THE PROBLEMS OF THE INFINITE AND THE CONTINUUM IN SOME MAJOR PHILOSOPHICAL SYSTEMS OF THE ENLIGHTENMENT

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ABSTRACT

THE PROBLEMS OF THE INFINITE AND THE CONTINUUM IN SOME MAJOR PHILOSOPHICAL SYSTEMS OF THE ENLIGHTENMENT

by Rolf A. George

The philosophers discussed in this dissertation are Leibniz, Berkeley, Bayle, Kant, and Bolzano. Its aim is to show that certain difficulties connected with infinite and continuous sets were recognized by these philosophers, and that their systems were, at least in part, designed in such a way that these difficulties did not arise in them. Notably the so-called Paradox of Galileo played a major role in this respect: Galileo had shown that a one-to-one correspondence can be established between integers and their squares, and Leibniz realized that it is a property of all infinite sets that they have subsets the members of which can be brought in such a biunivocal correspondence with the members of the original set. Up to Bolzano, this was held to contradict the "wholly reliable" Euclidean axiom that the whole is bigger than its part, and was used to prove that infinite sets are impossible. Berkeley, for one, was aware of precisely this problem when he developed a metaphysic in which infinite sets do not occur.

Leibniz' solution consisted in the following: He assumed that the number of monads constituting or "giving rise" to any given finite body is always larger than any finite number, but that it is not permissible to speak of these monads as forming a set. When we nevertheless speak of a given body as forming one thing, we are taking a liberty that is excusable in everyday discourse only for reasons of its pragmatic efficacy. Such statements cannot be tolerated in a language that endeavors an ultimately reliable description of the universe. For, if a given body were an infinite set of parts, then it would have contradictory properties, so Leibniz believed.

Kant asserted that a phenomenal object does not have an infinite number of parts already in it. Its being extended is the result of the form of outer sense. Hence it cannot be said that it has more parts than this sense distinguishes in it. However, an operative decomposition of such an object can be carried out, and a rule of reason guarantees that this decomposition has no final stage. But this is not the same as to say that the object is a set of infinitely many members, or a whole with infinitely many parts, and this distinction supposedly forestalls the arising of paradox.

Bolzano was the first to realize that the so-called

Paradox of Galileo is no paradox at all, but simply describes a common property of all infinite sets.

As concerns the constitution of continua the problem was that neither the assumption that a continuum ultimately consists of unextended parts, nor that it consists of extended parts seemed defensible. Against the former case it was argued that unextended parts, no matter how many, cannot make a finite extension, against the latter that extended parts are not ultimate, but are further divisible. Bayle held that none of the logical alternatives are defensible, so that no one need bother to change whatever opinion he happens to have on the subject.

Berkeley argued that there is no extension in objects, but only extension as perceived, and a phenomenal object cannot be said to have parts smaller than a _minimum sensibile_. Hence, any continuous shape will consist of a finite number of smallest particles, and the difficulty disappears. Berkeley held this to be one of the most important consequences of the "immaterial hypothesis". Kant's solution has already been sketched in connection with the Paradox of Galileo.

Leibniz' position was that there is no actual continuum, but that any continuum is an "intellectual construct". It must be analyzed as a concept, i.e. into simpler concepts, not into smaller parts. A part of a given continuous entity is considered to be a more complicated construct than the whole of which it is a part, since it must be described by reference to that whole. Thus continua are said to be "prior" to their parts. Leibniz consequently held that the quest for the ultimate spatial parts of continua is pointless.

Bolzano declared that in a continuum every point has a neighbor within any distance, no matter how small. This definition, although ultimately unsatisfactory, proved to be of great help in discovering various important properties of continuous sets.

The purpose of this dissertation is not to sketch the evolution of thought on the subjects of infinite and continuous sets, but to show how the problems connected with them were no less important in the development of Enlightenment philosophy than the epistemological predicaments customarily discussed in histories of philosophy.

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by

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R. A. G.

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INTRODUCTION

In 1662 Arnauld wrote the following about the problems of the infinite in his Port Royal Logic.[1]

> The most compendious way to the full extent of knowledge is not to toil ourselves in the search of that which is above us, and which we can never rationally expect to comprehend. Such are those questions that relate to the Omnipotency of God, which it would be ridiculous to confine within the narrow limits of our Understandings; and generally, as to whatever partakes of Infinity, and lies overwhelmed under the multitude of thoughts, contradicting one another.
>
> Hence may be drawn the most convenient and shortest solution of many questions, about which there will be no end of disputing, so long as Men are infected with the Itch of dispute, in regard they can never be able to arrive at any certain knowledge, whereby to assure and fix the Understanding. Is it possible for God to make a Body infinite in quantity, a movement infinite in swiftness, a multitude infinite in number? Is a number infinite even or odd? Is one infinite more extensive than another? He that should answer once for all, I know nothing of it, may be said to have made as fair a Progress in a moment, as he that had

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[1] (Antoine Arnauld) <u>Logic or the Art of Thinking</u>, Part 4, Ch. 1. (The quotation is taken from the 2nd edition of the English translation, London 1693, p. 390 F.)

> been beating his Brains twenty years, about these Niceties. The only difference between these Persons is, that he that drudges day and night about these Questions, is in the greatest danger of falling a degree lower than bare Ignorance; which is, to believe he knows that which he knows not at all.

Contrary to the spirit of this quotation. Arnauld then proceeds to give various demonstrations for the infinite divisibility of matter, which, following Descartes, he considers to be continuous.

Before Arnauld. Descartes had similarly proclaimed that there is not much point to an investigation of the infinite division of matter, since it must be considered to transcend our finite understanding,[2] and much the same reasons were advanced by Galileo when he cautioned "let us remember that we are dealing with infinites and indivisibles, both of which transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness."[3]

These quotations express a sentiment apparently quite widespread around the middle of the seventeenth

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[2]Descartes. Les Principes de la Philosophie. Part II. No. 35, Oeuvres. Ed. Victor Cousin, Paris 1824, vol. 3. p. 150.

[3]Galileo Galilei, Dialogues Concerning Two New Sciences, Ed. Henry Crew and Alfonso de Salvio. New York. 1914, p. 26.

century: a general resignation before the problems of infinite sets and the logic of the continuum. But this attitude changed very rapidly: a mere half century after the Port Royal Logic, Collier remarks scornfully about the passages in that work which pertain to our subject that it is indeed a sign "that our understandings are very weak and shallow, when such stuff as this shall not only pass for common sense, but even look like argument."[4] At the beginning of the eighteenth century, the appeal to the finitude of our understanding in discussing the apparently incomprehensible properties of infinite sets was no longer considered a philosophically tenable position. A vigorous attack upon these problems had been initiated by Leibniz, Bayle, Berkeley, Collier and others, and the belief that the difficulties of the continuum and the infinite must be capable of a rational solution, once gained, was never relinquished until such a solution finally was accomplished around the middle of the last century.

The present study is devoted to a discussion of some attempts on the part of various philosophers to find a solution to the problems of continuity and the infinite.

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[4]Arthur Collier, Clavis Universalis, Ed. Ethel Bowman, Chicago, 1909, p. 74.

In particular, Leibniz, Bayle, Berkeley, Collier, Kant and Bolzano will be discussed. I wish to establish the thesis that the philosophical systems developed by these philosophers had as one of their primary objectives the resolution of the indicated problems; and that this concern can, accordingly, be said to be largely responsible for the particular character which those philosophical systems took on.

While we have many historical accounts of mathematicians dealing with these and kindred problems, especially problems in analysis, hardly any investigations have been made into the role and importance of the problems of the infinite and the continuum for the nature and structure of metaphysical and epistemological systems in the seventeenth and eighteenth century. This is all the more surprising as the philosophical literature of that age contains so many references to these difficulties, so many proposals for their resolution that the sheer frequency of these remarks would suffice to legitimate them as among the persistent problems of philosophy in those centuries. A further fact which should have drawn the attention of historians to these philosophical labors is that the mathematicians of the seventeenth and eighteenth century tended to regard the problems of the continuum as "pre-mathematical": they were to be resolved through metaphysical speculation, much in the same way in which Plato wanted to establish the truth of the axioms of geometry through dialectics. This sentiment is described by Boyer in his Concepts of the Calculus:

> The attitude of most of the mathematicians of the seventeenth century...was that of doubt. They employed infinitesimals and the infinite on the assumption that they existed, and treated the continuous as though made up of indivisibles, the results being justified pragmatically by their consistency with Euclidean geometry. In any case the attitude was not that of unprejudiced postulation and definition, followed by logical deduction.[5]

One might add that the situation did not appreciably change during the eighteenth century, and only in the nineteenth, commencing with Bolzano, were the fundamental problems of analysis, of the continuum and of infinity attacked in a new spirit within the field of mathematics itself. Before that time, problems in the foundation of mathematics were relegated to the metaphysician. But it was not only the case that the mathematicians were hesitant to attack the foundation-problems of their science, and let their results justify their assumptions; the philosophers of the age felt

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[5]Carl B. Boyer, The Concepts of the Calculus, (New York), 1949, p. 305.

it incumbent upon themselves to supply what the mathematicians avoided. Kant speaks of having to bring about a reconciliation of geometry and metaphysics; Berkeley urges the mathematicians to join metaphysics to their mathematics; and Leibniz admonished the geometricians to leave well alone the problems of the continuum:"...the geometer does not need to encumber his mind with the famous puzzle of the composition of the continuum..."[6]

In a sense, the philosophy of mathematics was not only thought to have to describe mathematics, but also to have to establish its first principles. Of course, the notion that the latter should be the task of philosophy could arise only if the former was not properly performed In a way, all the philosophical investigations that were undertaken in this direction stem from misconceptions concerning the nature of mathematics. The nature of these misconceptions, their genesis and consequences embodied in

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[6]cf. George Berkeley. Of Infinites. The Works of George Berkeley, ed. A. A. Luce and T. E. Jessup, London 1948 ff., Vol. 4, pp. 235-238, p. 238. In the sequel, Berkeley will be quoted after this edition, which will be referred to simply as The Works of George Berkeley. G. W. Leibniz, Discourse on Metaphysics, section X, in Leibniz, Selections ed Philip P. Wiener, New York 1951. (In the sequel quoted as Wiener), p. 303.

these philosophical labors constitute the subject of the present discussion.

The methods of approaching the problems of the continuum were, for example, strongly influenced by the assumption that geometry is the science whose task it is to describe empirical space or phenomenal space. This had led some of the philosophers under discussion to regard the problem of the composition of the continuum as synonymous with the problem of the composition of visually continuous surfaces or lines. In others we find the problems of the continuum discussed in connection with the composition of matter or of space or of time. Thus our problem was discussed under various guises and one must not expect to find questions of mathematics divorced from considerations of physics or from discussions concerning the structure of our experience. After all, the enquiries concerning the nature of the continuum are found embedded in larger treatises, and it is characteristic of all these approaches that they claim to make the puzzlements of the continuum disappear if only the problem is seen in the right (empiricist or idealist or what not) context.

It is customary, in the introductory chapter of an investigation such as the present one, to state precisely the nature of the problem to be discussed in it. So far I have offered not much more than vague generalities. Evidently, in order to become more explicit I ought to offer a definition of such key terms as "continuum" and "infinite" if I wish to make the extent and concern of the present dissertation perfectly clear. In trying to do so, however, a perplexing difficulty presents itself: It has often been claimed, especially by mathematicians, that the first rigorous definition of "continuous" was given by Dedekind, and that before this momentous event the word "continuous" was associated with a more or less vague notion of hang-togetherness.[7] If this were so, then we would, strictly speaking, have no assurances that the philosophers we are about to investigate were in fact all dealing with the same problem (as was, say, Dedekind) when they spoke of the problem of the continuum. But the difficulty is not as severe as it might seem. While no precise definition of continuity was available, there was always the example of the geometrical line, the points on which were always considered to form a continuous set. Thus the problem was not simply to find a precise definition of a hitherto vague term,

[7]Cf. Boyer, op. cit, pp. 42, 291.

but to describe accurately the arrangement or order of the points in a line in such a way that, for example, the axioms of Euclid could be fulfilled. If it had been otherwise, then there could be no reason why the definitions advanced by Leibniz and later by Mach, namely that a line is continuous if between any two points on that line there lies another point, should have to be rejected.[8] We could simply treat this definition as a hortatory one and assume that it renders precise a concept that had so far been vague. But the point is that linear point sets can be produced which fulfill Leibniz' requirement but which do not allow a development of geometry in Euclid's sense. For example, conditions can be described in which "a straight line meets two straight lines, so as to make the interior

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[8]Leibniz defines "continuity" thusly: "There is continuous extension whenever points are assumed to be so situated that there are no two between which there is not an intermediate point." (Bertrand Russell, A Critical Exposition of the Philosophy of Leibniz, 2nd ed., London, 1937, p. 247). For Mach cf. Boyer, op. cit., p. 291: "The scientist Ernst Mach likewise regarded this property of denseness of an assemblage as constituting its continuity." (Die Principien der Warmelehre, historisch-kritisch entwickelt 2nd. ed. Leipzig, 1900, p. 71. Similar considerations apply to Bolzano's definition of 'continuity'. Bolzano claims that a point set is continuous if, and only if, the points are situated in such a way that "every single one of these points has at least one neighbor in the set within any distance, no matter how small. (Bernard Bolzano, Paradoxien des Unendlichen ed. Fritz Prihonsky, Hamburg 1955, (1st. ed. Leipzig 1851), p. 73).

angles on the same side of it taken together less than two right angles" where nevertheless the two straight lines do not "at length meet" because one of them runs through a gap in the other, i.e they do not have a point in common.[9]

The definitions advanced by Leibniz, Bolzano and Mach was at length discarded because they did not imply all the properties associated with the preanalytic notion of continuity. But this only goes to show that this preanalytic notion was a fairly precise one. What was lacking was a definition of "continuous" in terms of the components of the continuum, or, as the philosophers of the age were wont to put it, the problem of the composition of the continuum was unresolved However, the mere fact that the term was always applied to the same kinds of object assures us that different philosophers, when speaking about the continuum, were really dealing with the same problem. I have come to this conclusion in spite of the fact that the descriptions of the composition of the continuum vary from wildly metaphorical, as in the case of Galileo, to dryly sober in the case of Bolzano. Galileo writes in the <u>Dialogues Concerning</u>

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[9]Cf. Euclid's <u>Elements</u>, ed. Isaac Todhunter, London 1955, Axiom 12.

Two New Sciences.

> Having broken up a solid into many parts. having reduced it to the finest of powder and having resolved it into its infinitely small indivisible atoms, why may we not say that this solid has been reduced to a single continuum, perhaps a fluid like water or mercury or even a liquified metal?[10]

It must be noted that the fact that Galileo was unable to give an adequate description of the composition of the continuum did not prevent him from employing the term correctly, as his geometrical researches amply testify. It is this similarity in use that assures us that Galileo meant to deal with the same problems as later Bolzano and Cantor.

The situation is slightly more difficult in the case of the concept of infinity. Apparently the concept of infinity has always been surrounded with certain romantic notions. It is still customary to speak about somebody's "infinite compassion." Infinity seems to have always been considered by some people as an honorific attribute: Descartes speaks of God's infinite and man's finite mind, and similar examples could be adduced in great abundance.

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[10] Galileo, op. cit., p. 39f.

Fortunately, the sense in which the term "infinity" is used is usually made clear by the context. But let me give an example of the confusion that this ambiguity of the term frequently caused:

Collier, in his Clavis Universalis attempts to show, as Berkeley did before him, that there is no external matter. After having done this to his satisfaction, he tries to show that certain contradictions follow from the contrary assumption, namely that there is external matter. He writes: "External matter, as a creature, is evidently finite, and yet as external is evidently infinite, in the number of its parts, or divisibility of its substance."[11]

From this he concludes that a contradiction follows from the assumption that there is external matter, namely, such matter would have to be both finite and infinite But this contradiction is only apparent, since the word 'finite' is construed as the negation of an honorific attribute that has nothing to do with the mathematical notion of infinity. Collier puts it thus:[12]

[11]Arthur Collier, Clavis Universalis, ed. Ethel Bowman, Chicago 1909, p. 69.

[12]Ibid., Op. cit., p. 68.

"Infinite is to be absolute. finite. to be not absolute." Now since God is the only absolute being. infinity can properly be attributed only to Him and not to anything that depends for its being upon God's concurrence. In this sense external matter is said to be finite because it is a creature. Now the term 'infinite' in the first quotation above evidently denotes the number of members in a set and thus can clearly not be construed as the contradictory of the term "finite" in the same quotation. as Collier would have it. This example demonstrates amply the confusion caused by the ambiguity of 'infinite'.[13]

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[13]We have seen that it was frequently argued at the time that man cannot understand the composition of the continuum because his mind is finite. but that only God. whose mind is infinite can have an adequate understanding of it. Commentators have often argued that 'finite' and 'infinite' are here used in a non-quantitative way. They speak of the "qualitative" infinite. Hegel even attaches a value-discrimination to this distinction when he condemns the quantitative infinite as "<u>das schlechte Unendliche</u>". In the text I have argued that the two conceptions should be kept distinct, but it might be well to point out that there is a pretty obvious connection between these two concepts, especially among the Cartesians. I believe that (qualitative) infinity was thought to be that attribute of the understanding of God which allowed him to comprehend the nature of infinite sets. in particular the composition of the continuum. The postulation of this particular attribute of God's understanding would preserve, after a fashion. the rationality of the world, for if not even God could reconcile the apparently contradictory properties of infinite sets, then we must think of ourselves as somehow surrounded by inconsistent facts. In addition, 'infinite'

For obvious reasons I have therefore made it a policy to discuss only such occurrences of 'infinite' as seem to refer to the number of elements in a set. For this reason I have decided not to include a discussion of Spinoza. Spinoza, as far as I can discern, defines "finite after its kind" as a quantity that can always be increased. He says "A thing is called <u>finite after its kind</u>, when it can be limited by another thing of the same nature; for instance a body is called finite because we always conceive another, greater body."[14] A body is finite, then, if something can be added to it, and it would seem that a body is infinite if nothing can be added to it. Or, as Bolzano puts it, Spinoza believed "only that to be infinite which is not capable of further increase, or to which nothing can be added."[15]

Bolzano points out that many other philosophers and

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was of course also used as an honorific term. However, the relation between the "qualitative" and the "quantitative" infinite in 17th century philosophy would bear some further investigation.

[14] Benedict Spinoza, <u>Ethics</u>, Definition 21; in: <u>The Chief Works of Benedict de Spinoza</u>, ed. R. H. M. Elwes, N. Y., 1951.

[15] Bernard Bolzano, <u>op. cit.</u>, § 12, No. 2.

mathematicians adhered to this definition of 'infinite'. In the present paper I shall make no attempt to account for any theories of this sort, i.e.. I shall restrict myself to only such accounts as do not regard an infinite number as the greatest thinkable number, or give similar definitions. As we shall presently see, writers who did not treat infinity as an honorific attribute and who did not adhere to a definition similar to that of Spinoza were able to indicate, quite some time ago, several important properties of infinite sets.

One difficulty seems to permeate all the discussions of the continuum that we are about to study, and that is the sense of 'compose' in the question "of what is the continuum composed?" and we may just as well give it some thought at this juncture. Apparently the continuum was thought to be "composed" of something in the sense in which a brick wall is composed of bricks. The bricks are the parts of which the brick wall is composed, but there are no components that make up a continuum in this way. Certainly, if we consider a continuous line, the points in this line do not "compose" the line in this sense. We know that if we have more bricks, then we can make a bigger wall, but a larger number of points does not make a longer line than a smaller number of points. Likewise, in a brick wall, a brick has immediate neighbors (provided there is more than one brick in the wall). This does not hold for the points in a line. That the continuum would exhibit such features was a fact not generally reckoned with, and the confusion surrounding this concept can in a large part be explained by assuming that what was sought after were parts of the continuum that would compose it like bricks compose a wall. The abandonment of this preconception was one of the most important conditions, but evidently also one of the conditions most difficult to attain, before an adequate description of the continuum could be given.

At this place a few remarks are also in order concerning the method of the subsequent enquiries. Our task is the exposition, at least in part, of certain philosophical systems. For these expositions I have adopted the following strategies: I attempt to identify the problems which these systems were designed to resolve, and try to delineate the way in which this resolution took place in each case. At the danger of repeating myself needlessly, I wish to emphasize again that the problems of the infinite and the composition of the continuum hold a place of great eminence among the problems to be resolved in the systems which will be discussed. This procedure of identifying the initial problems of philosophical systems seems to have certain advantages over other modes of exposition. It makes clear, for one thing, that the philosophy of the enlightenment must not be thought of as so much idle speculation: it was to a great extent just as serious an endeavor to resolve the problems indicated as were the later and more successful investigations of mathematicians. Another advantage of this mode of exposition would seem to lie in the following: if it can be shown that, for example, Leibniz adopted the division between intelligible and phenomenal world with the explicit purpose of resolving certain logical difficulties, then the affirmation of this bifurcation among later idealists, who can think of no problem to be resolved thereby, is just purposeless, ludicrous speculation, a senseless repetition of a philosophical distinction the import of which had been entirely lost.

It is a disgrace for the historians of philosophy that the term 'idealist' should be applied indifferently to both Leibniz and those who aped him. But a distinction between the two can obviously only be found if one is aware of the problem which precipitated the development of Leibniz' system, and the sad lack of such problems, for example, with Fichte. The much touted "historical method" would never be capable of finding this distinction, concentrating as it does on the lines of development of philosophical ideas.

Let this suffice as an introduction. I shall take the liberty to make further remarks concerning the mode of my exposition in the body of this dissertation.

CHAPTER I

LEIBNIZ

In discussing the philosophy of Leibniz, formidable difficulties present themselves. Leibniz did not leave a definitive treatment of his philosophy, and his incidental expositions are almost always adapted to the capacity of his correspondent or his immediate audience, or are the outgrowth of polemics over certain restricted points. In view of these facts it is necessary for the commentator to adopt a strategy which will organize the material for him and allow him to weave it into a coherent whole. In keeping with a general practice, a historical approach to Leibniz is most frequently chosen and the "development" of his philosophy is discussed. Various influences upon him are cited and characteristic alterations of delivered opinions are noted. Under this viewpoint, Leibniz appears as a great synthesizer of previously held philosophical opinions. Latta, for example, notes correctly that "the philosophical work of Leibniz was an endeavor to reconcile the notion of substance as _continuous_ with the contrary notion of substance as consisting of indivisible elements".[16] But while he recognizes the importance of the subject to Leibniz, he thinks of him as primarily interested in affecting a synthesis of two positions that were in fact held by his predecessors, namely Descartes on one hand and the atomists on the other. A much more blunt statement of the view that Leibniz was, in the main, a synthesizer of previously held philosophical views is found in Maziarz' Philosophy of Mathematics. Maziarz writes: "His (Leibniz') countless references to ancient, medieval, and contemporary thinkers...reveal his basic trend and tendency: to compromise and synthesize, to remove individual differences, to harmonize speculative opposition in his own center of perspective."[17] The viewpoint here characterized can easily be substantiated through quotations from Leibniz himself. Thus in the Nouvaux Essais, Leibniz writes: "This system appears to unite Plato and Democritus, Aristotle and Descartes, the Scholastics with the Moderns, theology and ethics with reason. It seems to

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[16] Robert Latta, Introduction to: Leibniz, The Monadology, ed. R. Latta, 1st edition, Oxford 1898, p. 28.

[17] Edward A. Maziarz, The Philosophy of Mathematics, New York, 1950, p 58.

take the best from all sides..."[18] It must be noted that for a critical exposition of Leibniz it is really of secondary importance to discover the historical sources of various parts of Leibniz philosophy and to discuss the psychological motivation for the development of his system. It must be noted, and will be substantiated in the sequel, that the views which Leibniz synthesized were each of them held irrefutable by Leibniz without much regard for their historical origin. Thus it is of primary importance to undertake a reconstruction of his system, and it is only of secondary philosophical interest to trace historical origins and psychological motivations. I believe that in thus reconstructing Leibniz' system better justice is done to the spirit of Enlightenment philosophy, which took its metaphysical problems just as seriously as we take problems of mathematics or methodology, and which was not given to the transaction of speculations out of piety for received opinions. Thus, in this chapter I do not wish to advance a psychological or historical thesis concerning the genesis of Leibniz' philosophy. Rather, I wish to point out that he was aware of certain problems connected with the con-

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[18]Leibniz, New Essays Concerning Human Understanding, ed. Alfred G. Langley, La Salle, 1949, p. 66 (Book 1, Ch. 1).

tinuum. and that his philosophical system can be envisaged as being designed. to a large extent, in order to resolve these problems. Thus I shall not argue that the problems of the continuum were the psychological starting point of his speculations. On the other hand. the present chapter can be taken as an attempt to show that there are no inner-systematic reasons against this assumption. However. the establishment of the psychological thesis itself would require a good deal of additional detective work which I do not consider to be the responsibility of the philosophical commentator.

A different strategy for the exposition of Leibniz' system has been chosen by Bertrand Russell. Properly, Russell does not pay much attention to the historical origin of Leibniz' views. but treats his philosophy, at least in its speculative parts, as a coherent system and claims that he attempts to develop it from a small set of premises: These premises are:

> I Every proposition has a subject and a predicate, II. A subject may have predicates which are qualities existing at various times (such a subject is called a substance). III. True propositions not asserting existence at particular times are necessary and analytic. but such as assert existence at particular times are contingent and synthetic. The latter depend upon final causes. IV The ego is a

> substance. V. Perception yields knowledge of an external world, i.e., of existents other than myself and my states.[19]

One might expect that Russell would develop Leibniz' system from these premises with the aid of the laws of logic. But he does not give us the treat of demonstrating Leibniz' philosophy as an axiomatic system, and one might excuse his self-deception on this count by pointing out that his book on Leibniz preceded his logical work. He claims, however. that "the first four of the above premises...lead to the whole or nearly the whole of the necessary propositions of the system."[20]

I believe that a reconstruction of Leibniz' philosophy as an axiomatic system would not present to us the spectacle of a large number of theorems following from a half dozen or so axioms. as one might expect from the philosophical system of a man with Leibniz' logical acumen. Notice that Russell merely claims to have identified the "principal" premises of the system, although he fails to make clear in what sense he takes certain premises to be more important than others.

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[19]Bertrand Russell. A Critical Exposition of the Philosophy of Leibniz. 2nd ed., London 1937, p. 4.

[20]Ibid.

In keeping with the general procedures set forth in the introduction I shall attempt to give an exposition of parts of Leibniz' philosophy by a method different from the above indicated two. I believe that Leibniz' major problem lay in the fact that he adopted a number of propositions each of which he had to consider as well established, but which together seemed to form an inconsistent set. It seems that through an identification of these propositions a reasonable understanding of the problems of Leibniz' system can be provided. I shall therefore proceed to point out what these propostions were and we shall see that, in order to harmonize them, a large part of Leibniz' system had to be developed. In identifying the propositions in question I shall occasionally refer to the history of their discovery, but I want this taken as incidental information. I do not wish to appear to make the same error that I have criticized in Maziarz.

Leibniz knew that the members of an infinite collection can be brought in a one to one correspondence with the members of a proper subset of that collection. This fact was already known to Galileo. who wrote in the Dialogues Concerning Two New Sciences:

Salv. Very well; and you also know that just as the products are called squares, so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore, if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?

