ON FINITE COMMUTATIVE SEMIGROUPS HAVING A GROUP-LIKE PROPERTY

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ABSTRACT

ON FINITE COMMUTATIVE SEMIGROUPS HAVING A GROUP-LIKE PROPERTY

by Richard L. Gantos

In the study of finite commutative groups, it is a fact that every homomorphic image of a finite commutative group G can be imbedded in G. Moreover, if $\theta(G)$ is a homomorphic image of G, it itself has this same property. This thesis investigates when a finite commutative semigroup S will have these same imbedding properties. That is, we try to determine the structure of S so that every homomorphic image of S can be imbedded in S -- from now on referred to as property P -- and all homomorphic images of S also satisfy property P.

Chapter 1 deals with finite commutative one-idempotent (unipotent) semigroups S which belong to the following class of semigroups:

$\mathcal{C}$ is the collection of all finite commutative unipotent semigroups such that

(1) if $S \varepsilon \mathcal{C}$, then S has property P and

(2) if $S \varepsilon \mathcal{C}$, then $\theta(S) \varepsilon \mathcal{C}$, where θ is a homomorphism on S.

Let S be a finite commutative unipotent semigroup, e its idempotent, G its unique maximal subgroup, $(S \supset G)$,

and $R = (S-G) \cup e$. The following necessary conditions are determined in order that $S \varepsilon \mathcal{C}$:

(1) If $S \varepsilon \mathcal{C}$, then R is a nilpotent semigroup of S of class $q > 1$ with $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R$.

(2) If $S \varepsilon \mathcal{C}$, then $|R^{q-m}| = m+1$ and $|S^{q-m}| = |G| + m$, where $m = 1, 2, \ldots, q-2$ and q is the nilpotent class of R.

(3) If $S \varepsilon \mathcal{C}$, then the nilpotent class q of R must satisfy the inequality $1 < q \leq 4$.

From (1), (2), and (3) we are led to the complete structure of $S \varepsilon \mathcal{C}$:

$S \varepsilon \mathcal{C}$ if and only if either S is a group, $S \varepsilon \mathcal{C}_2$, $S \varepsilon \mathcal{C}_3$, or $S \varepsilon \mathcal{C}_4$.

The classes $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ consist of finite commutative semigroups S such that

(1) $S = G \cup R$, $G \cap R = e$, where G is the maximal group in S, e is the idempotent in S, and $R = (S-G) \cup e$,

(2) $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R$, and

(3) R is nilpotent of class $q = 2, 3$ or 4, respectively.

In the case of the classes $\mathcal{C}_3$ and $\mathcal{C}_4$, further conditions are imposed on the nilpotent subsemigroup R.

Chapter 2 deals with finite semilattices and finite semilattices of groups. In a finite semilattice E (a finite commutative semigroup every element of which is idempotent) we have a partial ordering $\leq$ defined by:

$e \leq f$ if and only if $ef = fe = e$.

Let $\mathcal{H}$ be the collection of finite commutative semigroups such that

(1) if $S \varepsilon \mathcal{H}$, then S has property P and

(2) if $S \varepsilon \mathcal{H}$, then $\theta(S) \varepsilon \mathcal{H}$, where θ is a homomorphism on S.

Some of the theorems in Chapter 2 pertaining to this class $\mathcal{H}$ are the following:

If $E \varepsilon \mathcal{H}$ is a finite semilattice which contains elements a, b, c such that $a = bc$, $b > a$, $c > a$ then

(1) $e < b$ or $e < c$ for every $e \varepsilon E$ where the dimension of e is less than or equal to the dimension of a,

(2) the number of incomparable elements $e \varepsilon E$, with e incomparable to a and having dimension less than or equal to a, is at most one.

A finite semilattice of groups is a finite commutative semigroup S such that

(1) $S = \bigcup_{i=1}^{n} G_i$, where G_i is the maximal subgroup of S containing the idempotent e_i,

(2) $G_i \cap G_j = \emptyset$ for $i \neq j$, and

(3) $G_i G_j = G_k$ where $e_i e_j = e_k$.

Some theorems on finite semilattices of groups:

If $S \varepsilon \mathcal{H}$ is a finite semilattice of groups, then $gG_i = \{g\}$ for all $g \varepsilon G_1$ (where $e_1 \leq e$ for every idempotent e in S) and $i = 2, 3, \ldots, n$.

If $S \varepsilon \mathcal{H}$ is a finite chain semilattice of groups, then $g_i g_j = g_i$, for all $g_i \varepsilon G_i$, $g_j \varepsilon G_j$, where $e_i < e_j$, $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n$.

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Richard L. Gantos

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Dedicated to

Ted and Haseebie Gantos

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Notation Used in Chapters 1 and 2

Square brackets are used for reference to the bibliography.

Let A and B be sets.

$A \subset B$ (or $B \supset A$) means A is properly contained in B.

$A \subseteq B$ (or $B \supseteq A$) means $A \subset B$ or $A = B$.

$A-B$ means the set of elements of A which are not in B.

$|A|$ means the cardinal number of the set A.

$\emptyset$ denotes the empty set.

If A and B are subsets of a semigroup S, then AB means $\{ab | a \in A, b \in B\}$.

If ρ is an equivalence relation on a set X, and if $(a,b) \in \rho$, then we write $a \rho b$ and say that a and b are ρ-equivalent, and that they belong to the same ρ-class.

If ρ is a congruence relation on a semigroup S, then S/ρ denotes the factor semigroup of S modulo ρ. S/I denotes the Rees factor semigroup of S modulo an ideal I.

$\cong$ means "isomorphic".

If θ is a homomorphism on S and $R \subset S$, then $\theta | R$ shall mean θ restricted to R.

Introduction

The aim of this dissertation is to study finite commutative semigroups which have a property inherited by finite commutative groups. It is evident that every finite commutative group G has the property, we shall call it property P, that all its homomorphic images can be imbedded in G. That is, for each homomorphic image H of G there is a subgroup of G isomorphic with H. Moreover H has property P. We shall investigate the structure of finite commutative semigroups S for which (1) S has property P and (2) $\theta(S)$ has property P, where θ is a homomorphism of S.

As in group theory, every homomorphic image of a semigroup S can be realized (differing only by an isomorphism) as a factor semigroup. We therefore need to introduce the concept of congruences. A relation ρ on a semigroup S is said to be <u>compatible</u> if $a \rho b$ (a, b in S) implies $ac \rho bc$ and $ca \rho cb$ for every c in S. By a <u>congruence</u> on S we mean an equivalence relation on S which is compatible.

Denote by S/ρ the set of all equivalence classes of S mod ρ. Let K_1 and K_2 be members of S/ρ. Let $a_1, a_2 \varepsilon K_1$ and let $b_1, b_2 \varepsilon K_2$. From $a_1 \rho a_2$ we have $a_1b_1 \rho a_2b_1$. From $b_1 \rho b_2$ we have $a_2b_1 \rho a_2b_2$. The transitivity of ρ gives us that $a_1b_1 \rho a_2b_2$.

Therefore the set product K_1K_2 of K_1 and K_2 is contained in a unique equivalence class K_3 mod ρ. Because of this property one may define in a natural way an operation in S/ρ. Since this operation does not coincide with the operation of multiplication of subsets, we shall use the sign o for denoting the result of this operation.

Suppose that K_1, K_2, K_3 are three equivalence classes mod ρ such that $K_1K_2 \subset K_3$. In S/ρ we put

$$(1)\ K_1 \circ K_2 = K_3.$$

Since, for any $K_1, K_2, K_3 \in S/\rho$

$$(K_1 \circ K_2) \circ K_3 \supset (K_1K_2)K_3 = K_1K_2K_3$$

$$K_1 \circ (K_2 \circ K_3) \supset K_1(K_2K_3) = K_1K_2K_3$$

the operation o is associative in S/ρ.

Definition. For a congruence ρ on the semigroup S, the set S/ρ of all equivalence classes mod ρ, considered relative to (1), is a semigroup, called the factor semigroup of the semigroup S modulo ρ.

Assign to each element $a \in S$ the equivalence class K of S/ρ which contains it. We obtain a mapping of S onto the factor semigroup S/ρ. This mapping is a homomorphism; it is called the natural homomorphism of S onto S/ρ. Therefore every factor semigroup of a semigroup S is a homomorphic image of S. The following theorem (Clifford and Preston [1]) shows conversely that every homomorphic image of S is isomorphic with a factor semigroup of S. Therefore, if we do not distinguish between isomorphic semigroups, the study of homomorphic images can be replaced by the study of congruences on S.

(Main Homomorphism Theorem). Let θ be a homomorphism of a semigroup S onto T. Let $a \rho b$ (a, b in S) if and only if $\theta(a) = \theta(b)$. Then ρ is a congruence on S and if we denote the natural homomorphism of S onto S/ρ by ρ^*, there is an isomorphism β of S/ρ onto T such that $\beta\rho^* = \theta$.

An important example which will be of constant application throughout this dissertation is the following.

Let I be an ideal of a semigroup S. Define $a \rho b$ (a, b in S) if and only if either $a = b$ or else both a and b belong to I. ρ is called the Rees congruence modulo I. The equivalence classes of S modulo ρ are the ideal I itself and all the one element sets $\{x\}$ with x in $S-I$. The ideal I, as an equivalence class of S modulo ρ, is of course the zero element of the semigroup S/ρ. One may represent S/ρ as a semigroup obtained from S by identifying with one another all the elements of the ideal I. We shall write S/I instead of S/ρ, and we call S/I the Rees factor semigroup of S modulo I.

Now suppose S is a finite commutative semigroup.

It has an idempotent element e. (i.e. $e^2 = e$). In fact some power of every element of a finite semigroup is idempotent. This was first shown by Frobenius (Über endliche Gruppen, Sitzungsber. Preuss. Akad. Berlin, 1895). Let E be the set of idempotents of S. For each e in E, let S_e be the set of all x in S such that $x^n = e$ for some positive integer n. Then $S_e \cap S_f = \emptyset$ if $e \neq f$ in E, and $S = \bigcup_{e \in E} S_e$. Each S_e is a subsemigroup of S containing e but no other idempotent and $S_e S_f \subseteq S_{ef}$ for all e, f in E. S_e is called a maximal one-idempotent (or unipotent) subsemigroup of S. All of the above facts can be found in either Clifford and Preston [1] or Lyapin [2].

Because of this decomposition of a finite commutative semigroup we first studied finite unipotent commutative semigroups which have property P and whose homomorphic images also have property P. We are able to completely describe the products in such semigroups for this case. That is, we are able to determine the fine structure of finite unipotent commutative semigroups having property P and whose homomorphic images have property P. This result is stated in Theorem 1.16.

We next considered the case when the maximal unipotent semigroups S_e have only the idempotent e as an element. In this case, every element of S is idempotent. Such semigroups are called bands. Finally we close with the possibility that the maximal unipotent semigroups are groups.

Chapter 1

Finite Commutative Unipotent Semigroups

In this chapter we shall characterize those finite commutative unipotent semigroups S which have the property that every homomorphic image of S can be imbedded in S and moreover all its homomorphic images have the same property.

1.1 Basic Properties of Unipotent Semigroups

We develop in this section some basic concepts of finite unipotent commutative semigroups. Some of these concepts can be found in either Clifford and Preston [1] or Lyapin [2].

<u>Definition</u>. A two-sided ideal M of a semigroup S is said to be <u>universally minimal</u> in S if it is contained in every ideal of S.

<u>Lemma 1.1</u>. If a semigroup S possesses a two-sided ideal G that is a group, then G is the universally minimal ideal of S.

<u>**Proof**</u>. Let L be an arbitrary left ideal of S. Since G is a two-sided ideal of S we have $GL \subseteq GS \subseteq G$

and since L is a left ideal it follows $GL \subseteq SL \subseteq L$. Moreover $G(GL) = G^2L \subseteq GL$ which implies GL is a left ideal of G. However G is a group and a group has no ideals other than itself so we must have $GL = G$. Therefore $G = GL \subseteq L$ and thus G is contained in every left ideal of S. A similar argument shows G is contained in every right ideal of S.

Lemma 1.2. A semigroup S cannot have more than one two-sided ideal that is a group.

Proof. Suppose G_1 and G_2 are two-sided ideals of S which are also groups. By Lemma 1, G_1 and G_2 are both universally minimal ideals of S. Hence we would have $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$ so that $G_1 = G_2$.

Theorem 1.3. Let S be a finite unipotent commutative semigroup, let e denote its unique idempotent element and let

$$G = \{a \in S \mid ea = a \text{ and } xa = e \text{ for some } x \in S\}.$$

Then

1) $G = eS = Se = eSe$.

2) G is the unique maximal non-empty commutative subgroup of S.

3) G is the two-sided ideal which is universally minimal in S.

4) $S^n = G$ for some positive integer n.

Proof. Since $e \in G$ then G is nonempty. Let $x \in S$. The cyclic subsemigroup $\langle x \rangle$ (i.e. the set of all positive powers of x) is finite so there exists positive integers m and r such that $x^{m+r} = x^r$ and $\langle x \rangle = \{x, x^2, \ldots, x^{m+r-1}\}$. Moreover the set $K_x = \{x^r, x^{r+1}, \ldots, x^{m+r-1}\}$ is a cyclic subgroup of S. Its identity element x^k, $r \leq k \leq r+m-1$, is an idempotent element in S and hence $x^k = e$, for S is unipotent. Therefore for every $x \in S$ there exists a positive integer k such that $x^k = e$.

1) We show $G = eS$. First of all $G = eG \subseteq eS$. Let $x \in eS$. Therefore $x = ey$ where $y \in S$, $ex = x$ and there exists a positive integer k such that $x^k = e$. If $k > 1$, then $e = x^k = x^{k-1}x$ so that $x \in G$. If $k = 1$, then $x = e$ and again $x \in G$. Therefore $eS \subseteq G$ and equality follows.

Moreover $G = eS = Se = eSe$ since S is commutative.

2) We show first that G is a group. Let g_1, $g_2 \in G$. Then there exist elements u_1 and u_2 in S with $e = g_1u_1 = g_2u_2$, $g_1e = g_1$ and $g_2e = g_2$. Therefore

$$(g_1g_2)(u_2u_1) = g_1(g_2u_2)u_1 = g_1eu_1 = (g_1e)u_1 = g_1u_1 = e,$$
$$(g_1g_2)e = g_1(g_2e) = g_1g_2$$

and we have $g_1g_2 \in G$. The element $e \in G$ is the identity element of G. Let $g \in G$. There exists $u \in S$ such that $e = gu$ and $g = ge$. Now

$$e = e^2 = (gu)e = g(ue) = (ge)(ue) = g(eue)$$

and $eue \in G$ since $G = eSe$. Therefore each element $g \in G$ has an inverse element $g^{-1} = eue$ in G and it follows G is a subgroup of S.

Moreover G is a maximal subgroup of S. Suppose H is a subgroup of S. Since S is unipotent it follows that the identity element of H is the identity element e of G. Therefore

$$H = eHe \subseteq eSe = G.$$

3) From part (1) we have $SG = S(eSe) = (Se)(Se) \subseteq Se = G$ and thus $GS \subseteq G$. Therefore G is a two-sided ideal of S. By Lemma 1.1, G is universally minimal in S. The uniqueness of G follows from Lemma 1.2.

4) S^m is a two-sided ideal of S for every positive integer m. Therefore, by part (3), we have $S^m \supseteq G$ for every positive integer m and $S \supseteq S^2 \supseteq \ldots \supseteq S^m \supseteq \ldots \supseteq G$. Since S is finite it follows $S^n = S^{n+1} = \ldots = S^{n+k} = \ldots$ for some integer n and for all k. Now S^n is a subsemigroup of S and $e \in S^n$ since $e = e^n \in S^n$. Let $u \in S^n$ with $u \neq e$. Now $u \in S$ so there exists a positive $k > 1$ such that $u^k = e$. Therefore $u^k = u^{k-1}u = e$ and $u^{k-1} \in S^n$ so that u^{k-1} is the inverse element of u in S^n. We have S^n a subgroup of S and from part (2) it follows $S^n \subseteq G$. We have shown $G \subseteq S^n$, so $G = S^n$.

<u>Lemma 1.4</u> . If S is a finite unipotent abelian

semigroup, then S/ρ is a finite unipotent abelian semigroup for every congruence ρ on S.

<u>Proof</u>. Since S is finite, it partitions into a finite number of equivalence classes mod ρ so that S/ρ is finite. Since S is abelian it follows $K_1K_2 = K_2K_1$ for any two arbitrary ρ-classes. Thus, by definition of the operation $\circ$ in S/ρ, it follows $K_1 \circ K_2 = K_2 \circ K_1$ and S/ρ is abelian.

Let $K_0 = \{x \mid x \rho e\}$, where e is the idempotent of S. Let $u \in K_0$ and $v \in K_0$. Then $u \rho e$ and $v \rho e$ so that $(uv) \rho e$. Therefore $uv \in K_0$ and we have $K_0K_0 \subseteq K_0$ which implies $K_0 \circ K_0 = K_0$; i.e. K_0 is an idempotent element in S/ρ. Suppose K is an equivalence class mod ρ and $K \circ K = K$. This implies $KK \subseteq K$ and for $x \in K$ we have $x^2 \rho x$. Using the compatibility of the congruence ρ sucessively on $x^2 \rho x$, we have $x^m \rho x$ for all positive integers m. But there exists a positive integer k such that $x^k = e$. Hence $e \rho x$ so that $x \in K_0$. Thus $K \subseteq K_0$ and since both are equivalence classes mod ρ it follows that $K = K_0$. Therefore S/ρ has exactly one idempotent.

1.2 Semigroups of Class $\mathscr{C}$

Throughout this section S will denote a finite commutative unipotent semigroup, e its unique idempotent element, G its unique maximal subgroup which is the ideal universally minimal in S (the existence of which was determined in Theorem 1.3) and finally R will denote the set of elements of S not in G along with e; i.e. $R = (S-G) \cup \{e\}$. Then $S = G \cup R$ where $G \cap R = e$.

