{N}(B_3, T'))$ is a P.L. n-ball, (Y^n, B_1^1) and (Y^n, B_2^1) are (3,1)-ball pairs with $n \geq 4$ and hence by E. C. Zeeman [17] there exists a P.L. homeomorphism $\bar{h} : Y^n \to Y^n$ such that $\bar{h}(B_2^1) = B_1^1$ and $\bar{h}|_{\dot{Y}^n} = 1_{\dot{Y}^n}$. Extend $\bar{h}$ to a P.L. homeomorphism $h : R^n \to R^n$ by the identity and note $h(C) \cap B_1^1 = \emptyset$ and $h|_{\dot{\tau}^n \cup \dot{\bar{\tau}}^n} = 1_{\dot{\tau}^n \cup \dot{\bar{\tau}}^n}$. Now choose a fine subdivision T'' of T' such that $N(h(B_2^1), T'') \cap h(C) = \emptyset$.

Then $Z^n = \mathrm{Cl}(X^n - \mathring{N}(h(B^1_2), T''))$ is a P.L. n-ball with $h(C) \subset Z^n$ such that $h(C) \cap \dot{Z}^n = C \cap \dot{Z}^n = v$. Hence there exists a homeomorphism $\bar{g} : Z^n \to Z^n$ such that $\bar{g} \circ h(C) \subset N_\varepsilon(\tau^n)$ and $\bar{g}|_{\dot{Z}^n} = 1_{\dot{Z}^n}$. Finally extend $\bar{g}$ to $g : R^n \to R^n$ by the identity and hence obtain the required homeomorphism $k = g \circ h : R^n \to R^n$. □

<u>Definition 1.5</u>: An <u>arc</u> <u>$A = pq$</u> is the homeomorphic image of D^1 where p and q correspond to the image of -1 and 1 respectively.

<u>Theorem 1.6</u>: Let M^n be a connected n-manifold such that

(i) C_1 and C_2 are connected compact subsets of $\mathring{M}^n$,

(ii) (D_1, f_1) and (D_2, f_2) are disjoint subsets of M^n,

(iii) $C_i \subset D_i$ for $i = 1,2$,

and (iv) U is a connected open neighborhood of $C_1 \cup C_2$ in $\mathring{M}^n$.

Then there exists an n-euclidean domain (D, f) in M^n such that $C_1 \cup C_2 \subset D \subset U$.

<u>Proof</u>: <u>Case I</u> For $n = 1$ proof is trivial.

<u>Case II</u> For $n \geq 4$. Since U is open and connected there exists an n-cell B^n with bicollared boundary and arcs $A_i = p_i q_i$ for $i = 1,2$ contained in U such that $B^n \cap C_i = \emptyset$, $A_i \cap C_i = P_i$, $A_i \cap B^n = q_i$ and $A_i \subset D_i$ for $i = 1,2$. Also by Lemma 1.3 there exists an $(n-1)$ cell B^{n-1} with bicollared boundary in $\dot{B}^n \cap D$, and a homeomorphism $g : R^n \to R^n$

such that $g\circ f_1|_{B^{n-1}}(B^{n-1}) = \sigma^{n-1}$ an $(n-1)$-simplex in R^n and $v*\sigma^{n-1}\cap g\circ f_1(C_1\cup A_1) = g\circ f_1(q_1) \in \overset{\circ}{\sigma}{}^{n-1}$ for some $v \in R^n$. Since B^{n-1} is bicollared in $\dot{B}^n$, $Cl(B^n - B^{n-1})$ is an n-cell by [2] hence we can extend $(g\circ h_1)^{-1}|_{\sigma^{n-1}}$ to a homeomorphism $k : v*\sigma^{n-1} \to B^n$ such that $k(v) = q_2$. We now construct a homeomorphism $h : R^n \to R^n$ such that $h\circ f_2(C_2\cap A_2)\cap[(v*\sigma^{n-1})\cup g\circ f_2(C_1\cup A_1)] = h\circ f_2(q_2) = v$.

For the construction of h we choose an n-simplex $\tau^n = v*\eta^{n-1} \subset (v*\sigma^{n-1} - \sigma^{n-1})$ such that $(v*\sigma^{n-1},\eta^{n-1})$ is a $(n,n-1)$-ball pair, η^{n-1} is parallel to σ^{n-1} and $k(\tau^n) \subset D_2$. We note since B^n is bicollared k can be extended to a collar of $v*\sigma^{n-1}$, it then follows that $f_2\circ k(\tau^n)$ is an n-cell with bicollared boundary. Hence by Lemma 1.1 there exists a homeomorphism $h_1 : R^n \to R^n$ such that $h_1\circ f_2\circ k|_{\tau^n} = 1_{\tau^n}$. W.l.o.g. we may assume $f_2(\overset{\circ}{A}_2)$ is in the unbounded component of $f_2(C_2)$ by simply modifying f_2 if necessary. Hence $\tau^n - v$ is in the unbounded component of $R^n - h_1\circ f_2(C_2\cup A_2)$. Then by Lemma 1.4 with $h_1\circ f_2(C_2\cup A_2)$ replacing C and setting $\epsilon = \min\{dist(v,g\circ f_1(A_1\cup C_2)), dist(v,v*\sigma^{n-1} - \tau^n)\}$ there exists a homeomorphism $h_2 : R^n \to R^n$ such that $h_2|_{\tau^n} = 1_{\tau^n}$ and $h_2\circ h_1\circ f_2(C_2\cup A_2) \subset N_\epsilon(v)$. Hence $h_2\circ h_1\circ f_2(C_2\cup A_2)\cap v*\sigma^{n-1} = v$ and we set $h = h_2\circ h_1$.

