SELF-COMPLEMENTARY GRAPHS: THEIR STRUCTURAL PROPERTIES AND ADJACENCY MATRICES

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
RICHARD ADDISON GIBBS
1970

This is to certify that the

thesis entitled

SELF-COMPLEMENTARY GRAPHS:
THEIR STRUCTURAL PROPERTIES
AND ADJACENCY MATRICES

presented by

Richard Addison Gibbs

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

J. Sutherland Frame
Major professor

Date April 29, 1970

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ABSTRACT

SELF-COMPLEMENTARY GRAPHS: THEIR STRUCTURAL PROPERTIES AND ADJACENCY MATRICES

By

Richard Addison Gibbs

In this thesis we examine some structural properties and adjacency matrices of self-complementary graphs. We first describe a basic algorithm which will produce all self-complementary graphs, and from which most of the main results are derived. We show that any self-complementary graph contains a collection of induced subgraphs, each isomorphic to the "smallest" self-complementary graph, P_3. A useful adjacency matrix is introduced and for certain self-complementary graphs we exhibit a decomposition of such matrices which facilitates the calculation of their eigenvalues. Finally, some results are presented concerning the possible degrees of the points of a self-complementary graph.

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By

Richard Addison Gibbs

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1970

To

SANDI, BONNIE, NANCY,

RICHARD JR. or SALLY

and

THE FOLKS

ACKNOWLEDGMENTS

I am deeply indebted to Professor J. Sutherland Frame for his patience, inspiration and timely suggestions throughout the research and preparation of this thesis. I also wish to thank Professor E. M. Palmer for his encouragement and interest.

Special thanks is due to Professor J. J. Seidel who, although he doesn't know it, started it all.

TABLE OF CONTENTS

LIST OF FIGURES AND TABLES

LIST OF FIGURES AND TABLES (cont'd.)

INTRODUCTION

Heretofore, self-complementary graphs have been the subject of little investigation. For a given positive integer n, algorithms for constructing all self-complementary graphs having n points have been developed, and a formula for the number of such graphs has been derived. Other results have been obtained, chiefly concerning special types of self-complementary graphs: regular, quasi-regular and cyclic. Also, the characteristic polynomials of the standard 0,1 adjacency matrices of special self-complementary graphs have been studied. However, structural properties of self-complementary graphs have not often been considered.

In this thesis, we present a new algorithm for constructing all self-complementary graphs and use it to obtain several results concerning the structural characteristics of these graphs. In addition, a different adjacency matrix is defined and we show that the eigenvalues of the matrices corresponding to special self-complementary graphs are relatively easy to find. Some properties of the eigenvalues of these matrices are presented. Finally, results are given concerning the degrees of the points of self-complementary graphs.

CHAPTER I

SURVEY OF THE LITERATURE

Only two articles on self-complementary graphs themselves and two on their enumeration have been listed in the Mathematical Reviews. In 1963, Read [4] presented formulas which enumerate, for a given n, the number of non-isomorphic self-complementary graphs having n points. In 1962 and 1963 articles were published by Sachs [6] and Ringel [5], respectively, in which each presented an algorithm for constructing all self-complementary graphs having a given number of points. In addition, Ringel proved that if G is self-complementary, then the diameter of G is 2 or 3. Sachs' paper, which is more comprehensive than Ringel's, contains several results concerning special self-complementary graphs, including the matrix properties referred to in the introduction. The results of Ringel and Sachs will be acknowledged as they appear below.

In 1969 Palmer [3], using Read's results, discovered a simple asymptotic formula for the number of self-complementary graphs having a given number of points. His results are referred to in Appendix B.

CHAPTER 2

DEFINITIONS AND NOTATION

A graph $G = G(V,E)$ consists of a finite set V together with a subset, E, of $V^{(2)}$, the set of unordered pairs of distinct elements of V. V is called the set of vertices or points of G, E is called the set of edges or lines. A graph G is connected if, for any two points v_i and v_j, there exists in E a sequence of edges

$$(v_{i_1},v_{i_2}),\ (v_{i_2},v_{i_3})\ldots$$

where $v_{i_1} = v_i$ and $v_{i_k} = v_j$ for some k. If $(v_1,v_2) \in E$ then the edge (v_1,v_2) is said to join v_1 and v_2, v_1 and v_2 are adjacent in G, and v_1 and v_2 are endpoints of (v_1,v_2). The number of points to which a point v is adjacent is the degree of v, written deg(v). If G has points 1,2,...,n, the sequence

$$\deg(1),\ \deg(2),\ldots,\ \deg(n)$$

is called the degree sequence of G. If G has degree sequence

$$k,k,\ldots,k$$

G is regular of degree k.

Two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ are isomorphic, written $G_1 \cong G_2$, if there is a one-to-one map σ from V_1

onto V_2 such that

$$(v,w) \in E_1$$

if and only if

$$(\sigma(v), \sigma(w)) \in E_2.$$

σ is called an <u>isomorphism</u> of G_1 onto G_2 and we will write

$$\sigma(G_1) = G_2.$$

If $G_1 \cong G_2$ and both have n points we may think of V_1 and V_2 as being the set $\{1,2,\ldots,n\}$. Then σ is simply a permutation of the symbols 1,2,...,n. For example, if G_1 and G_2 are as in Figure 2.1

Figure 2.1 Two isomorphic graphs.

then the permutation

$$\sigma = (14)(235)$$

is an isomorphism of G_1 onto G_2. (We will assume throughout that all permutations are represented as the product of disjoint cycles.)

If $G_1 = G_2 = G$ then the permutation σ such that

$$\sigma(G) = G$$

is called an <u>automorphism</u> of G. An automorphism of a graph G, then, is a relabeling of its points which preserves adjacency. For example, the permutation

$\sigma = (12)(35)$

is an automorphism of the graph G_2 above. The set of all automorphisms of a graph G, together with the operation of permutation multiplication, is a group, $\Gamma(G)$, called the automorphism group of G.

The complement, $\bar{G}$, of a graph G is the graph with the same point set as G such that two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G. For example, the graphs below are complements of one another.

Figure 2.2 Complementary graphs.

If $G \cong \bar{G}$ then G is self-complementary. There are no self-complementary graphs having two or three points. There is one self-complementary graph having four points, which we shall denote P_3.

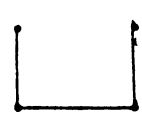


Figure 2.3 P_3, the self-complementary graph having four points.

There are two self-complementary graphs having five points.

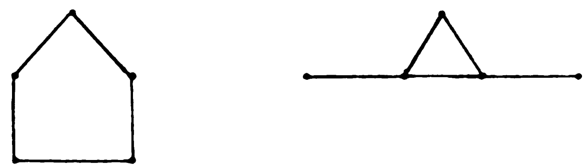


Figure 2.4 The two self-complementary graphs having five points.

The graph having n points and all pairs of points adjacent is called the <u>complete</u> graph with n points and is denoted K_n. It is easy to see that K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ lines.

A <u>subgraph</u> $H(V_1, E_1)$ of a graph $G(V,E)$ is any subset V_1 of V together with any subset E_1 of E such that if $(v,w) \in E_1$ then $v \in V_1$ and $w \in V_1$. A subgraph of G <u>induced</u> by the subset $V_1 \subset V$ is the subgraph with point set V_1 and <u>all</u> lines of G having both endpoints in V_1.

For example, the figure below presents a graph, subgraph and an induced subgraph.

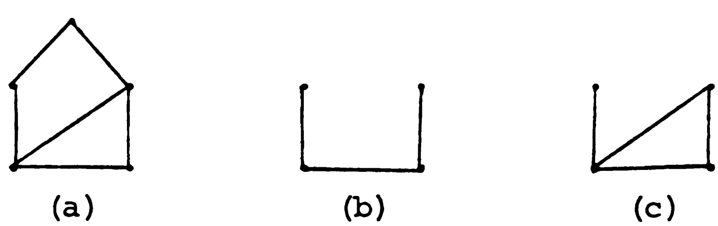


Figure 2.5 a) A graph, b) a subgraph, c) an induced subgraph.

There are several matrices which can be assigned to a graph. The standard 0,1 adjacency matrix has its i-j entry 1 if i and j are adjacent, 0 otherwise. The adjacency matrix which we shall use was found by Seidel [7] to have advantages over the standard 0,1 adjacency matrix. It is particularly well suited for the study of self-complementary graphs. The adjacency matrix, $A(G) = (a_{ij})$, of a graph G having points labeled 1,2,...,n has as its i-j entry

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \text{ is not adjacent to } j \\ -1 & \text{if } i \text{ is adjacent to } j. \end{cases}$$

If G is the graph in Figure 2.6

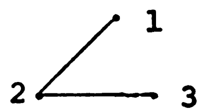


Figure 2.6 A three point graph.

then

$$A(G) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Since $a_{ij} = a_{ji}$, the matrix A(G) is symmetric and has real eigenvalues.

Other definitions and notations will be given as the need arises. For a complete lexicon of graph theory vocabulary, see Harary [2]. A general review of basic matrix theory can be found in Finkbeiner [1].

The end of a proof will be denoted by the symbol '□'.

