GUARANTEED PERFORMANCE ROBUST GAIN-SCHEDULING CONTROL WITH UNCERTAIN SCHEDULING PARAMETERS

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ABSTRACT

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One of the main objectives in control theory is to develop control strategies and synthesis conditions that not only guarantee closed-loop stability but also achieve guaranteed performance. In this research, novel Robust Gain-Scheduling (RGS) control synthesis conditions are developed for Linear Parameter-Varying (LPV) systems. In contrast to the conventional gain-scheduling synthesis methods, the scheduling parameters are assumed to be inexactly measured. This is a practical assumption since measurement noise is unavoidable in practical engineering applications.

The contributions of this dissertation are the characterization of novel synthesis conditions in terms of Parametrized Linear Matrix Inequalities (PLMIs) and Parametrized Bilinear Matrix Inequalities (PBMIs) for designing RGS controllers with guaranteed stability and performance. Multi-simplex modeling approach is utilized to model the scheduling parameters and their uncertainties in a convex domain. Synthesis conditions for RGS State-Feedback (SF), full-order Dynamic Output-Feedback (DOF), and Static Output-Feedback (SOF) controllers are developed in a unified framework. Matrix coefficient check approach is used to relax the PLMIs conditions into finite dimensional set of Linear Matrix Inequalities (LMIs) to obtain the optimal or suboptimal controller. The resulting controller not only ensures robustness against scheduling parameters uncertainties but also guarantees closed-loop performance under these uncertainties in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance. By the virtue of introducing extra slack variables, controller synthesis is independent of Lyapunov variables, that assures improved performance and viability for multi-objective controller synthesis without introducing additional conservativeness. Since PBMIs problems are non-tractable in general, numerical algorithm is developed to solve the PBMIs conditions. Numerical illustrative examples and comparisons with the existing approaches confirm that the developed control approach outperforms the existing ones.

Furthermore, experimental validation of the developed RGS controllers has been conducted on the test bench of the Electric Variable Valve Timing (EVVT) actuator of automotive engines. Engine speed and vehicle battery voltage are used as noisy scheduling parameters. The experiments are performed at MSU Automotive Controls Lab at a room temperature of 25°. Experimental results demonstrate the effectiveness of the developed approach.

To My Dad with love

ACKNOWLEDGEMENTS

Getting your Ph.D. degree done is exactly like you reached the peak of a high mountain after you spent a hard journey to climb it. The analogy is really valid because when you climb a mountain you will never find a clear paved way to get there. You have to break rocks to make your own way. Additionally, sometimes as you go through your way, unexpected wind blow up and pull you back many steps behind. Then you have to take a deep breath, look for a better way, and proceed climbing again. In sum, Ph.D. and mountain climbing can not be done without passion and persistence.

Over the past four years, I have received support, encouragement, and assistance from many people around me to get this dissertation done. First, I would like to express my deep gratitude and appreciation to my advisor, *Professor Guoming Zhu*, for his guidance, advise, and support during my Ph.D. study at Michigan State University. He shared with me his wide diverse knowledge and made considerable effort to make my life easier. In my opinion, I will never ever meet a person has sweet heart like him.

I would like to thank *Professor Jonguen Choi* for the useful discussions during first year of my study. Great thanks to *Professor Hassan Khalil* and *Professor Ranjan Mukherjee* for serving on my guidance committee. I have gained a wealth of knowledge from the lectures taught by *Professor Hassan Khalil* through my study years.

Last, but not least, I would like to express sincere gratitude to my family for their persistent encouragement and support over the rough years. They believed in me and made me believe in myself during some painful times.

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KEY TO ABBREVIATIONS

ASP	Actual Scheduling Parameter
BMI	Bilinear Matrix Inequality
DC	Direct Current
DOF	Dynamic Output-Feedback
EVVT	Electric Variable Valve Timing
GS	Gain-Scheduling
IC	Internal Combustion
ISOFD	Iterative Static Output-Feedback Design
LFT	Linear Fractional Transformation
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
LTV	Linear Time-Varying
LTI	Linear Time-Invariant
MSP	Measured Scheduling Parameter
PDLF	Parameter-Dependent Lyapunov Function
PLMI	Parametrized Linear Matrix Inequality
PBMI	Parametrized Bilinear Matrix Inequality
qLPV	quasi-Linear Parameter Varying
RGS	Robust Gain-Scheduling
ROLMIP	Robust LMI Parser
SF	State-Feedback
SOS	Sum-Of-Squares
SV	Slack Variable
SOF	Static Output-Feedback
TDC	Top Dead Center

CHAPTER 1

INTRODUCTION

"In the black-and-white robust versus gain-scheduled control world, a gray area should be created that allows to trade-off closed-loop performance with robustness against uncertainty in the scheduling parameter." Jan De Caigny

1.1 Background

Linear Parameter-Varying (LPV) systems are a large class of dynamical systems for which the future evolution of the states depends on the current states of the system plus some additional (time-varying) parameters called *scheduling parameters*. These systems emerged from non-linear system theory and became one of the most successful directions in the post-modern control era. In the past few decades, *classical* (or conventional) gain-scheduling control approach had been successfully applied to wide variety control applications for nonlinear and time-varying systems. The classical approaches can be generally described as divide and conquer techniques, where the control problem of nonlinear systems is decomposed into a finite number of linear subproblems [4]. The major difficulty, at that time, was the lack of general theory for analyzing stability of LPV systems and for efficiently designing gain-scheduled control laws. Due to the absence of a concrete theory for analysis and synthesis, the classical gain-scheduled control methods come with no guarantees on stability, performance or robustness, as pointed out in the pioneering work of Shamma and Athans [5, 6, 7]. As a result of these shortcomings, analysis and synthesis theories have been persistently considered and revisited by the control community over the past twenty-five years, and a continuing effort is evident to develop solid theory that guarantee stability and performance of

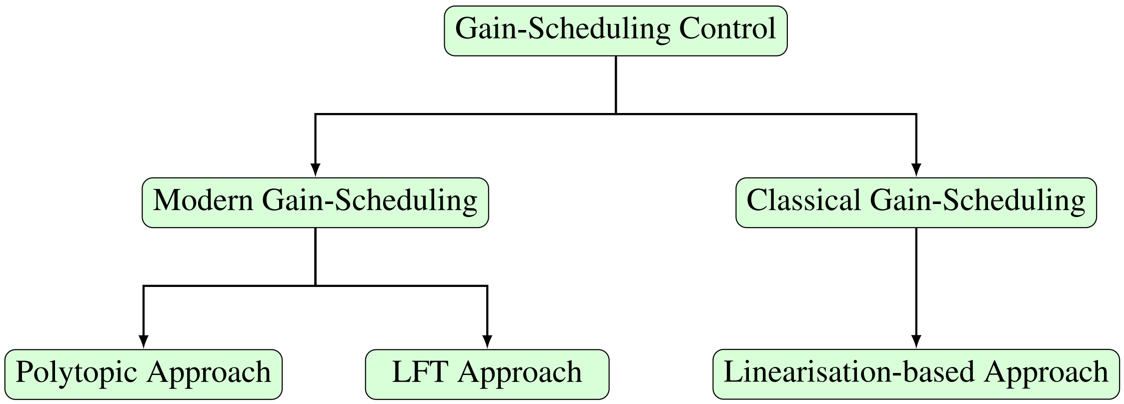


Figure 1.1: Classification of Gain-scheduling control.

LPV systems. Consequently, clear differences are made in the literatures between *classical* and the so-called *modern* gain-scheduling approach (see Figure 1.1). In the classical approach, the design procedure to obtain gain-scheduling controller consists of the following *ad hoc* steps. Initially, a family of local Linear Time-Invariant (LTI) models is determined by selecting different operating points of the dynamical system that cover the entire range of parameters variations. Then, local LTI controllers are designed for each LTI model individually. Next, based on the values of the parameters (measured or estimated on-line), schedule the local controllers using some interpolation (or switching) methods. Finally, extensive simulations are conducted to check and verify closed-loop stability and performance. Thus, the classical gain-scheduling approach has the following critical drawbacks

- Exhaustive and costly simulations and validations are mandatory because ad hoc steps are used in the design procedure.
- It is a challenging task to guarantee stability and performance globally when interpolating (or switching) over a finite family of separately designed (local) controllers.
- Since classical approaches rely on local griding of the operating domain, such approaches imply a sever risk to miss critical system configurations.
- More importantly, these techniques implicitly assume that the scheduling parameters are

frozen in time and ignore the non-stationary nature of parameter variations. In other words, the designed controller does not provide any guarantees in the face of rapid changes in the scheduling parameters. These phenomena represent a major source of failure and may destroy the overall control scheme.

In response to these shortcomings, modern gain-scheduling approaches emerged as a promising alternative and received a considerable attention in control community. Generally, they offer capabilities to handle the whole operating domain without recourse to grid the parameter space. Furthermore, robust stability and performance are guaranteed against parameter variations. And as a key ingredient, they offer an indisputable degree of computational and operational simplicity since the controller can be synthesized directly without using any sort of griding scheme. More concretely, modern LPV problems are convex and amenable to LMI computations, the latter being supported by efficient and reliable software tools. Altogether, this makes these (modern) techniques an excellent candidates for practical engineering applications.

Modern GS approaches can be classified further into two distinct categories. Linear Fractional Transformation (LFT) based structure that use small-gain theory approach, and polytopic structure that based on Lyapunov theory approach. The following overview gives a short survey on the main developments in literature for both the LFT and polytopic structure, without an in-depth discussion.

Polytopic LPV structure (see Figure 1.2) starts from state-space representation of the system and applies Lyapunov's direct method (see Khalil [8]) to derive analysis and synthesis conditions. One of the most critical issues in the polytopic approach is the parametrization of the Lyapunov function (as a function of scheduling parameters) used to establish stability and performance. Initially, many of the researchers adopted the concept of *quadratic stability* where constant Lyapunov matrix is considered because this choice results in numerically tempting and tractable optimization problems [9, 10]. In [9], sufficient conditions were derived for the existence of output-feedback controller that stabilizes closed-loop system exponentially for arbitrarily fast parameter variations. The existence conditions were in the form of a feasibility problem with infinite constraints. Al-

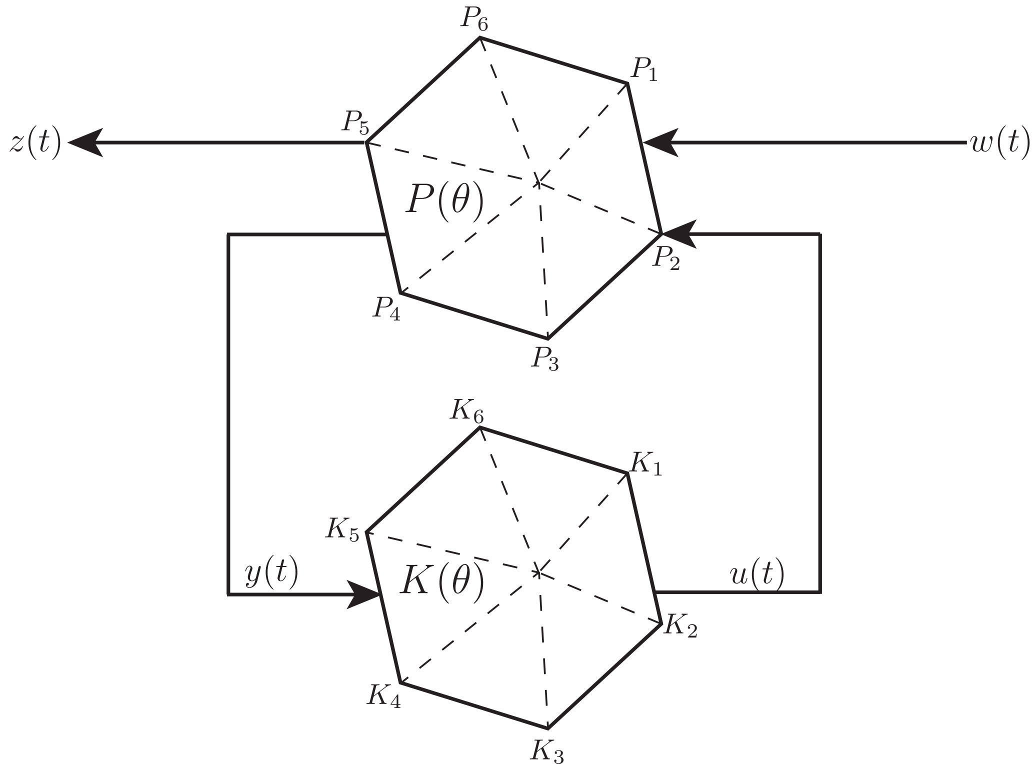


Figure 1.2: LPV system in polytopic structure.

though there is, in general, no systematic method to solve this problem, simplifications can be made for some specific classes of LPV models. For affine LPV models with parameter values belonging to a convex polytope, the solvability conditions reduce to a feasibility problem with a finite number of LMI constraints. Specifically, it is sufficient to evaluate the constraints associated with the vertices of the polytope of parameter values since this ensures that the constraints hold for every parameter value within the polytope [11]. However, as pointed out in [12], quadratic stability approach leads to conservative results since it assumes that the rate of changes of the scheduling parameters are infinite. Consequently, many researchers studied Parameter-Dependent Lyapunov Functions (PDLF) to alleviate the conservatism associated with the quadratic stability-based approach [13, 14, 15].

On the other hand, Packard [16] developed the first LPV control design via LFT structure (see Figure 1.3) using small-gain theory [17] for discrete-time systems. Then, Apkarian and Gahinet

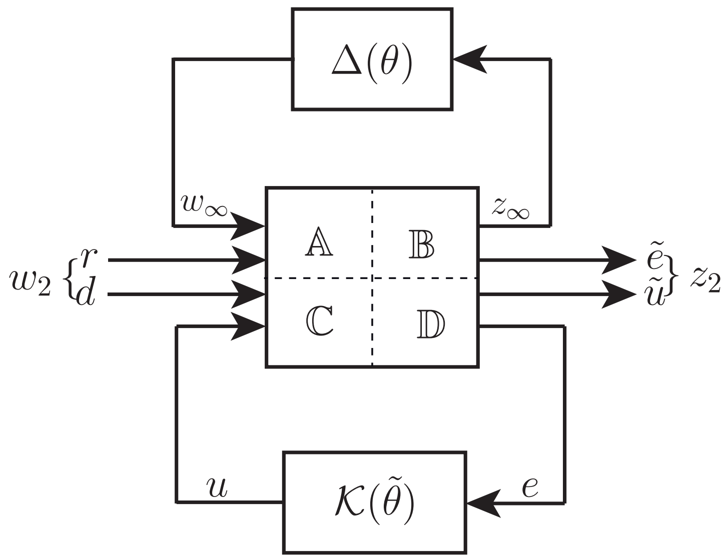


Figure 1.3: LPV system in LFT structure.

extend the work by developing a unifying LMI approach for synthesizing dynamic output-feedback GS controllers for both continuous- and discrete-time LPV systems with $\mathscr{H}_\infty$ performance [18]. For plant models with parameter-dependent LFT structure, the scaled small-gain solvability conditions can be reformulated equivalently as a numerically tractable convex feasibility problem with a finite number of LMIs. Although the approach prescribed in these papers ([16, 18]) is very attractive and fully characterized in terms of a finite number of LMIs, it suffers from conservativeness due to structured scaling matrices. However, such conservativeness can be reduced by the new method based on non-structured (full-block) scaling matrices developed by Scherer [19].

To conclude, this dissertation is concerned with polytopic structure GS synthesis methods. In direct contrast to the literatures mentioned above, the scheduling parameters are assumed to be polluted by noise, which is relatively new topic in the field of GS control as will be illustrated shortly in Section 1.3 and Section 1.4.

1.2 LPV, LTV, and LTI Systems

The terminology *linear parameter-varying* was first introduced in [20] to distinguish LPV systems from both Linear Time-Invariant (LTI) and Linear Time-Varying (LTV) systems. Generally, an

LPV system is a system that can be governed by the following state-space representation

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) &= C(\theta(t))x(t) + D(\theta(t))u(t), \end{aligned} \tag{1.1}$$

where $\theta(t)$ is a time-varying vector of plant parameters belong to a known set, and the matrices $A(\theta(t))$, $B(\theta(t))$, $C(\theta(t))$, $D(\theta(t))$ are functions of $\theta(t)$. A common assumption in the LPV system theory is that scheduling parameters are *unknown* during controller synthesis stage, however, they are available in real-time (by measurement or estimation) for gain-scheduling. Clearly, for a frozen parameters $(\theta(t) = constant)$, the LPV system in (1.1) turns into an LTI system, i.e.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.2}$$

Thus, the distinction between LTI and LPV is clear since LPV systems are non-stationary systems.

On the other hand, the distinction between LPV and LTV is less apparent. Recall that LTV plant is any linear system governed by state equations of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{1.3}$$

where the state-space matrices $A(t), B(t), C(t), D(t)$ are time-varying matrices. It is worth noting that for any given scheduling parameter trajectory, $\theta(t)$, the dynamic (1.1) represents LTV system but the reverse is not true, since the LTV system in (1.3) is completely known in advance. Thus, theoretical treatment of LPV and LTV systems is not the same from analysis and synthesis perspective.

In the same context, this dissertation provide synthesis conditions for gain-scheduling controllers with guaranteed performance in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms. Since $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms are well defined for LTI systems, special care need to be taken when dealing with these performance indices in the LPV framework. However, we use $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms here with slightly abused terminology so that the reader can easily grasp our problem setting by simple analogy to LTI systems. The strict definition of the control problem will be given later in Chapter 3 since necessary definitions and notations need to be introduced in the next chapter.

It is worth mentioning that the terms LPV and parameter-dependent systems (and gain-scheduling in the case of controller) are used interchangeably in this dissertation to refer to a system in the structure of (1.1).

1.3 Motivation of the Work

The main motivation behind Gain-Scheduling (GS) control is the direct extension of the well established linear control design tools to nonlinear and time-varying systems. The field of linear parameter-varying systems has evolved rapidly in the last two decades and became one of the most promising framework for modern industrial control with a growing number of applications (see [21] for a recent survey). Although scheduling parameters are unknown during controller design stage, it is implicitly assumed that they are available for on-line measurement to be used for control adaptation. The significance of this control strategy is attributed to the fact that the dynamics of many physical systems can be efficiently modeled as a function of a time-varying parameters. Moreover, a wide class of nonlinear systems can be represented as quasi-LPV (qLPV) systems [22] that exploit the simplicity of linear control theory instead of sophisticated nonlinear design methodologies. Practical examples that proved the effectiveness of gain scheduling control include spacecrafts [23], Hypersonic vehicles [24], wind turbines [25], automotive engines [26], robotic manipulators [27], active magnetic bearings [28], and miscellaneous mechatronic systems [29, 30, 31].

A common assumption considered in the vast majority of the existing works is that an exact measurement of scheduling parameters is available in real-time for controller scheduling. Generally, this assumption is not true for practical applications. Since uncertainties in scheduling parameters are unavoidable, perfect measurement is impossible to obtain. Due to this measurement noise, discrepancy always exist between the Actual Scheduling Parameters (ASPs) and the Mea-

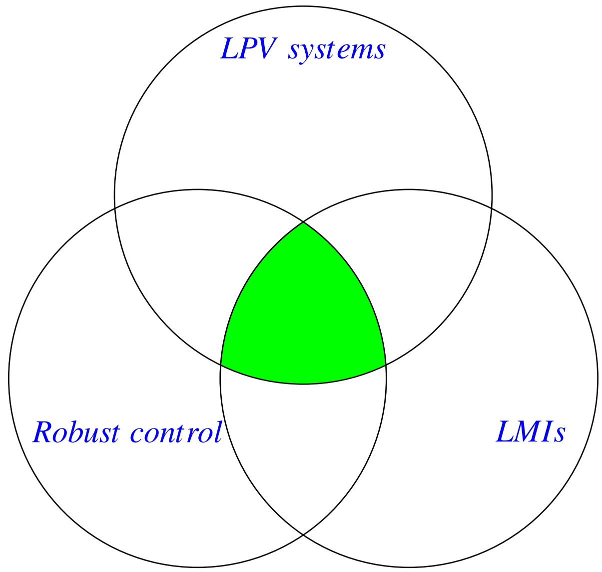


Figure 1.4: Topic of the dissertation.

sured Scheduling Parameters (MSPs). This discrepancy not only leads to performance degradation but could also lead to instability problems. In other words, when applying the controller designed using traditional techniques to a practical application, the closed-loop performance will be worse than the expected theoretical performance since measurement noise in the scheduling parameters had not been considered during controller synthesis stage. Furthermore, the overall stability of the system could be lost because the mismatch between the ASPs and the MSPs. Therefore, this control problem is one of the most important control design problems in the community of gain-scheduling control and LPV systems.

Motivated by the importance of this problem, this dissertation deals with gain-scheduling control with guaranteed performance subject to uncertain scheduling parameters. Thus, the topic of this dissertation is well illustrated by the Venn diagram shown in Figure 1.4 that represents the intersection of the following areas, LPV systems, robust control, and LMIs. As a result of this intersection, *Robust Gain-Scheduling* (RGS) techniques arise in order to not only cope with this type of uncertainty but also to guarantee closed-loop performance.

1.4 Literature Survey

The vast majority of the available work in gain-scheduling control literature assume perfect knowledge of scheduling parameters [9, 10, 32, 11, 13, 14, 33, 15, 34]. Although there are many attempts in literature to address uncertainties in scheduling parameters theoretically, this problem is still undisclosed and barely investigated. In [35], output-feedback synthesis conditions are derived with the assumption that only some of the scheduling parameters are available for feedback control without considering uncertainties in the scheduling parameters. The first work that address uncertainties in the scheduling parameters explicitly is proposed by Daafouz *et al.* [2]. In this paper, gain-scheduling synthesis conditions that guarantee a prescribed performance level in the presence of uncertainties in the scheduling parameters are derived. However, the whole approach presented in [2] is impractical since uncertainties are modeled to be proportional to the values of the scheduling parameters, which is not common to any measurement system. Furthermore, the synthesis conditions are very sensitive to the uncertainty bound. After [2], several papers that addressed the same control problem have been published by Sato *et al.* [36, 37, 1, 38]. Synthesis conditions for state-feedback [36] and dynamic output-feedback controllers [37] are derived with noisy scheduling parameters. However, in [36, 37] quadratic stability (constant Lyapunov matrix) approach is used for controller synthesis. As pointed out in [12], such approach are extremely conservative and certain systems are not even quadratically stabilizable. To alleviate this problem, parameter-dependent Lyapunov function approach was used to synthesize scheduling controllers in [1] for state-feedback, and in [38] for dynamic output-feedback as a remedy for quadratic stability approach. While PDLF approach reduce conservativeness associated with quadratic stability approach, but it introduces a serious implementation drawback. Thus, the developed controller requires not only the real-time measurement of the scheduling parameters, but also requires their derivatives to be available on-line as well. Hence, the synthesized controller is not *practically valid* [13]. From practical view point, the derivatives of the scheduling parameters cannot be obtained in real-time due to the fact that derivative is very sensitive to measurement noise. Furthermore, some of the system matrices are restricted to be independent on the varying parameters in [38] in order to synthesize a controller.

Considering the exist literature, the objectives of this research is to overcome the drawbacks associated with the existing results by developing novel synthesis conditions to synthesize RGS controllers with guaranteed performance under noisy scheduling parameters.

1.5 Specific Contributions

The contributions of this dissertation can be summarized as follow:

1. Characterization of PLMIs synthesis conditions for synthesizing RGS state-feedback controller with guaranteed $\mathscr{H}_2$ performance in Chapter 4.

2. In the same chapter, RGS state-feedback synthesis conditions with guaranteed $\mathscr{H}_\infty$ performance are developed.

3. In Chapter 5, novel conditions in terms of Parametrized Bilinear Matrix Inequalities (PBMIs) have been derived to synthesize RGS Dynamic Output-Feedback (DOF) controller with guaranteed $\mathscr{H}_2$.

4. Similarly, synthesis conditions in terms of PBMIs has been characterized to synthesize RGS DOF controller with guaranteed $\mathscr{H}_\infty$ in Chapter 5. It is worth mentioning that the synthesis conditions of the RGS DOF can handle the case where the time-varying parameters affecting both the state matrix and the control input matrix. This is one of the contributions of Chapter 5 since in literature only state matrix was allowed to be affected by the time-varying parameters.

5. Development of an efficient numerical algorithm to solve the PBMIs conditions iteratively.

6. Novel conditions in terms of PLMIs has been derived in Chapter 6 to synthesize RGS Static Output-Feedback (SOF) controller with guaranteed $\mathscr{H}_2$. These conditions are utilize the two-

stage design approach to synthesize state-feedback scheduling controller in the first stage, then, using this controller in the second stage to synthesize the RGS SOF controller. The RGS SOF controller is synthesized independently of any of the open-loop matrices or Lyapunov matrix, therefore, with this novel design the time-varying parameters could affect all the open-loop matrices without any restrictions.

7. Characterizations of synthesis conditions to synthesize RGS Static Output-Feedback (SOF) controller with guaranteed $\mathscr{H}_\infty$ performance. Similarly, these conditions utilize the two-stage design approach mentioned above to synthesize RGS SOF controller.

8. Experimental validation of the RGS controllers on the test bench of Electric Variable Valve Timing (EVVT) actuator is given in Chapter 7. Engine speed and vehicle battery voltage are used as noisy scheduling parameters.

1.6 Organization

Figure 1.5 shows a road map of the dissertation's chapters. This dissertation is organized as follows: notations, definitions, and multi-simplex modeling approach are given in Chapter 2. Readers are recommended to read Chapter 2 before proceeding to other chapters since it represents the basic building block for modeling the time-varying parameters. Mathematical formulations of the RGS control problem and the proposed solution approach are outlined in Chapter 3. In this chapter, a general framework is presented to handle uncertainties in the scheduling parameters. RGS State-Feedback (SF) PLMIs synthesis conditions are presented in Chapter 4 with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances. Numerical examples, simulations, and comparisons with other approaches from literature are presented at the end of Chapter 4. In Chapter 5, PBMIs synthesis conditions for RGS DOF controllers with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances are developed along with the numerical algorithm necessary to solve the PBMIs conditions. Then, the RGS SOF synthesis conditions are developed in Chapter 6. The synthesis approach of the SOF utilizes the two-stage design approach, where in the first stage SF scheduling controller is designed to be used in the second stage for synthesizing

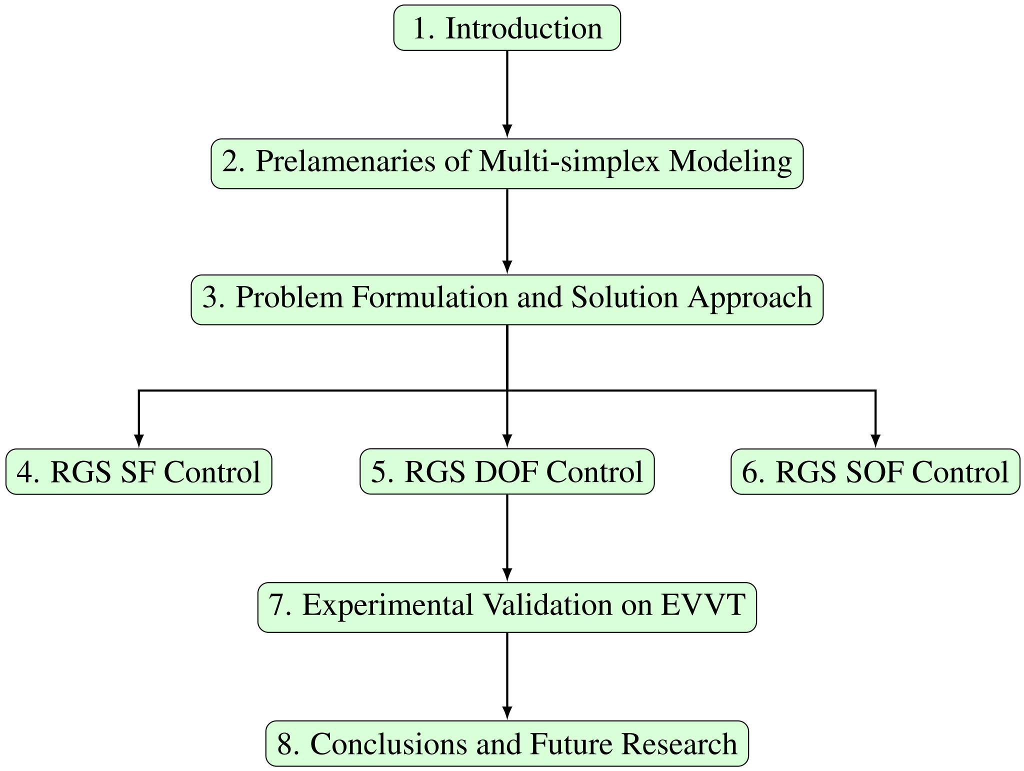


Figure 1.5: Dissertation's chapters road map.

SOF controller. Chapter 7 presents the experimental study of applying RGS controller on Electric Variable Valve Timing (EVVT) system test bench of automotive engine with validation. Finally, Chapter 8 presents conclusions and recommendations for future research. Fundamentals of LMIs for LTI systems are given in the appendix.

CHAPTER 2

PRELIMINARIES OF MULTI-SIMPLEX MODELING

The aim of this chapter is to briefly introduce necessary notations associated with the multi-simplex modeling approach that represents the foundation of this dissertation. This chapter does not present any theoretical contributions but it is included here to present notations, terminologies, and definitions that are used throughout this dissertation. Most of the definitions and terminologies used in this chapter can be found in [39, 40, 41].

2.1 Notations

Notations used in this dissertation are fairly standard. The positive definiteness of a matrix A is denoted by $A > 0$. $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers, respectively. The symbol $\star$ is used to represents the transpose of the off-diagonal matrix block. $trace(A)$ denotes the trace of the matrix A, which represents the sum of diagonal elements of the matrix A. I_n is used to refer to identity matrix of size $n \times n$. Zero matrix of size $n \times p$ is referred to as $\mathbf{0}_{n \times p}$. These subscripts will be omitted when the size of the corresponding matrix can be inferred from the context. The transpose of matrix A is refereed to as A'; and $A + (\bullet)' = A + A'$. Other notations will be explained in due course.

2.2 Definitions and Terminologies

Definition 2.1. *Unit-simplex[39]: a unit-simplex is defined as follows*

$$\Lambda_\ell := \left\{ \alpha(t) \in \mathbb{R}^\ell : \sum_{i=1}^{\ell} \alpha_i(t) = 1,\ \alpha_i(t) \geq 0,\ i = 1, 2, \cdots, \ell \right\},$$

where the variable $\alpha_i(t)$ varies in the unit-simplex Λ_ℓ that has ℓ vertices.

Definition 2.2. *Multi-simplex[40]: a multi-simplex Λ is the Cartesian product of a finite number* *of q simplexes, where*

$$\Lambda_{N_1} \times \Lambda_{N_2} \times \cdots \times \Lambda_{N_q} = \prod_{i=1}^{q} \Lambda_{N_i} := \Lambda.$$

The dimension of the multi-simplex Λ is defined as the index $N = (N_1, N_2, \cdots, N_q)$ and for simplicity of notation, $\mathbb{R}^N$ denotes for the space $\mathbb{R}^{N_1+N_2+\cdots+N_q}$. Thus, any variable $\alpha(t)$ in the multi-simplex domain Λ can be decomposed as $(\alpha_1(t), \alpha_2(t), \cdots, \alpha_q(t))$, and each $\alpha_i(t)$, belonging to a unit-simplex Λ_{N_i}, can be further decomposed as $(\alpha_{i1}(t), \alpha_{i2}(t), \cdots, \alpha_{iN_i}(t))$ for $i = 1, 2, \cdots, q$.