Let U_1 be a neighborhood of $g\circ f_1(C_1\cup A_1)$ such that $Cl(U_1) \subset g\circ f_1(D_1\cap U)$, $Cl(U_1)\cap h\circ f_2(A_2\cup C_2) = \emptyset$ and $Cl(U_1)\cap v*\sigma^{n-1} \subset v*\sigma^{n-1} - \tau^n$. Also let U_2 be a neighborhood of $h\circ f_2(C_2\cup A_2)$ such that $Cl(U_2) \subset h\circ f_2(D_2\cap U)$, $Cl(U_2)\cap Cl(U_1) = \emptyset$ and $Cl(U_2)\cap v*\sigma^{n-1} \subset v*\eta^{n-1} = \tau^n$.

We define a homeomorphism

$$f' : [(v*\sigma^{n-1}) \cup (Cl(U_1) - (v*\sigma^{n-1})^\circ) \cup (Cl(U_2) - (v*\sigma^{n-1})^\circ)] \to M^n$$

by $f'|_{v*\sigma^{n-1}} = k$,

$$f'|_{Cl(U_1)-(v*\sigma^{n-1})^\circ} = (g\circ f_1)^{-1}|_{Cl(U_1)-(v*\sigma^{n-1})^\circ},$$

and $f'|_{Cl(U_2)-(v*\sigma^{n-1})^\circ} = (h\circ f_2)^{-1}|_{Cl(U_2)-(v*\sigma^{n-1})^\circ}$.

Note f^1 is well-defined since $k|_{\sigma^{n-1}} = (g\circ f_1)^{-1}|_{\sigma^{n-1}}$ and $k^{-1}(h\circ f_2)^{-1}|_{\tau^n} = k^{-1}\circ f_2^{-1}\circ h_1^{-1}\circ h_2^{-1}|_{\tau^n} = k^{-1}\circ f_2^{-1}\circ h_1^{-1}|_{\tau^n} = 1_{\tau^n}$.

If we let W be the interior of the component of $Cl(U_1) \cup Cl(U_2) \cup (v*\sigma^{n-1})$ containing $g\circ f_1(C_1 \cup A_1) \cup h\circ f_2(C_2 \cup A_2)$, then the desired n-euclidean domain (D, f) is given by $D = f'(V)$ and $f = (f'|_V)^{-1}$.

<u>Case III</u> For $n = 2$ or 3. Let $g : D^{n-1} \times I \to U$ be an embedding such that $g(D^{n-1} \times 0) \subset D_1 - C_1$, $g(D^{n-1} \times 1) \subset D_2 - C_2$, and $g(D^{n-1} \times I)$ is an n-cell with bicollared boundary. It follows that there exists an $\epsilon > 0$ such that $g(D^{n-1} \times [0, \epsilon)) \subset D_1 - C_1$ and $g(D^{n-1} \times [1-\epsilon, 1)) \subset D_2 - C_2$ are n-cells with bicollared boundary. Hence homeomorphisms $h_1, h_2 : R^n \to R^n$ exist such that $h_1\circ f_1\circ g(D^{n-1} \times 0) = \sigma_1^{n-1}$, $h_2\circ f_2\circ g(D^{n-1} \times 1) = \sigma_2^{n-1}$, where σ_1^{n-1} and σ_2^{n-1} are (n-1)-simplexes in R^n, $h_1\circ f_1(D_1) \subset H_n^-$ and $h_2\circ f_2(D_2) \subset H_n^+$ ($H_n^+ = \{x \in R^n \mid x = (x_1, \ldots, x_n)$ and $x_n > 0\}$), and $h_1\circ f_1\circ g(D^{n-1} \times 0) \cup h_2\circ f_2\circ g(D^{n-1} \times 1) = \sigma_1^{n-1} \cup \sigma_2^{n-1}$ is contained in the unbounded component of $R^n - [h_1\circ f_1(C_1) \cup h_2\circ f_2(C_2)]$. Let

$\ell : D^{n-1}\times I \to R^n$ be an embedding, obtained by "swelling" a polyhedral arc between σ_1 and σ_2, such that $\ell(D^{n-1}\times I)$ is a polyhedron, $\ell|_{D^{n-1}\times 0} = \sigma_1^{n-1}$, $\ell|_{D^{n-1}\times 1} = \sigma_2^{n-1}$ and $\ell(D^{n-1}\times I)\cap[h_1\circ f_1(C_1)\cup h_2\circ f_2(C_2)] = \emptyset$. By modifying h_1 if necessary we may assume $\ell^{-1}\circ h_i\circ f_i\circ g|_{\dot{D}^{n-1}\times(i-1)} : D^{n-1}\times(i-1) \to D^{n-1}\times(i-1)$ for $i = 1,2$ both preserve or reverse orientation. Then by Fisher [8] there exists an isotopy $H'' : \dot{D}^{n-1}\times I \to \dot{D}^{n-1}\times I$ such that

$$(\dagger)\quad H''|_{\dot{D}^{n-1}\times(i-1)} = \ell^{-1}\circ h_i\circ f_i\circ g|_{\dot{D}^{n-1}\times(i-1)} \quad \text{for } i = 1,2.$$

Let $H' : (D^{n-1}\times I)^{\cdot} \to (D^{n-1}\times I)^{\cdot}$ be a homeomorphism defined by $H'|_{\dot{D}^{n-1}\times I} = H''$ and $H'|_{D^{n-1}\times(i-1)} = \ell^{-1}\circ h_i\circ f_i\circ g$ for $i = 1,2$. H' is well defined by $(\dagger)$. Since $D^{n-1}\times I$ is an n-cell and H' is defined on $(D^{n-1}\times I)^{\cdot}$ there exists a homeomorphism $H : D^{n-1}\times I \to D^{n-1}\times I$ which is an extension of H'.