Simp. Most certainly.

Salv. If I should ask further how many squares there are, one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.

Simp. Precisely so.

Salv. But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers, because every number is the root of some square. This being granted we must say that there are as many squares as there are numbers, because they are just as numerous as their roots, and all the numbers are roots.[21]

Galileo was quite puzzled with his discovery and he assumes that the difficulty can in part be resolved by assuming that "the attributes 'equal', 'greater', and 'less', are not applicable to infinite, but only to finite.

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[21] Galileo Galilei, Dialogues Concerning Two New Sciences. Translated and ed. by Henry Crew and Alfonso De Salvio. N. Y., 1914, p. 32.

quantities."[22] Any difficulty that might remain must be attributed to our finite understanding which cannot entirely cope with the problems of the infinite.[23]

Leibniz was familiar with this result. He occasionally claims to have discovered it himself. Thus he writes: "Many years ago I proved that the number or sum of all numbers involves a contradiction (the whole would equal the part)."[24] In the Nouvaux Essais he writes: "But there is no infinite number, neither line nor other infinite quantity, if these are understood as veritable wholes, as is easy to show."[25] Here he obviously alludes to the same result. It is noteworthy that Leibniz realized that the paradox of Galileo not only applies to discrete sets such as the set of natural numbers, for which alone Galileo presents a proof, but that the paradox is seen to apply also to such point sets as lines, which are here considered as

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[22] Ibid

[23] See Introduction, p. 2.

[24] Letter to Bernoulli, 1698, Wiener, op. cit., p. 99.

[25] Leibniz, Nouvaux Essais, p. 161, (Bk. II, Ch. XVII)

just another kind of infinite quantity. The clearest statement of the "Paradox of Galileo" is the following:

> The number of all numbers implies a contradiction, which I show thus: To any number there is a corresponding number equal to its double Therefore the number of all numbers is not greater than the number of even numbers, i.e. the whole is not greater than its part.[26]

It is perhaps not unfair to Leibniz to state the proposition in question as "If there is an infinite collection, then it has a proper subset which can be brought in a one to one correspondence with it". This proposition is clearly presupposed in the above presented arguments, especially in the last one. But the arguments are of <u>modus tollens</u> form: they deny the consequent, namely that the part ever is as great as the whole, in order to deny the antecedent, namely that there are infinite collections.

Thus we have discovered a second proposition which Leibniz considered established beyond all doubt, namely that the whole is greater than its part. This is the ninth of Euclid's axioms.[27] Leibniz frequently acknowledges his unconditional acceptance of this proposition as for example

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[26]Russell, <u>op. cit</u>., p. 244.

[27]Euclid, <u>The Elements</u>, ed. Issac Todhunter. London, 1955, p. 6.

in the Nouvaux Essais: "Euclid says that the whole is greater than its part, a statement which is wholly trustworthy".[28] On another occasion, Leibniz attempts to deduce the principle of whole and part from a definition. He writes: "If a part of a quantity is equal to the whole of another quantity, then the first is called the greater, the second the smaller. Whence the whole is greater than the part."[29] However, when Leibniz establishes that there would be as many even numbers as integers, he does not make use of the provisions of his definition, for, clearly, the set of integers has a part which is equal in number to the set of even numbers and that is the set of even numbers itself. Hence by the above definition, the set of even numbers must be called smaller than the set of integers. Moreover, when we think of the set of integers as equal with the set of even numbers, then the whole of which the even numbers are a part must be greater than the set of integers. But that whole is the set of integers itself. Hence, by the above definition, the set of integers is greater than itself, which makes it impossible to consider

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[28] Leibniz, Nouvaux Essais, p. 471.

[29] Metaphysical Foundations of Mathematics. Wiener, op. cit., p. 205.

"greater than" as irreflexive, a very odd usage indeed The upshot of this discussion is that Leibniz had very pressing reasons to deny the existence of infinite sets, If there are no infinite sets, then the above definition becomes, of course, quite acceptable, since then the principle of whole and part would generally hold.

There are numerous other passages in which Leibniz asserts the principle of whole and part, and occasionally he calls it an axiom.[30] The two propositions considered so far, namely the paradox of Galileo and the principle of whole and part could be reconciled with one another, but will jointly lead to the consequence that there are no infinite sets. The following, however, seems to assert the existence of infinite sets and thus forced Leibniz to all manner of speculation as to how it could be reconciled with the first mentioned two propositions. It can be put thus: There is external matter and it is infinitely divisible. It has infinitely many parts.

Obvious difficulties arise at once. Leibniz had asserted that we cannot make any statements about infinite

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[30]Specimen Dynamicum, Wiener, op. cit., p. 130.

sets considered as "veritable units" but that we can only speak of infinite sets as what he calls distributive wholes. Thus while we may form a sentence of the form "Every even number has such and such a property", we are not supposed to write "the class of even numbers is such and such". This restriction becomes problematic once the assumption is made that every piece of external matter forms a whole of infinitely many parts. Then a sentence like "this stone is of such and such a nature" violates the rule that prohibits our speaking of infinite sets as veritable wholes.

It seems then that the three propositions, namely the paradox of Galileo, the principle of whole and part, and the assertion of the actual infinite subdivision of any given piece of matter form an inconsistent set. This inconsistency is part of what Leibniz calls the "labyrinth of the continuum". Actually, Leibniz does not consider external matter to be continuous, but his denial of the real continuity of matter is alreay part of the solution of the problem, as we shall see in the sequel. The reason why the indicated problem must nevertheless be considered to

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[30] _Specimen Dynamicum_, Wiener, _op. cit._, p. 130.

belong to the so-called labyrinth of the continuum is that many of Leibniz predecessors subscribed to the notion that matter is continuous <u>and therefore</u> infinitely divisible. Leibniz agrees that matter is so divisible. but his views on the continuity of matter are rather more complex and shall be discussed later.

That Leibniz believed in the actual infinite is shown in the following passage and numerous others.

> I am so much for the actual infinite that instead of admitting that nature abhors its, as is commonly said, I hold that it affects nature everywhere in order to indicate the perfections of its author. So I believe that every part of matter is, I do not say divisible, but actually divided, and consequently the smallest particle should be considered as a world full of an infinity of creatures.[31]

That there should be external matter at all was apparently not seriously questioned by Leibniz, at least not in his later years. Russell remarks that only in his earlier years did Leibniz regard the existence of matter as a problem, but that later "he so far forgot his earlier unresolved doubts that, when Berkeley's philosophy appeared. Leibniz had no good word for it. 'The man in Ireland', he

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[31] <u>Specimen calculi universalis</u>. Wiener, <u>op. cit.</u>, p. 99.

writes, 'who impugns the reality of bodies. seems neither to give suitable reasons, nor to explain himself sufficiently. I suspect him to be one of that class of men who wish to be known by their paradoxes.'[32]

The difficulty then is the following: there are everywhere in nature infinite sets. Every piece of matter can be thought of as such an infinite set. But this assumption is not compatible with Leibniz' other results, namely that there is a contradiction in the concept of an infinite set. Since the problem here indicated is connected, historically, with the assumption that matter is continuous, let us call it the first problem of the continuum. Its resolution, i.e. the reconciliation of the three propositions so far identified requires the generation of a large part of Leibniz' system. and we can therefore think of these propositions as "nuclear" propositions: the system was generated for the most part to make their simultaneous assertion tenable.

In saying that the paradox of Galileo, the principle of whole and part, and the assumption of the actual infinite were the propositions for the sake of which a large

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[32] Russell, op. cit., p. 72.

part of the system was developed. I do not wish to give the impression that these propositions should be considered as axioms. Rather, they have a relation to the fundamental assumptions of the system similar to that of the better known theorems of arithmetic to their axioms: they are believed first and only subsequently axioms or premises are sought from which they follow.

Before I try to show the ways in which Leibniz attempts his resolution, let me point to a second problem connected with the continuum. This second difficulty had already engaged Descartes. It is the following: The ultimate parts of the continuum, or of matter when viewed as continuous, can be neither points nor extended particles (so it was argued). Not the former because extension cannot be built up, it was believed, out of unextended parts, not the latter, because extended parts or extended atoms are not ultimate but can be further divided.[33] Now the number of parts, whatever their nature, must be infinite since a finite number of divisions of a continuous quantity will always result in extended parts which can be further divided. Thus all that can be said is that any

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[33] Cf. Descartes, Principles, Part II, Principle 20.

continuous quantity consists of an infinite number of parts of undetermined nature. This is where Descartes rests the issue. He argues that we must not doubt this division, although we cannot comprehend it.[34] The reason why Descartes did not want this division to be put in question is, of course, that he wanted Euclidean geometry to be applicable without restriction to extended objects, and this geometry demands, among other things, that any line, no matter how short, must be divisible. Let us call the problem here indicated the second problem of the continuum. I think that what Leibniz calls the "labyrinth of the continuum" consists of the two problems here identified. It will be noted that what I have called "the first problem of the continuum" attaches to all infinite sets, while the second has to do with continuous sets only. We can take Leibniz' word that his philosophy was to a large extent designed to resolve both these problems. In the beginning of the _Theodicee_ he writes:

> There are two famous labyrinths, in which our reason often goes astray: the one relates to the great question of _liberty_ and _necessity_, especially in regard of the production and origin of _evil_; the other consists in the

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[34] Cf., _Ibid_., Part II, Principle 35.

> discussion of _continuity_ and the _indivisible points_ which appear to be its elements. and this question involves the consideration of the _infinite_. The former of these perplexes almost all the human race, the latter claims the attention of philosophers alone.[35]

Latta comments about this:

> Leibniz makes the _Theodicee_ an investigation of the meaning of liberty and necessity, while in others of his writings he offers a solution of the problem which he describes as the special perplexity of the philosophers.[36]

Let us now proceed to sketch that part of Leibniz' system which is required for the solution of the problems of the continuum.

For reasons that need not concern us at this point, Leibniz assumed that every well-formed proposition is constituted of one subject and one predicate. Relations are recognized as useful but merely "ideal things". In considering the relation "greater than" which is to hold between M and L, he states that "is greater than L" can be considered an accident of M, "is less than M" can be considered an accident of L. but he also realizes that there is a third way of analyzing this proposition. namely by treating the relation "as something abstracted from both".

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[35] Latta, _op. cit._, p. 21.
[36] _Ibid_.

In this case

> It cannot be said that both of them. L and M together are the subject of such an accident; for if so, we should have an accident in two subjects with one leg in one and the other in the other; which is contrary to the notion of accidents. Therefore we must say that this relation, in this third way of considering it, is indeed <u>out of</u> the subjects; but being neither a substance, nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful.[37]

Thus, while "is greater than M" refers to a real accident, "is greater than" refers to an "ideal" thing. It seems that the relations owe this shadow of an existence to the pragmatic justification that "their consideration is useful", and indeed the knowledge of some of the properties of relations is indispensable for certain calculations Nevertheless, the proper analysis of a statement such as "M is greater than L" resolves it into one subject and one predicate.

Thus the only sentences which are considered strictly meaningful are ones that consist of one subject and of an absolute predicate. Let us consider for the moment only singular statements of this kind. Leibniz held that such statements fall into two classes: those

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[37]George Martin Duncan (ed.). <u>The Philosophical Works of Leibniz</u>, New Haven 1890, pp. 266f. Cf. Russell, <u>op cit</u>., pp. 12f.

that are about what he calls true or substantial units, and those whose subject refers to aggregates. Now the statements about true units have a certain metaphysical prerogative over the others: all others are meaningful if and only if they can be replaced by statements about true units. But what is a true or substantial unit? Leibniz writes: "Substantial unity calls for a thoroughly indivisible being. naturally indestructible."[38] Objects which are not so indestructible are said to have no real existence. "What is not really _one_ being (_un_ etre) is not really a _being_ (un _etre_)".[39] From true or substantial units Leibniz distinguishes "beings by aggregation". As examples he cites armies, flocks of sheep, piles of stone. Moreover, any extended physical object is considered such a being by aggregation. He says that:

> A block of marble is no more a thoroughly single substance than would be the water in a pond with all the fish included, even when all the water and all the fish were frozen; or any more than a flock of sheep, even when all the sheep were tied together so that they could only walk in step and that one could not be touched without producing a cry from all.[40]

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[38]Letter to Arnauld, in: George R. Montgomery, (ed.) Leibniz, _Discourse on Metaphysics_, _Correspondence with Arnauld_. _and Monadology_, La Salle, 1902.

[39]Russell, _op. cit._, 242, Montgomery, _op cit_., 192.

[40]Letter to Arnauld, Montgomery, _op. cit_., p. 161.

It is the mind of the observer alone that gives unity to such beings by aggregation. "This mass of substances", he says, does not form in truth one substance. This is a result to which the soul by its perception and its thought gives its last achievement of unity."[41] Now Leibniz holds that statements about aggregates can frequently be replaced by statements about the components of such aggregates. He points out:

> It seems that what constitutes the essence of a being by aggregation consists only on the mode of the being of its component elements. For example, what constitutes the essence of an army? It is simply the mode of being of the men who compose it.[42]

Apparently, Leibniz was working on a method whereby all statements about couples, trios, etc. can be translated into statements about the members of such couples, trios, etc.[43] Now if such a translation can always be afforded, then no harm can come from asserting something about an aggregate, since such an assertion can always be replaced by an equivalent set of statements about the members of the

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[41]Nouvaux Essais, p. 235 (Bk II. Ch. XXIV).

[42]Letter to Arnauld, Montgomery, op. cit., p. 190.

[43]Ad specimen calculi universalis addenda, cf. R. M. Yost, Leibniz and Philosophical Analysis, Berkeley 1954, p. 11.

aggregate. The question whether or not the aggregate has existence is then a purely metaphysical one, and in denying such existence, Leibniz shows himself as a nominalist.[43] On the other hand, if there is a group of statements about aggregates which cannot be translated, in principle, into statements about true units, then the issue becomes somewhat more complicated, especially if any important statements, such as the enunciations of physics or geometry are included in this latter category. For all statements which are in principle incapable of translation into statements about true units must be considered meaningless in the strictest sense, although practically speaking they may be rather useful. For example, an assertion about a couple of stones can perhaps be translated into statements about the stones in that couple, but these latter statements are themselves about aggregates, the stones being composed, according to Leibniz, of an infinity of parts. Consequently

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[43] In Leibniz, no clear distinction seems to be drawn between the whole-part relation and the class membership relation. When I use the term 'aggregate' I wish to use this term verbally so as to apply to wholes as well as classes. Now Leibniz seems to have desired the existence of classes as well as that of wholes composed of more than one metaphysically simple individual.

the statements about the stones are no more about true units than were the statements about the couple. Now if a set of statements about true units can be found equivalent to the statements about the stones, then these latter statements are shown to be meaningful in the ultimate metaphysical sense, otherwise they are not. But for such a translation it is not sufficient to develop a method whereby statements about finite aggregates can be replaced by statements about the members of such aggregates, rather, if a precise and strictly meaningful language about physical objects is to be developed, then a method of analysis must be brought forth with which infinite aggregates can be similarly handled. This is a consequence of the assumption of the actual infinite. Does Leibniz hold that such an analysis can, at least in principle, be carried out? I believe that this question must be answered in the negative. But in order to establish this result we will have to examine Leibniz' views on matter.

Assuming that a statement about a stone is to be analyzed, what can we say about the ultimate constituents of that stone, and why should it be impossible to replace the original statement by a series, (perhaps an infinite series) about the ultimate parts of that stone? First of all, what are the ultimate parts of which a physical object is composed? These units cannot be particles of matter or material atoms since "every part of matter is, I do not say divisible, but actually divided, and consequently the smallest particle should be considered as a world full of an infinity of creatures."[45] This amounts to saying that while we can separate spatially the members of a class of physical objects, we cannot carry out a spatial separation down to the ultimate constituents. Thus while it is possible to divide, ad infinitum, any piece of matter, such a division will not ultimately produce the units out of which that piece of matter is constituted. Rather, the piece of matter is given rise to by entities that do not partake of spatial characteristics at all. Properly speaking, the ultimate constituents of matter cannot even be said to have location in space:

> Space is the order of coexisting phenomena, as time is the order of successive phenomena. There is no nearness or distance, whether spatial or absolute, among Monads, and to say that they are collected together in one point or dispersed throughout space, is to make use of certain fictions of our mind, by which we try to represent to ourselves in imagination what cannot be

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[45] Specimen calculi universalis, Wiener, op. cit., p. 99.

imagined but only understood.[46]

In short, the true units or monads to which the subjects of properly constructed sentences must refer, are not bits of matter and are not amenable to any characterization in terms of spatial properties. Rather, matter "results from" and is constituted by, the ultimate individuals or monads. "Strictly speaking, matter is not composed of constitutive unities, but results from them, for matter or extended mass is nothing but a phenomenon founded on things, like the rainbow or the parhelion, and all reality belongs only to unities."[47] Thus the analysis of statements about physical objects runs into considerable trouble since the monads are contained in a physical object not in the same way in which an element of a set is "in" that set. Thus it would seem that the rules developed for the analysis of statements about couples, trios, etc. become inapplicable not only because the sets here considered are infinite, but because of the peculiar mode of containment of monads in their objects. This has as a consequence

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[46]Letter to Des Bosses, Latta, op. cit., p. 221.

[47]Russell, op. cit., p. 243.

that the properties of physical objects must. for the most part. be considered emergent with respect to the properties of the monads contained "in" that object, i.e. they are not predictable from the properties of the contained monads.

This can be demonstrated in the following way. In the Monadology Leibniz introduces a classification of monads into souls and other monads. the souls being distinguished by more distinct perceptions and by the possession of memory.[48] Presumably. inanimate bodies. such as stones. do not contain souls among their constituting monads. Now in a letter to De Volder. Leibniz writes that the essence of the soul is to represent bodies.

One gathers from the context that this is a distinguishing characteristic of souls. i.e. that other monads do not represent bodies. On the other hand. monads are defined as units of perception. Moreover. it is asserted that all monads mirror the universe and therefore in particular also the body in which they are housed. One might conclude that. if all the perceptions of any one monad in any body were known. one would then be able to deduce the properties which such a body would have when considered as

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[48] Monadology, No. 19, Wiener, op. cit.. pp. 536f.

a physical or phenomenal object. But this does not have to be the case. If a stone, for example, does not contain any souls among its monads, then it does not contain any monad which perceives the stone spatially in the way in which souls perceive it. That is to say, since the modes of perceiving the universe differ in the different classes of monads, it may not be possible to deduce all the phenomenal properties that the stone has in the experience of a soul from the way in which it is perceived by the "bare monads" in the stone.

Outside of perceptions each monad is said to have materia prima. Materia prima is said to be that element by virtue of which bodies resist penetration and locomotion. As far as I can see, impenetrability and inertia are the only properties that are found in monads as well as in the bodies which arise from monads and hence are the only properties in bodies which are clearly not emergent with respect to the monads out of which these bodies are constituted.[49]

There is further evidence for the thesis just proposed. In the Nouvaux Essais, Leibniz points out:

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[49] Wiener, op. cit., p. 161.

> It is to be observed that matter, taken as a complete being is nothing but a collection or what results from it, and that every real collection presupposes simple substances or real unities, and when we consider further what belongs to the nature of these real unities, i.e. perception and its consequences, we are transferred, so to speak, into another world, that is into the intelligible world of substances, whereas before we were only among the phenomena of the senses.[50]

The locution that we are "transferred ..into another wo world" suggests that Leibniz wished to distinguish as sharply as possible between the phenomenal and the intelligible world. But elsewhere he is even more explicit:

> I believe the true <u>criterion</u> as regards objects of sense is the connection of phenomena, i.e. the connection of what happens in different times and places, and the experience of different men, who are themselves, in this respect, very important phenomena to one another...But it must be confessed that all this certainty is not of the highest degree....For it is not impossible, metaphysically speaking, that there should be a dream connected and lasting as the life of a man.[51]

Now if the characteristics of bodies were not emergent with respect to the properties of the monads which constitute the bodies, then the particular certainty which Leibniz expresses with respect to his metaphysical scheme would be transferred to statements about phenomena, the

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[50] <u>Nouvaux Essais</u>, p. 428.

[51] <u>Ibid</u>., p. 422.

latter being deducible from the former. However, since statements about the world of experience have only <u>moral certainty</u>, the characteristics of matter as experienced must be, for the most part, emergent relative to the properties of the monads in question. On the other hand, wherever matter is experienced, and wherever such an experience occurs in an orderly connection with other experiences, Leibniz assumes monads to be present "in" the matter so experienced. This I take as the meaning of the dictum that physical objects are <u>phenomena bene fundata</u>. <u>Bene fundata</u> does not mean then that the properties of physical objects are all inferable from the properties of their constituting monads, rather, it is to say that physical objects are not mere illusions, but that, wherever a physical object is present, there are monads.

It appears that in Leibniz' system statements about physical objects were thought to have a function and character akin to those attributed to value judgments by the early positivists; the usefulness of value judgments was never denied, but they were treated as some sort of useful nonsense, irreducible to observation statements, i.e. irreducible to the most fundamental kinds of assertion that the positivists thought could be made.

Similarly with statements about physical objects in Leibniz: we must make them in order to get around in the world. In point of fact. Leibniz is most emphatic in stating that the metaphysical analysis of matter contributes nothing to the development of the science of physics and is in no way required to put physics or common talk about physical objects on a sure footing. Leibniz writes:

> I grant that the consideration of these forms (i.e. substantial forms or monads) is of no service in the details of physics and ought not to be employed in the explanation of particular phenomena.[52]

and

> The physicist can explain his experiments, now using simpler experiments already made, now employing geometrical or mechanical demonstrations without any need of the general considerations which belong to another sphere, and if he employs the cooperation of God, or perhaps of some soul or animating force. or something else of a similar nature, he goes out of his path quite as much as that man, who, when facing an important practical question, would wish to enter into profound argumentations regarding the nature of destiny and of our liberty.[53]

This points again at the wide gap between the phenomenal world of matter and physics and the intelligible

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[52] Discourse on Metaphysics, No. X, Wiener, op. cit., p. 302.

[53] Ibid., p. 303.

world of substances and metaphysics. But why did Leibniz introduce this bifurcation?

In Descartes we already find a similar distinction. There, experience is divided into the veridical experience of primary qualities and the illusionary experience of what are called secondary qualities. Descartes undertook this distinction partly in order to explain certain oddities of direct experience: auditory experiences without external stimuli, light sensations without external stimulation, warm and cold sensations simultaneously from the same external source etc. Leibniz, however, had different reasons for distinguishing a phenomenal world from an intelligible one. It will be noted that in Leibniz the bifurcation is much more thorough, in that experiences of extension and duration are also referred to the perceptor rather than the perceived. While Descartes' distinction was to resolve epistemological puzzles, in Leibniz the differentiation is made in order to overcome logical difficulties, in particular both the indicated problems of the continuum. How is this result attained?

We have seen that statements about phenomenal objects cannot be translated into statements about true units or monads. Nevertheless, Leibniz assures us that it is perfectly proper for the scientist to employ modes of speech that are improper. metaphysically speaking. This assurance, it must be noted again, is not based on the assumption that such improper forms of speech can be replaced by equivalent metaphysically correct expressions, rather. it hinges on the fact that the phenomenal world has a certain order in its own right, that the phenomena occur in orderly sequences and can be described in a practically satisfactory way without recourse to a description of the constituting monads. But this pragmatic justification does not guarantee that a description of the phenomenal world will always be free of contradictions. Thus, if bodies are phenomenally continuous, as Leibniz admits, then they have infinitely many parts and thus have proper parts with just as many parts as themselves. This contradicts an accepted axiom. Leibniz argues that the extended when conceived through itself alone. i.e. when considered as a merely phenomenal object. contains a contradiction. In order to prevent the arising of this contradiction, all statements about physical (phenomenal) objects are declared meaningless in the strictest metaphysical sense. Hence strictly speaking. the contradiction cannot be stated.

To summarize the results attained to this point: Leibniz introduces a distinction of phenomenal and intelligible world in order to remove a logical difficulty. For, if any piece of material substance is considered as a "veritable whole", then a contradiction arises since this piece of matter is (at least phenomenally) continuous and has therefore infinitely many parts. Consequently, under this assumption it would have a proper part with just as many parts as itself, which contradicts Euclid's ninth axiom. Hence a piece of matter, or any extended substance cannot, according to Leibniz, form a "veritable whole", its wholeness is merely illusionary and depends on the particular mode of representation which characterizes souls. Now any statement which presumes that a piece of matter forms a whole, i.e. which makes an assertion about such a piece of matter is, strictly speaking, nonsense, so that the threatening contradiction cannot be stated in the precise language which requires that the subjects of sentences must refer to veritable wholes.

While it cannot be said that Leibniz developed a sort of type theory, the strategy is strikingly similar to that employed by Russell. Russell, too, disallows certain statements which, at first sight seem quite inoccuous, for example "the class of cats is not a cat", and his stipulation to consider as nonsense utterly respectable sentences of the natural language in order to keep contradictions from arising seems already to have been employed by Leibniz. But further parallels are hardly justifiable. For one thing. Leibniz' theory contains a good deal of metaphysics. His entities "of lowest type", the monads, are described in some detail, while in Russell's theory it is not further specified of what particular character the individuals on the lowest level are, or even whether there is an absolutely lowest type level. Furthermore. Leibniz' theory is distinguished from that of Russell by the fact that statements which violate the prescriptions for the precision language of monads nevertheless can be justified pragmatically, which allows on the one hand free use of the language of phenomena and on the other hand explains the ultimate insufficiency and inconsistency of this language by its essential imprecision.

It seems that what we have called the first problem of the continuum is resolved in a very interesting way by the provisions which have been discussed above. But I have already pointed out that the so-called labyrinth of the continuum actually consisted of two separate problems. The second problem had to do with composition of the continuum and its ultimate elements. Now in the intelligible world of monads no continuum is to be found. Leibniz points out:

> In actuals there is nothing but discrete quantity, namely the multitude of monads or simple substances, which is greater than any number whatever in any aggregate whatever that is sensible or corresponds to phenomena.[54]

Thus, continuity can be attributed only to phenomenal objects, and investigations of these phenomenal objects must be carried out in the imprecise language that is inevitable with them. However, one might object that the problem of continuity is really quite divorced from considerations of what is real, and that it has only to do with certain order types or with a discussion of the kind of entity that fulfills the axioms of geometry. Under this view the latter is also a purely abstract consideration which is totally divorced from any question concerning the nature of reality. I think that these objections are quite well taken. Only, in the 17th and 18th century geometry was not usually considered an axiomatic system in the modern sense, but a description of space. For Leibniz this meant that geometry had the task of describing the

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[54]Russell, op. cit., pp. 245f.

phenomenal world in certain of its aspects.

A word of caution must be uttered at this point. "Phenomenal" for Leibniz did not mean the same as "perceivable". When he defines in a passage already quoted in the introduction a continuous line as one where there is a point between any two points. he is speaking about the phenomenal world. But this does not mean that there is a visually distinguishable point between any two other visually distinguishable points. Rather. this dictum describes a non-perceivable feature of lines; in other words. not all characteristics of phenomenal objects are perceptible. This assumption. soon to be removed by Berkeley. seems to make it more accurate to say that the phenomenal world is ordered according to the principles of geometry. and that we can perceive only the grosser features of this arrangement. This means that the theorems of geometry. although descriptive of phenomenal space, are not arrived at through the empirical observation of the phenomenal world. Rather. it is asserted apodictically that the phenomenal world conforms to the theorems of geometry. even in those parts that are beyond observation. Leibniz points out that "in order for there to be any regularity and order in Nature. the physical must be constantly in harmony with the geometrical."[54a]

[54a] Letter to Varignon. Wiener, <u>op. cit</u>.. p. 185.