We will first determine several necessary conditions for S to have the property that all its homomorphic images can be imbedded in S. By imposing on S the added condition that all homomorphic images of S also have this property we will be able to completely determine the structure of S.

It is known that every finite commutative group satisfies these properties. Therefore in the following discussions we will take $S = G \cup R$, $R \cap G = e$ and $S \supset G$ (or equivalently, $R \supset \{e\}$).

<u>Definition</u>. A semigroup T is said to have <u>property P</u> if and only if T/ρ can be imbedded in T for every congruence ρ on T.

<u>Definition</u>. A semigroup T with zero element 0 is said to be <u>nilpotent</u> if there exists a positive integer $n > 1$ such that $T^n = \{0\}$. It is said to have <u>class n</u>, $(n > 1)$, if n is the least positive integer such that $T^n = \{0\}$. If the class of T is $n = 2$ we will call T a <u>null</u> semigroup.

Lemma 1.5 . Every finite unipotent commutative semigroup T with zero element 0 is nilpotent.

Proof. To show T is nilpotent we need to show $T^m = \{0\}$ for some positive integer m. Since T is a finite unipotent commutative semigroup, then it follows from Theorem 1.3 that it has a unique maximal subgroup K which is an ideal universally minimal in T and moreover $T^n = K$ for some positive integer n. But $\{0\}$ is an ideal of T contained in every ideal of T and since K is universally minimal it follows $K = \{0\}$. Therefore $T^n = \{0\}$.

Theorem 1.6 . If S has property P, then

1) R is a finite unipotent commutative subsemigroup of S with e as its zero element.

2) $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R$.

3) R is nilpotent of class $q > 1$.

Proof. 1) Since G is a proper ideal of S we can form S/G the Rees factor semigroup of S modulo G. By Lemma 1.4 we have S/G is a finite unipotent commutative semigroup. Its elements, the equivalence classes modulo G, are every one-element set $\{a\}$ where $a \varepsilon S-G$ and G itself. The element G in S/G is a zero element for S/G and $|S/G| = |S|-|G|+1 = |R|$.

Now, by hypothesis, S/G can be imbedded in S so there exists subsemigroup T of S with zero element

z and $|T| = |S| - |G| + 1$. The element z is an idempotent in S so it follows $z = e$. Also $G \cap T = \{e\}$ for e is the identity element of G and the zero element of T. But $T \subseteq S = G \cup R$ and $G \cap T = \{e\}$ implies $T \subseteq R$. However $|T| = |R|$ so that $T = R$.

Therefore we have shown part (1), namely, R is a finite unipotent commutative subsemigroup of S with e its zero element.

We have $er = e$ for every $r \in R$ since e is the zero element of R. Let $g \in G$, $g \neq e$ and let $r \in R$. There exists a positive integer m such that $g^m = e$ and therefore g is the unique inverse element of g^{m-1}. On the other hand, $gr \in G$ for G is an ideal of S and $g^{m-1}(gr) = g^m r = er = e$. Hence gr is also an inverse of g^{m-1} in G so we must have $gr = g$.

By Lemma 1.5, since R is a finite unipotent commutative semigroup with zero element, we have that R is nilpotent of class $q > 1$.

<u>Lemma 1.7</u> . If S has property P, then

(1) $S^m = G \cup R^m$ and $G \cap R^m = e$ for all positive integers m.

(2) $S^m \supset G$ for $m = 1, 2, \ldots, (q-1)$ and $S^m = G$ for $m \geq q$, where q is the nilpotent class of R.

(3) S/S^m is a nilpotent semigroup of class m, $m = 2, 3, \ldots, q$.

(4) $|(S/S^m)^n| = |S^n|-|S^m|+1$ where n and m are positive integers such that $1 \leq n \leq m \leq q$.

<u>Proof</u>. To verify $S^m = G \cup R^m$ we apply induction on m. The decomposition of our finite unipotent commutative semigroup S, namely $S = G \cup R$ and $G \cap R = e$, verifies $S^m = G \cup R^m$ for $m = 1$.

By Theorem 1.6, $GR = G$ and consequently $GR^m = G$ for all positive integers m. Assume $S^m = G \cup R^m$. Then

$$S^{m+1} = S^m S = (G \cup R^m)(G \cup R) = G \cup R^m G \cup GR \cup R^{m+1}$$
$$= G \cup R^{m+1}.$$

Also $G \cap R^m = e$ since e is the identity element of G and zero element of R.

Also by Theorem 1.6, R is nilpotent. Let q be its class. We have $S^m = G \cup R^m$ for any positive integer m so that $G \subseteq S^m$ for any positive integer m. If $G = S^n$ for some natural number n, then it follows, from $G \cap R^n = e$, that $R^n = e$. Hence, $n \geq q$.

We now proceed to show that S/S^m is nilpotent of class $m > 1$. S/S^m is a finite unipotent commutative semigroup with zero element; its zero element being the ideal S^m itself. Lemma 1.5 gives us that S/S^m is nilpotent. Let $S^* = S/S^m$. Denote its zero element by 0^* and the remaining elements by $x^* = \{x\}$, where $x \in S-S^m = R-R^m$. Indeed $(S^*)^m = 0^*$ for $(S-S^m)^m \subseteq S^m$. We next assert that $(S^*)^{m-1} \neq 0^*$. Since $R \supset R^2 \supset \ldots \supset R^{m-1} \supset R^m \supset \ldots \supset R^q = e$ we can select $x \in (R^{m-1}-R^m)$.

Then $x = x_1 x_2 \ldots x_{m-1}$ where $x_i \in R-R^2$ $(i = 1, 2, \ldots, m-1)$. However $x^*, x_1^*, \ldots, x_{m-1}^*$ are non-zero elements of S^* and $x^* = x_1^* \circ x_2^* \circ \ldots \circ x_{m-1}^*$. Consequently, $x^* \in (S^*)^{m-1}$ and $x^* \neq 0^*$. But $(S^*)^m = 0^*$ and $(S^*)^{m-1} \neq 0^*$ imply S^* is nilpotent of class m.

It remains to show that $|(S/S^m)^n| = |S^n|-|S^m|+1$. By part (3), this equation obviously holds for $n = m$. Using an argument similar to the one above, one can easily note that there is a one-to-one correspondence between the non-zero elements x^* in $(S/S^m)^n$ and the elements x in $S^n - S^m$. Hence $|(S/S^m)^n| = |S^n|-|S^m|+1$.

It has been shown, thus far, that if S is to have property P, it is necessary that R be a nilpotent semigroup of class $q > 1$. We now would like to determine the order of the ideals R^m where $m = 2, 3, \ldots, q-1$. To do this we need to introduce the following concept.

Definition. An element u in S is said to have a factorization in S^m, $(m = 2, 3, \ldots, q)$ if and only if $u \in S^m$. Two factorizations of u in S^m, $u = x_1 x_2 \ldots x_m = y_1 y_2 \ldots y_m$, are said to be distinct if and only if the sets $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ are unequal. The number of distinct factorizations of u in S^m will be denoted by $\beta(u;m)$.

<u>Lemma 1.8</u> . If S has property P, then (1) $|R^{q-1}| = 2$ and (2) $|S^{q-1}| = |G|+1$ where q is the nilpotent class of R, $q > 2$.

<u>Proof</u>. By Theorem 1.6, R is nilpotent of class q so that $R^{q-1} \supset R^q = \{e\}$ and consequently $|R^{q-1}| \geq 2$. Suppose $|R^{q-1}| > 2$. We can find at least two distinct elements in R^{q-1} both different from the zero element e in R. Choose $u \in (R^{q-1}-\{e\})$ such that $\beta(u;q-1)$ is maximal. Let v be any element in $(R^{q-1}-\{e\})$ with $u \neq v$. Clearly $\beta(u;q-1) \geq \beta(v;q-1) \geq 1$.

Let us consider that partition of the semigroup S determined by the disjoint subsets $K_0 = S^q = G$, $K = \{u,v\}$ and all one element subsets $K_x = \{x\}$ where $x \in S-(S^q \cup \{u,v\})$. Let ρ be the equivalence relation induced by this partition. Moreover we can show that ρ is a congruence on S. Suppose $a\,\rho\,b$, where a and b are elements in S such that $a \neq b$. Then we must have both a and b belong to $K_0 = S^q$ or both a and b belong to $K = \{u,v\}$. If the former occurs then, since $S^q = G$ is an ideal of S, it follows $xa\,\rho\,xb$ for all $x \in S$. If the latter occurs, we would have $a = u$ and $b = v$ or $a = v$ and $b = u$. However $\{u,v\} \subseteq R^{q-1} \subseteq S^{q-1}$ and hence xu and xv belong to S^q for every $x \in S$. Therefore $xu\,\rho\,xv$ which implies $xa\,\rho\,xb$ for every $x \in S$. We have verified that the equivalence relation ρ induced by the disjoint subsets K_0, K

and the one-element sets $K_x = \{x\}$ is indeed a congruence.

We form the factor semigroup S/ρ. Let us denote its elements (equivalence classes mod ρ) by K_0, K and all $K_x = \{x\}$ where $x \in S-(S^q \cup \{u,v\})$. S/ρ is a finite unipotent commutative semigroup and, since S^q is an ideal of S, the idempotent K_0 of S/ρ is a zero element of S/ρ.

We first show that $K \in (S/\rho)^{q-1}$. Now $u \in R^{q-1}-\{e\}$ which implies $u = u_1u_2\ldots u_{q-1}$ where each $u_i \in R-R^2$. Indeed $u_i \in S-(S^q \cup \{u,v\})$ so that each of the one-element subsets $K_{u_i} = \{u_i\}$ $(i = 1, 2, \ldots, q-1)$ of S are non-zero elements of S/ρ different from the element K. However

$$K_{u_1}K_{u_2}\ldots K_{u_{q-1}} = \{u_1u_2\ldots u_{q-1}\} = \{u\} \subset \{u,v\}$$

which implies $K_{u_1} \circ K_{u_2} \circ \ldots \circ K_{u_{q-1}} = K$. Hence $K \in (S/\rho)^{q-1}$.

Next we show that $(S/\rho)^q = K_0$. This result together with $K \in (S/\rho)^{q-1}$ tells us that S/ρ is nilpotent of class q. Let $A \in (S/\rho)^q$. Then $A = A_1 \circ A_2 \circ \ldots \circ A_q$ where $A_i \in S/\rho$ $(i = 1, 2, \ldots, q)$. If any $A_i = K_0$ we would have $A = K_0$ so we may assume $A_i \neq K_0$ for all $i = 1, 2, \ldots, q$. This implies that each A_i, as a subset of S, is either a one-element set $\{x_i\}$, where $x_i \in S-(S^q \cup \{u,v\})$ or the set $\{u,v\}$. In any case, the set product $A_1A_2\ldots A_q$ must be a subset of S^q. Consequently, by the definition of the operation $\circ$

in S/ρ, it follows $A = A_1 \circ A_2 \circ \ldots \circ A_q = K_0$ and we have $(S/\rho)^q = K_0$.

Since $K \varepsilon (S/\rho)^{q-1}$, then K has a factorization in $(S/\rho)^{q-1}$. Let $\beta(K;q-1)$ be the number of distinct factorizations of K in $(S/\rho)^{q-1}$. We now claim that $\beta(u;q-1) + \beta(v;q-1) \leq \beta(K;q-1)$. To show this we first shall show that distinct factorizations of both u and v in R^{q-1} give rise to distinct factorizations of K in $(S/\rho)^{q-1}$. Suppose $u = x_1x_2\ldots x_{q-1} = y_1y_2\ldots y_{q-1}$ are two distinct factorizations of u in R^{q-1}; i.e. the set $\{x_1, x_2, \ldots, x_{q-1}\}$ and the set $\{y_1, y_2, \ldots, y_{q-1}\}$ are unequal. Now $x_i \varepsilon (R-R^2)$ and $y_i \varepsilon (R-R^2)$ for $i = 1, 2, \ldots, q-1$, since $u \varepsilon R^{q-1}-\{e\}$. Consequently each of the one-element subsets of S, $K_{x_1} = \{x_1\}$, $K_{x_2} = \{x_2\}$, $\ldots$, $K_{x_{q-1}} = \{x_{q-1}\}$; $K_{y_1} = \{y_1\}$, $K_{y_2} = \{y_2\}$, $\ldots$, $K_{y_{q-1}} = \{y_{q-1}\}$ is a non-zero element of S/ρ. Moreover the sets $\{K_{x_1}, \ldots, K_{x_{q-1}}\}$ and $\{K_{y_1}, \ldots, K_{y_{q-1}}\}$ are unequal. But, from

$$K_{x_1}K_{x_2}\ldots K_{x_{q-1}} = \{x_1x_2\ldots x_{q-1}\} = \{u\} \subset \{u,v\}$$

and

$$K_{y_1}K_{y_2}\ldots K_{y_{q-1}} = \{y_1y_2\ldots y_{q-1}\} = \{u\} \subset \{u,v\}$$

it follows $K = K_{x_1} \circ K_{x_2} \circ \ldots \circ K_{x_{q-1}} = K_{y_1} \circ K_{y_2} \circ \ldots \circ K_{y_{q-1}}$. This implies K has two distinct factorizations in $(S/\rho)^{q-1}$. Using the same argument, one can obtain distinct factorizations for K in $(S/\rho)^{q-1}$ from distinct factorizations of v in R^{q-1}. On the other hand, since

$u \neq v$, a factorization of u in R^{q-1} and a factorization of v in R^{q-1} will give rise to two distinct factorizations of K in $(S/\rho)^{q-1}$. This together with the above clearly imply that $\beta(u;q-1)+\beta(v;q-1) \leq \beta(K;q-1)$.

Up to this point we have shown that S/ρ is a nilpotent semigroup of class q and moreover there exists a non-zero element K of S/ρ which has $\beta(K;q-1) \geq \beta(u;q-1)+\beta(v;q-1)$ factorizations in $(S/\rho)^{q-1}$.

Since S has property P, then S/ρ can be imbedded in S. Hence there exists a nilpotent subsemigroup T of S of class q and in T there is an element t which has $\beta(K;q-1)$ factorizations in T^{q-1}. But $T \cap G = e$ for e is the zero element of T and the identity element of G. Hence $T \subseteq R$ and $t \neq e$. Therefore $t \in R^{q-1}-\{e\}$. However t has at least $\beta(K;q-1)$ factorizations in R^{q-1} and since $\beta(K;q-1) \geq \beta(u;q-1)+\beta(v;q-1) > \beta(u;q-1)$ we have a contradiction with the choice of u. Hence we must have $|R^{q-1}| = 2$. This proves part (1) of our lemma.

By Lemma 1.7, $S^{q-1} = G \cup R^{q-1}$ and $G \cap R^{q-1} = \{e\}$. Therefore $|S^{q-1}| = |G|+|R^{q-1}|-1 = |G|+1$. This completes the proof of the lemma.

The condition that S has property P has given us the conclusion that the subsemigroup R is nilpotent of class $q > 1$ and moreover the ideal R^{q-1} contains exactly one non-zero element. By further imposing on S the condition that all homomorphic images of S also have property P we shall show that $|R^{q-2}| = 3$, $|R^{q-3}| = 4$, ..., $|R^2| = q-1$. This result will be used repeatedly in determining those semigroups S which have property P and whose homomorphic images have property P.

<u>Definition</u>. Let $\mathcal{C}$ be the collection of finite commutative unipotent semigroups such that

(1) if $S \varepsilon \mathcal{C}$, then S has property P and

(2) if $S \varepsilon \mathcal{C}$, then $\theta(S) \varepsilon \mathcal{C}$ for all homomorphisms θ on S.

<u>Lemma 1.9</u> . If $S \varepsilon \mathcal{C}$, then $R \varepsilon \mathcal{C}$ and $G \varepsilon \mathcal{C}$ where G is the unique maximal subgroup of S and R is the nilpotent subsemigroup of S such that $S = G \cup R$ and $G \cap R = \{e\}$.

<u>Proof</u>. Let $S \varepsilon \mathcal{C}$ and let G be its maximal subgroup with identity element e. Then, by Theorem 1.6, $S = G \cup R$, $G \cap R = e$, R is the nilpotent subsemigroup of class $q > 1$ and $gr = g$ for all $g \varepsilon G$ and $r \varepsilon R$.

Since G is a finite commutative group it follows G and $\theta(G)$, θ a homomorphism of G, both have property P. Hence $G \varepsilon \mathcal{C}$.

Let $\theta: S \rightarrow R$ be the transformation of S into R defined by

$$\theta(x) = \begin{cases} x & \text{if } x \in R \\ e & \text{if } x \in G \end{cases}$$

θ is indeed single-valued and onto R. Since $gr = g$ for every $g \in G$ and $r \in R$ it follows θ is a homomorphism of S onto R. Since $S \in \mathcal{C}$ we have $R \in \mathcal{C}$.

Theorem 1.10 . If $S \in \mathcal{C}$, then

1) $|R^{q-m}| = m+1$ for $m = 1, 2, \ldots, q-2$ and

2) $|S^{q-m}| = |G| + m$ for $m = 1, 2, \ldots, q-2$, where q is the nilpotent class of R.

Proof. We have $|S^{q-m}| = |G \cup R^{q-m}| = |G| + |R^{q-m}| - 1$. Hence to show (2) we need only show $|R^{q-m}| = m+1$. By Lemma 1.9, $S \in \mathcal{C}$ implies $R \in \mathcal{C}$. Therefore it suffices to prove this result for finite nilpotent unipotent commutative semigroups $R \in \mathcal{C}$. Suppose the statement of the theorem is false. Let $R \in \mathcal{C}$ be a minimal counterexample. Let the nilpotent class of R be $q > 2$. By Lemma 1.8, $|R^{q-1}| = 2$ since R has property P. Therefore there exists a positive integer m, where $1 < m \leq q-2$, such that $|R^{q-n}| = n+1$ for $n = 1, 2, \ldots, m-1$ but $|R^{q-m}| \neq m+1$. Since $R^{q-m} \supset R^{q-(m-1)}$, then $|R^{q-m}| > |R^{q-(m-1)}| = m$. Consequently, $|R^{q-n}| = n+1$ for $n = 1, 2, \ldots, m-1$ but $|R^{q-m}| > m+1$.