CHAPTER 3

THE CONSTRUCTION ALGORITHM

Let $|L(G)|$ be the number of lines of graph G. If G has n points then

$$|L(G)| + |L(\overline{G})| = |L(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$$

If G is self-complementary then

$$|L(G)| = |L(\overline{G})|$$

so that

$$|L(G)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}\ .$$

Hence either n or (n-1) must be divisible by 4. This proves

<u>Theorem</u> 3.1. If $G \cong \overline{G}$ and G has n points then

$$n \equiv 0 \text{ or } 1 \pmod 4.$$

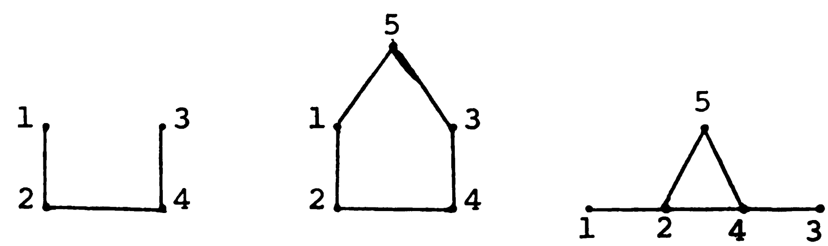
We show below, by developing an algorithm, that if $n \equiv 0$ or 1 (mod 4) then there exist self-complementary graphs having n points. Before discussing this construction algorithm we mention a property of self-complementary graphs which follows immediately from the definition.

<u>Theorem</u> 3.2. If $G \cong \overline{G}$ then G is connected.

Proof: It is easy to show, and well known, that G and $\overline{G}$ cannot both be disconnected. □

Let $G \cong \overline{G}$ have n points labeled from 1 to n. From the definition, we know that there exists a permutation, σ, of the symbols $1,2,\ldots,n$ such that $\sigma(G) = \overline{G}$. We will call σ a complementing permutation or anti-automorphism of G. For example, particular complementing permutations corresponding to the four and five point self-complementary graphs are given in Figure 3.1.

Graph



complementing permutations	(1234)	(1234) (5)	(1234) (5)
	(1432)	(2453) (1)	(1432) (5)

Figure 3.1 Self-complementary graphs and complementing permutations.

Notice that non-isomorphic graphs may have the same complementing permutation and that a particular graph may have more than one complementing permutation. The structure of a complementing permutation is described in the following theorem due, independently, to Ringel [5] and Sachs [6].

Theorem 3.3. Suppose that $\sigma(G) = \overline{G}$ and that G has n points.

1) If $n \equiv 0 \pmod 4$ then each cycle of σ has length divisible by 4.

2) If $n \equiv 1 \pmod 4$ then σ has exactly one cycle of length 1 and all other cycles have length divisible by 4.

Proof: If ρ is any complementing permutation of G then the pair of points (i,j) has the opposite adjacency relation of the pair $(\rho(i),\rho(j))$. Hence ρ can fix no pair of points. Therefore σ can have no transpositions and at most one 1-cycle. Note that any odd power of σ, such as σ^{2k+1}, is also a complementing permutation of G. Thus σ can have no cycles of length 2k+1 or 4k+2 because σ^{2k+1} will have at least 2k+1 1-cycles in the former case and at least 2k+1 transpositions in the latter. Therefore all cycles in σ, with the possible exception of a single 1-cycle, have lengths divisible by 4. $\square$

As an immediate corollary we have

Corollary 3.1. If $n = 4k+1$ then G has a least one point of degree 2k.

Proof: The point fixed by any complementing permutation must be adjacent to exactly half of the remaining points. $\square$

It was observed in the proof of Theorem 3.2 that if $\sigma(G) = \overline{G}$ then $\sigma^{2k+1}(G) = \overline{G}$. It follows that

$$\{G \mid \sigma(G) = \overline{G}\} \subseteq \{G \mid \sigma^{2k+1}(G) = \overline{G}\}.$$

As the labeling of the points of G is immaterial, it is apparent that, if σ and ρ have the same cycle structure, then

$$\{G \mid \sigma(G) = \overline{G}\} = \{G \mid \rho(G) = \overline{G}\}.$$

Now, we can eliminate all odd prime power divisors of the cycle lengths of σ by raising σ to the product of such odd powers. This new permutation σ' will have all its cycle lengths powers of 2. (Of course, none can be of length 2.) Therefore, successive odd powers of σ' will have the same cycle structure as σ'. Consequently, if we can find, for permutations on n symbols,

$$(1) \qquad \bigcup_{\sigma'} \{G \mid \sigma'(G) = \overline{G}\}$$

where the union is taken over all possible cycle structures where the cycles have 2-power lengths, we will have found all self-complementary graphs having n points. We remark that a particular graph may occur in several of the individual sets. For example, four of the graphs in Appendix A have an 8-cycle as well as a two 4-cycle complementing permutation.

The following algorithm, suggested by the algorithm of Ringel [5], provides a method for constructing (but not uniquely) all self-complementary graphs having a given complementing permutation σ with cycles having 2-power lengths. This algorithm is basic to the entire thesis and will be referred to often.

<u>Construction Algorithm</u>. Assume, without loss of generality, that the symbols in σ are numbered consecutively from 1 to n, and that the cycles are of non-decreasing lengths $4k_1, 4k_2, \ldots$ (except for a possible 1-cycle (n) at the end). Here, of course, $k_1, k_2, \ldots$ are powers of 2. We focus our attention on the following symbols: the symbols

$$2, 3, \ldots, 2k_1 + 1$$

of the first cycle, the first $4k_1$ symbols of each other cycle, and the symbol n if (n) is a 1-cycle. This set will be called the <u>range</u> of the symbol 1. (We will construct a graph G with points labeled $1, \ldots, n$ and therefore we identify the symbols in σ with the points of G.) Consider the (unordered) pairs (1,j) where j is in the range of 1. For each pair we arbitrarily decide whether or not 1 and j will be adjacent in G. Once this choice has been made, the same choice must apply to the pairs

$$(\sigma^{2i}(1), \sigma^{2i}(j)) \qquad i = 1, 2, \ldots, 2k_{i_j}$$

where j is in a cycle of length $4k_{i_j}$. If $j = n = 4k + 1$ let $i = 1, 2, \ldots, 2k_1$. The opposite adjacency relation must apply to the pairs

$$(\sigma^{2i-1}(1), \sigma^{2i-1}(j))$$

where i is as above.

This completes the first stage of the algorithm. We next reduce the problem by replacing the permutation σ by the simpler permutation σ^*, on $n-4k_1$ symbols, obtained from σ by deleting its first cycle. Now, apply the procedure outlined above to σ^*. Then delete its first cycle and continue until no cycles remain. The procudure will terminate since σ has a finite number of symbols. This completes the algorithm. $\square$

We must now show that, in fact, a well-defined graph is determined and that it is self-complementary.

<u>Theorem</u> 3.4. As a result of performing the Construction Algorithm:

1) the adjacency relation between points is well-defined,

2) every pair of points is assigned an adjacency relation,

3) the graph thus constructed is self-complementary.

<u>Proof</u>: 1) The pair $(1,j)$ cannot be sent to itself by an odd power of σ because clearly $\sigma^{2i-1}(j) \neq j$, and if $\sigma^{2i-1}(1) = j$ then j is the symbol $2i$ in the first cycle and $\sigma^{2i-1}(j) \equiv 4i-1 \pmod{4k_1} \equiv -1 \pmod 4 \neq 1$. Thus the pair $(1,j)$ can never be assigned simultaneous adjacency and non-adjacency. The same argument applies to the pairs $(\sigma^i(1), \sigma^i(j))$ and carries over for all stages of the Construction Algorithm.

2) It is clear that, once the first point of each

cycle has its adjacencies determined with all the points which follow it, all adjacencies of the graph are determined. For the remainder of this thesis we adopt the notation

$$(2) \qquad \alpha(i,j) = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \text{ is not adjacent to } j \\ -1 \text{ if } i \text{ is adjacent to } j. \end{cases}$$

Thus $\alpha(i,j) = a_{ij}$ is the i-j entry of the adjacency matrix A(G). By the Construction Algorithm,

$$(3) \qquad \alpha(\sigma^k(i), \sigma^k(j)) = (-1)^k \alpha(i,j).$$

If $j \leq 4k_1$, that is if j is in the first cycle, then letting $i = 1$ and $j = 4k_1 + 1 - j$ in (3) we see that

$$\alpha(1,j) = (-1)^{j-1} \alpha(1,4k_1 + 2 - j)$$

Hence, all adjacencies in the first cycle are determined if we know the adjacencies between 1 and

$$2, 3, \ldots, 2k_1 + 1$$

The adjacencies between 1 and the first $4k_1$ points in any other cycle determine all adjacencies between the first cycle and that cycle. This is because the adjacencies between 1 and the first $4k_1$ points form a pattern of period $4k_1$ since k_1 divides k_j for $j > 1$.

We have shown, then, that the adjacencies among points in the first cycle and the adjacencies between the points of the first cycle and all other points of the graph are determined once the adjacencies between 1 and the points in its range are given. Repeating the argument for each stage of the Construction Algorithm proves that all adjacencies of the graph are determined.

3) Now that we have shown that the adjacencies are unique and that all possible adjacencies are determined, it follows immediately that the graph constructed from the Construction Algorithm is self-complementary, because the given permutation σ is an isomorphism. That is, $\sigma(G) = \overline{G}$. □

We have observed that every self-complementary graph having n points possesses a complementing permutation σ with cycles having 2-power lengths. Hence, for a given such permutation, as the various arbitrary choices of initial adjacencies are chosen, the Construction Algorithm will produce, possibly with repetition, all self-complementary graphs having σ as a complementing permutation. That is, it is possible to find

$$\{G \mid \sigma'(G) = \overline{G}\}$$

so that, according to (1), all self-complementary graphs having n points can be determined.

We illustrate the application of the Construction Algorithm on a permutation σ having 9 symbols, to construct a self-complementary graph having 9 points. Let

$$\sigma = (1234)(5678)(9).$$

The range of 1 is

2,3,5,6,7,8,9.