Definition 2.3. *Homogeneous Polynomial: Given a unit-simplex Λ_N of dimension $N \in \mathbb{N}$, a polynomial $p(\alpha)$ defined on $\mathbb{R}^N$ of degree $g \in \mathbb{N}$ is called homogeneous if all of its monomials have the same total degree g.*

Example 2.1. *Let $\alpha \in \Lambda_3$, then the following polynomial $p(\alpha) = 5\alpha_1^4 + \alpha_1^2\alpha_3^2 - 2\alpha_2^3\alpha_3 + 6\alpha_1\alpha_2\alpha_3^2$ is a homogeneous polynomial of degree $g = 4$.*

Definition 2.4. *Λ-Homogeneous Polynomial: Given a multi-simplex Λ of dimension $N \in \mathbb{N}^q$, a polynomial $p(\alpha)$ defined on $\mathbb{R}^N$ of degree $g \in \mathbb{N}^q$ is called Λ-homogeneous if, for any given integer i_0, with $1 \leq i_0 \leq q$, and for any given $\alpha_i \in \mathbb{R}^{N_i}$, for $1 \leq i \neq i_0 \leq q$, the partial application $\alpha_{i_0} \in \mathbb{R}^{N_{i_0}} \longmapsto P(\alpha)$ is a homogeneous polynomial in α_{i_0}.*

Example 2.2. *Let $\alpha \in \Lambda$, with $\Lambda = \Lambda_2 \times \Lambda_3$, then the following polynomial $p(\alpha) = \alpha_{11}^3\alpha_{22} - 3\alpha_{11}\alpha_{12}^2\alpha_{23} - \alpha_{12}^3\alpha_{21} + 6\alpha_{11}^2\alpha_{12}\alpha_{22}$ is a Λ-homogeneous polynomial of partial degree $g = (3, 1)$.*

Definition 2.5. *Partial degree is the degree of a parameter-dependent matrix that depends on multi-simplex parameters which is used to define the individual degree of each unit-simplex inside the multi-simplex domain. For a unit-simplex, g is scalar; while in the multi-simplex domain, g is a vector representing the degrees of each unit-simplex inside the multi-simplex. Thus, the number of elements of vector g is the same as the number of individual simplexes inside the multi-simplex.*

Lemma 2.1. *(Binomial Expansion) For a given nonnegative integer $g \in \mathbb{N}$ and two given numbers a and b*

$$(a+b)^g = \sum_{j=0}^{g} \frac{g!}{j!(g-j)!} \, a^{g-j} \, b^j$$

Lemma 2.2. *(Expansion of powers of sums of N numbers) For a given nonnegative integer g and a given vector x of N numbers*

$$\left(\sum_{i=1}^{N} x_i\right)^g = \sum_{k \in \mathscr{Q}(N,g)} \frac{g!}{\pi(k)} \, x^k \tag{2.1}$$

where $\mathscr{Q}(N,g)$ is the set of N-tuples obtained from all possible combinations of N nonnegative integers $k_i, i = 1,2,\cdots,N$, with sum $k_1 + k_2 + \cdots + k_N = g$ and $\pi(k) = (k_1!)(k_2!)\cdots(k_N!)$, such that

$$\mathscr{Q}(N,g) = \left\{ k \in \mathbb{N}^N : \sum_{i=1}^{N} k_i = g \right\}.$$

The number of elements in $\mathscr{Q}(N,g)$ is given by

$$\mathscr{R}(N,g) := card\, \mathscr{Q}(N,g) = \frac{(N+g-1)!}{g!(N-1)!},$$

with *card* $\mathscr{Q}(N,g)$ refers to the cardinality of $\mathscr{Q}(N,g)$.

2.3 Polynomial Completion and Homogenization

Definition 2.6. *(Λ_N-completion of a polynomial) Given a unit-simplex Λ_N of dimension $N \in \mathbb{N}$ and a polynomial $p(\alpha)$ defined on $\mathbb{R}^N$, the Λ_N-completion of $p(\alpha)$, denoted $comp_{\Lambda_N}(p(\alpha))$, is the (unique) homogeneous polynomial of minimal degree equal to $p(\alpha)$ on Λ_N.*

The Λ_N-completion of $p(\alpha)$ can be easily constructed using Λ_N-homogenization procedure.

Definition 2.7. *(Λ_N-homogenization) For $\alpha \in \Lambda_N$ and a given monomial $m(\alpha)$ of degree $d \in \mathbb{N}$, the Λ_N-homogenization of degree $g \in \mathbb{N}$ of $m(\alpha)$ is obtained by multiplying $m(\alpha)$ with*

$$\left(\sum_{i=1}^{N} \alpha_i\right)^g = 1,$$

then, using Lemma 2.2, can be written as homogeneous polynomial

$$1 = \left(\sum_{i=1}^{N} \alpha_i\right)^g = \sum_{k \in \mathscr{Q}(N,g)} \frac{g!}{\pi(k)}\, \alpha^k.$$

Now, the Λ_N-completion of $p(\alpha)$ can be easily constructed as follows. Let $p(\alpha)$ consist of M monomials of respective degree d_ℓ, for $\ell = 1, 2, \cdots, M$, and let $g = \max_{\ell} d_\ell$. Then, the minimal degree Λ_N-completion of $p(\alpha)$ is obtained by applying a Λ_N-homogenization of degree $g - d_\ell$ to each monomial of $p(\alpha)$.

Example 2.3. *For $\alpha \in \Lambda_3$ the Λ_N completion of $p(\alpha) = \alpha_1^3 + 2\alpha_1\alpha_2 - 5$ is obtained as*

$$comp_{\Lambda_N}(p(\alpha)) = \alpha_1^3 + 2\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \alpha_3) - 5(\alpha_1 + \alpha_2 + \alpha_3)^3.$$

Naturally, the definitions of completion and homogenization can be easily extended to the multi-simplex case, as shown in the following definitions.

Definition 2.8. *(Λ-completion of a polynomial) Given a multi-simplex Λ of dimension $N \in \mathbb{N}^N$ and a polynomial $p(\alpha)$ defined on $\mathbb{R}^N$, the Λ-completion of $p(\alpha)$, denoted $comp_\Lambda(p(\alpha))$, is the (unique) Λ-homogeneous polynomial of minimal degree equal to $p(\alpha)$ on Λ.*

As in the unit-simplex case, the Λ-completion of $p(\alpha)$ can be constructed using Λ-homogenization.

Definition 2.9. *(Λ-homogenization) For $\alpha \in \Lambda$ and a given monomial $m(\alpha)$ of degree $d \in \mathbb{N}^q$, the Λ-homogenization of degree $g \in \mathbb{N}^q$ of $m(\alpha)$ is obtained by multiplying $m(\alpha)$ with*

$$\prod_{i=1}^{q} \left(\sum_{j=1}^{N_i} \alpha_{i,j}\right)^{g_i} = 1$$

using Lemma 2.2, this equation is equal to Λ-homogeneous polynomial

$$1 = \prod_{i=1}^{q} \left(\sum_{j=1}^{N_i} \alpha_{i,j}\right)^{g_i} = \sum_{k \in \mathscr{Q}(N,g)} \frac{\pi(g)}{\pi(k)}\, \alpha^k.$$

Now, the Λ-completion of $p(\alpha)$ can be easily constructed as follows. Let $p(\alpha)$ consist of M monomials of respective degree $d_\ell \in \mathbb{N}^q$, for $\ell = 1, 2, \cdots, M$, and let g be the minimal vector of q natural numbers such that $g \succeq d_\ell$, for $\ell = 1, 2, \cdots, M$. Then, the minimal degree Λ-completion of $p(\alpha)$ is obtained by applying a Λ-homogenization of degree $g - d_\ell$ to each monomial of $p(\alpha)$.

Example 2.4. *Consider the* Λ*-completion of the polynomial*

$$p(\alpha) = -9\alpha_{1,1}^2 - 5\alpha_{1,2}\alpha_{2,1}\alpha_{2,2} + 2\alpha_{2,3}^3$$

where $\alpha \in \Lambda = \Lambda_2 \times \Lambda_3$*. Since the polynomial degree of the three monomials is* $d_1 = (2,0), d_2 = (1,2)$ *and* $d_3 = (0,3)$*, the degree of the* Λ*-homogenization is obtained as* $g = (2,3)$ *and consequently*

$$comp_\Lambda(p(\alpha)) = -9\alpha_{1,1}^2(\alpha_{2,1}+\alpha_{2,2}+\alpha_{2,3})^3 - 5\alpha_{1,2}\alpha_{2,1}\alpha_{2,2}(\alpha_{1,1}+\alpha_{1,2})(\alpha_{2,1}+\alpha_{2,2}+\alpha_{2,3}) + 2\alpha_{2,3}^3(\alpha_{1,1}+\alpha_{1,2})^2 \quad (2.2)$$

2.4 Homogeneous Polynomial Lyapunov Matrix

In order to provide a systematic procedure to generate sufficient LMI conditions of increased precision, a quadratic Lyapunov function $v(x(t)) = x(t)'P(\alpha(t))x(t)$ is defined, with [1]

$$P(\alpha) = \sum_{k \in \mathscr{Q}(N,g)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N} P_{k_1k_2\cdots k_N} = \sum_{k \in \mathscr{Q}(N,g)} \alpha^k P_k, \qquad k = k_1k_2\cdots k_N, \tag{2.3}$$

where $P_k \in \mathbb{R}^{n\times n}$ is a matrix-valued coefficient and α^k is the corresponding monomial with homogeneous of degree $g \in \mathbb{N}$.

Example 2.5. *Consider a homogeneous polynomial matrix of degree* $g = 3$ *with two vertices* $(N = 2)$*, then the possible combinations of the partial degrees are* $\mathscr{Q}(N,g) = \mathscr{Q}(2,3) = \{03, 12, 21, 30\}$*, so* $\mathscr{R}(2,3) = 4$ *corresponding to the generic polynomial form*

$$P(\alpha) = \alpha_2^3 P_{03} + \alpha_1\alpha_2^2 P_{12} + \alpha_1^2\alpha_2 P_{21} + \alpha_1^3 P_{30}.$$

for $g = 0 \Longrightarrow P(\alpha) = P_0$,

$$\text{for } g = 1,\ N = 2 \Longrightarrow \mathscr{Q}(N,g) = \left\{ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right\} \Longrightarrow P(\alpha) = \alpha_1 P_{01} + \alpha_2 P_{10}$$

[1] Sometimes the dependency on t will be omitted for notational simplicity.

$$g=2,\ N=2, \Longrightarrow \mathscr{Q}(N,g)=\left\{\begin{array}{cc} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{array}\right\} \Longrightarrow P(\alpha)=\alpha_1^2 P_{02}+\alpha_1\alpha_2 P_{11}+\alpha_2^2 P_{20}$$

$$g=3,\ N=2, \Longrightarrow \mathscr{Q}(N,g)=\left\{\begin{array}{cc} 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{array}\right\} \Longrightarrow P(\alpha)=\alpha_2^3 P_{03}+\alpha_1\alpha_2^2 P_{12}+\alpha_1^2\alpha_2^1 P_{21}+\alpha_1^3 P_{30}$$

2.5 Generality of the Modeling Approach

This section illustrates the generality of the multi-simplex Λ and the corresponding Λ-homogeneous polynomial parameterization. It is shown that polytopic, affine, and polynomial parameterizations can be recovered as special cases of the homogeneous polynomial parameterization.

2.5.1 Scheduling variables parameterization

1. **Polytopic parametrization** It is easy to note that a matrix with the following representation

   $$A(\alpha(t))=\sum_{i=1}^{N}\alpha_i A_i,$$

   with $\alpha(t)$ in the unit-simplex Λ_N of dimension $N\in\mathbb{N}$ is a special case of the general parameterization (2.3) by choosing the multi-simplex $\Lambda=\Lambda_N$ and the degree of the Λ-homogeneous polynomial $g=1$.

2. **Affine parametrization** A matrix with the following affine structure on q bounded variables $-\bar{\theta}_i\leq\theta_i(t)\leq\bar{\theta}_i$ for $i=1,2,\cdots q$,

   $$A(\theta)=A_0+\sum_{i=1}^{q}\theta_i A_i,$$

can be written as a Λ-homogeneous matrix-valued polynomial by defining

$$\alpha_{i1}(t) = \frac{\theta_i(t) + \bar{\theta}_i}{2\bar{\theta}_i}, \quad \alpha_{i2}(t) = 1 - \alpha_{i1}(t)$$

$$\alpha_i(t) = (\alpha_{i1}(t), \alpha_{i2}(t)) = \left(\frac{\theta_i(t) + \bar{\theta}_i}{2\bar{\theta}_i}, \frac{\bar{\theta}_i - \theta_i(t)}{2\bar{\theta}_i} \right) \qquad for\ i = 1, 2, \cdots, q,$$

such that $\alpha_{i1} \geq 0, \alpha_{i2} \geq 0$, and $\alpha_{i1} + \alpha_{i2} = 1$. Consequently, $\alpha = \left(\alpha_1, \alpha_2, \cdots, \alpha_q\right)$ takes values inside the multi-simplex Λ of dimension $(2, 2, \cdots, 2) \in \mathbb{N}^q$. A Λ-homogeneous polynomial $\hat{A}(\alpha)$, defined over this multi-simplex, equal to $A(\theta)$ can be constructed as follows

$$A(\theta) = A_0 - \sum_{i=1}^{q} \bar{\theta}_i A_i + \sum_{i=1}^{q} 2\bar{\theta}_i A_i \alpha_{i1} = \tilde{A}(\alpha)$$

Obviously, $\tilde{A}(\alpha)$ has a degree one for each variable α_{i1}. Therefore, the Λ-completion $\hat{A}(\alpha) = comp_{\Lambda}(\tilde{A}(\alpha))$ is a homogeneous polynomial of degree $g = (1, 1, \cdots, 1) \in \mathbb{N}^q$, defined over the multi-simplex Λ of dimension $N = (2, 2, \cdots, 2) \in \mathbb{N}^q$ equal to $A(\theta)$.

3. **Polynomial parametrization** A matrix with the following polynomial structure of degree g on a bounded variable $-\bar{\theta} \leq \theta(t) \leq \bar{\theta}$

$$A(\theta) = \sum_{k=0}^{g} \theta^k A_k$$

can be rewritten as a homogeneous matrix-valued polynomial as follows. First, define $\alpha \in \Lambda$ as

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = \left(\frac{\theta(t) + \bar{\theta}}{2\bar{\theta}}, \frac{\bar{\theta} - \theta(t)}{2\bar{\theta}} \right)$$

such that $\alpha_1(t) \geq 0, \alpha_2(t) \geq 0$, and $\alpha_1(t) + \alpha_2(t) = 1$. Since $\theta(t) = 2\bar{\theta}\alpha_1(t) - \bar{\theta}$, it is clear that

$$A(\theta(t)) = \sum_{k=0}^{g} \left(2\bar{\theta}\alpha_1(t) - \bar{\theta}\right)^k A_k$$

Using Lemma 2.1, it can be written as

$$A(\theta(t)) = \sum_{k=0}^{g} \left(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (2\bar{\theta})^j \ \alpha_1^j(t) \ \left(-\bar{\theta}\right)^{k-j} \right) A_k,$$

which yields, after reordering the terms,

$$A(\theta(t)) = \sum_{j=0}^{g} (2\bar{\theta})^j \alpha_1(t)^j \left(\sum_{k=j}^{g} \frac{k!}{j!(k-j)!} \; (-\bar{\theta})^{k-j} \; A_k \right) = \tilde{A}(\alpha(t))$$

It is clear that the $g+1$ monomial terms of $\tilde{A}(\alpha)$ respectively have degree j in α_1, for $j = 0, 1, \cdots, g$. Consequently, the Λ_2-completion $\hat{A}(\alpha(t)) = comp_{\Lambda_2}\tilde{(}A(\alpha(t)))$ can be obtained by applying a Λ_2-homogenization of degree $g-j$ to each monomial term j, for $j = 0, 1, \cdots, g$. This yields a Λ_2-homogeneous polynomial $\hat{A}(\alpha(t))$ of degree $g \in \mathbb{N}$, defined over the unit-simplex Λ_2, that is equal to $A(\theta(t))$.

Example 2.6. *Consider the following LPV system with polynomial dynamic matrix*

$$\dot{x}(t) = \begin{bmatrix} 0 & 1-\theta_2(t) \\ \theta_2(t)\theta_3(t) - 1 & \theta_1^2(t) - 2 \end{bmatrix} x(t) \tag{2.4}$$

with the following bounds,

$$-1 \leq \theta_1(t) \leq 1, \quad -1 \leq \theta_2(t) \leq 1, \quad -0.5 \leq \theta_3(t) \leq 0.5$$

This system can be modeled using multi-simplex approach with three simplexes

- *Simplex#1 to model $\theta_1(t)$ with two vertices $N_1 = 2$ and $g_1 = 2$, $\mathscr{Q}(N_1, g_1) = \begin{Bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{Bmatrix}$, thus, $\tilde{\alpha}(t) = (\tilde{\alpha}_1(t), \tilde{\alpha}_2(t)) \in \Lambda_2$.*
- *Simplex#2 to model $\theta_2(t)$ with 2-vertices $N_2 = 2$ and $g_2 = 1$, $\mathscr{Q}(N_2, g_2) = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$, thus, $\hat{\alpha}(t) = (\hat{\alpha}_1(t), \hat{\alpha}_2(t)) \in \Lambda_2$.*
- *Simplex#3 to model $\theta_3(t)$ with 2-vertices $N_3 = 2$ with $g_3 = 1$, $\mathscr{Q}(N_3, g_3) = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$, thus, $\check{\alpha}(t) = (\check{\alpha}_1(t), \check{\alpha}_2(t)) \in \Lambda_2$.*

The matrix $A(\theta)$ can be written as

$$A(\theta(t)) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \theta_1^2(t) + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \theta_2(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \theta_2(t)\theta_3(t) \quad (2.5)$$

Now express scheduling variables in terms of the multi-simplex variables to obtain $\tilde{A}(\alpha(t))$ by substituting the following relationships in (2.5)

$$\theta_1(t) = 2\bar{\theta}_1\tilde{\alpha}_1(t) - \bar{\theta}_1 = 2\tilde{\alpha}_1(t) - 1, \quad \textit{with } \tilde{\alpha}_1(t) + \tilde{\alpha}_2(t) = 1$$

$$\theta_2(t) = 2\bar{\theta}_2\hat{\alpha}_1(t) - \bar{\theta}_2 = 2\hat{\alpha}_1(t) - 1, \quad \textit{with } \hat{\alpha}_1(t) + \hat{\alpha}_2(t) = 1$$

$$\theta_3(t) = 2\bar{\theta}_3\check{\alpha}_1(t) - \bar{\theta}_3 = \check{\alpha}_1(t) - 0.5, \quad \textit{with } \check{\alpha}_1(t) + \check{\alpha}_2(t) = 1$$

and simplifying to get

$$\begin{aligned} A(\theta) = A(\tilde{\alpha}, \hat{\alpha}, \check{\alpha}) = {} & \begin{bmatrix} 0 & 2 \\ -0.5 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \tilde{\alpha}_1(t) + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \tilde{\alpha}_1^2(t) \\ & + \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \hat{\alpha}_1(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \check{\alpha}_1(t) + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \hat{\alpha}_1(t)\check{\alpha}_1(t). \end{aligned} \quad (2.6)$$

Now Λ-completion of (2.6) *can be easily constructed using Λ-homogenization procedure such that* $\Lambda = \Lambda_2 \times \Lambda_2 \times \Lambda_2$,

Term#1: The 1st term should be multiplied by

$$\begin{aligned} (\tilde{\alpha}_1^2 + \tilde{\alpha}_1\tilde{\alpha}_2 + \tilde{\alpha}_2^2)(\hat{\alpha}_1 + \hat{\alpha}_2)(\check{\alpha}_1 + \check{\alpha}_2) = {} & \tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1 + \tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_2 + \tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_1 + \tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_2 + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_1 \\ & + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_2 + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_1 + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_2 + \tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_1 + \tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_2 + \tilde{\alpha}_2^2\hat{\alpha}_2\check{\alpha}_1 + \tilde{\alpha}_2^2\hat{\alpha}_2\check{\alpha}_2 \end{aligned}$$

Term#2: The second term should be homogenized as

$$\begin{aligned} \tilde{\alpha}_1(\tilde{\alpha}_1 + \tilde{\alpha}_2)(\hat{\alpha}_1 + \hat{\alpha}_2)(\check{\alpha}_1 + \check{\alpha}_2) = {} & \tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1 + \tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_2 + \tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_1 + \alpha_1^2\hat{\alpha}_2\check{\alpha}_2 + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_1 \\ & + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_2 + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_1 + \tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_2 \end{aligned}$$

Term#3: The 3rd term should be homogenized as

$$\tilde{\alpha}_1^2(\hat{\alpha}_1 + \hat{\alpha}_2)(\check{\alpha}_1 + \check{\alpha}_2) = \tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1 + \tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_2 + \tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_1 + \tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_2$$

Term#4: The 4th term will be homogenized

$$\hat{\alpha}_1(\tilde{\alpha}_1^2+\tilde{\alpha}_1\tilde{\alpha}_2+\tilde{\alpha}_2^2)(\check{\alpha}_1+\check{\alpha}_2)=\tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_2+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_2$$
$$+\tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_1+\alpha_2^2\hat{\alpha}_1\check{\alpha}_2$$

Term#5: The 5th term will be homogenized

$$\check{\alpha}_1(\tilde{\alpha}_1^2+\tilde{\alpha}_1\tilde{\alpha}_2+\tilde{\alpha}_2^2)(\hat{\alpha}_1+\hat{\alpha}_2)=\tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_1+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_1$$
$$+\tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_2^2\hat{\alpha}_2\check{\alpha}_1$$

Term#6: The 6th term will be homogenized

$$\hat{\alpha}_1\check{\alpha}_1(\tilde{\alpha}_1^2+\tilde{\alpha}_1\tilde{\alpha}_2+\tilde{\alpha}_2^2)=\tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_1+\tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_1$$

and so on. Now, the matrix $A(\theta(t))$ in system (2.4)*, can be written in the homogenized multi-simplex variables $\alpha(t)=(\tilde{\alpha}(t),\hat{\alpha}(t),\check{\alpha}(t))$ as follow*

$$A(\theta)=A(\tilde{\alpha},\hat{\alpha},\check{\alpha})=\tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_1A_1+\tilde{\alpha}_1^2\hat{\alpha}_1\check{\alpha}_2A_2+\tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_1A_3+\tilde{\alpha}_1^2\hat{\alpha}_2\check{\alpha}_2A_4$$
$$+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_1A_5+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_1\check{\alpha}_2A_6+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_1A_7+\tilde{\alpha}_1\tilde{\alpha}_2\hat{\alpha}_2\check{\alpha}_2A_8$$
$$+\tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_1A_9+\tilde{\alpha}_2^2\hat{\alpha}_1\check{\alpha}_2A_{10}+\tilde{\alpha}_2^2\hat{\alpha}_2\check{\alpha}_1A_{11}+\tilde{\alpha}_2^2\hat{\alpha}_2\check{\alpha}_2A_{12}=A(\alpha)$$

with the following vertix matrices (of the multi-simplex):

$$A_1=\begin{bmatrix}0&0\\-0.5&-1\end{bmatrix},\quad A_2=\begin{bmatrix}0&0\\-1.5&-1\end{bmatrix},\quad A_3=\begin{bmatrix}0&2\\-1.5&-1\end{bmatrix},$$
$$A_4=\begin{bmatrix}0&2\\-0.5&-1\end{bmatrix},\quad A_5=\begin{bmatrix}0&0\\-1&-6\end{bmatrix},\quad A_6=\begin{bmatrix}0&0\\-3&-6\end{bmatrix},$$
$$A_7=\begin{bmatrix}0&4\\-3&-6\end{bmatrix},\quad A_8=\begin{bmatrix}0&4\\-1&-6\end{bmatrix},\quad A_9=\begin{bmatrix}0&0\\-0.5&-1\end{bmatrix},$$
$$A_{10}=\begin{bmatrix}0&0\\-1.5&-1\end{bmatrix},\quad A_{11}=\begin{bmatrix}0&2\\-1.5&-1\end{bmatrix},\quad A_{12}=\begin{bmatrix}0&2\\-0.5&-1\end{bmatrix}$$

Note that the number of vertices of the multi-simplex is given by

$$\mathscr{R}(N,g)=\mathscr{R}((2,2,2),(2,1,1))=\prod_{i=1}^{3}\mathscr{R}(N_i,g_i)=3\times2\times2=12$$

2.5.2 Scheduling variables dependency

Modeling uncertainty domain when one scheduling parameter depends on another scheduling parameter can be easily done via multi-simplex modeling approach. The following example is borrowed from [42] to illustrate this idea. Consider a system depending on two scheduling parameters $\theta_1(t)$ and $\theta_2(t)$ that take values in the region indicated by Figure 2.1a with the gray-shaded area. Depending on the available information, this region can be modeled in a different way. Consider the two following situations:

1. The scheduling parameters are bounded by

$$1 \leq \theta_1(t) \leq 4, \qquad 0 \leq \theta_2(t) \leq 5.$$

2. The second scheduling parameter depends on the first one such that,

$$1 \leq \theta_1(t) \leq 4, \qquad \underline{\theta}_2(\theta_1(t)) \leq \theta_2(t) \leq \bar{\theta}_2(\theta_1(t))$$

with

$$\underline{\theta}_2(\theta_1(t)) = \begin{cases} -0.5\theta_1(t) + 2.5, & if\ 1 \leq \theta_1(t) \leq 2 \\ -1.5\theta_1(t) + 4.5, & if\ 2 \leq \theta_1(t) \leq 3 \\ 3\theta_1(t) - 9, & if\ 3 \leq \theta_1(t) \leq 4 \end{cases} \tag{2.7}$$

$$\bar{\theta}_2(\theta_1(t)) = \begin{cases} -0.5\theta_1^2(t) - \theta_1(t) + 3.5, & if\ 1 \leq \theta_1(t) \leq 3 \\ -\theta_1(t) + 8, & if\ 3 \leq \theta_1(t) \leq 4 \end{cases}$$

In the first case, the multi-simplex can be modeled by treating both scheduling parameters independently as in the approach presented in Example 2.6, yielding

$$\alpha(t) = ((\alpha_{1,1}, \alpha_{1,2}), (\alpha_{2,1}, \alpha_{2,2})) = \left(\left(\frac{\theta_1(t) - 1}{3}, \frac{4 - \theta_1(t)}{3} \right), \left(\frac{\theta_2(t)}{5}, \frac{5 - \theta_2(t)}{5} \right) \right)$$

by taking values in the multi-simplex of dimension $N = (2,2)$, where $\alpha_{1,2} = 1 - \alpha_{1,1}$, $\alpha_{2,2} = 1 - \alpha_{2,1}$, and $\alpha(t) \in \Lambda$. The boundary of the resulting region can be represented in the $(\alpha_{1,1}, \alpha_{2,1})$-space, as shown in the red dashed lines in Figure 2.1b. Since only the bounds of the scheduling

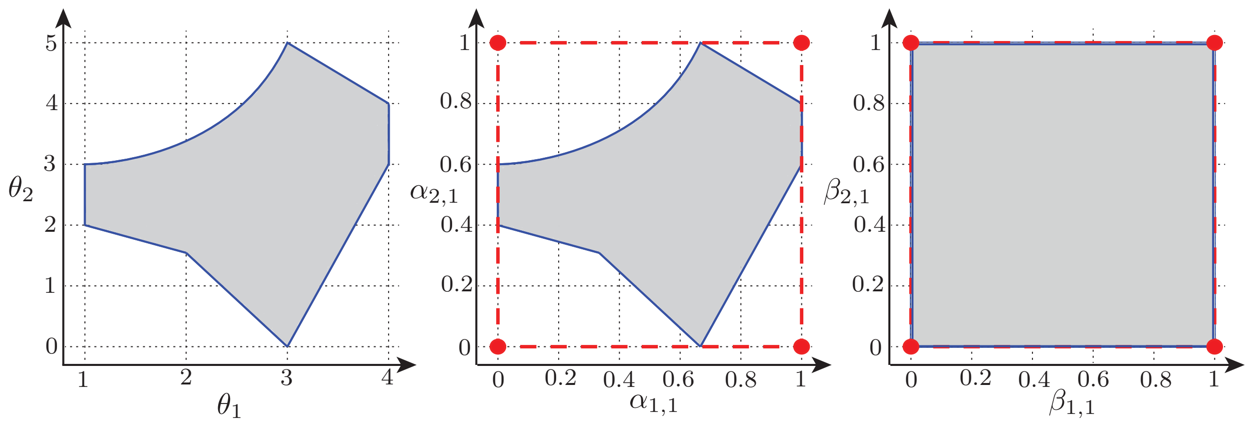


Figure 2.1: Comparison between different modeling approach

parameter are used, all points inside this (red-dashed) box are being considered in this model. This leads to conservativeness since the gray-shaded area is the actual region where $(\alpha_{1,1}, \alpha_{2,1})$ can assume values based on the bounds (2.7).

In the second case , an exact representation of the region in the multi-simplex can be obtained by observing that the lower and upper bound on $\theta_2(t)$ are functions of $\theta_1(t)$, as given in (2.7). Using these bounds, the parameter $\beta(t)$ (where $\beta(t) \in \Lambda$) can be defined as

$$\beta(t) = ((\beta_{1,1}, \beta_{1,2}), (\beta_{2,1}, \beta_{2,2})) = \\ \left(\left(\frac{\theta_1(t) - 1}{3}, \frac{4 - \theta_1(t)}{3} \right), \left(\frac{\theta_2(t) - \underline{\theta}_2(\theta_1(t))}{\bar{\theta}_2(\theta_1(t)) - \underline{\theta}_2(\theta_1(t))}, \frac{\bar{\theta}_2(\theta_1(t)) - \theta_2(t)}{\bar{\theta}_2(\theta_1(t)) - \underline{\theta}_2(\theta_1(t))} \right) \right)$$

taking values in the multi-simplex of dimension $N = (2,2)$. The red-dashed square in Figure 2.1c shows the boundary of the region in the $(\beta_{1,1}, \beta_{2,1})$-space. In this case, the square coincides with the actual region (the gray shaded area).

This example shows the effectiveness of the multi-simplex modeling approach to utilize all the available information about scheduling parameters to reduce conservativeness as much as possible [42].

2.6 Summary

This chapter introduced the notations and definitions of the multi-simplex modeling and homogeneous polynomials parametrization used throughout this dissertation. It is well-known that the multi-simplex domain is a generalized representation of the unit-simplex. These notations and definitions will be utilized in the next chapters to model the time-varying parameters and the associated uncertainties.

CHAPTER 3

PROBLEM FORMULATION AND SOLUTION APPROACH

In this chapter, problem formulation of gain-scheduling controller synthesis with uncertain scheduling parameters is presented in a unified framework. Then, based on the concepts given in Chapter 2, steps of the solution approach are given in this chapter as well. Generally, the solution approach consists of six stages. First, a convex change of variables is to be performed to convert the scheduling parameters and the associated uncertainties from their original parameter domain into (convex) multi-simplex domain. Then, the rates of variation of the scheduling parameters and uncertainties are modeled in a convex set in the second stage as well. The third stage includes derivation of the PLMIs/PBMIs synthesis conditions. The proofs of the PLMIs/PBMIs synthesis conditions will be given in the next three chapters. However, short description of PLMIs will be given in this chapter. Then, a relaxation scheme is used to relax the infinite dimensional constraints into finite set of LMIs constraints. Matrix coefficient check relaxation method will be illustrated in this chapter as it is used to relax the PLMIs conditions in this dissertation. Once a feasible solution is obtained, the controller coefficients recovered via inverse transformation. Finally, the controller is implemented using these coefficients by utilizing the Measured Scheduling Parameters (MSPs).

3.1 Problem Formulation

Consider the following LPV system

$$\mathscr{S}_{OL} := \begin{cases} \dot{x}(t) & = A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t) \\ z(t) & = C_z(\theta(t))x(t) + D_{zu}(\theta(t))u(t) + D_{zw}(\theta(t))w(t) \\ y(t) & = C_y(\theta(t))x(t) + D_{yw}(\theta(t))w(t), \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{nu}$ is the control input, $w(t) \in \mathbb{R}^{nw}$ is the disturbance input, $z(t) \in \mathbb{R}^{nz}$ is the controlled output, and $y(t) \in \mathbb{R}^{ny}$ is the measured output. The system matrices have the following compatible dimensions $A(\theta(t)) \in \mathbb{R}^{n\times n}$, $B_u(\theta(t)) \in \mathbb{R}^{n\times nu}$, $B_w(\theta(t)) \in$

$\mathbb{R}^{n \times n_w}$, $C_z(\theta(t)) \in \mathbb{R}^{n_z \times n}$, $D_{zu}(\theta(t)) \in \mathbb{R}^{n_z \times n_u}$, $D_{zw}(\theta(t)) \in \mathbb{R}^{n_z \times n_w}$, $C_y(\theta(t)) \in \mathbb{R}^{n_y \times n}$, and $D_{yw}(\theta(t)) \in \mathbb{R}^{n_y \times n_w}$. $\theta(t)$ is a real vector containing the time-varying scheduling parameters, where

$$\theta(t) = \left[\theta_1(t), \theta_2(t), \cdots, \theta_q(t)\right]', \tag{3.2}$$

and q represents the number of scheduling parameters. The system matrices in (3.1) are assumed to be affine parameter-dependent, i.e., each of the system matrices can be represented by the following parametrization

$$A(\theta(t)) = A_0 + \sum_{i=1}^{q} \theta_i(t) A_i.$$

The scheduling parameters in (3.2) are assumed to be inexactly measured (corrupted with noise) denoted by $\tilde{\theta}(t)$, such that

$$\begin{aligned}
\tilde{\theta}(t) &= \left[\tilde{\theta}_1(t), \tilde{\theta}_2(t), \cdots, \tilde{\theta}_q(t)\right]', \\
\delta(t) &= \left[\delta_1(t), \delta_2(t), \cdots, \delta_q(t)\right]', \\
\tilde{\theta}(t) &= \theta(t) + \delta(t),
\end{aligned}$$

or in the scalar form,

$$\tilde{\theta}_i(t) = (\theta_i(t) + \delta_i(t)), \qquad i = 1, 2, \cdots, q, \tag{3.3}$$

where $\delta_i(t)$ represents uncertainty of the i-th scheduling parameter and $\theta_i(t)$ is the true value. These scheduling parameters and its uncertainties are assumed to be independent on each other and they varies within the following known bounds (see Figure 3.1)

$$-\bar{\theta}_i \leq \theta_i(t) \leq \bar{\theta}_i, \quad -\bar{\delta}_i \leq \delta_i(t) \leq \bar{\delta}_i, \quad i = 1, 2, \cdots, q. \tag{3.4}$$

Furthermore, these parameters are assumed to have bounded rates of variation

$$-b_{\theta_i} \leq \dot{\theta}_i(t) \leq b_{\theta_i}, \quad -b_{\delta_i} \leq \dot{\delta}_i(t) \leq b_{\delta_i}, \quad i = 1, 2, \cdots, q. \tag{3.5}$$

Without loss of generality, these bounds are assumed to be symmetric. Note that (3.4) and (3.5) are not restrictive, since (3.4) can always be achieved by change of variables; while (3.5) represents a realistic hypothesis because the rates of variation of the parameters are naturally limited in practical engineering applications.