Consider the Rees factor semigroup R/R^{q-1}. By Lemma 1.7 we know that it is nilpotent of class $q-1$ and moreover by (4) of this same lemma we have

$$|R/R^{q-1}| = |R|-2+1 = |R|-1,$$

$$|(R/R^{q-1})^{q-n}| = |R^{q-n}|-2+1 = n \quad \text{for} \quad n = 2, \ldots, m-1$$

and $$|(R/R^{q-1})^{q-m}| = |R^{q-m}|-2+1 > m.$$

However $R/R^{q-1} \varepsilon \mathcal{C}$ since it is a homomorphic image of R. Therefore R/R^{q-1} is also a counterexample to our desired result which has order less than the order of R. This contradicts the minimality of R so our assumption is not valid.

<u>Corollary 1.11</u>. If $S \varepsilon \mathcal{C}$ and q is the nilpotent class of the subsemigroup R of S, then

$$|S/S^{q-1}| = |R|-1 \quad \text{and} \quad |(S/S^{q-1})^{q-m}| = m$$

where $m = 1, 2, \ldots, q-2$.

Proof. By Theorem 1.10 we have $|S^{q-1}| = |G|+1$ and $|S^{q-m}| = |G|+m$ for $m = 2, 3, \ldots, q-2$. Then

$$|S/S^{q-1}| = |S|-|S^{q-1}|+1 = (|G|+|R|-1)-(|G|+1)+1 = |R|-1.$$

Also by Lemma 1.7 we have

$$|(S/S^{q-1})^{q-m}| = |S^{q-m}|-|S^{q-1}|+1 = m,$$

where $m = 1, 2, \ldots, q-2$.

<u>Theorem 1.12</u>. If $S \varepsilon \mathcal{C}$ and if q is the nilpotent class of the subsemigroup R, then $1 < q \leq 4$.

Proof. Suppose $q \geq 5$. R being nilpotent of class $q \geq 5$ gives the following chain of ideals

$$R \supset R^{q-3} \supset R^{q-2} \supset R^{q-1} \supset R^q = \{e\}.$$

By Theorem 1.10 it follows $|R^{q-1}| = 2$, $|R^{q-2}| = 3$ and

$|R^{q-3}| = 4$. Therefore we can choose non-zero distinct elements v, u, and w from $R-\{e\}$ such that

$$R^{q-1}-R^q = \{v\}, \quad R^{q-2}-R^{q-1} = \{u\} \text{ and}$$
$$R^{q-3}-R^{q-2} = \{w\}.$$

That is, $R^q = \{e\}$, $R^{q-1} = \{e,v\}$, $R^{q-2} = \{e,u,v\}$, and $R^{q-3} = \{e,w,u,v\}$.

$v \in R^{q-1}-R^q$ implies $v = r_1r_2\ldots r_{q-1}$ where $r_i \in R-R^2$ for $i = 1, 2, \ldots, q-1$. Consider the $(q-1)$ elements $a_m = \prod_{i=1, i\neq m}^{q-1} r_i$ where $m = 1, 2, \ldots, q-1$. First of all, each a_m is an element of R^{q-2}. Since $v = r_m a_m$ and since $v \in R^{q-1}-R^q$ it follows that a_m ($m = 1, 2, \ldots, q-1$) is not an element of R^{q-1}; for otherwise $v = r_m a_m \in R\cdot R^{q-1} = R^q$ which cannot happen. Therefore $a_m \in R^{q-2}-R^{q-1}$ for all $m = 1, 2, \ldots, q-1$. Hence $a_m = u$ for $m = 1, 2, \ldots, q-1$ so that $v = r_m u$, $m = 1, 2, \ldots, q-1$, and $u = \prod_{i=1, i\neq m}^{q-1} r_i$ for each $m = 1,2,\ldots, q-1$. However

$$u = \prod_{i=1, i\neq m}^{q-1} r_i = r_n \prod_{i=1, i\notin\{m,n\}}^{q-1} r_i = r_n\cdot a_{n,m} \quad (n \neq m)$$

Clearly $a_{n,m} \in R^{q-3}$ for all n and m, with $n \neq m$, $n = 1, 2, \ldots, q-1$ and $m = 1, 2, \ldots, q-1$. Since $u = r_n a_{n,m}$ and since $u \in R^{q-2}-R^{q-1}$ it follows $a_{n,m} \in R^{q-3}-R^{q-2}$ for the above possible values of n and m. Therefore $a_{n,m} = w$ for $n \neq m$, $n = 1, 2, \ldots, q-1$ and $m = 1, 2, \ldots, q-1$. We have thus shown that

$$v = \prod_{i=1}^{q-1} r_i = r_m u \; , \; u = \prod_{i=1, i\neq m}^{q-1} r_i = r_n w \text{ and } w = \prod_{i=1, i\notin\{m,n\}}^{q-1} r_i$$

where $m = 1, 2, \ldots, q-1$ and $n = 1, 2, \ldots, q-1$.

We show further that

$$v = r_1^{q-1} = r_2^{q-1} = \ldots = r_{q-1}^{q-1}$$

$$(12.1) \qquad u = r_1^{q-2} = r_2^{q-2} = \ldots = r_{q-1}^{q-2}$$

$$w = r_1^{q-3} = r_2^{q-3} = \ldots = r_{q-1}^{q-3}$$

First of all, $u = r_1 w = r_1(\prod_{i=1}^{q-3} r_i) = r_1^2 \prod_{i=2}^{q-3} r_i$. By finite induction we can derive

$$(12.2) \qquad u = r_1^k(\prod_{i=2}^{q-k-1} r_i) \text{ for } k = 1, 2, \ldots, q-3.$$

We have already shown the truth of (12.2) for $k = 2$. Suppose $u = r_1^{k-1}\prod_2^{q-k} r_i$. Now $r_1^{k-1}\prod_2^{q-k-1} r_i \varepsilon R^{q-3}$ and since

$$(r_1^{k-1}\prod_2^{q-k-1} r_i)r_{q-k} = r_1^{k-1}\prod_2^{q-k} r_i = u$$

it follows $r_1^{k-1}\prod_{i=2}^{q-k-1} r_i \varepsilon R^{q-3}-R^{q-2}$. This implies $w = r_1^{k-1}\prod_{i=2}^{q-k-1} r_i$ and

$$u = r_1 w = r_1(r_1^{k-1}\prod_{i=2}^{q-k-1} r_i) = r_1^k\prod_{i=2}^{q-k-1} r_i.$$

Formula (12.2) when $k = q-3$ yields $u = r_1^{q-3}r_2$. Therefore $r_1^{q-3} \varepsilon R^{q-3}-R^{q-2}$ which implies $w = r_1^{q-3}$. This in turn implies $u = r_1 w = r_1^{q-2}$ and $v = r_1 u = r_1^{q-1}$. Since R is commutative then by a mere relabeling we have certainly verified (12.1).

We next show that $r_1 = r_2 = \ldots = r_{q-1}$. Suppose two of the r_i are distinct; say $r_1 \neq r_2$. Consider the Rees factor semigroup S/S^{q-1}. It is a nilpotent semigroup of class $q-1$, $|S/S^{q-1}| = |R|-1$ and moreover, by Corollary 1.11 we have

$$|(S/S^{q-1})^{q-1}| = 1, \ |(S/S^{q-1})^{q-2}| = 2 \text{ and}$$

$$|(S/S^{q-1})^{q-3}| = 3.$$

Since S/S^{q-1} can be imbedded in S, there exists a subsemigroup T of S such that

(12.3) T is nilpotent of class $q-1$,

(12.4) $|T| = |R|-1$, and

(12.5) $|T^{q-1}| = 1$, $|T^{q-2}| = 2$ and $|T^{q-3}| = 3$.

Actually T is a subsemigroup of R; for e is the zero element of T and hence $T \cap G = e$. Condition (12.5) implies we can choose two distinct elements $\underline{t}$ and t^* in T such that $T^{q-2}-T^{q-1} = \{\underline{t}\}$ and $T^{q-3}-T^{q-2} = \{t^*\}$. Moreover $t\underline{t} = e$ for all $t \in T$. Also $\underline{t} \in T^{q-2} \subseteq R^{q-2}$ $= \{e, u, v\}$; hence we have either $\underline{t} = u$ or $\underline{t} = v$. Since we are assuming $r_1 \neq r_2$ and since $|T| = |R|-1$ we have either $r_1 \in T$ or $r_2 \in T$. Therefore either $r_1^{q-2} \in T^{q-2}$ or $r_2^{q-2} \in T^{q-2}$. But $u = r_1^{q-2} = r_2^{q-2}$, so we will always have $u \in T^{q-2}$. However $u \notin R^{q-1}$ which implies $u \notin T^{q-1}$. Hence $u \in T^{q-2}-T^{q-1}$ which yields $u = \underline{t}$. On the other hand, since either $r_1 \in T$ or $r_2 \in T$ we should have either $e = r_1\underline{t} = r_1 u$ or $e = r_2\underline{t} = r_2 u$. But we know $v = r_1 u = r_2 u$ and hence we have our contradiction. Consequently we have $r_1 = r_2 = \cdots = r_{q-1}$ so that

$$(12.6) \qquad v = r_1^{q-1} = r_1 u, \quad u = r_1^{q-2} = r_1 w, \quad w = r_1^{q-3}.$$

Our next step is to show that $u = r_1^{q-2}$ is the only factorization of u in R^{q-2}. That is, if $u = s_1 s_2 \cdots s_{q-2}$, where $s_i \in R-R^2$, is a factorization of u in R^{q-2}, then we shall show that $s_1 = s_2 = \dots = s_{q-2} = r_1$. Assume at least one of the above $s_i \neq r_1$; say $s_1 \neq r_1$.

We have $u = s_1 s_2 \dots s_{q-2}$, where $s_i \in R-R^2$, and $s_1 \neq r_1$. The elements $b_m = \prod_{\substack{i=1 \\ i\neq m}}^{q-2} s_i$, $(m = 1, 2, \dots, q-2)$, are elements of R^{q-3}. Moreover $u = s_m b_m$, for $m = 1, 2, \dots, q-2$, and since $u \in R^{q-2}-R^{q-1}$ it follows $b_m \in R^{q-3}-R^{q-2}$ for $m = 1, 2, \dots, q-2$. Therefore $w = b_m$ for $m = 1, 2, \dots, q-2$. However

$$s_1 w = s_1 b_1 = s_1 \prod_{i=2}^{q-2} s_i = s_1 s_2 \dots s_{q-2} = u$$

and

$$u = s_1 w = s_1 \prod_{i=1}^{q-3} s_i = s_1^2 \prod_{i=2}^{q-3} s_i.$$

Using finite induction in the same manner as employed in formula (12.2), we obtain

$$(12.7) \qquad u = s_1^k \prod_{i=2}^{q-k-1} s_i, \quad k = 1, 2, \dots, q-3.$$

For $k = q-3$ in (12.7), we find that $u = s_1^{q-3} s_2$. Hence s_1^{q-3} is an element of $R^{q-3}-R^{q-2}$ so that $w = s_1^{q-3}$. This implies $u = s_1 w = s_1^{q-2}$ and $v = r_1 u = r_1(s_1 w) = s_1(r_1 w) = s_1 u = s_1^{q-1}$.

Again we consider the Rees factor semigroup S/S^{q-1}. As before we have the existence of a subsemigroup T of R which has the properties (12.3), (12.4) and (12.5). Note that r_1 is not in T; for otherwise $v = r_1^{q-1} \in T^{q-1} = \{e\}$ which is impossible. Since $s_1 \neq r_1$, $r_1 \notin T$ and $|T| = |R|-1$, it follows $s_1 \in T$. But this is certainly impossible for

$$v = s_1^{q-1} \varepsilon T^{q-1} = \{e\}.$$

Therefore $s_1 = s_2 = \dots = s_{q-2} = r_1$ which gives $u = r_1^{q-2}$ is the only factorization of u in R^{q-2}.

The above result gives us that $w = r_1^{q-3}$ is the only factorization of w in R^{q-3}; for suppose $w = s_1 s_2 \dots s_{q-3}$, where $s_i \varepsilon R-R^2$. Then

$$u = r_1 w = r_1(s_1 s_2 \dots s_{q-3}),$$

and $r_1(s_1 s_2 \dots s_{q-3}) \varepsilon R^{q-2}-R^{q-1}$. It follows from the unique factorization of u in R^{q-2} that $s_1 = s_2 = \dots = s_{q-3} = r_1$. Hence $w = r_1^{q-3}$ is the only factorization of w in R^{q-3}.

Up to this point our arguments have shown that if $S \varepsilon \mathcal{C}$ and $q \geq 5$, then we must have

$$v = r_1^{q-1} = r_1 u, \text{ where } R^{q-1}-R^q = \{v\},$$

$$(12.8) \qquad u = r_1^{q-2} = r_1 w, \text{ where } R^{q-2}-R^{q-1} = \{u\},$$

$$w = r_1^{q-3}, \qquad \text{where } R^{q-3}-R^{q-2} = \{w\},$$

$u = r_1^{q-2}$ is the only factorization of u in R^{q-2} and $w = r_1^{q-3}$ is the only factorization of w in R^{q-3}. We will proceed to show that this situation cannot occur so that our assumption that $S \varepsilon \mathcal{C}$ and $q \geq 5$ is false.

We again consider the Rees factor semigroup S/S^{q-1}. As before, it is nilpotent of class $q-1$, $|S/S^{q-1}| = |R|-1$, $|(S/S^{q-1})^{q-2}| = 2$ and $|(S/S^{q-1})^{q-3}| = 3$. We therefore know that the sets

$$(S/S^{q-1})^{q-2}-(S/S^{q-1})^{q-1} \text{ and } (S/S^{q-1})^{q-3}-(S/S^{q-1})^{q-2}$$

are one-element sets. It will help us to know exactly what these elements are and also the manner in which they are factored in S/S^{q-1}. The elements of S/S^{q-1} consist of the zero element $K_0 = S^{q-1}$ and all one-element sets $K_r = \{r\}$ where $r \in R-R^{q-1}$. Therefore $K_u = \{u\}$, $K_w = \{w\}$ and $K_{r_1} = \{r_1\}$ are all non-zero and distinct elements of S/S^{q-1}. However $u = r_1^{q-2}$ and $w = r_1^{q-3}$ imply

$$K_u = K_{r_1} \circ K_{r_1} \circ \dots \circ K_{r_1} = (K_{r_1})^{q-2} \varepsilon \, (S/S^{q-1})^{q-2}$$

and

$$K_w = K_{r_1} \circ K_{r_1} \circ \dots \circ K_{r_1} = (K_{r_1})^{q-3} \varepsilon \, (S/S^{q-1})^{q-3}.$$

Moreover, from $|(S/S^{q-1})^{q-2}| = 2$ and $|(S/S^{q-1})^{q-3}| = 3$, it follows

$$(S/S^{q-1})^{q-2} - (S/S^{q-1})^{q-1} = \{K_u\} \quad \text{and}$$
$$(S/S^{q-1})^{q-3} - (S/S^{q-1})^{q-2} = \{K_w\}.$$

$S \varepsilon \mathcal{C}$ implies S/S^{q-1} can be imbedded in S so there exists a subsemigroup T of R such that conditions (12.3), (12.4) and (12.5) are satisfied. From condition (12.5), we have the existence of two distinct elements $\underline{t}$ and t^* in T such that $T^{q-2}-T^{q-1} = \{\underline{t}\}$ and $T^{q-3}-T^{q-2} = \{t^*\}$. But since $\underline{t}$ and t^* are the isomorphic images of K_u and K_w in S/S^{q-1} respectively and since $K_u = K_{r_1}^{q-2}$ and $K_w = K_{r_1}^{q-3}$, it follows we have the further condition;

(12.9) there exists an element $t_1 \varepsilon T$ such that $\underline{t} = t_1^{q-2}$, $t^* = t_1^{q-3}$ and thus $\underline{t} = t_1 t^*$. Also

$t \cdot \underline{t} = e$ for every $t \in T$.

Since $\underline{t} \in T^{q-2} \subseteq R^{q-2} = \{e, u, v\}$ we have either $\underline{t} = u$ or $\underline{t} = v$. We show that neither situation is possible giving us our desired contradiction.

Case 1: Suppose $\underline{t} = v$. Now $t^* \in T^{q-3} \subseteq R^{q-3} = \{e, v, u, w\}$. But $\underline{t} \neq t^*$ so that $t^* = u$ or $t^* = w$. If $t^* = u$, then $t_1 t^* = \underline{t}$ implies $t_1 u = v$. But $v = t_1 u = t_1 r_1^{q-2} = (t_1 r_1^{q-3}) r_1$. This gives $t_1 r_1^{q-3} \in R^{q-2} - R^{q-1}$ so that $u = t_1 r_1^{q-3}$. But r_1^{q-2} is the only factorization of u in R^{q-2} so that $r_1 = t_1$. This would imply $u = t^* = t_1^{q-3} = r_1^{q-3} = w$ which is not possible. Hence $t^* = w$. But $w = r_1^{q-3}$ is the only factorization of w in R^{q-3} so that $t_1 = r_1$. Hence $r_1 \in T$ and thus $u = r_1^{q-2} \in T^{q-2}$. But $u \in R^{q-2} - R^{q-1}$ implies $u \in T^{q-2} - T^{q-1} = \{\underline{t}\}$. This again is impossible since $\underline{t} = v$.

Case 2: Suppose $\underline{t} = u$. Now $\underline{t} = t_1^{q-2}$ so that $u = t_1^{q-2} \in R^{q-2}$. However r_1^{q-2} is the only factorization of u in R^{q-2} so we must have $t_1 = r_1$. Therefore $r_1 \in T$ and then, by condition (12.9), it follows $v = r_1 u = r_1 \underline{t} = e$ which again is impossible.

We shall now describe three classes of finite unipotent commutative semigroups in which the structure of these semigroups can be completely determined. We will show later that these three types of semigroups are the only finite unipotent commutative semigroups in the class $\mathcal{C}$.