For 5, the range is

6,7,9.

Let $\alpha(i,j)$ be as in (2) and let us choose the values

$$\alpha(1,j) = \begin{cases} -1 & \text{for } j = 2,5,8 \\ 1 & \text{for } j = 3,6,7,9 \end{cases}$$

$$\alpha(5,j) = \begin{cases} -1 & \text{for } j = 6,9 \\ 1 & \text{for } j = 7 \end{cases} .$$

We shall represent this adjacency assignment schematically as in Figure 3.2.

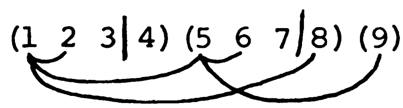


Figure 3.2 An adjacency scheme for a self-complementary graph.

In the scheme, if a line is not drawn from an initial point to a point in its range, non-adjacency is implied. From the Construction Algorithm we infer that

$$\alpha(1,4) = 1$$

$$\alpha(2,j) = \begin{cases} -1 & \text{for } j = 1,4,7,8,9 \\ 1 & \text{for } j = 3,5,6 \end{cases}$$

$$\alpha(3,j) = \begin{cases} -1 & \text{for } j = 4,6,7 \\ 1 & \text{for } j = 1,2,5,8,9 \end{cases}$$

$$\alpha(4,j) = \begin{cases} -1 & \text{for } j = 2,3,5,6,9 \\ 1 & \text{for } j = 1,7,8 \end{cases}$$

$$\alpha(5,8) = 1$$

$$\alpha(6,j) = \begin{cases} -1 & \text{for } j = 5,8 \\ 1 & \text{for } j = 7,9 \end{cases}$$

$$\alpha(7,j) = \begin{cases} -1 & \text{for } j = 8,9 \\ 1 & \text{for } j = 5,6 \end{cases}$$

$$\alpha(8,j) = \begin{cases} -1 & \text{for } j = 6,7 \\ 1 & \text{for } j = 5,9 \end{cases}$$

The graph G, determined by the Construction Algorithm, is given in Figure 3.3.

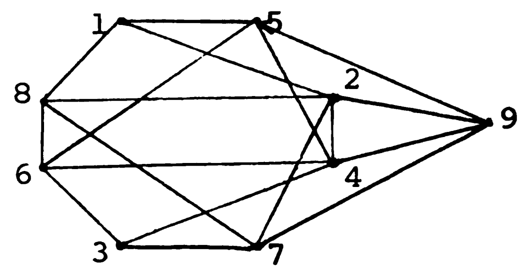


Figure 3.3 The self-complementary graph constructed from the scheme in Figure 3.2.

There are several important corollaries which follow immediately from Theorem 3.3, the Construction Algorithm, and Theorem 3.4. Assume throughout that G has n points and that $\sigma(G) = \overline{G}$.

Corollary 3.2. The set of points in any subset of the cycles of σ will induce a self-complementary subgraph of G.

Proof: σ, restricted to the symbols in the chosen cycles, will, by the Construction Algorithm, produce a self-complementary graph. □

In the example above, the sets

$$\{1,2,3,4,9\} \text{ and } \{5,6,7,8\}$$

induce the subgraphs shown in Figure 3.4.

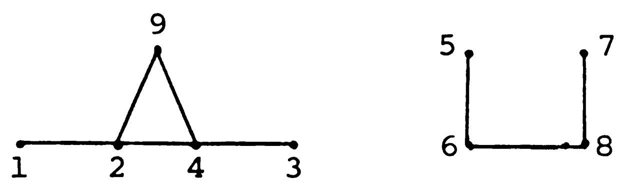


Figure 3.4 Subgraphs induced by cycles of a complementing permutation.

Corollary 3.3. In any cycle of σ with length greater than 1, the points alternate in degree, the sum of the consecutive degrees being n-1.

Proof: If i_1 is adjacent (on the left, say) to i_2 in a cycle of σ then $\sigma(i_1) = i_2$. Hence

$$\deg(i_1) \text{ in } G = \deg(i_2) \text{ in } \bar{G} = n-1-(\deg(i_2) \text{ in } G). \square$$

In the example above, the first cycle has degrees 3,5 and the second has degrees 4,4.

Corollary 3.4. Let G have point set V. If $S \subset V$ contains all points of certain degrees, and T is the set of points of degrees complementary to those in S, then $S \cup T$ induces a self-complementary subgraph of G.

Proof: By Corollary 3.3 the set of points in $S \cup T$ is fixed under σ and must therefore constitute a subset of the cycles of σ. Hence, by Corollary 3.2, the induced subgraph is self-complementary. □

Note that if $S \cap T = \phi$ then S and T have the same number of points, say m, and there are $\frac{m^2}{2}$ lines between the sets (half the possible number).

Corollary 3.5. If H is any subgraph of G induced by the point set $S \subset V$ and we interpret $\sigma(H)$ to be the graph induced by the point set $\{\sigma(i) \mid i \in S\}$, then

$$\sigma(H) \cong \overline{H}.$$

In particular if H is a self-complementary induced subgraph of G then

$$\sigma^i(H) \cong H \qquad i = 1,2,\ldots$$

Proof: i_1 and i_2 in S are adjacent in H if and only if $\sigma(i_1)$ and $\sigma(i_2)$ are non-adjacent in $\sigma(H)$. □

Corollary 3.6: The set of automorphisms and anti-automorphisms of G form a group in which $\Gamma(G)$ is a (normal) subgroup of index 2. It follows that G has as many automorphisms as anti-automorphisms.

Proof: We know that $\Gamma(G)$ is a group. Since $\sigma^2(G) = G$ it follows that $\sigma^{-1}(G) = \overline{G}$ (and that $\sigma(\overline{G}) = G$). Also, if $\rho(G) = \overline{G}$ then $\sigma\rho$ and $\rho\sigma$ belong to $\Gamma(G)$. □

Corollary 3.7. $\Gamma(G)$ is a non-trivial group.

Proof: Since σ has order divisible by 4, σ^2 cannot be the identity permutation. But $\sigma^2 \in \Gamma(G)$. □

The properties of a self-complementary graph having $n = 4k+1$ points can be determined directly from a self-complementary (induced) subgraph of 4k points. In fact, if $\sigma(G) = \overline{G}$ and G has $n = 4k+1$ points then from Corallary 3.1, the point n has degree 2k. Moreover, by the Construction Algorithm, n is joined to alternate points in each cycle of σ. By Corollary 3.2, if we remove the point n and its 2k lines we will obtain a self-complementary graph having 4k points. Conversely, if we start with a self-complementary graph G having 4k points and complementing permutation ρ, we can easily construct all self-complementary graphs having 4k+1 points and having G as a 4k point induced subgraph: simply connect the new point to alternate points in each cycle of ρ. By the Construction Algorithm, this new graph will be self-complementary. Observe in Figure 3.1 that the two five point graphs are obtained from the four point graph with complementing permutation (1234) by joining 5 to 1 and 3, and to 2 and 4, respectively.

Hence we shall restrict our attention in the remainder of the thesis to self-complementary graphs having 4k points.

CHAPTER 4

THE DECOMPOSITION THEOREM

In this chapter we show that every self-complementary graph having 4k points possesses a collection of k disjoint induced subgraphs isomorphic to P_3.

First, we investigate some properties of self-complementary graphs which possess a complementing permutation consisting of a single cycle. Graphs of this type have been considered by Sachs [6]. The present results go beyond those of Sachs, however Sachs observed part 2) of the next theorem.

Theorem 4.1. Let $\sigma(G) = \bar{G}$ where

$$\sigma = (1\ 2\ \dots\ 4k)$$

then

1) each odd-labeled point of G is adjacent to exactly k even-labeled points and each even-labeled point is adjacent to exactly k odd-labeled points

2) G has points of two degrees: for some r such that

$$k \leq r \leq 3k-1$$

there are 2k points of degree r and 2k points of degree 4k-1-r. Moreover, for every such r, at least one self-complementary graph G with $\sigma(G) = \overline{G}$ exists having 2k points of degree r and 2k points of degree 4k-1-r.

Proof: 1) Using the $\alpha(i,j)$ notation of (2) in Chapter 3 we obtain

$$\alpha(1,2i) = (-1)^{4k+1-2i}\, \alpha(\sigma^{4k+1-2i}(1),\ \sigma^{4k+1-2i}(2i))$$

$$= -\ \alpha(4k+2-2i,1)$$

$$= -\ \alpha(1,4k+2-2i).$$

Hence 1 is adjacent to 2i if and only if it is not adjacent to 4k+2-2i. Therefore, 1 is adjacent to exactly half of the 2k even-labeled points. By the Construction Algorithm, this implies that every odd-labeled point is adjacent to exactly k even-labeled points. Similarly we have

$$\alpha(2,2i+1) = (-1)^{4k+1-2i}\, \alpha(\sigma^{4k+1-2i}(2),\ \sigma^{4k+1-2i}(2i+1))$$

$$= -\ \alpha(4k+4-(2i+1),2)$$

$$= -\ \alpha(2,4k+4-(2i+1))$$

so that 2, and hence any even-labeled point, is adjacent to exactly k of the odd-labeled points.