Let U_i be a connected neighborhood of $h_i\circ f_i(C_i)$ in $h_i\circ f_i(U\cap D)$ such that $\emptyset \neq Cl(U_i)\cap\ell(D^{n-1}\times I) \subset \ell(\mathring{D}^{n-1}\times I)$ for $i = 1,2$ and note $U_1\cap U_2 \subset H_n^+\cap H_n^- = \emptyset$. We define an embedding

$$f' : [\ell(D^{n-1}\times I)\cup(U_1-(\ell(D^{n-1}\times I)^\circ)\cup(U_2-k(D^{n-1}\times I)^\circ)] \to U$$

by $f'|_{\ell(D^{n-1}\times I)} = g\circ H^{-1}\circ\ell^{-1}|_{\ell(D^{n-1}\times I)}$ and $f'|_{U_i-(\ell(D^{n-1}\times I))^\circ} = (h_i\circ f_i)^{-1}$ for $i = 1,2$. f' is well defined since

$$g\circ H^{-1}\circ\ell^{-1}|_{\ell(D^{n-1}\times(i-1))} = g\circ(\ell^{-1}\circ h_i\circ f_i\circ g)^{-1}\circ\ell^{-1}|_{\ell(D^{n-1}\times(i-1))}$$

$$= g\circ g^{-1}\circ f_i^{-1}\circ h_i^{-1}\circ\ell\circ\ell^{-1}|_{\ell(D^{n-1}\times(i-1))}$$

$$= (h_i\circ f_i)^{-1}|_{\ell(D^{n-1}\times(i-1))} \quad \text{for } i = 1,2.$$

If V is the interior of the component of $U_1 \cup U_2 \cup \ell(D^{n-1} \times I)$ containing $\ell(D^{n-1} \times I)$ then the desired n-euclidean domain (D,f) is given by $D = f'(V)$ and $f = (f')^{-1}|_{f'(V)}$. □

<u>Corollary 1.7</u>: Let $\{(D_i,f_i)\}_{i=1}^{m}$ be a pairwise disjoint collection of n-euclidean domains in M^n, $C_i \subset D_i$ connected and compact for $i = 1,2,\ldots,m$, and U a connected open subset of $\overset{\circ}{M}{}^n$ containing $C = \bigcup_{i=1}^{m} C_i$. Then there exists an n-euclidean domain (D,f) such that $C \subset D \subset U$.

<u>Definition 1.8</u>: A topological space <u>X</u> <u>is</u> <u>o-dimensional</u> if for every $x \in X$ there exists arbitrarily small neighborhoods of x with empty boundary.

<u>Theorem 1.9</u>: Let U be an open connected neighborhood of a o-dimensional compact set C in the interior of a topological n-manifold M^n. Then there exists an n-euclidean domain (D,f) such that $C \subset D \subset U$.

<u>Proof</u>: For each $x \in C$ there exists an open n-cell neighborhood O_x of M^n and an open <u>and</u> closed V_x in C such that $x \in V_x \subset O_x \subset U$. Since C is compact there exists a covering $\{V_{x_i}\}_{i=1}^{m}$ of C. We define $U_j = V_{x_j} - \bigcup_{i=1}^{j-1} V_{x_i}$ for $j = 1,2,\ldots,m$ and note $\{U_j\}_{j=1}^{m}$ is a pairwise disjoint o-dimensional compact covering of C and hence by theorem 3 of Osborne [13] there exists a collection of pairwise disjoint arcs $\{A_i\}_{i=1}^{m}$ such that $U_i \subset A_i \subset O_{x_i} \cap U$. Since M^n is normal there exists a pairwise disjoint collection of open subsets of M^n, $\{W_i\}_{i=1}^{m}$ such that $A_i \subset W_i \cap U \subset W_i$ for $i = 1,2,\ldots,m$;

hence $\{W_i \cap O_{x_i} \cap U\}_{i=1}^m$ is a pairwise disjoint collection of open subsets of U such that $W_i \cap O_{x_i} \cap U$ embeds in R^n and $U_i \subset A_i \subset W_i \cap O_{x_i} \cap U$ for $i = 1,\dots,m$. Letting D_i denote the component of A_i in $W_i \cap O_{x_i} \cap U$ for $i = 1,2,\dots,m$ it follows from Corollary 1.7 there exists an n-euclidean domain (D,f) such that $C \subset D \subset U$. □

Corollary 1.10: Any two disjoint compact o-dimensional subsets in the interior of a connected topological n-manifold M^n lie in disjoint homeomorphic n-euclidean domains.

Proof: Let C_1, C_2 be the o-dimensional subsets and U_1 and U_2 disjoint open connected neighborhoods of C_1 and C_2 respectively in $\overset{\circ}{M}{}^n$. Then by Theorem 1.9 there exists disjoint n-euclidean domains (D_1,f_1) and (D_2,f_2) such that $C_i \subset D_i \subset U_i$ for $i = 1,2$. By D. Tondra [16] there exists an open connected subset $\tilde{U}$ of R^n such that $f_1(D_1) = \overset{\infty}{\underset{i=1}{(M)}} X_i$ and $f_2(D_2) = \overset{\infty}{\underset{i=1}{(M)}} Y_i$ where the unions are monotone, and X_i, Y_i and $\tilde{U}$ are homeomorphic. Since $f(C_i)$ for $i = 1,2$ are compact there exists an $N \in Z$ such that $f(C_1) \subset X_N$ and $f(C_2) \subset Y_N$. Hence the required n-euclidean domains are (D_i^*, f_i^*) for $i = 1,2$ where $D_1^* = f_1^{-1}(X_N)$, $D_2^* = f_2^{-1}(Y_N)$, $f_1^* = f_1|_{f^{-1}(X_N)}$ and $f_2^* = f_2|_{f_2^{-1}(Y_N)}$. □

Corollary 1.11: Any o-dimensional compact subset in the interior of a connected topological n-manifold M^n lies on the boundary of a k-cell in $\overset{\circ}{M}{}^n$ for $k = 1,2,\dots,n$.

Proof: Theorem 1.9 and Osborne [13]. □

<u>Definition 1.12</u>: A o-dimensional subset of a topological n-manifold is called <u>locally flat</u> if it lies on a locally flat closed 1-cell.

<u>Corollary 1.13</u>: Any compact o-dimensional set C in the interior of a connected topological n-manifold M^n lies in an open n-cell O^n except for perhaps a locally flat o-dimensional subset C' of C.