Thus the fundamental assumption that this universe is well arranged, "the best of all possible", leads Leibniz to conclude that geometry must apply without restriction to the physical world, or, to be more specific, that Euclidean geometry thus applies. Now the lines, surfaces and bodies discussed in this geometry are said to have the property of continuity: "Geometry is but the science of the continuous"[55] Leibniz points out. Now since geometry and physics are in harmony, physical bodies must also be considered continuous, and this harmony, which is considered a priori certain, would not obtain "if wherever geometry requires some continuation, physics would allow a sudden interruption".[56]

It must constantly be borne in mind that geometry does not describe the world as it is, but only as it appears, or, as Leibniz puts it on another occasion: "continuous quantity is something ideal, which belongs to possibles and to actuals considered as possibles."[57] It may have seemed to Leibniz that the eviction of the problem of the continuum from the realm of actuals robbed it of

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[55] Ibid.

[56] Ibid.

[57] Russell, op. cit., p. 246.

its metaphysical sting: there simply _is_ no continuous entity and thus the question as to the composition of the continuum -- if there were one -- becomes rather "academic". But this did not prevent Leibniz from addressing himself to this problem: namely what sort of thing is it that is demanded by Euclidean geometry or, to put it otherwise, what is the composition of the phenomenal continuum of physical bodies.

According to Leibniz, when a representation of a continuous quantity takes place in a mind, there is no corresponding continuous quantity "out there". This follows from the distinction of the phenomenal from the intelligible. Hence all that can be said when a mind has such a representation is that a "notion" is present in that mind which may be occasioned through, or occur coincidentally with, the presence of monads. Now one cannot speak of the parts of a notion in the same sense in which one speaks of the parts of an object in everyday discourse. A distinction must be made between what may be called the analysantia of a notion and the parts of an object. This Leibniz points out in the following passage:

> Several people who have philosophized, in mathematics, about the point and unity, have become confused, for want of distinguishing

> between resolution into notions and division into parts. Parts are not always simpler than the whole, though they are always less than the whole.[58]

It seems to be Leibniz' contention that any given continuum, being nothing but a notion, has no parts which are simpler than itself. This, it seems to me, amounts to saying, that "continuum" cannot be defined through terms denoting entities that are contained "in" continua. This conviction, of course, is bound up with the notion that in a definition the definiens must always consist of concepts that are simpler in some sense than the concept in the definiendum. Leibniz seems to have assumed that parts of a continuum are never simpler notions than the original continuum itself, supposedly because they have to be described as 1/2 or 1/4 of the original quantity, etc. Thus the question as to what the phenomenally simple constituents of the continuum are, is invalid if what is asked for are entities simpler than the continuum itself; there are no such things.

Nevertheless, divisions of a continuum can, of course, be made. But, Leibniz contends "there are no

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[58] Russell, op. cit., p. 246.

divisions in it but such as are made by the mind, and the part is posterior to the whole".[59] This amounts to saying that in a continuous quantity any assignable number of divisions can be carried out, but the resultant parts will never be the constituents of the continuous quantity, if by "constituents" we mean parts that give rise to, or are simpler than, the whole which they compose. This is the strategem by which Leibniz wants to resolve the problem of the composition of the continuum: the continuum is declared not to have parts that can be said to compose it. Rather, its parts are posterior to the whole. However. one might justifiably ask what the difference is with respect to the paradox of Galileo between infinite sets whose elements are defined through the set and sets that are defined through its elements. It seems that this distinction makes no difference whatever. Thus. if a line has infinitely many parts which are posterior to the whole. the paradox arises just as much as it does if the parts are anterior to the whole. It seems to me that it is not even necessary fully to understand the anterior-posterior distinction here introduced by Leibniz in order to see that the stragegem

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[59] Russell, op. cit.. p 245.

is of no avail. The paradox of Galileo, successfully defeated in the intelligible world rears its ugly head in another world.

However, I believe that Leibniz can be defended. If the divisions of a continuum are, so to speak, man made, then one cannot say that a continuum has any more parts than are actually produced, and this number is, of course, always finite. If this is what Leibniz had in mind, then we may say that the problems of the continuum are, after a fashion, removed. At any rate, the solutions which Leibniz offers and which I have attempted to indicate on the preceding pages are more ingenious and comprehensive than any subsequent ones down to the time of Bolzano.

One of the questions that remains to be resolved is how Leibniz justified the fact that geometricians, he himself not excluded, availed themselves freely of the notions that a line is infinitely subdivisible, that it contains an infinite number of infinitesimals etc. The justification for these procedures is entirely pragmatical, as I have indicated above, and seemed to satisfy Leibniz fully In the <u>New Essays</u>, Leibniz writes:

> You (Philalethes -- Locke) deceive yourself in wishing to imagine an absolute space, which is an infinite whole composed of parts;

> there is none such. it is a notion which implies a contradiction. and these infinite wholes. and there opposed infinitesimals, are used only in the calculations of geometers. just like the imaginery roots of algebra.[60]

Of course, that these calculations are successful is patent to every observer, and therefore Leibniz seemed to believe the fictions contained in the calculations must be condoned. Some commentators seem to believe that Leibniz, in claiming that there really are no infinitesimals already had an inkling of the foundations of the calculus that were to be laid much later with Couchy and others. Boyer. in his Concepts of the Calculus writes:

> A rather inexact tradition would impute to Leibniz a belief in actually infinitesimal magnitudes. However Leibniz himself. in a letter written about two months before his death, said emphatically that he 'did not believe at all that there are magnitudes truly infinite or truly infinitesimal'. These conceptions he regarded as 'fictions useful to abbreviate and to speak universally'.[61]

The preceding investigation points at the sense in which this remark is to be taken. Of course there are no infinitesimals in reality. The intelligible, real world

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[60] Nouvaux Essais., p. 163.

[61] Carl B. Boyer, op. cit.. p. 219.

contains no such things. But as far as the phenomenal world is concerned they can be justified, so Leibniz seemed to believe, just as much or just as little as other geometrical entities, such as continuous lines. Thus the repudiation of infinitesimals was not brought about through predominantly mathematical considerations, but followed from a certain metaphysical position, which in turn, it must be pointed out again, was adopted in order to overcome certain logico-mathematical difficulties.

Let me now briefly indicate the significance of the foregoing investigation. I have attempted to show that Leibniz' metaphysical system was adopted, to a large extent, in order to overcome certain logical difficulties. If I have succeeded in doing this, I have at the same time shown that here is a system of metaphysics in which certain seeming logical paradoxes cannot arise. and I have reason to suppose that other metaphysical systems of the same and subsequent eras were developed with similar purposes in mind

Of course I have not attempted to give an exegesis of Leibniz' entire system, but a great number of its features have been discussed, and a great number of others would seem to follow immediately from the points mentioned.

For example in the beginning of the Monadology Leibniz writes:

> There is also no way of explaining how a monad can be altered or changed in its inner being by any other creature. for nothing can be transposed within it. nor can there be conceived in it any internal movement which can be excited. augmented, or diminished within it. as can be done in composites. where there is change among the parts. The monads have no windows through which anything can enter or depart.[62]

Thus it can be seen that the assumption of actual independence of the created monads from one another is an outgrowth of the view that the monads have no parts. an assumption which we saw above to be indispensible for the resolution of the problems of the continuum. The lack of windows in monads. in turn. requires the introduction of the notion of the preestablished harmony in order to explain the succession and presence of representations within the individual monads. Thus other parts of Leibniz' system, although not themselves required to resolve the problems of the continuum, would seem to follow at least in Leibniz' opinion, from the assertions which were made in order to keep those problems from arising. This gives us all the

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[62] Monadology, No. 7, Wiener, op. cit.. pp. 533f.

more reason for supposing that those parts of the system which are connected with the problems of the continuum are the ones which are closest to the problem for the sake of which the whole scheme was developed.

The foregoing pages have shown Leibniz' philosophy to be a very complicated and intricate system. In the sequel. I shall examine solutions to the problems of the continuum offered by Bayle and Berkeley which are much more radical and also much simpler than Leibniz'

CHAPTER II

GEORGE BERKELEY

I now wish to discuss the ways in which Berkeley dealt with the problems of the composition of the continuum. To this end it is necessary to remove certain prevalent misconceptions concerning Berkeley's philosophy and its development. For example, Berkeley's interest in mathematics has sometimes been doubted despite the fact that a very large proportion of his studies are concerned with mathematical topics. A certain amount of mathematical interest has sometimes been conceded to him, but his mathematical ability has usually been considered extraordinarily poor. Hand in hand with these attitudes has gone the notion that Berkeley's motives for developing his immaterialist philosophy were largely of a moralist or theological character. Occasionally, too, Berkeley is considered to be one of the philosophers, in a long line of system builders, who do not have much regard for extraphilosophical problems or for the application of a philosophical scheme to the world at large.

All of these interpretations I hold to be mistaken, and I will try in the sequel to disprove them in detail. In addition, I wish to show that the resolution of the problem of the continuum. as far as we can now discern, was one of the chief motivations for developint his immaterialistic metaphysics. The consistent application of his principles led him to develop a geometry that is very odd, indeed, but the mere fact that the application of his principles is consistent have led me to consider his mathematical ability with greater respect than seems customarily the case among his commentators.

Let us consider first the frequently held view that Berkeley's immaterialism was inspired mainly by moral and theological desires. This notion is bluntly, if not very aptly, expressed by Mr. Butler, who points out in his Philosophy of Berkeley.

> To Berkeley's mind it (i.e. the doctrine of an external material world) is the main prop for scepticism, atheism and materialism. As a good Bishop he is therefore eager to establish the view that only spirits and their ideas exist.[63]

Mr. Butler has here expressed in a singularly forthright manner what seems to me a widely held opinion concerning the origin of Berkeley's system. Ever since Kant (Critique

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[63] Benjamin Butler. Philosophy of Berkeley, Boston, 1957, p. 2.

themselves, since their doctrine has certain inconsistent consequences which reduce the whole system to absurdity and should leave the adherent of such a system suspended in a state of doubt, i.e. should make him a skeptic. It must be borne in mind that Berkeley's arguments contra skepticism can be employed by any scientist who replaces a given scientific generalization by another because the former lead to inconsistencies or was inconsistent with certain other statements believed to be true. But the fact that a scientist can use such an argument, and the fact that Berkeley did use it, explains in no way the form and content of the respective theories.

As to the second point, namely that Berkeley's immaterialism was advanced because the contrary doctrine supports atheism, it must be agreed that Berkeley believed that a major result of his philosophy is the demonstration of the untenability of atheism -- he gives a well-known proof for the existence of God and in general expounds this feature of his philosophy generously and repeatedly. But one cannot sanely assume that this demonstration was an only, or even a major, motive for the development of "the immaterial hypothesis." Moreover, many Christian apologists have made it their aim to disprove atheism

of Pure Reason B 71) mocking critics of Berkeley have felt called upon to refer to him as "the good Berkeley" or "the good Bishop Berkeley". It may not be out of place to point at the exceptional inappropriateness of this faintly supercilious epithet where Berkeley's philosophical insight is discussed. It must furthermore be pointed out that Berkeley only became a bishop more than two decades after the _Principles_ were published. To be sure. at the time of the publication of the _Principles_ he was ordained, but the ordination was a routine part of the _academic_ career of the fellows at Trinity.

Let us consider Butler's explanation of the origin of Berkeley's immaterialism in some detail. First it is said that the contrary doctrine, namely that there is an external material world. furnishes support for skepticism. One might be lead to think that skepticism was rampant at Berkeley's time, and that he set out to develop a theory that would not give rise to it. But such clearly was not the case. The first dialogue between Hylas and Philonous makes it amply clear that the man alleged to be a skeptic was Berkeley himself and that, by way of a counter attack he attempts to establish that those adhering to the doctrine of an outside material world _ought to be_ skeptics without thereby being led to a philosophy of immaterialism. The third point made by Mr. Butler, namely that Berkeley advocated his immaterialism because it would disprove materialism is too trivial to need any comment. In summary, the notion that Berkeley was posed against materialism because of certain religious and moral convictions concerning its connection with skepticism and atheism gives no reasonable explanation of the development of Berkeley's system.

More serious students of Berkeley have attempted to "explain" the genesis of his system by pointing to his dependence upon Locke and Malebranche [64] These eminently worthy pieces of historical scholarship suffer from the one-sidedness of treating philosophy as something like an intramural affair, a viewpoint that is perhaps appropriate for a discussion of the German Idealists, but certainly out of place where Enlightenment philosophers are under discussion. There can be no question that Berkeley was influenced by both Locke and Malebranche, but the mere tracing of lines of historical development will not answer the question why Berkeley differed so significantly from both

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[64] Cf. R. I. Aaron, "Locke and Berkeley's Commonplace Book", Mind, N.S. Vol. 40, 1931, pp. 439-459. A. A. Luce, Berkeley and Malebranche, London, 1934.

these philosophers. Obviously, there were certain difficulties which, to his mind, neither of these systems resolved. A reconstruction of the context of the development of Berkeley's philosophy can therefore not be complete without an identification of these problems.

A third group of commentators who concerned themselves with the genesis of Berkeley's system is formed by those philosophers who are reluctant to search at all for a reason behind the adoption of the immaterial hypothesis, who treat this adoption as some sort of spontaneous, unconsidered act. This orientation leads to the assumption that Berkeley's immaterialism must be considered as something like an unmotivated fabrication whose author subsequently tried, with considerable difficulty, to reconcile it with his experience and the body of knowledge that he had inherited. Johnston expresses this viewpoint most clearly:

> No harsh Socratic maieutic was needed to bring it (the New Idea) to the birth; it came to light easily and almost imperceptibly, and as we scan the sentences in which Berkeley indicated the process, it is easy to sympathize with his joy and surprise as he gazes at the child of his mind.[65]

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[65] G. A. Johnston, The Development of Berkeley's Philosophy, London, 1923, p. 20.

This naive view has at once the most adverse consequences as regards the appreciation of Berkeley's system. Immaterialism becomes an aperçu, a mere playful invention, which can hardly be justified in view of its results upon geometry. Johnston writes:

> His willingness to throw overboard the solid achievements of the established geometry simply because they did not accord with an aperçu of his own does not encourage to rate his mathematical ability very highly.[66]

I believe that Johnston's approach to Berkeley's philosophy is misguided. His way of considering philosophical systems would have been appropriate for a discussion of some German philosophers of the early Nineteenth Century whose systems can perhaps be described as free creations. It is known that Schelling occasionally produced philosophical systems so rapidly that some of his products were superceded by others before they had a chance to appear in print. The editor of one of the editions of Hegel's Philosophy of Law attacks critics of Hegel who point out that Hegel's system does not agree with the facts by saying that "systems can only be refuted through other systems."[67] Implicit in this statement is the notion that

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[66] Op. cit., p. 90.

[67] Hegel, Werke, 2nd. ed., Berlin 1840, Vol. 8, Ed. Eduard Gans, p. XIV.

system-building philosophers are not concerned with problems found in the sciences, in mathematics, or in everyday life, and that therefore an appeal to facts cannot lead to a refutation of their systems. For the philosophers of the Enlightenment it can generally be said that their systems had the purpose of resolving certain problems which are encountered in everyday experience, in the sciences or in mathematics. It is, therefore, of the utmost importance that the commentator become aware of the extraphilosophical problems which are the *raison d'être* and mainspring of these philosophical systems. A deficiency in this respect is a very serious failure. Let me remark here parenthetically that Cartesianism was thrown into its deepest crisis when it was discovered that a reasonable system of *physics* must assume that a force is required to change the direction of a motion. The philosophers of that age took this extraphilosophical discovery seriously enough to abandon all attempts of rescuing Descartes' unsatisfactory attempts at explaining the interaction between mind and body.

Johnston's psychological thesis of spontaneous creation has its systematic counterpart in the notion that Berkeley's fundamental assumptions have none or "few" arguments in its favor. This view is expressed by C. R. Morris, who writes:

"He (Berkeley) expounded his system boldly and shortly, offering in the first place very few arguments in support of it except that of its obviousness."[68] Since Morris never states what Berkeley does in the second place, and since he never states the few arguments aside from obviousness, we must assume that he was unable to find any. It seems likely that Morris cannot find any such arguments because he is, so to speak, looking in the wrong direction.

Let me insert at this point a general consideration as to what kind of argument may be required in order to make a philosophical system or thesis plausible, and where one should look to find "arguments in favor" of a given thesis. It seems that what Morris is looking for are antecedently established propositions of which the thesis or system is a deductive consequence. But this procedure, although frequently employed by commentators. is not always in order. since it is not always possible to produce arguments of this sort. But if no such premises are offered, it does not. therefore. follow that the system in question

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[68] C. R. Morris. Locke, Berkeley, Hume, London, 1931, p. 65.

has none or "few" arguments in its favor.[69]

I submit that Berkeley's system is to be envisaged in much the same way as a high level scientific generalization Its support is to be found in its deductive consequences and their truth rather than in premises from which it follows. There is much evidence in Berkeley's writings to the effect that he wanted his philosophy to be considered in this light. In the Philosophical Commentaries, entry no. 207, he states:

> My end is not to deliver Metaphysiques altogether in a General Scholastique way, but in some measure to accommodate them to the Sciences and show how they may be useful in Optiques, Geometry etc.[70]

Generally, the statements in which Berkeley attempts to justify the immaterial hypothesis always stress the fact that the consequences of his thesis are more desirable, explain more, or are less paradoxical, than the consequences of the rival materialist thesis. For example, Berkeley's

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[69](Of course. I do not wish to claim that the procedure criticised apropos of Berkeley's system is always out of order. If a philosopher claims that the evidence for his system rests with a number of "self-evident" propositions, then his claim will have to be examined. But Berkeley never argues in this vein).

[70]A. A. Luce and T. E. Jessup, (eds.), The Works of George Berkeley, London, 1948, Vol. I, p. 27.

claim that the problems of the continuum do not arise in his system is well enough known. It must here be emphasized that this alleged disappearance of the problems of the continuum is one of the results of the immaterial hypothesis and is as such offered in support of the thesis.

The analogy between certain philosophical systems and scientific generalizations will help us to clarify another point. Scientific generalizations are generally put forth in an attempt to remove certain problems, to explain certain lower level propositions. These problems, or the assertion of such propositions, can be said to occasion the higher level generalization. While the occasioning of such problems is, perhaps, of small logical importance, its recognition is indispensible for the reconstruction of the context of discovery. For the philosophy of science such reconstructions may be dispensible. Not so in the history of philosophy. Berkeley is a case in point. On a successful reconstruction of the context of his discovery of the immaterial hypothesis hinges the answer to such questions as "was the immaterial hypothesis an _aperçu_ that was only subsequently and with difficulty reconciled with experience and mathematics?" or "was it a hypothesis specifically introduced in order to resolve certain difficulties in our account of experience or mathematics?" I am convinced that the latter is the case, and that the problem of the continuum was the particular occasion for Berkeley's introduction of the immateralist hypothesis.

It is now incumbent upon me to demonstrate in what way the problem of the continuum influenced the development of Berkeley's metaphysics and to what solution the problem was brought. Berkeley's solution is familiar in kind to those of some modern analytic philosophers: he declares the problem of the continuum to be a pseudo-problem. Continuous quantities had been described as being infinitely divisible i.e. as having more than a finite number of parts. Galileo had shown, and Berkeley knew, that infinite sets have proper subsets with as many members as themselves. That Berkeley was familiar with this result is attested to in the following passage from the Philosophical Commentaries. "The infinite divisibility of matter makes the half have an equal number of equal parts with the whole."[71] This is at variance with Euclid's ninth axiom. Hence the difficulty. In Berkeley it is resolved, as we shall see, through the assumption that there are no

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[71]Entry 322. The Works of George Berkeley, Vol. I, p. 39.

entities which fit the description given above of continuous quantities. Hence, if there are no entities of this sort, then the question what it is that they are composed of is invalid, is a pseudo-question.

Berkeley's solution may here be sketched in a few words. The problem is construed as being concerned with the composition of extended sensible objects. These can be said to be continuous in appearance, "continuous" here referring to a certain phenomenal characteristic. I hesitate to hazard a definition of "continuous", but by way of example it can be said that a figure is visually continuous if it is uniformly colored. Similar criteria may be set up for phenomenal continuity in the tactile field etc. and eventually one may arrive at a definition of phenomenal continuity in general. However, since Berkeley nowhere attempts to produce such a definition, a reconstruction of his system runs into rather formidable difficulties on this point. We have to be satisfied with a rather vague notion of phenomenal continuity.

According to Berkeley, any phenomenally continuous figure, line, etc., is composed of a finite number of smallest phenomenal units, called _minima sensibilia_. _Minima sensibilia_ can be divided into _minima visibilia_, tin

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minima tangibilia etc. Since space and extension are constructions dependent only upon the senses of sight and touch, the _minima visibilia_ and _tangibilia_ are the only kinds of smallest unit of perception with which we need be concerned at this point.

It is a consequence of the principle that _esse est percipi_ that entities smaller than _minima sensibilia_ cannot be said to exist. Now since the number of _minima sensibilia_ in a figure is finite, and since no figure can be said to have parts smaller than a _minimum sensibile_ it cannot be asserted within the framework of Berkeley's system that any figure has more than a finite number of parts. Thus the paradox of Galileo cannot arise. Moreover, in Berkeley's system geometry is described as a science whose task it is to describe the various relations obtaining between continuous figures and lines, etc. Geometry is thereby made an empirical science. It is to discuss certain aspects of sensory experience. It does not concern itself with infinitely divisible lines and figures, but only with the phenomenal pseudocontinuum. It seems that Berkeley took seriously the theory that geometry is concerned with describing the space of our experience, and it is perfectly justifiable to say that this space contains nowhere any

subspace of which it can be empirically ascertained that it it has infinitely many parts.

A serious criticism of Berkeley's views on this subject can only proceed from the assumption that it is not the aim of geometry to describe empirical space. The construction of such a geometry, of course, is in part arbitrary and, in particular, it can be so constructed that it speaks of spaces, surfaces and lines which have infinitely many parts. It speaks well for Berkeley's mathematical ability that he was willing to countenance such a geometry, but, since it would largely consist of assertions not referring to matters of empirical fact, he places against it a moral injunction, to wit, that it is not worthwhile to investigate matters without practical value. If difficulties and inconsistencies arise in such a geometry, they can no longer be said to arise from an inability to analyze matters of empirical fact. We could summarize Berkeley's opinions on this subject by saying that it was of small concern to him that some mathematicians have constructed a system called geometry, the difficulties of which they were unable to resolve. On the other hand, he claims that:

> Whatever is useful in geometry, and

> promotes the benefit of human life, does still remain firm and unshaken on our Principles; that science considered as practical will rather receive advantage than any prejudice from what has been said...For the rest, though it should follow that some of the more intricate and subtle parts of Speculative Mathematics may be paired off without any prejudice to truth, yet I do not see what damage will be thence derived to mankind.[72]

The above gives in essence Berkeley's solution to the problem of the continuum. In summary, what is considered is not the continuum as later defined e.g. by Dedekind, but a phenomenal continuum not further defined. Any figure that is continuous in this sense is said to consist of a finite number of minima sensibilia, smaller than which nothing can be. This result follows from the principle that esse est percipi. This being the case there can be no actual infinite, and the paradox of Galileo cannot arise with respect to actuals. Thus geometry as the science of empirical space is vindicated. Against "speculative geometry" Berkeley places a moral injunction. We thus find in Berkeley the same attitude that we could observe in Leibniz: the problems of the continuum were felt to be

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[72]Principles of Human Knowledge, No. 131. Wherever Berkeley has provided a numbering or the paragraphs, I shall quote by paragraph number, without stating edition or page.

severe only as long as they arose in connection with a description of the world. Not nearly as much significance was attached to them as problems of pure mathematics.

After having thus outlined Berkeley's solution, I will now endeavor to demonstrate that among those problems which Berkeley initially attempted to solve with his immaterial hypothesis the problems of the continuum and of the infinite divisibility of matter have a prominent place.

Let us, first of all, turn to a passage in the Principles in which Berkeley sets forth some of the "innumerable consequences, highly advantageous to true philosophy as well as to religion" which flow from his principles. He points out:

> If by distinguishing the real existence of unthinking things from their being perceived, and allowing them a subsistence of their own, out of the minds of spirits, no one thing is explained in nature, but on the contrary a great many inexplicable difficulties arise; if the supposition of matter is barely precarious, as not being grounded in so much as a single reason; if its consequences cannot endure the light of examination and free enquiry, but screen themselves under the dark and general pretence of infinites being incomprehensible; if withal the removal of this matter be not attended with the least evil consequence; if it be not even missed in the world, but everything as well, nay much easier conceived without it; if, lastly, both sceptics

> and Atheists are forever silenced upon supposing only spirits and ideas, and this scheme of things is perfectly agreeable to both Reason and Religion: methinks we may expect it should be firmly embraced, though it were proposed only as an <u>hypothesis</u>, and the existence of matter had been allowed possible: which yet I think we have evidently demonstrated that it is not.[73]

I think that we can expect this passage not only to supply us with arguments for the acceptance of the immaterial hypothesis, but also with a reason why it was adopted in the first place. Three such reasons are given here. First, the contrary assumption, namely that there is matter independent of mind, leads to difficulties with the infinite and thereby to contradictions concerning the composition of the continuum. Secondly, the adoption of the immaterial hypothesis has no evil consequences. And thirdly, the adoption of this hypothesis will silence the sceptics and atheists forever. Clearly, not much weight can be attached to the second of these reasons. No philosopher will concede that his system has evil consequences, and one can hardly be expected to accept inoccuous nonsense merely because it is harmless. The third of the stated reasons has customarily been brought forth to explain

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[73]Berkeley, <u>Principles of Human Knowledge</u>, No. 133.

Berkeley's motivation for developing his metaphysics. However. while it stands to reason that Berkeley would not have accepted or developed a doctrine that runs counter to the tenets of received religion. I believe that it sufficiently established that the fortification of religious arguments was a welcome consequence rather than the problem originally to be resolved. There remains the third point mentioned, and I believe that from a consideration of Berkeley's early writings it can be shown that the immaterial hypothesis was adopted primarily in order to cope with the problem of the composition of the continuum.

Some questions of chronology have to be considered at this juncture Three of Berkeley's early writings have to be discussed to settle the issue. namely his Arithmetica and Miscellanea Mathematica.[74] his small essay Of Infinites[75] and his Commonplace Book or as Luce has renamed it. Philosophical Commentaries.[76] The composition of all three of

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[74]The Works of George Berkeley, Ed. A. A. Luce and T. E. Jessop, London 1948 ff., Vol. IV, pp. 159ff. (Where-ever Berkeley has not supplied paragraph numbers. I shall quote according to the Luce - Jessop edition).

[75]Ibid.. Vol. IV. pp. 235-238.