2) As σ is a single cycle, by Corollary 3.3, there are only two degrees of points in any graph G such that $\sigma(G) = \overline{G}$. These degrees alternate among the symbols in σ so that the odd-labeled points have the same degree, say r, and the even-labeled points have the complementary degree

$4k-1-r$. By 1) we have

$$k \leq r$$

and

$$k \leq 4k-1-r.$$

that is,

$$k \leq r \leq 3k-1.$$

Let r be given such that

$$k \leq r \leq 3k-1.$$

We indicate how to construct a graph G such that $\sigma(G) = \overline{G}$ and the odd-labeled points have degree r. Since 1 is adjacent to k even-labeled points, we must join 1 to exactly $r-k$ odd-labeled points. Since

$$\begin{aligned}\alpha(1,2i+1) &= (-1)^{4k-2i}\,\alpha(\sigma^{4k-2i}(1),\ \sigma^{4k-2i}(2i+1)) \\ &= \alpha(4k+1-2i,1) \\ &= \alpha(1,4k+2-(2i+1))\end{aligned}$$

it follows that, if 1 is joined to s of the odd-labeled points

$$3,5,\ldots,2k-1$$

it is also joined to s of the points

$$2k+3,\ 2k+5,\ldots,\ 4k-1.$$

The adjacency relation between 1 and $2k+1 = 4k+2-(2k+1)$ implies no other adjacency. If $r-k$ is even, join 1 to $\frac{r-k}{2}$

of the points

$3,5,\ldots,2k-1,$

to any subset of the points

$2,4,\ldots,2k$

and apply the Construction Algorithm. The graph constructed will have the odd-labeled points of degree r, the even-labeled points of degree 4k-1-r. If r-k is odd, join 1 to the point 2k+1, to $\frac{r-k-1}{2}$ of the points

$3,5,\ldots,2k-1$

and to any subset of the points

$2,4,\ldots,2k$

and apply the Construction Algorithm. The graph constructed will, as above, have the odd-labeled points of degree r, the even-labeled points of degree 4k-1-r. $\square$

Remark: Sachs calls a graph G quasi-regular if $G \cong \overline{G}$ has 4k points, 2k of degree 2k-1 and 2k of degree 2k. By the Construction Algorithm, quasi-regular graphs can be constructed for all k. If we construct a new graph G' from G by adding a new point, of degree 2k, joined to the points of G of degree 2k-1, then G' will have 4k+1 points and will be regular of degree 2k. Conversely, given a regular self-complementary graph, it must have 4k+1 points, for some k, and be regular of degree 2k. If, for a given complementing permutation, we remove the fixed point and its 2k lines, a quasi-regular self-complementary graph having 4k points will remain.

We now present the fundamental lemma which will lead to the Decomposition Theorem, one of the main results of the thesis.

Lemma 4.1. Let $\sigma(G) = \overline{G}$, $\sigma = (1\ 2\ \dots\ 4k)$. G contains a subgraph isomorphic to P_3 induced by the set of points

$$\{1,2,2i-1,2i\}$$

for some i such that

(1) $$2 \le i \le k+1.$$

Proof: Let i be the first positive integer for which

$$\alpha(1,2i) = -\alpha(1,2).$$

Then the subgraph induced by the set

$$\{1,2,2i-1,2i\}$$

is isomorphic to P_3. Because, if

$$\alpha(1,2) = -1 = \alpha(2i-1,2i)$$

then

$$\alpha(1,2i-2) = -1$$

and

$$\alpha(2,2i-1) = 1.$$

Also

$$\alpha(1,2i-1) = -\alpha(2,2i)$$

so the subgraph induced by the set

$$\{1,2,2i-1,2i\}$$

will be one of those in Figure 4.1.

Figure 4.1 The two possible P_3 subgraphs if $\alpha(1,2) = -1$.

If, on the other hand,

$$\alpha(1,2) = 1 = \alpha(2i-1,2i)$$

then the subgraph induced by the set

$$\{1,2,2i-1,2i\}$$

will be one of those in Figure 4.2.

Figure 4.2 The two possible P_3 subgraphs if $\alpha(1,2) = 1$.

To show that such an i, as in (1), exists, observe that if

$$\alpha(1,2i) = \alpha(1,2) \qquad i = 2,3,\ldots,k$$

then, from the proof of 1), Theorem 4.1,

$$\alpha(1,2k+2) = \alpha(1,4k+2 - (2k)) = -\alpha(1,2)$$

so that the subgraph induced by the set

$$\{1,2,2k+1,\ 2k+2\}$$

must be isomorphic to P_3.□

We are now in a position to present the main result of this Chapter.

Theorem 4.2. (The Decomposition Theorem).
If $G \cong \overline{G}$ has 4k points then G possesses a collection of k disjoint induced subgraphs each isomorphic to P_3.

Proof: Let σ be a complementing permutation of G whose cycle lengths are powers of 2. (We know from Chapter 3 that at least one such permutation exists.) By Corollary 3.2 the points in each cycle of σ induce a self-complementary subgraph of G. It thus suffices to show that these subgraphs have the desired decomposition.

Consider a cycle of σ of length 2^m. For convenience of notation, let the cycle be

$$(1\ 2\ 3\ \ldots\ 2^m).$$

Suppose, according to Lemma 4.1, that 2i is the first even-labeled point such that

$$\{1,2,2i+1,2i+2\}$$

induces a P_3 subgraph of G. Consider the set of odd integers

$$(2) \qquad \{\sigma^{2ti}(1)\} = \{a_t\} \qquad t = 0,1,2,\ldots \quad .$$

If

$$i = 2^r \cdot s \;, \quad s \text{ odd}$$

then, since

$$t = 2^{m-1-r}$$

is the smallest integer such that

$$2ti \equiv 0 \pmod{2^m},$$

the set (2) will contain 2^{m-1-r} distinct odd integers. Thus 2^{m-2-r} distinct pairs of odd integers

$$(1, \sigma^{2i}(1)), \ (\sigma^{4i}(1), \sigma^{6i}(1)), \ \ldots$$

are determined.

If $r > 0$, select an odd integer $j < 2^m$ which is <u>not</u> in $\{a_t\}$ and obtain a new set

$$\left\{ \sigma^{2ti}(j) \right\} \qquad t = 0,1,2,\ldots \quad .$$

This set will determine, as before, 2^{m-2-r} new disjoint pairs of odd integers. We continue until <u>every</u> odd integer less than 2^m belongs to some set. 2^r such sets will be constructed, thus determining 2^{m-2} disjoint pairs of odd integers. For each pair

$$(2j+1, \sigma^{2i}(2j+1))$$

thus determined, we associate the following pair of even integers

$$(2j+2, \sigma^{2i}(2j+2))$$

and observe that the subgraph induced by the set

$$\left\{2j+1,\ 2j+2,\ \sigma^{2i}(2j+1),\ \sigma^{2i}(2j+2)\right\}$$

is isomorphic to P_3. This is because the above point set is the image of the set

$$\left\{1,\ 2,\ 2i+1,\ 2i+2\right\}$$

under σ^{2j}, so that Corollary 3.5 applies. Hence a cycle of σ with length 2^m furnishes a set of 2^{m-2} disjoint induced subgraphs isomorphic to P_3. Therefore, assembling such a set for each cycle, we obtain a collection of k disjoint induced subgraphs of G, each isomorphic to P_3. $\square$

For example, let G be determined by the adjacency scheme in Figure 4.3.

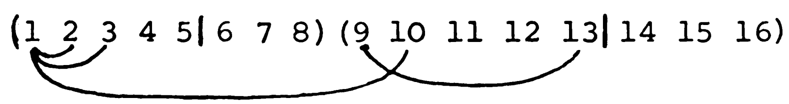


Figure 4.3 An adjacency scheme.

Since, from Theorem 4.2, i = 1 for the first cycle and i=2 for the second, the set, as in (2), for the first cycle is $\{1, 3, 5, 7\}$ and for the second cycle the sets are $\{9, 13\}$ and $\{11, 15\}$. Thus the first cycle produces the subgraphs in Figure 4.4 and the second produces those in Figure 4.5.

Figure 4.4 The two P_3 subgraphs furnished by the first cycle.

Figure 4.5 The two P_3 subgraphs furnished by the second cycle.

CHAPTER 5

ADJACENCY MATRICES

In this chapter it is shown that adjacency matrices corresponding to certain self-complementary graphs have a special form which simplifies the calculation of their eigenvalues.

We first derive some additional properties of the adjacency matrix defined in Chapter 2.

Theorem 5.1. If G has 2k points and λ is an eigenvalue of A(G) then $\frac{\lambda^2-1}{2}$ is an algebraic integer. In particular, we conclude that

1) A(G) is non-singular
2) If λ is an integer then λ is odd
3) If λ^2 is an integer then λ^2 is odd
4) If a graph H has 2k+1 points then the rank of A(H) is at least 2k.

Proof: Consider the matrix $A^2(G)$. Each diagonal entry is 2k-1. The off-diagonal entries, being the inner-products of two rows of A(G), consist of the sum of 2k-2 (+1)'s and (-1)'s and hence are even integers. Therefore the matrix

$$M = \frac{A^2(G)-I}{2}$$

has integer entries and thus has algebraic integer eigenvalues. That is, if λ is an eigenvalue of A(G) then

$$\frac{\lambda^2-1}{2}$$

is an algebraic integer. Since $-\frac{1}{2}$ is not an algebraic integer, $\lambda^2 \neq 0$. Therefore A(G) is non-singular. Statements 2) and 3) follow immediately. If H has 2k+1 points then deletion of any row and its corresponding column from A(H) leaves an adjacency matrix for a graph having 2k points which is, by 1), non-singular. Statement 4) then follows. $\square$

Let G with points 1,2,...,n have adjacency matrix A(G). If $G \cong G_1$ and the points of G_1 are labeled 1,2,...,n then we know that there is a permutation σ of the symbols 1,2,...,n such that

$$\sigma(G) = G_1.$$

Corresponding to σ, there is an nxn permutation matrix $P = (p_{ij})$ defined by

$$p_{ij} = \begin{cases} 1 \text{ if } \sigma(i) = j \\ 0 \text{ otherwise.} \end{cases}$$

The equation

$$\sigma(G) = G_1$$

becomes, in matrix form,

$$P^T A(G) P = A(G_1).$$

Conversely, if $Q = (q_{ij})$ is any nxn permutation matrix, then $Q^T A(G) Q$ will be the adjacency matrix of a graph, G_1, isomorphic to G. In fact, if ρ is a permutation on 1,2,...,n with

$$\rho(i) = j \text{ if and only if } q_{ij} = 1$$

then

$$\rho(G) = G_1.$$

This proves

Theorem 5.2. If G and G_1 are graphs having n points 1,2,...,n then $G \cong G_1$ if and only if there exists an nxn permutation matrix P such that $P^T A(G) P = A(G_1)$.