<u>Proof</u>: Let (D,f) be an n-euclidean domain given by Theorem 1.9 and N^n a connected P.L. compact n-manifold neighborhood of $f(C)$ in $f(D)$. Then by M. Brown [4] $N^n = P^n \cup R$ where P^n is an open n-cell, R is an (n-1)-complex in some triangulation of N^n, and R is disjoint from P^n. But then $f(C) \cap R = \bigcup_{\sigma^{n-1} \in R} (C \cap \sigma^{n-1})$ is the finite union of tame o-dimensional compact sets and hence by Osborne [13] is tame. Therefore there exists a locally flat arc containing $f(C) \cap R$ in $\mathring{N}^n$. But then we are finished by setting $O^n = f^{-1}(O^n)$ and $C' = f^{-1}(f(C) \cap R)$. □

<u>Corollary 1.14</u>: Any compact o-dimensional set C in a topological n-manifold M^n has an open neighborhood which embeds in R^n.

<u>Proof</u>: Let $N^n = M^n \cup_f \dot{M}^n \times [0,1)$ where $f : \dot{M}^n \to \dot{M}^n \times 0$ is defined by $f(x) = x \times 0$ for all $x \in \dot{M}^n$. Then N^n is a topological n-manifold without boundary, hence by <u>Theorem 1.9</u> there exists an n-euclidean domain (D,f) such that $C \subset D \subset N^n$. But then $D - (N^n - \mathring{M}^n)$ is an open subset of M^n which contains C and embeds in R^n. □

CHAPTER II

P.L. NEIGHBORHOODS OF ARCS IN TOPOLOGICAL MANIFOLDS

This chapter is concerned with the construction of P.L. manifold neighborhoods of arcs in topological manifolds. We begin with the following definition.

<u>Definition 2.1</u>: <u>M^n is a P.L. n-manifold</u> if there exists a homeomorphism $t : |K| \to M^n$ where K is a finite simplicial complex in which the star of each vertex is P.L. homeomorphic to an n-simplex. In this case we may write $M^n = (K,t)$ and call $|K|$ a <u>combinatorial n-manifold</u>.

The next two lemmas are well known facts in P.L. topology.

<u>Lemma 2.2</u>: If two combinatorial n-manifolds M_1^n, M_2^n are attached by a P.L. homeomorphism h along P.L. (n-1)-manifolds in their boundaries then the adjunction space $M_1^n \cup_h M_2^n$ is a combinatorial n-manifold.

<u>Lemma 2.3</u>: Let U be an open neighborhood of a compact set C in a combinatorial n-manifold M^n. Then there exists a combinatorial n-submanifold $K^n \subset U$ such that

(i) $C \subset K^n$

(ii) $K \cap \dot{M}^n$ is a combinatorial (n-1)-manifold containing $C \cap \dot{M}$ in its interior.

<u>Lemma 2.4</u>: Let A be an arc in the interior of a

topological n-manifold M^n. Then $A \subset (D_1,f_1) \cup (D_2,f_2)$ where (D_i,f_i) for $i = 1,2$ are n-euclidean domains in M^n.

<u>Proof</u>: Let C be a countable dense subset of A, then C lies in an open n-cell O^n by [6]. But $A - O^n$ is a compact o-dimensional set and hence lies in an n-euclidean domain by Theorem 1.9. □

We now introduce special notation to be used for the rest of this chapter. Let $A = pq$ be an arc in the interior of an n-manifold M^n. If B^n is a closed n-cell neighborhood of $x \in A$ then denote by p'_{B^n}, the "first" point in A that meets B^n and; denote by q'_{B^n}, the last point of A that meets B^n and note that $A = pp'_{B^n}xq'_{B^n}q$. Also let $S^i_{B^n} = p'_{B^n}x - \mathring{B}^n$ and $S^f_{B^n} = xq'_{B^n} - \mathring{B}^n$.

<u>Lemma 2.5</u>: Let

(i) $M^n = D_1 \cup D_2$ be an n-manifold without boundary such that D_1 and D_2 are open in M^n,

(ii) $A = pq$ an arc in M^n,

(iii) $x \in A \cap D_1 \cap D_2$,

(iv) C a compact set in M^n such that $C \cap \{x\} = \emptyset$.

Then there exists an $\varepsilon > 0$ so that for any closed n-cell neighborhood B^n of x such that diam $(B^n) < \varepsilon$ we have

(v) $B^n \cup S^i_{B^n} \cup S^f_{B^n} \subset D_1 \cap D_2$

and (vi) $C \cap (B^n \cup S^i_{B^n} \cup S^f_{B^n}) = \emptyset$.

Proof: Since A is locally connected there exists a connected neighborhood U of x in A such that $U \subset A \cap [(D_1 \cap D_2) - C]$ and we note that U is a subarc of A. Hence there exists an open set V in $[(D_1 \cap D_2) - C]$ such that $A \cap V = U$. If we let $\varepsilon = \text{dist}(x, M^n - V)$ then for any closed n-cell neighborhood B^n of x such that diameter $(B^n) < \varepsilon$ we have $B^n \subset V$, $B^n \cap A \subset U$, $p'_{B^n} x q'_{B^n} \subset [(D_1 \cap D_2) - C]$, $B^n \cup S^i_{B^n} \cup S^f_{B^n} \subset [(D_1 \cap D_2) - C]$, hence conditions (v) and (vi). □

Theorem 2.6: Let A be an arc in the interior of a topological n-manifold M^n. Then there exists a P.L. n-manifold N^n such that $A \subset \mathring{N}^n \subset \mathring{M}^n$.

Proof: By Lemma 2.4 we may assume $M^n = D_1 \cup D_2$ where (D_i, f_i) are n-euclidean domains for $i = 1,2$. Let $\{x_i\}_{i=0}^{\ell+1}$ be an increasing ordered set of points in A (with respect to a parameterization of A) from $x_0 = p$ to $x_{\ell+1} = q$ such that the subarcs $x_i x_{i+1} \subset D_{i \bmod 2+1}$ for $i = 0,1,\ldots,\ell$ and $x_i \in D_1 \cap D_2$ $i = 1,2,\ldots,\ell$. We may also assume $q = x_{\ell+1} \in D_1 \cap D_2$ by extending A if necessary.