[76]Ibid.. Vol. I. pp. 1-139.

these probably falls into the years 1704-1708. The *Arithmetica* and *Miscellanea Mathematica* were published in 1707. but supposedly composed three years earlier. *Of Infinites* is a small paper that was presumably read before a philosophical discussion group. It is undated. The *Philosophical Commentaries* are a diary in which notes are taken. obviously in preparation of the writing of the *New Theory of Vision* and the *Principles*. There are no dates in this diary either. The immaterial hypothesis is found almost at the outset of it. and one can assume that it was started with the idea of developing the consequences of this thesis.

I think that it can be shown that the essay *Of Infinites* and the *Arithmetica etc*. preceded the diary and the inception of the immaterial hypothesis. and that Berkeley's concern with the questions of geometry and the continuum were not resultant upon the development of his thesis but preceded it. Luce writes the following about the *Arithmetica* and *Miscellanea Mathematica*:

> The work has the look of a fellowship thesis, specially designed to remove the impression that he disliked mathematics. We know from the *Philosophical Commentaries* that Berkeley about this time held very scornful views about some mathematics and some mathematicians, and if he did not keep those views to himself. College

> gossip would soon make him out ignorant of mathematics or prejudiced against them, and such gossip might weaken his chances of election.[77]

We see that even Luce, otherwise a very competent commentator, had considerable doubts whether Berkeley had any genuine interest in Mathematics. In this case the immaterial hypothesis is said to force him, for reasons having to do with his career, to undertake a study of mathematical subjects. But Luce's assumption clearly rests on very shaky grounds. We have no reason to mistrust Berkeley when he states in the introduction to the essay that most of the parts of the work "<u>triennium in scriniis delituerint</u>".[78] Now the fellowship which Berkeley wished to fill fell vacant in 1706 and was to be filled in 1707. We are hardly justified in assuming that he wrote an essay in 1704 in order to convince the dons in 1707 that he did not dislike mathematics. It seems much more natural to assume that the work was composed out of true interest for mathematical subjects. The publication date may have been decided upon for career reasons, but it is not likely that

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[77] Editor's Introduction to the essay in <u>The Works of George Berkeley</u>, Vol. IV, pp. 159f.

[78] <u>The Works of George Berkeley</u>, Vol. IV, p. 167.

the same holds for the content. The essay itself I do not wish to discuss here. It is nowhere concerned with the problems of the continuum. I merely mentioned it to lend emphasis to the theory that Berkeley showed interest in mathematics prior to the inception of the immaterial hypothesis.

Let us now turn to the second of the above mentioned writings. I think that a strong case can be made for the assumption that Of Infinites also preceeded the immaterial hypothesis. Luce again believes the contrary. Apparently, to some commentators Berkeley is unthinkable without immaterialism. Luce writes in his "Editors Introduction" to the essay:

> Here is no abstract mathematical problem accidentally connected with Berkeley's studies. It touches the heart of his philosophy, and is vitally connected with the massive argument for immaterialism.[79]

and in his Life of George Berkeley he points out:

> The mathematical doctrine of infinite divisibility was commonly regarded as furnishing evidence for the existence of matter; hence Berkeley's special interest in infinitesimals.[80]

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[79] Ibid., p. 233.

[80] A. A. Luce, The Life of George Berkeley, London, 1949, p. 36.

It is clear from these two quotations that Luce assumes the small essay to have been written as a result of Berkeley's concern with immaterialism. However, there is no indication whatever in the essay that Berkeley so much as toyed with the notions that were to form the core of his philosophical system. In fact, the theories expounded in the essay are in part incompatible with his later arguments, as I will now show.

Of Infinites shows Berkeley's disinclination to accept the thesis that the continuum is infinitely divisible, but beyond this it merely indicates in a vague way that the problems of the continuum might be removed through a certain terminological distinction. To begin with he quotes Locke to the effect that a distinction must be made between the idea of infinity of space and of space infinite. With Locke, Berkeley holds that we are able to form the former, but not the latter. He continues:

> Now if what Mr. Locke says were, mutatis mutandis applied to quantities infinitely small, it would, I doubt not, deliver us from that obscurity and confusion which perplexes otherwise very great improvements of the Modern Analysis. For he that, with Mr. Locke, shall duly weigh the distinction there is betwixt infinity of space and space infinitely great or small, and consider that we have an idea of the former, but none at all of the latter, will hardly go beyond his notions to talk of

> parts infinitely small or *partes infinitesimae* of finite quantities, and much less of *infinitesimae infinitesimarum*, and so on.[81]

Berkeley then points out that the "supposition of quantities infinitely small is not essential to the great improvements of the Modern Analysis"[82] and concludes the essay by saying:

> Now I am of the opinion that all disputes about infinites would cease and the consideration of quantities infinitely small no longer perplex Mathematicians, would they but join Metaphysics to their Mathematics, and condescend to learn from Mr. Locke what distinction there is betwixt infinity and infinite.[83]

Locke's notion was that we can have no idea of an infinitely large space or infinitely long time but that, by contrast, we can have an idea of infinitey, which is an "endless growing idea".[84] Similarly, and Berkeley repeats here only another contention of Locke's, we can have no idea of an infinitely small extension but only the idea of a perpetually diminishing or dividing of finite extensions which can never come to an end since any result

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[81] *The Works of George Berkeley*, Vol. IV, p. 235.

[82] *Ibid.*, p. 237.

[83] *Ibid.*

[84] Locke, *An Essay Concerning Human Understanding*, Bk. II, Ch. XVII, No. 7.

of such a division will still be a finite extension.[85]

Now the proof that Berkeley must have written this essay before he conceived of the immaterial hypothesis is simple enough. After all, the whole doctrine of the minima sensibilia amounts to saying that the process of division cannot be continued ad infinitum as Locke proposed and Berkeley suggested in Of Infinites. The suggestion that extension can be divided ad infinitum was one of the first doctrines to be rejected in the development of the immaterial philosophy as recorded in the Philosophical Commentaries. These considerations show the little essay quite at variance with Berkeley's later philosophy and indicate clearly that the thesis must be rejected that Berkeley concerned himself with the problem of infinite divisibility in this essay because he was trying to work out the consequences of the immaterial hypothesis. Luce himself states that the doctrine of infinite divisibility was supposed to furnish evidence for the existence of matter. How can we assume that Berkeley, the immaterialst, could embrace such a doctrine as he obviously does in Of Infinites? On the other hand, Of Infinites shows Berkeley

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[85] Cf. Ibid., Ch. XVII, No. 12.

actively engaged in an attempt to resolve the problems of the continuum. The above considerations should suffice not only to establish the essay as written before the Philosophical Commentaries were begun, but also to show that a solution for the problems of the continuum was actively sought by Berkeley before the inception of the immaterial hypothesis.

I have stated above that in the Philosophical Commentaries the immaterialist hypothesis is found almost at the outset. It is of great importance to notice, however, that at the very beginning of this diary Berkeley seems to assume that aside from phenomenal extension and time there is also a real extension and time. Now phenomenal extension quite clearly does not admit of infinite subdivision, and neither does phenomenal time. On the other hand, Berkeley seems to assume that real extension and time permit such a treatment, as I shall subsequently document. He left this position shortly afterwards in favor of his immaterialism. Now if we assume that real extension and time were rejected solely on the grounds that their existence cannot be empirically demonstrated or perhaps for no reason at all, then we cannot easily account for the particular form Berkeley's philosophy

took, since a rather simpler form of immaterialism suggests itself.

It is generally recognized that upon rejection of "real" extension and time Berkeley had to harmonize his notion of phenomenal extension with geometry, since the latter can now no longer be said to have "real" extension as its subject matter. But unlike the Gospels, immaterialism and geometry can be made to harmonize through the following device: assume that any idea of visual perception is produced by God with infinitely many parts of which our visual acuity can distinguish only a finite number. We can then pursue geometry as before, since we can always claim that the teachings of Euclid apply to our ideas as produced by God, although not as seen through us.

I do not wish to claim that this is a very reasonable philosophical stand to take, but I believe that a case can be made for a brand of immaterialism that allows for this sort of direct reconciliation with geometry. However, I believe that it would have been quite unacceptable for Berkeley, since it inherits all the problems of geometry, in particular the difficulties in connection with the infinite divisibility of the continuum. That is to say that if we assume that it is the removal of the latter which is Berkeley's main concern, then we can understand why Berkeley adopted his particular type of immaterialism: immaterialism does not in and by itself prevent the occurrence of the problems of the continuum, but immaterialism together with the doctrine of _minima sensibilia_, smaller than which nothing exists, avoids them.

That Berkeley's reasoning followed these lines can be shown through a discussion of the beginning sections of the _Philosophical Commentaries_.[86]

I have already pointed out that at the beginning of the 18th century the question of the composition of the continuum was rarely considered _in abstracto_, but was usually couched in terms of a problem concerning the nature of time or physical space. At the outset of the _Philosophical Commentaries_ Berkeley is concerned with precisely these problems viz., "what is the nature of time?" and "what is the nature of extension?". The fourth entry reads "time train of ideas succeeding each other", and the third, which

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[86] I shall quote the entries in the _Philosophical Commentaries_ according to the numbers given them by A. A. Luce in _The Works of George Berkeley_, Vol. I, pp. 1-139. I shall assume throughout that the order in which Luce gives the entries reflects correctly the temporal sequence of their composition.

is obviously also about time. says "Whether succession of ideas in ye divine intellect."[87] Apparently, the flow of time is here considered to be formed by a series of succeeding ideas. This notion of time is familiar from the _Principles_, where Berkeley says:

> For my own part, whenever I attempt to frame a simple idea of _time_, abstracted from the succession of ideas in my mind, which flows uniformly, and is participated by all beings, I am lost and embrangled in extricable difficulties. I have no notion of it at all: only I hear others say it is infinitely divisible, and speak of it in such a manner as leads me to harbor odd thoughts of my existence: since this doctrine lays one under the absolute necessity of thinking, either that he passes away innumerable ages without a thought, or else that he is annihilated every moment of his life; both which seem equally absurd.[88]

This quotation from the _Principles_ shows not only that the characterization of time sketched in the _Commentaries_ is retained in the _Principles_ but also that the difficulty was with the notion of the infinite divisibility itself. Berkeley recognizes that infinite divisibility presupposes infinitely many parts, and these he does not concede for finite extensions or time spans. He holds

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[87] _The Works of George Berkeley_, Vol. I, p. 9.

[88] _Principles of Human Knowledge_, No. 98.

that if there is any time-span between two thoughts, and if this time span is infinitely divisible, then it must be infinitely long. Thus the notion that I must pass innumerable ages without a thought. On the other hand, if for the spirit esse est percipere, then the assumption that a spirit does not think at any one time leads to the consequence that he is annihilated at that time. Hence Berkeley assumes that time for a spirit is measured in terms of succession of ideas in that spirit's mind. This makes time private, and it cannot be assumed that all beings participate in the same time. In the beginning of the Commentaries Berkeley does not seem to have been entirely satisfied with this making private of time.

So aside from the concept of time in the above sense he introduces the notion of duration. Entry No. 8 reads "duration infinitely divisible, time not so."[89] There is no definition given of "duration", but it stands to reason that by this term he meant the intersubjective Newtonian time in contradistinction to the phenomenal time discussed above. A similar dualism seems to be intended with respect to extension. Entry No. 11 says "Extension not infinitely

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[89]The Works of George Berkeley, Vol. I., p. 9.

divisible in one sense.[90] I would presume that Berkeley held at the time that there was another sense in which one could speak of extension as infinitely divisible. This means that time and $extension_1$ are not infinitely divisible, while duration and $extension_2$ are. Time and $extension_1$ would be phenomenal time and space, duration and $extension_2$ would be the abstract time and space of Newtonian physics. The puzzlements connected with the continuum can occur in the latter, but they are exorcised from the former, simply because neither phenomenal time nor phenomenal space form a continuum. But the problem is not thereby resolved. It is merely extradited from the phenomenal sphere, and retains all its severity in the realm of absolute space and time. It is not until these latter are denied any existence that the problem of the continuum is made to disappear. Entry No. 26 states "Infinite divisibility of extension does suppose ye external existence of extension, but the latter is false, ergo ye former also."[91] This terse modus tollens argument announces the solution that Berkeley gave to the problem of the continuum: he denied

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[90] Ibid.

[91] Ibid., p. 10.

that there is absolute space or duration, and assumed that the phenomenal time and space are not infinitely divisible.

However, the doctrine here brought forth is not without its difficulties, as Berkeley well realized. For one thing, its repercussions on classical geometry become at once apparent. Entry No. 29 says "Diagonal incommensurable with ye side Quaere how this can be in my doctrine?"[92]

This entry bespeaks the fact that at this point Berkeley still meant to bring about a reconciliation of his doctrine with classical geometry. The entry in fact asserts the incommensurability of the diagonal with the side, but the inconsistency of this theorem with his doctrine is realized. How does he resolve the inconsistency? A large number of entries deal with the problem,[93] but of especial interest are entries 263 and 264. 263 states: "Mem: To enquire most diligently Concerning the Incommensurability of Diagonal and side. Whether it does not go on the supposition of unit being divisilbe ad infinitum, i.e., of the extended thing spoken of being divisible ad

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[92]<u>Ibid</u>.

[93]See Nos. 249, 250, 258, 263, 264, 276, 340, 457, 469, 470, 481, 500, 510, 516.

infinitum..."[94] and the next entry states flatly: "The Diagonal is commensurable with the Side."[95]

Here the decision is made. To achieve consistency classical geometry is declared in error. Of course, this not only holds for particular theorems of geometry but for the whole mode of approach associated with Euclidian geometry. According to Berkeley geometry is to be transacted as an empirical science. This becomes evident in entry 249:

> Particular Circles may be squar'd, for the circumference being given, a Diameter may be found betwixt which and ye true there is not any perceivable difference, therefore there is no difference. Extension being a perception and perception not perceived is a contradiction, nonsense, nothing. In vain to allege the difference may be seen by Magnifying Glasses. For in that case there is ('tis true) a difference perceived but not between the same ideas but others much greater entirely different therefrom.[96]

This passage is remarkable not only because of the assertion concerning the ratio between the diameter and circumference of a given circle, but also because it explains how the assumption of external matter could lead

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[94] The Works of George Berkeley, Vol. I, p. 33.

[95] Ibid.

[96] Ibid., p. 31.

to the thesis of infinite divisibility. For Berkeley, the sensation that we have when we look at a circle through a magnifying glass cannot be said to be "of the same object" than when we look at the "same" circle with the naked eye. Hence magnifying glasses will not allow us to discern parts in a *minimum visibile* that we sensed in a previous sight experience with the naked eye. If we assumed the contrary, then we would have to agree that more powerful magnifying instruments might produce still larger images of the "same" object, and that we could then discern parts that were not seen before, etc. Unless an absolute limit of magnification can be demonstrated, this would lead to precisely the problems that the immaterial hypothesis was meant to remove.

The above analysis has shown that Berkeley's solution to the problem of the continuum depended crucially upon the notion of the *minimum sensibile* (*visibile*, *tangibile*, etc.). Whenever a specific problem of a geometrical nature arises, e.g., the squaring of a particular circle or the ascertainment of the length of the diagonal of a given square, then we ought to proceed, according to Berkeley, by counting the *minima sensibilia* out of which these figures are said to be composed. The *minima*

sensibilia, being the smallest perceptible units, have no perceptible parts and therefore, according to Berkeley, no parts at all. But since they are perceivables, they have some magnitude so that a line or a shape is always made up of a finite number of them. But a peculiar difficulty arises at this juncture. Warnock puts it in the following way:

> He (Berkeley) says that there is an idea nothing that is not actually discerned in it. But certainly a line drawn on paper is not _seen_ as composed of a definite number of points; it looks continuous. It is not only that one has no idea how to make even a reasonable guess at the _number_ of points in a line; it does not look as if it were made up of points at all. And should not Berkeley have concluded that it _is_ not made up of points?[97]

The difficulty here exposed cannot be resolved by referring to Berkeley's theory of abstraction. It is well enough known that he countances only one mode of "abstraction" and that it is the imagining of parts of complex ideas of perception. He says:

> I find indeed I have a faculty of imagining, or representing to myself, the ideas of those particular things I have perceived, and of variously compounding and dividing them...To be plain, I own myself able to abstract in one sense, as when I consider some particular parts

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[97] G. J. Warnock, _Berkeley_, London 1953, p. 219.

> or qualities separated from others, with which, though they are united in some object, yet it is possible they may really exist without them.[98]

As concerns the number of points in a line it is precisely the question how many parts there are to the line that can really exist (i.e., exist in perception) without the rest being around, and before this is ascertained no subdivision of the line into smallest parts can be carried out in imagination. As far as I can see, Berkeley nowhere suggests a solution to the difficulties here indicated. A solution might be found if some kind of operation were indicated through which the division of a line or surface into *minima visibilia* or *tangibilia* might be affected. But even then considerable oddities would remain.

Berkeley repeatedly uses "point" and "*minimum visibile*" in the same sense. For example in the *New Theory of Vision* he says:

> No exquisite formation of the eye, no peculiar sharpness of sight, can make it (i.e., the *minimum visibile*) less in one creature than another; for, it not being distinguishable into parts, nor in anywise, consisting of them it must necessarily be the same to all. For suppose it otherwise, and that the *minimum visibile* of a mite, for instance, be less than the *minimum visibile* of a man; the latter

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[98] *Principles of Human Knowledge*, Introduction, No. No. 10.

> therefore may, by detraction of some part, be made equal to the former. It doth therefore consist of parts, which is inconsistent with the notion of a _minimum visibile_ or point.[99]

Assume that, as a matter of empirical fact the smallest perceivable blotches of a certain color, say green, are always oblong. Now if we were to say that green _minima visibilia_ are always oblong, we would be at variance with Berkeley as quoted above, since it would be clearly false that such an object does not in anywise consist of parts. We can always find a statement that is true of one side of such a shape but not of the other, and we thereby gain a procedure of distinguishing parts in such a shape. If this holds and yet there is no perceptible green speck smaller than certain oblong shapes, we are forced to the conclusion that there are no green _minima visibilia_, because the shapes described are no _minima_ since there is a sense in which parts can be distinguished in them, and they cannot be subtracted from since they would then no longer be _visibilia_. Under the above assumptions a geometry in Berkeley's sense of green objects would be quite impossible, since no ascertainable relations of magnitude

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[99] _Essay towards A New Theory of Vision_, _The Works of George Berkeley_, Vol. I., pp. 159-239, No. 80., p. 204.

obtain between green objects, magnitudes being compared by counting _minima sensibilia_. But this is clearly false. We may, after all, measure the length of green lines as well as the length of lines of any other color. Thus the assumption that there are parts in the smallest visible green specks must be false. In order to salvage the rest of the system we shall have to say that green _minima visibilia_ are oblong. The consequences of this assumption, however, are quite strange. For one thing, green geometry would differ from the geometry of shapes of other colors. It is conceivable, moreover, that a green square may be produced the length of whose side is one _minimum sensibile_. The difference between the length of the diagonal of this square and a _minimum visibile_ is smaller than one minimum visibile, hence there is no difference. Another square may be produced whose diagonal has the length of two minima visibilia precisely. The difference between the length of its sides and one minimum visible is smaller than a minimum visibile, hence there is no difference. In the first case $\sqrt{2}$ was equal to 1, in the second case it was equal to 2. In other squares it will vary between these two values. One can safely assume that this is wreaking more havoc in geometry than Berkeley intended, but if Geometry is to become an empirical science, it will have to put up with all manner of empirical conditions. Surely, since a set of statements that accurately describes the world or part of it is always consistent, Berkeley's geometry will be consistent. But one must hope that Berkeley's method is not the only one of removing the apparent inconsistencies of the continuum: Were it not for the fact that an inconsistent geometry always entails Berkeley's geometry, one might even be tempted to put up with inconsistencies.

A consideration of the above sort has some merit in that it throws doubt on the alleged generality of geometrical proofs in Berkeley's system. A geometrical proof, according to Berkeley, is conducted on some particular figure, but it becomes general, i.e., holds for all figures which have all those properties that are explicitly mentioned in the given proof. In all other aspects they may vary from the figure used in the demonstration.[100] But it seems now that if Berkeley's program of turning geometry into an empirical science is to be taken seriously, certain empirical properties of the paradigm figures other than those customarily advanced in geometrical demonstrations might have to be considered. It is a problem that

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[100] Principles of Human Knowledge, Introduction.

must be empirically investigated whether or not these empirical properties, e.g., color, do or do not make a difference. Only within the limits indicated by such an empirical investigation may we say that "any theorem may become universal in its use."[101]

Let me include here a brief summary. I think that I have shown the reasons for the adoption of the immaterialist hypothesis to lay with the fact that it avoids the problems of the continuum. To paraphrase Berkeley: on one hand, no demonstration can be given for the existence of external matter, but on the other hand, its rejection removed the difficulties of the infinite.[102] But Berkeley's solution is obtained at the expense of having to put up with a system of geometry of a rather strange nature. However, I believe that any attempt to construe geometry as an empirical science in Berkeley's sense will run into similar difficulties, a very good reason not to consider geometry an empirical science.

With the proposal of immaterialism and the postulation of a new basis for geometry Berkeley's task was not

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[101] *Ibid.*, No. 128.

[102] *Ibid.*, No. 133.

completed. He had to concern himself with the obvious fact that mathematical analysis, working with the assumption of infinite divisibility of lines, etc., and with the assumption of the existence of infinitesimals had been overwhelmingly successful. According to Berkeley, analysis as then conceived was premised upon erroneous assumptions. This is at first dogmatically asserted in the Philosophical Commentaries, the Principles and the Dialogues Between Hylas and Philonous, and later on proved in the Analyst.[103] In the Analyst Berkeley presents two sets of arguments against the Calculus. The first depends upon his own theory of abstraction as put forth in the introduction to the Principles. Query No. 4, at the end of the Analyst, asks "whether men may properly be said to proceed in a scientific method, without clearly conceiving the object they are conversant about...?"[104] and Query 9 reads "Whether mathematicians do not engage themselves in disputes and paradoxes concerning what they neither do nor can

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[103] The Analyst or A Discourse Addressed to an Infidel Mathematician, first printed in 1734, The Works of George Berkeley, Vol. IV, pp. 53-103.

[104] Ibid., p. 96.

conceive?"[105]

According to Berkeley, what can be conceived are ideas which must be considered in analogy to sense-data. In this sense, of course, infinitesimals or velocities in a point cannot be conceived. Therefore Berkeley's critical attitude. I believe that these objections had a rather salutory effect in that they made it quite clear that the foundations of analysis did not lay, and should not be sought, in the world of sense experience. But Berkeley's error is quite apparent: whether or not the symbols used in mathematical analysis in fact refer, or can be made to refer, to sensory experience is of no concern to the mathematician qua mathematician. It suffices that they be well defined, except for the primitive terms, and that all deductions proceed according to the laws of logic from the accepted axioms. Thus these objections of Berkeley's, while consistent with his philosophy and general approach to mathematics were really beside the mark.

But there is a second kind of objection raised in the Analyst which must be taken more seriously. They consist in detailed studies and criticism of particular

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[105] Ibid.

arguments put forth by Newton and Leibniz. Berkeley is able to show that there is at least a good deal of impreciseness contained in the proofs which he considers. Let me give one example:

> Let the quantity x flow uniformly, and be it proposed to find the fluxion of x^n. In the same time that x by flowing becomes x+o, the power x^n becomes $(x+o)^n$, i.e., by the method of infinite series
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> $$x^n + nox^{n-1} + \frac{nn - n}{2}\, oox^{n-2} + \text{etc.},$$
>
> and the increments
>
> $$o \text{ and } nox^{n-1} + \frac{nn-1}{2}\, oox^{n-2} + \text{etc.},$$
>
> are to one another as
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> $$1 \text{ to } nx^{n-1} + \frac{nn - n}{2}\, ox^{n-2} + \text{etc.}$$
>
> Let now the increment vanish, and their last proportion will be nx^{n-1}. But it should seem that this reasoning is not fair or conclusive. For when it is said, let the increments vanish, i.e., let the increments be nothing, or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, i.e., an expression got by virtue thereof is retained.[106]

Berkeley's reasoning is that the division undertaken in order to obtain the third line of the above proof presupposes that zero is not equal to zero, while later on,

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[106]*Ibid.*, No. 13, p. 71f.

when all but the first member of the infinite series are cancelled, zero _is_ assumed to be equal to zero. The point I wish to make here is not that all computations of the derivative of x^n are open to Berkeley's objection, but that in point of fact the status of the differential zero in the calculation was not clear at the time, and that some mathematicians went so far as to claim that for the purposes of the differential calculus a division by zero is permissible.

Berkeley brings forth arguments similar to the one above, against other methods of arriving at derivatives, and his criticisms are generally sound. The _Analyst_ has been said to mark "a turning point in the history of mathematical thought in Great Britain",[107] and Boyer points out "Berkeley's criticism of Newton's propositions was well taken from a mathematical point of view, and his objection to Newton's infinitesimal conceptions as self-contradictory was quite pertinent."[108] It must be noted that Berkeley's

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[107] Florian Cajori, _A History of the Conception of Limits and Fluxions in Great Britain from Newton to Woodhouse_, Chicago 1919, p. 89. cf. Also _The World of Mathematics_, ed. James R. Newman, N. Y. 1956, Vol. I, p. 286. For the importance of the _Analyst_ in the history of mathematics cf. also Gerhard Stammler, "Berkeley's Philosophie der Mathematik", _Kantstudien_, Erganzungsheft No. 55, Berlin, 1922.

[108] Carl B. Boyer, _The Concepts of the Calculus_, p. 226.

criticism was not so much directed against the mathematician as an "artisan" whose procedures are justified by their results, but against the mathematician qua logician. He says:

> To prevent all possibilities of your mistaking me, I beg leave to repeat and insist, that I consider the geometrical analyst as a logician, i.e., so far forth as he reasons and argues; and his mathematical conclusions, not in themselves, but in their premises; not as true or false, useful or insignificant, but as derived from such principles, and by such inferences. And, of as much as it may perhaps seem an unaccountable paradox that mathematicians should deduce true propositions from false principles, be right in the conclusion and yet err in the premises; I shall endeavor particularly to explain why this may come to pass, and show how error may bring forth truth, though it cannot bring forth science.[109]

The doctrine here indicated is that the results of computations in the differential calculus gain their truth and value from a cancellation of erros, a doctrine that we need not discuss here.

That Berkeley was correct in his objections was finally borne out when the foundations of analysis could be laid entirely without the introduction of actual infinitesimals solely through a consideration of limits. It must be pointed out, however, that the later theories

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[109] <u>The Analyst</u>, No. 20, p. 76f.

concerning the foundations of analysis, although doing justice to Berkeley's objections against Newton and Leibniz, did not incorporate any of his positive suggestions: none of them assumes that a line or surface ultimately consists of certain finitely extended particles.