A property of the eigenvalues of an adjacency matrix is given in the next theorem.

Theorem 5.3. If the graph G having n points has adjacency matrix A(G) with eigenvalues

$$a_1, \ldots, a_n$$

then

$$\sum_{i=1}^{n} a_i^2 = n(n-1) .$$

<u>Proof</u>: The diagonal entries of $A^2(G)$ are all n-1. Therefore

$$\text{trace } (A^2(G)) = n(n-1).$$

If a_i is an eigenvalue of A(G) then a_i^2 is an eigenvalue of $A^2(G)$. Since the trace of a matrix is the sum of its eigenvalues, the result follows. $\square$

Having observed these preliminary properties, which hold for any graphs, we now consider matrices and eigenvalues of self-complementary graphs. Since, for any graph G,

$$A(\overline{G}) = -A(G),$$

if $G \cong \overline{G}$, there exists a permutation matrix P such that

$$P^{-1}A(G)P = P^TA(G)P = -A(G).$$

Since A(G) is similar to -A(G), both have the same eigenvalues. But a_i is an eigenvalue of A(G) if and only if $-a_i$ is an eigenvalue of -A(G). We conclude that the eigenvalues of A(G) occur in opposable pairs (except for the eigenvalue $a_n = 0$ if n is odd). If G has 4k points we can list the eigenvalues as

$$\pm a_1, \pm a_2, \ldots , \pm a_{2k}.$$

If G has 4k+1 points, 0 must be an eigenvalue of multiplicity one by 4) of Theorem 5.1, and we can list the eigenvalues as

$$\pm a_1, \pm a_2, \ldots , \pm a_{2k}, 0.$$

From Theorem 5.3 we see that, for a self-complementary graph G having n points and adjacency matrix A(G) with non-zero eigenvalues

$$\pm a_1, \ldots, \pm a_{2k}, \quad k = [\tfrac{n}{4}]$$

we have

$$\sum_{i=1}^{2k} a_i^2 = \binom{n}{2} .$$

We summarize these results as

<u>Theorem</u> 5.4. Let $G \cong \overline{G}$ have n points and adjacency matrix A(G). Then

1) The non-zero eigenvalues of A(G) occur in opposable pairs. If

$$\pm a_1, \ldots, \pm a_{2k}, \quad k = [\tfrac{n}{4}]$$

are the non-zero eigenvalues of A(G) then

$$\sum_{i=1}^{2k} a_i^2 = \binom{n}{2} .$$

2) If G has 4k+1 points then A(G) is singular with rank 4k.

We now consider self-complementary graphs with the property that they possess a complementing permutation having cycles of equal length. If $G \cong \overline{G}$ has 4k points and possesses a complementing permutation σ whose cycles have equal length, then A(G) can be transformed into the direct sum of two matrices of dimension 2k, thus simplifying the determination of its eigenvalues.

We emphasize that any relabeling of the points of any graph G will give rise to an isomorphic graph G_1 with adjacency matrix similar to A(G). The same effect is obtained by relabeling rows and columns of the adjacency matrix rather than relabeling the points. If G has a complementing permutation consisting of the product of cycles of equal length, we will use this latter option to construct a matrix similar to A(G) whose eigenvalues are more readily determined.

Before proving it, we illustrate this process with an example. Let

$$\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$$

and let G be determined by the adjacency scheme in Figure 5.1.

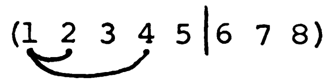


Figure 5.1 An adjacency scheme.

Then G is the graph in Figure 5.2.

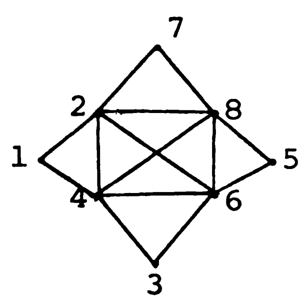


Figure 5.2 The graph constructed from the scheme.

It follows that

$$A(G) = \begin{bmatrix} 0 & - & + & - & + & + & + & + \\ - & 0 & + & - & + & - & - & - \\ + & + & 0 & - & + & - & + & + \\ - & - & - & 0 & + & - & + & - \\ + & + & + & + & 0 & - & + & - \\ + & - & - & - & - & 0 & + & - \\ + & - & + & + & + & + & 0 & - \\ + & - & + & - & - & - & - & 0 \end{bmatrix}$$

(where - means -1 and + means 1).

Note that $\sigma^2 = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8)$ is the automorphism rotating G through 90°. If we list the rows and columns of A(G) in the order

1, 5, 2, 6, 3, 7, 4, 8

we obtain a matrix A_1 which is similar to A.

$$A_1 = \begin{bmatrix} 0 & + & - & + & + & + & - & + \\ + & 0 & + & - & + & + & + & - \\ - & + & 0 & - & + & - & - & - \\ + & - & - & 0 & - & + & - & - \\ + & + & + & - & 0 & + & - & + \\ + & + & - & + & + & 0 & + & - \\ - & + & - & - & - & + & 0 & - \\ + & - & - & - & + & - & - & 0 \end{bmatrix}$$

This new A_1 can be described as a symmetric 4 x 4 matrix whose entries are 2 x 2 matrices of special form. On the diagonal we have

$$\pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and elsewhere we have

$$\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Let

$$s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$e = s^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

then

$$A_1 = \begin{bmatrix} s & -e+s & e+s & -e+s \\ -e+s & -s & e-s & -e-s \\ e+s & e-s & s & -e+s \\ -e+s & -e-s & -e+s & -s \end{bmatrix}$$

If we replace s by its eigenvalue 1 and $e = s^2$ by $1^2 = 1$ we obtain the 4 x 4 matrix

$$B_1 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -2 \\ 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{bmatrix}$$

If we replace s by its other eigenvalue -1 and $e = s^2$ by $(-1)^2 = 1$ we obtain the matrix

$$B_2 = \begin{bmatrix} -1 & -2 & 0 & -2 \\ -2 & 1 & 2 & 0 \\ 0 & 2 & -1 & -2 \\ -2 & 0 & -2 & 1 \end{bmatrix} .$$

Now, B_1 has eigenvalues ± 1, ± 3, B_2 has eigenvalues ± 3, ± 3, and the eigenvalues of A(G) are ± 1, ± 3, ± 3, ± 3, precisely those of B_1 and B_2. In fact, if we transform A_1 by the product RP where

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we obtain

$$(RP)^{-1} A_1 (RP) = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix}$$

where O is the 4 x 4 zero matrix.

This example illustrates the result of the main theorem of this Chapter.

<u>Theorem</u> 5.5. If a self-complementary graph G, having 4k points, possesses a complementing permutation consisting of the product of cycles of equal length then:

1) $A(G)$ is similar to a symmetric $2k \times 2k$ matrix whose entries are polynomials in the matrix $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The diagonal entries are s or $-s$ and all the other entries are $\pm e \pm s$.

2) If we form B_1 by letting $e = s = 1$ and B_2 by letting $e = -s = 1$ then the eigenvalues of $A(G)$ are those of B_1 and B_2.

Proof: 1) Let $\sigma(G) = \overline{G}$ where

$$\sigma = (1\ 2\ \dots\ 4k_1)(4k_1+1\ \dots\ 8k_1)\dots(\dots 4k).$$

Form A_1 from $A(G)$ by labeling its rows and columns in the order

$$1,\ 2k_1+1,\ 2,\ 2k_1+2,\dots,\ 2k_1,\ 4k_1$$

$$4k_1+1,\ 6k_1+1,\ 4k_1+2,\ 6k_1+2,\dots,\ 6k_1, 8k_1$$

$$\dots$$

$$4(k-k_1)+1,\ 4(k-k_1)+2k_1+1,\dots,\ 4k-2k_1,\ 4k\ .$$

For the entry at row i and column j put $\alpha(i,j)$. We focus our attention on the consecutive pairs

$$(1,\ 2k_1+1), (2,\ 2k_1+2),\dots,\ (4k-2k_1,\ 4k).$$

Consider the 2x2 submatrix formed by 2 of these pairs.

$$\begin{matrix} \vdots \\ \binom{t}{2k_1+t} \\ \vdots \end{matrix} \left[\begin{matrix} \dots & (m, 2k_1+m)\dots \\ & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ & \end{matrix} \right]$$

There are two cases.