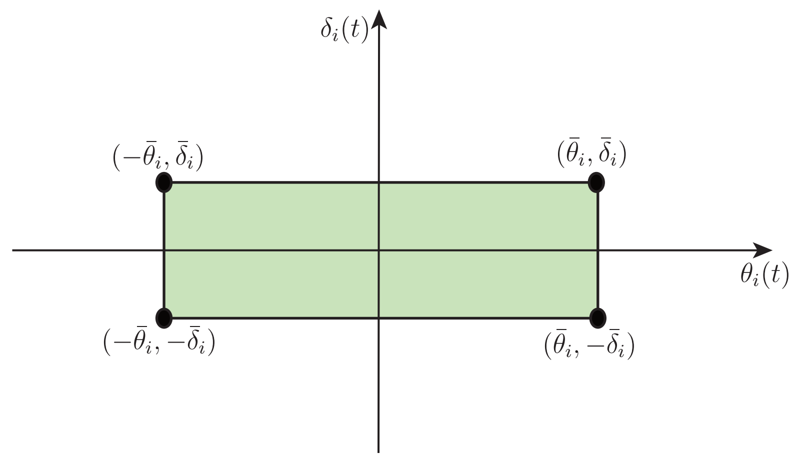


Figure 3.1: Uncertainty domain for measured scheduling parameter.

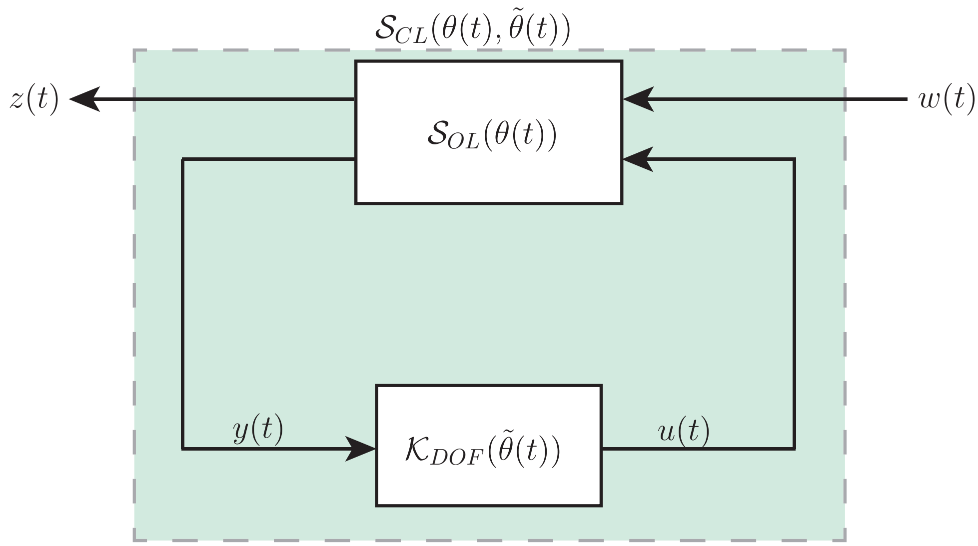


Figure 3.2: Closed-loop system with output-feedback gain-scheduling control.

The goal is to synthesize

1. RGS state-feedback controller of the form

$$u(t) = K(\tilde{\theta}(t))x(t), \tag{3.6}$$

to robustly stabilize the closed-loop system

$$\mathscr{S}_{CL} := \begin{cases} \dot{x}(t) & = \mathscr{A}(\theta(t), \tilde{\theta}(t))x(t) + \mathscr{B}(\theta(t), \tilde{\theta}(t))w(t), \\ z(t) & = \mathscr{C}(\theta(t), \tilde{\theta}(t))x(t) + \mathscr{D}(\theta(t), \tilde{\theta}(t))w(t). \end{cases} \tag{3.7}$$

with

$$
\begin{aligned}
\mathscr{A}(\theta(t),\tilde{\theta}(t)) &= A(\theta(t)) + B_u(\theta(t))K(\tilde{\theta}(t)),\\
\mathscr{B}(\theta(t),\tilde{\theta}(t)) &= B_w(\theta(t)),\\
\mathscr{C}(\theta(t),\tilde{\theta}(t)) &= C_z(\theta(t)) + D_{zu}(\theta(t))K(\tilde{\theta}(t)),\\
\mathscr{D}(\theta(t),\tilde{\theta}(t)) &= D_{zw}(\theta(t)),\\
C_y(\theta(t)) &= I,\\
D_y w(\theta(t)) &= \mathbf{0},
\end{aligned}
\tag{3.8}
$$

or

2. Dynamic Output-Feedback controller of the form

$$
\mathscr{K}_{DOF} := \begin{cases} \dot{x}_c(t) & = A_c(\tilde{\theta}(t))x_c(t) + B_c(\tilde{\theta}(t))y(t),\\ u(t) & = C_c(\tilde{\theta}(t))x(t), \end{cases}
\tag{3.9}
$$

to robustly stabilize the closed-loop system (see Figure 3.2)

$$
\mathscr{S}_{CL} := \begin{cases} \dot{\xi}(t) & = \mathscr{A}(\theta(t),\tilde{\theta}(t))\xi(t) + \mathscr{B}(\theta(t),\tilde{\theta}(t))w(t),\\ z(t) & = \mathscr{C}(\theta(t),\tilde{\theta}(t))\xi(t) + \mathscr{D}(\theta(t),\tilde{\theta}(t))w(t). \end{cases}
\tag{3.10}
$$

with $\xi(t) = \begin{bmatrix} x(t)' & x_c(t)' \end{bmatrix}'$, and

$$
\left[\begin{array}{c|c} \mathscr{A}(\theta(t),\tilde{\theta}(t)) & \mathscr{B}(\theta(t),\tilde{\theta}(t)) \\ \hline \mathscr{C}(\theta(t),\tilde{\theta}(t)) & \mathscr{D}(\theta(t),\tilde{\theta}(t)) \end{array}\right] = \left[\begin{array}{cc|c} A(\theta(t)) & \mathbf{0} & B_w(\theta(t)) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline C_z(\theta(t)) & \mathbf{0} & \mathbf{0} \end{array}\right]
$$

$$
+ \left[\begin{array}{cc} \mathbf{0} & B_u(\theta(t)) \\ I_n & \mathbf{0} \\ \hline \mathbf{0} & D_{zu}(\theta) \end{array}\right] \begin{bmatrix} A_c(\tilde{\theta}(t)) & B_c(\tilde{\theta}(t)) \\ C_c(\tilde{\theta}(t)) & \mathbf{0} \end{bmatrix} \left[\begin{array}{cc|c} \mathbf{0} & I_n & \mathbf{0} \\ C_y(\theta(t)) & \mathbf{0} & D_{yw}(\theta(t)) \end{array}\right].
$$

with

$$\begin{aligned}
\mathscr{A}(\theta(t),\tilde{\theta}(t)) &= \begin{bmatrix} A(\theta(t)) & B_u(\theta(t))C_c(\tilde{\theta}(t)) \\ B_c(\tilde{\theta}(t))C_y(\theta(t)) & A_c(\tilde{\theta}(t)) \end{bmatrix}, \\
\mathscr{B}(\theta(t),\tilde{\theta}(t)) &= \begin{bmatrix} B_w(\theta(t)) \\ B_c(\tilde{\theta}(t))D_{yw}(\theta(t)) \end{bmatrix}, \\
\mathscr{C}(\theta(t),\tilde{\theta}(t)) &= \begin{bmatrix} C_z(\theta(t)) & D_{zu}(\theta(t))C_c(\tilde{\theta}(t)) \end{bmatrix}, \\
\mathscr{D}(\theta(t),\tilde{\theta}(t)) &= D_{zw}(\theta(t)).
\end{aligned} \tag{3.11}$$

or

3. Static Output-Feedback controller of the form

$$u(t) = \mathscr{K}(\tilde{\theta}(t))y(t) \tag{3.12}$$

to robustly stabilizes the closed-loop system

$$\begin{aligned}
\dot{x}(t) &= \mathbb{A}(\theta,\tilde{\theta})x(t) + \mathbb{B}(\theta,\tilde{\theta})w(t) \\
z(t) &= \mathbb{C}(\theta,\tilde{\theta})x(t) + \mathbb{D}(\theta,\tilde{\theta})w(t) \\
\mathbb{A}(\theta,\tilde{\theta}) &:= A(\theta) + B_u(\theta)\mathscr{K}(\tilde{\theta})C_y(\theta) \\
\mathbb{B}(\theta,\tilde{\theta}) &:= B_w(\theta) + B_u(\theta)\mathscr{K}(\tilde{\theta})D_{yw}(\theta) \\
\mathbb{C}(\theta,\tilde{\theta}) &:= C_z(\theta) + D_{zu}(\theta)\mathscr{K}(\tilde{\theta})C_y(\theta) \\
\mathbb{D}(\theta,\tilde{\theta}) &:= D_{zw}(\theta) + D_{zu}(\theta)\mathscr{K}(\tilde{\theta})D_{yw}(\theta)
\end{aligned} \tag{3.13}$$

Note that the controller matrices in (3.6), (3.9), and (3.12) are assumed to have affine parametrization with respect to the Measured Scheduling Parameters (MSPs). In other words, those matrices

$K(\tilde{\theta}(t))$, $A_c(\tilde{\theta}(t))$, $B_c(\tilde{\theta}(t))$, $C_c(\tilde{\theta}(t))$, and $\mathscr{K}(\tilde{\theta}(t))$ are parameterized as follows

$$\begin{aligned}
K(\tilde{\theta}(t)) &= K_0 + \sum_{i=1}^{q} \tilde{\theta}_i(t) K_i, \\
A_c(\tilde{\theta}(t)) &= A_{c_0} + \sum_{i=1}^{q} \tilde{\theta}_i(t) A_{c_i}, \\
B_c(\tilde{\theta}(t)) &= B_{c_0} + \sum_{i=1}^{q} \tilde{\theta}_i(t) B_{c_i}, \\
C_c(\tilde{\theta}(t)) &= C_{c_0} + \sum_{i=1}^{q} \tilde{\theta}_i(t) C_{c_i}, \\
\mathscr{K}(\tilde{\theta}(t)) &= \mathscr{K}_0 + \sum_{i=1}^{q} \tilde{\theta}_i(t) \mathscr{K}_i.
\end{aligned} \tag{3.14}$$

Therefore, the goal is to obtain the controller coefficient matrices K_i, $(A_{c_i}, B_{c_i}, C_{c_i})$, and $\mathscr{K}_i$ for $i = 0, 1, \cdots, q$, such that the RGS controller can be implemented using only the MSPs $\tilde{\theta}_i$.

3.2 Solution Approach

The six stages of the solution approach of RGS synthesis problem is presented in Figure 3.3. Appropriate transformation is used in the first stage to convert the scheduling parameters and the uncertainties from their original parameter domain into a (convex) multi-simplex domain. Then, the rates of variations of the scheduling parameters and uncertainties are modeled in a convex set in the second stage. The third stage consists derivation of the PLMIs/PBMIs synthesis conditions. Matrix coefficient check relaxation scheme [39] is used to relax the infinite dimensional constraints into finite-dimensional constraints to solve the optimization problem. Once a feasible solution is obtained, inverse transformation (multi-simplex-to-affine) is used to obtain controller implementation coefficients that utilize the noisy scheduling parameters.

3.2.1 Affine to Multi-Simplex Transformation

The goal of this subsection is to develop a suitable change of variables to transform all the time-varying parameters (scheduling and uncertainties) from their original space into a convex multi-

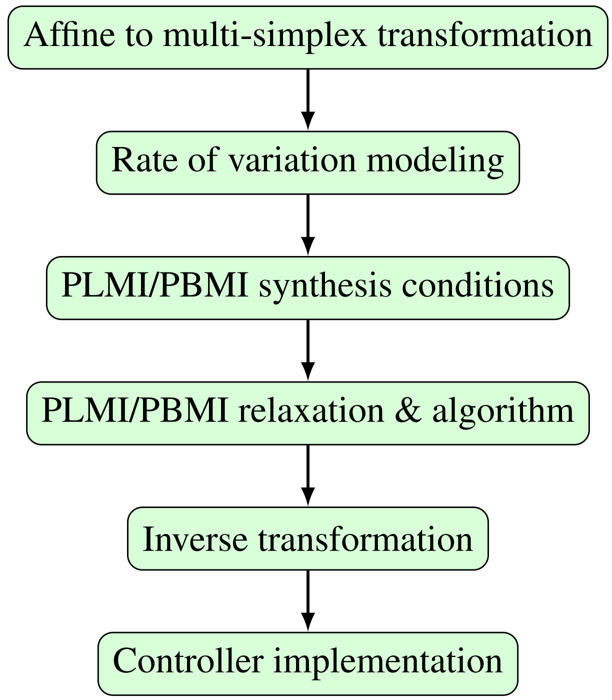


Figure 3.3: Six stages solution approach.

simplex domain. Suppose that the Actual Scheduling Parameters (ASPs) $\theta(t)$ are affected by time-varying measurement noise $\delta(t)$ as given by (3.3); and suppose further that $\theta(t)$, $\delta(t)$, $\dot{\theta}(t)$, and $\dot{\delta}(t)$ are bounded as defined in (3.4) and (3.5). Since $\delta_i(t)$ associated with each $\theta_i(t)$ needs to be modeled in a convex domain, two unit-simplexes for each MSP are used. Each unit-simplex has two vertices due to the fact that each parameter has upper and lower bounds as defined in (3.4). Thus, each of those (time-varying) parameters ($\theta_i(t)$ and $\delta_i(t)$) will be modeled independently in their own unit-simplexes. Following the approach depicted in [43], the ASP and their uncertainties can be modeled as follow:

1. Actual scheduling parameters $(\theta_i(t) \Rightarrow \breve{\alpha}_i(t))$,

$$\breve{\alpha}_{i1}(t) = \frac{\theta_i(t) + \bar{\theta}_i}{2\bar{\theta}_i} \;\; \Rightarrow \;\; \theta_i(t) = 2\bar{\theta}_i \breve{\alpha}_{i1}(t) - \bar{\theta}_i, \tag{3.15}$$

then,

$$\breve{\alpha}_{i2}(t) = 1 - \breve{\alpha}_{i1}(t) = 1 - \frac{\theta_i(t) + \bar{\theta}_i}{2\bar{\theta}_i} = \frac{\bar{\theta}_i - \theta_i(t)}{2\bar{\theta}_i},$$

where,

$$\check{\alpha}_i(t) = (\check{\alpha}_{i1}(t), \check{\alpha}_{i2}(t)) \in \Lambda_2, \ \ \forall i = 1, 2, \cdots, q,$$

$$\check{\alpha}(t) = (\check{\alpha}_1(t), \check{\alpha}_2(t), \cdots, \check{\alpha}_q(t)).$$

2. Uncertainties $(\delta_i(t) \Rightarrow \hat{\alpha}_i(t))$,

$$\hat{\alpha}_{i1}(t) = \frac{\delta_i(t) + \bar{\delta}_i}{2\bar{\delta}_i} \ \ \Rightarrow \ \ \delta_i(t) = 2\bar{\delta}_i\hat{\alpha}_{i1}(t) - \bar{\delta}_i, \tag{3.16}$$

then,

$$\hat{\alpha}_{i2}(t) = 1 - \hat{\alpha}_{i1}(t) = 1 - \frac{\delta_i(t) + \bar{\delta}_i}{2\bar{\delta}_i} = \frac{\bar{\delta}_i - \delta_i(t)}{2\bar{\delta}_i},$$

where,

$$\hat{\alpha}_i(t) = (\hat{\alpha}_{i1}(t), \hat{\alpha}_{i2}(t)) \in \Lambda_2, \quad \forall i = 1, 2, \cdots, q,$$

$$\hat{\alpha}(t) = (\hat{\alpha}_1(t), \hat{\alpha}_2(t), \cdots, \hat{\alpha}_q(t)).$$

Thus, using this change of variables, the original affine parameter-dependent system (3.1) as well as the gain-scheduling controllers (3.6), (3.9), and (3.12) can be converted from $\theta(t)$ and $\tilde{\theta}(t)$ into new multi-simplex variables $\check{\alpha}(t)$ and $\hat{\alpha}(t)$, respectively. Therefore, the multi-simplex variables $\alpha(t)$ can be defined as,

$$\tilde{\alpha}(t) = (\check{\alpha}_i(t), \hat{\alpha}_i(t)), \quad i = 1, 2, \cdots, q, \quad \alpha(t) \in \Lambda, \quad \text{where} \quad \Lambda = \underbrace{\Lambda_2 \times \Lambda_2 \times \cdots \times \Lambda_2}_{2q \text{ simplexes}}. \tag{3.17}$$

Considering the case that $q = 1$ (one scheduling parameter), $\check{\alpha}_1(t) = (\check{\alpha}_{11}(t), \check{\alpha}_{12}(t))$ and $\hat{\alpha}_1(t) = (\hat{\alpha}_{11}(t), \hat{\alpha}_{12}(t))$, the homogeneous terms in the multi-simplex variables can be written in terms of the new variables $\tilde{\alpha}(t) = (\check{\alpha}_{11}(t), \check{\alpha}_{12}(t), \hat{\alpha}_{11}(t), \hat{\alpha}_{12}(t))$.

For illustration purposes, the details of such transformation will be given for two cases, single (measured) scheduling parameter and two (measured) scheduling parameters. Then, a generalization for any number of scheduling parameters will be given as well.

3.2.1.1 Single scheduling parameter

For instance, let $Z(\tilde{\theta}(t))$ be any matrix of the controller variables given in (3.14). This matrix can be expressed affinely in terms of the MSPs as

$$Z(\tilde{\theta}(t)) = Z_0 + \tilde{\theta}_1(t)Z_1 = Z_0 + (\theta_1(t) + \delta_1(t))Z_1. \tag{3.18}$$

Substituting for $\theta_1(t)$ and $\delta_1(t)$ from (3.15) and (3.16) yields[1],

$$Z(\tilde{\theta}(t)) = Z_0 + (2\bar{\theta}_1\check{\alpha}_{11} - \bar{\theta}_1 + 2\bar{\delta}_1\hat{\alpha}_{11} - \bar{\delta}_1)Z_1 = Z(\tilde{\alpha}(t)),$$

and applying homogenization procedure [39] leads to

$$\begin{aligned} Z(\tilde{\alpha}(t)) = Z_0 &\overbrace{(\check{\alpha}_{11} + \check{\alpha}_{12})}^{1}\overbrace{(\hat{\alpha}_{11} + \hat{\alpha}_{12})}^{1} + [2\bar{\theta}_1\check{\alpha}_{11}\overbrace{(\hat{\alpha}_{11} + \hat{\alpha}_{12})}^{1} \\ &- (\bar{\theta}_1 + \bar{\delta}_1)\underbrace{(\check{\alpha}_{11} + \check{\alpha}_{12})}_{1}\underbrace{(\hat{\alpha}_{11} + \hat{\alpha}_{12})}_{1} + 2\bar{\delta}_1\hat{\alpha}_{11}\underbrace{(\check{\alpha}_{11} + \check{\alpha}_{12})}_{1}]Z_1. \end{aligned}$$

As a result, $Z(\tilde{\alpha}(t))$ is a parameter-dependent matrix that depends on time varying parameters inside the multi-simplex domain Λ [43]. In other words, the parameters bounds $\bar{\theta}$ and $\bar{\delta}$ are used to convert $Z(\tilde{\theta}(t))$ into $Z(\tilde{\alpha}(t))$. Thus, the matrix can be written in the homogenized terms as

$$Z(\tilde{\alpha}(t)) = \check{\alpha}_{11}\hat{\alpha}_{11}\mathscr{Z}_{1,1} + \check{\alpha}_{11}\hat{\alpha}_{12}\mathscr{Z}_{1,2} + \check{\alpha}_{21}\hat{\alpha}_{11}\mathscr{Z}_{2,1} + \check{\alpha}_{21}\hat{\alpha}_{21}\mathscr{Z}_{2,2} \;, \tag{3.19}$$

where the coefficients $\mathscr{Z}_{1,1}, \mathscr{Z}_{1,2}, \mathscr{Z}_{2,1}$ and $\mathscr{Z}_{2,2}$ can be generated using the bounds as

$$\begin{aligned} \mathscr{Z}_{1,1} &= Z_0 + (\bar{\theta}_1 + \bar{\delta}_1)Z_1, \\ \mathscr{Z}_{1,2} &= Z_0 + (\bar{\theta}_1 - \bar{\delta}_1)Z_1, \\ \mathscr{Z}_{2,1} &= Z_0 + (-\bar{\theta}_1 + \bar{\delta}_1)Z_1, \\ \mathscr{Z}_{2,2} &= Z_0 + (-\bar{\theta}_1 - \bar{\delta}_1)Z_1. \end{aligned} \tag{3.20}$$

3.2.1.2 Two scheduling parameters

$$Z(\tilde{\theta}(t)) = Z_0 + \tilde{\theta}_1(t)Z_1 + \tilde{\theta}_2(t)Z_2 = Z_0 + (\theta_1(t) + \delta_1(t))Z_1 + (\theta_2(t) + \delta_2(t))Z_2.$$

[1] Sometimes the dependency on t will be omitted for notational simplicity.

then,

$$Z(\tilde{\theta}) = Z_0 + (2\bar{\theta}_1\check{\alpha}_{11} - \bar{\theta}_1)Z_1 + (2\bar{\delta}_1\hat{\alpha}_{11} - \bar{\delta}_1)Z_1 + (2\bar{\theta}_2\check{\alpha}_{21} - \bar{\theta}_2)Z_2 + (2\bar{\delta}_2\hat{\alpha}_{21} - \bar{\delta}_2)Z_2 = Z(\tilde{\alpha}).$$

Homogenizing this equation,

$$\begin{aligned} Z(\tilde{\alpha}) = {} & Z_0 \underbrace{(\check{\alpha}_{11}+\check{\alpha}_{12})}_{1}\underbrace{(\hat{\alpha}_{11}+\hat{\alpha}_{12})}_{1}\underbrace{(\check{\alpha}_{21}+\check{\alpha}_{22})}_{1}\underbrace{(\hat{\alpha}_{21}+\hat{\alpha}_{22})}_{1} + \\ & [2\bar{\theta}_1\check{\alpha}_{11}\underbrace{(\hat{\alpha}_{11}+\hat{\alpha}_{12})}_{1}\underbrace{(\check{\alpha}_{21}+\check{\alpha}_{22})}_{1}\underbrace{(\hat{\alpha}_{21}+\hat{\alpha}_{22})}_{1}]Z_1 \\ & - (\bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2)\underbrace{(\check{\alpha}_{11}+\check{\alpha}_{12})}_{1}\underbrace{(\hat{\alpha}_{11}+\hat{\alpha}_{12})}_{1}\underbrace{(\check{\alpha}_{21}+\check{\alpha}_{22})}_{1}\underbrace{(\hat{\alpha}_{21}+\hat{\alpha}_{22})}_{1} \\ & + [2\bar{\delta}_1\check{\alpha}_{11}\underbrace{(\hat{\alpha}_{11}+\hat{\alpha}_{12})}_{1}\underbrace{(\check{\alpha}_{21}+\check{\alpha}_{22})}_{1}\underbrace{(\hat{\alpha}_{21}+\hat{\alpha}_{22})}_{1}]Z_1 \\ & + [2\bar{\theta}_2\check{\alpha}_{11}\underbrace{(\hat{\alpha}_{11}+\hat{\alpha}_{12})}_{1}\underbrace{(\check{\alpha}_{21}+\check{\alpha}_{22})}_{1}\underbrace{(\hat{\alpha}_{21}+\hat{\alpha}_{22})}_{1}]Z_2 \\ & + [2\bar{\delta}_2\check{\alpha}_{11}\underbrace{(\hat{\alpha}_{11}+\hat{\alpha}_{12})}_{1}\underbrace{(\check{\alpha}_{21}+\check{\alpha}_{22})}_{1}\underbrace{(\hat{\alpha}_{21}+\hat{\alpha}_{22})}_{1}]Z_2. \end{aligned} \tag{3.21}$$

As a result, $Z(\tilde{\alpha})$ is a parameter-dependent matrix with parameters in the multi-simplex Λ,

$$\begin{aligned} Z(\tilde{\alpha}) = {} & \check{\alpha}_{11}\hat{\alpha}_{11}\check{\alpha}_{21}\hat{\alpha}_{21}\mathscr{Z}_{1,1,1,1} + \check{\alpha}_{11}\hat{\alpha}_{11}\check{\alpha}_{21}\hat{\alpha}_{22}\mathscr{Z}_{1,1,1,2} + \check{\alpha}_{11}\hat{\alpha}_{11}\check{\alpha}_{22}\hat{\alpha}_{21}\mathscr{Z}_{1,1,2,1} \\ & + \check{\alpha}_{11}\hat{\alpha}_{11}\check{\alpha}_{22}\hat{\alpha}_{22}\mathscr{Z}_{1,1,2,2} + \check{\alpha}_{11}\hat{\alpha}_{12}\check{\alpha}_{21}\hat{\alpha}_{21}\mathscr{Z}_{1,2,1,1} + \check{\alpha}_{11}\hat{\alpha}_{12}\check{\alpha}_{21}\hat{\alpha}_{22}\mathscr{Z}_{1,2,1,2} \\ & + \check{\alpha}_{11}\hat{\alpha}_{12}\check{\alpha}_{22}\hat{\alpha}_{21}\mathscr{Z}_{1,2,2,1} + \check{\alpha}_{11}\hat{\alpha}_{12}\check{\alpha}_{22}\hat{\alpha}_{22}\mathscr{Z}_{1,2,2,2} + \check{\alpha}_{12}\hat{\alpha}_{11}\check{\alpha}_{21}\hat{\alpha}_{21}\mathscr{Z}_{2,1,1,1} \\ & + \check{\alpha}_{12}\hat{\alpha}_{11}\check{\alpha}_{21}\hat{\alpha}_{22}\mathscr{Z}_{2,1,1,2} + \check{\alpha}_{12}\hat{\alpha}_{11}\check{\alpha}_{22}\hat{\alpha}_{21}\mathscr{Z}_{2,1,2,1} + \check{\alpha}_{12}\hat{\alpha}_{11}\check{\alpha}_{22}\hat{\alpha}_{22}\mathscr{Z}_{2,1,2,2} \\ & + \check{\alpha}_{12}\hat{\alpha}_{12}\check{\alpha}_{21}\hat{\alpha}_{21}\mathscr{Z}_{2,2,1,1} + \check{\alpha}_{12}\hat{\alpha}_{12}\check{\alpha}_{21}\hat{\alpha}_{12}\mathscr{Z}_{2,2,1,2} + \check{\alpha}_{12}\hat{\alpha}_{12}\check{\alpha}_{22}\hat{\alpha}_{21}\mathscr{Z}_{2,2,2,1} \\ & + \check{\alpha}_{12}\hat{\alpha}_{12}\check{\alpha}_{22}\hat{\alpha}_{22}\mathscr{Z}_{2,2,2,2} \end{aligned} \tag{3.22}$$

where the matrices $\mathscr{Z}_{1,1,1,1}$, $\mathscr{Z}_{1,1,1,2}$, $\mathscr{Z}_{1,1,2,1}$, $\mathscr{Z}_{1,1,2,2}$, $\mathscr{Z}_{1,2,1,1}$, $\mathscr{Z}_{1,2,1,2}$, $\mathscr{Z}_{1,2,2,1}$, $\mathscr{Z}_{1,2,2,1}$, $\mathscr{Z}_{1,2,2,2}$, $\mathscr{Z}_{2,1,1,1}$, $\mathscr{Z}_{2,1,1,2}$, $\mathscr{Z}_{2,1,2,1}$, $\mathscr{Z}_{2,1,2,2}$, $\mathscr{Z}_{2,2,1,1}$, $\mathscr{Z}_{2,2,1,2}$, $\mathscr{Z}_{2,2,2,1}$, $\mathscr{Z}_{2,2,2,1}$, and $\mathscr{Z}_{2,2,2,2}$

can be generated as,

$$
\begin{aligned}
\mathscr{Z}_{1,1,1,1} &= Z_0 + \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,1,1,2} &= Z_0 + \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,1,2,1} &= Z_0 + \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,1,2,2} &= Z_0 + \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,2,1,1} &= Z_0 + \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,2,1,2} &= Z_0 + \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,2,2,1} &= Z_0 + \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{1,2,2,2} &= Z_0 + \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,1,1,1} &= Z_0 - \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,1,1,2} &= Z_0 - \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,1,2,1} &= Z_0 - \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,1,2,2} &= Z_0 - \bar{\theta}_1 Z_1 + \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,2,1,1} &= Z_0 - \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,2,1,2} &= Z_0 - \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 + \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,2,2,1} &= Z_0 - \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 + \bar{\delta}_2 Z_2, \\
\mathscr{Z}_{2,2,2,2} &= Z_0 - \bar{\theta}_1 Z_1 - \bar{\delta}_1 Z_1 - \bar{\theta}_2 Z_2 - \bar{\delta}_2 Z_2.
\end{aligned}
\tag{3.23}
$$

3.2.1.3 Multiple numbers of scheduling parameters

This procedure can be systematically extended to handle all the system matrices in (3.1) and controller matrices in (3.6), (3.9), and (3.12) to convert them into the multi-simplex variables $\tilde{\alpha}(t) = (\breve{\alpha}(t), \hat{\alpha}(t))$ for any number of scheduling parameters $q \geq 1$. The matrices $\mathscr{Z}_{j_1,j_2,\cdots,j_q,k_1,k_2,\cdots,k_q}$ in (3.23) for $j_1, j_2, \cdots, j_q, k_1, k_2, \cdots, k_q = 1,2$, can be written in a generalized form as

$$
\mathscr{Z}_{j_1,j_2,\cdots,j_q,k_1,k_2,\cdots,k_q} = Z_0 + \sum_{i=1}^{q} \left\{ (-1)^{j_i+1} \bar{\theta}_i + (-1)^{k_i+1} \bar{\delta}_i \right\} Z_i. \tag{3.24}
$$

Thus, it is worth mentioning that the synthesis variables that used to construct controller matrices in (3.6), (3.9), and (3.12) should be converted into the multi-simplex domain using the procedure described above. Therefore, the controller matrices can be written in terms of the multi-simplex parameters as $K(\tilde{\alpha}(t)), A_c(\tilde{\alpha}(t)), B_c(\tilde{\alpha}(t)), C_c(\tilde{\alpha}(t))$, and $\mathscr{K}(\tilde{\alpha}(t))$.