What were Berkeley's positive suggestions concerning the foundations of analysis? In his earlier writings he asserts that on the adoption of his principles none of the advantages of the modern analysis will be lost and he points out that mathematicians really deceive themselves when they think that they actually consider infinitesimals:

> Whatever mathematicians may think of Fluxions, or the Differential Calculus, and the like, a little reflexion will show them that, in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense. They may indeed call those little and almost insensible quantities Infinitesimals.... But at bottom this is all, they being in truth finite; nor does the solution of problems require the supposing of any other.[110]

In connection with this passage, emphasis must be put on the phrase that mathematicians "in working with those methods" do not conceive, etc. Berkeley seems to mean by this that the mathematician, as soon as he employs the methods of the calculus in order to produce a drawing or

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[110]<u>The Principles of Human Knowledge,</u> No. 132.

in order to solve a practical problem must finally speak of, and consider only, finitely extended quantities. Berkeley seems to suggest that the calculations which precede this final interpretation are actually dispensible, but his works contain no positive assertion on this point. In the absence of such assertions I surmise that Berkeley would be inclined to square circles and draw tangents to curves entirely by gross visual estimation, assuming, as it seems, that the results would not visually differ from carefully drawn tangents that are produced after preliminary calculation in the differential calculus. I can find no evidence or suggestion for any other procedure in Berkeley. But my surmise seems to be well in keeping with Berkeley's fundamental assumption that geometry, and likewise analysis, are empirical sciences that must ultimately subject themselves to the judgments and estimates of the senses.

In conclusion let me once again quote from the <u>Philosophical Commentaries</u>. Toward the end of the first part of that diary he writes: "The Mathematicians think there are insensible lines, about these they harangue, these cut in a point at all angles, these are divisible <u>ad infinitum.</u> We Irish men can conceive no such lines."[111]

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[111]No. 393, <u>The Works of George Berkeley</u>, Vol. I., p. 44.

And a little later "I Publish not this so much for anything else as to know whether other men have the same Ideas as we Irishmen."[112] As it turned out, some other men had different ideas.

[112] No. 398., *Ibid*.

CHAPTER III

PIERRE BAYLE

My concern with Bayle in this chapter will be twofold: I wish to investigate Bayle's contribution to the philosophical discussion of the problem of the continuum and of infinite sets, and secondly I want to make some comments on the matter of Bayle's influence upon Berkeley. I have violated the historical sequence by discussing Bayle after Berkeley, since it seemed more convenient to discuss Bayle's influence after the material on Berkeley had already been assembled.

Bayle's main work and the one for which he is best known is the Dictionaire,[113] and I shall confine my discussion to views contained in it. The Dictionary does not set forth what one could call a philosophical system. Rather, it discusses philosophical topics only incidentally and in connection with the biographies of the philosophers contained

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[113]Pierre Bayle, Dictionaire historique et critique, Rotterdam 1697. This is a two volume edition. A greatly amended second edition in four volumes was published in 1702. In the present paper, I shall quote Bayle after the second English edition of the Dictionary, London 1734-38. The annotations will be made by citing the name of the article in question, and, if required, the letter and number of the note in question.

in that work. Nevertheless, penetrating philosophical insight reveals itself in these occasional remarks, and a definite philosophical persuasion is quite apparent in them. The Dictionaire exercised considerable influence, and occupied a position of much greater importance in intellectual history than is the usual share of reference works.[114]

I think that it is safe to assume that the magnitude of Bayle's influence was, in part, due to the fact that he entered the then topical dispute about the delineation of the boundaries of faith and reason with extraordinarily persuasive arguments, arguments that soon earned him the name of a sceptic. Bayle's philosophical persuasion with respect to these matters is, in a sense, a continuation of

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[114]That the Dictionaire was widely used is shown by the fact that upon the investigation of 500 private libraries of the 18th century in France the work was found in 288. (See Selections from Bayle's Dictionary, ed. E. A. Beller and M. duP. Lee, Jr., Princeton 1952, p. XX). Forty-some years after his death, Berkeley's and his son's and grandson's library was sold. According to Popkin, (Richard A. Popkin, "Berkeley and Pyrrhonism", The Review of Metaphysics, Vol. V., 1951-52, pp. 223-246) a copy of Bayle's Dictionaire was contained in that library. Popkin bases his assertion on the authority of A. A. Luce, (cf. Luce's edition of Berkeley's Philosophical Commentaries, London 1944, p. 388) who in turn trusted his own reading of an article by Aaron, Mind, N.S. XLI, p. 465 ff, which discusses the content of the library auctioned off. Aaron mentions only Boyle's, not Bayle's works as contained in Berkeley's library. That the Encyclopedists paid great heed to Bayle's opinions is well known, but it is obviously erroneous to assume that his ideas were propagated exclusively through that circle.

that expressed in the Port Royal Logic.[115] In essence it is the denial of the possibility that a consistent description of reality can ever be achieved. In the Port Royal Logic it had been asserted that "there are some things which are incomprehensible in their manner, yet certain in their existency, we cannot comprehend how they are, however it is certain, they are."[116] and Bayle notes with approval that the Port Royalists had already pointed out that researches concerning the nature of the infinite have their only use in forcing the understanding "however unwilling, to own that some things exist though it is not capable of comprehending them... All the force of human understanding cannot comprehend the smallest atom of matter, and is obliged to own that it clearly sees that such an atom is infinitely divisible, without being able to see how that can be."[117]

This insufficiency of reason over against reality was supposed by Bayle as well as the Port Royal Logicians to lead the thinker to an unquestioning acceptance of the

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[115] See Introduction to this dissertation, p. 1f.

[116] (Antoine Arnauld), Logic or the Art of Thinking, p. 392.

[117] Bayle, op. cit., Article Zeno, End of remark G. Bayle took the quotation from the Port Royal Logic, loc. cit.

doctrines of religion, or at least to wrest from him the argument that these doctrines cannot be accepted because they are contrary to reason, since not even the description of physical reality can be carried out without finally leading to inconsistencies.

There is some doubt, however, whether or not Bayle was ultimately sincere with respect to the persuasion that I have delineated above: In the article _Leucippus_ Bayle makes the following statement:

> Leucippus, Epicurus, and the other Atomists might have guarded against several unanswerable objections, if they had bethought themselves of giving a soul to every atom... I know they could not have avoided all difficulties by ascribing it to them: they might still have been pressed with invincible objections. Yet there had been some glory in parrying a thrust here and there.[118]

This quotation is apt to substantiate to some extent the claim that I made above, namely that Bayle did not believe that an ultimately consistent and trustworthy metaphysic or science could be developed. When sufficiently hard pressed, any position would lead to absurd consequences, so Bayle contended. Now part of the task of the _Dictionaire_ clearly was to investigate various philosophical positions,

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[118] _Ibid_., Article _Leucippus_, end of note E.

not in order to accept one or the other, but to show that in the final analysis they are all untrustworthy. I do not wish to decide here, and it is quite unnecessary for the purpose at hand to do so whether this undertaking was designed to show forth the limitations of reason and to support the position of faith, or whether it was performed for the glory that there is dealing many a thrust. Thus, while the over-all aim might have been to display brilliance in criticism or to demonstrate the weakness of reason for the benefit of faith, the result is that the Dictionaire hardly ever expounds or establishes a doctrine without criticising or demolishing it in some other passage or perhaps even some other volume of the work.

As a case in point let us consider some arguments connected with naive realism and immaterialism as they occur in the Dictionaire. We find the following situation. In the article Anaxagoras, Bayle criticizes that philosopher who seemed to have assumed that the ultimate components of bodies are destructible. Bayle points out:

> Compound Bodies alone are born and die and pass through a thousand Vicissitudes of Generation and Corruption; but Principles retain their Nature unchangeably under all the Forms which are successively produced. Anaxagoras could not say this of HIS Principles.[119]

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[119] Ibid., Article Anaxagoras, Remark C, No. I.

and

> Wood, when destroyed by Fire, ceases not to exist as Matter, or extended Substance. Thus there is a great Defect in the System of Anaxagoras.[120]

A little later he criticizes Moreri for attributing a certain doctrine to Anaxagoras by exclaiming:

> Here is Earth, there Air and Water; here a Meadow, and there a Wood. Anaxagoras would have been more extravagant, than even the most absurd Visionary that was ever put in a Mad-House, had he entertained any Doubts about it.[121]

The point at issue was the opinion, which Moreri had attributed to Anaxagoras, that the universe is homogeneous. So in the last quoted passage Bayle did not want to defend the existence of external matter, but the heterogeneity of the universe. Nevertheless, the language is not that of a phenomenalist or immaterialist, but rather that of a realist, a position which was clearly expressed in the quotations preceding the last one, and which clearly underlies Bayle's criticism of Anaxagoras.

Now the doctrine that is here used, or at least presupposed, does not tally in the least with the immaterialism that is proposed in the articles <u>Pyrrho</u> and <u>Zeno</u>, as we

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[120] <u>Ibid</u>.

[121] <u>Ibid</u>., remark C, No. IX.

> that bodies are coloured, and yet it is a mistake. I ask, whether God deceives men with respect to those colours? If he deceives them in that respect, what hinders but he may deceive them with respect to extension. This latter illusion will not be less innocent, nor less consistent, than the former, with the most perfect being.[121]

And further on Bayle points out "God does not force you to say, that it does exist, but only to judge that you feel it, and that it appears to you to exist."[122]

While Bayle here merely puts in doubt the existence of external matter on the grounds that there is not a sufficient reason for believing it, he produces very forceful reasons for denying its existence in the article Zeno. Before proceeding to a discussion of these arguments, let me again caution against the assumption that these disquisitions are intended to establish the foundations for an immaterialist or idealistic philosophy. They are only part in the general plan of demonstrating the untenability of any system of metaphysics. Thus, while the existence of external matter is put in doubt in Pyrrho, and a demonstration against this existence is offered in Zeno, external matter is nonchalantly assumed in Anaxagoras, as

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[121] Ibid., Article Pyrrho, remark B.

[122] Ibid.

shall presently see. It is clear, then, that Bayle's critical endeavours do not proceed from a well formulated and systematically developed philosophical position, but that they use whatever means are available, or can be provided _ad hoc_, in order to demonstrate the untenability of any given philosophical system. It seems then that we cannot attribute to Bayle a philosophical position or system in the ordinary sense, but rather some sort of meta-position, which maintained the futility of any philosophical commitment. This explains, and perhaps was meant to excuse, the eclectic and inconsistent method of criticism found in the _Dictionaire_.

But let me now try to document the phenomenological or immaterialist views that are set forth in the articles _Pyrrho_ and _Zeno_. In _Pyrrho_ the existence of external matter is put in doubt, and in _Zeno_ space and external matter are denied outright, and forthful reasons are presented in support of this denial.

In _Pyrrho_, Bayle writes:

> I have...not one good proof for the existence of bodies. The only good proof they can give me for it, is, that God would deceive me, if he imprinted in my soul the ideas I have of body, if there were no bodies, but that proof is very weak; it proves too much. Ever since the beginning of the world all men, except, perhaps, one in two hundred millions, do firmly believe

we have seen. The demonstrations in Zeno are not expressions of what Bayle affirms, but propositions which he contemplates.

A peculiar feature of relation between Bayle and Berkeley now becomes apparent: if there was any influence of Bayle upon Berkeley, we cannot say that Berkeley was persuaded by Bayle's convictions, but rather, that Berkeley asserted some of the propositions which Bayle merely discussed. Thus any discussion of the relation between these two men will have to acknowledge that, while Berkeley probably accepted some of Bayle's demonstrations, he was convinced, unlike Bayle, that a consistent and rational account of reality can be given, and that the tenets of religion are not well served through attempts at demonstrating the fundamental irrationality of the world.

How could Berkeley have come to such a position if the the Dictionaire offered convincing arguments against every philosophical position which is discussed there? The answer is that the Dictionaire actually falls far short of achieving this goal. On one hand, there are very good arguments against the existence of external matter, but the immaterialism that is thus established is nowhere convincingly refuted. Rather, as in Anaxagoras, realism is assumed, but not proved to be tenable.

It must be noted that the position delineated in the articles Pyrrho and Zeno taken in itself does not constitute a philosophy that could properly be called sceptical. After all, the affirmation that there is no external matter is an affirmation. I believe that Bayle wanted to point out that there are very good arguments against naive realism, but that an immaterialist philosophy is not a good alternative. In fact, a decision between immaterialism and realism would either run counter to the arguments in Zeno, or contradict what everybody knows, namely that external matter "obviously" exists. Thus, no matter how we proceed, an inconsistency is immediately forthcoming.

It is to Berkeley's credit to have realized for the first time that the one alternative, namely epistemological realism, is not really supported by any arguments, a fact that Bayle completely overlooked. This is one of the reasons why any assertion that Bayle anticipated Berkeley must be accepted with great reservations.

I realize that the interpretation which I have here put on the passages in Bayle is not the customary one. Traditionally, the fictional personage who presents the above quoted passages in the article Pyrrho[123] is called

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[123] See above, pp. 117f.

the _Abbe Pyrrhonien_, indicating what is here stated is Pyrrhonist or sceptical position. Popkin asserts that the argument expresses "a brilliant conception of the ultimate in scepticism."[124] However, the viewpoint expounded by the _Abbe Pyrrhonien_ is not a sceptical, but an immaterialist or, if you will, phenomenological one. Notice again the passage "God does not force you to say that _it does exist_, but only to judge that you feel it, and that it appears to you to exist."[125] Again it is Berkeley's merit to have emphasized time and again against a welter of misunderstanding that the phenomenological viewpoint must not be confounded with the sceptical one, that they have nothing in common. It is, in a way, the most important point of his system. The arguments in _Pyrrho_ can be used for the support of a sceptical philosophy only, if a naive realism is maintained at the same time, and if the resulting inconsistency is turned into an argument for the whithholding of judgment.

It is now time to turn to an examination of the arguments which Bayle brings forth in support of his

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[124]Richard H. Popkin, "Pierre Bayle's Place in 17th Century Scepticism" in _Pierre Bayle, Le Philosophe de Rotterdam_, Paris 1959, pp. 1-19, p. 6.

[125]See above, p. 118.

"irrationalism". They are of two kinds. The arguments offered in _Pyrrho_ point, in a rather naive way, to the so-called mysteries of religion, pointing out that the propositions put forth there are some of them contrary to all reason and should nevertheless be accepted. But in _Pyrrho_ it is also suggested that extension is a secondary quality. If this assumption could be proved, i.e. if the external extended matter could be shown to be illusionary, then a contradiction with naive realism would arise. I think that in Bayle's opinion this would make both naive realism and phenomenalism doubtful, and would show that they are in no better position than the revelations of religion as far as plausibility is concerned. The argument for the non-existence of external matter is presented in _Zeno_. It is, as we shall see, of the _reductio ad absurdum_ type. Let me remark, in passing, that the employment of this type of argument is not in keeping with the general tenets of Bayle's undertaking: if the inconsistency of an assumption does not necessarily discredit it (see the assumptions of revealed religion), then _reductio ad absurdum_ is of little value.

Let us have a look at the argument. Bayle argues in support of Zeno's contention that there is no motion, and he writes the following:

> There is no extension, therefore there is no motion. The consequence is good, for what has no extension fills no space, and what fills no space cannot possibly pass from one place to another, and consequently move. This is incontestable; the difficulty is then to prove that there is no extension. Zeno might have argued thus: Extension cannot be composed either of mathematical points, or of atoms, or of parts divisible <u>ad infinitum</u>; therefore it's existence is impossible. The consequence seems certain, by reason it is impossible to conceive more than these three modes of composition in extension; wherefore the antecedent alone remains to be proved. A few words shall suffice as to mathematical points; for a man of meanest capacity may apprehend with the utmost evidence, if he is but a little attentive, that several nothingnesses of extension joined together will never make an extension...Wherefore...let us take it to be impossible, or at least inconceivable, that matter should be composed of them...
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> Nor is it less impossible or inconceivable that it should be composed of Epicurean atoms, that is, of extended and indivisible corpuscles; for every extension, how small soever, hath a right and a left side, an upper and lower side: therefore it is a conjunction of distinct bodies; and I may deny of the right side what I affirm of the left, for these two sides are not in the same place: a body cannot be in two places at once and consequently every extension which fills several parts of space contains several bodies... whence it follows that if there be an extension, its parts are divisible <u>in infinitum</u>. But on the other side, if they cannot be divisible in infinitum, we ought to conclude the existence of extension impossible, or at least incomprehensible.
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> An infinite number of parts of extension, each of which is extended, and distinct of all others, as well with respect to its entity as to the space which it fills, cannot be contained in a space one hundred thousand million times less than a

> hundred thousands of a barley corn....If there be no body but what contains an infinity of parts, it is evident that each particular part of extension is separated from all others by an infinity of parts, and that the immediate contact of two parts is impossible, wherefore the existence of extension necessarily requiring the immediate contact of its parts, and that immediate contact being impossible in an extension divisible *in infinitum*, it is evident that the existence of such an extension is impossible.[126]

Of the three arguments here presented, only the case against Epicurean atoms holds water. They cannot be the ultimate constituents of extended bodies since they are themselves constituted of parts. But while Bayle argues, in the case of these atoms, against their ultimacy, in the other two cases it is the mode of composition which provides the difficulties. Now none of these latter arguments are cogent, at least not unless important qualifications are added. In the first argument it was asserted that several points "joined together" will not make a finite extension. The difficulty seems to lie in the phrase "joined together". In the sequel we shall see that one of the most important advances in the analysis of continuous quantities was Bolzano's realization that an infinity of points, when "joined together" in a certain way, *can*

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[126]Bayle, *op. cit.*, Article *Zeno*, Remark G.

constitute a finite (continuous) extension. I take Bayle's remark to mean that several points, no matter how many, when joined together, no matter in what way, will not make a finite extension. This statement is clearly false. Thus the argument against points is not valid. I presume that it is precisely the intuitive accompaniment of such a phrase as "joined together", or the mental image that goes with it, which misled Bayle as it did so many of his contemporaries.

The argument against infinite divisibility is likewise invalid. First, it is not the case that an infinite number of finitely extended parts of extension cannot be contained in a finite extension. If a half inch is added to an inch, and a quarter inch to the sum of the two etc., the total will not exceed two inches, even if infinitely many of these finitely extended lengths are added in accordance with the law of the series. To be sure, I cannot actually produce all these different lengths first, and then join them together; all I can do is to give the law according to which they must be produced and joined, and I believe that it was this restriction which lead Bayle to believe that an infinity of finite extensions cannot be contained in some other finite extension.

Finally, it has been recognized for quite some time, that it does not make sense, given a point on a line, to speak of the "next" or neighborning point, as it also does not make sense, given a moment, to speak of the "next" moment. If two points are distinct, then there must be another point which lies between them, from which it follows that there must be an infinity of points which lie between them. I believe that this fact was first recognized and emphasized by Boscovich[127] and later on accepted by Bolzano, as I shall show. Bayle apparently recognized this feature of continuous sets, but it seemed so incredible to him that he preferred to reject the notion that the continuum is infinitely divisible. However, it must be noted that in the last argument given above the concept of extension is arbitrarily restricted by the assertion that it requires the immediate contact of its parts. If, as seems to be the case, by these parts are meant the ultimate parts or points, then this requirement would in fact lead to the elimination of continua, or extension, since two points

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[127] Ruggiero Guiseppe Boscovich, *Theoria Philosophiae Naturalis*, Vienna 1758, § 30-33. See Ernst Cassirer, *Das Erkenntnisproblem*, 3 Vols. 3rd. ed., Berlin 1922, Vol. 2, p. 510f.

that are in "immediate contact" are identical. It is therefore this arbitrary restriction which must be rejected, and may be rejected, since it can be shown that immediate contact of points is not necessary, or even possible, in the type of serial order which we call a continuum.

We see then that Bayle's entire argument derives its superficial plausibility merely from the difficulties connected with continuous sets, and, generally, from the counterintuitive features of infinite sets. However, as may be expected, arguments of this sort were the rule rather than the exception at Bayle's time, and they exerted a tremendous influence upon philosophical speculation.

It has been suggested that these discussions of Bayle's exercised considerable influence upon the development of Berkeley's philosophy, and it is this claim which I shall now examine.[128]

Popkin tries to rake together all the evidence that would support the supposition at issue. He points out that Berkeley's repeated and vigorous insistence that his

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[128] Both Popkin and Luce have asserted that Bayle strongly influenced Berkeley. Cf. Richard H. Popkin, "Berkeley and Pyrrhonism", The Review of Metaphysics, Vol. V., 1951-52, pp. 223-246 and A. A. Luce's note on p. 388 of his editio diplomatica of Berkeley's Philosophical Commentaries, London 1944.

philosophy is not a sceptical one can be interpreted as an attempt to distance himself from Bayle and the reputation that the latter had acquired. He points out that Berkeley had obviously acquainted himself with some of the arguments in Zeno. since entry no. 358 of the Commentaries reads "Malebranche's and Bayle's arguments do not seem to prove against Space. but only Bodies."[129] Considering the wide distribution and the popularity of the Dictionaire, and the material accumulated by Popkin, there seems little doubt that some influence took place, especially as regards the treatment of primary qualities. However. in their zeal for connecting Berkeley with Bayle. both Popkin and Luce have overlooked that there is a rather considerable disagreement between the two philosophers, especially as concerns the topic of this dissertation.[130]

After having presented his case in the form quoted above. Bayle summarizes his results in the following fashion:

> All those who argue an extension are determined in their choice of an hypothesis no otherwise than by the following principle: If there are but three ways of explaining a subject,

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[129] The Works of George Berkeley, Vol. I. p. 43.

[130] I have already pointed at another divergence concerning scepticism, above, pp. 119f.

> the truth of the third necessarily follows from the falsity of the other two. A Zenonist might tell those who choose one of these three hypotheses: you do not argue right, you make use of a disjunctive syllogism...The fault of your argumentation lies not in the form, but in the matter: you ought to lay aside your disjunctive syllogism, and make use of this hypothetical one: If extension existed, it would be composed either of Mathematical points, or of Physical points, or of parts divisible *in infinitum*. But it is not composed either of Mathematical points, nor of Physical points, nor of parts divisible *in infinitum*. Therefore it doth not exist.[131]

Compare this with entry no. 26 of the *Philosophical Commentaries*, which says: "Infinite divisibility of extension does suppose ye external existence of extension but the latter is false. ergo ye former also."[132] Now Luce asserts that "entry no. 26...relates the infinite divisibility to external extension exactly as the Zeno article does;"[133] and Popkin fully agrees with this result.[134] However, a moments consideration will show that such is not at all the case. For one thing, the consequent in Bayle's hypothetical syllogism is an alternation of three members.

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[131] Bayle, op. cit., Article *Zeno*, remark G.

[132] *The Works of George Berkeley*. Vol. I., p. 10.

[133] A. A. Luce, *loc. cit.*

[134] Richard H. Popkin, *op cit.*, p. 243.

One could however, assume that Bayle listed all three possibilities only in order to serve the tenets of his encyclopedic enterprise, and that the case against atoms was already so well established that Berkeley deemed their discussion superfluous. Furthermore, Berkeley might have realized that the infinite divisibility and constitution of points amounts to the same thing: But even if one were to agree to all this, a fundamental difference would still remain. If we consider only the case of infinite divisibility, Bayle's argument would read: 'if extension exists, then it must be infinitely divisible! But it is not infinitely divisible. Therefore it does not exist.' On the other hand, Berkeley's argument could be phrased thus: 'If something is infinitely divisible, then it exists externally. But nothing exists externally, therefore, nothing is infinitely divisible.' Thus, to claim that Berkeley relates infinite divisibility to external extension exactly as the Zeno article does is patently false. Rather, in Bayle the denial of external extension is the final result of the argument, in Berkeley it is a premise. Actually, Berkeley's argument is similar to the supporting proofs of Bayle's in which he attempts to demonstrate the impossibility of infinite division, but while Bayle's arguments suffer from logical inadequacies, Berkeley puts forth a metaphysical assumption which in fact, if true, would remove the difficulties concerning the composition of bodies, as I have shown above.

The differences between the two arguments can be summed up as follows: the _terminus ad quem_ of Bayle's demonstration is the proposition that extension does not exist. In order to show this, a (fallacious) proof is offered to the effect that extension is not infinitely divisible. In Berkeley's argument, the proposition to be proved is that infinite division is impossible. To show this, it is assumed as a hypothesis (_not proved_) that there is no external extension.

Thus, here again we must meet the claim that Berkeley was strongly influenced by Bayle with great reservations. Their arguments differ essentially as their aims were different. Berkeley was primarily concerned with the problems of the infinite, while Bayle's aim, at least in _Zeno_ was to discredit naive epistemological realism.

Nevertheless, it is a likely supposition that some influence took place. Let me conclude this chapter with another quotation from Bayle which tends to substantiate this latter assumption.

Since the same bodies are sweet to some men and bitter to others, it may reasonably be inferred that they are neither sweet nor bitter in their own nature, and absolutely speaking. The modern Philosophers, though they are no Sceptics, have so well apprehended the foundations of the epoch with relation to sounds, odours, heat, and cold, hardness and softness, ponderosity, and lightness, savours and colours etc. that they teach all these qualities are perceptions of our mind, and do not exist in the objects of our senses. Why should we not say the same thing of extension? If a being, void of colour, yet appears to us under a colour determined as to its species, figure and situation, why cannot a being, without any extension, be visible to us, under an appearance of determinate extension, shaped and situate in a certain manner?[135]

[135] Bayle, _op. cit._, Article _Zeno_, Remark H.

CHAPTER IV

IMMANUEL KANT

So far we have discussed three approaches to the problem of the composition of the continuu. and Kant was apparently familiar with all three of them. In the preface to the first edition of the Critique of Pure Reason he asserts that he has "found a way of guarding against errors which have hitherto set reason, in its non-empirical employment. at variance with itself"; and he continues: "I have not evaded its questions by pleading the insufficiency of human reason."[136]

The difficulties which set reason against itself are. of course. those which are recorded in the "antinomies". of which the second is of especial importance for our present investigation.[137] It seems to me that the above quotation

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[136] I shall quote from Norman Kemp Smith, A Translation of Kant's Critique of Pure Reason, London 1929, but, as customary. I shall give the pagination of the first (A) and second (B) editions of 1781 and 1787. All other works will be quoted after the Akademieausgabe, Berlin 1902 ff, but I shall consult and indicate already existing translations. The present quotation is from A 12.

[137] The Second Antinomy (Critique of Pure Reason, A 434. B 462) is stated thusly: "Thesis: Every composite

indicates Kant's familiarity with earlier approaches such as Bayle's which blaime the arising of contradictions in speculation upon the insufficiency of human reason.

According to a recent commentary, the problem of the constitution of matter and Bayle's account of the issue were frequently discussed in the Wolfian school, so that there is sufficient reason to suppose that Kant did not merely mean to attack a hypothetical position in the above quoted passage.[138]

But not only was Kant familiar with the school of thought that blamed the insufficiency of human reason for the arising of antinomies, he also seems to have known Berkeley quite well. However, it is doubtful that he envisaged Berkeley's philosophy as an attempt at solving or removing the problem of the constitution of (continuous) matter. He says of Berkeley:

> He (Berkeley) maintains that space, with all the things of which it is the inseparable condition, is something which is in itself impossible; and he therefore regards the things in space as merely imaginary entities (<u>Einbildungen</u>) Dogmatic idealism is unavoidable, if space is interpreted

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substance in the world is made up of simple parts, and nothing anywhere exists save the simple or what is composed of the simple. Antithesis: No composite thing in the world is made up of simple parts, and there nowhere exists in the world anything simple."

[138]Cf. Gottfried Martin, <u>Kant's Metaphysics and Theory of Science</u>, Manchester 1955, p. 47.

> as a property that must belong to things in themselves."[139]

If this passage is interpreted to say that Berkeley developed his philosophy because he recognized contradictions in the assumption of continuous bodies which fill space, then Kant recognized precisely the starting point of Berkeley's philosophy. I have some doubts about this. Nevertheless, Kant here, in a more or less vague fashion, seems to link Berkeley's problem with that of the second antinomy.