<u>Case</u> 1) If $t = m$ then we have a block on the diagonal of A_1 and so

$$a = d = 0.$$

Now,

$$b = \alpha(t,\ 2k_1+m)$$

and

$$c = \alpha(2k_1+t,m) = \alpha(\sigma^{2k_1}(t),\ \sigma^{2k_1}(2k_1+m))$$

$$= (-1)^{2k_1}\ \alpha(t,\ 2k_1+m) = \alpha(t,\ 2k_1+m)$$

$$= b.$$

Therefore

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm\, s\ .$$

Case 2) If $t \neq m$ then

$$a = \alpha(t,m)$$

and

$$b = \alpha(t,\ 2k_1+m).$$

Thus, since all cycles have the same length $4k_1$, we have

$$d = \alpha(2k_1+t,\ 2k_1+m) = \alpha(\sigma^{2k_1}(t),\ \sigma^{2k_1}(m))$$

$$= (1-1)^{2k_1}\ \alpha(t,m) = \alpha(t,m) = a$$

and

$$c = \alpha(2k_1 + t, m) = \alpha(\sigma^{2k_1}(t), \sigma^{2k_1}(2k_1+m)$$

$$= (-1)^{2k_1} \alpha(t, 2k_1+m) = \alpha(t, 2k_1+m) = b.$$

Therefore

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm e \pm s.$$

This completes the proof of 1).

2) The matrices e and s are simultaneously diagonalized by the matrix

$$r = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

In fact,

$$r^{-1}sr = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad r^{-1}(-s)r = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1) $$r^{-1}(e+s)r = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} , \quad r^{-1}(-e-s)r = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$r^{-1}(e-s)r = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} , \quad r^{-1}(-e+s)r = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} .$$

Let

$$R = \begin{bmatrix} r & & & \\ & r & & \\ & & \ddots & \\ & & & r \end{bmatrix}$$

with 2k diagonal blocks. Then

$$A_2 = R^{-1} A_1 R$$

will be composed of 2x2 blocks of the forms listed in (1). Those on the diagonal of A_2 are of the first two types whereas the off-diagonal blocks are of one or more of the other four types. It can be seen that the i-j entry of A_2 is zero whenever $i+j \equiv 1 \pmod{2}$. Therefore, if we transform A_2 by the 4k x 4k permutation matrix P whose columns are respectively, columns

$$1, 3, 5, \ldots, 4k-1, 2, 4, 6, \ldots, 4k$$

of the unit matrix of dimension 4k, we will obtain

$$P^{-1} A_2 P = \begin{bmatrix} C & O \\ O & D \end{bmatrix}$$

The entries of C are the "upper-left" entries of the 2x2 blocks in A_2 and the entries of D are the "lower-right" entries of those 2x2 blocks. Now, observe that, in the 2x2 matrices in (1), if we let $e = s = 1$ we obtain the upper-left entries and if we let $e = -s = 1$ we obtain the lower-right. Therefore $C = B_1$ and $D = B_2$. Finally, since A(G) is similar to

$$\begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix},$$

the eigenvalues of A(G) are those of B_1 along with those of B_2.□

Recalling that A(G) has opposable eigenvalues, we remark that it can be shown that B_1 and B_2 themselves have opposable eigenvalues.

In Theorem 5.1 we observed some properties of the eigenvalues of adjacency matrices of arbitrary graphs. For self-complementary graphs of the type considered in Theorem 5.5, more specific properties exist.

<u>Theorem</u> 5.6. Let G be as in Theorem 5.5. If λ is an eigenvalue of A(G) then

$$\frac{\lambda^2 - 1}{4}$$

is an algebraic integer. In particular, if λ^2 is an integer then $\lambda^2 \equiv 1 \pmod 4$

<u>Proof</u>: The matrices B_1 and B_2 in Theorem 5.5 have 1 and -1 on the diagonal and 0, 2 or -2 off the diagonal. Hence

$$B_1 = I + 2M_1$$

and

$$B_2 = I + 2M_2$$

where M_1 and M_2 have integral entries. Therefore the eigenvalues of M_1 and M_2 are algebraic integers. Now,

$$B_i^2 = I + 4M_i + 4M_i^2 \quad , \quad i = 1,2$$

and so

$$\frac{B_i^2 - I}{4} = M_i + M_i^2 \quad , \; i = 1,2.$$

If λ is an eigenvalue of A(G) then, by Theorem 5.5, λ is and eigenvalue of B_1 or B_2 so that

$$\frac{\lambda^2-1}{4} = x + x^2$$

where x is an eigenvalue of M_1 or M_2. In any case, x is an algebraic integer and so is $x^2 + x$. Clearly if λ^2 is an integer, then

$$\lambda^2 \equiv 1 \pmod 4$$

since the only rational algebraic integers are integers. □

It is suspected that the result of Theorem 5.6 holds for any self-complementary graph having $n \equiv 0 \pmod 4$ points.

The method of Theorem 5.5 and the result of Theorem 5.6 are applicable to all ten of the self-complementary graphs having eight points, as each possesses a complementing permutation consisting of two 4-cycles. In Appendix A we list the ten graphs and some of their complementing permutations, together with the eigenvalues of their adjacency matrices.

In concluding this chapter we observe that not all self-complementary graphs satisfy the conditions of Theorem 5.5. The following example presents a self-complementary graph having twelve points which possesses no complementing permutation consisting of three 4-cycles.

Let

$$\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)$$

and consider the adjacency scheme in Figure 5.3.

(1 2 3|4) (5 6 7 8 9|10 11 12)

Figure 5.3 An adjacency scheme.

The graph G constructed from this scheme is given in Figure 5.4, where points 2 and 4 are also connected to all of the points

5, 7, 9, 11

but the lines have not been drawn.

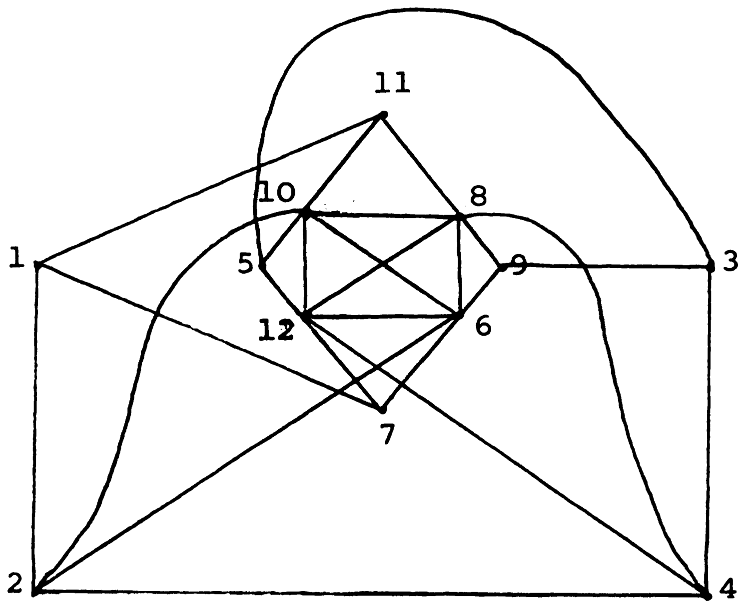


Figure 5.4 The graph constructed from the scheme.

If G has a complementing permutation σ' consisting of three 4-cycles then

$$\sigma' = \sigma\alpha$$

where α is an automorphism of G. However, the only automorphisms of G are

$$I,\ (5\ 9)(7\ 11)(6\ 10)(8\ 12),$$

$$(1\ 3)(2\ 4)(5\ 11\ 9\ 7)(6\ 12\ 10\ 8),$$

$$(1\ 3)(2\ 4)(5\ 7\ 9\ 11)(6\ 8\ 10\ 12)$$

and, if we multiply σ by any of these, the product will consist of a 4-cycle and an 8-cycle. Therefore G has no complementing permutation consisting of three 4-cycles.

CHAPTER 6

TYPE-SEQUENCES

Let G be a graph having n points and let p be a point of degree d. In $\overline{G}$, p will have degree n-d-1. For this reason it is seen that for each point of degree d in a self-complementary graph having 4k points there must correspond a point of complementary degree 4k-d-1. Moreover, if $G \cong \overline{G}$ has 4k points and $\sigma(G) = \overline{G}$ then, by Theorem 3.3 and Corollary 3.3, we see that there are an even number of points in G of any given degree. These two observations lead to a simple way of denoting the degree sequence of a self-complementary graph having 4k points. Since points of complementary degree exist in equal number, we need only specify the degrees less than 2k. Since each degree has even multiplicity, the degree sequence is described by a *type-sequence* of k integers less than 2k, each representing 2 points of that degree and 2 of the complementary degree. For example, a self-complementary graph having 8 points and degree sequence

1, 1, 3, 3, 4, 4, 6, 6

has type-sequence

1, 3

one with degree sequence

3, 3, 3, 3, 4, 4, 4, 4

has type-sequence

3, 3

etc. The type-sequences for all self-complementary graphs having eight points are given in Appendix A.

Will any set of k positive integers less than 2k be the type-sequence for some self-complementary graph? Not necessarily. Certain conditions must be satisfied by the set of integers.

Theorem 6.1. In the type-sequence of a self-complementary graph having 4k points, the integer i can occur at most i times.

Proof: We must show that if $G \cong \overline{G}$ has 4k points, then G has at most 2i points of degree i. (Of course, $i < 2k$.) Assume there are t points of degree i. By Corollary 3.4 and the note following it, there must be exactly $t^2/2$ lines connecting the t points of degree i to the t points of degree 4k-i-1. Since this number $t^2/2$ cannot exceed the number $t \cdot i$ of point-line incidences determined by the t points of degree i, we have

$$\frac{t^2}{2} \leq t \cdot i$$

or

$$t \leq 2i. \square$$

From Theorem 4.1, if $G \cong \overline{G}$ having 4k points possesses a single cycle complementing permutation then the points of G have exactly two degrees, r and 4k-1-r (assume $r < 2k$), with

$$k \leq r.$$

The preceeding theorem provides a more general result, applicable regardless of the cycle structure of a complementing permutation.