Remark 3.1. *Note that the open-loop system matrices in* (3.1) *are independent of the uncertainties $\delta_i(t)$. They are only depend on the ASPs $\theta(t)$. However, the same procedure described above can be used to transform them from the original parameter space $\theta(t)$ into multi-simplex space $\alpha(t)$ by imposing $\bar{\delta}_i = 0$ in* (3.24)*. In this case, for notational simplicity, new the multi-simplex variables $\alpha(t)$ is used instead of $\tilde{\alpha}(t)$ to distinguish variables that depend on ASPs from variables that depend on MSPs. Thus, the open-loop system matrices will be written in terms of the multi-simplex variables as*

$$A(\alpha(t)), B_w(\alpha(t)), B_u(\alpha(t)), C_z(\alpha(t)),\ D_{zu}(\alpha(t)),\ D_{zw}(\alpha(t)), C_y(\alpha(t)),\ and\ \ D_{yw}(\alpha(t)).$$

3.2.2 Rate of Variation Modeling

The objective of this subsection is to construct a new convex parameter space $\eta(t)$ to model the derivatives of the varying parameters in the convex domain. The rates of change of each parameter and uncertainty are assumed to be bounded as defined in (3.5) for all $t \geq 0$. Since each varying parameter belongs to a unit-simplex, it is clear that the following relation is satisfied

$$\dot{\alpha}_{i1}(t) + \dot{\alpha}_{i2}(t) = 0 \qquad i = 1, 2, \cdots, q. \tag{3.25}$$

Since $\alpha_i(t) \in \Lambda_2$, the time derivatives of the parameters α_i can assume values that modeled by a convex polytope Ω_i [44, 45]

$$\Omega_i = \left\{ \phi \in \mathbb{R}^2 : \phi = \sum_{k=1}^{2} \eta_{ik} H_i^{(k)},\ \sum_{k=1}^{2} H_i(k,j) = 0,\ \eta_i \in \Lambda_2 \right\},\ j = 1, 2,\ i = 1, 2, \cdots, 2q. \tag{3.26}$$

Given the bounds b_{θ_i} and b_{δ_i} in (3.5), $H_i^{(k)}$ represents the k-th column of matrix H_i. Since simplexes with two vertices have been considered for each varying parameter, as a direct consequence, the matrices H_i will have size of 2×2. Notice that, due to (3.25), the sum of the elements of each column of $H_i^{(k)}$ is zero. Consequently,

$$\dot{\alpha}(t) \in \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{2q} = \prod_{i=1}^{2q} \Omega_i. \tag{3.27}$$

Note that the relationship between the bounds of the rates of variations of the varying parameters $\dot{\theta}$ and $\dot{\delta}$, and the rates of changes of multi-simplex variables $\dot{\alpha}$ can be obtained using (3.5) and (3.15) as follows

$$\frac{-b_{\theta_i}}{2\bar{\theta}_i} \leq \dot{\alpha}_{i1}(t) \leq \frac{b_{\theta_i}}{2\bar{\theta}_i},$$

with $\dot{\alpha}_{i2}(t) = -\dot{\alpha}_{i1}(t)$ as the consequence of (3.25). Therefore, the transformation of the rates of variations from $\dot{\theta}(t)$ and $\dot{\delta}(t)$ into $\dot{\alpha}_i(t)$ is exact as well. As an example consider one scheduling parameter ($q = 1$) with the following bounds

$$-1 \leq \theta(t) \leq 1, \qquad -1 \leq \dot{\theta}(t) \leq 1,$$

the multi-simplex variables rate bounds are $-0.5 \leq \dot{\alpha}_{11}(t) \leq 0.5$; and considering (3.25) at the bounds of $\dot{\alpha}_{11}(t)$ and $\dot{\alpha}_{12}(t)$, one can easily construct the columns of the matrix H_1. Taking the convex combination of these columns [40], yields

$$\begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} \eta_{11} + \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} \eta_{12} = \underbrace{\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}}_{H_1} \begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix}.$$

Thus, the derivative of the parametric Lyapunov matrix that depend on a time-varying parameters in multi-simplex can be computed through this procedure as

$$\begin{aligned} \dot{P}(\alpha) = \frac{\partial P(\alpha)}{\partial \alpha}\dot{\alpha} = \sum_{i=1}^{2q}\sum_{j=1}^{2} \frac{\partial P(\alpha)}{\partial \alpha_{ij}}\dot{\alpha}_{ij} = \sum_{i=1}^{2q}\sum_{j=1}^{2} \frac{\partial P(\alpha)}{\partial \alpha_{ij}} \sum_{k=1}^{2} \eta_{ik} H_i(j,k) \\ = \frac{\partial P(\alpha)}{\partial \alpha_{ij}} (\eta_{i1} H_i(j,1) + \eta_{i2} H_i(j,2)) := \Pi(\alpha, \eta), \quad \eta_i \in \Lambda_2. \end{aligned} \tag{3.28}$$

3.2.3 PLMIs Conditions

It is known that synthesis problems of RGS and robust controllers with parametric uncertainties are frequently arise as optimization problem with PLMIs constraints. A PLMIs is an extension of an ordinary Linear Matrix Inequalities (LMIs) with the difference is that it is dependent on a time-varying parameter vector. Generally speaking, the solution to this type of LPV control problems is formulated as PLMIs, which is a special type of convex optimization problem. PLMIs are equivalent to infinite number of LMI constraints and, consequently, require infinite numerical computations to be solved directly. Consider, for example, the stability problem of finding a continuously differentiable parameter-dependent symmetric matrix $P(\alpha(t))$ for the following non-autonomous system $\dot{x}(t) = A(\alpha(t))x(t)$ with $(\alpha, \dot{\alpha}) \in \Lambda \times \Omega$

$$\begin{aligned} A(\alpha(t))'P(\alpha(t)) + P(\alpha(t))A(\alpha(t)) + \sum_{i=1}^{2q}\sum_{j}^{2} \frac{\partial P(\alpha(t))}{\partial \alpha_{ij}(t)}\dot{\alpha}_{ij}(t) < 0 \\ P(\alpha(t)) > 0. \end{aligned} \tag{3.29}$$

Clearly, inequality (3.29) is a PLMI with infinite dimensional space. For every value of the varying parameter $(\alpha, \dot{\alpha}) \in \Lambda \times \Omega$, the inequality (3.29) represents ordinary LMI.

Therefore, synthesis conditions of the RGS controllers presented in this dissertation are formulated as a convex optimization problem with PLMIs constraints. After all scheduling parameters and their uncertainties (with their rates of variations) are modeled to vary within convex sets (Λ, Ω), as illustrated by the procedure given in the previous two sub-sections, the detailed proofs and derivation of the synthesis PLMIs conditions will be the core of the next two chapters for $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance, respectively.

3.2.4 PLMIs Relaxation

As mentioned, the synthesis conditions of RGS state-feedback and the static output-feedback with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance are formulated in terms of PLMIs while RGS dynamic output-feedback are formulated as PBMIs. Although such conditions represents convex optimization problem (for

PLMIs) with infinite dimensional constraints, modern robust optimization techniques considerably strengthened this framework by providing rigorous ways for dealing with parameter-dependent LMIs [46, 47, 48]. In [12], brutal force griding method developed to divide parameter space to relax the infinite-dimensional optimization problem into a finite-dimensional problem. However, using this method, the number of the resulting LMI constraints grows rapidly as the number of scheduling parameters increases. Moreover, this method only provides an approximated solution which satisfies the LMI constraints at gridding points in the parameter space. Thus, the result from finite gridding points is unreliable. On the other hand, alternative approaches have been actively sought to turn PLMIs into a standard LMI problem by constructing their relaxation forms. In the case of LPV systems depending affinely on the scheduling parameter, vertex method was considered in [49] to determine constant Lyapunov functions satisfying affine parameter-dependent LMI. The solution is exact but it prevents the possibility of using parameter-dependent Lyapunov functions which lead to a conservative solution. In [50], convex covering techniques were applied to PLMIs to obtain parameter-dependent solutions. However, these methods often require large division numbers to achieve accurate results. Multi-convexity properties was imposed in [51] to provide a finite set of LMIs to solve PLMIs problems. More recently, many powerful theoretical and computational tools have been developed and applied successfully (by several researchers independently) to relax PLMIs problems into finite set of LMIs. Thus, the notion of Sum-Of-Square (SOS)-convexity has been proposed as a tractable approach for convexfing PLMIs based on SOS-decomposition [52]. Slack Variable (SV) approach [53], dilated LMI approach [54], and coefficient check approach using Pólya's theorem [41] are other powerful techniques that applied successfully in literature to relax PLMIs. The promising results of these relaxation methods overcomes the difficulty of solving PLMIs to synthesize gain-scheduling controllers as demonstrated in [55].

Therefore, the relaxation approach that was developed in [39] is adopted in this dissertation to relax the PLMIs conditions in Chapter 4, Chapter 5, and Chapter 6 since it supports PLMIs that depends on multi-simplex parameters. In [41], Oliveira *et al.* developed a systematic procedure to construct a family of finite-dimensional LMI relaxations for uncertain LTI systems in polytopic domain modeled by unit-simplex. Then, they extend the work in [39] for multi-simplex domain with time-varying parameters. For completeness, PLMIs relaxation procedure for the stability test of the time-varying system in (3.29) will be illustrated here. For convenience, the PLMIs are rewritten again here

$$A(\alpha(t))'P(\alpha(t))+P(\alpha(t))A(\alpha(t))+\sum_{i=1}^{2q}\sum_{j}^{2}\frac{\partial P(\alpha(t))}{\partial\alpha_{ij}(t)}\dot{\alpha}_{ij}(t)<0, \tag{3.30}$$

$$P(\alpha(t))>0, \tag{3.31}$$

with $(\alpha,\dot{\alpha})\in\Lambda\times\Omega$. For Λ-homogeneous matrices $P(\alpha(t))$ and $A(\alpha(t))$ of partial degrees $g=(g_1,g_2,\ldots,g_q)$ and $r=(r_1,r_2,\ldots,r_m)$ respectively, the total degree of the first two terms of inequality (3.30) is $\bar{g}=(g_1+r_1,g_2+r_2,\ldots,g_q+r_q)$. Thus, the main task is to homogenize accordingly the third term, i.e.

$$\sum_{i=1}^{2q}\sum_{j}^{2}\frac{\partial P(\alpha(t))}{\partial\alpha_{ij}(t)}\dot{\alpha}_{ij}(t).$$

The general expression for the derivative of the Lyapunov matrix $P(\alpha(t))$ with respect to the i-th component of the multi-simplex, $i=1,2,\ldots,q$ and then with respect to its j-th component, $j=1,2$ is given by

$$\begin{aligned}\frac{\partial P(\alpha(t))}{\partial\alpha_{ij}(t)} &= \sum_{k\in\mathscr{Q}(N,g)} k_{ij}\alpha_1^{k_1}\cdots\alpha_{i1}^{k_{i1}}\cdots\alpha_{ij}^{k_{ij}-1}\cdots\alpha_{i2}^{k_{i2}}\cdots\alpha_q^{k_q}P_k\\ &= \sum_{k\in\mathscr{Q}(N,g-e_{i|q})}\alpha^k\left((k+e_{i|q}\otimes e_{j|2})_{ij}P_{k+e_{i|q}\otimes e_{j|2}}\right)\end{aligned}$$

where by definition $e_{i|q}$ is the vector of dimension q with zero components, except 1 in the i-th position. To fit (on α) with the partial degrees $\bar{g}$, the following homogenization is necessary

$$\sum_{i=1}^{q}(\alpha_{i1}+\cdots+\alpha_{i2})^{r_i+1}\sum_{j=1}^{2}\frac{\partial P(\alpha(t))}{\partial \alpha_{ij}(t)} =$$

$$\sum_{i=1}^{q}\sum_{j=1}^{2}\sum_{k\in\mathscr{Q}(N,g-e_{i|q})}\alpha^{k}\left(\sum_{\substack{\hat{k}\in\mathscr{Q}(r+e_{i|q})\\ \hat{k}\preceq k}}\frac{(r_i+1)!}{\pi(\hat{k}_i)}\left((k-\hat{k}+e_{i|q}\otimes e_{j|2})_{ij}P_{k-\hat{k}+e_{i|q}\otimes e_{j|2}}\right)\right) \tag{3.32}$$

where $\pi(k_i)=(k_{i1}!)(k_{i2}!)$. Now the third term of (3.30) should be homogenized to become multi-affine on η. This is done as follows

$$\prod_{\substack{p=1\\ p\neq i}}^{q}(\eta_{p1}+\eta_{p2})\sum_{\ell=1}^{2}\eta_{i\ell}H_i(j,\ell)=\sum_{p_1=1}^{2}\cdots\sum_{p_i=1}^{2}\cdots\sum_{p_q=1}^{2}\eta_{1p_1}\cdots\eta_{ip_i}\cdots\eta_{qp_q}H_i(j,p_i). \tag{3.33}$$

Considering (3.32) and (3.33), the third term in the left-hand side of (3.30) can be equivalently written as

$$\sum_{i=1}^{q}\sum_{j=1}^{2}\frac{\partial P(\alpha)}{\partial \alpha_{ij}}\sum_{\ell=1}^{2}\eta_{i\ell}H_i(j,\ell)=\sum_{p_1=1}^{2}\cdots\sum_{p_i=1}^{2}\cdots\sum_{p_q=1}^{2}\eta_{1p_1}\cdots\eta_{ip_i}\cdots\eta_{qp_q}\times$$

$$\left(\sum_{k\in\mathscr{Q}(N,g+r)}\alpha^{k}\ \sum_{i=1}^{q}\sum_{j=1}^{2}\sum_{\substack{\hat{k}\in\mathscr{Q}(r+e_{i|q})\\ \hat{k}\preceq k}}\frac{(r_i+1)!}{\pi(\hat{k}_i)}\left((k-\hat{k}+e_{i|q}\otimes e_{j|2})_{ij}P_{k-\hat{k}+e_{i|q}\otimes e_{j|2}}\right)H_i(j,p_i)\right).$$

Now, note that

$$\prod_{p=1}^{q}(\eta_{p_1}+\eta_{p2})\left(A(\alpha)'P(\alpha)+P(\alpha)A(\alpha)\right)=$$

$$\sum_{p_1=1}^{2}\cdots\sum_{p_i=1}^{2}\cdots\sum_{p_q=1}^{2}\eta_{1p_1}\cdots\eta_{ip_i}\cdots\eta_{qp_q}\left(A(\alpha)'P(\alpha)+P(\alpha)A(\alpha)\right), \tag{3.34}$$

and finally, (3.30) can be tested since all terms have the same partial degrees for both α and η. The next lemma presents LMI relaxations for the robust stability (analysis) problem of the parameter-varying matrix $A(\alpha(t))$ for any pair $(\alpha,\dot{\alpha})\in\Lambda\times\Omega$.

Lemma 3.1. *[39] Let Λ be a multi-simplex of dimension $N = (2, 2, \ldots, 2)$. The Λ-homogeneous polynomial matrix $A(\alpha(t))$ of partial degrees $r = (r_1, r_2, \ldots, r_q)$ is robustly stable for any pair $(\alpha, \dot{\alpha}) \in \Lambda \times \Omega$, if there exists $g = (g_1, g_2, \ldots, g_q)$, $k \in \mathscr{Q}(N, g)$ and matrices $P_k = P_k' \in \mathbb{R}^{n \times n}$ such that for all $(i_1, i_2, \ldots, i_q) \in \{1,2\} \times \{1,2\} \times \cdots \times \{1,2\}$ the following LMIs are satisfied*

$$\begin{aligned} & P_k > \mathbf{0}_n, \\ & \Phi_k = \sum_{\substack{\tilde{k} \in \mathscr{Q}(N,r) \\ \tilde{k} \preceq k}} \left(A_{\tilde{k}}' P_{k-\tilde{k}} + P_{k-\tilde{k}} A_{\tilde{k}} \right) + \Psi_k < \mathbf{0}_n, \quad \forall k \in \mathscr{Q}(N, g+r) \end{aligned}$$

where

$$\Psi_k = \sum_{i=1}^{q} \sum_{j=1}^{2} \sum_{\substack{\hat{k} \in \mathscr{Q}(r+e_{i|q}) \\ \hat{k} \preceq k}} \frac{(r_i+1)!}{\pi(\hat{k}_i)} \left((k - \hat{k} + e_{i|q} \otimes e_{j|2})_{ij} P_{k-\hat{k}+e_{i|q} \otimes e_{j|2}} \right) H_i(j, p_i).$$

Proof. See [39]. □

It is clear from the previous example that the algebraic manipulation of the relaxed conditions requires a priori knowledge on the formation law of the monomials, which depends on the number of scheduling parameters and on the degree of the parametrization of the Lyapunov matrix $P(\alpha(t))$. As can be easily observed, the previous procedure deals with products between two parameter-dependent matrices. However, When the LMIs to be solved are more complex and have products involving three or more parameter-dependent matrices, the rules to compose the monomials become much more complicated. Moreover, each PLMI demands manipulation of different polynomials. Such task, as well as programming the resulting LMIs, is sophisticated, time-demanding and can be a source of programming errors. Therefore, a specialized RObust LMI Parser (ROLMIP)[2] [56] has been recently developed as a tool to perform such manipulation and LMI relaxation of the PLMIs. This package works jointly with the LMI parser YALMIP [57] and the LMI solver SeDuMi [58] that is used in this dissertation to obtain the optimal solution of

[2]Available for download at `http://www.dt.fee.unicamp.br/~agulhari/rolmip/rolmip.htm`

the convex optimization problems of the synthesis conditions that will be given in the next three chapters.

3.2.5 PBMI Algorithm

Since only the synthesis conditions of RGS DOF controller are formulated as PBMIs, it will be covered in Chapter 5.

3.2.6 Inverse Transformation

Once a feasible solution is obtained using the PLMIs conditions or the PBMI algorithm, inverse transformation is required to map the solution from multi-simplex domain $\tilde{\alpha}$ into original parameter space $\tilde{\theta}$. Since real-time implementation of the gain-scheduling controllers follows the structure of (3.18) that can be generalizes for a q-scheduling parameters as

$$Z(\tilde{\theta}(t)) = Z_0 + \sum_{i=1}^{q} \tilde{\theta}_i(t) Z_i. \tag{3.35}$$

In other words, the key is to calculate Z_j, for $j = 0, 1, 2, \cdots, q$, that required in real-time for controller implementations.

- One scheduling parameter:

$$\mathscr{Z}_{1,1} = Z_0 + (\bar{\theta}_1 + \bar{\delta}_1) Z_1, \tag{3.36}$$

$$\mathscr{Z}_{1,2} = Z_0 + (\bar{\theta}_1 - \bar{\delta}_1) Z_1, \tag{3.37}$$

$$\mathscr{Z}_{2,1} = Z_0 + (-\bar{\theta}_1 + \bar{\delta}_1) Z_1, \tag{3.38}$$

$$\mathscr{Z}_{2,2} = Z_0 + (-\bar{\theta}_1 - \bar{\delta}_1) Z_1. \tag{3.39}$$

  To obtain Z_0 and $Z_1$from the multi-simplex variables (after a feasible solution is found), adding (3.36), (3.37), (3.38) and (3.39) to obtain

$$Z_0 = \frac{1}{4}[\mathscr{Z}_{1,1} + \mathscr{Z}_{1,2} + \mathscr{Z}_{2,1} + \mathscr{Z}_{2,2}].$$

Then, $(3.36)+(3.37)-(3.38)-(3.39)$ to obtain

$$Z_1 = \frac{1}{4\bar{\theta}}[\mathscr{Z}_{1,1} + \mathscr{Z}_{1,2} - \mathscr{Z}_{2,1} - \mathscr{Z}_{2,2}].$$

- Two scheduling parameters: Equation (3.35) can be written as

$$Z(\tilde{\theta}(t)) = Z_0 + \tilde{\theta}_1(t)Z_1 + \tilde{\theta}_2(t)Z_2.$$

To obtain Z_0, Z_1, and Z_2 from the multi-simplex variables (after a feasible solution is found), adding all equations in (3.23) to obtain

$$\begin{aligned} Z_0 = \frac{1}{16}[&\mathscr{Z}_{1,1,1,1} + \mathscr{Z}_{1,1,1,2} + \mathscr{Z}_{1,1,2,1} + \mathscr{Z}_{1,1,2,2} + \mathscr{Z}_{1,2,1,1} \\ &+ \mathscr{Z}_{1,2,1,2} + \mathscr{Z}_{1,2,2,1} + \mathscr{Z}_{1,2,2,2} + \mathscr{Z}_{2,1,1,1} + \mathscr{Z}_{2,1,1,2} \\ &+ \mathscr{Z}_{2,1,2,1} + \mathscr{Z}_{2,1,2,2} + \mathscr{Z}_{2,2,1,1} + \mathscr{Z}_{2,2,1,2} + \mathscr{Z}_{2,2,2,1} + \mathscr{Z}_{2,2,2,2}]. \end{aligned}$$

Then, (the 1st 8 equations)-minus-(the last 8 equations) to obtain

$$\begin{aligned} Z_1 = \frac{1}{16\bar{\theta}_1}[&\mathscr{Z}_{1,1,1,1} + \mathscr{Z}_{1,1,1,2} + \mathscr{Z}_{1,1,2,1} + \mathscr{Z}_{1,1,2,2} + \mathscr{Z}_{1,2,1,1} \\ &+ \mathscr{Z}_{1,2,1,2} + \mathscr{Z}_{1,2,2,1} + \mathscr{Z}_{1,2,2,2} - \mathscr{Z}_{2,1,1,1} - \mathscr{Z}_{2,1,1,2} \\ &- \mathscr{Z}_{2,1,2,1} - \mathscr{Z}_{2,1,2,2} - \mathscr{Z}_{2,2,1,1} - \mathscr{Z}_{2,2,1,2} - \mathscr{Z}_{2,2,2,1} - \mathscr{Z}_{2,2,2,2}]. \end{aligned}$$

Then, (the 1st 2 eqs)-minus-(the 2nd 2 eqs)-plus-(the 3rd 2 eqs)-minus-(the 4th 2 eqs)-plus-... to obtain

$$\begin{aligned} Z_2 = \frac{1}{16\bar{\theta}_2}[&\mathscr{Z}_{1,1,1,1} + \mathscr{Z}_{1,1,1,2} - \mathscr{Z}_{1,1,2,1} - \mathscr{Z}_{1,1,2,2} + \mathscr{Z}_{1,2,1,1} \\ &+ \mathscr{Z}_{1,2,1,2} - \mathscr{Z}_{1,2,2,1} - \mathscr{Z}_{1,2,2,2} + \mathscr{Z}_{2,1,1,1} + \mathscr{Z}_{2,1,1,2} \\ &- \mathscr{Z}_{2,1,2,1} - \mathscr{Z}_{2,1,2,2} + \mathscr{Z}_{2,2,1,1} + \mathscr{Z}_{2,2,1,2} - \mathscr{Z}_{2,2,2,1} - \mathscr{Z}_{2,2,2,2}]. \end{aligned}$$

- Any number of scheduling parameters $i = 1, 2, \cdots, q$:

$$\boxed{Z_0 = \frac{1}{2^{2q}} \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \cdots \sum_{j_q=1}^{2} \sum_{k_i=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_q=1}^{2} \mathscr{Z}_{j_1,j_2,\cdots,j_q,k_1,k_2,\cdots,k_q}.} \tag{3.40}$$

$$\boxed{Z_i = \frac{1}{2^{2q}\bar{\theta}_i} \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \cdots \sum_{j_q=1}^{2} \sum_{k_i=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_q=1}^{2} (-1)^{j_i+i} \mathscr{Z}_{j_1,j_2,\cdots,j_q,k_1,k_2,\cdots,k_q}.} \tag{3.41}$$

3.3 Summary

This chapter presented mathematical formulation of the RGS controller synthesis problem with uncertain scheduling parameters. Then, a systematic solution procedure was given as well. The next three chapters deal with the synthesis conditions of RGS SF, DOF, and SOF controllers with guaranteed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance, respectively. Based on the procedure presented in this chapter, numerical examples, simulations, and comparisons with other work from literature are given in the next three chapters to demonstrate the effectiveness of the developed conditions.

CHAPTER 4

RGS STATE-FEEDBACK CONTROL

In this chapter, characterizations of synthesis conditions for RGS State-Feedback (SF) control with guaranteed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance subject to noisy scheduling parameters are developed. The organization of this chapter is as follows. First, RGS synthesis problem formulation with guaranteed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance are presented and then, PLMIs conditions for the synthesis of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ controllers are developed. Numerical examples, simulations, and comparisons with other approaches from literature is given to demonstrate the effectiveness of the developed conditions. Finally, a short summary is given in the last section.

4.1 SF Synthesis Problem

Consider the following LPV system

$$\mathscr{S}_{OL} := \begin{cases} \dot{x}(t) & = A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t), \\ z(t) & = C_z(\theta(t))x(t) + D_{zu}(\theta(t))u(t) + D_{zw}(\theta(t))w(t), \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{nu}$ is the control input, $w(t) \in \mathbb{R}^{nw}$ is the disturbance input, and $z(t) \in \mathbb{R}^{nz}$ is the controlled output. The system matrices have the following compatible dimensions $A(\theta(t)) \in \mathbb{R}^{n\times n}$, $B_u(\theta(t)) \in \mathbb{R}^{n\times nu}$, $B_w(\theta(t)) \in \mathbb{R}^{n\times nw}$, $C_z(\theta(t)) \in \mathbb{R}^{nz\times n}$, $D_{zu}(\theta(t)) \in \mathbb{R}^{nz\times nu}$, and $D_{zw}(\theta(t)) \in \mathbb{R}^{nz\times nw}$.

The goal is to synthesize a RGS state-feedback controller of the form

$$u(t) = K(\tilde{\theta}(t))x(t), \tag{4.2}$$

to stabilize the closed-loop system (see Figure 3.2)

$$\mathscr{S}_{CL} := \begin{cases} \dot{x}(t) & = \mathscr{A}(\theta(t), \tilde{\theta}(t))x(t) + \mathscr{B}(\theta(t), \tilde{\theta}(t))w(t), \\ z(t) & = \mathscr{C}(\theta(t), \tilde{\theta}(t))x(t) + \mathscr{D}(\theta(t), \tilde{\theta}(t))w(t). \end{cases} \tag{4.3}$$

where

$$\begin{aligned}
\mathscr{A}(\theta(t),\tilde{\theta}(t)) &= A(\theta(t)) + B_u(\theta(t))K(\tilde{\theta}(t)),\\
\mathscr{B}(\theta(t),\tilde{\theta}(t)) &= B_w(\theta(t)),\\
\mathscr{C}(\theta(t),\tilde{\theta}(t)) &= C_z(\theta(t)) + D_{zu}(\theta(t))K(\tilde{\theta}(t)),\\
\mathscr{D}(\theta(t),\tilde{\theta}(t)) &= D_{zw}(\theta(t)).
\end{aligned} \tag{4.4}$$

Furthermore, performance bounds in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ should be guaranteed as well.

The controller matrix in (4.2) is assumed to have affine parametrization with respect to the Measured Scheduling Parameters (MSPs). In other words, the controller matrix, $K(\tilde{\theta}(t))$, is parameterized as

$$K(\tilde{\theta}(t)) = K_0 + \sum_{i=1}^{q} \tilde{\theta}_i(t) K_i. \tag{4.5}$$

Therefore, the goal is to obtain the controller coefficients K_i for $i = 0, 1, 2, \cdots, q$, to implement the RGS controller by using only the MSPs $\tilde{\theta}_i$.

4.2 RGS $\mathscr{H}_2$ Control

Since LPV systems is the topic of this dissertation, the term "$\mathscr{H}_2$ norm" for LPV systems should be treated with care since the standard $\mathscr{H}_2$ control theory is originally developed for LTI systems. However, several definitions of $\mathscr{H}_2$ norm for LPV systems have been proposed in literature [59]. These definitions can essentially be divided into the following two groups,

- Deterministic interpretation where the exogenous input is modeled as unknown disturbance belongs to a bounded $\mathscr{L}_2$ energy set [60], and
- Stochastic interpretation based on the covariance of the output due to Gaussian white noise [61].

It is well-known that these definitions coincide for LTI systems, but not for the LPV case. In pursuing the extension of the second definition to the LPV case, care must be exercised since the output to stationary noise may no longer be stationary. This leads to two different interpretations based on whether the *average* or *worst-case* output variance are considered [59]. In this dissertation, the stationary white noise interpretation is used and the worst-case output variance is defined as the squared $\mathscr{H}_2$ norm for LPV systems. Since this norm can be calculated in terms of the Lyapunov matrix, such a choice of $\mathscr{H}_2$ norm interpretation appears to be most appropriate for the synthesis approach developed in this dissertation.

Problem 4.1. *Suppose that the scheduling parameters $\theta(t)$ are provided as $\tilde{\theta}(t)$ with uncertainty $\delta(t)$ as defined in* (3.3). *Suppose further that $\mathscr{D}(\theta(t), \tilde{\theta}(t)) = 0$ in* (4.3). *Converting all the open-loop system matrices and synthesis variables to the multi-simplex variables α or $\tilde{\alpha}$ instead of θ and $\tilde{\theta}$, respectively, using* (3.15) *and* (3.16). *For a given positive scalar ν, find a RGS state-feedback controller in the form of* (4.2) *to stabilize the closed-loop system* (4.3) *for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ and satisfy*

$$\sup_{(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega} \boldsymbol{E}\left\{\int_0^T z(t)' z(t) dt\right\} < \nu^2, \tag{4.6}$$

for the disturbance input $w(t)$ given by

$$w(t) = w_0 \delta(t)$$

where $\delta(t)$ is the Dirac's delta function and w_0 is a random variable satisfying

$$\boldsymbol{E}\left\{w_0 w_0'\right\} = I_k$$

and $\boldsymbol{E}\{\cdot\}$ denotes the mathematical expectation.

Lemma 4.1. *[62] Let $\mathscr{D}(\theta(t), \tilde{\theta}(t)) = 0$ in* (4.3). *For a given positive scalar ν, if there exist a continuously differentiable positive-definite matrix $P(\alpha) = P(\alpha)' \in \mathbb{R}^{n \times n}$ and parameter-dependent matrix $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z \times n_z}$ such that the following PLMIs are satisfied*

$$\begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})P(\alpha)+P(\alpha)\mathscr{A}(\alpha,\tilde{\alpha})'-\dot{P}(\alpha) & \star \\ \mathscr{B}(\alpha,\tilde{\alpha})' & -I \end{bmatrix} < \mathbf{0}, \tag{4.7}$$

$$\begin{bmatrix} P(\alpha) & \star \\ \mathscr{C}(\alpha,\tilde{\alpha})P(\alpha) & W(\alpha) \end{bmatrix} > \mathbf{0}, \tag{4.8}$$

$$trace(W(\alpha)) < \nu^2, \tag{4.9}$$

then the closed-loop system defined in (4.3) *is asymptotically stable for any pairs* $(\tilde{\alpha},\dot{\tilde{\alpha}}) \in \Lambda \times \Omega$ *and* (4.6) *is satisfied, where* $\mathscr{A}(\alpha,\tilde{\alpha})$, $\mathscr{B}(\alpha,\tilde{\alpha})$, *and* $\mathscr{C}(\alpha,\tilde{\alpha})$ *are the closed system matrices defined in* (4.3) *with* θ *and* $\tilde{\theta}$ *replaced by* α *and* $\tilde{\alpha}$ *using* (3.15) *and* (3.16).

In view of Lemma 4.1 and equation (4.6), ν represents an upper bound of the $\mathscr{H}_2$ norm of the closed-loop system (4.3). Before presenting PLMIs synthesis conditions Finsler's lemma will be given since it is used in the derivation of the synthesis conditions [63, 64].

Lemma 4.2. *Let* $\zeta \in \mathbb{R}^n$, $\Psi \in \mathbb{R}^{n\times n}$ *and* $V \in \mathbb{R}^{m\times n}$ *with* $rank(V) < n$, $V^{\perp}$ *such that* $VV^{\perp} = 0$. *Then, the following conditions are equivalent:*

i) $\zeta'\Psi\zeta < 0, \forall\, \zeta \neq 0\ : V\zeta = 0$;

ii) $V^{\perp\prime}\,\Psi\, V^{\perp} < 0$;

iii) $\exists\, \mu \in \mathbb{R} : \Psi - \mu V'V < 0$;

iv) $\exists\, X \in \mathbb{R}^{n\times m} : \Psi + XV + V'X' < 0$.

Theorem 4.1. *Let* $\mathscr{D}(\theta(t),\tilde{\theta}(t)) = 0$ *in* (4.3). *Given a scalar* $\nu > 0$ *and a sufficiently small scalar* $\varepsilon > 0$. *If there exist a continuously differentiable parameter-dependent matrix* $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, *parameter-dependent matrices* $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z\times n_z}$, $Z(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, $G(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ *for* *any pair* $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ *such that the following PLMIs are satisfied*

$$\begin{bmatrix} A(\alpha)G(\tilde{\alpha}) + B_u(\alpha)Z(\tilde{\alpha}) + (\bullet)' - \dot{P}(\alpha) & \star & \star \\ P(\alpha) - G(\tilde{\alpha}) + \varepsilon(A(\alpha)G(\tilde{\alpha}) + B_u(\alpha)Z(\tilde{\alpha}))' & -\varepsilon(G(\tilde{\alpha}) + G(\tilde{\alpha})') & \star \\ B_w(\alpha)' & \mathbf{0}_{n_w \times n} & -I_{n_w} \end{bmatrix} < \mathbf{0}_{2n+n_w}, \tag{4.10}$$

$$\begin{bmatrix} G(\tilde{\alpha}) + G(\tilde{\alpha})' - P(\alpha) & \star \\ C_z(\alpha)G(\tilde{\alpha}) + D_{zu}(\alpha)Z(\tilde{\alpha}) & W(\alpha) \end{bmatrix} > \mathbf{0}_{n+n_z}, \tag{4.11}$$

$$trace(W(\alpha)) < \nu^2. \tag{4.12}$$

Then, the gain-scheduling controller

$$K(\tilde{\alpha}) = Z(\tilde{\alpha})G(\tilde{\alpha})^{-1}, \tag{4.13}$$

stabilizes the closed-loop system with guaranteed $\mathscr{H}_2$ *performance bound* ν *satisfying* (4.6).