That Kant was familiar with Leibniz is commonplace and does not need much further substantiation. Kant was an avid student of Leibniz and began his philosophical career entirely within the tradition of the Leibniz-Wolff school. The problem of the composition of the continuum was much discussed in these circles and several of Kant's first philosophical essays concern themselves directly or indirectly with it.

Thus it is clear that Kant, in addressing himself to the problems of the second antinomy, consciously became part of a long line of attempts at a solution of this

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[139] Critique of Pure Reason, B 274.

problem.[140]

What distinguishes Kant from some of his predecessors, notably Bayle and the Port Royalists, and from many of his successors, is a wholesome fear of contradictions. Time and again he declares that one of the most important features of his Critical Philosophy is the avoidance of antinomies that have hitherto bedevilled speculation. Thus in the preface to the first edition of the _Critique of Pure Reason_ he bemoans the fact that through the uncritical acceptance of "principles which overstep all possible empirical employment... human reason precipitates itself into darkness and contradictions."[141] The _Critique_, of course, is to be the cure for this malady. In the preface to the second edition he makes a similar point,[142] and the body of the work contains many related remarks. Finally, an explicit statement of the

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[140]According to Erich Adickes, _Kant als Naturforscher_. 2 Vols., Berlin 1924/25, Vol. I, p. 172, Kant in the _Monadologia Physica_, Akademieausgabe, Vol. I, pp. 473-487, asks himself the question, "what is it that makes it happen that matter, stuff, occupies a space and cannot be removed from that space?" Adickes claims that it was Kant's "immeasurable merit" to have seen "the necessity for that question." On the evidence of our previously collected material it is evident that to claim originality for Kant on that score is patently absurd.

[141]_Critique of Pure Reason_, A VIII.

[142]_Ibid_., B XX.

difficulties and contradictions which beset reason in its "uncritical" use is found in the <u>Antinomies</u>.[143]

In the second half of the last century the importance of the Antinomies for the development of Kant's philosophy came to be realized. Alois Riehl, for one, pointed out that the essential features of Kant's philosophy sprang from the desire to avoid the occurrence of the antinomies, and Erdmann, Vaihinger, Adickes and many others accepted his opinion, so that now it is the subject of scarcely any further dispute.[144]

Thus I cannot claim any originality in elaborating this point again. Nevertheless, several reasons prompted me to include a consideration of Kant in the present dissertation. One is that previous commentators have relied largely on testimony contained in letters and posthumously found notes in order to establish the importance of the problem of the antinomies for Kant's philosophy, i.e. they have chosen a historical approach. I shall confine myself to a discussion of his published work in order to establish the same point.

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[143] <u>Ibid</u>., A 407-567, B 435-595.

[144] Cf. Klaus Reich in the introduction of his edition of Immanuel Kant, <u>De mundi sensibilis atque intelligibilis forma et principiis</u>. Hamburg (Meiner) 1958, pp. VIII ff

This procedure will have the advantage of making clear the inner-systematic importance of the problem of the antinomies, in particular the second antinomy, and will thus differ markedly from a mere ascertainment of historical sequence

A second reason why I thought it necessary to include a discussion of Kant lies in the fact that many commentators, particularly Riehl, do not seem to have been aware at all of the implications of the problem at hand. Thus Riehl feels justified in claiming that "opposite assertions, based on entirely different presuppositions, do not contradict one another".[145] and that

> the proofs for and against the infinite divisibility of matter are not conducted from the same or similar standpoints. The reasons offered on either side are not homogenious, so that there can be no real contradiction between them. The thesis is proved ongologically from the conception of a composite reality, while the antithesis is proved for perception from the idea of space [146]

Now clearly, for the occurrence of a contradiction it can be of no importance whatever how each of the contradictory statements has first been established. Nor did Kant doubt for a moment that he had contradictions on his hands, merely

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[145] Alois Riehl, Introduction to the Theory of Science and Metaphysics, London 1894, p. 270.

[146] Ibid., p. 271.

because thesis and antithesis had been established in different ways. But Riehl committed another major blunder. With reference to the third and fourth antinomy he points out:

> In order to understand the proof of the theses of these dynamical antinomies, it is necessary first to forget the doctrines established by Kant in the Transcendental Analytic. This evident contradiction in Kant's system (sic) can only be explained by assuming that the antinomies are the oldest part of the Critique, or rather that they preceded it.[147]

There could be no clearer testimony that Riehl's interest and ability were merely historical. not systematic. The passage shows a misunderstanding of the position of the antinomies in Kant's system while, at the same time, it establishes a historical thesis that is probably correct. It was Kant's precise objective first to establish in each antinomy both the thesis and the antithesis from a pre-critical viewpoint, i.e. forgetting the doctrines established in the Transcendental Analytic, and then to discuss and remove the ensuing difficulties by bringing to bear on them the findings of the preceding parts of Critique, i.e. the Transcendental Esthetic and Analytic. On the other hand, there is a good deal of historical testimony to vindicate

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[147] Ibid.

Riehl's thesis that the antinomies (possibly even in their final form) preceded much of the rest of the Critique.[148]

I find it generally the case that commentators either did not pay much attention to the systematic significance of the antinomies for the rest of the critical enterprise,[149] or else were determined largely by historical considerations and preoccupied with the establishment of such philological theses as the so-called patchwork theory, according to which the Critique of Pure Reason was pieced together out of a large number of preexisting writings.[150]

It is for all these reasons that I propose to investigate again the importance of the antinomies, in particular the second one, for the development and character of Kant's

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[148] A letter from Kant to Garve has been found dated the 21st of September, 1798, in which Kant says, "It was not the investigation as to the existence of God, but the antinomies of pure reason which first wakened me from a dogmatic slumber, and impelled me to a critique of reason itself." Cf. also the Prolegomena, beginning of § 50, Akademieausgabe, Vol. IV, p. 338.

[149] E.g. Norman Kemp Smith, A Commentary on Kant's Critique of Pure Reason, 2 New York 1950, and T. W. Weldon, Kant's Critique of Pure Reason, 2 Oxford 1958.

[150] Cf. Alois Riehl, Der philosophische Kritizismus und seine Bedeutung fur die positive Wissenschaft, 3 Vols., Leipzig 1876-87, The above quoted work, Introduction to the Theory of Science and Metaphysics (above p. 115) is a translation of the third volume of Der philosophische Kritizismus. The charge of preoccupation with historical and philological

philosophy.[151]

Before I proceed to analyze Kant's theories in detail. let me try to indicate briefly the epistemological status that Kant ascribed to his proposed solutions. It will be remembered that Berkeley described the main facet of his philosophy as the immaterial hypôthesis. In keeping with this characterization. supporting arguments were sought, as we have seen, not in propositions of which the immaterial hypothesis is a consequence. but in propositions which it entails. In opposition to this procedure. Leibniz delivers his system as a priori certain, and does not ordinarily attribute hypothetical character to it.

How did Kant regard his own doctrines? In the preface to the second edition of the Critique of Pure Reason, when describing in a preliminary way his "Copernican revolution" with the aid of which certain contradictions can be avoided.

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questions can be levelled even against Vaihinger, whose commentary is generally held to be above reproach. Hans Vaihinger, Kommentar zu Kant, 2 vols. 2nd. ed. Stuttgart 1922. It must be said, in vindication of Vaihinger, that his commentary does not proceed beyond the Transcendental Esthetic, and that he discusses our present problem in an avowedly historically oriented Excurs: "The Historical Origin of the Kantian Doctrine of Space and Time", Vol. II, p. 422ff which follows the systematic exposition of the Transcendental Esthetics.

[151] I have found the best discussion of the importance of the antinomies in Kant in Gottfried Martin, Kant's Metaphysics and Theory of Science, Manchester 1955.

he points out,

> The (Newtonian attraction) would have remained forever undiscovered if Copernicus had not dared, in a manner repugnant to common sense, (*widersinnisch*) but yet true, to seek the observed movements not in the heavenly bodies, but in the spectator. The change in point of view, analogous to this hypothesis, which is expounded in the *Critique*, I put forward in this preface as an hypothesis only, in order to draw attention to the character of these first attempts at such a change, which are always hypothetical. But in the *Critique* it will be proved apodeictically not hypothetically, from the nature of our representations of space and time and from the elementary concepts of the understanding.[152]

It seems that Kant considered the introduction of a proposition as a hypothesis merely as a preliminary matter, and that any hypothesis, in order to be permanently acceptable must be "proved". In this respect his outlook did not change from his early rationalist days to the time of the *Critiques*. In one of his earliest writings, the *Monadologia Physica*, he criticizes those approaches to natural philosophy which do not accept any information but what is immediately evident through experiment. He says: "*Ex hac sane via leges naturae exponere profecto possumus, legum originem et causas non possumus*."[153] Thus, through observation and experiments

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[152]*Critique of Pure Reason*, B XXIII.

[153]*Metaphysica cum geometriae iunctae usus in philosophia naturali, cuius specimen I continet monadologiam physicam*, Konigsberg 1756, *Akademieausgabe* Vol. I, pp. 473-487, p. 475.

laws can indeed be made evident, but their origin and causes cannot be found in this way. This seems to be an invitation to speculation, to the formation of hypotheses which would explain the laws found through experimentation - a perfectly legitimate enterprise. However, the body of the Monadologia supposedly proves every hypothesis which is introduced there. Needless to say, many of these proofs are invalid, but the attempt is there, just as in the Critique of Pure Reason

The difference between Kant and Berkeley could not be more striking. I am quite convinced that Berkeley did not think his hypothesis demonstrable in the same sense in which Kant sought to prove his. The immaterial hypothesis is said to be as good as its explanatory merit. Kant on the other hand, always thought of his doctrines as requiring some sort of deductive proof, or, to put it more strongly, he attempted to, and presumably believed that he did, prove all propositions which he offered. Interestingly enough, it is those parts of the Critique of Pure Reason which offer these proofs, namely the Esthetic and Analytic, which must be considered dated, although even they command interest at every turn.

In offering a reconstruction of parts of Kant's philosophy I shall do some violence to Kant's own intentions by neglecting, for the most part, the alleged proofs that he offers in support of his positions - otherwise I would have to discuss the entire Transcendental Esthetic and Analytic, a task much too formidable for the small scope of this paper. Thus I shall discuss certain of his propositions as though they had been intended merely to function as hypotheses.

The above considerations point to a remarkable and perhaps fundamental distinction between the British empiricist school and the continental rationalism and idealism. Concerning the epistemological status of his doctrines, Kant did not differ at all from his rationalist predecessors.

The earliest work in which Kant attempted to offer a reasonable solution to the problem of the constitution of matter was his Monadologia Physica. To understand fully the problem as it was envisaged in that little essay, let us first turn to a much later work, namely the Prolegomena to Any Future Metaphysics. Here Kant writes:

> It will always remain a noteworthy phenomenon in the history of philosophy that there was a time when even mathematicians, who were at the same time philosophers, began to doubt. though not the correctness of geometrical theorems as far as they concern mere space. but at least the objective validity (Gültigkeit) and application of this concept to nature. since they feared that a line in nature might consist of physical points, hence the true space in the object consist of simple parts, although the space which the geometrician has in mind cannot possible consist of them.[154]

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[154] Akademieausgabe, Vol. IV. pp. 253-383. § 13, Note I. p. 287f.

It is quite clear that the object of this criticism is Christian Wolff, who was a mathematician and at the same time a philosopher. Wolff had assumed that the physical world consists of bodies which are extended, and which are made up of simple elements called '<u>atomi naturae</u>' and sometimes also 'monads'. Wolff seems to have subscribed to Leibniz' views that whatever is extended in space, and hence composite, must consist of simple, indivisible elements. But while Leibniz categorically denies the spatiality of these simple elements, Wolff was not quite certain what character he should ascribe to them. He believed that his atoms could combine to produce larger extended bodies, which he believed to be continuous, but since he could not explain the nature of this composition, he claimed that human notions of the composition of matter must remain unclear. However, in one way or another, a finite number of atoms was believed to produce a body of perceptible size, and hence it was concluded that the atoms must themselves have a finite extension. Hence any piece of matter could be divided down to the atoms, but not further.

Geometrical space, on the other hand, was said to be infinitely divisible, hence, so it was argued, can have no simple parts. These considerations led Wolff to distinguish between physical or natural space (i.e. space occupied by a body), and geometrical or pure space.[155]

It is this distinction between two kinds of space that Kant has in mind when he claims that geometry and metaphysics are at variance, or at least were before his critical system. Once one realizes this distinction, and sees that geometry and metaphysics were thought to make conflicting assertions about space, one understands the urgency of the Kantian question 'How is it that geometry can apply to experience?' which, of course, means the same as 'How is it that geometry can be used in physical space when supposedly it describes accurately only its own special kind of space?' For Kant, the puzzlement consisted in the fact that the application of geometry was successful despite the conflict between the metaphysical and geometrical description of (occupied) space - for, that there was but one space, and that Wolffs distinction was untenable, Kant assumed from the beginning.

This is a preliminary discussion of the problems that Kant attempted to solve in the Monadologia Physica. The

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[155]Cf. Ueberweg, Grundriss der Geschichte der Philosophie, 3 Vols., Berlin 1905-1909, Vol. III, p. 228.

title of the little work already suggests its aim: Metaphysicae cum geometriae iunctae usus in philosophia naturali specimen. Both geometry and metaphysics are to be used jointly in natural philosophy. This means that on one hand the 'metaphysical' demonstration of the existence of simple bodies is accepted, while on the other hand the geometrical demonstration of the infinite divisibility of space is also granted, even if that space is occupied by a body.

First the term "monad" is introduced by definition: "Substantia simplex, monas dicta, est quae non constat pluralitate partium, quarum una absque aliis existere potest."[156] Then a theorem is offered, namely "Corpora constant monadibus".[157] This is to be proved by the statement that if all composition were suspended, then the remaining parts obviously do not have composition, and hence are simple. This alleged proof is similar to one which was time and again offered by Leibniz. Its difficulty lies in the notion of 'suspending composition'. I am not at all clear what could be meant by this, and I cannot see that this "proof" demonstrates in any way the existence of ultimate parts.

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[156] Akademieausgabe, Vol. I, p. 477.

[157] Ibid.

After thus having satisfied himself that there are ultimate parts to bodies - and Kant means finitely extended ultimate parts, as we shall see - he gives a demonstration that the space which a body occupies is infinitely divisible, and hence does not consist of simple parts: "Spatium, quod corpora implent, est in infinitum divisibile, neque igitur constat partibus primitivis atque simplicibus."[158] He offers the following proof: Consider a line ef[159] of "indefinite" length, i.e. one that can be extended at will, and let it be filled with monads. (linea partibus materiae primitivis conflata) Call such lines "physical lines". On ef draw at right angles another physical line cd, and likewise at right angles to ef another physical line ax of the same length as cd and different from cd. Now mark points on ef, call them g, h, i, k, etc. and connect these points through physical lines with c.

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[158] Ibid., p. 478.

[159] Notice that 'e' and 'f' do not denote points, but this is of no consequence for the proof. See diagram.

The point in which cg intersects ab is called o. Now any line which is drawn from a point to the right of g to c will intersect ab above o, and as we choose points farther and farther to the right on ef, the intersections approach a more and more closely. In fact, Kant concluded, since we can extend ef at will, we can make as many divisions in ao as we wish. According to Kant, this proves that ao and hence space is infinitely divisible, and hence cannot consist of simple parts.

As far as the mathematics of the situation is concerned, the proof did not offer any novelty, and Kant admits as much. Why then did he include it? The only innovation that it incorporates is the stipulation that the lines be "physical" lines, i e. filled with monads. Now it must be understood that the number of monads in any physical body is always considered finite, hence that a monad occupies a finite space (We shall see later on what it means for a body to be "filled" with monads). Clearly, this innovation makes the proof invalid, since the lines drawn between c and the points on ef have finite width, so that only a certain number of them can be accomodated in the proximity of c. Kant apparently did not realize this.

It is clear that the proof was to demonstrate something over and above the similar geometrical proof, and while in fact it proves nothing, we can nevertheless inquire what it was supposed to prove. This is actually a quite simple matter. If a certain finite number of monads were located on ab, then for any monad m there is always a point n on ef such that cn cuts through m. Or, to put it geometrically, for any finite part ij of ab there is a point n on ef such that the intersection of cn and ab lies between i and j. This is supposed to show that the space which a monad occupies is always divisible, although the monad itself is simple. Or, to put it otherwise: simple, finitely extended monads are located in a space which is infinitely divisible, although the monads themselves are not.

Clearly, the demonstration is directed against Wolff, who claimed that the space occupied by an indivisible body is itself indivisible, and who was therefore forced to introduce a distinction between geometrical and physical space.

This situation is completely misunderstood by Adickes, who writes:

> Only the space which they (the monads) occupy is divided to infinity. Of the monads, on the other hand, only a finite numbe is present. However, if this is the case, then there is no reason why the monads had to be dragged into the proofs of the infinite divisiblity of space.[159]

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[159] Adickes, op. cit., Vol. I, p. 135.

That there was a reason for considering the monads in this proof is indicated by the passage cited from the Prolegomena, in which Kant attacks the view that "the true space of the object" consists of simple parts. The demonstration therefore must have been intended not only to show that space is infinitely divisible, but that it is so divisible no matter whether it is occupied by a monad or not. Thus, the distinction between physical and geometrical space is abandoned.

But this distinction had served a purpose. If the space which a physical object occupied could not be divided beyond the smallest parts of this object, then, it was assumed, it did not make any sense to speak of matter as being divisible beyond its atoms. Infinite divisibility was restricted to geometrical space, which was considered a merely ideal construct, designed to satisfy the postulates of geometry In disregarding this attempt at solving the problem of the constitution of matter, Kant at once faced again the old problems that had been encountered, for example by Cordemoy. Let us see how these problems are dealt with.

Consider theorem IV of the Monadologia Physica. It states "Compositum in infinitum divisibile non constat partibus primitivis s. simplicibus."[160] This, we know now,

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[160] Akademieausgabe, Vol. I, p. 479.

needs the qualification that an infinitely divisible compound does not consist of extended primitive parts. But theorem IV gives rise to the corollary that if something does consist of simple parts, then it is not infinitely divisible; according to Proposition II, this holds of all bodies: "Corollarium: Corpus igitur quodlibet definito constat elementorum simplicium numero."[161] from which Kant concludes in Proposition V that the monad not only is in space, but occupies space, i.e. is extended, in spite of its postulated simplicity.[162]

Here then arises the old predicament of atomism. If the monad, or the atom, as case may be, is extended, how can it be called simple, or, in other words, how can a body be extended, but nevertheless indivisible?

Adickes has leveled the charge against Kant that he confused physical divisibility with spatial extension. This charge must be investigated; it will serve to bring the difficulties of the atomist position into clearer focus. From these considerations it will be seen first, that the problem

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[161] Ibid

[162] Ibid., p. 480.

is not nearly as simple as Adickes seems to have supposed. and second that Kant is not guilty of any confusion in this matter. Indeed. it will be shown that the second part of the Monadologia Physica was. in fact. designed to cope with the difficulties that arise from a distinction between extension and "physical divisibility".

We can best reach clarity about this matter by considering the various meanings of the term 'divisible'. In so doing I shall not investigate the difficulties that attach to this term insofar as it is a "dispositional predicate". It would be presumptuous to attack such a difficult problem in this context. Also. the term will be understood as having to do with division into spatial parts. Three meanings of 'divisible' can then be distinguished; namely. 1. x is divisible_1= Df. a law of the form 'if such and such is done to x, then x will fall into parts' is known. 2. x is divisible_2= Df. a law of the form 'if such and such is done to x. then x will fall into parts' is not self-contradictory. 3. x is divisible_3= Df. x occupies a finite space. This third kind of divisibility should actually be attributed only to geometrical objects. Here it is no longer a question of "taking something apart". One speaks of lines as being bisected i.e. divided by other lines. surfaces by lines etc We may say that one line is divided by another line if the two have a point in common. but both also have at least one point which they do not share. Thus. in this third sense of 'divisible', we may say that a line or a surface or body is $divisible_3$ if it consists of more than one point, i.e. if it is extended. Now since we are talking about division in space, I should think that if a body is $divisible_1$, then it is $divisible_2$, and if it is $divisible_2$, then it is $divisible_3$. However, both Descartes and Leibniz, as well as many others believed that not only does $divisibility_2$ imply $divisibility_3$, but that the converse also holds. An assertion frequently found in these authors is one to the effect that an object which is extended in space is also "divisible, at least in thought", that is to say, if x is extended in space, then a law of the form "if such and such is done to x, then x will fall into parts" is not self-contradictory. This then is the perennial predicament of the classical atomists: they must show that $divisibility_3$ does not imply $divisibility_2$. In fact, to be an atomist in the classical sense is to defend precisely this point. But atomists have not generally rested with the assertion that there are finitely extended particles of matter which are nevertheless so indivisible. They usually took it upon themselves to describe these particles in such a way that their view would become intuitively plausible, a task from which they might better have refrained. Thus they attached pseudo-phenomenal properties to their atoms: Cordemoy called his "infinitely hard", Lambert called his "solid", and Gassendi topped Lambert by ascribing "absolute solidity" to his atoms. Now none of these attributions can be said to add anything to the respective author's central thesis that there are extended, but indivisible$_2$ particles of matter. On the contrary, these embellishments were apt to cloud the issue. Leibniz for one did not think that the properties of infinite hardness or absolute solidity made it impossible to divide an atom "in thought", and he seems to have realized that the ascription of these properties did not add any force to, or make more plausible, the proofs which were offered for the existence of atoms in the first place.

Kant comes up against similar difficulties. He had inferred, as we saw, that space is infinitely divisible, and therefore does not consist of simple parts. Matter, it is assumed, does consist of simple parts, and therefore is not infinitely divisible. Hence the physical monads are said to "occupy" finite spaces: Kant speaks of the "space of the monad's presence". But if a monad occupies a finite space, why is it not divisible$_2$? Kant makes a rather clever suggestion by which the monad "occupies" a finite space in a certain sense, without thereby becoming divisible$_2$. He contends that the monad proper is indeed unextended, but that it is enveloped in a field of forces which do not allow other monads to approach beyond a certain limit. "*Monas spatiolum presentiae suae definit non pluralitate partium suarum substantialum, sed sphera activitatis, qua externas utrinque sibi praesentes arcet ab ulteriori ad se invicem appropinquatione*."[163] That is to say the monad does not "define" (circumscribe) the space of its presence through a plurality of its substantial parts, but through a sphere of "activity" which keeps the other monads from closer approach.

Proposition VII cautions that the radius of this sphere of activity must not be confounded with the radius of the monad proper.[164] The monad proper is altogether unextended. The force which Kant introduced, and which keeps monads from approaching one another beyond a certain limit is said to be the same force that others have called *impenetrability*.[165]

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[163] *Ibid*., p. 480.

[164] *Ibid*., p. 481.

[165] Cf. prop. VIII, *ibid*., p. 482.

Thus the notion of impenetrability or repelling force becomes crucial. This force guarantees that any given piece of matter is constituted of a finite set of monads, while allowing the monads themselves to be unextended. Thus, the monad "occupies" a space not by being literally present in it, but by exercising a force in it.

The repelling force which belongs to each monad is not considered constant within a certain sphere but is thought to decrease inversely proportional to the cube of the distance from the monad proper. In close proximity to the monad itself it is said to become infinite.

Much later, in the Metaphysische Anfangsgründe der Naturwissenschaft, Kant makes it clear[166] -- and in this respect his views did not change -- that he thought of the impenetrability of matter as what he calls "relative" impenetrability. That is to say matter, and thus also the monads are capable of being compressed to some extent. But since the repelling force increases toward infinity in close proximity of the monad proper, a total compression is assumed to be impossible. In the Anfangsgründe he contrasts "relative" impenetrability with "absolute" impenetrability, i.e. with the rigid

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[166] Akademieausgabe, Vol. IV, pp. 465-565, p. 502.

occupation of space. A theory under which a space is filled with absolutely impenetrable matter he calls a theory of mathematical occupation of space, his own he calls a dynamical theory.

The scheme presented in the Monadologia seems to work out quite well. The monad proper is not extended and therefore not divisible[2]. But it nevertheless occupies space. Hence the desired result is achieved.

But let us return again for a moment to the Anfangsgründe. While Kant's views had undergone considerable change between the Monadologia and this later work -- the Critique of Pure Reason falls between the two -- there is much procedural similarity between the two. A passage from the Anfangsgründe is helpful in explaining how Kant thought this kind of speculation to be justified. It seems that the desired results are achieved ultimately by way of definition. Thus, the system is developed in such a way that any critic eventually runs afoul of a definition. In the Anfangsgründe Kant makes the following point:

> Lambert and others called that property by virtue of which it (matter) occupies space 'solidity' (quite an ambiguous expression) and wanted it assumed for everything that exists (substance), at least as far as the world of outer awareness is concerned. According to their notions, the presence of something real in space should carry this resistance with it by virtue of its concept, hence

> by virtue of the law of contradiction, and should thus bring it about that nothing can be co-present with such a thing in the same space. However, the law of contradiction does not repel a piece of matter which approaches to occupy a certain space in which another is found. Only when I attribute to that which occupies space a force of repelling any mobile matter which approaches can I understand how there is a contradiction in (assuming) that a thing penetrates into a space occupied by another thing.[167]

It is patent that the attribution of force is no more successful in making impenetrability plausible, than is the attribution of solidity. In the final analysis, impenetrability is _postulated_ in both cases, or, to put it otherwise: the atom, matter, or monad, as case may be are _defined_ in such a way that the assumption of something else that occupies the same space with that atom, that monad or that piece of matter is contradictory, and it makes no difference whether the notion of force or that of solidity is called in to make the case plausible.

Essentially, the situation is no different in the _Monadologia_. Here, too, it is the postulated properties of physical monads that would make it contradictory to assume that a monad is divisible$_2$, and it would seem that Kant has overcome a major difficulty of atomism by postulating such

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[167] _Ibid_., p. 497.

properties for his monads as would make them "occupy" -- in a rather Pickwickian sense, to be sure -- a finite space, while they are nevertheless indivisible_2.

Aside from impenetrability or repelling force, Kant later on postulates a further property of monads: another force is introduced, namely gravitation, which is said to pull the monads together.[168]

As with impenetrability, Kant also assigns a definite intensity to gravitation: it is said to decrease inversely proportional to the square of the distance. I cannot understand the reasons that led Kant to settle on the cube of the distance for repulsion, and the square in this case, but the following results from them:[169] If the repulsion has a force of, say, 1000 units of one kind or another at distance 1 from the monad proper, and attraction the force of 100, then at distance 2, repulsion drops to 125, and attraction to 25. At distance 10, the respective forces balance one another, and this distance then describes the limit of the"sphere of activity" of the monad.

Adickes points out that Kant's spheres of activity

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[168] Proposition X, Akademieausgabe, Vol. I, p. 483.