Corollary 6.1. If $G \cong \overline{G}$ has 4k points with 2k points of degree $r < 2k$, then $r \geq k$.

Proof: The type-sequence of G has k numbers, all equal to r. By the theorem, r can occur at most r times. Therefore we must have

$$r \geq k. \square$$

There is an interesting inequality which must be satisfied by the sum of the numbers in a type-sequence.

Theorem 6.2. Let

$$d_1, d_2, \ldots, d_k$$

be the type-sequence of $G \cong \overline{G}$ having 4k points. Then

$$k^2 \leq \sum_{i=1}^{k} d_i \leq 2k^2 - k.$$

Proof: Since $d_i \leq 2k - 1$ for all i, we have

$$\sum_{i=1}^{k} d_i \leq k(2k - 1) = 2k^2 - k.$$

Consider the subgraph G_1 induced by the 2k points of degree less than 2k. The sum of the degrees of these 2k points is

$$2\sum_{i=1}^{k} d_i .$$

Since $2k^2$ lines join these points to the points of degree greater than 2k - 1 we have

$$0 \leq \left|L(G_1)\right| = \frac{2\sum_{i=1}^{k} d_i - 2k^2}{2} = \sum_{i=1}^{k} d_i - k^2.$$

Therefore

$$k^2 \leq \sum_{i=1}^{k} d_i \; . \square$$

The reader may verify these inequalities for k = 2 in Appendix A.

The inequalities in Theorems 6.1 and 6.2 are best possible. To see this, let $\sigma(G) = \overline{G}$ where σ is a single cycle. If the 2 degrees of the points of G are k and 3k - 1 then the type-sequence has k numbers, all equal to k. So $d_i = k$ for all i and

$$k^2 = \sum_{i=1}^{k} d_i .$$

On the other hand, if G is quasi-regular then $d_i = 2k - 1$ for all i and

$$\sum_{i=1}^{k} d_i = 2k^2 - k.$$

Suppose $G \cong \overline{G}$ has 4k points and r occurs r times in the type-sequence of G. Then G has 2r points of degree r. Consider the self-complementary subgraph of G induced by these 2r points and the 2r points of complementary degree 4k-r-1. Since $2r^2$ lines join these sets, all lines from the points of degree r must go to points of degree 4k-r-1. That is, the 2r points of degree r are joined to none of the other 4k-4r points of the graph. Therefore, the 2r points of degree 4k-r-1 are joined to <u>all</u> the other 4k-4r points of the graph. The degrees of these 4k-4r points must then be at least 2r+1. Since $r \leq k$ (G has at least 4r points) we have

$$4k-r-1 \geq 3r-1$$

and if $r \geq 2$ we have

$$3r-1 > 2r+1.$$

That is,

$$2r+1 < 4k-r-1$$

if $2 \leq r$. Thus we have proven

<u>Theorem</u> 6.3. Except for the case of P_3 ($r = k = 1$), if $G \cong \overline{G}$ with 4k points has 2r points of degree r ($r < 2k$) then

1) r is the minimum degree in G

and

2) the next higher degree in G is at least 2r+1.

For example, if $G \cong \overline{G}$ has 100 points, G has at most 36 points of degree 18 and if G has 36 points of degree 18, the next higher degree of any point in G is at least 37.

We have seen that, if $G \cong \overline{G}$ has 4k points, then, for a given $r < 2k$, G has an even number of points of degree r, the maximum number being 2r. The following question then arises. Is it possible, given a sufficient number of points, to find a $G \cong \overline{G}$ having exactly 2t points of degree r where $0 \leq t \leq r$? The answer is affirmative, as will be shown in the following theorem. First, observe that if G has points of degree r where r is the smaller of the complementary degrees, then the graph must have at least $2r + 2$ points if r is odd and at least $2r + 4$ points if r is even. Also, if there are 2t points of degree r, then G must have at least 4t points. We may now state our theorem.

<u>Theorem</u> 6.4. Let t and r be given positive integers with $t \leq r$. Given any

$$k \geq \max\ (t,\ 1+[r/2])$$

there exists a self-complementary graph G having 4k points with exactly 2t points of degree r.

Proof: The proof proceeds by induction on k. We first must prove the statement for $k = \max(t, 1+[r/2])$. There are two cases:

case 1) If $k = t$ then

$$t \leq r \leq 2t - 1$$

and by Theorem 4.1 we know there exists a $G \cong \overline{G}$ having 4t points, 2t of degree r and 2t of degree 4t-r-1.

case 2) If $k = 1+[r/2]$ then let σ be a permutation consisting of two cycles, one of length 4t and one of length $4 + 4[r/2] - 4t$. Construct a graph from the cycle of length 4t having the two degrees

$$x = r + 2t - 2 - 2[r/2]$$

and

$$4t - x - 1 = 2t + 2 + 2[r/2] - r - 1.$$

Note that if r is odd then $x = 2t - 1$ and if r is even then $x = 2t - 2$. Since we must have $x > 0$, we are excluding, for the time being, the case $t = 1$ when r is even. For $x > 0$, construct a graph from the $(4 + 4[r/2] - 4t)$-cycle with the two degrees

$$d_1 = 1 + [r/2] - t$$

and

$$d_2 = 3 + 3[r/2] - 3t - 1,$$

Note that $d_1 \neq r$, since $d_1 = r$ means that $r = 1 - 2t$ if r is odd, or $r = 2 - 2t$ if r is even, both of which are impossible. Now, we construct the desired graph, from the Construction

Algorithm, by joining all points of degree x in the first cycle with the $2 + 2[r/2] - 2t$ points of degree d_2 in the second cycle. The points of degree $4t - x - 1$ will therefore also be joined to these same points. The graph thus constructed will have the 4 distinct degrees

$$r,\ 4 + 4[r/2] - r - 1 > r,\ d_1 \neq r$$
$$\text{and } d_2 + 4t = 3 + 3[r/2] + t - 1.$$

There will then be at least 2t points of degree r. It only remains to show that $d_2 + 4t \neq r$, in order to prove that there are exactly 2t points of degree r. But, if $d_2 + 4t = r$ then $r = -2t - 1 < 0$ if r is odd, or $r = -4 - 2t < 0$ if r is even, both of which are absurd. We must finally consider the case $t = 1$ when r is even. Here, if $r > 2$, we consider a three cycle permutation consisting of two 4-cycles and one $(2r - 4)$-cycle. We construct two four point graphs from the two 4-cycles, and we construct a graph from the $(2r - 4)$-cycle having the two degrees $r/2 - 1$ and $3r/2 - 4$. Join the points of degree 2 in the first 4-cycle to the $r - 2$ points of degree $3r/2 - 4$ in the $(2r - 4)$-cycle and to none of the points in the second 4-cycle. Join the points of degree 2 in the second 4-cycle to all points of the $(2r - 4)$-cycle. Completing the construction will yield a self-complementary graph with six degrees of points, all distinct if $r \neq 4$. There will be 2 of degree r and 2 of degree $r + 3$ in the first 4-cycle, 2 of degree 3 and 2 of degree 2r from the second 4-cycle and there will be $r - 2$ of degree $r/2 + 1$ and $r - 2$ of degree $3r/2 + 2$ from the $(2r - 4)$-cycle. Since $r > 2$, it is clear that we have exactly $2 = 2t$ points of degree r. Lastly, if $r = 2$ and $t = 1$, a self-complementary graph having 8 points with type-sequence 2,3 is a graph with exactly two points of degree 2.

This anchors the induction. To complete the proof we now show that if there is a graph on $4n$ points with exactly $2t$ points of degree r then we can construct a graph having $4n + 4$ points with exactly $2t$ points of degree r. Note that we are assuming $r < 2n$. To construct the desired graph we simply add a 4-cycle to the complementing permutation of the graph on $4n$ points and join the points of the 4-cycle to all $2n$ points of degree $> 2n$ of the graph having $4n$ points. This will produce a self-complementary graph having $4n + 4$ points with exactly $2t$ points of degree r, since the points of the 4-cycle will have degrees $2n + 1$ and $2n + 2$, both $> r$, and those of the initial complementing permutation have their larger degrees (those $\geq 2n$) increased by 2 while the smaller degrees (those $< 2n$), including the $2t$ of degree r, remain unchanged. Thus the induction is completed. □

We conclude by remarking that the necessary conditions on the type-sequence of a self-complementary graph given in Theorems 6.1 and 6.2 are far from sufficient. For example, it can be shown that <u>no</u> type-sequence can contain the integers

$$2,\ 3,\ 4,\ 5$$

as a subset. The problem of determining necessary and sufficient conditions for a collection of numbers to be a type-sequence seems to be quite a difficult one.

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BIBLIOGRAPHY

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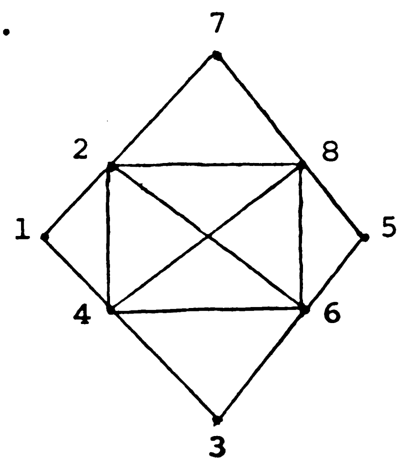
APPENDICES

APPENDIX A

EIGHT POINT SELF-COMPLEMENTARY GRAPHS

In Figure A.1 we list the ten self-complementary graphs having eight points as well as the eigenvalues of their adjacency matrices, their type-sequences, and some complementing permutations. Only graphs I, II, III and IV possess 8-cycle complementing permutations. Note that if λ is an eigenvalue and λ^2 is an integer then $\lambda^2 \equiv 1 \pmod 4$ as proven in Theorem 5.6.