Let G be any finite commutative group and let R_2 be a finite unipotent commutative null semigroup (i.e. nilpotent class 2). Identify the identity element of G with the zero element of R_2; denote this idempotent by e. Define $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R_2$. Let $S_2 = G \cup R_2$ where $G \cap R_2 = \{e\}$. Then S_2 is a finite unipotent commutative semigroup. For a given finite commutative group G, the structure of S_2 is completely determined. Let $\mathcal{C}_2$ be the collection of those finite unipotent commutative semigroups of the form S_2.

Let R_3 be a finite nilpotent unipotent commutative semigroup of class 3 such that

14.1) $|R_3^2| = 2$ and

14.2) R_3 has a nilpotent subsemigroup N of class 2 with $|N| = |R_3| - 1$.

Identify the zero element of R_3 with the identity of G. Let e denote this idempotent. Define $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R_3$. Let $S_3 = G \cup R_3$ where $G \cap R_3 = e$. Then S_3 is a finite unipotent commutative semigroup. Again its structure is completely determined for a given finite commutative group G. Let $\mathcal{C}_3$ be the collection of those finite unipotent commutative semigroups of form S_3. We can actually give the multiplication table for semigroups $S_3 \varepsilon \mathcal{C}_3$. Let $G = \{e, g_1, \ldots, g_n\}$, $R_3 = \{e, r_1, \ldots, r_m, v\}$ and $R_3^2 = \{e, v\}$:

S_3	e	$g_1 \cdots$	g_n	r_1	$r_2 \cdots$	$r_n \cdots$	r_m	v
e	Cayley Table for group G			e	$e \cdots$	$e \cdots$	e	e
g_1				g_1	$g_1 \cdots$	$g_1 \cdots$	g_1	g_1
$\vdots$				$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
g_n				g_n	$g_n \cdots$	$g_n \cdots$	g_n	g_n
r_1	e	$g_1 \cdots$	g_n	e	$e \cdots$	_ $\cdots$	e	e
r_2	e	$g_1 \cdots$	g_n	e	$e \cdots$	_ $\cdots$	e	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r_n	e	$g_1 \cdots$	g_n	_	_ $\cdots$	_ $\cdots$	_	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r_m	e	$g_1 \cdots$	g_n	e	$e \cdots$	_ $\cdots$	e	e
v	e	$g_1 \cdots$	g_n	e	$e \cdots$	$e \cdots$	e	e

The elements to be placed in the blank spaces occurring in the row and column of the element r_n are to be any element of $R_3^2 = \{e, v\}$ such that associativity and commutativity hold. The subsemigroup N of R_3 is $N = R_3 - \{r_n\}$.

Let R_4 be a finite nilpotent unipotent commutative semigroup of class 4 such that

15.1) $|R_4^2| = 3$ and $|R_4^3| = 2$,

15.2) Let $u \varepsilon R_4^2 - R_4^3$ and let $v \varepsilon R_4^3 - R_4^4$.

There exists exactly one element $r_1 \varepsilon R_4 - R_4^2$ such that $v = r_1^3$ and $u = r_1^2$. Moreover $u = r_1^2$ is the only factorization of u in R_4^2,

15.3) $r \cdot r_1 \varepsilon R_4^3$ for every $r \varepsilon R_4$ with $r \neq r_1$,

15.4) there is exactly one element $r_0 \varepsilon R_4 - R_4^2$ where $r_0 \neq r_1$, $v = r_0^2$ and rr_0 is the zero element of R for all $r \varepsilon R - \{r_0, r_1\}$.

15.5) $M = R_4 - \{r_0, r_1\}$ is a null subsemigroup of R_4.

Identify the zero element of R_4 with the identity element of G; denote this idempotent by e. Define $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R_4$. Let $S_4 = G \cup R_4$ with $G \cap R_4 = e$. Then S_4 is a finite unipotent commutative semigroup and its structure, for a given commutative group G, is also completely determined. Let $\mathcal{C}_4$ be the collection of those finite unipotent commutative semigroups of form S_4. Again we can actually give the Cayley multiplication table for $S_4 \varepsilon \mathcal{C}_4$. We do this below. Let $G = \{e, g_1, \ldots, g_n\}$, $R_4 = \{e, v, u, r_0, r_1, \ldots, r_m\}$, $R_4^2 = \{e, v, u\}$, $R_4^3 = \{e, v\}$ and $R_4^4 = \{e\}$;

S_4	e	g_1	…	g_n	r_0	r_1	r_2	…	r_m	u	v
e		Cayley Table			e	e	e	…	e	e	e
g_1		for			g_1	g_1	g_1	…	g_1	g_1	g_1
⋮		group G			⋮	⋮	⋮		⋮	⋮	⋮
g_n					g_n	g_n	g_n	…	g_n	g_n	g_n
r_0	e	g_1	…	g_n	v	_	e	…	e	e	e
r_1	e	g_1	…	g_n	_	u	_	…	_	v	e
r_2	e	g_1	…	g_n	e	_	e	…	e	e	e
⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮
r_m	e	g_1	…	g_n	e	_	e	…	e	e	e
u	e	g_1	…	g_n	e	v	e	…	e	e	e
v	e	g_1	…	g_n	e	e	e	…	e	e	e

The elements to be placed in the blank spaces occurring in the row and column for the elements r_0 and r_1 are to be any element of $R_4^3 = \{e, v\}$ such that the associative and commutative laws hold.

We shall first show that the classes $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are subclasses of the class $\mathcal{C}$. According to the next lemma it suffices to show that R_2, R_3 and R_4 are members of $\mathcal{C}$.

<u>Lemma 1.13</u> . Let R be a finite unipotent nilpotent commutative semigroup and let G be a finite commutative group whose identity element is identified

with the zero element of R where $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R$. If $R \varepsilon \mathcal{C}$ then $S = G \cup R \varepsilon \mathcal{C}$.

Proof. Let λ be a homomorphism of S onto S^*. Then $\lambda | R = R^*$ is a finite unipotent nilpotent commutative semigroup; its zero element is $e^* = \lambda(e)$ where e is the zero element of R. Also $\lambda | G = G^*$ is a finite commutative group with identity element $e^* = \lambda(e)$. Since $R \varepsilon \mathcal{C}$ and $G \varepsilon \mathcal{C}$ it follows $R^* \varepsilon \mathcal{C}$ and $G^* \varepsilon \mathcal{C}$. But $S^* = R^* \cup G^*$, $R^* \cap G^* = \{e^*\}$ and $g^* r^* = g^*$ for every $g^* \varepsilon G^*$ and $r^* \varepsilon R^*$. Hence S^* is a finite unipotent commutative semigroup. Since R^* and G^* can be imbedded in R and G respectively we have S^* can be imbedded in S. Hence S has property P.

Now R^* and G^* have property P so that one can apply the same argument on S^* as was applied on S to deduce that S^* has property P. Therefore $S \varepsilon \mathcal{C}$.

Theorem 1.14. $\mathcal{C}_2 \subseteq \mathcal{C}$ and $\mathcal{C}_3 \subseteq \mathcal{C}$.

Proof. Let $R_2 \varepsilon \mathcal{C}_2$. A homomorphic image of a finite unipotent nilpotent commutative semigroup of class 2 is either a trivial semigroup or is one of the same type. Hence $R_2 \varepsilon \mathcal{C}$. By Lemma 1.13, it follows $S_2 \varepsilon \mathcal{C}$. Hence $\mathcal{C}_2 \subseteq \mathcal{C}$.

Let $S_3 = G \cup R_3 \varepsilon \mathcal{C}_3$. We need only show $R_3 \varepsilon \mathcal{C}$. Now $R_3^3 = \{e\}$ and $R_3^2 - \{e\} = \{u\}$. Also $ru = e$ for every $r \varepsilon R_3$. Let $u = u_1 u_2$, where $u_i \varepsilon R_3 - \{e\}$ and $i = 1, 2$.

However $|N| = |R_3| - 1$ implies there is exactly one element $a \varepsilon R_3 - N$. u_1 and u_2 do not both belong to N; for otherwise $u = u_1u_2 = e$ which is not possible. Therefore either $u_1 = a$ or $u_2 = a$. We have shown that whenever $u = u_1u_2$, where $u_i \varepsilon R_3 - R_3^2$, then $u_1 = a$ or $u_2 = a$.

Let ρ be a congruence on R_3 and consider R_3/ρ. We shall show R_3/ρ belongs to the class $\mathcal{C}$. Let $K_e = \{x \varepsilon R_3 | x \rho e\}$, K_1, K_2, ..., K_m be the equivalence classes of R_3 mod ρ. K_e is the zero element of R_3/ρ.

Case 1. Suppose $a \varepsilon K_e$. Then $a \rho e$ implies $u_1 a \rho u_1 e$ and hence $u \rho e$. Therefore $u \varepsilon K_e$. Since a does not belong to K_i $(i = 1, 2, \ldots, m)$ it follows each class K_i $(i = 1, 2, \ldots, m)$, considered as a subset of R_3, is actually a subset of the subsemigroup N. But $N^2 = \{e\}$ so that $K_iK_j \subseteq NN = \{e\}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$. Thus $K_i \circ K_j = K_e$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$. From each class K_i, $i = 1, \ldots, m$, choose an element b_i. Let $T = \{e, b_1, \ldots, b_m\}$. Every b_i is an element of N so it follows $b_ib_j = e$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$. Consequently T is a subsemigroup of R_3. Define the transformation $\alpha : R_3/\rho \longrightarrow T$ as follows:

$\alpha(K_e) = e$ and $\alpha(K_i) = b_i$ for $i = 1, 2, \ldots, m$.

α is an isomorphism of R_3/ρ onto T so we have R_3/ρ

imbedded in R_3. Hence R_3/ρ has property P.

Now T is nilpotent of class 2 and therefore R_3/ρ is nilpotent of class 2. But we already know such finite unipotent nilpotent commutative semigroups have property P. Hence R_3/ρ has property P. Since R_3 and R_3/ρ have property P, it follows $R_3 \varepsilon \mathcal{C}$.

Case 2. Suppose a does not belong to K_e but $u \varepsilon K_e$. Let $a \varepsilon K_1$. Then each equivalence class K_i, $(i = 2, 3, \ldots, m)$, considered as a subset of R_3, is actually a subset of the subsemigroup N. Therefore $K_i \circ K_j = K_e$ for $i = 2, 3, \ldots, m$ and $j = 2, 3, \ldots m$. Moreover $K_1 \circ K_i = K_e$ for $i = 1, 2, \ldots, m$; for let $r_1 r_2 \varepsilon K_1 K_i$ where $r_1 \varepsilon K_1$ and $r_2 \varepsilon K_i$. Since $r_1 r_2 \varepsilon R_3^2 = \{e, u\}$ and since $u \varepsilon K_e$, it follows $r_1 r_2 \rho\, e$. Therefore $K_1 K_i \subseteq K_e$ so that $K_1 \circ K_i = K_e$, for $i = 1, 2, \ldots, m$. The elements of R_3/ρ have the following products:

$$K_e \circ K_i = K_e \text{ and } K_i \circ K_j = K_e \text{ for all } i \text{ and } j.$$

From each equivalence class K_i, $i = 2, 3, \ldots, m$, choose an element a_i. Let $T = \{e, u, a_2, \ldots, a_m\}$. Now $a_i a_j = e$ for $i = 2, \ldots, m$ and $j = 2, \ldots, m$ since they all are elements of N. Moreover $u a_i = e$ for all $i = 2, \ldots, m$. Therefore T is a subsemigroup of R_3. Define the transformation $\alpha : R_3/\rho \longrightarrow T$ as follows:

$$\alpha(K_e) = e, \quad \alpha(K_1) = u \text{ and } \alpha(K_i) = a_i \text{ for } i = 2, \ldots, m.$$

α is an isomorphism of R_3/ρ onto T so that R_3 has property P.

Moreover R_3/ρ has property P for it also is nilpotent of class 2. Thus we have $R_3 \varepsilon \mathcal{G}$.

Case 3. Suppose both a and u are not elements of K_e. If $a \rho u$, then $u_1 a \rho u_1 u$ which implies $u \rho e$. This cannot happen so that a and u belong to distinct equivalence classes. Let $a \varepsilon K_1$ and $u \varepsilon K_2$. Now each K_i, $i = 2, 3, \ldots, m$, are actually subsets of N. Therefore $K_i \circ K_j = K_e$ for $i = 2, \ldots, m$ and $j = 2, \ldots, m$. Also $K_1 \circ K_2 = K_e$ for $u \varepsilon K_2$ and $ru = e$ for every $r \varepsilon R_3$. However either $K_1 \circ K_j = K_e$ or $K_1 \circ K_j = K_2$ for $j \neq 2$. This follows from the fact that either $ar = e$ or $ar = u$ for $r \varepsilon R_3$. The products in R_3/ρ are

$K_e \circ K_i = K_e$ for $i = 1, \ldots, m$, $K_i \circ K_j = K_e$ for $2 \leq i \leq m$

$K_1 \circ K_2 = K_e$ and either $K_1 \circ K_j = K_e$ or

$K_1 \circ K_j = K_2$ for $j \neq 2$.

From each class K_j, $j = 3, 4, \ldots, m$, choose one element a_j. Let $T = \{e, a, u, a_3, a_4, \ldots, a_m\}$. Since $T^2 \subseteq R_3^2 = \{e, u\}$, it follows T is a subsemigroup of R_3. Define $\alpha : R_3/\rho \longrightarrow T$ as follows:

$\alpha(K_e) = e$, $\alpha(K_1) = a$, $\alpha(K_2) = u$, $\alpha(K_i) = a_i$ for

$i = 3, 4, \ldots, m$.

One can check that α is an isomorphism of R_3/ρ onto T. Hence R_3 has property P.

By studying the multiplication in R_3/ρ one can determine that R_3/ρ will be either nilpotent of class 2 or will be a semigroup of the same type as R_3. Hence R_3/ρ has property P.

These cases all yield the same result; namely $R_3 \varepsilon \mathcal{C}$. By Lemma 1.13, it follows $S_3 \subseteq \mathcal{C}$ and hence $\mathcal{C}_3 \subseteq \mathcal{C}$.

<u>Theorem 1.15</u> . $\mathcal{C}_4 \subseteq \mathcal{C}$.

<u>Proof</u>. Let $R_4 \varepsilon \mathcal{C}_4$ and let ρ be a congruence on R_4 and consider R_4/ρ. Since $R_4 \varepsilon \mathcal{C}_4$ it has properties (15.1) thru (15.5) introduced earlier. Let e, v, u, r_1 and r_0 be the elements of R_4 as defined in properties (15.1) thru (15.5). The equivalence classes mod ρ will be denoted by K_e, K_1, K_2, ..., K_m, where $K_e = \{x \varepsilon R_4 | x \rho e\}$. We again need to consider several cases.

Case 1. Suppose $r_1 \varepsilon K_e$. Then $r_1^2 \rho e$ and $r_1^3 \rho e$ so that u and v both belong to K_e. Consequently, the equivalence class K_e contains $R_4^2 = \{e, v, u\}$. Hence $K_i \circ K_j = K_e$ for all i = 1, ..., m and j = 1, ..., m. That is, R_4/ρ is nilpotent of class 2. Now $|K_e| \geq 4$ so that $m \leq |R_4|-4$. But $|M| = |R_4|-2$ where $M = R_4 - \{r_0, r_1\}$. Hence for each equivalence class K_i (i = 1, ..., m) we can select an element a_i, where $a_i \neq a_j$ if $i \neq j$, from the set M. Let $T = \{e, a_1, ..., a_m\}$. Property 15.5 gives

$a_i a_j = e$ for all i and j. Therefore T is a subsemigroup of R_4. Define $\alpha : R_4/\rho \longrightarrow T$ by $\alpha(K_e) = e$ and $\alpha(K_i) = a_i$ for $i = 1, \ldots, m$. α is an isomorphism of R_4/ρ onto T. Therefore R_4 has property P.

Since R_4/ρ is nilpotent of class 2, it follows R_4/ρ has property P. Thus $R_4 \varepsilon \mathcal{C}$.

Case 2. Suppose r_1 does not belong to K_e but $u \varepsilon K_e$. $u \varepsilon K_e$ implies $r_1 u \rho r_1 e$ so that $v \rho e$. Hence $v \varepsilon K_e$. Therefore the set K_e contains the ideal $R_4^2 = \{e, v, u\}$, and, as in the above case, we must have $K_i \circ K_j = K_e$, where $i = 1, \ldots, m$ and $j = 1, \ldots, m$. But $|K_e| \geq 3$ so that $m \leq |R_4| - 3$. Hence $m < |M|$ and consequently for each equivalence class K_i $(i = 1, 2, \ldots, m)$ we can select an element a_i, with $a_i \neq a_j$ when $i \neq j$, from the set M. Let $T = \{e, a_1, a_2, \ldots, a_m\}$. T is a subsemigroup of R_4, by property (15.5). Define $\alpha : R_4/\rho \longrightarrow T$ by $\alpha(K_e) = e$ and $\alpha(K_i) = a_i$ for $i = 1, 2, \ldots, m$. α is an isomorphism of R_4/ρ onto T and we conclude that R_4 has property P.

As in the previous case, R_4/ρ is nilpotent of class 2 so that R_4/ρ has property P.

Case 3. Suppose r_1 and u do not belong to K_e but $v \varepsilon K_e$. Suppose $r_1 \rho u$. Then $r_1^2 \rho r_1 u$ which implies $u \rho v$. But this is impossible so that r_1 and u belong to distinct equivalence classes. Let $r_1 \varepsilon K_1$ and $u \varepsilon K_2$. Since $r_1^2 = u$ it follows $K_1 \circ K_1 = K_2$.