[169] Cf. Adickes, op. cit., p. 175.

are literally spheres, in the sense that the monads cannot come any closer to one another than is permitted by the circumference of these spherical entities, but that they also cannot stay any further apart.[170] He finds some difficulty in the fact that Kant did not want to tolerate "empty spaces" within a body.[171] On this Adickes comments: "difficulties arise from the fact that his monads all have the same spherical volume. If they touch one another with this, small empty spaces must form in which the forces of repulsion do not have the preponderance."[172] However, I fail to see why these small spaces deserve to be called empty. They can be no more nor less empty than the spaces included <u>in</u> the ideal spheres. Kant nowhere suggests that for a monad to occupy space its repelling forces must outweigh the attractive ones. Rather, he has developed a theory according to which matter consists indeed of monads which are loosely scattered in space, which are themselves unextended and hence indivisible, but which have extended areas of activity and thereby "fill" space. To say that monads fill a given space here merely

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[170] <u>Ibid</u>., p. 161.

[171] Cf. proposition VIII, <u>ibid</u>., p. 482.

[172] <u>Ibid</u>.

means that the forces of attraction and repulsion of the monads have values higher than some given value for any point in that space. The value can be set in such a way that the "pockets" between the spheres can be considered filled.

This then is Kant's first attempt at a solution of the problem of the composition of matter. The position he takes here is frequently called one of dynamism as opposed to atomism, but this distinction appears to me to be rather artificial. I prefer to think of Kant's essay as an attempt to describe atoms as both extended and indivisible[2]. This way of looking at Kant's Monadologia Physica shows very clearly that he was not guilty of confounding spatial extension and what Adickes call "physical divisibility". The entire problem of the Monadologia arises from this distinction, as I hope to have shown.

Kant's attempt is highly noteworthy, and finds its place in this dissertation as an attempt at reviving atomism. But what is the significance of the Monadologia physica in relation to the problem of the composition of the continuum itself? It seems that the little work constitutes neither an advance, nor a regression in this respect. The problem is not even considered. I believe that Kant was completely unaware of, or did not recognize, the difficulties concerning the continuum of abstract space itself; and this in spite of his familiarity with Leibniz. He does not at this point betray any knowledge of the paradox of Galileo, nor of the fact that between any two points in a continuum there is another point, both of which were known and provided considerable conceptual difficulties for some thinkers of that age. Kant nowhere shows any apprehension when he asserts that any finite space is infinitely divisible, nor does he seem to realize the implications of that stand. Rather, here as later on, he adheres to a definition of continuity that is clearly inadequate, but which in Kant never becomes the subject of any enquiry. This definition is "A quantum is continuous which is not composed of simples."[173] Many of the views espoused in the monadologia physica Kant never altered in his later years, despite the profundity of the change of his views from his precritical to his critical period. However, this does not mean that the stand

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[173] In this form the definition is found in Kant's inaugural dissertation De mundi sensibilis atque intelligibilis forma et principiis, Akademieausgabe, Vol. II, pp. 385-419. § 14.4, p. 399. The translation is taken from John Handyside: Kant's Inaugural Dissertation and Early Writings on Space, Chicago 1929, p. 54. I shall use Handyside's translation throughout the discussion of Kant's dissertation. Actually, the above definition does not occur in the Monadologia Physica. Instead a weaker statement is introduced: Compositum in infinitum divisibile non constat partibus primitivis s. simplicius. (Akademieausgabe, Vol. I, p. 479).

taken in the <u>Monadologia</u> remained unproblematic to him. This early piece poses a number of problems which urgently demand resolution. The most outstanding of these is that matter as described in the <u>Monadologia</u> is not the same as it appears to the senses - it is a congeries of monads and their forces, while it appears, so Kant would hold, as continuous. I do not think that Kant was very much disconcerted by this fact when he wrote the <u>Monadologia</u>. The custom to regard all sensory experience as confused was too well entrenched among rationalists. I believe that it was Kant's encounter with Hume that forced upon him the recognition that the testimony of the senses cannot profitably be discounted in all cases when epistemological or metaphysical problems arise. Thus Hume's influence made him recognize the problematic aspects of his earlier views.

The matter of Hume's influence upon Kant has often been discussed, and Kant frequently acknowledges his indebtedness to Hume, especially in the introduction to the <u>Prolegomena</u>. This seems to be at variance with the views that I have stated above, namely that the cosmological antinomies were the most important factor in the development of Kant's later philosophy.[174]

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[174]The problem here is not Hume's influence upon Kant

Unfortunately, the historical evidence on this point is not sufficient to decide the case. Kant sometimes attributes the awakening from his "dogmatic slumber" to his encounter with Hume, and at other times to the antinomies. What I wish to establish now is that the cosmological antiomies would not have arisen, had it not been for Kant's acquaintance with Hume. The point is that the antinomies do not seem to have been recognized before. As long as experience as a source of evidence is eschewed, the antithesis to both cosmological antinomies cannot be established. In the Monadologia we find a geometry which demands only a continuous space, and a metaphysics which demands finitely extended ultimate constituents of matter, but no evidence is recognized which would demand that the same matter for which metaphysics entails a finite number of constituents, should also have an infinite number of ultimate parts. Now if it was Hume who taught Kant the principles of empiricism, then it was also Hume who made Kant recognize the problem of the antinomies. Thus the dispute between Paulsen and Riehl toward the end of

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as far as the latter's analysis of causality is concerned. Influence in this respect is well nigh undeniable. Rather, the point at issue is whether or not Kant's first writing of his critical period, the Dissertation, which is not concerned with an analysis of causality, is also partially indebted to Hume.

the last century, whether it was Hume or the antinomies which prompted the development of the critical viewpoint is rather pointless.[175]

Let us consider now how Kant's encounter with empiricism could have led him to recognize certain problematic aspects of his earlier philosophy. After the encounter with Hume's philosophy, Kant no longer discounts sensory experience lightly, but on the other hand, Hume's influence did not reach deep enough to make him discard the rationalist scheme advanced in the Monadologia. If such a half-way position is taken, an account must be made of the discrepancy between the rationalist account of matter on one hand, and the sensory experience of matter on the other. Here lies the first problem. A second one becomes apparent when we consider that in the Monadologia geometry was accepted as an ultimately trustworthy description of space. Now Kant must have realized that geometry in its practical application has to do with the world of sensory experience. But if sensory experience is very likely to be misleading, then geometry - whose certainty was unquestioned - could not derive its validity from such experience. But if sensory

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[175] For this dispute cf. Klaus Reich, Introduction to Kant, De mundi etc., Hamburg 1958, pp. VII ff.

experience is very likely to be misleading, then geometry - whose certainty was unquestioned - could not derive its validity from such experience. On the other hand, geometry could also not be validated through a metaphysics whose principles run counter to the doctrines of geometry. In the Monadologia, metaphysics and geometry were considered, as it were, as two independent and equally certain branches of speculation whose reconciliation is attempted. But as Kant's attention was drawn to the importance of sensory experience, the problem of the application of geometry to sensory experience and its validation -- not as a piece of speculation but as an applied science -- becomes of pivotal importance.

That geometry could in fact be applied to empirical objects was, of course, commonplace; but if geometry applied universally and unrestrictedly to experienced objects, and if these objects were thought to be continuous, then they could not have simple extended parts, at least not as experienced objects. But bodies as such did have simple parts, as had supposedly been shown in the monadologia physica.

In summary, the difficulties that Kant realized after his encounter with Hume stemmed largely from the fact that sensory experience, relegated to insignificance in the early writings, rose in status, and that geometry was recognized not to apply to things as metaphysics showed them to be, but to the world of sensory experience. We may say that it is to Hume's credit that Kant recognized, and attempted to resolve, the unfinished business of the monadologia physica.

This task was undertaken in the inaugural dissertation of 1770, De mundi sensibilis atque intelligibilis forma et principiis.[176] Thus the Dissertation, as later the Transcendental Esthetics, was meant to resolve two problems at the same time, namely the divergence between the metaphysical and empirical accounts of matter, and the validation of geometry. Moreoever, the suggestions made to this end are very similar to those put forth in the Transcendental Esthetics. In both accounts it is clearly stated that a distinction must be made between things as they appear and things as they really are[177] and, in addition, the Dissertation endeavors for the first time to explain why things appear as they do. As far as the perception of objects in space is concerned, it is asserted that:

> The concept of space is a pure intuition... It is not put together from sensations, but is the fundamental form of all outer sensation. This

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[176] Akademieausgabe, Vol. II, pp. 385-419. See also note above.

[177] Ibid., p. 392, Handyside, op. cit., p. 44.

> pure intuition can be readily observed in the axioms of geometry, and in every mental construction of postulates or of problems.[178]

and

> Space is not something objective and real, neither substance, nor accident, nor relation, but subjective and ideal; and, as it were, a schema, issuing by a constant law from the nature of the mind, for the co-ordinating of all outer sensa whatsoever.[179]

This theory does not only offer a ground for the validation of geometry, but it also explains why geometry is applicable to the world of experience:

> Thus, as regards all properties of space which are demonstrated from a hypothesis not invented, but intuitively given as being a subjective condition of all phenomena, nature is meticulously conformed to the rules of geometry, and only in accordance with them can nature be revealed to the senses.[180]

Here as later, Kant explicitly rejects the notion that geometry is derived from experience, and thus places himself in direct opposition to Hume He points out:

> Unless the concept of space had been given originally through the nature of the mind, the use of geometry in natural philosophy would be very unsafe; for it would be possible to doubt whether the notion of space obtained from experience will

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[179] Ibid., p. 403. Handyside op. cit., p. 61.

[180] Ibid., p. 404, Handyside, op. cit., p. 63.

> sufficiently harmonize with nature, the determinations from which it has been abstracted being perchance denied.[181]

In fact, the theory which would derive geometry from experience is the only rival account of the validation of geometry discussed in the Dissertation. It would indeed explain why geometry applies to experience: because it is abstracted from it. But it lacks the desired universality and certainty which geometry *de facto* possesses, since any generalization derived from experience has only a more or less high level of confirmation, and can be invalidated "if the determinations from which it has been abstracted...(are) perchance denied."

The theory of geometry so far expounded is well known, being identical in substance with that put forth in the Transcendental Esthetics. It differs widely from the considerations offered in the *monadologia*. It must be noted that in this new account sensuous knowledge is no longer regarded as necessarily confused.

> It is wrong to regard the sensitive as that which is more confusedly known, and the intellectual as that of which our knowledge is distinct...for

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[181] *Ibid*., p. 404f., Handyside, *op. cit*.. p. 63. Cf. David Hume, *A Treatise of Human Nature*. ed. Selby - Bigge, Oxford 1955, p. 71.

> that matter, the sensitive may be very distinct, and the intellectual extremely confused. This is shown, on the one hand by geometry, the prototype of all sensitive knowledge, and on the other hand by metaphysics, the instrument of all things intellectual.[182]

In the monadologia, the admission of the distinctness of sensitive knowledge would have caused considerable difficulties. While the monadologia assumes that differences exist between the appearance and the real character of things, the difference was put to the confusion of sensory experience. Now, the difference is still recognized, but it is said to be due to the fact that all sensitive knowledge is subject to certain clearly recognizable forms. However, if objects appear different from what they really are, how can the recognition of their phenomenal character be called "knowledge"? Since things are different from their appearance, does not the old charge that sensitive knowledge is confused still hold? Would one not have to admit that beliefs formed about phenomena are deceiving, since the things in themselves do not agree with these beliefs? Kant answers this objection in an exemplary fashion by observing that

> to take judgments about what is known by sense, the truth of the judgment consists in the agreement of its predicate with the given subject. But the

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[182] Ibid., p. 394f., Handyside op. cit., p. 47f.

> concept of the subject, so far as it is a phenomenon, can be given only by its relation to the sensitive faculty of knowledge; and it is by the same faculty that sensitively observable predicates are also given. Hence it is clear that the representations of subject and predicate arise according to common laws, and so allow a perfectly true knowledge.[183]

Thus the senses are capable of delivering perfectly precise knowledge about phenomena. The passage does not need any further explanation but its decidedly modern flavor deserves to be noted.

However, now the difficulties come to a head. If sensitive knowledge can be precise, and intellectual knowledge can also be precise, then the accounts of the composition of matter derived from both these sources must either be in agreement, or, if they are not, an explanation must be forthcoming why they differ. Unfortunately, the accounts are not in agreement. Matter as perceived must obey the laws of geometry, which asserts continuity of its bodies, and to be continuous, according to Kant, is to have no ultimate parts, although the continuous is a compound.[184] On the other hand, matter when considered as a compound and apart from the conditions of its sensory perception

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[183] Ibid., p. 397, Handyside, op. cit., p. 51.

[184] Ibid., p. 399, Handyside op. cit., p. 54.

must contain simple parts: "...the intellect proves that given a complex of substances, there are given also elements of composition, i.e. simples."[185]

It must be noted that in these two conflicting accounts we have a statement of the second antimony. In order to see how a resolution of this antinomy is attempted in the <u>Dissertation</u>, we must turn to the opening passages of that work.

Kant here makes a distinction between the operative and the conceptual formation of the concept of a whole. Since he does not distinguish between classes and wholes, he claims that the composition in the latter case is achieved through the class concept, but in the case of the operative formation (synthesis) of the concept of a whole, the process depends on the conditions of time, since it requires the successive addition of part to part.

Thus, to venture an example, the conceptual formation of a certain whole (class) e.g the class of brown horses and brown cows, is achieved through the class concept, i.e. "product of the class of brown things with the sum of the class of horses and the class of cows." The operative formation of the concept of that whole proceeds by "taking account" in

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[185] <u>Ibid</u>., p. 415. Handyside, <u>op. cit</u>., p. 79.

one form or another, of each member of that class, where the whole finally emerges as "the class of all those things of which I have here taken account (which I have added together)".

For our purposes more important than the case of composition is the similar situation that obtains with respect to decomposition. Here again Kant contrasts a conceptual decomposition with an operative decomposition.

> Given a complex of substances, we easily reach the idea of simples by completely removing the intellectual notion of composition; for it is simples that remain when all conjunction is abolished. But according to the laws of intuitive knowledge, the case is different; composition is not removed completely save by a regress from the given whole to all possible parts whatsoever, in other words, by an analysis which again rests on conditions of time.[186]

Such a decomposition can be carried out fully only if the process can be terminated within a finite time. But the continuum is of such a character that its decomposition cannot thus be brought to an end.

> But since in the <u>continuous</u> quantum the regress from the whole to its possible parts... finds no end...the analysis...will be impossible of completion. The whole cannot, in conformity with the laws of intuition, be apprehended by an exhaustive division of parts.[187]

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[186] *Ibid.*, p. 387, Handyside, *op. cit.*, p. 36.

[187] *Ibid.*, p. 388. Handyside, *op. cit.*, p. 36.

In order to understand the force of this argument, it must be recalled that both space and time are given as pure intuitions and that, moreover, they are the forms under which phenomena appear. This means that the conceptual decomposition of space or time, and of objects given in space and time cannot be achieved; or, to put it otherwise, it is not permissible to argue that an object given in intuition, i.e. under the form of space. consists of simple parts, if it is complex. This argument holds only for complexes given, not through intuition, but through the intellect. Thus, as far as objects in space are concerned, one cannot "reason out" what their ultimate parts are, nor that they have ultimate parts at all. and the reason for this is that the primary properties of space and time are, as it were, non-discursive. Kant points out: "All primary properties of these concepts (i.e. space and time) are beyond the jurisdiction of reason, and so cannot in any way be intellectually explained."

Hence, if a division of space, or of an object in space is to be undertaken, or if it is to be made out of what space or objects in space are composed, one must avail oneself of that method which is appropriate to objects given in

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[188] _Ibid_., p. 405. Handyside, _op. cit_., p. 64.

intuition, i.e. operative decomposition. Thus, according to Kant, operative decomposition is the only way in which anything can be made out about the parts of a continuum. But such a decomposition can never come to an end in such a case, and this is precisely what characterizes continua. The reason why this decomposition cannot come to an end lies in the following: Every division of a body or spatial region requires a finite time. Hence, if the dividing game has been going on only for a finite time the parts of the body or spatial region which result upon a series of divisions will always be finitely extended. Therefore, these parts are further divisible. Kant presumably reasoned that if the ultimate parts of the continuum were unextended, then they could be reached only through a process of division extending over an infinite length of time. But the experience of an infinite time, i.e. a length of time consisting of infinitely many finite time-spans, is impossible, for, if I start at any given moment, there is never a moment at which I can say that now I have experienced such a time span. Hence to speculate what the ultimate constituents of the continuum are is unwarranted, since the speculation transcends all possible verification.

In the <u>Dissertation</u> Kant merely asserts that it is false that the continuum has ultimate parts The above considerations, however, have shown that under his theory he should have asserted that it is meaningless to say that the continuum has ultimate parts. This position is expressed in the Critique of Pure Reason, in the proof of the antithesis of the second antimony. There Kant says:

> The second proposition of the antithesis that nowhere in the world does there exist anything simple, is intended to mean only this, that the existence of the absolutely simple cannot be established by any experience or perception, either outer or inner, and that the absolutely simple is therefore a mere idea, the objective reality of which can never be shown in any possible experience, and which, as being without an object, has no application in the explanation of the appearances.[189]

We may then say that because of the above described limitations no operation can produce an intuition of ultimate parts of the continuum. Thus, since ultimate parts of the continuum cannot be known, a continuous body or surface or line cannot be apprehended through the apprehension of its ultimate parts.

It appears that although intuition provides us with the notion of space as continuous, as well as with the knowledge of objects in space, it is not therefore capable of delivering also all the implications of the concept of

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[189]Critique of Pure Reason, A 436f, B 464f.

continuity which it has brought about. One such implication is that "in a continuous quantum the regress from the whole to its possible parts...finds no end," which means that the continuum cannot be represented in intuition as the sum of its parts. This leads to a rather thorny problem. We have seen that the discussion of the problems of the continuum is outside of the domain of reason, and now it is asserted that intuition, too, is incapable of ascertaining much about the continuous. As a matter of fact, it is asserted that some of the properties of the continuum, e.g. its divisibility *ad infinitum* is contrary to all intuitive expectations, i.e. it is counter-intuitive.

If this is the case, then continuous quantities ought to be accepted as intuitively given, and one should forego any analysis of that which is so given. But in point of fact, an analysis of sorts had already been undertaken when it was pointed out that a division of a continuous quantity cannot come to an end.

It seems now that there are no grounds on which this belief can be held, since both intuition and reason show themselves incapable of substantiating such an assertion. Here is the passage in which it is asserted that the infinite division of the continuum cannot be apprehended in intuition, but in which it is also asserted that this fact does not marshal against the belief in this infinite division:

> Since the unrepresentable and the impossible are commonly regarded as meaning the same thing, and since representation of the concepts of continuity and infinitude in accordance with the laws of intuitive knowledge is clearly impossible, we see how it comes about that these concepts are usually rejected...It is, however, of the greatest importance that those who follow this highly perverse line of argument should be warned that in so doing they fall into a most serious error. Whatever is opposed to the laws of understanding and reason is wholly impossible; but that which (as being an object of pure reason) is at variance only with the laws of intuitive apprehension is not necessarily impossible. For this disagreement between sensibility and intellect...shows no more than this, that the mind is frequently unable to follow out in the concrete, and translate into intuitions, abstract ideas which it has received from the intellect.[190]

The passage is remarkable for its defense of counter-intuitivity, but on the face of it it seems to raise more issues than it settles. For one thing, it appears that all of a sudden the continuous is characterized as arising from the intellect, moreover it is indicated that this is the reason why we have difficulties in intuiting it.

If this is what is meant, then the above passage clearly introduces an inconsistency into the system of the Dissertation. But I think that an interpretation is possible

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[190] Akademieausgebe Vol. II, pp. 388f Handyside, op. cit., pp. 37f.

which avoids this drastic course. To begin with, let me reiterate that according to Kant, continua can be given only in intuition. It is the attempt to <u>describe</u> these continua which soon introduces counter-intuitive aspects, i.e., it is somewhere in this description that "abstract ideas...(are) ...received from the intellect" which the intuition cannot follow out in the concrete. The only feature in the description of continua which cannot be intuited is the fact that they are capable of being divided indefinitely. Thus, while reason cannot tell us what the ultimate parts of the continuum are, or whether there are any such, it proves that upon any operative division of any given continuum finitely extended, divisible parts remain. That is to say, reason does not give us the representation of continua, it is not even responsible for the arising of the concept of the continuum in general, but it does elucidate certain aspects of this concept which are incapable of being intuited. While this view of the matter is not very clearly expressed in the <u>Dissertation</u>, there can be no doubt that it is adopted in the <u>Critique of Pure Reason</u>. Section eight of the Antinomy of Pure Reason introduces what is called the Regulative Principle of Pure Reason in its Application to the Cosmological Ideas. There it is pointed out:

> The principle of reason is thus only a _rule_, prescribing a regress in the series of the conditions of given appearances, and forbidding it to bring the regress to a close by treating anything at which it may arrive as absolutely unconditioned... Thus it is a principle of reason serving as a _rule_, postulating what we ought to do in the regress, but _not anticipating_ what is present _in the object as it is in itself, prior to all regress_. Accordingly, I entitle it a _regulative_ principle of reason...[191]

The regress that is generated through a series of divisions of a given continuum falls under this rule of reason: "If we divide a whole which is given in intuition, we proceed from something conditioned to the conditions of its possibility."[192] That this rule of reason prescribes a process which cannot, in its entirety, be represented in intuition becomes clear from the following passage:

> When...we have in mind the transcendental division of an appearance in general, the question how far it may extend does not await an answer from experience; it is decided by a principle of reason which prescribes that, in the decomposition of the extended, the empirical regress, in conformity with the nature of this appearance, be never regarded as absolutely completed.[193]

Thus, reason tells us that the operative decomposition of continuous entities will never come to an end, and this

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[191] _Critique of Pure Reason_ A 508f, B 536f.

[192] _Ibid._, A 523, B 551.

[193] _Ibid._, A 527, B 555.

is the counter-intuitive aspect which reason introduces in the analysis of the concept of the continuum.

However, while reason provides us with this rule, it cannot make out anything else about continua. To summarize, since the concept of continuity has its origin not in the intellect, but in intuition, reason cannot ascertain the composition of the continuum. But reason dictates that any division of the continuum will result in finitely extended parts, no matter how many divisions are successively undertaken. On the other hand, we cannot intuit such an unending sequence of divisions, since they would demand an infinite time, of which an intuition is impossible.

This is the point to which, according to Kant, the analysis of the continuum can proceed, and any further pronouncement about it, as I have tried to show is held to be meaningless.

It is clear from the above that a continuous quantity is not to be thought of as a complex of substances (for as such it would alwyas have to composed of simples) but only as a something which is given in intuition. Thus the intellect, which demands that <u>its</u> objects consist of simples cannot pass judgment on the composition of such continuous quantities. On the other hand, things in themselves, according to Kant, must fulfill the requirement that, if they are complex, they must consist of a number of simples, either finite or infinite. This position is retained in the Critique of Pure Reason, although neither there not in Dissertation a proof is offered, unless one were to consider the following passage a proof:

> The assertion that if all compositeness of matter be thought away nothing at all will remain, does not appear to be compatible with the concept of a substance which is meant to be the subject of all compositeness, and which must persist in the elements of the composite, even although the connection in space, whereby they constitute a body, be removed. But while this is true of a thing in itself, as thought through a pure concept of the understanding, it does not hold of that which we entitle substance in the field of appearance. For this latter is not an absolute subject, but only an abiding image of sensibility; it is nothing at all save as an intuition, in which unconditionedness is never to be met with.[194]

I interpret this passage in the following way: by substance is here meant something which is absolutely simple. Any complex of substances depends for its existence on the existence of the (simple) substances of which it is composed. Thus it is analytic that a complex of substances consists of, and depends for its existence on, simples. On the other hand, not all complexes are complexes of substances, and hence

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[194] Critique of Pure Reason, A 525, B 554.

the conclusion that all complexes consist of simples is fallacious The latter holds in particular for continua.

Here then lies Kant's solution to the problem of the continuum: As in Leibniz, an intelligible world of noumena is distinguished from a phenomenal world. Continua occur only in the latter, but since they have their origin in intuition, not much can be made out about them discursively. But intuition, being subject to the limitations of time is also incapable of breaking down a continuum to its ultimate parts, so that it becomes meaningless to speak of the ultimate parts of the continuum. Noumena, or things in themselves cannot be said to be continuous. Of them it can therefore confidently be asserted that, if they are composite, they have simple parts.

A word of caution is here in place. When it is asserted that a series of divisions of a given continuum can, in principle, be infinitely extended, i.e. that the operation of division cannot have a final member, then this is not to mean that there is an infinity of parts already given in that continuum. Any intuition is always given as a whole, in one act of intuition. In the successive steps of its operative decomposition more and more of its parts are realized. But since a given continuum is nothing but an intuition, and nothing but one intuition, we cannot say that these parts were already there before the operative decomposition began. Thus it can be said that a continuum can be divided ad infinitum, but not that it "has" infinitely many parts.[195]

This point is made so often, from the Dissertation to the Metaphysische Anfangsgründe, and Kant is so insistent upon it that it is quite obvious that he attached a good deal of importance to it. And justifiably so, for he regarded the assumption of an actual infinite in a finite object as inconsistent. In the Anfangsgründe he flatly asserts that "there is no real infinite number (of parts) in the object (which would be an explicit contradiction)."[196]

In the Critique of Pure Reason he tries to give a ground for this assertion by saying:

> In the case of an organic body conceived as organized in infinitum the whole is represented as already divided into parts, and as yielding to us, prior to all regress, a determinate and yet infinite number of parts. This, however, is self-contradictory. This infinite involution (Entwicklung) is regarded as an infinite (that is never to be completed) series, and yet at the same time as completed in a complex (Zusammennehmung).[197]

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[195] Cf. Critique of Pure Reason, A 513f, B 541f; and A 524f, B 552f.

[196] Akademieausgabe, Vol. IV, p. 507.

[197] Critique of Pure Reason, A 527, B 555.

That this argument is fallacious can easily be shown. In order to avoid problems that arise from higher orders of infinity let us consider the set of rational numbers larger than or equal to 1 and smaller than or equal to 2. This set has 1 and 2 as its smallest and largest member respectively, and it contains infinitely many members. A one to one correspondence between it and the set of positive integers can be demonstrated. But to argue that there cannot be infinitely many rational numbers between 1 and 2 _because_ the set of integers has no last member is clearly a _non sequitur_. But this is what Kant's alleged proof amounts to.

One could perhaps surmise that Kant's denial of the simultaneous infinite was brought about through Leibniz' contention that there is no infinite number, which we have discussed above. In that case we would find in Kant again some influence of Galileo's paradox. This surmise receives some lilelyhood from the fact that Leibniz frequently asserts that an infinite number of parts in an object is impossible, while only occasionally stating his reasons for this assertion. Be this as it may, Kant's denial of the actual, simultaneous infinite can be said to have had profound influence upon the structure of his system, whatever his reasons for this denial might have been.

The theory so far discussed is that represented in the *Dissertation*. In my exposition I have made use of material from the *Critique of Pure Reason* only for elucidatory purposes, and I feel justified in having done so since I cannot perceive any fundamental difference between the theories of the *Dissertation* on one hand, and the *Transcendental Esthetics* and the discussion of the first two antinomies on the other. The first *Critique* goes beyond the dissertation in that it gives a clearer statement of the antinomies to be resolved, but the resolution, at least of the first two antinomies, is undertaken entirely within the framework already provided by the *Dissertation*. What is left for us to do, then, is to consider the Second Antinomy, and to see how, precisely, its resolution is attempted. The Second Antinomy is stated thusly:

> Thesis: Every composite substance in the world is made up of simple parts, and nothing anywhere exists save the simple or what is composed of the simple.
>
> Antithesis: No composite thing in the world is made up of simple parts, and there nowhere exists in the world anything simple.[198]

By the term 'world' is here meant 'the sum of all appearances',[199] which, according to Kant, makes the thesis

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[198] *Critique of Pure Reason*, A 434, B 462.