Proof. Using slack variable approach, additional optimization variable $U(\tilde{\alpha})$ can be introduced via Finsler's lemma to decouple the dynamic matrix $A(\alpha)$ from Lyapunov matrix $P(\alpha)$, hence using *iv*) in Lemma 4.2, inequality (4.7) can be written

$$\Xi(\alpha) + U(\tilde{\alpha})V(\alpha) + V(\alpha)'U(\tilde{\alpha})' < \mathbf{0}, \tag{4.14}$$

where,

$$\Xi(\alpha) := \begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & \mathbf{0} \\ P(\alpha) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}, \; U(\tilde{\alpha}) := \begin{bmatrix} G(\tilde{\alpha})' & \mathbf{0} \\ Y(\tilde{\alpha})' & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}, \; V(\alpha) := \begin{bmatrix} A(\alpha)' & -I & \mathbf{0} \\ B_w(\alpha)' & \mathbf{0} & -I \end{bmatrix},$$

such that $V(\alpha)^{\perp'}\,\Xi(\alpha)\,V(\alpha)^{\perp} < \mathbf{0}$, with $V(\alpha)^{\perp'} = [I \quad A(\alpha) \quad B_w(\alpha)]$. Therefore, substituting these relations into (4.14),

$$\begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & \mathbf{0} \\ P(\alpha) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} G(\tilde{\alpha})' & \mathbf{0} \\ Y(\tilde{\alpha})' & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} A(\alpha)' & -I & \mathbf{0} \\ B_w(\alpha)' & \mathbf{0} & -I \end{bmatrix}$$
$$+ \begin{bmatrix} A(\alpha) & B_w(\alpha) \\ -I & \mathbf{0} \\ \mathbf{0} & -I \end{bmatrix} \begin{bmatrix} G(\tilde{\alpha}) & Y(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} < \mathbf{0}. \quad (4.15)$$

At this end, it is important to impose particular structure to the slack variable $U(\tilde{\alpha})$ to maintain convex parametrization. Therefore, setting $Y(\tilde{\alpha}) = \varepsilon G(\tilde{\alpha})$ is sufficient to keep convexity of (4.15), where ε is a scalar used as an extra degree of freedom to perform line search for performance improvement [65]. Substituting for $A(\alpha)$ as closed-loop matrix $\mathscr{A}(\alpha,\tilde{\alpha})$ given by (4.4) with α and $\tilde{\alpha}$ replacing θ and $\tilde{\theta}$, respectively, and the change of variable $Z(\tilde{\alpha}) = K(\tilde{\alpha})G(\tilde{\alpha})$ yields

$$\begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & \mathbf{0} \\ P(\alpha) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} G(\tilde{\alpha})'\mathscr{A}(\alpha,\tilde{\alpha})' & -G(\tilde{\alpha})' & \mathbf{0} \\ \varepsilon G(\tilde{\alpha})'\mathscr{A}(\alpha,\tilde{\alpha})' & -\varepsilon G(\tilde{\alpha})' & \mathbf{0} \\ B_w(\alpha)' & \mathbf{0} & -I \end{bmatrix}$$
$$+ \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & \varepsilon\mathscr{A}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & B_w(\alpha) \\ -G(\tilde{\alpha}) & -\varepsilon G(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -I \end{bmatrix} < \mathbf{0},$$

that directly lead to (4.10). Multiplying (4.11) from left by $[\mathscr{C}(\alpha,\tilde{\alpha}) \quad -I]$ and by its transpose from right with $\mathscr{C}(\alpha,\tilde{\alpha})$ in (4.4) to obtain

$$W(\alpha) > \mathscr{C}(\alpha,\tilde{\alpha})P(\alpha)\mathscr{C}(\alpha,\tilde{\alpha})'$$

with Schur complement, (4.8) can be recovered. The PLMI (4.12) ensures that ν is the guaranteed cost (upper bound) of the $\mathscr{H}_2$ norm of the closed-loop system. □

4.3 RGS $\mathscr{H}_\infty$ Control

Problem 4.2. *Suppose that the scheduling parameters $\theta(t)$ are provided as $\tilde{\theta}(t)$ with uncertainty $\delta(t)$. Converting all the open-loop system matrices and synthesis variables to the multi-simplex variables α or $\tilde{\alpha}$ instead of θ and $\tilde{\theta}$, respectively, using* (3.15) *and* (3.16)*. For a given positive scalar γ_∞, find a RGS state-feedback controller in the form of* (4.2) *to stabilize the closed-loop system* (4.3) *for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ and satisfy*

$$\sup_{(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t))\in\Lambda\times\Omega} \quad \sup_{w\in\mathscr{L}_2, w\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma_\infty. \tag{4.16}$$

The next lemma will be used in the derivations of the PLMIs synthesis conditions.

Lemma 4.3. *[66] If there exists a continuously differentiable parameter-dependent symmetric positive-definite matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ such that the following PLMI satisfied*

$$\begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})P(\alpha) + (\bullet)' - \dot{P}(\alpha) & \star & \star \\ \mathscr{C}(\alpha,\tilde{\alpha})P(\alpha) & -I_{n_z} & \star \\ \mathscr{B}(\alpha,\tilde{\alpha})' & \mathscr{D}(\alpha,\tilde{\alpha})' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0}, \tag{4.17}$$

the closed-loop system (4.3) *is asymptotically stable with* (4.16) *satisfied, where $\mathscr{A}(\alpha,\tilde{\alpha})$, $\mathscr{B}(\alpha,\tilde{\alpha})$, $\mathscr{C}(\alpha,\tilde{\alpha})$ and $\mathscr{D}(\alpha,\tilde{\alpha})$ are the closed system matrices defined in* (4.3) *with θ and $\tilde{\theta}$ replaced by α and $\tilde{\alpha}$ using* (3.15) *and* (3.16)*.*

Theorem 4.2. *Given a scalar $\gamma_\infty > 0$ and a sufficiently small scalar $\varepsilon > 0$. If there exist a continuously differentiable parameter-dependent matrix $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, parameter-dependent matrices $Z(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, $G(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ such that the following PLMI satisfied*

$$\begin{bmatrix} A(\alpha)G(\tilde{\alpha}) + B_u(\alpha)Z(\tilde{\alpha}) + (\bullet)' - \dot{P}(\alpha) & \star & \star & \star \\ P(\alpha) - G(\tilde{\alpha}) + \varepsilon(A(\alpha)G(\tilde{\alpha}) + B_u(\alpha)Z(\tilde{\alpha}))' & -\varepsilon(G(\tilde{\alpha}) + G(\tilde{\alpha})') & \star & \star \\ C(\alpha)G(\tilde{\alpha}) + D_u(\alpha)Z(\tilde{\alpha}) & \varepsilon C(\alpha)G(\tilde{\alpha}) + \varepsilon D_u(\alpha)Z(\tilde{\alpha}) & -I_{n_z} & \star \\ B_w(\alpha)' & \mathbf{0}_{n_w\times n} & D_w(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0}. \tag{4.18}$$

Then, the gain-scheduling controller

$$K(\tilde{\alpha}) = Z(\tilde{\alpha})G(\tilde{\alpha})^{-1},$$

stabilizes the closed-loop system with guaranteed $\mathscr{H}_\infty$ performance bound γ_∞ satisfying (4.16).

Proof. Additional slack variable $U(\tilde{\alpha})$ can be introduced to inequality (4.17) using Lemma 4.2 to decouple the dynamic matrix from Lyapunov matrix $P(\alpha)$. With $\mathscr{A}(\alpha,\tilde{\alpha})$ and $\mathscr{C}(\alpha,\tilde{\alpha})$ are the closed-loop matrices, inequality (4.17) can be written

$$\Xi(\alpha) + \Phi(\tilde{\alpha}) < 0$$

with

$$\Xi(\alpha) = diag\left(\begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) \\ P(\alpha) & 0 \end{bmatrix}, \begin{bmatrix} -I_{n_z} & D_w(\alpha) \\ D_w(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix}\right)$$
$$= \begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & 0 & 0 \\ P(\alpha) & 0 & 0 & 0 \\ 0 & 0 & -I_{n_z} & D_w(\alpha) \\ 0 & 0 & D_w(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} \tag{4.19}$$

and

$$\Phi(\tilde{\alpha}) = U(\tilde{\alpha})V(\alpha) + V(\alpha)'U(\tilde{\alpha})' \tag{4.20}$$

where

$$U(\tilde{\alpha}) = \begin{bmatrix} G(\tilde{\alpha})' & I \\ \varepsilon G(\tilde{\alpha})' & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad V(\alpha) = \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})' & -I & \mathscr{C}(\alpha,\tilde{\alpha})' & 0 \\ 0 & 0 & 0 & B_w(\alpha) \end{bmatrix}$$

such that $V(\alpha)^{\perp'}\Xi(\alpha)V(\alpha)^{\perp} < 0$, with

$$V(\alpha)^{\perp} = \begin{bmatrix} I & 0 \\ \mathscr{A}(\alpha,\tilde{\alpha})' & \mathscr{C}(\alpha,\tilde{\alpha})' \\ 0 & I \\ 0 & 0 \end{bmatrix}.$$

Substituting $U(\tilde{\alpha})$ and $V(\alpha)$ in (4.20),

$$\Phi(\tilde{\alpha}) = U(\tilde{\alpha})V(\alpha) + V(\alpha)'U(\tilde{\alpha})' =$$

$$\begin{bmatrix} G(\tilde{\alpha})' & I \\ \varepsilon G(\tilde{\alpha})' & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})' & -I & \mathscr{C}(\alpha,\tilde{\alpha})' & 0 \\ 0 & 0 & 0 & B_w(\alpha) \end{bmatrix}$$

$$+ \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha}) & 0 \\ -I & 0 \\ \mathscr{C}(\alpha,\tilde{\alpha}) & 0 \\ 0 & B_w(\alpha)' \end{bmatrix} \begin{bmatrix} G(\tilde{\alpha}) & \varepsilon G(\tilde{\alpha}) & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix},$$

$$\Phi(\tilde{\alpha}) = \begin{bmatrix} G(\tilde{\alpha})'\mathscr{A}(\alpha,\tilde{\alpha})' & -G(\tilde{\alpha})' & G(\tilde{\alpha})'\mathscr{C}(\alpha,\tilde{\alpha})' & B_w(\alpha) \\ \varepsilon G(\tilde{\alpha})'\mathscr{A}(\alpha,\tilde{\alpha})' & -\varepsilon G(\tilde{\alpha})' & \varepsilon G(\tilde{\alpha})'\mathscr{C}(\alpha,\tilde{\alpha})' & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & \varepsilon\mathscr{A}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & 0 & 0 \\ -G(\tilde{\alpha}) & -\varepsilon G(\tilde{\alpha}) & 0 & 0 \\ \mathscr{C}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & \varepsilon\mathscr{C}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & 0 & 0 \\ B_w(\alpha)' & 0 & 0 & 0 \end{bmatrix},$$

$$
\Phi(\tilde{\alpha}) = \begin{bmatrix} G(\tilde{\alpha})'\mathscr{A}(\alpha,\tilde{\alpha})' + \mathscr{A}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & -G(\tilde{\alpha})' + \varepsilon\mathscr{A}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & G(\tilde{\alpha})'\mathscr{C}(\alpha,\tilde{\alpha})' & B_w(\alpha) \\ \varepsilon G(\tilde{\alpha})'\mathscr{A}(\alpha,\tilde{\alpha})' - G(\tilde{\alpha}) & -\varepsilon G(\tilde{\alpha})' - \varepsilon G(\tilde{\alpha}) & \varepsilon G(\tilde{\alpha})'\mathscr{C}(\alpha,\tilde{\alpha})' & 0 \\ \mathscr{C}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & \varepsilon\mathscr{C}(\alpha,\tilde{\alpha})G(\tilde{\alpha}) & 0 & 0 \\ B_w(\alpha)' & 0 & 0 & 0 \end{bmatrix}, \quad (4.21)
$$

then, adding (4.19) and (4.21) and substituting for closed-loop matrices with the change of variables $K(\tilde{\alpha}) = Z(\tilde{\alpha})G(\tilde{\alpha})^{-1}$, inequality (4.18) can be obtained.

□

4.4 Extension to Unmeasurable Parameters

Theorem 4.1 and Theorem 4.2 address the design problem of RGS state-feedback controllers in which all the time-varying parameters of the plant are measurable on-line. However, there are many cases where some of the parameters are available for real-time measurement and others are not. In such a case, the unmeasurable parameters can be treated as plant uncertainties, then the synthesized gain-scheduling controller will be independent of these parameters and the resulting closed-loop system should be robust against these parameters as well.

To illustrate, the following notation should be defined for partial degrees in the multi-simplex domain: (i) $g = (g_1, g_2, \cdots, g_{2q})$ is a vector representing the partial degrees associated with Lyapunov matrix $P(\alpha)$ [40]. Thus, g_i represents the degree of Lyapunov matrix associated with the *i-th* unit-simplex. (ii) $s = (s_1, s_2, \cdots, s_{2q})$ is a vector representing the partial degrees associated with the synthesis matrices $G(\tilde{\alpha})$ and $Z(\tilde{\alpha})$ in Theorem 4.1. Therefore, choosing $s = (1, 1, \cdots, 1)$, rational controller can be synthesized since the synthesis matrices have affine dependence on the varying parameters. On the other hand, a robust controller (i.e., parameter-independent) can be designed by setting $s = (0, 0, \cdots, 0)$. Thus, proper selection of partial degrees of Lyapunov matrix and synthesis variables has significant influence not only on the achieved performance but also on the parametrization of the synthesized controller (robust, affine, or rational) [40].

As a rule of thumb, all the partial degrees of the synthesis variables (s) should be chosen the same for gain-scheduling control, however, setting some of these degrees to zero results in a partially scheduled controller. This remarkable feature could be exploited efficiently to handle the case of unmeasurable scheduling parameters. In other words, the synthesis variables $G(\tilde{\alpha})$ and $Z(\tilde{\alpha})$ will be independent of these parameters (corresponding to the zero-degree simplexes), leading to the notion of *selective gain-scheduling* controllers that was addressed for the first time in [40]. Suppose for example an LPV system that depends on two time-varying parameters, one of them is measurable (the first one) and the second one is not (uncertain). Then by setting the partial degrees of the synthesis variable as $s = (1,0)$ results in a controller that is dependent on the first parameter and independent on the second one. This selectivity feature can also be used to investigate the impact of each of the scheduling parameter independently on the achievable performance [40].

Due to presence of the additional slack variable $G(\tilde{\alpha})$ in Theorem 4.1, the controller will be synthesized using this variable instead of Lyapunov matrix $P(\alpha)$. Therefore, the partial degrees of the synthesis variable s and Lyapunov matrix g could be different. Thus, robust controllers can be designed with parameter-dependent Lyapunov matrix that improves the controller performance considerably. Corollary 4.1 will utilize this feature to synthesize robust controllers that will be used in Section 4.5 for performance comparison with gain-scheduling controllers.

Therefore, suppose that all scheduling parameters are unavailable for real-time measurement. By setting the partial degrees of the synthesis variables $G(\tilde{\alpha})$ and $Z(\tilde{\alpha})$ to be a zero vector, i.e. $s = (0,0,\cdots,0)$, robust controller (parameter-independent) is synthesized since in this case $G(\tilde{\alpha}) = G$ and $Z(\tilde{\alpha}) = Z$. In other words, Theorem 4.1 treats robust controller synthesis as a special case.

Corollary 4.1. *Given a scalar $\nu > 0$ and a sufficiently small positive scalar $\varepsilon > 0$. If there exist a continuously differentiable parameter-dependent matrix $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, parameter-dependent matrix $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z\times n_z}$, constant matrices $Z \in \mathbb{R}^{n_u\times n}$ and $G \in \mathbb{R}^{n\times n}$ for any* *pairs* $(\alpha(t),\dot{\alpha}(t)) \in \Lambda \times \Omega$ *such that the following PLMIs are satisfied*

$$\begin{bmatrix} A(\alpha)G + B_u(\alpha)Z + (\bullet)' & \star & \star \\ P(\alpha) - G + \varepsilon(A(\alpha)G + B_u(\alpha)Z)' & -\varepsilon(G+G') & \star \\ B_w(\alpha)' & \mathbf{0}_{n_w \times n} & -I_{n_w} \end{bmatrix} < \mathbf{0}_{2n+n_w},$$

$$\begin{bmatrix} G + G' - P(\alpha) & \star \\ C_z(\alpha)G + D_{zu}(\alpha)Z & W(\alpha) \end{bmatrix} > \mathbf{0}_{n+n_z},$$

$$trace(W(\alpha)) < \nu^2,$$

Then, the robust controller

$$K = ZG^{-1},$$

stabilizes the closed-loop system with guaranteed $\mathscr{H}_2$ *performance bounded* ν *satisfying* (4.6).

Proof. The proof can be done in a similar way to that of Theorem 4.1. □

Corollary 4.2. *Given a scalar* $\gamma_\infty > 0$ *and a sufficiently small scalar* $\varepsilon > 0$. *If there exist a continuously differentiable parameter-dependent matrix* $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, *constant matrices* $Z \in \mathbb{R}^{n_u \times n}$, $G \in \mathbb{R}^{n\times n}$ *for any pair* $(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ *such that the following PLMI satisfied*

$$\begin{bmatrix} A(\alpha)G + B_u(\alpha)Z + (\bullet)' & \star & \star & \star \\ P(\alpha) - G + \varepsilon(A(\alpha)G + B_u(\alpha)Z)' & -\varepsilon(G+G') & \star & \star \\ C(\alpha)G + D_u(\alpha)Z & \varepsilon C(\alpha)G + \varepsilon D_u(\alpha)Z & -I_{n_z} & \star \\ B_w(\alpha)' & \mathbf{0}_{n_w\times n} & D_w(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0}_{2n+n_z+n_w}.$$

Then, the robust controller

$$K = ZG^{-1},$$

stabilizes the closed-loop system with guaranteed $\mathscr{H}_\infty$ *performance bound* γ_∞ *satisfying* (4.16).

Proof. The proof can be done in a similar way to that of Theorem 4.2 □

Remark 4.1. *As have been mentioned in Subsection 3.2.4, matrix coefficient check relaxation method [41, 39] with ROLMIP [56] is used solve the conditions of Theorem 4.1 and Theorem 4.2 to obtain the optimal controller.*

4.5 Numerical Examples

The objective of the numerical examples presented in this section is to illustrate the effectiveness of the developed conditions. To facilitate comparisons with other methods, two illustrative examples have been borrowed from literature, one for state-feedback and the other is for dynamic output-feedback controller. The routines for theses examples are implemented in MATLAB® (R2013a) using a computer equipped with Intel Quad Core *i*5 (2.4 GHz) processor, 4 GB RAM and Windows 7 (64-bit) operating system.

Example 4.1: Consider the following LPV system [67],[1]. It represents the dynamics of the mechanical system with two-masses and two-springs,

$$\left[\begin{array}{c|c|c} A(\theta) & B_u(\theta) & B_w(\theta) \\ \hline C_z(\theta) & D_{zu}(\theta) & \end{array}\right] = \left[\begin{array}{cccc|c|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & -\theta_1(t) & 0 & 0 & 1 \\ 2 & -2 & 0 & -2\theta_1(t) & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & \theta_2(t) & \end{array}\right],$$

with the following bounds,

$$0.5 \leq \theta_1(t) \leq 3.5, \quad 0.5 \leq \theta_2(t) \leq 1.5, \quad |\dot{\theta}_q(t)| \leq \kappa,$$
$$|\delta_q(t)| \leq \zeta, \quad |\dot{\delta}_q(t)| \leq 10 \times \zeta, \quad q = 1,2.$$

After applying the procedure developed in Section 3.2 to convert the system matrices into multi-simplex domain, Theorem 4.1 is used to synthesize gain-scheduling controller for this system with the given bounds. Table 4.1 illustrates the guaranteed $\mathscr{H}_2$ performance bound ν with different uncertainty bounds (ζ). As expected, ν is influenced by the uncertainty bound ζ associated with the scheduling parameters $\theta_q(t)$ for $q = 1,2$. Corollary 4.1 is used to synthesize robust state-feedback controllers which achieves closed-loop performance presented in the last row of the Table 4.1.

Table 4.1: Guaranteed $\mathscr{H}_2$ performance: Theorem 4.1.

ζ	κ			
	0.001	0.01	0.1	1
0.1	0.092	0.093	0.097	0.129
0.2	0.173	0.173	0.178	0.157
0.5	0.276	0.276	0.278	0.285
1	0.278	0.279	0.280	0.287
2	0.282	0.283	0.286	0.295
Robust	0.282	0.283	0.286	0.295

Table 4.2: Guaranteed $\mathscr{H}_2$ performance: method of [1].

ζ	κ			
	0.001	0.01	0.1	1
0.1	0.070	0.077	0.139	0.147
0.2	0.132	0.141	0.192	0.201
0.5	0.258	0.263	0.299	0.300
1	0.284	0.289	0.300	0.300
2	0.300	0.300	0.300	0.300
Robust	0.300	0.300	0.300	0.300

Note that, as the uncertainty bound (measurement noise) increases, the achieved $\mathscr{H}_2$ performance of the gain-scheduling controller deteriorates and approaches the performance provided by the robust controller. For example, when $\zeta = 2$ the achieved performance is the same as the performance provided by the robust controller. This is a logical observation since as the uncertainty of the scheduling parameter increases, the measurement will be unreliable for scheduling the controller anymore. In this case, there is no benefit for the gain-scheduling controller over the robust one since the achievable performance of the two designs is the same. Table 4.2 from [1] is given here to facilitate comparison with our controllers. Although the two methods achieve competitive results, it can be noted that the controllers synthesized via Theorem 4.1 achieve better performance than the controllers of method [1] for large uncertainty bound ζ and rates of variations of the parameters κ. Figure 4.1 illustrates the achieved performance as function of ζ and κ. This figure demonstrates that as ζ and κ increase, the performance of gain-scheduling controllers approach the performance of the robust controller. A line search for ε with a linear-grid of 200 points between

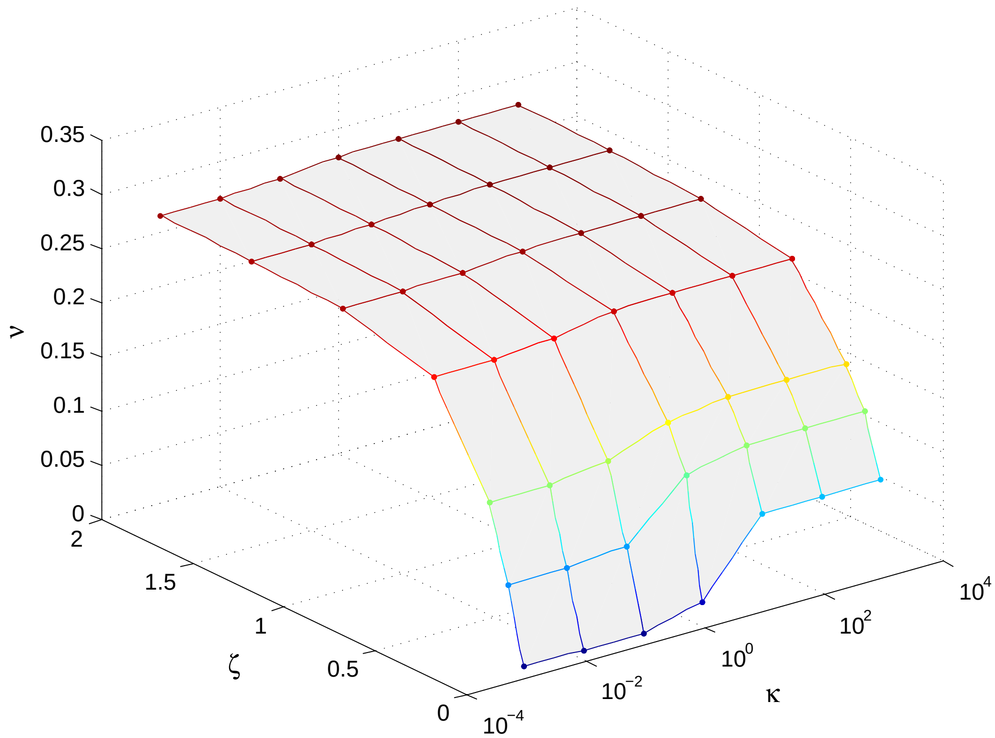


Figure 4.1: $\mathscr{H}_2$ guaranteed cost.

10^{-3} and 10^{-1} has been conducted and is shown in Figure 4.2.

Example 4.2: Consider the following LPV system [1],

$$A(\theta(t)) = \begin{bmatrix} 25.9 - 60\theta(t) & 1 \\ 20 - 40\theta(t) & 34 - 64\theta(t) \end{bmatrix}, B_u = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$$B_w = \begin{bmatrix} -0.03 \\ -0.47 \end{bmatrix}, C_z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, D_{zw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, D_{zu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The varying parameter $\theta(t)$ has the following bounds $0 \le \theta(t) \le 1$, $|\dot{\theta}(t)| \le \kappa$, with measurement uncertainty bound $|\delta(t)| \le \zeta$, and $|\dot{\delta}(t)| \le 10 \times \zeta$.

Theorem 4.2 is used to synthesize state-feedback RGS $\mathscr{H}_\infty$ controller for this example. The guaranteed $\mathscr{H}_\infty$ performance is shown in Table 4.3, as expected, the performance deteriorates as the uncertainty bound δ and the rates of change of the parameters κ increase. Robust controller has been synthesized as well to study the achieved performance compared to the performance of gain-scheduling controllers for different noise bounds. The performance of the robust controller is

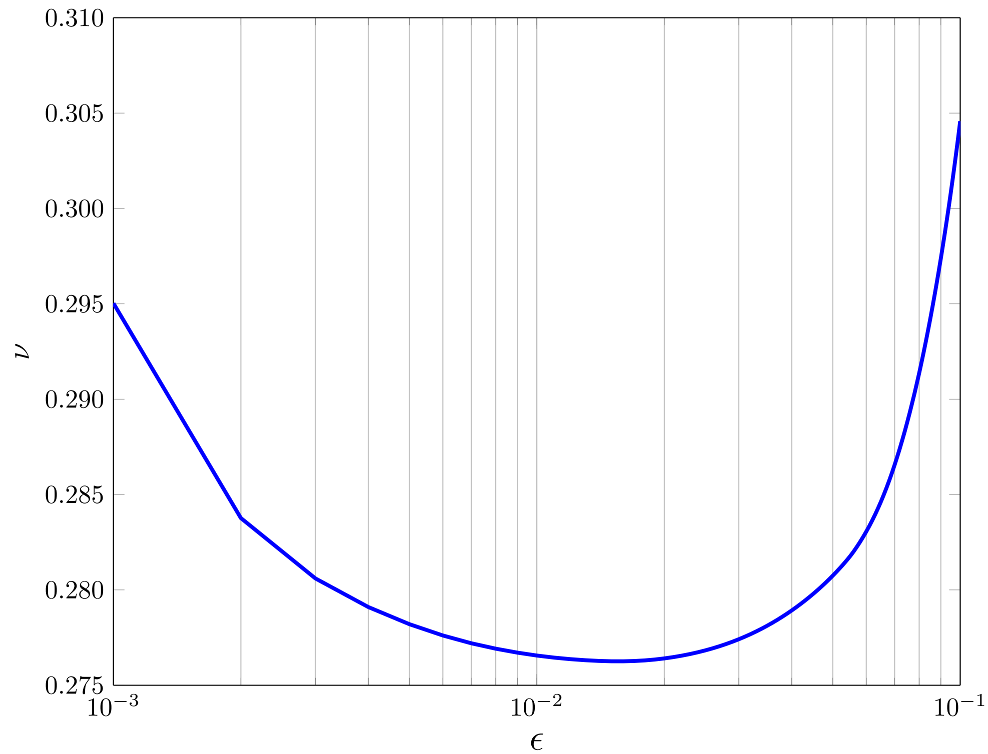


Figure 4.2: Line search for ε to obtain the optimal controller for $\zeta = 0.5$ and $\kappa = 0.01$.

shown in the last row of Table 4.3. While the gain-scheduling controller shows better performance for low range of uncertainty size, it provides no improvement over the robust controller for $\zeta \geq 0.5$. Thus, for $\zeta \geq 0.5$, there is no point to implement gain-scheduling controller as demonstrated in Figure 4.3 that shows the guaranteed $\mathscr{H}_\infty$ performance as function of ζ and κ. This is a natural expectation, since for a larger noise bounds the measurement would be unreliable for controller scheduling. Table 4.4 shows the results of [1] for the same example. Comparing these two tables, it can be observed that our approach achieves very competitive results with those associated with [1]. A line search for ε with a linear-grid of 350 points between 10^{-4} and 10^{-1} has been conducted and is shown in Figure 4.4 for $\zeta = 0.2$ with different rates of change (κ).

Simulation study has been conducted for this example to illustrate robustness of the synthesized controller against the mismatch between the ASPs and MSPs. A scheduling parameter that is defined as $\theta(t) = 0.5 + 0.5\sin(0.2t)$ and a noisy version of this signal ($\tilde{\theta}(t)$) are both shown in Figure 4.5A. A random noise with bounds $|\delta(t)| \leq 0.075$ and $|\dot{\delta}(t)| \leq 1$ has been intentionally added

Table 4.3: $\mathscr{H}_\infty$ Guaranteed cost γ_∞ using Theorem 4.2.

ζ	κ			
	0.001	0.01	0.1	1
0.01	0.583	0.584	0.589	0.652
0.1	0.718	0.718	0.727	0.793
0.2	0.791	0.791	0.792	0.795
0.5	0.795	0.795	0.795	0.795
Robust	0.795	0.795	0.795	0.795

Table 4.4: $\mathscr{H}_\infty$ Guaranteed cost γ_∞ using method of [1].

ζ	κ			
	0.001	0.01	0.1	1
0.01	0.480	0.489	0.594	0.795
0.1	0.612	0.622	0.732	0.795
0.2	0.752	0.763	0.795	0.795
0.5	0.795	0.795	0.795	0.795
Robust	0.795	0.795	0.795	0.795

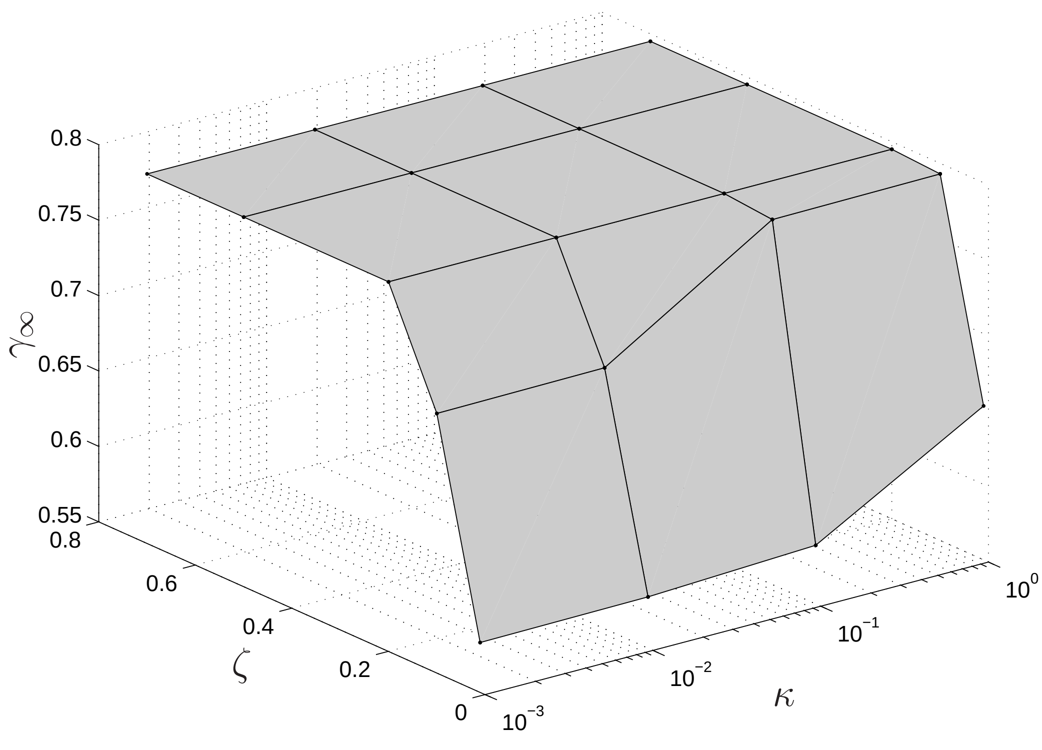


Figure 4.3: $\mathscr{H}_\infty$ guaranteed performance.

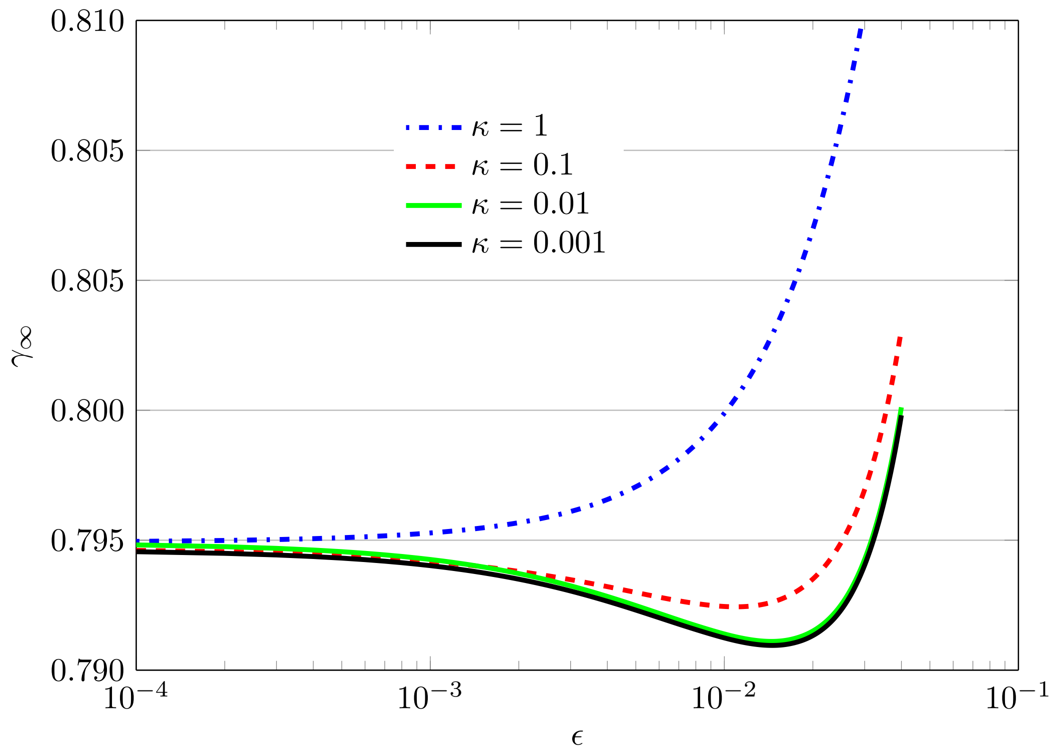


Figure 4.4: $\mathscr{H}_\infty$ performance vs. ε with $\zeta = 0.2$.

to the ASP to imitate measurement noise. Then, Theorem 4.2 is used to synthesize controller (with $\varepsilon = 0.001$) at the vertices of the multi-simplex domain. Then equation (3.24) is utilized to compute the coefficient matrices of the controller in (4.5) as

$$K_0 = [1.5940 \quad -258.245], \qquad K_1 = [1.0760 \quad -11.5690],$$

which is used to implement the controller in real-time. To simulate the closed-loop system, an $\mathscr{L}_2$ disturbance signal defined by $w(t) = \exp(-0.4t)$ is generated as disturbance input. The responses to this disturbance for both cases are shown in Figure 4.5B. Clearly, the noise amplitude in the response (associated with the noisy scheduling parameter) is much less than the noise amplitude in the MSP. This simulation result not only shows good robustness against measurement noise but also good disturbance attenuation.