On the other hand $K_i \circ K_j = K_e$ except when $i = j = 1$. This follows from the fact that $K_iK_j \subseteq R_4^3 \subseteq K_e$. Again $m \leq |M|$ so for each equivalence class K_i, $i = 3, \ldots, m$, we can select an element a_i ($a_i \neq a_j$ if $i \neq j$) from the set M. Let $T = \{e, r_0, v, a_3, a_4, \ldots, a_m\}$. T is a subsemigroup. Define $\alpha : R_4/\rho \longrightarrow T$ by $\alpha(K_e) = e$, $\alpha(K_1) = r_0$, $\alpha(K_2) = v$ and $\alpha(K_i) = a_i$ for $i = 3, \ldots, m$. One can check to see that α is an isomorphism of R_4/ρ onto T. Therefore R_4 has property P.

By studying the multiplication in R_4/ρ, one can determine that it is a nilpotent semigroup of class 3 and moreover of the same type as R_3. That is, it satisfies properties (14.1) and (14.2). Hence R_4/ρ has property P.

Case 4. Suppose r_1, u and v do not belong to K_e. Then we must have r_1, u and v belong to three distinct ρ-classes; say $r_1 \varepsilon K_1$, $u \varepsilon K_2$ and $v \varepsilon K_3$. Moreover r_0 belongs to some ρ-class distinct from K_1, K_2 and K_3. For each equivalence class K_i, where $i = 5, 6, \ldots, m$, we select an element a_i ($a_i \neq a_j$ if $i \neq j$) from the set $(M-\{u,v\})$. Let $T = \{e, r_0, r_1, u, v, a_5, a_6, \ldots, a_m\}$. T is a subsemigroup of R_4 since $T^2 \subseteq R_4^2 \subseteq T$. Define $\alpha : R_4/\rho \longrightarrow T$ as follows:

$$\alpha(K_e) = e, \quad \alpha(K_1) = r_1, \quad \alpha(K_2) = u, \quad \alpha(K_3) = v,$$
$$\alpha(K_4) = r_0 \quad \text{and} \quad \alpha(K_i) = a_i \text{ for } i = 5, 6, \ldots, m.$$

α can be checked to show it is an isomorphism of R_4/ρ onto T. Therefore R_4 has property P.

The multiplication in T shows that R_4/ρ is a semigroup of the same type as R_4. That is, it has properties (15.1) thru (15.5). Hence R_4/ρ has property P.

In each of the above cases, we have shown that if $S_4 \varepsilon \mathcal{C}_4$, then $R_4 \varepsilon \mathcal{C}$. Hence $S_4 \varepsilon \mathcal{C}$ and we have $\mathcal{C}_4 \subseteq \mathcal{C}$.

We are now ready to prove the main result of this section. That is, we shall now characterize those finite unipotent commutative semigroups S where $S \varepsilon \mathcal{C}$.

<u>Theorem 1.16</u>. $S \varepsilon \mathcal{C}$ if and only if either

1) S is a finite commutative group,

2) $S \varepsilon \mathcal{C}_2$,

3) $S \varepsilon \mathcal{C}_3$, or

4) $S \varepsilon \mathcal{C}_4$.

<u>Proof</u>. If S is a finite commutative group then $S \varepsilon \mathcal{C}$. Theorem 1.14 and Theorem 1.15 show that when S is a member of the class $\mathcal{C}_2$, $\mathcal{C}_3$, or $\mathcal{C}_4$, then $S \varepsilon \mathcal{C}$.

Suppose $S \varepsilon \mathcal{C}$ and S is not a group. Let e be its idempotent, let G be its maximal group and let $R = (S-G) \cup \{e\}$. By Theorem 1.6, R is a finite unipotent nilpotent commutative subsemigroup of S of class $q > 1$ and $gr = g$ for every $g \varepsilon G$ and $r \varepsilon R$. Moreover, from Theorem 1.12, it follows $1 < q \leq 4$. We now determine the structure of S when $q = 2$, $q = 3$ and $q = 4$.

Case 1. If $q = 2$, then R is nilpotent of class 2 and clearly $S \varepsilon \zeta_2$.

Case 2. Suppose $q = 3$. By Theorem 1.10, it follows $|S^2| = |G|+1$ and $|R^2| = 2$. Consider the Rees factor semigroup S/S^2. It is nilpotent of class 2 and, from Corollary 1.11, it follows $|S/S^2| = |R|-1$. But $S \varepsilon \zeta$ implies S/S^2 can be imbedded in S. Hence there exists a subsemigroup N of S such that N is nilpotent of class 2, e is the zero element of N and $|N| = |R|-1$. But $N \cap G = e$ so that $N \subseteq R$. Therefore R is a nilpotent unipotent commutative semigroup of class 3 such that $|R^2| = 2$ and there is a nilpotent subsemigroup N of class 2 with $|N| = |R|-1$. Therefore R has the properties (14.1) and (14.2) so that $S \varepsilon \zeta_3$.

Case 3. Suppose $q = 4$. By Theorem 1.10, it follows $|R^2| = 3$ and $|R^3| = 2$. We proceed to show that R satisfies (15.2), (15.3), (15.4) and (15.5). Let $R^2 - R^3 = \{u\}$ and $R^3 - R^4 = \{v\}$; that is, $R^2 = \{e, v, u\}$ and $R^3 = \{e, v\}$.

$v \varepsilon R^3$ implies $v = r_1 r_2 r_3$ where $r_i \varepsilon R-R^2$. Then $r_1 r_2$, $r_1 r_3$ and $r_2 r_3$ are non-zero elements of $R^2 - R^3$

so it follows $u = r_1r_2 = r_1r_3 = r_2r_3$. Consequently, $v = r_1u = r_1r_1^2 = r_1^3$. Likewise $v = r_2^3 = r_3^3$ and we have $v = r_1^3 = r_2^3 = r_3^3$ and $u = r_1^2 = r_2^2 = r_3^2$.

We now show that $r_1 = r_2 = r_3$. Suppose two of these elements r_i are distinct; say $r_1 \neq r_2$. Consider the Rees factor semigroup S/S^3. It is nilpotent of class 3 and by Corollary 1.11, $|S/S^3| = |R|-1$ and $|(S/S^3)^2| = 2$. Since $S \varepsilon \mathcal{C}$ it follows that S/S^3 can be imbedded in S. Thus there is a nilpotent subsemigroup T of class 3 in S with

$$(16.1)\ |T| = |R|-1,$$

$$(16.2)\ |T^2| = 2.$$

Actually T is a subsemigroup of R. By (16.2), there is an element $t^* \varepsilon T$ with $T^2-T^3 = \{t^*\}$. From $T^3 = \{e\}$ and $t^* \varepsilon T^2$ it follows $t \cdot t^* = e$ for all $t \varepsilon T$. Moreover $t^* \varepsilon T^2 \subseteq R^2 = \{e, v, u\}$ implies $t^* = u$ or $t^* = v$. We are assuming $r_1 \neq r_2$ so it follows, from $|T| = |R|-1$, that either $r_1 \varepsilon T$ or $r_2 \varepsilon T$. Hence either $r_1^2 \varepsilon T^2$ or $r_2^2 \varepsilon T^2$. Since $u = r_1^2 = r_2^2$ then we must always have $u \varepsilon T^2$. But $u \notin T^3$ which gives us $u \varepsilon T^2-T^3$. Hence $u = t^*$.

On the other hand, since either $r_1 \varepsilon T$ or $r_2 \varepsilon T$ we have either $e = r_1t^* = r_1u = v$ or $e = r_2t^* = r_2u = v$. This indeed is impossible so that our assumption is false. Hence $r_1 = r_2 = r_3$ which implies $v = r_1^3$ and $u = r_1^2$.

Next we show that $u = r_1^2$ is the only factorization of u in R^2. That is, if $u = s_1 s_2$, where $s_i \varepsilon R-R^2$, is a factorization of u in R^2, then we show that $s_1 = s_2 = r_1$. Assume that at least one of the $s_i \neq r_1$; say $s_1 \neq r_1$. Now $v = r_1 u = r_1 s_1 s_2$ implies $r_1 s_1$, $r_1 s_2$ and $s_1 s_2$ are elements of R^2-R^3. Hence $u = r_1 s_1 = r_1 s_2 = s_1 s_2$ and $v = s_1 u = s_1(r_1 s_1) = r_1 s_1^2$. But $v = r_1 s_1^2$ implies $s_1^2 \varepsilon R^2-R^3$ so that $u = s_1^2$ and $v = s_1^3$. Again we consider the Rees factor semigroup S/S^3. As before we have the existence of a semigroup T of R which is nilpotent of class 3 and satisfies properties (16.1) and (16.2). Note that $r_1 \notin T$; for otherwise $v = r_1^3 \varepsilon T^3 = \{e\}$. Therefore $s_1 \varepsilon T$ since $|T| = |R|-1$ and $s_1 \neq r_1$. But again we have a contradiction since $v = s_1^3 \varepsilon T^3 = \{e\}$. Thus $u = r_1^2$ is the only factorization of u in T^2. We have just shown that R has property (15.2).

Since $rr_1 \neq u$ unless $r = r_1$, it follows $r \cdot r_1 \varepsilon \{e, v\} = R^3$ for every $r \varepsilon R$ with $r \neq r_1$. This shows that R has property (15.3)

We proceed to show that R has property (15.4). Again we consider S/S^3. We know $|(S/S^3)^2| = 2$ so that the set

$$(S/S^3)^2 - (S/S^3)^3$$

is a one-element set. It will help us to know exactly the element in this set and how it is factored in S/S^3. The elements of S/S^3 are the zero element $K_0 = S^3$

and all one-element sets $K_r = \{r\}$ where $r \varepsilon R-R^3$. Hence $K_u = \{u\}$ and $K_{r_1} = \{r_1\}$ are elements of S/S^3. Since $u = r_1^2$ it follows $K_u = K_{r_1} \circ K_{r_1} \varepsilon (S/S^3)^2$. Moreover $K_u = K_{r_1}^2$ is the only factorization of K_u in $(S/S^3)^2$ since $u = r_1^2$ is the only factorization of u in R^2. Therefore

$$(S/S^3)^2 - (S/S^3)^3 = \{K_u\}.$$

Since S/S^3 can be imbedded in S, it follows there exists a nilpotent subsemigroup T of class 3 in R which satisfies conditions (16.1) and (16.2). Condition (16.2) gives the existence of an element $t^* \varepsilon T$ with $T^2-T^3 = \{t^*\}$. Moreover, since t^* is the isomorphic image of K_u, it follows there exists $t_0 \varepsilon T$ with $t^* = t_0^2$. Also $t^* = t_0^2$ is the only factorization of t^* in T^2.

From $t^* \varepsilon T^2 \subseteq R^2 = \{e, u, v\}$, it follows $t^* = u$ or $t^* = v$. If $t^* = u$, then $t_0 = r_1$ for r_1^2 is the only factorization of u in R^2. Then $v = r_1^3 \varepsilon T^3 = \{e\}$ which clearly is not possible. Therefore $t^* = v$ so that $v = t_0^2$ and this is the only factorization of v in T^2. But $r_1 \notin T$ so that $T = R-\{r_1\}$. Hence $rt_0 = e$ for all $r \varepsilon R-\{t_0, r_1\}$. We just have proven property (15.4) holds in R by taking $r_0 = t_0$.

Let $M = R-\{r_1, r_0\}$. Let a and b be elements of M. The fact that r_1^2 is the only factorization of u in R^2 shows that $ab = e$ or $ab = v$. Since $a \neq r_1$

and $b \neq r_1$, it follows both a and b belong to T. But then $v \neq ab$ for r_0^2 is the only factorization of v in T^2. Hence $ab = e$ for every a and b in M. Therefore M is nilpotent of class 2 and we have shown R has property (15.5).

Since R satisfies properties (15.1) thru (15.5) it follows $S \varepsilon \mathcal{C}_4$.

Theorem 1.16 completely characterizes those finite unipotent commutative semigroups which have property P and whose homomorphic images also have property P. In our above results we did not need that all homomorphic images of S also have property P, but only those homomorphic images of the form S/S^k have property P. It is still an open question as to whether this condition is also necessary.

Chapter 2

Finite Commutative Semigroups of Class $\mathcal{H}$

The semigroups which will be discussed in this chapter are finite commutative bands and finite commutative semigroups which are unions of groups. For such semigroups S we will again impose the property P (recall that S has property P if and only if S/ρ can be imbedded in S for all congruences ρ in S). We will try to determine the fine structure for the above mentioned semigroups S which have property P and whose homomorphic images also have property P.

2.1 Certain Properties of Finite Semilattices

In this section we will define a semilattice (introduced by Klein-Barmen [6]) and develop several basic properties.

Let E be a finite commutative band. Recall that a band is a semigroup every element of which is idempotent. Consider the relation $\leq$ on the band E defined by $e \leq f$ $(e, f$ in $E)$ if and only if $ef = fe = e$. If $e \leq f$ we say that e is under f and that f is over e. The relation $\leq$ on E is a partial ordering of E. That is, $\leq$ is a reflexive, antisymmetric and transitive relation on E. To see that $\leq$ is a partial ordering on E, let $e, f, g \in E$. (1) $e^2 = e$ and hence $e \leq e$. (2) If $e \leq f$ and $f \leq e$, then $ef = fe = e$ and $fe = ef = f$, consequently $e = f$. (3) If $e \leq f$ and $f \leq g$, then $ef = fe = e$ and $fg = gf = f$, and hence

$$ge = eg = (ef)g = e(fg) = ef = e.$$

Therefore $e \leq g$. We shall call $\leq$ the <u>natural partial ordering</u> of E.

We define a <u>meet-semilattice</u> as follows. Let X be a partially ordered set. An element b of X is called a lower bound of a subset Y of X if $y \geq b$ for every y in Y. A lower bound b of Y is a greatest lower bound or meet of Y if $b \geq c$ for every lower bound c of Y. If Y has a meet in X, it is clearly unique. A partially ordered set X is called a meet-semilattice if every two-element subset $\{a,b\}$ of X has a meet in X; consequently every finite subset of X has a meet. The meet of $\{a, b\}$ is denoted by $a \wedge b$.

A commutative band E is a meet-semilattice with respect to the natural partial ordering of E. The meet, $a \wedge b$, of two elements a and b of E is just their product ab. From $(ba)a = ba^2 = ba$ and $(ab)b = ab^2 = ab$, we see that $ab \leq a$ and $ab \leq b$.

Suppose $c \leq a$ and $c \leq b$. Then $(ab)c = a(bc) = ac = c$, and similarly $c(ab) = c$, whence $c \leq ab$. This shows that the meet of $\{a, b\}$ is precisely ab.

It is evident that the converse is true. That is, every meet-semilattice is a commutative band with respect to the meet operation. In this thesis, the term <u>semilattice</u> will mean meet-semilattice and consequently we will use the term semilattice as synonymous with commutative band.

We know that every finite subset of a semilattice has a meet. Therefore, if E is a finite semilattice it follows E itself has a greatest lower bound or meet; we denote it by z. Then $z \leq e$ for every $e \in E$ so that $ze = ez = z$. Clearly z is unique and hence z is the zero element of E.

<u>Lemma 2.1</u> . If E is a finite semilattice and a, b, c and d are elements of E, then

(1) $a \geq b$ and $c \geq d$ imply $ac \geq bd$, and

(2) $a \geq b$ implies $xa \geq xb$ for every $x \in E$.

<u>Proof</u>: (1) $a \geq b$ and $c \geq d$ gives $ab = ba = b$ and $cd = dc = d$, respectively. Therefore $(ac)(bd) = (ab)(cd) = bd$ so that $ac \geq bd$.

(2) $(xa)(xb) = x^2(ab) = xb$ so that $xa \geq xb$ for every $x \in E$.

In this section and the following one, E will always denote a finite semilattice, z its zero element and "$\geq$" the natural partial ordering of E. We now define several concepts which are needed in the following arguments.

<u>Definition</u>. Let a and b be two arbitrary elements of E. If $a \geq b$ or $a \leq b$, a and b are said to be <u>comparable</u>; in the opposite case, a and b are said to be <u>incomparable</u> elements, which is expressed by $a \| b$. Further symbols used are $<$ and $>$, signifying $a < b$ or $b > a$, if $a \leq b$ but $a \neq b$.

<u>Definition</u>. If for a pair of elements a and b of E, $a < b$ holds and there is no element x in E such that $a < x < b$, then it is said the element a is <u>covered</u> by b (or b covers a). This situation is expressed by the symbol $a \ll b$ (or $b \gg a$). Accordingly, $a \underline{\ll} b$ will symbolize that "b either covers or equals a"; in short, "b at most covers a".

<u>Definition</u>. A subset T of E is said to be a <u>chain</u> (or simply ordered set) if and only if for every pair a, b in T, either $a \geq b$ or $b \geq a$. By the <u>length</u> of a chain T consisting of $m+1$ elements (that is, being of the form $x_0 < x_1 < \ldots < x_m$) we shall mean

the non-negative integer m. A chain T from a to b (a, b in T), of length m, is of the form $a = x_0 > x_1 > \ldots > x_m = b$. It is said to be a <u>maximal chain</u> from a to b, if x_i covers x_{i+1} ($i = 0, 1, \ldots, m-1$). (That is, being of the form $a = x_0 \gg \ldots \gg x_m = b$.)

<u>Definition</u>. An element a of E is said to have <u>dimension d</u> (written $d(a)$), where d is the length of the longest maximal chain from a to z (the zero element of E). We will make the convention $d(z) = 0$.

<u>Lemma 2.2</u> . (1) For every a and b in E, $a > b$, there exists a maximal chain from a to b.

(2) If $a > b$, then, for the non-negative integers $d(a)$ and $d(b)$, $d(a) > d(b)$.

<u>Proof</u>. If $a \gg b$, we then have a maximal chain from a to b. If a does not cover b, then there exists an element x in E such that $a > x > b$. Consider the set $T(a, b)$ of all comparable elements y in E with $a > y > b$. $T(a, b)$ is a non-empty and finite chain. Choose $x_1 \varepsilon T(a, b)$ with $x_1 \geq y$ for all $y \varepsilon T(a, b)$. Then $a \gg x_1 > b$. If x_1 covers b, we would have a maximal chain from a to b. If not, consider the set $T(x_1, b)$ of all comparable elements y in E with $x_1 > y > b$ and choose x_2 in $T(x_1, b)$ such that $x_2 \geq y$ for all $y \varepsilon T(x_1, b)$. We then have

$a >> x_1 >> x_2 > b$.