We will construct inductively a sequence of P.L. manifold pairs in M^n; $(N_1^*, L_1^*) = (K_1, H_1; t_1), \ldots, (N_{\ell+1}^*, L_{\ell+1}^*) = (K_{\ell+1}, H_{\ell+1}; t_{\ell+1})$ such that for $i = 1,\ldots,\ell+1$

(1) K_i is a combinatorial n-manifold,

(2) H_i is a combinatorial n-cell in K_i,

(3) L_i^* is a bicollared n-cell in $D_1 \cap D_2$ such that $x_i \in \mathring{L}_i^*$,

(4) $p p'_{L_i^*} \cup S^i_{L_i^*} \subset \mathring{N}_i^*$,

(5) $(S^f_{L^*_i} \cup q'_{L^*_i}q) \cap N^*_i = (S^f_{L^*_i} \cup q'_{L^*_i}q) \cap \dot{N}^*_i = (S^f_{L^*_i} \cup q'_{L^*_i}q) \cap \dot{L}^*_i$,

and (6) $\{x_{i+1}, \ldots, x_{\ell+1}\} \subset q'_{L^*_i}q$.

<u>Step I</u> Construction of $(N^*_1, L^*_1) = (K_1, H_1; t_1)$.

Choose a combinatorial n-manifold $|J^n_1| \subset f_1(D_1)$ containing $f_1(x_0x_1)$ in its interior with mesh so fine that

(i) $f_1(x_1) \in \sigma^n_1$ where $\sigma^n_1 \in J^n_1$ is an n-simplex

and (ii) diam $(f_1^{-1}(\sigma^n_1)) < \varepsilon_1$ where ε_1 is given by Lemma 2.5 with $M^n = D_1 \cup D_2$,

$A = A$,

$x = x_1$,

and $C = x_0 \cup x_2 q$.

Hence (iii) $f_1^{-1}(\sigma^n_1) \cup S^i_{f_1^{-1}(\sigma^n_1)} \cup S^f_{f_1^{-1}(\sigma^n_1)} \subset (D_1 \cap D_2) - (x_0 \cup x_2 q)$.

Next we obtain a combinatorial n-manifold $|\overline{K}^n_1| \subset |J^n_1|$ by Lemma 2.3 with

$M^n = |J^n_1| - \mathring{\sigma}^n_1$,

$C = f_1(pp'_{f_1^{-1}(\sigma^n_1)} \cup S^i_{f_1^{-1}(\sigma^n_1)})$,

and $U = (|J^n_1| - \mathring{\sigma}^n_1) \cap f_1[D_1 - (S^f_{f_1^{-1}(\sigma^n_1)} \cup q'_{f_1^{-1}(\sigma^n_1)}q)]$.

Hence (iv) $f_1(pp'_{f_1^{-1}(\sigma^n_1)} \cup S^i_{f_1^{-1}(\sigma^n_1)}) \subset |\overline{K}^n_1|$,

(v) $|K^n_1| \subset (|J^n_1| - \mathring{\sigma}^n_1) \cap f_1[D_1 - (S^f_{f_1^{-1}(\sigma^n_1)} \cup q'_{f_1^{-1}(\sigma^n_1)}q)]$,

and (vi) $|\overline{K}^n_1| \cap \dot{\sigma}^n_1$ is a combinatorial (n-1) manifold with $f_1(pp'_{f_1^{-1}(\sigma^n_1)} \cup S^i_{f_1^{-1}(\sigma^n_1)}) \cap \dot{\sigma}^n_1$ in its interior.

By Lemma 2.2 with $h = 1_{|\overline{K}^n_1| \cap \sigma^n_1}$ and (vi) we have

(vii) $|K_1| \overset{\text{def.}}{=\!=} |\overline{K}_1^n| \cup \sigma_1^n$ is a combinatorial n-manifold.

Now we set

$$\text{(viii)} \begin{cases} t_1 = f_1^{-1}|_{\sigma_1^n} \\ H_1 = \sigma_1^n \\ N_1^* = t_1(|K_1|) \\ L_1^* = t_1(\sigma_1^n) \end{cases}$$

Condition (1) is satisfied by (vii); condition (2) by (viii) and (vii); condition (3) is satisfied since f_1^{-1} is defined on a neighborhood of σ_1^n, (iii), and (i); condition (4) is satisfied by (iv) and (vi); condition (5) is satisfied by (v) and (vii); and condition (6) is satisfied by (iii) and (v).

<u>Step II</u> Suppose $(N_1^*, L_1^*) = (K_1, H_1; t_1), \ldots, (N_r^*, L_r^*) = (K_r, H_r; t_r)$ for $r \leq \ell$ have been chosen to satisfy conditions (1) through (6). Since L_r^* is a bicollared n-cell in $D_1 \cap D_2$ there exists a homeomorphism $h_r : R^n \to R^n$ such that $h_r \circ f_{r\bmod 2+1} \circ t_r : H_r \to R^n$ is a P.L. embedding. Let $g_r = h_r \circ f_{r\bmod 2+1}$ and choose a combinatorial n-manifold $|J_{r+1}^n| \subset g_r(D_{r\bmod 2+1})$ with mesh so fine that

(i) $g_r(x_r x_{r+1} \cup L_r^*) \subset |\mathring{J}_{r+1}^n|$,

(ii) $g_r(x_{r+1}) \in \mathring{\sigma}_{r+1}^n$ where $\sigma_{r+1}^n \in J_{r+1}^n$ is an n-simplex,

and (iii) $\operatorname{diam}[g_r^{-1}(\sigma_{r+1}^n)] < \epsilon_{r+1}$ where ϵ_{r+1} is given by Lemma 2.5 with

$M^n = D_1 \cup D_2$,

$A = A$,

$x = x_{r+1}$,

and $\quad C = N^*_r \cup x_{r+2}q$.