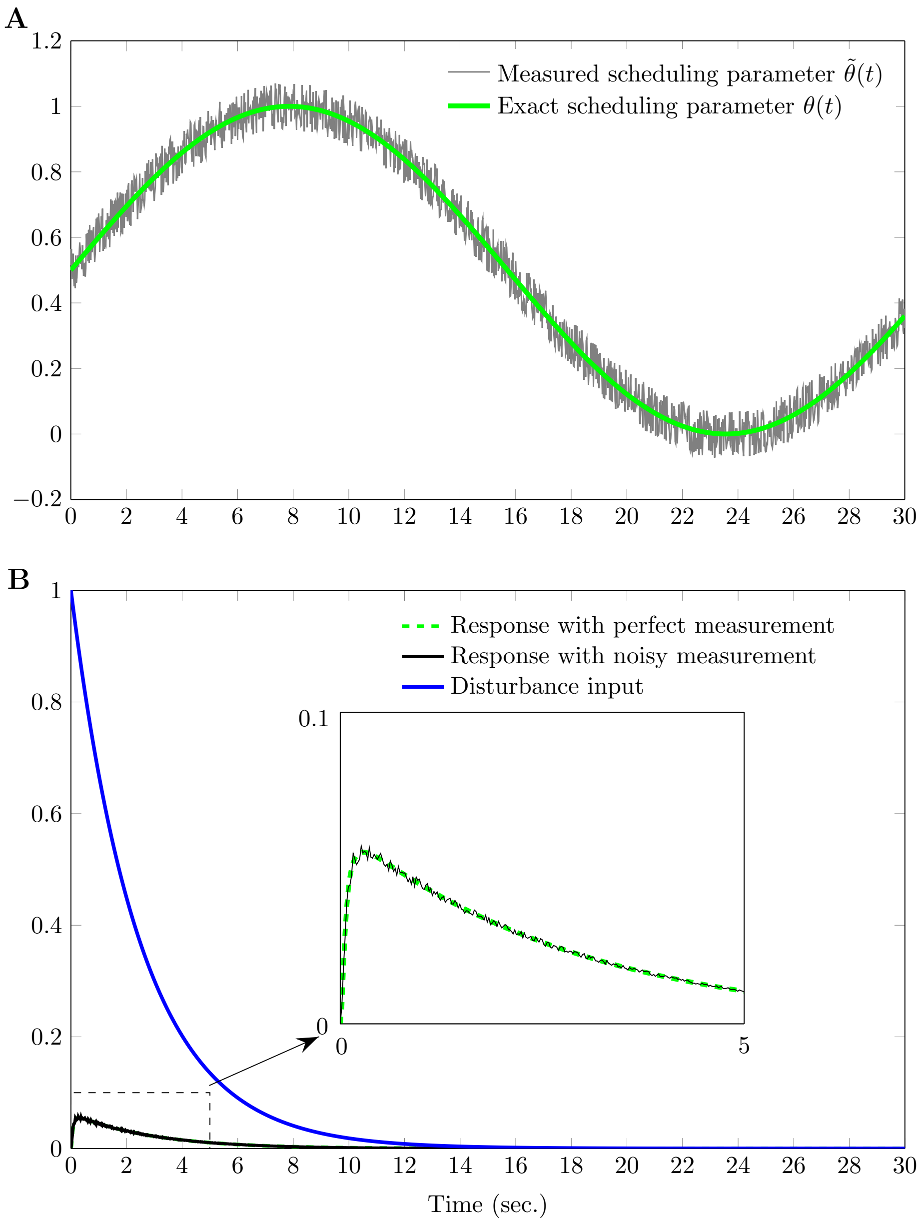


Figure 4.5: Simulation : A) Measured and exact scheduling parameters, B) Disturbance attenuation responses associated with exact and noisy scheduling parameter.

4.6 Summary

RGS State-feedback controller synthesis conditions for LPV systems were developed in this chapter. These conditions guarantee $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance subject to uncertain scheduling parameters. Synthesis conditions are formulated in terms of PLMIs with a single line search parameter

ε. By the virtue of slack variable approach, the formulation of both controllers, $\mathscr{H}_2$ and $\mathscr{H}_\infty$, are independent of Lyapunov matrix. Therefore, considerable performance achievement can be attained with the developed conditions. Numerical examples and simulation results are given as well. Therefore, the synthesis conditions encompass robust (parameter-independent) controller synthesis as a special case. Comparison study is conducted with the existing approaches from literature as well. Simulations and comparison results show that the synthesized controllers not only show robustness against uncertainties in the scheduling parameters, but also achieve improved performance.

CHAPTER 5

RGS DYNAMIC OUTPUT-FEEDBACK CONTROL

In this chapter, characterization of synthesis conditions in terms of PBMIs are derived for continuous-time polytopic LPV systems with noisy scheduling parameters. Both, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ full-order dynamic output-feedback controllers are investigated. Since the synthesis conditions are formulated as PBMIs, numerical algorithm has been developed to solve these conditions iteratively. Illustrative examples, simulations, and comparisons with other approaches from literature are also included. A short summary is given in the last section.

5.1 DOF Synthesis Problem

Consider the following open-loop system

$$\mathscr{S}_{OL} := \begin{cases} \dot{x}(t) &= A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t) \\ z(t) &= C_z(\theta(t))x(t) + D_{zu}(\theta(t))u(t) + D_{zw}(\theta(t))w(t) \\ y(t) &= C_y(\theta(t))x(t) + D_{yw}(\theta(t))w(t), \end{cases} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(t) \in \mathbb{R}^{n_y}$ is the measured output. The system matrices have the following compatible dimensions $A(\theta(t)) \in \mathbb{R}^{n \times n}$, $B_u(\theta(t)) \in \mathbb{R}^{n \times n_u}$, $B_w(\theta(t)) \in \mathbb{R}^{n \times n_w}$, $C_z(\theta(t)) \in \mathbb{R}^{n_z \times n}$, $D_{zu}(\theta(t)) \in \mathbb{R}^{n_z \times n_u}$, $D_{zw}(\theta(t)) \in \mathbb{R}^{n_z \times n_w}$, $C_y(\theta(t)) \in \mathbb{R}^{n_y \times n}$, and $D_{yw}(\theta(t)) \in \mathbb{R}^{n_y \times n_w}$.

The goal is to synthesize full-order dynamic output-feedback controller of the form

$$\mathscr{K}_{DOF} := \begin{cases} \dot{x}_c(t) &= A_c(\tilde{\theta}(t))x_c(t) + B_c(\tilde{\theta}(t))y(t) \\ u(t) &= C_c(\tilde{\theta}(t))x(t) \end{cases} \tag{5.2}$$

to stabilize the closed-loop system

$$\mathscr{S}_{CL} := \begin{cases} \dot{\xi}(t) & = \mathscr{A}(\theta(t),\tilde{\theta}(t))\xi(t) + \mathscr{B}(\theta(t),\tilde{\theta}(t))w(t) \\ z(t) & = \mathscr{C}(\theta(t),\tilde{\theta}(t))\xi(t) + \mathscr{D}(\theta(t),\tilde{\theta}(t))w(t) \end{cases} \tag{5.3}$$

with $\xi(t) = \begin{bmatrix} x(t)' & x_c(t)' \end{bmatrix}'$ and

$$\left[\begin{array}{c|c} \mathscr{A}(\theta(t),\tilde{\theta}(t)) & \mathscr{B}(\theta(t),\tilde{\theta}(t)) \\ \hline \mathscr{C}(\theta(t),\tilde{\theta}(t)) & \mathscr{D}(\theta(t),\tilde{\theta}(t)) \end{array}\right] = \left[\begin{array}{cc|c} A(\theta(t)) & \mathbf{0} & B_w(\theta(t)) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline C_z(\theta(t)) & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$+ \left[\begin{array}{cc} \mathbf{0} & B_u(\theta(t)) \\ I_n & \mathbf{0} \\ \hline \mathbf{0} & D_{zu}(\theta) \end{array}\right] \begin{bmatrix} A_c(\tilde{\theta}(t)) & B_c(\tilde{\theta}(t)) \\ C_c(\tilde{\theta}(t)) & \mathbf{0} \end{bmatrix} \left[\begin{array}{cc|c} \mathbf{0} & I_n & \mathbf{0} \\ C_y(\theta(t)) & \mathbf{0} & D_{yw}(\theta(t)) \end{array}\right].$$

where

$$\begin{aligned} \mathscr{A}(\theta(t),\tilde{\theta}(t)) &= \begin{bmatrix} A(\theta(t)) & B_u(\theta(t))C_c(\tilde{\theta}(t)) \\ B_c(\tilde{\theta}(t))C_y(\theta(t)) & A_c(\tilde{\theta}(t)) \end{bmatrix}, \\ \mathscr{B}(\theta(t),\tilde{\theta}(t)) &= \begin{bmatrix} B_w(\theta(t)) \\ B_c(\tilde{\theta}(t))D_{yw}(\theta(t)) \end{bmatrix}, \\ \mathscr{C}(\theta(t),\tilde{\theta}(t)) &= \begin{bmatrix} C_z(\theta(t)) & D_{zu}(\theta(t))C_c(\tilde{\theta}(t)) \end{bmatrix}, \\ \mathscr{D}(\theta(t),\tilde{\theta}(t)) &= D_{zw}(\theta(t)). \end{aligned} \tag{5.4}$$

Furthermore, performance[1] bounds in terms of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms are guaranteed.

Lemma 5.1. *For a given parameter-dependent symmetric matrix* $\Pi(\alpha)$ *and matrices* $\Psi_1(\alpha)$ *and* $\Psi_2(\alpha)$ *with compatible dimensions. If one of the two following conditions holds*

$$\begin{bmatrix} \Pi(\alpha) & \star \\ [\Psi_1(\alpha) \quad H(\alpha)\Psi_2(\alpha)] & -H(\alpha) \end{bmatrix} < \mathbf{0}, \tag{5.5}$$

[1]Note that in the $\mathscr{H}_2$ control problem, the feed-through matrix of the closed-loop system should be zero, i.e. $\mathscr{D}(\theta,\tilde{\theta}) = \mathbf{0}$.

$$\begin{bmatrix} \Pi(\alpha) & \star \\ [H(\alpha)\Psi_1(\alpha) \quad \Psi_2(\alpha)] & -H(\alpha) \end{bmatrix} < \mathbf{0} \tag{5.6}$$

for some parameter-dependent symmetric positive-definite matrix $H(\alpha)$, the condition

$$\Pi(\alpha) + \begin{bmatrix} \mathbf{0} & \star \\ \Psi_2(\alpha)'\Psi_1(\alpha) & \mathbf{0} \end{bmatrix} < \mathbf{0}, \tag{5.7}$$

holds.

Proof. Applying Schur compliment to (5.5) yields

$$\Pi(\alpha) + \begin{bmatrix} \Psi_1(\alpha)' \\ \Psi_2(\alpha)'H(\alpha)' \end{bmatrix} H(\alpha)^{-1} \begin{bmatrix} \Psi_1(\alpha) & H(\alpha)\Psi_2(\alpha) \end{bmatrix} < \mathbf{0}$$

that can be written as

$$\Pi(\alpha) + \begin{bmatrix} \mathbf{0} & \star \\ \Psi_2(\alpha)'\Psi_1(\alpha) & \mathbf{0} \end{bmatrix} < - \begin{bmatrix} \Psi_1(\alpha)'H(\alpha)^{-1}\Psi_1(\alpha) & \mathbf{0} \\ \mathbf{0} & \Psi_2(\alpha)'H(\alpha)\Psi_2(\alpha) \end{bmatrix}. \tag{5.8}$$

Since the RHS is negative-definite, (5.7) holds. The proof for (5.6) can be done in a similar manner. □

5.2 DOF $\mathscr{H}_2$ Control

Problem 5.1. *Suppose that the scheduling parameters $\theta(t)$ are provided as $\tilde{\theta}(t)$ with uncertainty $\delta(t)$ as defined in* (3.3). *Suppose further that $\mathscr{D}(\theta(t), \tilde{\theta}(t)) = 0$ in* (5.3). *Converting all the open-loop system matrices and synthesis variables to the multi-simplex variables α or $\tilde{\alpha}$ instead of θ and $\tilde{\theta}$, respectively, using* (3.15) *and* (3.16). *For a given positive scalar ν, find a RGS dynamic output-feedback controller in the form of* (5.2) *to stabilize the closed-loop system* (5.3) *for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ such that*

$$\sup_{(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega} \boldsymbol{E} \left\{ \int_0^T z(t)'z(t)dt \right\} < \nu^2, \tag{5.9}$$

for the disturbance input $w(t)$ given by

$$w(t) = w_0 \delta(t)$$

where $\delta(t)$ is the Dirac's delta function and w_0 is a random variable satisfying

$$\boldsymbol{E}\left\{w_0 w_0'\right\} = I_k$$

and $\boldsymbol{E}\{\cdot\}$ denotes the mathematical expectation.

The next theorem provides PLMIs conditions for synthesizing RGS dynamic output-feedback controller of the form (5.2) that is robust to the measurement uncertainties with a guaranteed $\mathscr{H}_2$ performance.

Assumption 5.1. *The measurement matrix in* (5.1) *is constant matrix, i.e. $C_y(\theta) = C_y$.*

Theorem 5.1. *Consider the system defined in* (5.1). *Given a scalar $\nu > 0$ and a sufficiently small scalar $\varepsilon > 0$, there exists a gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in the form of* (5.2) *such that the closed-loop system $\mathscr{S}_{CL}$ in* (5.3) *is asymptotically stable with a guaranteed $\mathscr{H}_2$ performance bound ν satisfying* (5.9), *if there exist continuously differentiable parameter-dependent matrices $0 < \mathscr{P}_{11}(\alpha) = \mathscr{P}_{11}(\alpha)' \in \mathbb{R}^{n\times n}$, $0 < \mathscr{P}_{22}(\alpha) = \mathscr{P}_{22}(\alpha)' \in \mathbb{R}^{n\times n}$, $\mathscr{P}_{21}(\alpha) \in \mathbb{R}^{n\times n}$, and parameter-dependent matrices $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z\times n_z}$, $R(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $S(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $T(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_1(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_2(\tilde{\alpha}) \in \mathbb{R}^{n\times n_y}$, $K_3(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, and positive-definite parameter-dependent matrix $H(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ satisfying the following PLMIs*

$$\left[\begin{array}{c|c} \Pi_2(\alpha,\tilde{\alpha}) & \star \\ \hline [\Psi_1(\tilde{\alpha}) \quad H(\tilde{\alpha})\Psi_2(\tilde{\alpha})] & -H(\tilde{\alpha}) \end{array}\right] < \mathbf{0}_{5n+n_w} \tag{5.10}$$

or

$$\left[\begin{array}{c|c} \Pi_2(\alpha,\tilde{\alpha}) & \star \\ \hline [H(\tilde{\alpha})\Psi_1(\tilde{\alpha}) \quad \Psi_2(\tilde{\alpha})] & -H(\tilde{\alpha}) \end{array}\right] < \mathbf{0}_{5n+n_w} \tag{5.11}$$

with

$$
\begin{aligned}
\Psi_1(\tilde{\alpha}) &= [\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + \Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) \qquad \mathbf{0}_{n\times n}],\\
\Psi_2(\tilde{\alpha}) &= \begin{bmatrix} S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_w} \end{bmatrix},
\end{aligned}
\tag{5.12}
$$

$$
\Delta A(\alpha,\tilde{\alpha}) := A(\tilde{\alpha}) - A(\alpha), \quad \Delta B_u(\alpha,\tilde{\alpha}) := B_u(\tilde{\alpha}) - B_u(\alpha)
$$

and

$$
trace(W(\alpha)) < \nu^2 \tag{5.13}
$$

$$
\begin{bmatrix} \mathscr{P}_{11}(\alpha) & \star \\ \mathscr{P}_{21}(\alpha) & \mathscr{P}_{22}(\alpha) \end{bmatrix} > \mathbf{0}_{n\times n}, \tag{5.14}
$$

$$
\begin{bmatrix} R(\tilde{\alpha}) + R(\tilde{\alpha})' - \mathscr{P}_{11}(\alpha) & \star & \star \\ I_n + T(\tilde{\alpha}) - \mathscr{P}_{21}(\alpha) & S(\tilde{\alpha}) + S(\tilde{\alpha})' - \mathscr{P}_{22}(\alpha) & \star \\ C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)K_3(\tilde{\alpha}) & C_z(\alpha) & W(\alpha) \end{bmatrix} > \mathbf{0}_{2n+n_z} \tag{5.15}
$$

$$
\Pi_2(\alpha,\tilde{\alpha}) =: \left[\begin{matrix} A(\alpha)R(\tilde{\alpha}) + B_u(\alpha)K_3(\tilde{\alpha}) + (\bullet)' - \dot{\mathscr{P}}_{11}(\alpha) & \star \\ A(\alpha)' + K_1(\tilde{\alpha}) - \dot{\mathscr{P}}_{21}(\alpha) & S(\tilde{\alpha})A(\alpha) + K_2(\tilde{\alpha})C_y + (\bullet)' - \dot{\mathscr{P}}_{22}(\alpha) \\ \mathscr{P}_{11}(\alpha) - R(\tilde{\alpha}) + \varepsilon(R(\tilde{\alpha})'A(\alpha)' + K_3(\tilde{\alpha})'B_u(\alpha)') & \mathscr{P}_{21}(\alpha)' - I_n + \varepsilon K_1(\tilde{\alpha})' \\ \mathscr{P}_{21}(\alpha) - T(\tilde{\alpha}) + \varepsilon A(\alpha)' & \mathscr{P}_{22}(\alpha) - S(\tilde{\alpha}) + \varepsilon(A(\alpha)'S(\tilde{\alpha})' + C_y'K_2(\tilde{\alpha})') \\ B_w(\alpha)' & B_w(\alpha)'S(\tilde{\alpha})' + D_{yw}(\alpha)'K_2(\tilde{\alpha})' \end{matrix}\right.
$$

$$
\left.\begin{matrix} \star & \star & \star \\ \star & \star & \star \\ -\varepsilon(R(\tilde{\alpha}) + R(\tilde{\alpha})') & \star & \star \\ -\varepsilon(T(\tilde{\alpha}) + I_n) & -\varepsilon(S(\tilde{\alpha}) + S(\tilde{\alpha})') & \star \\ \mathbf{0}_{n_w\times n} & \mathbf{0}_{n_w\times n} & -I_{n_w} \end{matrix}\right], \tag{5.16}
$$

for any pairs $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$. *Then, the matrices of the gain-scheduling dynamic output-*

feedback controller $\mathscr{K}_{DOF}$ in (5.2) *can be obtained as follows*

$$
\begin{aligned}
&C_c(\tilde{\alpha}) = K_3(\tilde{\alpha})F(\tilde{\alpha})^{-1},\\
&B_c(\tilde{\alpha}) = X(\tilde{\alpha})^{-1}K_2(\tilde{\alpha}),\\
&A_c(\tilde{\alpha}) = X(\tilde{\alpha})^{-1}[K_1(\tilde{\alpha}) - S(\tilde{\alpha})A(\tilde{\alpha})R(\tilde{\alpha}) - S(\tilde{\alpha})B_u(\tilde{\alpha})K_3(\tilde{\alpha})F(\tilde{\alpha})\\
&\qquad\qquad\qquad\qquad\qquad\qquad - X(\tilde{\alpha})K_2(\tilde{\alpha})C_yR(\tilde{\alpha})]F(\tilde{\alpha})^{-1},
\end{aligned}
\tag{5.17}
$$

where $F(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ and $X(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ can be obtained by taking any full-rank matrix factorization of $X(\tilde{\alpha})F(\tilde{\alpha}) = T(\tilde{\alpha}) - S(\tilde{\alpha})R(\tilde{\alpha})$.

Proof. To simplify notations in the proof, closed-loop system matrices $\mathscr{A}(\alpha,\tilde{\alpha})$, $\mathscr{B}(\alpha,\tilde{\alpha})$, and $\mathscr{C}(\alpha,\tilde{\alpha})$ in Lemma 4.1 will be denoted as $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, respectively. Using slack variable approach, additional optimization variables $U(\tilde{\alpha})$ can be introduced to inequality (4.7) via Finsler's Lemma [64] to decouple the dynamic matrix $\mathscr{A}$ from Lyapunov matrix $P(\alpha)$. This leads to the following sufficient condition of (4.7)

$$
\Xi(\alpha) + U(\tilde{\alpha})V(\tilde{\alpha}) + V(\tilde{\alpha})'U(\tilde{\alpha})' < \mathbf{0}, \tag{5.18}
$$

where,

$$
\Xi(\alpha) := \begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & \mathbf{0} \\ P(\alpha) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix},\ U(\tilde{\alpha}) := \begin{bmatrix} G(\tilde{\alpha})' & \mathbf{0} \\ Y(\tilde{\alpha})' & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix},\ V(\alpha) := \begin{bmatrix} \mathscr{A}' & -I & \mathbf{0} \\ \mathscr{B}' & \mathbf{0} & -I \end{bmatrix},
$$

such that $V(\tilde{\alpha})^{\perp'}\,\Xi(\alpha)\,V(\tilde{\alpha})^{\perp} < \mathbf{0}$ and $V(\tilde{\alpha})^{\perp'} = [I \quad \mathscr{A} \quad \mathscr{B}]$. Therefore, substituting these relations into (5.18) leads to,

$$
\begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & \mathbf{0} \\ P(\alpha) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} G(\tilde{\alpha})' & \mathbf{0} \\ Y(\tilde{\alpha})' & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}\begin{bmatrix} \mathscr{A}' & -I & \mathbf{0} \\ \mathscr{B}' & \mathbf{0} & -I \end{bmatrix} + \begin{bmatrix} \mathscr{A} & \mathscr{B} \\ -I & \mathbf{0} \\ \mathbf{0} & -I \end{bmatrix}\begin{bmatrix} G(\tilde{\alpha}) & Y(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} < \mathbf{0}. \tag{5.19}
$$

Note that it is important to impose particular structure to the slack variable $U(\tilde{\alpha})$ to maintain convex parametrization. Therefore, setting $Y(\tilde{\alpha}) = \varepsilon G(\tilde{\alpha})$ is sufficient to keep (5.19) convex, where $\varepsilon > 0$ is a scalar used as an extra degree of freedom to perform line search for reducing conservativeness [65]. This yields

$$\begin{bmatrix} -\dot{P}(\alpha) & P(\alpha) & \mathbf{0} \\ P(\alpha) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} G(\tilde{\alpha})'\mathscr{A}' & -G(\tilde{\alpha})' & \mathbf{0} \\ \varepsilon G(\tilde{\alpha})'\mathscr{A}' & -\varepsilon G(\tilde{\alpha})' & \mathbf{0} \\ \mathscr{B}' & \mathbf{0} & -I \end{bmatrix} + \begin{bmatrix} \mathscr{A}G(\tilde{\alpha}) & \varepsilon\mathscr{A}G(\tilde{\alpha}) & \mathscr{B} \\ -G(\tilde{\alpha}) & -\varepsilon G(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -I \end{bmatrix} < \mathbf{0},$$

that can be written as

$$\begin{bmatrix} \mathscr{A}G(\tilde{\alpha}) + (\bullet)' - \dot{P}(\alpha) & \star & \star \\ P(\alpha) - G(\tilde{\alpha}) + \varepsilon(G(\tilde{\alpha})'\mathscr{A}') & -\varepsilon(G(\tilde{\alpha}) + G(\tilde{\alpha})^T) & \star \\ \mathscr{B}' & \mathbf{0}_{n_w \times n} & -I_{n_w} \end{bmatrix} < \mathbf{0}_{2n+n_w}. \tag{5.20}$$

Since block$(2,2)$ of (5.20) implies $G(\tilde{\alpha}) + G(\tilde{\alpha})' > 0$, the matrix $G(\tilde{\alpha})$ is invertible and can be partitioned as,

$$G(\tilde{\alpha}) = \begin{bmatrix} R(\tilde{\alpha}) & G_1(\tilde{\alpha}) \\ F(\tilde{\alpha}) & G_2(\tilde{\alpha}) \end{bmatrix},$$

$$J(\tilde{\alpha}) := G(\tilde{\alpha})^{-1} = \begin{bmatrix} S(\tilde{\alpha})' & J_1(\tilde{\alpha}) \\ X(\tilde{\alpha})' & J_2(\tilde{\alpha}) \end{bmatrix}.$$

Define the following non-singular congruence transformation matrices,

$$Q_g(\tilde{\alpha}) := \begin{bmatrix} R(\tilde{\alpha}) & I \\ F(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}, \qquad Q_v(\tilde{\alpha}) := \begin{bmatrix} I & S(\tilde{\alpha})' \\ \mathbf{0} & X(\tilde{\alpha})' \end{bmatrix}, \tag{5.21}$$

such that,

$$G(\tilde{\alpha})Q_v(\tilde{\alpha}) = Q_g(\tilde{\alpha}), \qquad J(\tilde{\alpha})Q_g(\tilde{\alpha}) = Q_v(\tilde{\alpha}). \tag{5.22}$$

In order to guarantee that congruence transformations in (5.21) have full-rank, block matrices $F(\tilde{\alpha})$ and $X(\tilde{\alpha})$ should be non-singular. If this is not the case, small perturbation of $F(\tilde{\alpha})$ and $X(\tilde{\alpha})$ in terms of norms can always be done such that $F(\tilde{\alpha}) + \Delta F(\tilde{\alpha})$ and $X(\tilde{\alpha}) + \Delta X(\tilde{\alpha})$ are invertible.

The Lyapunov matrix can be partitioned as

$$P(\alpha) := \begin{bmatrix} P_{11}(\alpha) & \star \\ P_{21}(\alpha) & P_{22}(\alpha) \end{bmatrix} > \mathbf{0},$$

and

$$Q_v(\tilde{\alpha})'P(\alpha)Q_v(\tilde{\alpha}) := \mathscr{P}(\alpha) = \begin{bmatrix} \mathscr{P}_{11}(\alpha) & \star \\ \mathscr{P}_{21}(\alpha) & \mathscr{P}_{22}(\alpha) \end{bmatrix} > \mathbf{0}. \tag{5.23}$$

Let

$$\mathscr{T}_1(\tilde{\alpha}) = \begin{bmatrix} Q_v(\tilde{\alpha}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_v(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_w} \end{bmatrix},$$

Multiplying (5.20) by $\mathscr{T}_1(\tilde{\alpha})$ from right and by $\mathscr{T}_1(\tilde{\alpha})'$ from left and using (5.22) lead to

$$\begin{bmatrix} Q_v'(\tilde{\alpha})\mathscr{A}Q_g(\tilde{\alpha}) + (\bullet)' - \dot{\mathscr{P}}(\alpha) & \star & \star \\ \mathscr{P}(\tilde{\alpha}) - Q_v'(\tilde{\alpha})Q_g(\tilde{\alpha}) + \varepsilon(Q_v'(\tilde{\alpha})\mathscr{A}Q_g(\tilde{\alpha}))' & -\varepsilon(Q_v'(\tilde{\alpha})Q_g(\tilde{\alpha}) + (\bullet)') & \mathbf{0} \\ \mathscr{B}'Q_v(\tilde{\alpha}) & \mathbf{0} & -I_{n_w} \end{bmatrix} < \mathbf{0}, \tag{5.24}$$

Then, substituting closed-loop matrices (5.4) into (5.24) and considering (5.21) and (5.23) with the following relationships,

$$Q_v(\tilde{\alpha})'Q_g(\tilde{\alpha}) = \begin{bmatrix} R(\tilde{\alpha}) & I \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix}, \quad T(\tilde{\alpha}) := S(\tilde{\alpha})R(\tilde{\alpha}) + X(\alpha)F(\tilde{\alpha}),$$

$$Q_v(\tilde{\alpha})'\mathscr{A}Q_g(\tilde{\alpha}) = \begin{bmatrix} A(\alpha)R(\tilde{\alpha}) + B_u(\alpha)K_3(\tilde{\alpha}) & A(\alpha) \\ K_1(\tilde{\alpha}) & S(\tilde{\alpha})A(\alpha) + K_2(\tilde{\alpha})C_y \end{bmatrix} := \Phi(\tilde{\alpha}),$$

$$Q_v(\tilde{\alpha})'\mathscr{B} = \begin{bmatrix} B_w(\alpha) \\ S(\tilde{\alpha})B_w(\alpha) + K_2(\tilde{\alpha})D_{yw}(\alpha) \end{bmatrix},$$

inequality (5.24) can be written as

$$\begin{bmatrix} \Phi(\tilde{\alpha})+\Phi(\tilde{\alpha})'-\begin{bmatrix} \dot{\mathscr{P}}_{11}(\alpha) & \dot{\mathscr{P}}_{21}(\alpha)' \\ \dot{\mathscr{P}}_{21}(\alpha) & \dot{\mathscr{P}}_{22}(\alpha) \end{bmatrix} & \star & \star \\ \mathscr{P}(\alpha)-\begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix}+\varepsilon\Phi(\tilde{\alpha})' & -\varepsilon\left(\begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix}+(\bullet)'\right) & \star \\ \begin{bmatrix} B_w(\alpha)' & B_w(\alpha)'S(\tilde{\alpha})'+D_{yw}(\alpha)'K_2(\tilde{\alpha})' \end{bmatrix} & \mathbf{0}_{n_w\times 2n} & -I_{n_w} \end{bmatrix} < \mathbf{0}_{4n+n_w}, \tag{5.25}$$

where $K_1(\tilde{\alpha})$, $K_2(\tilde{\alpha})$, and $K_3(\tilde{\alpha})$ are intermediate controller variables defined as

$$\left[\begin{array}{c|c} K_1(\tilde{\alpha}) & K_2(\tilde{\alpha}) \\ \hline K_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] := \begin{bmatrix} X(\tilde{\alpha}) & S(\tilde{\alpha})B_u(\tilde{\alpha}) \\ \mathbf{0} & I \end{bmatrix}\begin{bmatrix} A_c(\tilde{\alpha}) & B_c(\tilde{\alpha}) \\ C_c(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}\begin{bmatrix} F(\tilde{\alpha}) & \mathbf{0} \\ C_yR(\tilde{\alpha}) & I \end{bmatrix}+\begin{bmatrix} S(\tilde{\alpha}) \\ \mathbf{0} \end{bmatrix}A(\tilde{\alpha})\begin{bmatrix} R(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}. \tag{5.26}$$

Since controller matrices should only depend on the MSPs, the open-loop matrices in (5.26) are allowed to depend on the multi-simplex variables $\tilde{\alpha}$ not α. However, $A(\tilde{\alpha})$ and $B_u(\tilde{\alpha})$ can be written as

$$\begin{aligned} A(\tilde{\alpha}) &= A(\tilde{\alpha})+A(\alpha)-A(\alpha) = A(\alpha)+\Delta A(\alpha,\tilde{\alpha}) \\ B_u(\tilde{\alpha}) &= B_u(\tilde{\alpha})+B_u(\alpha)-B_u(\alpha) = B_u(\alpha)+\Delta B_u(\alpha,\tilde{\alpha}), \end{aligned} \tag{5.27}$$

where $\Delta A(\alpha,\tilde{\alpha}) := A(\tilde{\alpha})-A(\alpha)$, and $\Delta B_u(\tilde{\alpha},\alpha) := B_u(\tilde{\alpha})-B_u(\alpha)$. Hence, this substitution allows us to construct controller matrices based only on the MSPs. Therefore, Substituting (5.27) into (5.26) to obtain

$$\left[\begin{array}{c|c} K_1(\tilde{\alpha}) & K_2(\tilde{\alpha}) \\ \hline K_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \bar{K}_1(\tilde{\alpha}) & \bar{K}_2(\tilde{\alpha}) \\ \hline \bar{K}_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] + \Sigma \tag{5.28}$$

where

$$\Sigma := \begin{bmatrix} S(\tilde{\alpha})\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha})+S(\tilde{\alpha})\Delta B_u(\alpha,\tilde{\alpha})C_c(\tilde{\alpha})F(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and

$$\left[\begin{array}{c|c} \bar{K}_1(\tilde{\alpha}) & \bar{K}_2(\tilde{\alpha}) \\ \hline \bar{K}_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] := \begin{bmatrix} X(\tilde{\alpha}) & S(\tilde{\alpha})B_u(\alpha) \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} A_c(\tilde{\alpha}) & B_c(\tilde{\alpha}) \\ C_c(\tilde{\alpha}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} F(\tilde{\alpha}) & \mathbf{0} \\ C_y R(\tilde{\alpha}) & I \end{bmatrix} + \begin{bmatrix} S(\tilde{\alpha})A(\alpha)R(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Substituting $C_c(\tilde{\alpha}) = \bar{K}_3(\tilde{\alpha})F(\tilde{\alpha})^{-1} = K_3(\tilde{\alpha})F(\tilde{\alpha})^{-1}$ into Σ yields

$$\Sigma = \begin{bmatrix} S(\tilde{\alpha})\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})\Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{5.29}$$