By finite induction, we get a chain of elements $a, x_1, x_2, \ldots, x_n, \ldots$ such that

$$a >> x_1 >> x_2 >> \cdots >> x_n >> \cdots > b.$$

Since E is finite, it follows we get an element x_m which covers b and hence a maximal chain $a >> x_1 >> \cdots >> x_m >> b$ from a to b.

(2) By part (1), there exists a maximal chain from a to b, say $a = a_0 >> a_1 >> \cdots >> a_m = b$. If $b = z$ we clearly have $d(a) > d(b)$. For $a > b$ and $b \neq z$, choose a maximal chain from b to z having length $d(b) = n$; say $b = b_0 >> b_1 >> \cdots >> b_n = z$. Then

$$a = a_0 >> a_1 >> \cdots >> b >> b_1 >> \cdots >> b_n = z$$

is a chain from a to z having length $m+n$. Since $m \geq 1$, it follows $d(a) \geq m+n > n = d(b)$. This completes the proof of the lemma.

A homomorphic image of a finite semilattice (commutative band) is clearly a finite semilattice (commutative band). If θ is a homomorphism of a finite semilattice E onto E^*, we shall use the same symbol "$\leq$" to denote the natural partial ordering in the semilattice E^*. We end this section by developing some inequalities involving the dimension function of elements a in E and $\theta(a)$ in E^*.

<u>Lemma 2.3</u> . Let E be a finite semilattice and θ a homomorphism of E. Then $d(a) \geq d(\theta(a))$ for every $a \in E$.

<u>Proof</u>. Let θ be a homomorphism of E onto E^*, $a \in E$ and $a^* = \theta(a) \in E^*$. Let z^* be the zero element in E^* $(z^* = \theta(z))$ and denote $d(a^*) = m$. If $m = 0$, then $d(a) \geq d(a^*) = 0$ since $d(a)$ is a non-negative integer.

Suppose $m > 0$. Then we can find a maximal chain in E^* from a^* to z^* having length m;

$$a^* = a_0^* >> a_1^* >> \ldots >> a_m^* = z^* .$$

Consider the following subsets of E:

$$\theta^{-1}(a_i^*) = \{x \in E \mid \theta(x) = a_i^*\}, \quad \text{where } i = 0, 1, \ldots, m.$$

Since θ is a homomorphism of E onto E^*, it follows $\theta^{-1}(a_i^*)$ is a non-empty subsemigroup of E and $\theta^{-1}(a_i^*) \cap \theta^{-1}(a_j^*) = \emptyset$ for $i \neq j$. Moreover

$$\theta^{-1}(a_i^*)\theta^{-1}(a_{i+1}^*) \subseteq \theta^{-1}(a_{i+1}^*) \quad (i = 0,\ 1,\ \ldots,\ m-1).$$

Indeed if $x \in \theta^{-1}(a_i^*)$ and $y \in \theta^{-1}(a_{i+1}^*)$, then $\theta(x) = a_i^*$ and $\theta(y) = a_{i+1}^*$, respectively. But

$$\theta(xy) = \theta(x)\theta(y) = a_i^* a_{i+1}^* = a_{i+1}^*$$

so that $xy \in \theta^{-1}(a_{i+1}^*)$.

Now $a \in \theta^{-1}(a_0^*)$. Choose $y_1 \in \theta^{-1}(a_1^*)$. Then $a_1 = ay_1 \in \theta^{-1}(a_0^*)\theta^{-1}(a_1^*) \subseteq \theta^{-1}(a_1^*)$. Consequently $a > a_1$. Choose $y_2 \in \theta^{-1}(a_2^*)$. Then $a_2 = a_1 y_2 \in \theta^{-1}(a_2^*)$ and $a > a_1 > a_2$. Continuing in this manner we obtain a chain $a > a_1 > a_2 > \ldots > a_{m-1}$, where $a_i \in \theta^{-1}(a_i^*)$. Now $z \in \theta^{-1}(a_m^*)$

and $z = a_{m-1}z$. Thus we have constructed a chain

$$a > a_1 > a_2 > \cdots > a_{m-1} > a_m = z$$

from a to z of length m. Since $d(a)$ is the length of the longest maximal chain, it follows $m \leq d(a)$ and we have proven the lemma.

Definition. An element $a \varepsilon E$ is called an atom if and only if a covers the zero element of E. (That is, $a >> z$.)

Note. An element $a \varepsilon E$ is an atom if and only if $d(a) = 1$. Below we introduce some very important subsets of E.

Definition. Let a be an arbitrary element of E. We denote the set of all elements x in E satisfying the inequality $x \leq a$ by $(a]$ and the set of all x in E satisfying $x \geq a$ by $[a)$.

Lemma 2.4 . Let a be an element of E. Then

(1) $[a)$ is a subsemigroup of E and

(2) $(a]$ is an ideal of E.

Proof. (1) Let $x, y \varepsilon [a)$. Then $x \geq a$ and $y \geq a$ so that $xy \geq a^2 = a$. Hence $xy \varepsilon [a)$ which shows $[a)$ is a subsemigroup of S.

(2) Let $e \varepsilon E$ and $x \varepsilon (a]$. Then $x \leq a$ and

$xe \leq x \leq a$ so it follows $xe \in (a]$. Thus $(a]$ is an ideal of E.

<u>Lemma 2.5</u> . Let A be the set of all atoms in the semilattice E. Then $B = A \cup z$ is an ideal of E. Moreover if λ is the natural homomorphism of E onto E/B, then $d(\lambda(a)) = d(a)-1$ for every $a \in E-B$.

<u>Proof</u>. Let $p \in A$. That is, let p be an atom in E. Then $(p] = \{z, p\}$ and, by Lemma 2.4, it is an ideal of E. However $B = \bigcup_{p \in A}(p]$ so it is also an ideal of E. Consequently we can consider the Rees factor semigroup of E modulo B. The ideal B is the zero element of E/B which we will denote by K_z. The remaining elements of E/B are one-element sets $\{x\}$ where $x \in E-B$. We denote these elements by $K_x = \{x\}$. Let λ be the natural homomorphism of E onto E/B. Then

$$\lambda(a) = \begin{cases} K_a = \{a\} & \text{if } a \in E-B \\ K_z & \text{if } a \in B . \end{cases}$$

Let $a \in E-B$. Then $\lambda(a) = K_a \in E/B$. By Lemma 2.3, $d(\lambda(a)) \leq d(a)$. Since a has $d(a) > 1$, then there is a maximal chain from a to z having length $d(a) = m$;

$$a = a_0 \gg a_1 \gg \ldots \gg a_{m-2} \gg a_{m-1} \gg a_m = z.$$

Now a_{m-1} is an atom of E so that $a_{m-1} \in B$. But $a_i \notin B$ for $i = 0, 1, \ldots, m-2$. Hence $K_a = \{a\}$,

$K_{a_1} = \{a_1\}, \ldots, K_{a_{m-2}} = \{a_{m-2}\}$ are non-zero elements in E/B and moreover

$$K_a = K_{a_0} \gg K_{a_1} \gg \ldots \gg K_{a_{m-2}} > K_z.$$

Indeed, if there exists $K \varepsilon E/B$ such that $K_{a_i} \geq K \geq K_{a_{i+1}}$ $(i = 0, 1, \ldots, m-3)$, then $K = \{x\}$ where $x \varepsilon E-B$ and

$$K_{a_i} \circ K = K \quad \text{and} \quad K \circ K_{a_{i+1}} = K_{a_{i+1}}.$$

Consequently $K_{a_i}K = \{a_i x\} \subseteq \{x\}$ and $KK_{a_{i+1}} = \{xa_{i+1}\} \subseteq \{a_{i+1}\}$. This implies $a_i x = x$ and $xa_{i+1} = a_{i+1}$ which yields $a_i \geq x \geq a_{i+1}$. Since $a_i \gg a_{i+1}$ we must have $a_i = x$ or $a_{i+1} = x$. Therefore $K = K_{a_i}$ or $K = K_{a_{i+1}}$ so it follows $K_{a_i} \gg K_{a_{i+1}}$ for $i = 0, 1, \ldots, m-3$. Therefore we have a chain from $\lambda(a) = K_a$ to $\lambda(z) = K_z$ in E/B of length $m-1 = d(a) - 1$. Hence $d(\lambda(a)) \geq d(a)-1$ and more precisely $d(a)-1 \leq d(\lambda(a)) \leq d(a) = m$.

Suppose $d(\lambda(a)) = d(a) = m$. Then we have a maximal chain in E/B from $\lambda(a) = K_a$ to K_z,

$$(5.1)\ \lambda(a) = K_a = K_0 \gg K_1 \gg \ldots \gg K_{m-1} \gg K_z$$

of length m. Since $K_i \neq K_z$ for $i = 0, 1, \ldots, m-1$ it follows $K_i = \{a_i\}$ where $a_i \varepsilon E-B$, $i = 0, 1, \ldots, m-1$. But (5.1) implies $a = a_0 \gg a_1 \gg \ldots \gg a_{m-1}$. Now $a_{m-1} \notin B$ and there exists an atom $a_m \varepsilon B$ such that $a_{m-1} > a_m$. Hence we have

$$(5.2)\ a = a_0 \gg a_1 \gg \ldots \gg a_{m-1} > a_m > z$$

which is a chain in E from a to z having length $m+1$.

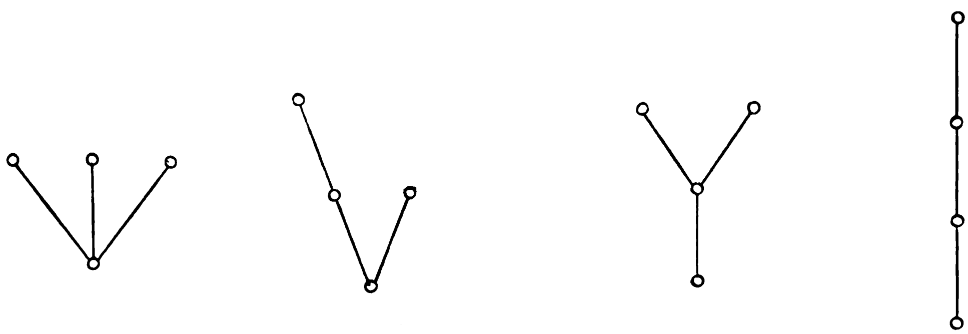
But this is impossible since the longest maximal chain from a to z has length $d(a) = m$. Therefore $d(\lambda(a)) \neq m$ so that $d(\lambda(a)) = d(a)-1$ and we have proven our lemma.

<u>Lemma 2.6</u> . Let M be a subsemigroup of E which contains the zero element z of E. Then the number of atoms in M is less than or equal to the number of atoms in E.

<u>Proof</u>. Since $z \varepsilon M \subseteq E$, it follows z is the zero element of M. Thus if p is an atom of M, then $p >> z$. Let n denote the number of atoms in M and let $a_1, a_2, \ldots, a_n$ be the distinct atoms in M. For each a_i, form the set $T(a_i, z)$, of all comparable elements $e \varepsilon E$ such that $a_i \geq e > z$. $T(a_i, z)$ is a non-empty chain. From each chain $T(a_i, z)$ choose a minimal element e_i; i.e. $e_i \leq e$ for every $e \varepsilon T(a_i, z)$. Then we have $a_i \geq e_i > z$ for $i = 1, 2, \ldots, n$. However the e_i $(i = 1, \ldots, n)$ are all distinct; for if $e_i = e_j$, then $a_i \geq e_i$ and $a_j \geq e_i$ which in turn implies $z = a_i a_j \geq e_i^2 = e_i$, a contradiction.

Thus each atom a_i in M $(i = 1, \ldots, n)$ gives rise to an atom e_i in E $(i = 1, \ldots, n)$ so that n is less than or equal to the number of atoms in E. This completes the proof of Lemma 2.6.

<u>Note</u>. It might help the reader, when reading through a proof involving finite semilattices, to actually represent the semilattice by a diagram. To obtain a diagram of a semilattice E (Every non-void finite partly-ordered set can be represented by a diagram, [Birkoff [9], Theorem 4]), represent each element a of E by a small circle C_a in the plane of the drawing (denoted by the letter a), such that if $a < b$ $(a, b \in E)$, the circle C_b is above C_a; now consider each pair of elements x, y in E for which $x \ll y$ and connect circles C_x, C_y representing such pairs of elements by segments. The resulting figure is the diagram of the set E with respect to the natural partial ordering on E. Below are four diagrams representing all (non-isomorphic) semilattices of order 4.



2.2 Finite Semilattices in the Class $\mathcal{H}$

We now begin to determine the structure of finite semilattices E which have property P and whose homomorphic images also have property P.

<u>Definition</u>. Let $\mathcal{H}$ be the collection of all finite commutative semigroups S such that

(1) if $S \varepsilon \mathcal{H}$, then S has property P and

(2) if $S \varepsilon \mathcal{H}$, then $\theta(S) \varepsilon \mathcal{H}$ for all homomorphisms θ on S.

Suppose E is a finite chain semilattice; that is, every pair of elements of E are comparable. Let θ be a homomorphism of E onto the finite semilattice E*. Recall that we use the same order relation used in E to denote the natural partial ordering of E*. If $x \leq y$, (x, y in E), then $\theta(x) = \theta(xy) = \theta(x)\theta(y)$ so that $\theta(x) \leq \theta(y)$. Therefore a homomorphism θ on E preserves the natural partial ordering in E. Hence, if E is a chain of length m, it follows E* is a chain of length at most m. Thus it follows finite chain semilattices are members of the class $\mathcal{H}$.

<u>Definition</u>. An element a of a semilattice E is said to be <u>reducible</u> if there exists in E elements a_1, a_2 such that

(1) $a = a_1 a_2 \quad (a_1, a_2 > a)$.

If some a has no decomposition at all of the form (1), it is said to be <u>irreducible</u>.

If the semilattice E has at least two atoms a_1 and a_2, then clearly its zero element z is reducible. Every non-zero element of a finite chain semilattice is irreducible. We now develop several necessary conditions for a semilattice E, which contains a reducible element a, to be in the class $\mathcal{H}$. If a is reducible in E, then there exist elements a_1, a_2 in E such that $a = a_1a_2$ with $a_1 > a$ and $a_2 > a$. Since E is finite we can find elements b and c in E such that $a_1 \geq b >> a$ and $a_2 \geq c >> a$. However $bc \leq a_1a_2 = a$ and $bc \geq a \cdot a = a$ so that $bc = a$. Consequently we shall study finite semilattices $E \in \mathcal{H}$ which contain an element a with $a = bc$ $(b >> a,\ c >> a)$.

<u>Lemma 2.7</u> . Let E be a finite semilattice with $E \in \mathcal{H}$. Let a, b, and c be elements of E such that a is an atom in E and $a = bc$ $(b >> a,\ c >> a)$. Then $p < b$ or $p < c$ for every atom p in E.

<u>Proof</u>. To show $p < b$ or $p < c$ for every atom p in E we need to show $pb = p$ or $pc = p$. Assume the contrary. That is, suppose there exist four elements a, b, c, p in $E \in \mathcal{H}$ such that

$$(7.1) \begin{cases} a \text{ and } p \text{ are atoms in } E, \\ a = bc \text{ with } b >> a \text{ and } c >> a, \text{ but} \\ pb \neq p \text{ and } pc \neq p \text{ .} \end{cases}$$

Clearly these elements are mutually distinct. Consider the maximal chains

$$b \gg a \gg z \quad \text{and} \quad c \gg a \gg z$$

and the ideals

$$(a] = \{z, a\}, \quad (b] = \{x \in E \mid x \leq b\} \quad \text{and}$$
$$(c] = \{x \in E \mid x \leq c\}.$$

The element p is not an element of either $(b]$ or $(c]$, since $bp \neq p$ and $cp \neq p$. Moreover z, a, b belong to $(b]$ and z, a, c belong to $(c]$. We need to study two separate cases.

Case 1. Suppose $(b] = \{z, a, b\}$ and $(c] = \{z, a, c\}$. Then $(b] \cap (c] = (a]$. Let q be the number of atoms in E. Then $2 \leq q \leq |E|-2$, since a and p are atoms in E but b and c are not atoms in E. Let us form the Rees factor semigroup $E/(a]$. We denote its zero element (namely the ideal $(a]$) by K_z and the remaining elements by $K_x = \{x\}$, where $x \in E-(a]$. The Rees factor semigroup $E/(a]$ is a finite semilattice with

$$|E/(a]| = |E|-1.$$

Since p, b and c are distinct elements of E not in $(a]$, it follows $K_p = \{p\}$, $K_b = \{b\}$ and $K_c = \{c\}$ are non-zero elements of $E/(a]$. We first will verify that K_b and K_c are atoms in $E/(a]$; that is, K_b covers K_z and K_c covers K_z. Suppose we have $K_b \geq K > K_z$ where $K \in E/(a]$. Then K is a non-zero element of $E/(a]$ so that $K = \{h\}$, where $h \in E-(a]$

and moreover $K_b \circ K = K$ and $K \circ K_z = K_z$. Consequently, $K_bK = \{bh\} \subseteq \{h\}$ which implies $bh = h$. Therefore $h \varepsilon (b] = \{z, a, b\}$ and since $h \neq a$ and $h \neq z$ (for h is not in $(a]$) we must have $h = b$. Indeed $K_b = K$ which proves K_b is an atom in $E/(a]$. The same type of argument can be applied on K_c.

Our next step is to show that each atom e in E $(e \neq a)$ gives rise to an atom K_e in $E/(a]$. Now e is not an element of $(a]$ so that $K_e = \{e\}$ is a non-zero element of $E/(a]$. Suppose $K_e \geq K > K_z$, where $K \varepsilon E/(a]$. Then $K = \{x\}$ where $x \varepsilon E-(a]$, and $K_e \circ K = K$, $K \circ K_z = K_z$. Consequently $ex = x$. Since $x \neq z$ we have $e \geq x > z$. But e covers x so that $e = x$. Therefore $K_e = K$ and we have K_e is an atom in $E/(a]$.