Note $\; N^*_r \cap \{x_{r+1}\} = \emptyset \;$ by condition (5) and (6). Hence

$$\text{(iv)} \quad g_r^{-1}(\sigma^n_{r+1}) \cup S^{i}_{g_r^{-1}(\sigma^n_{r+1})} \cup S^{f}_{g_r^{-1}(\sigma^n_{r+1})} \subset$$

$(D_1 \cap D_2) - (N^*_r \cup x_{r+2}q)$.

We now obtain a combinatorial n-manifold $\;|\overline{K}^n_{r+1}| \subset |J^n_{r+1}|$

by Lemma 2.3 with

$$M^n = |J^n_{r+1}| - (\mathring{\sigma}^n_{r+1} \cup g_r(L^*_r)),$$

$$C = g_r(S^f_{L^*_r} \cup q'_{L^*_r} p'_{g_r^{-1}(\sigma^n_{r+1})} \cup S^i_{g_r^{-1}(\sigma^n_{r+1})}),$$

and $\quad U = [\,|J^n_{r+1}| - (\mathring{\sigma}^n_{r+1} \cup g_r(L^*_r))] \cap g_r[D_{r\bmod 2+1} - [\mathrm{Cl}(N^*_r - L^*_r) \cup S^f_{g_r^{-1}(\sigma^n_{r+1})} \cup q'_{g_r^{-1}(\sigma^n_{r+1})} q]]$.

Hence $\quad$ (v) $\quad g_r(S^f_{L^*_r} \cup q'_{L^*_r} p_{g_r^{-1}(\sigma^n_{r+1})} \cup S^i_{g_r^{-1}(\sigma^n_{r+1})}) \subset |\overline{K}^n_{r+1}|$,

(vi) $\quad |\overline{K}^n_{r+1}| \subset [\,|J^n_{r+1}| - (\mathring{\sigma}^n_{r+1} \cup g_r(L^*_r)] \cap$

$g_r[D_{r\bmod 2+1} - [\mathrm{Cl}(N^*_r - L^*_r) \cup S^f_{g_r(\sigma^n_{r+1})} \cup q'_{g_r^{-1}(\sigma^n_{r+1})} q]]$,

(vii) $\quad |\overline{K}^n_{r+1}| \cap \dot{\sigma}^n_{r+1} \quad$ is a combinatorial (n-1)-manifold

with $\; g_r(S^f_{L^*_r} \cup q'_{L^*_r} p'_{g_r^{-1}(\sigma^n_{r+1})} \cup S^i_{g_r^{-1}(\sigma^n_{r+1})}) \cap \dot{\sigma}^r_{r+1} \;$ in its interior,

and (viii) $\; |\overline{K}^n_{r+1}| \cap g_r(L^*_r)^{\cdot} = |\overline{K}^n_{r+1}| \cap g_r(N^*_r) \;$ is a combinatorial

(n-1)-manifold with $\; g_r(S^f_{L^*_r} \cup q'_{L^*_r} p'_{g_r^{-1}(\sigma^n_{r+1})} \cup S^i_{g_r^{-1}(\sigma^n_{r+1})} \cap g_r(L^*_r))^{\cdot}$

in its interior.

Then by Lemma 2.2 with $\; h = 1_{|\overline{K}^n_{r+1}| \cap \sigma^n_{r+1}} \;$ we have

(ix) $\quad |\overline{\overline{K}}{}^n_{r+1}| \overset{\text{def.}}{=\!=\!=} |\overline{K}^n_{r+1}| \cup \sigma^n_{r+1} \quad$ is a combinatorial n-manifold and by Lemma 2.2 we have

(x) $\quad |K_{r+1}| \overset{\text{def.}}{=\!=\!=} |\overline{\overline{K}}{}^n_{r+1}| \cup_{(g_r t_r)^{-1}|_{|\overline{\overline{K}}{}^n_{r+1}| \cap g_r(L^*_r)}} \quad$ is a combinatorial n-manifold.

Finally we set

$$t_{r+1} = t_r \cup g_r^{-1}$$
$$H_{r+1} = \sigma^n_{r+1}$$
$$N^*_{r+1} = t_{r+1}|K_{r+1}|$$
$$L^*_{r+1} = t_{r+1}(\sigma^n_{r+1}).$$

Condition (1) is satisfied by (x); condition (2) is satisfied by (xi), (x) and (ix); condition (3) is satisfied since g_r^{-1} is defined on a neighborhood of σ^n_{r+1}, (iv) and (ii); condition (4) is satisfied by (v), (vii) and (viii); condition (5) is satisfied by (vi), (ix) and (x); and condition (6) is satisfied by (iv) and (vi).

The induction is now complete and to finish the proof we note $A \subset pp_{L^*_{\ell+1}} \cup S^i_{L^*_{\ell+1}} \cup L^*_1 \subset N^*_{\ell+1}$ by condition (4) and (2) with $i = \ell+1$. □

<u>Corollary 2.7</u>: Let A be an arc in the interior of a topological n-manifold M^n. Then A lies in an open n-cell O^n except for perhaps a locally flat compact o-dimensional set C.

<u>Proof</u>: By Theorem 2.6 we may assume that M^n is a combinatorial n-manifold with a triangulation T. By M. Brown [4] $|T| = R \cup O^n$ where O^n is an open n-cell and $R \subset T^{(n-1)}$ is disjoint from O^n. We may also assume by adjusting A slightly if necessary that $A \cap R$ is a compact o-dimensional set or empty. Hence A lies in O^n except for perhaps a compact o-dimensional set $C = A \cap R$. □

<u>Corollary 2.8</u>: An arc A in the interior of a topological n-manifold lies in the union of two open n-cells.

<u>Proof</u>: This follows immediately from Corollary 2.7 and the fact that a locally flat compact o-dimensional set lies in an open n-cell. □

CHAPTER III

A CODIMENSION ONE APPROXIMATION FOR SPHERES

In 1957 Bing [1] proved that a 2-sphere embedded in S^3 can be pointwise approximated by a flat 2-sphere. A generalization of Bing's theorem to higher dimensions seems difficult. But in this chapter we prove the following: Let $f : S^{n-1} \to S^n$ be an embedding, N a neighborhood of $f(S^{n-1})$ and $n \neq 4$, then there exists a P.L. embedding $g : S^{n-1} \to S^n$ such that $g(S^{n-1}) \subset N$ and $[f] \sim [g] \in \Pi_{n-1}(N)$. We begin by first proving the following theorem which approximates a sphere separation of two compact sets in S^n by a P.L. one.