Substituting (5.29) into (5.28), and then into (5.25) and noting that $\bar{K}_2 = K_2$ and $\bar{K}_3 = K_3$ leads to

$$\begin{bmatrix} \Phi(\tilde{\alpha}) + \Phi(\tilde{\alpha})' - \begin{bmatrix} \dot{\mathscr{P}}_{11}(\alpha) & \dot{\mathscr{P}}_{21}(\alpha)' \\ \dot{\mathscr{P}}_{21}(\alpha) & \dot{\mathscr{P}}_{22}(\alpha) \end{bmatrix} & \star & \star \\ \mathscr{P}(\alpha) - \begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + \varepsilon\Phi(\tilde{\alpha})' & -\varepsilon\left(\begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + (\bullet)'\right) & \star \\ \begin{bmatrix} B_w(\alpha)' & B_w(\alpha)'S(\tilde{\alpha})' + D_{yw}(\alpha)'K_2(\tilde{\alpha})' \end{bmatrix} & \mathbf{0}_{n_w\times 2n} & -I_{n_w} \end{bmatrix} + \left[\begin{array}{c|c} \mathbf{0}_{2n\times 2n} & \star \\ \hline \Psi_2(\tilde{\alpha})'\Psi_1(\tilde{\alpha}) & \mathbf{0}_{2n+n_w\times 2n} \end{array}\right] < \mathbf{0}_{4n+n_w}.$$

Note that the above inequality is in the form of (5.7) of Lemma 5.1 with

$$\Psi_2(\tilde{\alpha})'\Psi_1(\tilde{\alpha}) = \begin{bmatrix} S(\tilde{\alpha})\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})\Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n_w\times n} & \mathbf{0}_{n_w\times n} \end{bmatrix}$$

that can be factorized into

$$\Psi_1(\tilde{\alpha}) = [\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + \Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) \quad \mathbf{0}_{n\times n}],$$
$$\Psi_2(\tilde{\alpha}) = \begin{bmatrix} S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_w} \end{bmatrix}.$$

that directly leads to (5.10) or (5.11). Since (4.8) can be written as

$$\begin{bmatrix} G(\tilde{\alpha}) + (\bullet)' - P(\alpha) & \star \\ \mathscr{C}(\tilde{\alpha})G(\tilde{\alpha}) & W(\alpha) \end{bmatrix} > \mathbf{0}_{2n+n_z}, \tag{5.30}$$

multiplying (5.30) by $\mathscr{T}_2(\tilde{\alpha})$ from the right and by $\mathscr{T}_2(\tilde{\alpha})'$ from the left with

$$\mathscr{T}_2(\tilde{\alpha}) = \begin{bmatrix} Q_v(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & I_{n_z} \end{bmatrix},$$

leads to

$$\begin{bmatrix} Q_v(\tilde{\alpha})'Q_g(\tilde{\alpha}) + (\bullet)' - Q_v(\tilde{\alpha})'P(\alpha)Q_v(\tilde{\alpha}) & \star \\ \mathscr{C}(\tilde{\alpha})Q_g(\tilde{\alpha}) & W(\alpha) \end{bmatrix} > \mathbf{0}_{2n+n_z}.$$

Considering (5.21) and (5.22) and substituting

$$\mathscr{C}(\tilde{\alpha})Q_g(\tilde{\alpha}) = \begin{bmatrix} C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)K_3(\tilde{\alpha}) & C_z(\alpha) \end{bmatrix},$$

result

$$\begin{bmatrix} \begin{bmatrix} R(\tilde{\alpha}) & I \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + (\bullet)' - \begin{bmatrix} \mathscr{P}_{11}(\alpha) & \mathscr{P}_{21}(\alpha)' \\ \mathscr{P}_{21}(\alpha) & \mathscr{P}_{22}(\alpha) \end{bmatrix} & \star \\ \begin{bmatrix} C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)K_3(\tilde{\alpha}) & C_z(\alpha) \end{bmatrix} & W(\alpha) \end{bmatrix} > \mathbf{0}_{2n+n_z},$$

which leads to (5.15). On the other hand, solving (5.26) for the variables $K_1(\tilde{\alpha})$, $K_2(\tilde{\alpha})$, and $K_3(\tilde{\alpha})$ yields the following relations

$$K_1(\tilde{\alpha}) = X(\tilde{\alpha})A_c(\tilde{\alpha})F(\tilde{\alpha}) + S(\tilde{\alpha})A(\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})B_u(\tilde{\alpha})C_c(\tilde{\alpha})F(\tilde{\alpha}) + K_2(\tilde{\alpha})C_yR(\tilde{\alpha}),$$

$$K_2(\tilde{\alpha}) = X(\tilde{\alpha})B_c(\tilde{\alpha}),$$

$$K_3(\tilde{\alpha}) = C_c(\tilde{\alpha})F(\tilde{\alpha}).$$

Controller matrices can be solved in the following order $C_c(\tilde{\alpha})$, $B_c(\tilde{\alpha})$, and $A_c(\tilde{\alpha})$, leading to (5.17). □

Remark 5.1. *Theorem 5.1 addresses the general case when the uncertain terms $\Delta A(\alpha,\tilde{\alpha})$ and $\Delta B_u(\alpha,\tilde{\alpha})$ are both included (for a fixed C_y)[2]. To my best knowledge, this is a more general theory that handles this problem since only $\Delta A(\alpha,\tilde{\alpha})$ was considered in literature [2, 38, 3, 37]. The next two corollaries are special cases of Theorem 5.1.*

Corollary 5.1. *Suppose that the input matrix in* (5.1) *is constant, i.e. $B_u(\theta) = B_u$. Given a scalar $\nu > 0$ and a sufficiently small scalar $\varepsilon > 0$, there exists a gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in the form of* (5.2) *such that the closed-loop system $\mathscr{S}_{CL}$ in* (5.3) *is asymptotically stable with a guaranteed $\mathscr{H}_2$ performance bound ν satisfying* (4.6)*, if there exist continuously differentiable parameter-dependent matrices $0 < \mathscr{P}_{11}(\alpha) = \mathscr{P}_{11}(\alpha)' \in \mathbb{R}^{n\times n}$, $0 < \mathscr{P}_{22}(\alpha) = \mathscr{P}_{22}(\alpha)' \in \mathbb{R}^{n\times n}$, $\mathscr{P}_{21}(\alpha) \in \mathbb{R}^{n\times n}$, and parameter-dependent matrices $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z\times n_z}$, $R(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $S(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $T(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_1(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_2(\tilde{\alpha}) \in \mathbb{R}^{n\times n_y}$, $K_3(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, and positive-definite parameter-dependent matrix $H(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ satisfying* (5.10) *or* (5.11) *with*

$$\begin{aligned}
\Psi_1(\tilde{\alpha}) &= \begin{bmatrix} R(\tilde{\alpha}) & \mathbf{0}_{n\times n} \end{bmatrix}, \\
\Psi_2(\tilde{\alpha}) &= \begin{bmatrix} \Delta A(\alpha,\tilde{\alpha})'S(\tilde{\alpha})' + \Delta C_y(\alpha,\tilde{\alpha})'K_2(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_w} \end{bmatrix}, \\
\Delta A(\alpha,\tilde{\alpha}) &:= A(\tilde{\alpha}) - A(\alpha), \quad \Delta C_y(\alpha,\tilde{\alpha}) := C_y(\tilde{\alpha}) - C_y(\alpha)
\end{aligned} \tag{5.31}$$

and the conditions (5.13)*,* (5.14)*, and* (5.15) *are satisfied with $\Pi_2(\alpha,\tilde{\alpha})$ given by* (5.16) *for any pairs $(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t)) \in \Lambda\times\Omega$. Then, the matrices of the gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in* (5.2) *can be obtained using* (5.17) *by replacing $B_u(\tilde{\alpha})$ and C_y with B_u and $C_y(\tilde{\alpha})$, respectively.*

Remark 5.2. *The parameter-dependent matrix $H(\tilde{\alpha})$ in Theorem 5.2 can be viewed as a scaling matrix to over-bound the uncertain terms $\Delta A(\alpha,\tilde{\alpha})$ and $\Delta B_u(\alpha,\tilde{\alpha})$ (or $\Delta A(\alpha,\tilde{\alpha})$ and $\Delta C_y(\alpha,\tilde{\alpha})$*

[2]Ideally, the general problem with all uncertain terms including $\Delta A(\alpha,\tilde{\alpha})$, $\Delta B_u(\alpha,\tilde{\alpha})$, and $\Delta C_y(\alpha,\tilde{\alpha})$ should be solved, but unfortunately, it ends up with a highly nonlinear term in the synthesis conditions such that no existing algorithm can be used to solve this problem with guaranteed convergence. Therefore, the case with either constant C_y or B_u are solved in this chapter.

in Corollary 5.1). The introduction of this weighting matrix improves the achieved performance significantly.

Remark 5.3. *Again, based on* (5.10)*,* (5.11)*, and* (5.12)*, two different formulations of Theorem 5.2 can be obtained (similarly, based on* (5.10)*,* (5.11)*, and* (5.31)*, two different formulations of Corollary 5.1 can be obtained). At this stage, it is not clear which formulation achieves better performance. Therefore, for a given design problem, these two formulations should be tried to obtain the best possible performance.*

Corollary 5.2. *Suppose that $B_u(\theta)$ and $C_y(\theta)$ are constant matrices in* (5.1)*. Given a scalar $\nu > 0$ and a sufficiently small scalar $\varepsilon > 0$, there exists a gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in the form of* (5.2) *such that the closed-loop system $\mathscr{S}_{CL}$ in* (5.3) *is asymptotically stable with a guaranteed $\mathscr{H}_2$ performance bound ν satisfying* (5.9)*, if there exist continuously differentiable parameter-dependent matrices $0 < \mathscr{P}_{11}(\alpha) = \mathscr{P}_{11}(\alpha)' \in \mathbb{R}^{n\times n}$, $0 < \mathscr{P}_{22}(\alpha) = \mathscr{P}_{22}(\alpha)' \in \mathbb{R}^{n\times n}$, $\mathscr{P}_{21}(\alpha) \in \mathbb{R}^{n\times n}$, and parameter-dependent matrices $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z\times n_z}$, $R(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $S(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $T(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_1(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_2(\tilde{\alpha}) \in \mathbb{R}^{n\times n_y}$, $K_3(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$ and positive-definite parameter-dependent matrix $H(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ satisfying* (5.10) *or* (5.11) *with $\Psi_1(\tilde{\alpha})$ and $\Psi_2(\tilde{\alpha})$ given by*

$$\begin{aligned} \Psi_1(\tilde{\alpha}) &= \begin{bmatrix} R(\tilde{\alpha}) & \mathbf{0}_{n\times n} \end{bmatrix}, \\ \Psi_2(\tilde{\alpha}) &= \begin{bmatrix} \Delta A(\alpha,\tilde{\alpha})'S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_w} \end{bmatrix}. \end{aligned} \tag{5.32}$$

or

$$\begin{aligned} \Psi_1(\tilde{\alpha}) &= \begin{bmatrix} \Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) & \mathbf{0}_{n\times n} \end{bmatrix}, \\ \Psi_2(\tilde{\alpha}) &= \begin{bmatrix} S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_w} \end{bmatrix}, \end{aligned} \tag{5.33}$$

and the conditions (5.13)*,* (5.14)*, and* (5.15) *are satisfied with $\Pi_2(\alpha,\tilde{\alpha})$ given by* (5.16) *for any pairs $(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t)) \in \Lambda\times\Omega$. Then, the matrices of the RGS dynamic output-feedback controller $\mathscr{K}_{DOF}$ in* (5.2) *can be obtained using* (5.17) *by replacing $B_u(\tilde{\alpha})$ with B_u.*

Proof. The proof will be omitted here since it can be shown in a similar way to the proof of Theorem 5.1. □

Table 5.1: Possible formulations for Corollary 5.2.

Formulation A	Formulation B	Formulation C	Formulation D
(5.10), (5.32)	(5.10),(5.33)	(5.11),(5.32)	(5.11),(5.33)

Remark 5.4. *Based on* (5.10)*,* (5.11)*,* (5.32)*, and* (5.33)*, four different formulations of Corollary 5.2 can be obtained as illustrated in Table 5.1. Similarly, it is not clear which formulation achieves the best performance. Therefore, for a given design problem, all formulations should be tried to obtain the best possible performance.*

5.3 DOF $\mathscr{H}_\infty$ Control

Problem 5.2. *Suppose that the scheduling parameters $\theta(t)$ are provided as $\tilde{\theta}(t)$ with uncertainty $\delta(t)$. Converting all the open-loop system matrices and synthesis variables to depend on the multi-simplex variables α or $\tilde{\alpha}$ instead of θ and $\tilde{\theta}$, respectively, using* (3.15) *and* (3.16)*. For a given positive scalar γ_∞, find a RGS dynamic output-feedback controller in the form of* (5.2) *to stabilize the closed-loop system* (5.3) *for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ such that*

$$\sup_{(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t))\in\Lambda\times\Omega} \; \sup_{w\in\mathscr{L}_2, w\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma_\infty. \tag{5.34}$$

Assumption 5.2. *The measurement matrix in* (5.1) *is constant matrix, i.e. $C_y(\theta) = C_y$.*

Theorem 5.2. *Consider the system defined in* (5.1)*. Given a scalar $\gamma_\infty > 0$ and a sufficiently small scalar $\varepsilon > 0$, there exists a gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in the form of* (5.2) *such that the closed-loop system $\mathscr{S}_{CL}$ in* (5.3) *is asymptotically stable with a guaranteed $\mathscr{H}_\infty$ performance bound γ_∞ satisfying* (5.34)*, if there exist continuously differentiable parameter-dependent matrices $0 < \mathscr{P}_{11}(\alpha) = \mathscr{P}_{11}(\alpha)' \in \mathbb{R}^{n\times n}$, $0 < \mathscr{P}_{22}(\alpha) = \mathscr{P}_{22}(\alpha)' \in \mathbb{R}^{n\times n}$, $\mathscr{P}_{21}(\alpha) \in \mathbb{R}^{n\times n}$, and parameter-dependent matrices $R(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $S(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $T(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_1(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_2(\tilde{\alpha}) \in \mathbb{R}^{n\times n_y}$, $K_3(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, and positive-definite parameter-dependent matrix $H(\tilde{\alpha}) \in$*

$\mathbb{R}^{n\times n}$ *satisfying the following PLMIs*

$$\left[\begin{array}{c|c} \Pi_\infty(\alpha,\tilde{\alpha}) & \star \\ \hline [\Psi_1(\tilde{\alpha}) \quad H(\tilde{\alpha})\Psi_2(\tilde{\alpha})] & -H(\tilde{\alpha}) \end{array}\right] < \mathbf{0}_{5n+n_z+n_w} \tag{5.35}$$

or

$$\left[\begin{array}{c|c} \Pi_\infty(\alpha,\tilde{\alpha}) & \star \\ \hline [H(\tilde{\alpha})\Psi_1(\tilde{\alpha}) \quad \Psi_2(\tilde{\alpha})] & -H(\tilde{\alpha}) \end{array}\right] < \mathbf{0}_{5n+n_z+n_w} \tag{5.36}$$

with

$$\begin{gathered} \Psi_1(\tilde{\alpha}) = [\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + \Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) \qquad \mathbf{0}_{n\times n}], \\ \Psi_2(\tilde{\alpha}) = \begin{bmatrix} S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_z} & \mathbf{0}_{n\times n_w} \end{bmatrix}, \\ \Delta A(\alpha,\tilde{\alpha}) := A(\tilde{\alpha}) - A(\alpha), \quad \Delta B_u(\alpha,\tilde{\alpha}) := B_u(\tilde{\alpha}) - B_u(\alpha) \end{gathered} \tag{5.37}$$

$$\begin{bmatrix} \mathscr{P}_{11}(\alpha) & \star \\ \mathscr{P}_{21}(\alpha) & \mathscr{P}_{22}(\alpha) \end{bmatrix} > \mathbf{0}_{n\times n},$$

$$\Pi_\infty(\alpha,\tilde{\alpha}) := \begin{bmatrix} A(\alpha)R(\alpha)+B_u(\alpha)K_3(\alpha)+(\bullet)'-\dot{\mathscr{P}}_{11}(\alpha) & \star & \star & \star & \star & \star \\ K_1(\alpha)+A(\alpha)'-\dot{\mathscr{P}}_{21}(\alpha) & S(\alpha)A(\alpha)+K_2(\alpha)C_y+(\bullet)'-\dot{\mathscr{P}}_{22}(\alpha) & \star & \star & \star & \star \\ \mathscr{P}_{11}(\alpha)-R(\alpha)+\varepsilon(R(\alpha)'A(\alpha)'+K_3(\alpha)'B_u(\alpha)') & \mathscr{P}_{21}(\alpha)'-I_n+\varepsilon K_1(\alpha)' & -\varepsilon(R(\alpha)+R(\alpha)') & \star & \star & \star \\ \mathscr{P}_{21}(\alpha)-T(\alpha)+\varepsilon A(\alpha)' & \mathscr{P}_{22}(\alpha)-S(\alpha)+\varepsilon(A(\alpha)'S(\alpha)'+C_y'K_2(\alpha)') & -\varepsilon(T(\alpha)+I_n) & -\varepsilon(S(\alpha)+S(\alpha)') & \star & \star \\ C_z(\alpha)R(\alpha)+D_{zu}(\alpha)K_3(\alpha) & C_z(\alpha) & \varepsilon(C_z(\alpha)R(\alpha)+D_{zu}(\alpha)K_3(\alpha)) & \varepsilon C_z(\alpha) & -I_{n_z} & \star \\ B_w(\alpha)' & B_w(\alpha)'S(\alpha)'+D_{yw}(\alpha)'K_2(\alpha)' & \mathbf{0}_{n_w\times n} & \mathbf{0}_{n_w\times n} & D_{zw}(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix}, \tag{5.38}$$

for any pairs $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$. *Then, the matrices of the gain-scheduling dynamic output-feedback controller* $\mathscr{K}_{DOF}$ *in* (5.2) *can be obtained as follows*

$$\begin{aligned} C_c(\tilde{\alpha}) &= K_3(\tilde{\alpha})F(\tilde{\alpha})^{-1}, \\ B_c(\tilde{\alpha}) &= X(\tilde{\alpha})^{-1}K_2(\tilde{\alpha}), \\ A_c(\tilde{\alpha}) &= X(\tilde{\alpha})^{-1}[K_1(\tilde{\alpha}) - S(\tilde{\alpha})A(\tilde{\alpha})R(\tilde{\alpha}) - S(\tilde{\alpha})B_u(\tilde{\alpha})K_3(\tilde{\alpha})F(\tilde{\alpha}) \\ &\qquad - X(\tilde{\alpha})K_2(\tilde{\alpha})C_yR(\tilde{\alpha})]F(\tilde{\alpha})^{-1}, \end{aligned} \tag{5.39}$$

where $F(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ *and* $X(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ *can be obtained by taking any full-rank matrix factorization of* $X(\tilde{\alpha})F(\tilde{\alpha}) = T(\tilde{\alpha}) - S(\tilde{\alpha})R(\tilde{\alpha})$.

Proof. Following the proof of Theorem 4.2, inequality (4.18) can be written in terms of the closed-loop matrices as

$$\begin{bmatrix} \mathscr{A}G(\tilde{\alpha}) + (\bullet)' - \dot{P}(\alpha) & \star & \star & \star \\ P(\alpha) - G(\tilde{\alpha}) + \varepsilon(G(\tilde{\alpha})'\mathscr{A}') & -\varepsilon(G(\tilde{\alpha}) + G(\tilde{\alpha})') & \star & \star \\ \mathscr{C}G(\tilde{\alpha}) & \varepsilon\mathscr{C}G(\tilde{\alpha}) & -I_{n_z} & \star \\ \mathscr{B}' & \mathbf{0}_{n_w\times n} & \mathscr{D}' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0}_{2n+n_w}, \tag{5.40}$$

where for notations simplicity in the proof, closed-loop system matrices $\mathscr{A}(\alpha,\tilde{\alpha})$, $\mathscr{B}(\alpha,\tilde{\alpha})$, $\mathscr{C}(\alpha,\tilde{\alpha})$ and $\mathscr{D}(\alpha,\tilde{\alpha})$ in Lemma 4.3 will be denoted as $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$, respectively.

Define

$$P(\alpha) := \begin{bmatrix} P_{11}(\alpha) & \star \\ P_{21}(\alpha) & P_{22}(\alpha) \end{bmatrix} > \mathbf{0}.$$

Block$(2,2)$ of (5.40) implies $G(\tilde{\alpha}) + G(\tilde{\alpha})' > 0$. Therefore, the matrix $G(\tilde{\alpha})$ is invertible and can be partitioned as,

$$G(\alpha) = \begin{bmatrix} R(\tilde{\alpha}) & G_1(\alpha) \\ F(\tilde{\alpha}) & G_2(\alpha) \end{bmatrix}, \qquad V(\tilde{\alpha}) := G(\tilde{\alpha})^{-1} = \begin{bmatrix} S(\tilde{\alpha})' & V_1(\alpha) \\ X(\tilde{\alpha})' & V_2(\alpha) \end{bmatrix}.$$

Define the following non-singular congruence transformation matrices,

$$Q_g(\tilde{\alpha}) = \begin{bmatrix} R(\tilde{\alpha}) & I \\ F(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}, \qquad Q_v(\tilde{\alpha}) \; = \begin{bmatrix} I & S(\tilde{\alpha})' \\ \mathbf{0} & X(\tilde{\alpha})' \end{bmatrix}, \tag{5.41}$$

such that,

$$G(\tilde{\alpha})Q_v(\tilde{\alpha}) = Q_g(\tilde{\alpha}), \qquad V(\tilde{\alpha})Q_g(\tilde{\alpha}) = Q_v(\tilde{\alpha}). \tag{5.42}$$

In order to guarantee that congruence transformations in (5.41) have full-rank, block matrices $F(\tilde{\alpha})$ and $X(\tilde{\alpha})$ should be non-singular. If this is not the case, small perturb of $F(\tilde{\alpha})$ and $X(\tilde{\alpha})$ with sufficiently small matrices in terms of norms can always be done such that $F(\tilde{\alpha}) + \Delta F(\tilde{\alpha})$ and $X(\tilde{\alpha}) + \Delta X(\tilde{\alpha})$ are invertible. Define also

$$Q_v(\tilde{\alpha})'P(\alpha)Q_v(\tilde{\alpha}) := \mathscr{P}(\alpha) = \begin{bmatrix} \mathscr{P}_{11}(\alpha) & \star \\ \mathscr{P}_{21}(\alpha) & \mathscr{P}_{22}(\alpha) \end{bmatrix} > \mathbf{0}. \tag{5.43}$$

Multiplying (5.40) by $\mathscr{T}_1(\tilde{\alpha})$ from right and by $\mathscr{T}_1(\tilde{\alpha})'$ from left with,

$$\mathscr{T}_1(\tilde{\alpha}) = \begin{bmatrix} Q_v(\tilde{\alpha}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_v(\tilde{\alpha}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n_w} \end{bmatrix},$$

and considering (5.42) and (5.43) yields

$$\begin{bmatrix} Q_v'(\tilde{\alpha})\mathscr{A}Q_g(\tilde{\alpha}) + (\bullet)' - \dot{\mathscr{P}}(\alpha) & \star & \star & \star \\ \mathscr{P}(\alpha) - Q_v'(\tilde{\alpha})Q_g(\tilde{\alpha}) + \varepsilon(Q_v'(\tilde{\alpha})\mathscr{A}Q_g(\tilde{\alpha}))' & -\varepsilon(Q_v'(\tilde{\alpha})Q_g(\tilde{\alpha}) + (\bullet)') & \mathbf{0} & \star \\ \mathscr{C}Q_g(\tilde{\alpha}) & \varepsilon\mathscr{C}Q_g(\tilde{\alpha}) & -I_{n_z} & \star \\ \mathscr{B}'Q_v(\tilde{\alpha}) & \mathbf{0} & \mathscr{D}' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0}, \tag{5.44}$$

Then, substituting closed-loop matrices (5.4) and considering (5.41) with the following relations,

$$Q_v(\tilde{\alpha})'Q_g(\tilde{\alpha}) = \begin{bmatrix} R(\tilde{\alpha}) & I \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix},$$

$$T(\tilde{\alpha}) := S(\tilde{\alpha})R(\tilde{\alpha}) + X(\tilde{\alpha})F(\tilde{\alpha}),$$

$$Q_v(\tilde{\alpha})'\mathscr{A}Q_g(\tilde{\alpha}) = \begin{bmatrix} A(\alpha)R(\tilde{\alpha}) + B_u(\alpha)K_3(\tilde{\alpha}) & A(\alpha) \\ K_1(\tilde{\alpha}) & S(\tilde{\alpha})A(\alpha) + K_2(\tilde{\alpha})C_y \end{bmatrix} := \Phi(\tilde{\alpha}),$$

$$Q_v(\tilde{\alpha})'\mathscr{B} = \begin{bmatrix} B_w(\alpha) \\ S(\tilde{\alpha})B_w(\alpha) + K_2(\tilde{\alpha})D_{yw}(\alpha) \end{bmatrix},$$

$$\mathscr{C}Q_g(\tilde{\alpha}) = \begin{bmatrix} C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)K_3(\tilde{\alpha}) & C_z(\alpha) \end{bmatrix},$$

$$\mathscr{D} = D_{zw}(\alpha),$$

inequality (5.44) can be written as

$$\begin{bmatrix} \Phi(\tilde{\alpha}) + \Phi(\tilde{\alpha})' - \begin{bmatrix} \dot{\mathscr{P}}_{11}(\alpha) & \dot{\mathscr{P}}_{21}(\alpha)' \\ \dot{\mathscr{P}}_{21}(\alpha) & \dot{\mathscr{P}}_{22}(\alpha) \end{bmatrix} & \star & \star & \star \\ \mathscr{P}(\alpha) - \begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + \varepsilon\Phi(\tilde{\alpha})' & -\varepsilon\left(\begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + (\bullet)'\right) & \star & \star \\ \begin{bmatrix} C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)K_3(\tilde{\alpha}) & C_z(\alpha) \end{bmatrix} & \Phi_1(\tilde{\alpha}) & -I_{n_z} & \star \\ \begin{bmatrix} B_w(\alpha)' & B_w(\alpha)'S(\tilde{\alpha})' + D_{yw}(\alpha)'K_2(\tilde{\alpha})' \end{bmatrix} & \mathbf{0}_{n_w \times 2n} & D_{zw}(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0}_{4n+n_z+n_w}, \tag{5.45}$$

with

$$\Phi_1(\tilde{\alpha}) = \varepsilon \begin{bmatrix} C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)\bar{K}_3(\tilde{\alpha}) & C_z(\alpha) \end{bmatrix},$$

where $K_1(\tilde{\alpha})$, $K_2(\tilde{\alpha})$, and $K_3(\tilde{\alpha})$ are intermediate controller variables defined as

$$\left[\begin{array}{c|c} K_1(\tilde{\alpha}) & K_2(\tilde{\alpha}) \\ \hline K_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] := \begin{bmatrix} X(\tilde{\alpha}) & S(\tilde{\alpha})B_u(\tilde{\alpha}) \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} A_c(\tilde{\alpha}) & B_c(\tilde{\alpha}) \\ C_c(\tilde{\alpha}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} F(\tilde{\alpha}) & \mathbf{0} \\ C_y R(\tilde{\alpha}) & I \end{bmatrix} + \begin{bmatrix} S(\tilde{\alpha}) \\ \mathbf{0} \end{bmatrix} A(\tilde{\alpha}) \begin{bmatrix} R(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}. \tag{5.46}$$

Since controller matrices should only depend on the Measured Sheduling Parameters (MSPs), open-loop matrices in (5.46) need to depend on the multi-simplex variables $\tilde{\alpha}$ not α. However, $A(\tilde{\alpha})$ and $B_u(\tilde{\alpha})$ can be written as

$$\begin{aligned} A(\tilde{\alpha}) &= A(\tilde{\alpha}) + A(\alpha) - A(\alpha) = A(\alpha) + \Delta A(\alpha, \tilde{\alpha}) \\ B_u(\tilde{\alpha}) &= B_u(\tilde{\alpha}) + B_u(\alpha) - B_u(\alpha) = B_u(\alpha) + \Delta B_u(\alpha, \tilde{\alpha}), \end{aligned} \tag{5.47}$$

where $\Delta A(\alpha,\tilde{\alpha}) := A(\tilde{\alpha}) - A(\alpha)$, and $\Delta B_u(\tilde{\alpha},\alpha) := B_u(\tilde{\alpha}) - B_u(\alpha)$. Hence, this substitution allows us to construct controller matrices based only on the MSPs. Therefore, substituting (5.47) into (5.46) to obtain

$$\left[\begin{array}{c|c} K_1(\tilde{\alpha}) & K_2(\tilde{\alpha}) \\ \hline K_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \bar{K}_1(\tilde{\alpha}) & \bar{K}_2(\tilde{\alpha}) \\ \hline \bar{K}_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] + \Sigma \tag{5.48}$$

where

$$\Sigma := \begin{bmatrix} S(\tilde{\alpha})\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})\Delta B_u(\alpha,\tilde{\alpha})C_c(\tilde{\alpha})F(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and

$$\left[\begin{array}{c|c} \bar{K}_1(\tilde{\alpha}) & \bar{K}_2(\tilde{\alpha}) \\ \hline \bar{K}_3(\tilde{\alpha}) & \mathbf{0} \end{array}\right] = \begin{bmatrix} X(\tilde{\alpha}) & S(\tilde{\alpha})B_u(\alpha) \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} A_c(\tilde{\alpha}) & B_c(\tilde{\alpha}) \\ C_c(\tilde{\alpha}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} F(\tilde{\alpha}) & \mathbf{0} \\ C_yR(\tilde{\alpha}) & I \end{bmatrix} + \begin{bmatrix} S(\tilde{\alpha})A(\alpha)R(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Substituting $C_c(\tilde{\alpha}) = \bar{K}_3(\tilde{\alpha})F(\tilde{\alpha})^{-1} = K_3(\tilde{\alpha})F(\tilde{\alpha})^{-1}$ into Σ yields

$$\Sigma = \begin{bmatrix} S(\tilde{\alpha})\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})\Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{5.49}$$

Substituting (5.49) into (5.48), and then into (5.45) and noting that $\bar{K}_2 = K_2$ and $\bar{K}_3 = K_3$ leads to

$$\begin{bmatrix} \Phi(\tilde{\alpha}) + (\bullet)' - \begin{bmatrix} \dot{\mathscr{P}}_{11}(\alpha) & \dot{\mathscr{P}}_{21}(\alpha)' \\ \dot{\mathscr{P}}_{21}(\alpha) & \dot{\mathscr{P}}_{22}(\alpha) \end{bmatrix} & \star & \star & \star \\ \mathscr{P}(\alpha) - \begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + \varepsilon\Phi(\tilde{\alpha})' & -\varepsilon\left(\begin{bmatrix} R(\tilde{\alpha}) & I_n \\ T(\tilde{\alpha}) & S(\tilde{\alpha}) \end{bmatrix} + (\bullet)'\right) & \star & \star \\ \begin{bmatrix} C_z(\alpha)R(\tilde{\alpha}) + D_{zu}(\alpha)K_3(\tilde{\alpha}) & C_z(\alpha) \end{bmatrix} & \Phi_1(\tilde{\alpha}) & -I_{n_z} & \star \\ \begin{bmatrix} B_w(\alpha)' & B_w(\alpha)'S(\tilde{\alpha})' + D_{yw}(\alpha)'K_2(\tilde{\alpha})' \end{bmatrix} & \mathbf{0}_{n_w \times 2n} & D_{zw}(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix}$$
$$+ \left[\begin{array}{c|c} \mathbf{0}_{2n\times 2n} & \star \\ \hline \Psi_2(\tilde{\alpha})'\Psi_1(\tilde{\alpha}) & \mathbf{0}_{2n+n_z+n_w\times 2n} \end{array}\right] < \mathbf{0}_{4n+n_z+n_w},$$

Note that the above equation is in the form of (5.7) of Lemma 5.1 with

$$\Psi_2(\tilde{\alpha})'\Psi_1(\tilde{\alpha}) = \begin{bmatrix} S(\tilde{\alpha})\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})\Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n_z\times n} & \mathbf{0}_{n_z\times n} \\ \mathbf{0}_{n_w\times n} & \mathbf{0}_{n_w\times n} \end{bmatrix}$$

which can be factorized into

$$\Psi_1(\tilde{\alpha}) = [\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) + \Delta B_u(\alpha,\tilde{\alpha})K_3(\tilde{\alpha}) \qquad \mathbf{0}_{n\times n}],$$
$$\Psi_2(\tilde{\alpha}) = \begin{bmatrix} S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_z} & \mathbf{0}_{n\times n_w} \end{bmatrix}.$$

that directly leads to (5.35) or (5.36).

On the other hand, solving (5.46) for the variables $K_1(\tilde{\alpha})$, $K_2(\tilde{\alpha})$, and $K_3(\tilde{\alpha})$ yields the following relationships

$$K_1(\tilde{\alpha}) = X(\tilde{\alpha})A_c(\tilde{\alpha})F(\tilde{\alpha}) + S(\tilde{\alpha})A(\tilde{\alpha})R(\tilde{\alpha}) + S(\tilde{\alpha})B_u(\tilde{\alpha})C_c(\tilde{\alpha})F(\tilde{\alpha}) + K_2(\tilde{\alpha})C_yR(\tilde{\alpha}),$$
$$K_2(\tilde{\alpha}) = X(\tilde{\alpha})B_c(\tilde{\alpha}),$$
$$K_3(\tilde{\alpha}) = C_c(\tilde{\alpha})F(\tilde{\alpha}).$$

Controller matrices can be solved in the following order $C_c(\tilde{\alpha})$, $B_c(\tilde{\alpha})$, and $A_c(\tilde{\alpha})$, which leads to (5.39). □

The next two corollaries are special cases of Theorem 5.2. The proofs of these corollaries are omitted since it follows same steps as the proof of Theorem 5.2.