I.



Complementing permutations:

(1 2 3 4 5 6 7 8)

(1 6 5 2)(3 4 7 8)

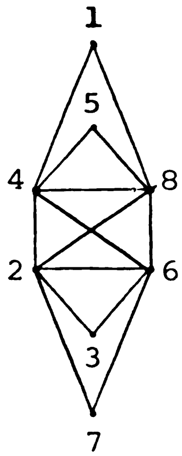
Type-sequence: 2,2

Eigenvalues: $\pm$ 1, $\pm$ 3, $\pm$ 3, $\pm$ 3

Figure A.1 Eight point self-complementary graphs.

Figure A.1 (cont'd.)

II.



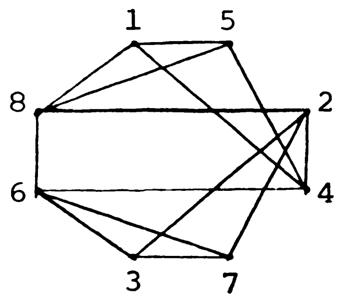
Complementing permutations:

(1 2 3 4 5 6 7 8)
(1 2 3 4) (5 6 7 8)

Type-sequence: 2,2

Eigenvalues: ± 1, ± 1, ± 3, $\pm\sqrt{17}$

III.



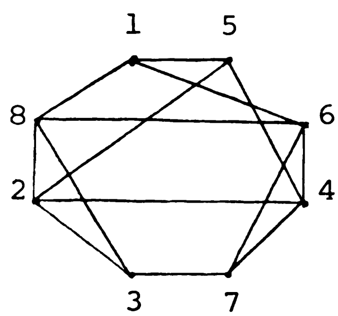
Complementing permutations:

(1 2 3 4 5 6 7 8)
(1 2 3 8) (4 5 6 7)

Type-sequence: 3,3

Eigenvalues: ± 1, ± 1, ± 1, ± 5

IV.



Complementing permutations:

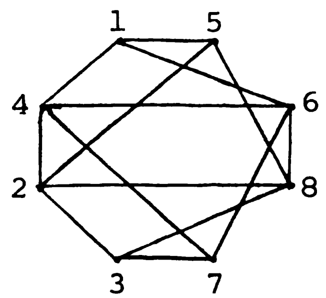
(1 2 3 4 5 6 7 8)
(1 4 5 8) (2 7 6 3)

Type-sequence: 3,3

Eigenvalues: ± 1, ± 3, ± 3, ± 3

Figure A.1 (cont'd.)

V.



Complementing permutations:

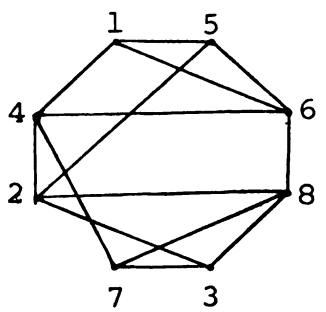
(1 2 3 4) (5 6 7 8)

(1 2 5 6) (3 4 7 8)

Type-sequence: 3,3

Eigenvalues: $\pm 1, \pm 1, \pm 3, \pm\sqrt{17}$

VI.



Complementing permutations:

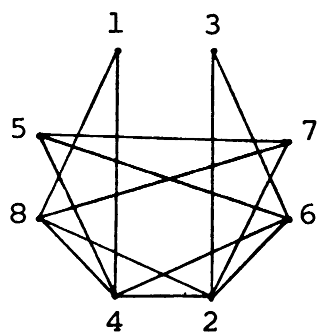
(1 2 3 4) (5 6 7 8)

(1 4 3 2) (5 8 7 6)

Type-sequence: 3,3

Eigenvalues: $\pm 1, \pm 1, \pm\sqrt{13 \pm 4\sqrt{5}}$

VII.



Complementing permutations:

(1 2 3 4) (5 6 7 8)

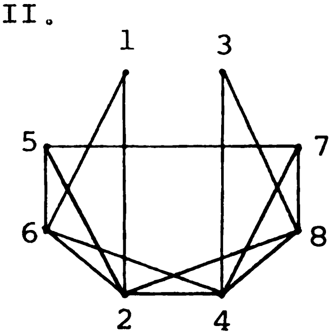
(1 4 3 2) (5 8 7 6)

Type-sequence: 2,3

Eigenvalues: $\pm\sqrt{5}, \pm\sqrt{5}, \pm\sqrt{5}, \pm\sqrt{13}$

Figure A.1 (cont'd.)

VIII.



Complementing permutations:

(1 2 3 4)(5 6 7 8)

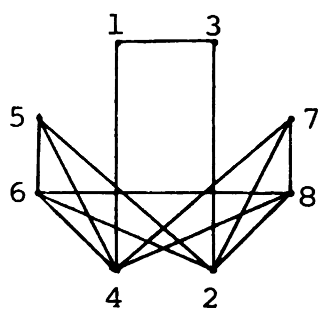
(1 4 3 2)(5 8 7 6)

Type-sequence: 2,3

Eigenvalues: $\pm\sqrt{5}$, $\pm\sqrt{5}$

$\pm$ $(2 + \sqrt{5})$, $\pm$ $(2 - \sqrt{5})$

IX.



Complementing permutations:

(1 2 3 4)(5 6 7 8)

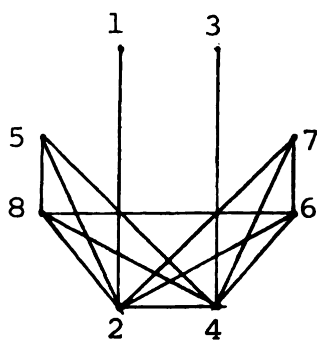
(1 4 3 2)(5 8 7 6)

Type-sequence: 2,3

Eigenvalues: $\pm\sqrt{5}$, $\pm\sqrt{5}$

$\pm$ $(2 + \sqrt{5})$, $\pm$ $(2 - \sqrt{5})$

X.



Complementing permutations:

(1 2 3 4)(5 6 7 8)

(1 2 3 4)(5 8 7 6)

Type-sequence: 1,3

Eigenvalues: $\pm\sqrt{5}$, $\pm\sqrt{5}$

$\pm$ $(2 + \sqrt{5})$, $\pm$ $(2 - \sqrt{5})$

APPENDIX B

COUNTS OF SELF-COMPLEMENTARY GRAPHS

We have seen that self-complementary graphs exist having n points for all positive $n \equiv 0$ or $1 \pmod 4$. The exact number, $\bar{g}_n$, of self-complementary graphs having n points was given in 1963 by Read [4].

If $n = 4N$ then

$$\bar{g}_n = \sum_{(k)} \frac{2^R}{\prod_{s=1}^{N} s^{k_s} \cdot k_s!}$$

where $R = 2\sum_{s=1}^{N} k_s(sk_s-1) + 4\sum_{1\leq\alpha<\beta\leq N} k_\alpha k_\beta \cdot d(\alpha,\beta)$,

the summation is taken over all partitions $(k) = (k_1, k_2, \ldots, k_N)$ of k where $\sum_{s=1}^{N} sk_s = N$, and $d(\alpha,\beta)$ is the greatest common divisor of α and β.

If $n = 4N+1$ the formula is the same except that

$$R = \sum_{s=1}^{N} k_s(2sk_s-1) + 4\sum_{1\leq\alpha<\beta\leq N} k_\alpha k_\beta \cdot d(\alpha,\beta) \quad .$$

In 1969 Palmer [3] found a simple asymptotic formula for approximating the number $\bar{g}_n$. It is the greatest integer in the term in Read's formula corresponding to the partition $(k) = (N,0,0,\ldots,0)$ and is denoted $\bar{g}_{n,0}$. For $n = 4N$ we have

$$\bar{g}_{n,0} = \left[\frac{2^{2N^2-2N}}{N!}\right]$$

and for $n = 4N+1$ we have

$$\bar{g}_{n,0} = \left[\frac{2^{2N^2-N}}{N!}\right] .$$

The first few values of $n, \bar{g}_n$ and $\bar{g}_{n,0}$ are given below in Table B.1.

Table B.1 Exact and asymptotic counts of self-complementary graphs.

n	4	5	8	9	12	13	16	17
$\bar{g}_n$	1	2	10	36	720	5,600	703,760	11,220,000
$\bar{g}_{n,0}$	1	2	8	32	682	5,465	699,050	11,184,811

Palmer proves that

$$\frac{\bar{g}_{n,0}}{\bar{g}_n} \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty .$$

It is of interest to note that, although the number of self-complementary graphs having a given number of points may seem large, there are in fact relatively few. Let g_n be the number of graphs having n points. It can be shown that

$$\frac{\bar{g}_n}{g_n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

As an indication

$$g_8 = 12,346$$

and

$$g_9 = 274,668.$$

Moreover, if we restrict our attention to graphs having $n \equiv 0$ or $1 \pmod 4$ points and $\frac{1}{2}\binom{n}{2}$ lines it happens that, even in this smaller set, self-complementary graphs are scarce. Let $g_{n,k}$ be the number of graphs having n points and k lines. It can be shown that, for $n \equiv 0$ or $1 \pmod 4$ and $k = \frac{1}{2}\binom{n}{2}$,

$$\frac{\bar{g}_n}{g_{n,k}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

As an indication

$$g_{8,14} = 1,648$$

and

$$g_{9,18} = 34,040.$$