<u>Theorem 3.1</u>. Let $f : S^{n-1} \to S^n$ be a topological embedding with $n > 4$ such that $f(S^{n-1})$ separates two compact subsets A and B. Then there exists a P.L. embedding $g : S^{n-1} \to S^n$ such that $g(S^{n-1})$ separates A and B.

<u>Proof</u>: Let D denote the closure of a complementary domain of $f(S^{n-1})$ containing A. Since D is a compact absolute retract there exists compact P.L. n-manifolds with connected boundaries N_1, N_2 and N_3 which are neighborhoods of D such that $N_i \subset \mathring{N}_{i+1}$ $(i = 1,2,3)$, any polyhedron of dimension $\leq n-3$ in N_2 is contained in a P.L. n-cell in N_3, $N_1 \subset N_2$ and $N_2 \subset N_3$ are null homotopic, and for $i = 1,2,3$ $N_i \subset S^n - B$ (the proof of this statement may be found in section 2 of M. L. Curtis and D. R. McMillan's paper [5]). Since $f(S^{n-1})$ is a 1-connected absolute neighborhood retract

there exists a neighborhood V of $f(S^{n-1})$ such that $f(S^{n-1}) \subset V$ induces the trivial map on Π_1 and $V \subset N_1 - A$. Also let N be a P.L. n-manifold neighborhood with connected boundary in $D \cup V$. Hence by Lemma 3.2 of [9] we can do surgery on $\dot{N}$ in V to obtain a closed P.L. (n-1) manifold $\dot{M}$ in V which is 1-connected and we let M denote the P.L. n-manifold which $\dot{M}$ bounds in N_1 and note that $A \subset \mathring{M}$.

We now show the pair $\lambda = (\mathring{N}_2, \mathring{N}_2 - M)$ is 2-connected and thus obtain a homeomorphism $h : N_2 \to N_2$ such that $h(M) \cap N_2^{(2)} = \emptyset$ and $h|_{\dot{N}_2} = 1_{\dot{N}_2}$ by Stalling's engulfing theorem [14].

Let $k : (\Delta^1, \dot{\Delta}^1) \to \lambda$ be a P.L. representative of a class in $\Pi_1(\lambda)$. By the van Kampen theorem M is simply connected and hence it easily follows that k represents the trivial class in $\Pi_1(\lambda)$.

Now let $\ell : (\Delta^2, \dot{\Delta}^2) \to \lambda$ be a representative of a class in $\Pi_2(\lambda)$. W.l.o.g. we may assume ℓ is a P.L. homeomorphism and that $L = \ell^{-1}(\ell(\Delta^2) \cap M)$ is a compact P.L. 2-manifold in the interior of Δ^2 which is contained in some subdivision of Δ^2. Choose a regular neighborhood E of L in $\mathring{\Delta}^2$ such that $\ell(E-L)$ is contained in an "outer" collar C of M where $C \cup M \subset \mathring{N}^1$ and $\ell(\dot{\Delta}^2) \cap C = \emptyset$. Since $\ell(E) \subset M \cup C$ and $M \cup C$ is 1-connected we can extend ℓ to ℓ' : $(\Delta^2 \times 0 \cup E^{(1)} \times I) \to N_2$ such that $\ell'(b,t) = \ell(b)$ for all $b \in \dot{B}$ and $t \in I$, $\ell'(E^{(1)} \times I) \subset M \cup C$, and $\ell'(E^{(1)} \times 1) \subset C$. By the van Kampen theorem C is 1-connected; hence we can extend ℓ' to ℓ'' : $(\Delta^2 \times 0 \cup E^{(1)} \times I \cup E \times 1) \to N_2$ such that $\ell''(E \times 1) \subset \mathring{C}$. Since $\ell''(E \times 0 \cup E^{(1)} \times I \cup E \times 1) \subset M \cup C \subset N_1$ and

$N_1 \subset N_2$ is null homotopic we can extend ℓ'' to ℓ''' : $(\Delta^2 \times 0 \cup E \times I) \to N_2$. Finally since $\ell'''(e,t) = \ell(e)$ for all $e \in \dot{E}$ and $t \in I$ we can extend ℓ''' to $\ell^{(iv)} : \Delta^2 \times I \to N_2$ by defining $\ell^{(iv)}(x,t) = \ell(x)$ for all $x \in \Delta^2 - E$ and $t \in I$. Therefore $\ell^{(iv)}$ is a homotopy rel $\dot{\Delta}^2$ between ℓ and $\ell^{(iv)}|_{\Delta^2 \times 1}$ where $\ell^{(iv)}|_{(\Delta^2 \times 1)} (\Delta^2 \times 1) \subset N_2 - M$, hence ℓ represents the trivial class in $\Pi_2(\lambda)$ and λ is 2-connected.

Now extend $h : N_2 \to N_2$ to a P.L. homeomorphism $h' : N_3 \to N_3$ by the identity and let F denote the complex consisting of the 2-skeleton of N_2 together with all the simplexes in $N_3 - \mathring{N}_2$. Then we have $h'(M) \cap F = \emptyset$. If G denotes the complementary (n-3)-skeleton to F in the first derived subdivision of N_3 then there exists a P.L. n-cell O^n in N_3 containing G since dimension $G \leq n-3$ and $G \subset N_2$. We note that $\mathring{O}^n$ is a neighborhood of G and that $h'(M) \cap F = \emptyset$ hence by Theorem 8.1 of [15] there exists a P.L. homeomorphism $h'' : N_3 \to N_3$ such that h'' is fixed on F and G and $h'(M) \subset h''(O^n)$. But then $A \subset M = (h')^{-1}(h'M) \subset (h')^{-1} \circ h''(O^n) \subset S^n - B$ and $(h')^{-1} \circ h''(\dot{O}^n)$ is a P.L. (n-1)-sphere which separates A and B. □

We now prove two lemmas which are used in the main theorem.