Our above arguments show that to each atom $e \varepsilon E$ $(e \neq a)$, there corresponds an atom K_e in $E/(a]$. Certainly, distinct atoms e in E correspond to distinct atoms K_e in $E/(a]$. Moreover, K_b and K_c are also distinct atoms in $E/(a]$ which are different from the above atoms K_e. If we let q^* denote the number of atoms in $E/(a]$, then our above remarks show that $q^* \geq q+1$, where q is the number of atoms in E.

Since $E \varepsilon \mathcal{H}$, there is a subsemigroup N of E isomorphic to $E/(a]$. Therefore N is a finite subsemilattice of E, with zero element $\underline{z}$, $|N| = |E|-1$

and the number of atoms in N is greater than or equal to $q+1$. Indeed $z = \underline{z}$; for we know $z \leq \underline{z}$ and if $z \notin N$ it follows a and p belong to N (for $|N| = |E|-1$) so that $z = ap \in N$, a contradiction. Therefore N is a subsemigroup of E, having z as its zero element, and the number of atoms of N is greater than or equal to $q+1$. From Lemma 2.6, this situation cannot occur and hence the situation in Case 1 is impossible.

Case 2. We consider the remaining possibility; namely $(b] \supset \{z, a, b\}$ and $(c] \supset \{z, a, c\}$. Let $Q(b) = \{x \in E \mid b \gg x \text{ and } x \neq a\}$ and $Q(c) = \{x \in E \mid c \gg x,\ x \neq a\}$. From $pb \neq p$ and $pc \neq p$ it follows p is not an element of either $Q(b)$ or $Q(c)$. Let

$$R(b) = \bigcup_{x \in Q(b)} (x] \quad \text{and} \quad R(c) = \bigcup_{x \in Q(c)} (x] .$$

Both $R(b)$ and $R(c)$ are unions of ideals so that each is itself an ideal. Let $W = R(b) \cup R(c)$. W is precisely the ideal of elements of E which are under the elements b and c with the exception of the element a. That is, if $y < b$, or $y < c$ and $y \neq a$, then $y \in W$. Also a, b, c and p are not members of W.

We form the Rees factor semigroup E/W. As before, K_z will denote its zero element and $K_y = \{y\}$, where $y \in E-W$, its remaining elements. We have $K_a = \{a\}$, $K_b = \{b\}$, $K_c = \{c\}$ and $K_p = \{p\}$ are nonzero elements of E/W. Moreover

$$K_c \circ K_b = K_a, \quad K_b \circ K_p \neq K_p, \quad K_p \circ K_c \neq K_c,$$

$K_b >> K_a$, $K_c >> K_a$ and K_a, K_p are atoms in E/W.

Form the following ideals in E/W;

$(K_b] = \{K \varepsilon E/W | K \leq K_b\}$ and $(K_c] = \{K \varepsilon E/W | K \leq K_c\}$.

Now $(K_b] \supseteq \{K_z, K_a, K_b\}$ and $(K_c] \supseteq \{K_z, K_a, K_c\}$. Suppose $K \varepsilon (K_b]$ and $K \neq K_z$. Then $K_z < K \leq K_b$ and $K = \{h\}$ with $h \varepsilon E-W$. But $K_b \geq K$ implies $hb = h$ and $z < h \leq b$. But $h \notin W$ and h being under b gives either $h = b$ or $h = a$. In either case $K = K_a$ or $K = K_b$ so that we have shown $(K_b] \subseteq \{K_z, K_a, K_b\}$. Therefore $(K_b] = \{K_z, K_a, K_b\}$ and using the same type of argument one can show $(K_c] = \{K_z, K_a, K_c\}$.

From $E \varepsilon \mathcal{H}$ and E/W being a homomorphic image of E, it follows $E/W \varepsilon \mathcal{H}$. Consequently, E/W is a finite semilattice in the class $\mathcal{H}$ which has four elements K_a, K_b, K_c, K_p which satisfy the same properties as those in (7.1) and moreover has the condition that $(K_b] = \{K_z, K_a, K_b\}$ and $(K_c] = \{K_z, K_a, K_c\}$; that is, the same conditions that E satisfied under Case 1. Hence we again have our desired contradiction. This completes the proof of Lemma 2.7.

The previous lemma stated a necessary condition for $E \varepsilon \mathcal{H}$ under the assumption that E contains an atom which is reducible. By applying this result and finite induction it is possible to obtain the same necessary condition for any reducible element in E. This is accomplished in the following theorem.

<u>Theorem 2.8</u> . Let E be a finite semilattice with $E \varepsilon \mathcal{H}$. Let a, b, c be elements of E such that $a = bc$, $b >> a$ and $c >> a$. Then $e < b$ or $e < c$ for every element $e \varepsilon E$ such that $0 \leq d(e) \leq d(a)$.

<u>Proof</u>. This theorem is proved by induction on the dimension of the reducible element a. Suppose $d(a) = 1$. If $d(e) = 0$ we have $e = z$, the zero element of E, and the theorem holds trivially for this case. If $d(e) = 1$, then e is an atom in E. Also a is an atom in E, since $d(a) = 1$. Consequently we can apply Lemma 2.7 to obtain our desired conclusion for the case $d(a) = 1$.

Assume the truth of the statement of the theorem for all finite semilattices in $\mathcal{H}$ which contain elements a, b, and c such that $a = bc$, $b >> a$, $c >> a$, and $1 \leq d(a) < n$.

Let E be a finite semilattice in $\mathcal{H}$ and suppose there exist elements a, b, c and e in E such that $a = bc$ $(b >> a$, $c >> a)$, $d(e) \leq d(a) = n$, $eb \neq e$ and $ec \neq e$.

We again need to consider two separate cases.

Case 1. Suppose $d(e) > 1$. Let A be the set of all atoms in E, let $B = A \cup z$ and form the Rees factor semigroup E/B. (This is valid for, by Lemma 2.5, B is an ideal of E.) Denote the zero element of E/B by K_z and its remaining elements by $K_y = \{y\}$ where $y \varepsilon E-B$. K_a, K_b, K_c and K_e are non-zero elements of E/B such that $K_a = K_b \circ K_c$ $(K_b >> K_a, K_c >> K_a)$, $K_b \circ K_e \neq K_e$ and $K_c \circ K_e \neq K_e$. Since $a, e \varepsilon E-B$, then, by Lemma 2.5, it follows $d(K_e) = d(e)-1$ and $d(K_a) = d(a)-1$. Therefore

$$d(K_e) = d(e)-1 \leq d(a)-1 = d(K_a)$$

and

$$d(K_a) = d(a)-1 < d(a) = n .$$

From $E \varepsilon \mathcal{H}$ we have $E/B \varepsilon \mathcal{H}$. Moreover E/B has three elements K_a, K_b, K_c which satisfy precisely the conditions of our induction hypothesis. Since $d(K_a) < n$ and $d(K_e) \leq d(K_a)$, it follows from our induction hypothesis that $K_e \circ K_b = K_e$ and $K_e \circ K_c = K_e$. This contradicts what we have above so that our assumption is false in the case $d(e) > 1$.

Case 2. Suppose $d(e) = 1$. Consequently we must have $d(a) > 1$; for otherwise we would contradict Lemma 2.7. Let A be the set of all atoms in E, $B = A \cup z$ and $R = B-\{e\}$. R is an ideal in E for

$$R = \bigcup_{x \in B-e} (x] .$$

Form the Rees factor semigroup E/R. Since a, b, c and e are not members of the ideal R, it follows $K_a = \{a\}$, $K_b = \{b\}$, $K_c = \{c\}$ and $K_e = \{e\}$ are non-zero elements of E/R. Moreover $K_a = K_b \circ K_c$, $K_b \gg K_a$, $K_c \gg K_a$, $K_e \circ K_b \neq K_e$ and $K_e \circ K_c \neq K_e$.

We first assert that $d(K_a) = d(a)-1 = n-1$. Recall we always have $d(K_a) \leq d(a) = n$. Since $d(a) > 1$, there exists a maximal chain from a to z

$$a = a_0 \gg a_1 \gg \cdots \gg a_{n-2} \gg a_{n-1} \gg a_n = z$$

of length n. From $be \neq e$ it follows $a_{n-1} \neq e$. But a_{n-1} is an atom in B; consequently $a_{n-1} \in R$. On the other hand, $a_i \notin R$ for $i = 0, 1, \ldots, n-2$. Therefore $K_a = \{a\}$, $K_{a_1} = \{a_1\}, \ldots, K_{a_{n-2}} = \{a_{n-2}\}$ are $(n-1)$ distinct non-zero elements of E/R and moreover

$$K_a \gg K_{a_1} \gg \cdots \gg K_{a_{n-2}} > K_z.$$

This implies we will have a maximal chain from K_a to K_z in E/R of length at least $(n-1)$. Hence $d(K_a) \geq n-1$. Suppose $d(K_a) = n$. Then there is a maximal chain from K_a to K_z in E/R,

$$K_a = K_{c_0} \gg K_{c_1} \gg \cdots \gg K_{c_{n-1}} \gg K_{c_n} = K_z ,$$

of length n. From $K_{c_i} \neq K_z$ for $i = 0, 1, \ldots, n-1$, it follows $K_{c_i} = \{c_i\}$ $(i = 0, 1, \ldots, n-1)$ where $c_i \in E-R$. The above chain induces a chain in E from a to z, namely

$$a = c_0 \gg c_1 \gg \cdots \gg c_{n-1} > z.$$

However there exists an atom $y \in R$ such that $c_{n-1} > y > z$. Consequently

$$a = c_0 \gg c_1 \gg \cdots \gg c_{n-1} > y > z.$$

Thus we will have a maximal chain from a to z of length at least $(n+1)$. This is not the case, for $d(a) = n$. Thus our assumption is false and $d(K_a) = n-1$.

Now $E/R \varepsilon \mathcal{H}$ and E/R has four elements K_a, K_b, K_c and K_e of the same nature as those in Case 1. Moreover $1 = d(e) \leq d(K_a) = d(a)-1 < n$. Our induction hypothesis applies and we obtain the same type of contradiction that occurred in Case 1.

The arguments used in Case 1 and Case 2 complete the proof of this theorem.

The previous theorem gives us a rather strong necessary condition for a finite semilattice E, admitting a non-zero reducible element a, to be in the class $\mathcal{H}$. Let a be a reducible element of E. We know that every element $e \varepsilon E$, $d(e) \leq d(a)$, is under one of the covers of a. We shall show that the number of incomparable elements $e \varepsilon E$, with $d(e) \leq d(a)$ and $e \| a$, can be at most one. Before doing this, we need some further properties.

<u>Lemma 2.9</u>. Let $E \varepsilon \mathcal{H}$ be a finite semilattice which has elements a, b, c such that a is an atom in E, $a = bc$ $(b \gg a, c \gg a)$. Let A be the set of atoms of E.

1) If $e \varepsilon A-a$, and $e \leq b$, then for every $x \geq e$ we have either $x \geq b$ or $x \leq b$.

2) If $e \varepsilon A-a$ and $e \leq c$, then for every $x \geq e$ we have either $x \geq c$ or $x \leq c$.

<u>Proof</u>. Suppose $e \varepsilon A-a$, $e \leq b$, $x \geq e$ but x is neither above b nor under b. First of all, x is not under c; for otherwise we have $e \leq x \leq c$ which contradicts that a is the atom in E under both b and c. Also x is not above c. Suppose $x \geq c$. Then $x \geq xb \geq e \circ e = e$ and $b \geq xb \geq e$. But $xb \neq z$, $xb \neq b$, and $xb \neq x$. Therefore $b > bx \geq bc = a$. Since $b >> a$, it follows $bx = a$ and consequently $a = bx \geq e > z$. This is not possible since a is an atom. Moreover x is neither above a nor under a. Indeed, suppose $a \leq x$. We have $a = ba \leq bx < b$ which implies $a = bx \geq e^2 = e$. This again is not possible.

Let $J = \{y \varepsilon (x] \mid y << x\}$ and consider the ideal $L = \bigcup_{y \in J} (y]$. First note that $L \subseteq (x]$ and b, c, a are not elements of $(x]$ (our above remarks verify this statement) and hence not elements of L. Also $x \notin L$.

Form the Rees factor semigroup E/L. It has $K_x = \{x\}$, $K_b = \{b\}$, $K_c = \{c\}$ and $K_a = \{a\}$ as non-zero elements. It is immediate that

$$K_b \circ K_c = K_a, \quad K_x \circ K_b = K_z, \quad K_x \circ K_c = K_z,$$

$$K_b >> K_a, \quad K_c >> K_a \quad \text{and } K_a \text{ is an atom in } E/L.$$

Moreover K_x is an atom in E/L since all elements under x belong to L.

Since $E/L \varepsilon \mathcal{H}$ and since there exist elements

K_a, K_b, $K_c \varepsilon E/L$ with $K_a = K_b \circ K_c$, $K_b >> K_a$, $K_c >> K_a$ and K_a an atom in E/L, it follows Lemma 2.7 applies; namely every atom in E/L is under K_b or K_c. However K_x is an atom of E/L which does not have this property.

We have therefore shown if $e \leq b$, $e \leq x$ then $x \geq b$ or $x \leq b$. A similar argument can be applied to show part (2) of this lemma.

Lemma 2.10. Let $E \varepsilon \mathfrak{H}$ be a finite semilattice which has elements a, b, c where a is an atom in E, $a = bc$ $(b >> a, c >> a)$. Then a is the only atom in E which admits such a decomposition.

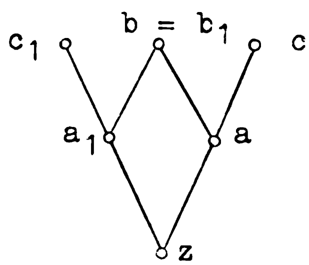
Proof. Suppose $a_1 \varepsilon E$ is an atom different from a which also has such a decomposition. That is, there exist elements b_1, c_1 in E with $a_1 = b_1 c_1$ $(b_1 >> a_1$, $c_1 >> a_1)$. We can apply Lemma 2.7 on both elements a and a_1. Consequently, either $a_1 \leq b$ or $a_1 \leq c$ and either $a \leq b_1$ or $a \leq c_1$. We have four separate cases to consider. (1) $a \leq b_1$ and $a_1 \leq b$, (2) $a \leq b_1$ and $a_1 \leq c$, (3) $a \leq c_1$ and $a_1 \leq b$, and (4) $a \leq c_1$ and $a_1 \leq c$. In case (1) we have $b \geq bb_1 \geq a \cdot a = a$ and $b_1 \geq bb_1 \geq a_1 a_1 = a_1$. But $b >> a$ and $b_1 >> a_1$ so in order that both the above inequalities hold simultaneously we must have $bb_1 = b = b_1$.

Using the same arguments for the remaining three cases, one can show that these cases yield the results

$$cb_1 = c = b_1, \quad bc_1 = b = c_1 \text{ and } \quad cc_1 = c = c_1,$$

respectively.

In any event, we must have the two sets $\{b, c\}$ and $\{b_1, c_1\}$ have an element in common. It suffices to assume that $b = b_1$. We have the situation $b \gg a$, $c \gg a$, $b \gg a_1$, $c_1 \gg a_1$, a and a_1 atoms in E such that $a = bc$ and $a_1 = bc_1$. By Lemma 2.9, since a_1 is an atom, $a_1 \in A-a$, $a_1 \leq b$ and $a_1 \leq c_1$, it follows $c_1 \geq b$ or $c_1 \leq b$. However both situations are not possible since c_1 and b cover a_1. (Of course, unless $c_1 = b$ which in turn is not possible from $c_1 b = a_1$).



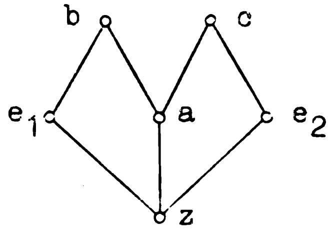
This ends the proof of our lemma.

<u>Lemma 2.11</u>. Let $E \in \mathcal{H}$ be a finite semilattice which has elements a, b, c such that a is an atom in E, $a = bc$ $(b \gg a, c \gg a)$. Let A be the set of atoms of E and $N = A-a$. Then either N is an empty set or $|N| = 1$.

<u>Proof</u>. Suppose $|N| \geq 2$. Note that from $N \neq \emptyset$ we can apply Lemma 2.7 to conclude that b and c are the only elements of E which cover a. Also, by Lemma 2.7, we know that each element of N is under either b or c. We therefore have two separate cases to consider.

Case 1. Suppose there exist elements $e_1, e_2 \in N$

such that $e_1 \leq b$ and $e_2 \leq c$. Form the Rees factor



semigroup $E/(a]$. It has $K_b = \{b\}$, $K_c = \{c\}$ as non-zero elements. We first show that there is a one-to-one correspondence between the atoms in N and the atoms of $E/(a]$.

Suppose e is an atom of E with $e \varepsilon N$. Then $K_e = \{e\}$ is a non-zero element of $E/(a]$. Suppose there is an element $K \varepsilon E/(a]$ for which $K_e \geq K > K_z$. Then $K = \{x\}$ where $x \varepsilon E-(a]$ and $K_e \circ K = K$. This implies $ex = x$ so that $e \geq x > z$. Since e is an atom, it follows $e = x$ which in turn implies $K_e = K$. Thus K_e is an atom in $E-(a]$. Note that this implies K_b and K_c are non-zero elements of $E/(a]$ which are not atoms.

Suppose $K_e = \{e\}$ is an atom in $E/(a]$, where $e \varepsilon E-(a]$. Let x be an element of E such that $e \geq x > z$. If $x = a$, then $e > a > z$. Consequently, $e \geq b$ or $e \geq c$ since b and c are the only elements which cover a. Therefore $e \geq b \geq a$ or $e \geq c \geq a$ and this, in turn, implies $K_e \geq K_b$ or $K_e \geq K_c$. We know K_b and K_c are not atoms in $E/(a]$ so we contradict the fact that K_e is an atom in $E/(a]$. Hence $x \neq a$. But $x \neq a$ and $e \geq x > z$ implies $x \notin (a]$. Therefore $K_x = \{x\}$ is a non-zero element of $E/(a]$ and $K_e \geq K_x > K_z$.