[199] Cf. *Critique of Pure Reason*, A 419, B 447.

false. The thesis would hold only of things in themselves, but these do not belong to the world defined as the sum of appearances.

On the other hand, the antithesis is intended to come out true, with the understanding that 'world', again, means 'the world of appearances.' Thus the conflict is removed since the thesis is true only for things in themselves, the antithesis only for appearances.

A difficulty arises from the fact that Kant seems to claim, on at least one occasion, that both thesis and antithesis are false. He points out that "the suit in which reason is implicated...has been <u>dismissed</u> as resting, on both sides, on false presuppositions."[200]

Now we have seen that the thesis is clearly meant to be false, owing to the restriction of 'world' to appearances. The antithesis, however, is meant to be true, if the term 'world' is used in the same restricted sense; it amounts to the claim that among appearances simples can nowhere be found.

I believe that what Kant assails as resting on false presuppositions is not the position stated in the antithesis, nor, for that matter, that stated in the thesis, but doctrines

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[200]<u>Critique of Pure Reason</u>, A 530, B 558.

that would arise, if by 'world' were meant, not 'the sum of all appearances, but 'the sum of all there is'. In this case, the thesis would be false because it is not considered to hold of appearances, and the antithesis would be false, since it supposedly does not hold of things in themselves. Thus, a conflict arises only if the claims, either for thesis or antithesis, are overextended, as would be the case if the term 'world' were meant to refer to all there is.

We have seen above what Kant's reasons were for accepting the doctrine of the thesis for things in themselves, and that of the antithesis for appearances, but a short summary is here in place.

In the early Kant the problem of the composition of matter was construed as a problem about the intelligible world. Kant assumed then that any piece of matter, considered in itself, is made up of a finite number of ultimate parts, while the space in which such matter is located is infinitely divisible. Since appearances do not agree with the metaphysical description of matter, it must be assumed that sensory experience is confused. This doctrine was put in jeopardy by the realization that geometry applies unrestrictedly to objects as they appear, and since geometry demanded the infinite divisibility of its lines and bodies, the same would have to hold of any and all objects in space. The unrestricted application of geometry to spatial objects was considered justified through the assumption that space is the form of all outer experience, and that the axioms of geometry are an accurate description of the pure intuition of space. As long as the doctrine of infinite divisibility applied only to purely geometrical entities it was perhaps not as pressing to the metaphysician as now, when this same doctrine was to hold of any objects in space. The solution was made more difficult through the assumption that it is contradictory to assume an infinity of parts in a finite object. The conflict was finally resolved through the introduction of the notion of operative decomposition: bodies in appearance are originally always given as _one_, and they have only as many actual parts as are in fact realized through successive steps of operative decomposition. That there is no last step in this decomposition is guaranteed by a so-called rule of reason. Thus bodies are said to be divisible _ad infinitum_ without therefore having an infinity of _actual_ parts. This whole scheme required that bodies are not something in themselves, because only if they are not things in themselves can it be assumed, so Kant thought, that the parts do not precede the actual division. If they were things in themselves an infinite division in the object would have to be assumed, which was held contradictory. Kant's theory was not only meant to resolve the problems of the continuum as far as the constitution of (continuous) bodies is concerned, but it was intended also to settle the issue for continua as they occur in pure geometry. For these the same considerations would hold, since they, too, are objects that occur in intuition, albeit in pure intuition.

Thus we can see how the doctrine of the Transcendental Esthetics is a proposal which, in Kant's opinion, would satisfactorily resolve the problems of the composition of the continuum.

CHAPTER V

BERNARD BOLZANO

In the preceding chapters I have attempted to show that some well-known philosophical systems of the Seventeenth and Eighteenth century were constructed in such a way that in them certain problems connected with the continuum and with infinite sets did not arise. In all those systems such problems were of central importance, as I hope to have shown, but they were by no means the only problems to be discussed: they were always transacted in a larger context. We saw that they arose in connection with the problem of the constitution of matter, of space or of time, and that their solution was attempted through metaphysical or epistemological considerations.

In Bolzano this is, for the first time, no longer the case. Bolzano addresses himself explicitly and directly to the problem of infinite sets in general, and of continuous sets in particular. Even through a casual perusal of Bolzano's work one becomes impressed with the forthrightness and comparative precision with which he states his problems and attacks them. It seems that this very fact earned him the contempt of many of his contemporaries. I believe that this is largely due to the influence of Kant's philosophy. Bolzano actually addresses himself to several of the problems with which Kant was originally concerned, and for which Kant supplied his own particular solutions. But it was Kant's solutions, rather than his problems, that directed the philosophical enquiries of his recognized successors. A passage from a review of Bolzano's Wissenschaftslehre bears testimony to this:

> The standpoint of the author is throughout the old one, which is called strictly objective, also dogmatic, in contradistinction to the contemporary one which is based on the psychological self-consciousness of the thinking mind.[201]

In the face of much derogatory twaddle, Bolzano continued to deliver his philosophy from the outmoded objective standpoint, and one of the fruits of his labor is his epochmaking, though by no means definitive Paradoxes of the Infinite.[202]

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[201] P. Menelaos (probably a nom de plum), Review of Bolzano's Wissenschaftslehre, Zeitschrift für Philosophie und Katholische Theologie, Heft 25. Quoted from Dr. Bolzano und seine Gegner, Sulzbach (Seidel) 1839, pp. 157f.

[202] Bernard Bolzano, Paradoxien des Unendlichen, (ed. Fritz Prihonsky), Hamburg (Felix Meiner) 1955, 1st. ed. Leipzig 1851. Translated as Paradoxes of the Infinite (translated and ed. by Donald A. Steele, S.J.), London (Routledge and Kegan Paul) 1950. This work was published posthumously (Bolzano died in 1848), and there is a likelyhood that Prihonsky did some editing which, at least on

This work is known chiefly for the fact that in it Bolzano states again several examples of the Paradox of Galileo, and generally embraces the position that all infinite sets have subsets to which they stand in biunivocal correspondence. Bolzano thought that he had, for the first time, discovered this property:

> We now pass on to consider a very remarkable peculiarity which can occur in the relation between two sets <u>when both are infinite</u>. Properly speaking it always occurs, but to the disadvantage of our insight into many a truth of metaphysics as well as physics and mathematics, it has hitherto been overlooked. Even now, when I come to state it, it will sound so paradoxical that we shall do well to spend some time over its investigation. I assert the following: When two sets are both infinite, then they can stand in such a relation to one another that:

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one occasion, led to a palpable falsification. Bolzano himself had prepared for publication a manuscript on the theory of functions which was published in 1930 (<u>Functionenlehre</u>, Prague 1930), in which he gives an explicit example of a continuous, non-differentiable function, while the paradoxes declare (Footnote § 37) that all continuous functions are differentiable, except for isolated points. The discrepancy was first discovered by Jasek. Cf. Introduction to <u>Paradoxes of the Infinite</u>, p. 54). It was generally held at the time that all continuous functions are differentiable, so that it can be unde stood how Prihonsky, a man of limited mathematical background could have added an "explanatory" footnote to that effect. On the other hand, in the present paper we are concerned with doctrines, the ingenuity and novelty of which point clearly to Bolzano as their author. Moreover, there are a large number of points of agreement between the <u>Paradoxien</u> and the <u>Wissenschaftslehre</u> (Bernard Bolzano, <u>Wissenschaftslehre</u>, 4 vols., Sulzbach (Seidel) 1837). In particular, the definition of "continuum" is identical in both accounts.

(i) it is possible to couple each member of the first set with some member of the second in such a way that, on the one hand, no member of either set fails to occur in one of the couples; and on the other hand, not one of them occurs in two or more of the couples; while at the same time

(ii) one of the two sets can comprise the other as a mere part of itself.[203]

Now we have already seen that Bolzano was mistaken in assuming that he was the first to notice this relation, since both Leibniz and Berkeley were aware of it in its generality, while Galileo knew at least an example.[204] What is new is that Bolzano did not think that this property makes the existence of infinite sets impossible; in other words, he no longer regards the sweeping acceptance of Euclid's part-whole axiom as justified.

This attitude of Bolzano's, of course, rests on his recognition that the assertion of the biunivocal correspondence of an infinite set with one of its proper subsets does not introduce any obvious contradiction into mathematics, provided that Euclid's axiom is properly understood. Bolzano's merit lies, then, not in his having discovered this feature

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[203]Bolzano, <u>Paradoxien</u>, § 20, p. 27 f.

[204]Jourdain is consequently mistaken when he writes that "this curious property of infinite aggregates was first noted by Bernard Bolzano" (Philip E. B. Jourdain (ed.) <u>Contributions to the Founding of the Theory of Transfinite Numbers</u>, by Georg Cantor, New York (Dover) 1915, editor's introduction, p. 41).

of infinite sets, but in that he, for the first time, placed enough trust in the procedures of mathematical enquiry simply to accept it as a fact, rather than to confront it with bewilderment as his philosophical precursors had done. Thus we must criticize Fraenkel for assuming that Bolzano's Paradoxien were to be "a catalogue of, as it were lamentable, paradoxes which condemns itself to fruitlessness."[205]

Actually, the Paradoxien are a discussion of alleged paradoxes, of which the Paradox of Galileo is one. But their whole point is that here are only seeming paradoxes which a precise account of infinite sets will dispel. Lamentable is at most the fact that other philosophers found paradoxes were there are none.

Bolzano makes it quite clear where the oddity of the Paradox of Galileo stems from:

> As I am far from denying, an air of paradox clings to these assertions; but its sole origin is to be sought in the circumstance that the mutual relation which we find between two sets when we can pair off their parts (members) with the previously mentioned result suffices in every case where these sets are finite to establish their perfect equinumerosity of members.[206]

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[205] Abraham A. Fraenkel, Mengenlehre und Logik, Erfahrung und Denken, Vol. II, Berlin (Dunker und Humblot), 1959, p. 10.

[206] Bolzano, Paradoxien, § 22, p. 31.

Let us consider, for the moment, only part of the import of this statement. What is asserted is that it sounds paradoxical to claim that infinite sets are in biunivocal correspondence to proper substes of themselves, but only because such a relation cannot obtain with finite sets. The latter fact is seen as an obstacle to the acceptance of the former, probably because we do not have any direct acquaintance of an intuitive or sensory sort with infinite sets.

A more important feature of the above passage is that it adds to the assertion that all infinite sets stand in this relation to some of their proper subsets the converse of this assertion, namely that only infinite sets have this property. However, this characteristic of infinite sets, namely that all and only infinite sets can be brought in a one-to-one correspondence with proper subsets of themselves, is not taken to be the definitive characteristic of infinite sets; Bolzano chose a different definition: "I shall call an infinite mulitiude one that is larger than any finite magnitude, i.e. one of which any finite set represents only a part."[207]

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[207] Bolzano, Paradoxien, § 9, p. 6.

I can see no objection against this procedure, which first defines 'finite', and then employs the term 'finite' in the definition of 'infinite', provided that a satisfactory definition of 'finite' is given. Bolzano gives this definition in the following passage:

> Let us consider a series whose first term is an individual of the species A, and whose every subsequent term is derived from its predecessor by joining a fresh individual with the equal of that predecessor so as to form a sum. Then clearly all terms which occur in this sequence, with exception of the first, which is a mere individual of the species A, will be multitudes of the species A. Such multitudes I call finite [208]

This definition poses more problems than it solves. Setting aside minor inadequacies of expression, what is intended is obviously the following: Consider a set A of objects. Choose any member of A, for example k, and form the unit set $\{k\}$ of k. Then either take k, and form a set consisting of k and some other element of A, say l. Or else pick a set of members of A which has as many elements as $\{k\}$, for example $\{m\}$, and form a set out of its members together with precisely one other member of A. In general, if N is a given set in the series, the next set is formed either by taking the elements of N together with precisely one other element of A, or else by taking a set N' which is "equal"

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[208] Ibid.. § 8, pp. 5 ff.

to N and which is also a subset of A, and adding to the elements of N' another element of A. Clearly, the procedure depends upon a definition of 'equal', which is nowhere given, a fact that leads to other serious shortcomings, as we shall presently see.

All sets in the series so generated are finite in the general, preanalytic acceptation of that term, but we do not therefore get a precise definition of 'finite', since obviously there are sets which neither occur in the sequence, nor fulfill the criteria for infinite sets. To remedy this situation, we would have to consider not just one set (species) A, but all sets there are, and for each of these, except for the unit sets, we would have to form more than one series, since under the above described procedure for example $\{1\}$, which is ordinarily considered a finite set, would nowhere occur in the series we were describing.

But even if all these amendations were made, we would still require a definition of equality, a proof that the choices here described can always be made and assurances that the consideration of all sets does not, in this case, lead to inconsistencies. I cannot here undertake a detailed discussion of these matters, but the above considerations show that Bolzano's definition of 'finite set' is not tenable in the form in which it is given.

To return to the discussion of infinite sets: We saw that Bolzano had recognized that all and only infinite sets stand in biunivocal correspondence to some of their proper subsets. It will be recalled that Cantor ascribed equal "power" (Machtigkeit) to sets that stand in such a relation to one another. Nowadays we generally call such sets equivalent. Presumably, Cantor chose the word 'power' in order to avoid such expressions as 'equal', or 'as large as', or 'having as many members as', which might merely have involved him in fruitless quibbles. Cantor, however, nowhere asserts that of two sets which have equal power, one can be larger than the other, no matter how "obvious" this might seem. Bolzano had not reached this stage. He points out that if we pair off the members of two finite sets, and none remains in either set, then none of the two sets is larger than the other. Not so with infinite sets. Here, he claims, two sets can be in one-to-one correspondence, even though one is larger than the other.[209]

> The mere fact...that two sets A and B are so related that every member a of A corresponds by some rule to some member b of B in such wise that

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[209] Ibid., § 22, p. 31.

> the set of couples (a+b) contains every member of A or B once and only once, never justifies us... to infer the equality of the two sets with respect to the multiplicity of their members <u>if these sets are infinite</u>.[210]

The "obvious" examples are the rational numbers between 0 and 5 and 0 and 12 respectively, and the points on a bounded line and on a proper part of that line. Although in both cases biunivocal correspondences can be established, Bolzano holds that there are "more" rational numbers between 0 and 12 than there are between 0 and 5. and a similar case is made for the two lines.

One might think that Bolzano perhaps differed only in terminology from the now generally accepted position. But in order to establish this conclusively, one would have to find out what, precisely, Bolzano meant by 'equal in number', 'larger' etc. To judge from his examples, what he had in mind is the following: If A is a subset of B, then A is smaller (has fewer members) than B. If A and B have identical members, then they are equal. If this is all that can be supplied in definition of 'equal' and 'smaller', then we are forced to the assumption that certain sets cannot be compared with respect to the number of their members; and Bolzano in fact asserts "whether there are

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[210] <u>Ibid</u>. § 21, p. 30.

more triangles or more syllogisms is indeed a question to which no answer can be given other than that one does not know how to compare these two infinite sets."[211]

The undesirability of Bolzano's approach becomes clear when we ask whether there are "more" (in Bolzano's sense) negative integers smaller than 0 than there are positive integers larger than 1. While he would assert that there are "more" positive integers larger than 0 than there are larger than 1, he cannot meaningfully answer the first question at all, for if he were to pair off all negative integers smaller than -1 with their positive counterparts, in order to have -1 left over, he would have established equinumerosity through one-to-one correspondence, a procedure that he denies himself explicitely.[212]

Bolzano himself did not always heed his own injunction against assuming equinumerosity for biunivocally corresponding infinite sets. On one occasion he flatly asserts that there are as many square of integers as there are integers,[213]

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[211] Bernard Bolzano, Wissenschaftslehre, Vol. 1, p. 438f.

[212] Cf. Bolzano, Paradoxien, § 22, p. 31.

[213] Ibid., § 33, p. 54.

and on another he claims that there are as many circular surfaces as there are circumferences.[214]

Let me now proceed to a discussion of Bolzano's views on continuity. He gives the same definition of 'continuous' in both the Wissenschaftslehre and the Paradoxien. He states that a continuum is present when, and only when, there is "a set (Inbegriff) of simple objects (of points in time or space, or even of substances) which are so situated that every single one of them has at least one neighbor for every distance, however small."[215]

In spite of its shortcomings, this definition enabled Bolzano to recognize certain important properties of the continuum, and to dispel mistaken notions about its composition. He points out that it had been known all along that the existence of extended entities cannot be explained, without circularity, by claiming that they are composed of parts that are themselves already extended. Nevertheless, he says, a contradiction was thought to be hidden in the assumption that extended entities could be composed of parts that are simple (points in time or space). He continues that the

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[214]Bolzano, Wissenschaftslehre, Vol. I, p. 439.

[215]Bolzano, Paradoxien § 38, p. 73. Cf. also Wissenschaftslehre, Vol. III p. 252.

reason why this last assumption was considered contradictory was sometimes given in the claim that a property which is lacking in the parts cannot be found in the whole. This Bolzano rejects on the grounds that every whole _must_ have several properties which are not found in any of its parts. According to Bolzano, a second objection was seen in the fact that any two points in time or space must always have a distance between them, and thus cannot form a continuum. This Bolzano takes to be a _non-sequitur_. He agrees that between any two points in time or space there must always lie another point, in fact infinitely many other points, but he denies that there is a contradiction in this assumption. All it proves is that a finite set of points can never form a (continuous) extension. Even an infinite set of points does not always form a continuum; it does not if the points do not occur in the proper order (Anordnung).[216]

Against the charge that all this cannot be grasped, he replies that "indeed, it cannot be grasped with the fingers, and not be observed with the eyes, but it is recognized through the understanding, recognized as something that

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[216] Cf. Bolzano, _Paradoxien_, § 38, p. 72 f.

necessarily must be so and cannot be otherwise."[217]

Bolzano's description of continua, despite the insufficiency of his definition,[218] constitutes considerable progress in more than one respect. To begin with, it is finally realized that continua can be described as point sets, and this realization is occasioned by a reliance upon the trustworthiness of purely logical analysis that had hitherto not been equalled.

In the sequel, Bolzano describes several properties of continuous sets, but they are generally ones which continuous sets share with merely dense sets. Of particular merit is the distinction that he draws for the first time between open, half open, and closed intervals.[219]

Before I proceed to a discussion of the cosmology which Bolzano bases on his theory of infinite sets, let me

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[217] _Ibid._, p. 74.

[218] Cantor already pointed out that according to Bolzano's definition of 'continuity', two entirely separate bodies, e.g. two spheres, could form _one_ continuum. Cf. George Cantor, in _Mathematische Annalen_, Vol. XXI, p. 576, also Hans Hahn's note to § 38 of the _Paradoxien_, _ibid._, p. 148.

[219] Bolzano, _Paradoxien_, § 41, p. 82.

briefly recapitulate the main points that have so far been made in this chapter.

We saw that Bolzano realized that biunivocal correspondence of infinite sets with their proper subsets does not demonstrate that infinite sets are impossible, but rather must be accepted as a general characteristic of all and only such sets. On the other hand, Bolzano does not consider the existence of such a correspondence between two sets as a proof for their "numerical equality". However, 'numerical equality' is not defined, so that one cannot make out how such equality is to be ascertained. A further shortcoming of Bolzano's exposition was found in the fact that no acceptable definition of 'finite set' is provided, the definition given by Bolzano being totally unsatisfactory. We may say that Bolzano's account despite its general precision suffers from his lassitude in definition at crucial points. On one occasion he writes that "if ∞ is an infinite set, then $\frac{\infty}{2}$, $\frac{\infty}{4}$, $\frac{\infty}{8}$ are also infinite sets,"[220] without having given any indication what the expressions $\frac{\infty}{2}$ etc. are to mean.

We then considered his definition of 'continuity'

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[220] Ibid, § 38, p. 74.

which was found to be too wide. Notwithstanding this fact, Bolzano's analysis of the continuum must be considered a major advance in the field. This holds especially for his description of continua as points sets.

Bolzano fully realized that a description of the properties of infinite sets does not in itself furnish a proof for the existence of such sets, and he consequently gave a number of examples of such sets. In doing so he applied the curious ontological bifurcation that he had already developed in the Wissenschaftslehre, i.e. he first shows that there are infinite sets of objects "that do not have reality", and then he attempts to prove that there also are infinite sets of "things that do have reality". In the former class he mentions the set of all propositions (Sätze an sich), the set of integers, the set of rational numbers, but also time and space, which he considers infinite point-sets that are not actual, although they are determinations of the actual.[221] It is interesting to notice that Bolzano realized that in accordance with his theory of time as a continuous set of moments it is false to say that there is a moment which is the next to any given moment.[222]

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[221] Ibid., §§ 13-17, pp. 13-24.

[222] Cf. Bolzano, Wissenschaftslehre, Vol. I, p. 405.

Among those things that do have reality, Bolzano also finds infinite sets, e.g. the set of all determinations (Bestimmungen) of a given object in any finite time. This set is indeed infinite as can easily be shown since the set of all properties of any given object is infinite at any given moment.

A much more interesting example is given with physical bodies, which are said to be continuous and to consist of an infinity of unextended atoms. Among these, some are said to be the dominant atoms, others subordinate. The former determine the characteristics of any given body, and are finite and loosely scattered within that body, while the latter form a sort of "ether". Where no dominant atoms are present, we have mere ether, while in the bodies it acts as a sort of "fill".

I find remarkable not so much this theory itself (although it was revolutionary enough to embrace the notion of the actual infinite in such a forthright fashion) but the relentlessness with which intuitively startling consequences are drawn from these assumptions. One is that:

> the same set of substances which at this moment fills this cubic foot can at another time be distributed in a million times larger space and at still another time be compressed to one thousandth of its original volume without is being the case that during the expansion any point in

> the larger space was empty, or during the compression any point had to accommodate two or more atoms.[223]

Another remarkable insight concerns the limits of bodies. By the limit of a body Bolzano means the set of its outermost atoms. From the above considerations it follows that if two bodies are immediate neighbors, then, if one of them has a limit in the just defined sense, the other cannot have such a limit where the two bodies border to one another, which, I believe, is the same as to say that two disjoint closed intervals cannot be neighbors since there must always be some points between them.[224]

Let this suffice as a brief sketch of Bolzano's theory of infinite sets and the continuum. It can be seen that Bolzano's work constitutes an enormous step forward to a final solution of the problems here discussed, although his own exposition is by no means flawless in every detail. The points of difference between Bolzano and his philosophical precursors as concerns the approach to these difficulties are obvious: Bolzano relied wholly upon logical analysis, developing his modest cosmological scheme subsequent to,

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[223]Bolzano, Paradoxien, § 59, p. 117.

[224]Ibid., § 66, pp. 125 ff.

and dependent upon, a solution to the problems of the continuum and of infinite sets that satisfied him. He realized for the first time that the assumption of infinite sets does not lead to inconsistencies, and, in particular, that the so-called Paradox of Galileo does not constitute such an inconsistency. He saw, furthermore, that continua can be described as point-sets, one of whose properties it is that between any two of their points there lies another point. His predecessors, who did not have these insights, felt obliged to circumvent the seemingly paradoxical properties of infinite and continuous sets by all manner of epistemological and metaphysical assumptions, a procedure from which Bolzano't logical acumen saved him.

CONCLUDING REMARKS

With the advent of modern analytic philosophy, much aspersion was cast upon all but a few products of classical philosophy. The positivist meaning criterion, for example, even in its later and much weakened forms would, in fact, reduce most older philosophems to nonsense. One of the consequences of this increased rigor was that the word "metaphysical" together with the systems that it describes fell into general disrepute.

When I became acquainted with, and found myself accepting, many of the procedures of this analytic philosophy, I became concerned whether classical philosophy ought to hold more interest for the modern student than a museum-piece and, if so, how a well reasoned and honest system of "metaphysics" might be distinguished from an irrelevant and corrupt one. The second problem can probably be met through careful interpretation, or, if you will, reconstruction of the systems in question, but my concern was not only whether the reconstruction of some philosophy would result in a system of statements which are meaningful under some of the present stringent meaning criteria, but also whether there is a communality of purpose to be found between contemporary

investigations and those of our philosophical precursors: there had to be some assurance that, after the reconstruction had sifted out the pseudo-problems, more than a collection of trivia would remain.

The foregoing dissertation was addressed to the second of these concerns. I have not undertaken to provide a reconstruction of the systems which I have considered, since each such reconstruction would have to take into account a philosophical system in its entirety. I felt justified in taking this course since the importance and scope of the problems which I found treated in these systems can be recognized in them even though various of their aspects were not clarified. I found that many facets of the philosophies which I have considered had to do, in some way, with the disquieting effect that the recognition of some of the properties of infinite and continuous sets tend to have upon those who concern themselves with such matters. In pursuing my task, I found myself discussing more historical detail than had initially been my intention, and also omitting certain topics related to the general purpose of this paper, which would be very worthwhile subjects for further inquiries. Of these, the following seem to me the most interesting.

I have repeatedly stated that oftentimes the justification of a philosophical system is to be sought in the propositions which it entails, rather than in some allegedly self-evident assertions which it contains or seems to presuppose. I have claimed that inasmuch as this viewpoint is adopted, a philosophical system is to be evaluated in much the same way as a scientific theory or hypothesis. But there is an important difference which I did not discuss: the systems which I have considered have none of them observable consequences, and it remains to be investigated what kind of consequence it is that supports them. At this point it seems to me that each of them should be envisaged as some kind of metatheory meant to provide certain rules which a scientific system or theory can violate only at the penalty of being inconsistent. I am far from clear on this point. However, I should be satisfied if the merit is seen that lies in judging a philosophical system by its consequences, but also if the problems are recognized that arise in connection with such an approach.

A second point that would warrant some further attention is of a more speculative nature: we have seen that Bolzano was in the possession of two insights which his predecessors had been denied, namely he realized that the Paradox of Galileo is not really a paradox, and that the sum of a finite or infinite number of extensionless parts does not have to be extensionless. The question arises what form the philosophical systems of his predecessors would have taken, had they been familiar with these results. In accordance with the results of this dissertation, I believe that the differences would have been profound and extensive to the point that any speculation about this issue is condemned to fruitlessness, irrespective of whether we take our hypothetical assumption in a psychological or systematic sense.

A third, and more important, unfinished issue is the following: We have seen, for example, that both Kant and Berkeley disclaimed the existence of actual infinite sets, at least for the world of appearances. It remains to be investigated whether they are not otherwise committed, without intending it, to the assumption of such sets. In general, since the aim of this paper was not a reconstruction of the systems under discussion, the question never arose whether these systems are consistent. I hope that any findings contradicting my results can be put to such undiscovered inconsistencies.

The point that I hope to have made in the foregoing dissertation -- and this may seem trivial to some -- is simply this: that the philosophers whom I have investigated were as far removed from purposeless speculation as from mysticism, that they were attentive in the presence of logic and fact, and that the modern analytic philosopher is far from having outgrown the problems with which they concerned themselves.

It is ironical that those who proclaim themselves the successors of these great men have often failed even to grasp their problems, while many analytic philosophers, unaware of the problems involved, have denounced a tradition of which they are today's representatives.

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