Lemma 3.2: Let $f : S^{n-1} \to S^n - \{p,q\}$ be an embedding such that $f(S^{n-1})$ separates p and q, then $[f]$ generates $\Pi_{n-1}(S^n - \{p,q\})$.

<u>Proof</u>: Since $f(S^{n-1})$ separates p and q there exists a strong deformation retract of $S^n-\{p,q\}$ onto $f(S^{n-1})$. If $r : S^n-\{p,q\} \to f(S^{n-1})$ denotes the retraction in the last stage of the deformation and $i : f(S^{n-1}) \subset S^n-\{p,q\}$ is the inclusion map then $r_* : \Pi_*(S^n-\{p,q\}) \to \Pi_*(f(S^{n-1}))$ is an isomorphism and $i_* : \Pi_*(f(S^{n-1})) \to \Pi_*(S^n-\{p,q\})$ is the inverse of r_*. Since $f' : S^{n-1} \to S^n-\{p,q\}$, defined by $f'(x) = f(x)$ for all $x \in S^{n-1}$, is a representative of a generator of $\Pi_{n-1}(f'S^{n-1}))$ we have $i_*[f'] = [i\circ f'] = [f] \in H_{n-1}(S^n-\{p,q\})$ is a generator of $\Pi_{n-1}(S^n-\{p,q\})$. □

<u>Lemma 3.3</u>: Let $f : S^{n-1} \to S^n-\{p,q\}$ be an embedding such that $f(S^{n-1})$ separates p and q, and N a neighborhood of $f(S^{n-1})$ in S^n. Then there exists a neighborhood W of $f(S^{n-1})$ in N such that if C is any compact set in W there is a map $h : S^n-\{p,q\} \to N$ such that $h|_{f(S^{n-1})\cup C} = 1_{f(S^{n-1})\cup C}$.

<u>Proof</u> Since N is an absolute neighborhood retract there exists an $\varepsilon > 0$ such that if $f,g : X \to N$ (X arbitrary) are such that dist $(f(x),g(x)) < \varepsilon$ for all $x \in X$ then $f \sim g$ in N. Let $r : S^n-\{p,q\} \to f(S^{n-1})$ be a retraction, $V = \{x \in N \mid \text{dist}\,(x,r(x)) < \varepsilon\}$ be an open neighborhood of $f(S^{n-1})$ in N, M a connected compact P.L. n-manifold neighborhood of $f(S^{n-1})$ in V and $W = \overset{\circ}{M}$. If we let $f = 1_M : M \to M$ and $g = r : M \to M$ we have dist $(1_M(x), r(x)) < \varepsilon$ for all $x \in M$ hence $1_M \sim r$ in N, say

$H : M\times I \to N$ such that $H|_{M\times 0} = 1_M$ and $H|_{M\times 1} = r$. Now let $k : M \to I$ be a Urysohn function such that $k(C\cup f(S^{n-1})) = 0$ and $k(\dot{M}) = 1$, and $\ell : M \to \text{Graph } k \subset M\times I$ a map defined by $\ell(x) = (x,k(x))$ for all $x \in M$. Finally define $h : S^n-\{p,q\} \to N$ by $h(x) = \begin{cases} r(x) & \text{if } x \in \mathrm{Cl}(S^n-(M\cup\{p,q\})) \\ H\circ\ell(x) & \text{if } x \in M \end{cases}$. We first note h is continuous for if $x \in \dot{M}$, $H\circ\ell(x) = H(x,k(x)) = H(x,1) = r(x)$; $h(S^n-\{p,q\}) \subset N$ since $r(S^n-(M\cup\{p,q\})) = f(S^{n-1}) \subset N$ and $H\circ\ell(M) \subset H(M\times I) \subset N$; and if $x \in f(S^{n-1})\cup C$ then $h(x) = H\circ\ell(x) = H(x,k(x)) = H(x,0) = 1(x)$. Therefore h is the required map. □

We now prove the main theorem.

<u>Theorem 3.4</u>: Let $f : S^{n-1} \to S^n$ be a topological embedding of the $(n-1)$-sphere in the n-sphere with $n \neq 4$ and N a neighborhood of $f(S^{n-1})$ in S^n. Then there exists a P.L. embedding $g : S^{n-1} \to S^n$ such that $g(S^{n-1}) \subset N$ and $[f] = [g] \in \Pi_{n-1}(N)$.

<u>Proof</u>: Let C and D be the complementary domains of $f(S^{n-1})$ in S^n, $A = C\cap(S^n-N)$ and $B = D\cap(S^n-N)$. Note if A (or B) is empty then any P.L. embedding of S^{n-1} into $\mathrm{Cl}(C)$ (or $\mathrm{Cl}(D)$) suffices since $\mathrm{Cl}(C)$ and $\mathrm{Cl}(D)$ are contractible. Now apply Theorem 3.1 to A and B to obtain a P.L. embedding $g : S^{n-1} \to S^n$ such that $g(S^{n-1})$ separates A and B, and f and g are pointed maps of $S^{n-1} \to S^n$ (note that Theorem 3.1 could have been stated for pointed maps and spaces by slightly adjusting the image of g in $S^n-(A\cup B)$).

Now by Lemma 3.2 we have $[f] = \pm[g] \in \Pi_{n-1}(S^n-\{p,q\})$ and we may assume, by preceeding g with an orientation reversing homeomorphism of S^{n-1} if necessary, that $[f] = [g]$. If $H : S^{n-1}\times I \to S^n-\{p,q\}$ is the homotopy between f and g in $S^n-\{p,q\}$ then letting $C = g(S^{n-1})$ in Lemma 3.3 we obtain an $h : S^{n-1}-\{p,q\} \to N$ such that $h\circ H : (S^n-\{p,q\})\times I \to (S^n-\{p,q\})$ is a homotopy between f and g in N. Hence g is the required P.L. embedding. $\square$

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