Since K_e is an atom, it follows $e = x$. Thus e is an atom of E.

In the above two paragraphs we have shown there is a one-to-one correspondence between the atoms e of E, $e \in N$, and the atoms K_e of $E/(a]$. Therefore the number of atoms in $E/(a]$ is $|N|$.

We next assert that $E/(a]$ has no atom $K_{a*} = \{a*\}$ for which there exist elements $K_{b*} = \{b*\}$, $K_{c*} = \{c*\}$ such that $K_{a*} = K_{b*} \circ K_{c*}$ $(K_{b*} \gg K_{a*}, K_{c*} \gg K_{a*})$; for otherwise $a*$ would be an atom in E, $b* \gg a*$, $c* \gg a*$ and $a* = b*c*$. From Lemma 2.10 we must have $a = a*$ and hence $b = b*$, $c = c*$. However $a = a*$ is impossible, since $a* \notin (a]$. This proves our desired assertion.

From $E \in \mathcal{H}$ we must have $E/(a]$ can be imbedded in E. Therefore there exists a sub-semilattice E_1 of E which satisfies

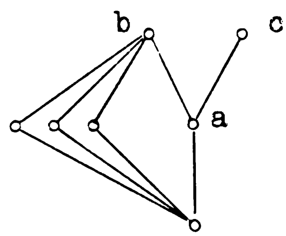
(11.1) $|E_1| = |E|-1$,

(11.2) number of atoms in E_1 is $|N|$, and

(11.3) there exists no atom $f \in E_1$ and elements g, h in E_1 such that $f = gh$, $h \gg f$ and $g \gg f$.

Suppose there is an atom $p \in E$ and $p \notin E_1$. From $|E_1| = |E|-1$, $b \neq p$, $c \neq p$, it follows $b, c \in E_1$. But E_1 is a subsemigroup of E so that $a = bc \in E_1$. This cannot occur for we would contradict property (11.3). Thus every atom of E is an element (and, hence, an atom) in E_1. However the number of atoms in E is $|N|+1$ (since $N = A-a$) which contradicts property (11.2). This completes the proof for this case.

Case 2. Suppose all atoms $e \in N$ are under the element b. Then the only atom in E under c is the atom a and every atom of E is under b.



Form the Rees factor semigroup $E/(a]$. Again $K_b = \{b\}$, $K_c = \{c\}$ are non-zero elements of $E/(a]$. It is necessary to show that K_c is an atom in $E/(a]$. Suppose $K_c \geq K_x > K_z$, where $K_x = \{x\}$ with $x \in E-(a]$. But $K_c \geq K_x$ implies $c \geq x > z$. Since c covers a and no other element $(x \neq a)$ it follows $c = x$. Therefore $K_x = K_c$ and we have shown K_c is an atom in $E/(a]$.

Using arguments exactly like the one employed in Case 1, one can show that there is a one-to-one correspondence between the atoms in N and the atoms in $E/(a]$. (Clearly K_b is not an atom in $E/(a]$). Hence the number of atoms of $E/(a]$ is equal to $|N|+1$ which is the number of atoms in E.

As in Case 1, we can apply Lemma 2.10 to assure us that no atom K_{a*} in $E/(a]$ has a reducible decomposition.

Let K_e be an atom in $E/(a]$ such that $K_e \neq K_c$. Then e is an atom of E, $e \neq a$. Hence $e \leq b$ which in turn implies $K_e \leq K_b$. Hence every atom of $E/(a]$, with the exception of K_c, is under the element K_b. It is clear that $d(K_b) = d(b) \geq 2$.

Since $E \in \mathcal{H}$, then $E/(a]$ can be imbedded in E. Therefore there exists a sub-semilattice E_1 of E such that

(11.4) $|E_1| = |E|-1$,

(11.5) number of atoms in E_1 is precisely the number of atoms in E,

(11.6) there exists an atom $e_0 \in E_1$ and an element $e_1 \in E_1$, where $d(e_1) = d(b)$, such that every atom $p \in E_1$, $p \neq e_0$, satisfies $p < e_1$ but $e_0 \not< e_1$, and

(11.7) there is no atom $p \in E_1$ which admits a decomposition of the form $p = x_1 x_2$ $(x_1 \gg p,\ x_2 \gg p)$.

Let p be an atom of E with $p \notin E_1$. Then, since $|E_1| = |E|-1$, $b \neq p$, $c \neq p$, it follows b, c are elements in E_1. Therefore $a = bc \in E_1$ which is not possible by property (11.7). Hence every atom of E is an atom of E_1, and since they are equal in number it follows all the atoms of E_1 are precisely the atoms of E. From $|N| \geq 2$, there is an atom $p \in E$ (and thus in E_1) such that $p \neq a$ and $p \neq e_0$. From (11.6), $p \leq e_1$. Also, by hypothesis, we have $p \leq b$. Lemma 2.9 applies to give us $e_1 \geq b$ or $e_1 \leq b$. But $d(e_1) = d(b)$ and so

$b = e_1$. However b has the property that every atom of E (and hence E_1) is under b. This contradicts that there is an atom $e_0 \varepsilon E_1$ such that $e_0 \not\leq e_1 = b$. This completes the proof of the lemma.

The previous lemma essentially states if we have a finite semilattice E in the class $\mathcal{H}$ which has a reducible atom a, then E will have at most one other atom p, $p \neq a$. The next theorem will give a similar conclusion removing the hypothesis that a is an atom. We will show that if a is reducible, then the number of incomparable elements $e \varepsilon E$ with $d(e) \leq d(a)$, $e \| a$, is at most one.

<u>Theorem 2.12</u>. Let $E \varepsilon \mathcal{H}$ be a finite semilattice which has elements a, b, c such that $a = bc$, $b \gg a$ and $c \gg a$. Let M be the set of incomparable elements $e \varepsilon E$ such that $e \| a$ and $d(e) \leq d(a)$. Then either M is an empty set or $|M| = 1$.

<u>Proof</u>. We prove this theorem by induction on the dimension of the reducible element a. Lemma 2.11 shows that the theorem is true for $d(a) = 1$.

Assume the truth of the statement of the theorem for all finite semilattices in $\mathcal{H}$ which contain elements a, b, and c having the above properties with $d(a) < n$. Let E be a finite semilattice in $\mathcal{H}$, let a, b, c be elements in E such that $a = bc$ $(b >> a,\ c >> a,\ d(a) = n)$ and let M be the set of incomparable elements $e \varepsilon E$ $(e \| a,\ d(e) \leq d(a) = n)$ such that $|M| \geq 2$. Consequently, M contains at least two elements e_1 and e_2 such that e_1, e_2, a are mutually incomparable and $d(e_1) \leq d(a)$, $d(e_2) \leq d(a)$.

We need to consider three separate cases.

Case 1. Suppose $d(e_1) > 1$ and $d(e_2) > 1$. Let A be the set of atoms in E, let $B = A \cup z$ and form the Rees factor semigroup E/B. We apply exactly the same arguments used in the proof of Theorem 2.8 (Case 1) to show that this case is impossible.

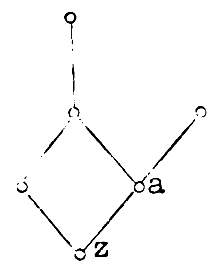
Case 2. Suppose $d(e_1) = 1$ and $d(e_2) = 1$. Then $d(a) > 1$; for otherwise we would contradict Lemma 2.11. Let A be the set of atoms in E, $B = A \cup z$ and $R = B - \{e_1, e_2\}$. R is an ideal in E for

$$R = \bigcup_{x \in R} (x] .$$

Again we can apply the same arguments that were used in the proof of Theorem 2.8 (Case 2) to show that this situation cannot occur.

Case 3. Suppose $d(e_1) = 1$ and $d(e_2) > 1$. Since $d(a) \geq d(e_2)$, it follows $d(a) > 1$. Let A be the set of all atoms in E, $B = A \cup z$ and $R = B - e_1$. We can form the Rees factor semigroup E/R and apply arguments similar to the one in Theorem 2.8 (Case 2) to show that this case also is not possible.

In this section we have developed two rather strong necessary conditions (namely, the statements of Theorem 2.8 and Theorem 2.12) when a finite semilattice E, having a reducible element, will be a member of the class $\mathcal{H}$. However we have not been able to completely classify the finite semilattices in the class $\mathcal{H}$. Below is a diagram of a semilattice which satisfies our necessary conditions but does not belong to $\mathcal{H}$.



2.3 Finite Semilattices of Groups in $\mathcal{H}$

This section will deal briefly with the study of finite commutative semigroups $S \in \mathcal{H}$ whose maximal unipotent subsemigroups are groups. That is, if we let E denote its set of idempotents (E is a finite semilattice), G_e the maximal unipotent subsemigroup of S containing the idempotent $e \in E$ but no other, then

$$S = \bigcup_{e \in E} G_e, \quad G_e \cap G_f = \emptyset \quad \text{if} \quad e \neq f \quad \text{in} \quad E,$$

$$G_e G_f \subseteq G_{ef} \quad \text{for all } e,\ f \quad \text{in} \quad E,$$

and G_e is a group for all $e \in E$. For such semigroups S, we shall use the abbreviated expression, "S is a semilattice of groups". For the case when E is a chain, we will say that S is a <u>chain semilattice of groups</u>.

The purpose of this section is to determine necessary conditions when a finite semigroup S, which is a semilattice of groups, will be in the class $\mathcal{H}$.

Throughout this section we will use the following notation: S will denote a finite semilattice of groups, $E = \{e_1, e_2, \ldots, e_n\}$ its set of idempotents and G_m, $(m = 1, 2, \ldots, n)$, the group having identity element e_m. Then $S = \bigcup_{i=1}^{n} G_i$, $G_i G_j \subseteq G_k$ for $e_i e_j = e_k$ and $G_i \cap G_j = \emptyset$.

<u>Lemma 2.13</u>. Let $S \in \mathcal{H}$ be a finite semilattice of groups, and e_1 the zero element of E. Then

$$gG_i = \{g\}$$

for all $g \in G_1$ and $i = 2, 3, \ldots, n$.

<u>Proof</u>. First of all, for $e_i < e_j$ we must have $G_i G_j = G_i$. Let e_1 be the zero element of the finite semilattice $E = \{e_1, e_2, \ldots, e_n\}$. From

$$G_1 S = G_1\left(\bigcup_{i=1}^{n} G_i\right) = \bigcup_{i=1}^{n} G_1 G_i \subseteq \bigcup_{i=1}^{n} G_1 = G_1 ,$$

it follows G_1 is an ideal of S. Therefore, we are able to form the Rees factor semigroup S/G_1. Denote its zero element (the ideal G_1) by K_{e_1} and its remaining elements by $K_x = \{x\}$ where $x \in S - G_1$. Note that

$$K_{e_2} = \{e_2\}, \ K_{e_3} = \{e_3\}, \ \ldots, \ K_{e_n} = \{e_n\}$$

are idempotents in S/G_1. Moreover, these and the idempotent K_{e_1} are precisely all the idempotents of S/G_1. Consequently, the number of idempotents in S/G_1 is $|E| = n$.

Since $S \varepsilon \mathcal{H}$ it follows S/G_1 can be imbedded in S. Hence there exists a subsemigroup T of S such that (1) $|T| = |S/G_1| = |S|-|G_1|+1$, (2) T has a zero element and (3) T contains $|E| = n$ idempotents.

From (3), we must have $E \subseteq T$ and in particular $e_1 \varepsilon T$. Since e_1 is the zero element of E and since T has a zero element, it follows e_1 is the zero element for T. However $G_1 \cap T = e_1$; for e_1 is the identity element of G_1. But $T \subseteq S$, $T \cap G_1 = e_1$ and $G_1 \cap G_j = \emptyset$ for $j = 2, \ldots, n$, all imply that $T \subseteq (S-G_1) \cup e_1$. On the other hand, $|T| = |S|-|G|+1$ $= |(S-G_1) \cup e_1|$, which implies $T = (S-G_1) \cup e_1$. That is, $T = \bigcup_{i=2}^{n} G_i \cup e_1$.

Since e_1 is the zero element of T we have $e_1 g_m = e_1$ for all $g_m \varepsilon G_m$, $m = 2, \ldots, n$. Let $g_1 \neq e_1$ be an arbitrary element of G_1, and g_m an arbitrary element of G_m $(m = 2, \ldots, n)$. Then

$$g_1 g_m = (g_1 e_1) g_m = g_1(e_1 g_m) = g_1 e_1 = g_1 \ .$$

This proves the lemma.

<u>Theorem 2.14</u>. Let $S \varepsilon \mathcal{H}$ be a finite chain semilattice of groups. Then

$$g_i g_j = g_i$$

for all $g_i \in G_i$, $g_j \in G_j$ where $e_i < e_j$, $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n$.

Proof. It suffices to show that $e_i g_j = e_i$ for all $g_j \in G_j$ with $e_i < e_j$; for if this is true, then

$$g_i g_j = (g_i e_i) g_j = g_i (e_i g_j) = g_i e_i = g_i.$$

Because S is a chain semilattice of groups, we have $E = \{e_1 << \ldots << e_n\}$.

From Lemma 2.13, we have $e_1 g_j = e_1$ for all $g_j \in G_j$ and $e_1 < e_j$. Suppose $e_i g_j = e_i$ for all $g_j \in G_j$, $e_i < e_j$ where $i = 1, 2, 3, \ldots, m-1$. We show $e_m g_j = e_m$ for all $g_j \in G_j$ where $e_m < e_j$.

Define $a \rho b$ if and only if $a = b$ or $a, b \in G_m$. ρ is an equivalence relation and moreover we claim it is a congruence on S. Let $a \rho b$, $a \neq b$, and let $x \in S$. Then $x \in G_k$, for some integer k; (a) If $e_k < e_m$, then, from our assumption, we have $e_k a = e_k$ and $e_k b = e_k$. Therefore

$$xa = (xe_k)a = x(e_k a) = xe_k = x .$$

Likewise $xb = x$. Consequently, $xa \rho xb$. (b) If $e_k \geq e_m$, then $G_k G_m \subseteq G_m$ so that xa and xb both belong to G_m. Thus $xa \rho xb$.

Let $S^* = S/\rho$. Its elements are the group G_m itself and all one-element sets $\{x\}$ where $x \in S - G_m$. We shall denote the one-element sets $\{x\}$ by x^* and the equivalence class G_m by e_m^*. Define

$$E^* = \{e_1^*, e_2^*, \ldots, e_n^*\} ,$$
$$G_i^* = \{x^* \varepsilon S^* | x \varepsilon G_i, \ i \neq m\} ,$$
$$G_m^* = \{e_m^*\} .$$

E^* is the set of idempotents of S^*. From $e_i^* = \{e_i\}$ for all $i \neq m$, it follows $e_1^* << e_2^* << \ldots << e_{m-1}^*$. However $e_m^* \circ e_j^* = e_m^*$ for $j > m$ since $e_j G_m \subseteq G_m$. This gives $e_1^* << e_2^* << \ldots << e_m^* << \ldots << e_n^*$.

The map $\tau : G_i \longrightarrow G_i^*$ $(i \neq k)$ defined by $\tau(x) = x^*$ is clearly an isomorphism of G_i onto G_i^*. Therefore G_i^*, $(i = 1, 2, \ldots, n)$, are the maximal groups of S^* containing the idempotents e_i^*. $(G_m^* = \{e_m^*\}.)$ Also $G_i^* \cap G_j^* = \emptyset$ $(i \neq j)$ since $G_i \cap G_j = \emptyset$ $(i \neq j)$.

Evidently the semigroup S^* has the decomposition, $S^* = \bigcup_{i=1}^{n} G_i^*$, $G_i^* \cap G_j^* = \emptyset$ and $G_i^* \circ G_j^* \subseteq G_k^*$ where $e_i^* \circ e_j^* = e_k^*$.

Let $g_j^* \varepsilon G_j^*$ and let e_i^* be the identity element of G_i^* where $e_i^* < e_j^*$. For $i = 1, 2, \ldots, m-1$, we have $e_i^* = \{e_i\}$; consequently, by our assumption, $e_i^* \circ g_j^* = e_i^*$. Since $G_m g_j \subseteq G_m$, it follows $e_m^* \circ g_j^* = e_m^*$ for all $g_j^* \varepsilon G_j^*$ where $e_m^* < e_j^*$. The above shows that

$$e_i^* \circ g_j^* = e_i^* \text{ for all } g_j^* \varepsilon G_j, \quad e_i^* < e_j^*$$
$$\text{where } i = 1, 2, \ldots, m.$$

Since $S \varepsilon \mathcal{H}$ we can imbed S^* in S. Hence there exists a subsemigroup H of S which itself is a finite chain semilattice of groups satisfying, (14.1) $H = \bigcup_{i=1}^{n} H_i$, $H_i \cap H_j = \emptyset$ $(i \neq j)$, where H_i is maximal group of H containing the idempotent f_i,

(14.2) $f_1 << f_2 << \ldots << f_n$ where the f_i are all the idempotents of H ,

(14.3) $|H_i| = |G_i|$ for $i \neq m$ and $|G_m| = 1$, and

(14.4) $f_i h_j = f_i$ for all $h_j \varepsilon H_j$, $f_i < f_j$, $i = 1, 2, \ldots, m$.

Now $E = \{e_1, e_2, \ldots, e_n\}$ and $f_i \varepsilon E$ for all i. Therefore $E = \{f_1, f_2, \ldots, f_n\}$. But (14.2) gives $f_i = e_i$ for $i = 1, 2, \ldots, n$. Then $e_i \varepsilon H_i \cap G_i$ for all i, so it follows $H_i \subseteq G_i$. From (14.3), we have $H_i = G_i$ for $i \neq m$; of course $H_m = e_m$. Then, using (14.4), we conclude

$$e_i g_j = e_i \text{ for all } g_j \varepsilon G_j, \; e_i < e_j, \; i = 1, 2, \ldots, m.$$

This completes the proof of the theorem.

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