Corollary 5.3. *Suppose that the input matrix in* (5.1) *is constant, i.e.* $B_u(\theta) = B_u$. *Given a scalar* $\gamma_\infty > 0$ *and a sufficiently small scalar* $\varepsilon > 0$, *there exists a gain-scheduling dynamic output-feedback controller* $\mathscr{K}_{DOF}$ *in the form of* (5.2) *such that the closed-loop system* $\mathscr{S}_{CL}$ *in* (5.3) *is asymptotically stable with a guaranteed* $\mathscr{H}_\infty$ *performance bound* γ_∞ *satisfying* (5.34), *if there exist continuously differentiable parameter-dependent matrices* $0 < \mathscr{P}_{11}(\alpha) = \mathscr{P}_{11}(\alpha)' \in \mathbb{R}^{n\times n}$, $0 < \mathscr{P}_{22}(\alpha) = \mathscr{P}_{22}(\alpha)' \in \mathbb{R}^{n\times n}$, $\mathscr{P}_{21}(\alpha) \in \mathbb{R}^{n\times n}$, *and parameter-dependent matrices* $R(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$,

$S(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $T(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_1(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_2(\tilde{\alpha}) \in \mathbb{R}^{n\times n_y}$, $K_3(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, and positive-definite parameter-dependent matrix $H(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ satisfying (5.35) *or* (5.36) *with*

$$
\begin{aligned}
\Psi_1(\tilde{\alpha}) &= [R(\tilde{\alpha}) \qquad \mathbf{0}_{n\times n}],\\
\Psi_2(\tilde{\alpha}) &= \begin{bmatrix} \Delta A(\alpha,\tilde{\alpha})'S(\tilde{\alpha})' + \Delta C_y(\alpha,\tilde{\alpha})'K_2(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_z} & \mathbf{0}_{n\times n_w} \end{bmatrix},\\
\Delta A(\alpha,\tilde{\alpha}) &:= A(\tilde{\alpha}) - A(\alpha), \quad \Delta C_y(\alpha,\tilde{\alpha}) := C_y(\tilde{\alpha}) - C_y(\alpha),
\end{aligned}
$$

and $\Pi_\infty(\alpha,\tilde{\alpha})$ given by (5.38) *for any pairs $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$. Then, the matrices of the gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in* (5.2) *can be obtained as in* (5.39) *by replacing $B_u(\tilde{\alpha})$ and C_y with B_u and $C_y(\tilde{\alpha})$, respectively.*

Corollary 5.4. *Suppose that the input and measurement matrices in* (5.1) *are constant, i.e. $B_u(\theta) = B_u$ and $C_y(\theta) = C_y$. Given a scalar $\gamma_\infty > 0$ and a sufficiently small scalar $\varepsilon > 0$, there exists a gain-scheduling dynamic output-feedback controller $\mathscr{K}_{DOF}$ in the form of* (5.2) *such that the closed-loop system $\mathscr{S}_{CL}$ in* (5.3) *is asymptotically stable with a guaranteed $\mathscr{H}_\infty$ performance bound γ_∞ satisfying* (5.34)*, if there exist continuously differentiable parameter-dependent matrices $0 < \mathscr{P}_{11}(\alpha) = \mathscr{P}_{11}(\alpha)' \in \mathbb{R}^{n\times n}$, $0 < \mathscr{P}_{22}(\alpha) = \mathscr{P}_{22}(\alpha)' \in \mathbb{R}^{n\times n}$, $\mathscr{P}_{21}(\alpha) \in \mathbb{R}^{n\times n}$, and parameter-dependent matrices $R(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $S(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $T(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_1(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$, $K_2(\tilde{\alpha}) \in \mathbb{R}^{n\times n_y}$, $K_3(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n}$, and positive-definite parameter-dependent matrix $H(\tilde{\alpha}) \in \mathbb{R}^{n\times n}$ satisfying* (5.35) *or* (5.36) *with*

$$
\begin{aligned}
\Psi_1(\tilde{\alpha}) &= [R(\tilde{\alpha}) \qquad \mathbf{0}_{n\times n}],\\
\Psi_2(\tilde{\alpha}) &= \begin{bmatrix} \Delta A(\alpha,\tilde{\alpha})'S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_z} & \mathbf{0}_{n\times n_w} \end{bmatrix},
\end{aligned}
\tag{5.50}
$$

or

$$
\begin{aligned}
\Psi_1(\tilde{\alpha}) &= [\Delta A(\alpha,\tilde{\alpha})R(\tilde{\alpha}) \qquad \mathbf{0}_{n\times n}],\\
\Psi_2(\tilde{\alpha}) &= \begin{bmatrix} S(\tilde{\alpha})' & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_z} & \mathbf{0}_{n\times n_w} \end{bmatrix},\\
\Delta A(\alpha,\tilde{\alpha}) &:= A(\tilde{\alpha}) - A(\alpha),
\end{aligned}
\tag{5.51}
$$

Table 5.2: Possible formulations for Corollary 5.4.

Formulation A	Formulation B	Formulation C	Formulation D
(5.35), (5.50)	(5.35),(5.51)	(5.36),(5.50)	(5.36),(5.51)

and $\Pi_\infty(\alpha, \tilde{\alpha})$ *given by* (5.38) *for any pairs* $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$. *Then, the matrices of the gain-scheduling dynamic output-feedback controller* $\mathscr{K}_{DOF}$ *in* (5.2) *can be obtained using* (5.39) *by replacing* $B_u(\tilde{\alpha})$ *with* B_u.

Remark 5.5. *Similarly, based on* (5.35), (5.36), (5.50), *and* (5.51), *four different formulations of Corollary 5.4 can be obtained as illustrated in Table 5.2. Similarly, it is not clear which formulation achieves the best performance. Therefore, for a given design problem, all formulations should be tried to obtain the best possible performance.*

5.4 PBMI Algorithm

Due to the multiplications between decision variables ($H(\tilde{\alpha})R(\tilde{\alpha})$ or $H(\tilde{\alpha})S(\tilde{\alpha})$ in the synthesis conditions), Theorem 5.1 and Theorem 5.2 are formulated as PBMIs in terms of time-varying parameters inside multi-simplex domain. This type of synthesis problem can be viewed as a special type of non-convex optimization problem. In other words, PBMIs are equivalent to infinite dimensional BMI constraints which is numerically non-tractable. To solve this problem, numerical algorithm (Algorithm 1) is developed to solve this type of optimization problem. This algorithm assumes the possibility of solving PLMIs (for fixed ε) which is indeed possible with the advent of powerful theoretical and computational tools [41, 68, 69]. Therefore, ROLMIP [56] is used to implement the PBMI algorithm (Algorithm 1) to obtain the sub-optimal controller. Although this algorithm does not guarantee the convergence to the global optima, significant conservativeness reduction with a few iterations can be expected since ν (or γ_∞) is monotonically non-increasing as will be illustrated in the next section.

Algorithm 1: Parameter-Dependent Bilinear Matrix Inequality Algorithm

Initialization:

- Set $i = 0$, $H_0(\tilde{\alpha}) = I_n$.
- Given $H_0(\tilde{\alpha})$, minimize ν under the PLMI conditions to obtain $S(\tilde{\alpha})$ and ν.
- Set $S_0(\tilde{\alpha}) = S(\tilde{\alpha})$ and $\nu_0 = \nu$.
- Set i_{max} and $Tolarance$.
- Set $i = i+1$.

while $i < i_{max}$ OR $|\nu_i - \nu_{i-1}| > Tolarance$ **do**

- Given $S_{i-1}(\tilde{\alpha})$, minimize ν under the PLMI conditions to obtain $H(\tilde{\alpha})$ and ν.
- Set $H_i(\tilde{\alpha}) = H(\tilde{\alpha})$ and $\nu_i = \nu$.
- Set $i = i+1$.
- Given $H_{i-1}(\tilde{\alpha})$, minimize ν under the PLMI conditions to obtain $S(\tilde{\alpha})$ and ν.
- Set $S_i(\tilde{\alpha}) = S(\tilde{\alpha})$ and $\nu_i = \nu$.

end

5.5 Numerical Examples

The following LPV system has been studied many times in literature [70, 71, 72, 46, 73, 2, 37, 38]. It is representative example to show the advantages of the synthesis conditions developed in this chapter and to make comparisons with the existing methods.

Example 5.1: The state-space model that represents the dynamics of the pitch-axis motion for a missile system is given by

$$\left[\begin{array}{c|c|c} A(\theta) & B_u(\theta) & B_w(\theta) \\ \hline C_z(\theta) & D_{zu}(\theta) & \\ \hline C_y(\theta) & & D_{yw}(\theta) \end{array}\right] = \left[\begin{array}{cc|c|c} -0.89-0.89\theta(t) & 1 & -0.119 & 0.01 \\ -142.6-178.25\theta(t) & 0 & -130.8 & 0 \\ \hline 0 & 1 & 1 & \\ \hline -1.52 & 0 & & 0.01 \end{array}\right],$$

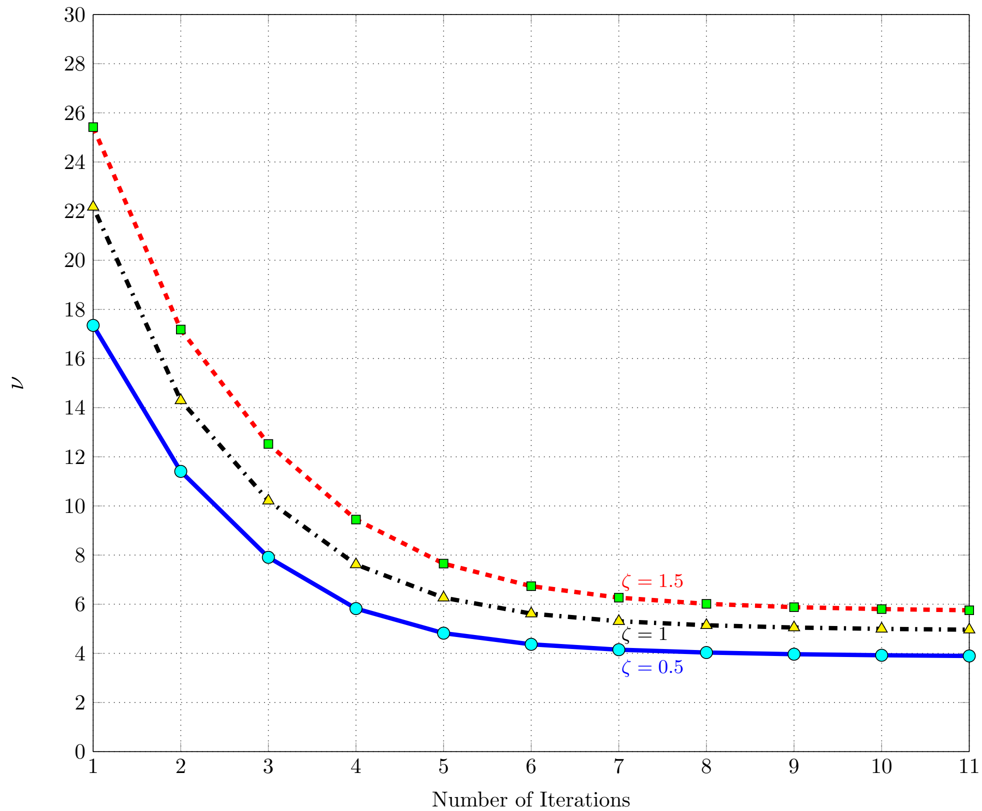


Figure 5.1: Algorithm convergence for different bounds of measurement noise (with $\varepsilon = 0.02$).

with the bounds

$$|\theta(t)| \leq 1, \qquad |\dot{\theta}(t)| \leq 1.$$

The uncertainty $\delta(t)$ of the MSP is bounded by

$$|\delta(t)| \leq \zeta, \qquad |\dot{\delta}(t)| \leq 10 \times \zeta.$$

DOF $\mathscr{H}_2$ control: This example is used to synthesize RGS DOF control with guaranteed $\mathscr{H}_2$ performance. The four formulations of Corollary 5.2 are solved for this example with Algorithm 1 and the results are shown in Table 5.3. To reduce numerical burdens, few points linearly gridded over a logarithmic scale in the interval $[10^{-1}\ ,\ 10^{-4}]$ are used for ε for this example. As have been mentioned earlier in Remark 5.3, it is difficult to judge a priori which formulation achieves best performance, however, from Table 5.3 it is clear that Formulation C achieves the best performance among the four formulations for this design example.

Table 5.3: Comparison of the guaranteed $\mathscr{H}_2$ bound ν for Corollary 5.2. ε is the number given in parentheses $(\cdot)$. Actual closed-loop $\mathscr{H}_2$-norm is given by the number between the square brackets $[\cdot]$.

ζ	Formulation A	Formulation B	Formulation C	Formulation D
0	**0.213** (0.02) [0.165]	**0.232** (0.0013) [0.184]	**0.214** (0.02) [0.165]	**0.387** (0.08) [0.238]
0.01	**0.910** (0.06) [0.354]	**1.025** (0.013) [0.558]	**0.895** (0.05) [0.372]	**0.917** (0.05) [0.366]
0.05	**1.613** (0.02) [0.744]	**1.995** (0.07) [0.435]	**1.613** (0.02) [0.745]	**1.636** (0.04) [0.534]
0.1	**2.068** (0.02) [0.888]	**2.511** (0.07) [0.428]	**2.068** (0.02) [0.892]	**2.159** (0.04) [0.604]
0.2	**2.669** (0.02) [1.040]	**3.760** (0.01) [1.022]	**2.669** (0.02) [1.048]	**3.558** (0.005) [1.166]
0.3	**3.112** (0.02) [1.127]	**4.396** (0.04) [1.427]	**3.110** (0.02) [1.139]	**3.986** (0.005) [1.382]
0.4	**3.478** (0.02) [1.196]	**5.559** (0.08) [1.647]	**3.476** (0.02) [1.219]	**4.467** (0.005) [1.463]
0.5	**3.803** (0.02) [1.233]	**6.380** (0.08) [1.712]	**3.837** (0.02) [1.684]	**4.879** (0.005) [1.521]
1	**5.052** (0.02) [1.397]	**6.810** (0.008) [2.855]	**4.910** (0.02) [1.956]	**5.970** (0.005) [1.910]
2	**6.819** (0.02) [1.470]	**7.127** (0.008) [1.925]	**6.319** (0.02) [2.221]	**7.757** (0.005) [2.017]

The algorithm convergence is shown in Figure 5.1 for different bounds on measurement noise. It demonstrates the effectiveness of the iterative procedure developed in Algorithm 1. Clearly, even with a few iterations, significant performance improvement can be achieved.

To demonstrate the advantage of the synthesized controllers ($\mathscr{K}_{DOF}$), comparisons with other existing methods are conducted. Figure 5.2 and Table 5.4 show the guaranteed $\mathscr{H}_2$ performance bounds for the synthesized controllers ($\mathscr{K}_{DOF}$) and other controllers from literature. As ζ increases, the controller provided by [2] shows considerable sensitivity to the uncertainty bounds with its $\mathscr{H}_2$ performance deteriorating exponentially. The maximum uncertainty size for which method [2] provide a feasible solution is $\zeta = 0.48$. When $\zeta > 0.48$ no stabilizing controller can be found using the conditions in [2] while the conditions of Theorem 5.2 provide controllers for

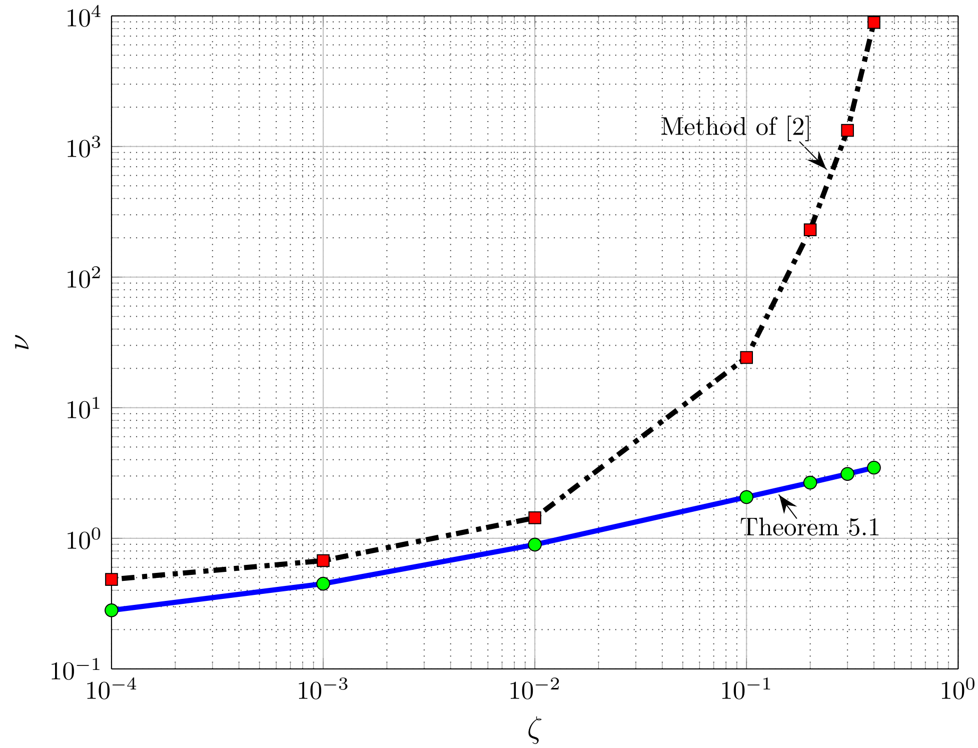


Figure 5.2: Comparison of $\mathscr{H}_2$ guaranteed performance vs. uncertainty size between the developed conditions and the method of [2].

a much wider range of ζ with improved performance bounds. Figure 5.2 illustrates the $\mathscr{H}_2$ performance of both controllers over the feasible range of [2] on a log-log scale. This comparison demonstrates the good improvement of the proposed method over that of [2].

Another comparison between $\mathscr{K}_{DOF}$ versus controllers synthesized using method in [3] (see Figure 5.3) for a specified range of uncertainty bounds. For small uncertainty size ($\zeta < 0.35$), the proposed method achieves a little worse performance, but as ζ increases, the proposed method outperforms the method in [3]. For instance, when $\zeta = 2$, the achieved $\mathscr{H}_2$ performance bound is $\nu = 6.319$ for $\mathscr{K}_{DOF}$ while the controller associated with [3] achieves $\nu = 36.918$.

Furthermore, closed-loop simulations are carried out with Actual Scheduling Parameter (ASP) defined by $\theta(t) = \sin(0.25t)$ and their bounds $|\theta(t)| \leq 1$, and $|\dot{\theta}(t)| \leq 1$. Measurement noise with bounds given by $|\delta(t)| \leq 0.2$, and $|\dot{\delta}(t)| \leq 2$ are added intentionally to the scheduling parameter. The synthesized dynamic output-feedback controller achieves $\nu = 2.669$ with $\varepsilon = 0.001$. Figure 5.4A

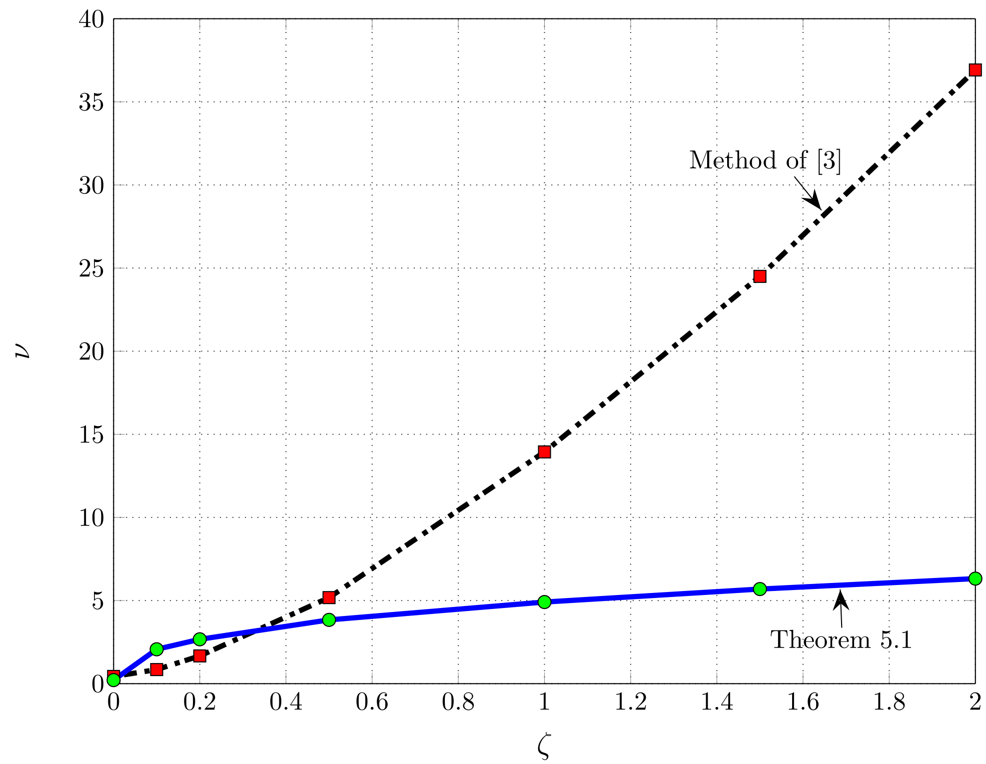


Figure 5.3: Comparison of the guaranteed $\mathscr{H}_2$ performance vs. uncertainty bound between the developed conditions and method of [3].

Table 5.4: Comparison of guaranteed $\mathscr{H}_2$ performance with other methods from literature.

	ζ						
	0	0.01	0.2	0.5	1	1.5	2
Corollary 5.2	0.214	0.895	2.669	3.837	4.910	5.691	6.319
Method of [2]	0.484	1.436	230.240	–	–	–	–
Method of [3]	0.434	0.459	1.669	5.177	13.941	24.500	36.918

– means no feasible solution.

shows the Measured Scheduling Parameter (MSP) and the ASP, respectively. $\mathscr{L}_2$ disturbance signal given by $w(t) = 15\exp(-0.3t)\sin(0.3t)$ is generated as disturbance input to the closed-loop system. Figure 5.4B illustrates system response to the disturbance input corresponds to the MSP. These simulations show not only good robustness to the measurement noise in scheduling parameter but also robustness against external disturbance.

DOF $\mathscr{H}_\infty$ control: The same example is used to synthesize RGS DOF controller with guaranteed $\mathscr{H}_\infty$ performance. The four formulations of Corollary 5.4 are solved with Algorithm 1 and the

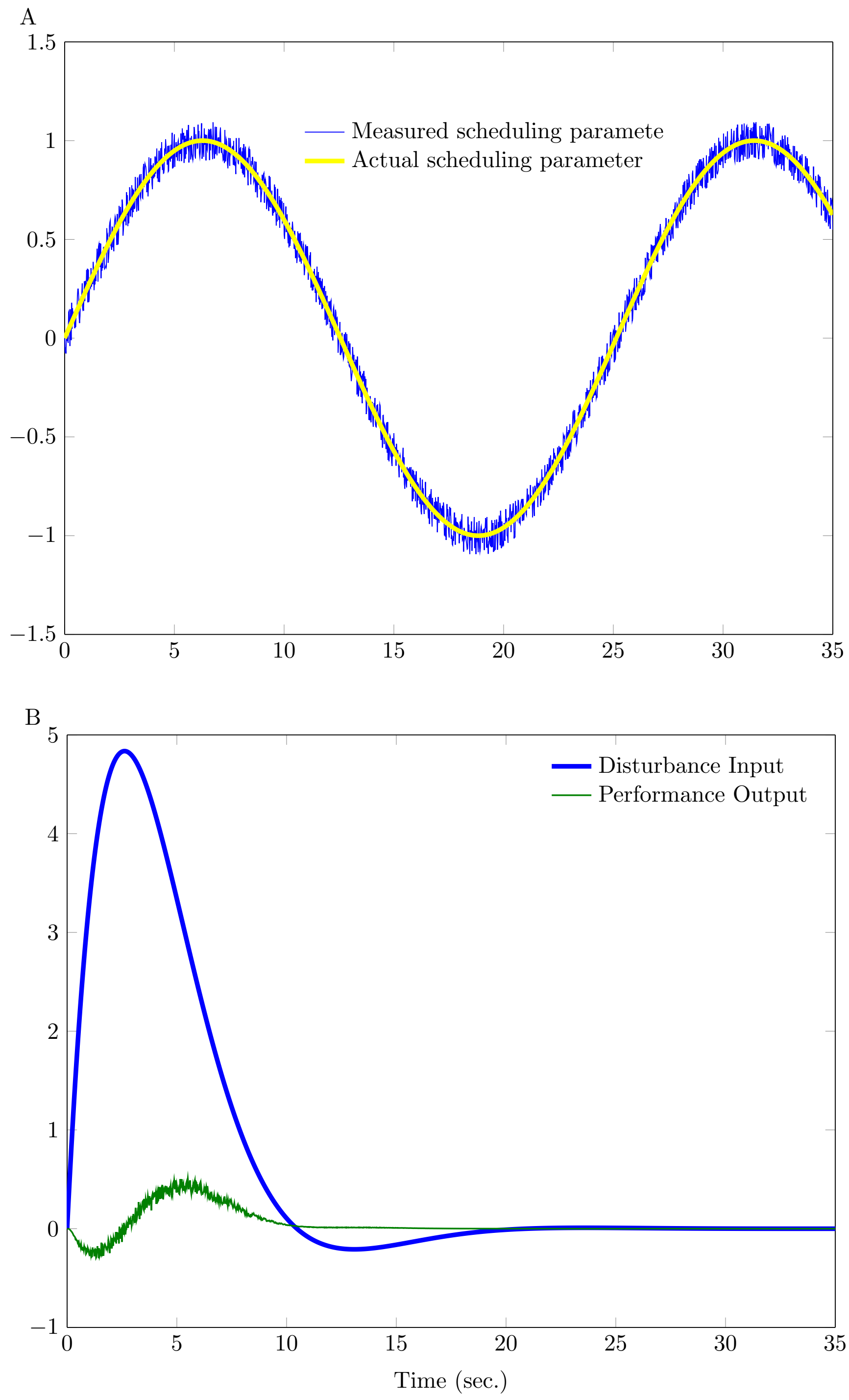


Figure 5.4: Simulation: A) Measured and actual scheduling parameters, B) Disturbance attenuation.

results are shown in Table 5.5. Similarly, to reduce numerical burdens, few points linearly gridded over a logarithmic scale in the interval $[10^{-1}\ ,\ 10^{-4}]$ are used for ε. Again, it is not clear to

Table 5.5: Comparison of the guaranteed $\mathscr{H}_\infty$ bound γ_∞ for Corollary 5.2. ε is the number given in parentheses $(\cdot)$. Actual closed-loop $\mathscr{H}_\infty$-norm is given by the number between the square brackets $[\cdot]$.

ζ	Formulation A	Formulation B	Formulation C	Formulation D
0	**0.116** (0.00207) [0.115]	**0.116** (0.005) [0.115]	**0.116** (0.005) [0.115]	**0.116** (0.005) [0.114]
0.01	**0.135** (0.00207) [0.113]	**0.118** (0.005) [0.114]	**0.118** (0.005) [0.113]	**0.118** (0.005) [0.114]
0.05	**0.157** (0.00207) [0.120]	**0.124** (0.005) [0.114]	**0.124** (0.005) [0.114]	**0.125** (0.005) [0.113]
0.1	**0.178** (0.00207) [0.117]	**0.138** (0.005) [0.116]	**0.139** (0.005) [0.115]	**0.139** (0.005) [0.115]
0.2	**0.213** (0.00207) [0.130]	**0.165** (0.005) [0.117]	**0.165** (0.005) [0.130]	**0.170** (0.005) [0.120]
0.3	**0.245** (0.00207) [0.157]	**0.193** (0.005) [0.119]	**0.194** (0.005) [0.146]	**0.196** (0.007) [0.143]
0.4	**0.276** (0.00207) [0.188]	**0.220** (0.005) [0.164]	**0.221** (0.002) [0.172]	**0.229** (0.007) [0.169]
0.5	**0.306** (0.00207) [0.219]	**0.246** (0.005) [0.187]	**0.248** (0.005) [0.196]	**0.303** (0.007) [0.207]
1	**0.466** (0.00207) [0.368]	**0.395** (0.008) [0.288]	**0.375** (0.005) [0.286]	**0.408** (0.009) [0.296]
2	**0.853** (0.00207) [0.685]	**0.844** (0.006) [0.691]	**0.712** (0.05) [0.117]	**0.801** (0.07) [0.189]

identify which formulation achieves best performance, therefore all formulations should be tried to achieve the best possible performance.

Additionally, these controllers are compared with controllers synthesized via the conditions of [37]. Figure 5.5 illustrates this comparison that demonstrate good performance improvement of the controllers synthesized using Theorem 5.2 over the controllers of method [37]. For small uncertainty size ($\zeta < 0.1$), competitive results can be obtained, but as ζ increases, our controllers outperform the controllers in [37].

Another comparison between controllers synthesized via Theorem 5.2 (with $\varepsilon = 0.001$) and controllers based on the conditions in [2] in Table 5.6. Similarly, as ζ increases, the synthesized controllers shows good performance improvement over controllers provided by [2].

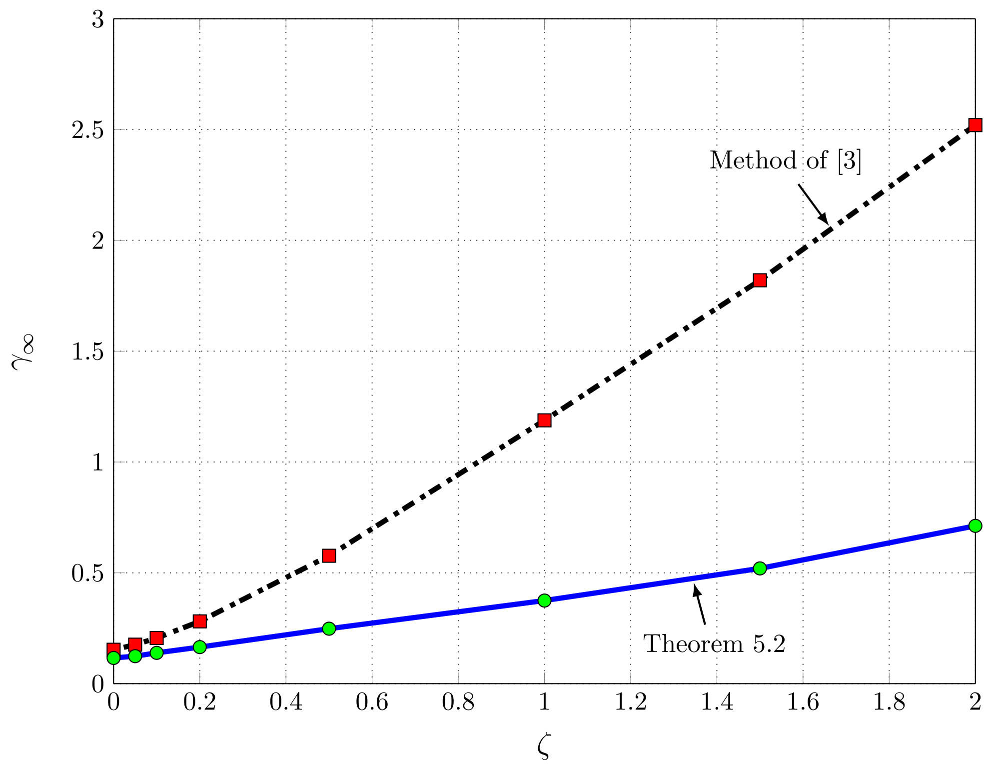


Figure 5.5: Comparison of guaranteed $\mathscr{H}_\infty$ performance between Theorem 5.2 and [3].

Table 5.6: Comparison of guaranteed $\mathscr{H}_\infty$ performance with method of [2].

ζ	Theorem 5.2	Method of [2]
0.01	0.118	0.063
0.1	0.139	0.569
0.2	0.165	1.577
0.3	0.194	3.922
0.4	0.221	11.30
0.48	0.242	339.88
1	0.375	–
2	0.712	–

– means no feasible solution.

Closed-loop simulation are conducted with ASP defined by $\theta(t) = \cos(0.25t)$ and their bounds $|\theta(t)| \leq 1$, and $|\dot{\theta}(t)| \leq 1$. Measurement noise with bounds given by $|\delta(t)| \leq 0.2$, and $|\dot{\delta}(t)| \leq 2$ are added intentionally to the scheduling parameter. Figure 5.6A shows the MSP and the ASP, respectively. $\mathscr{L}_2$ disturbance signal given by $w(t) = 15\exp(-0.3t)\sin(0.3t)$ is generated as disturbance input to the closed-loop system. Figure 5.6B shows system response to the disturbance input cor-

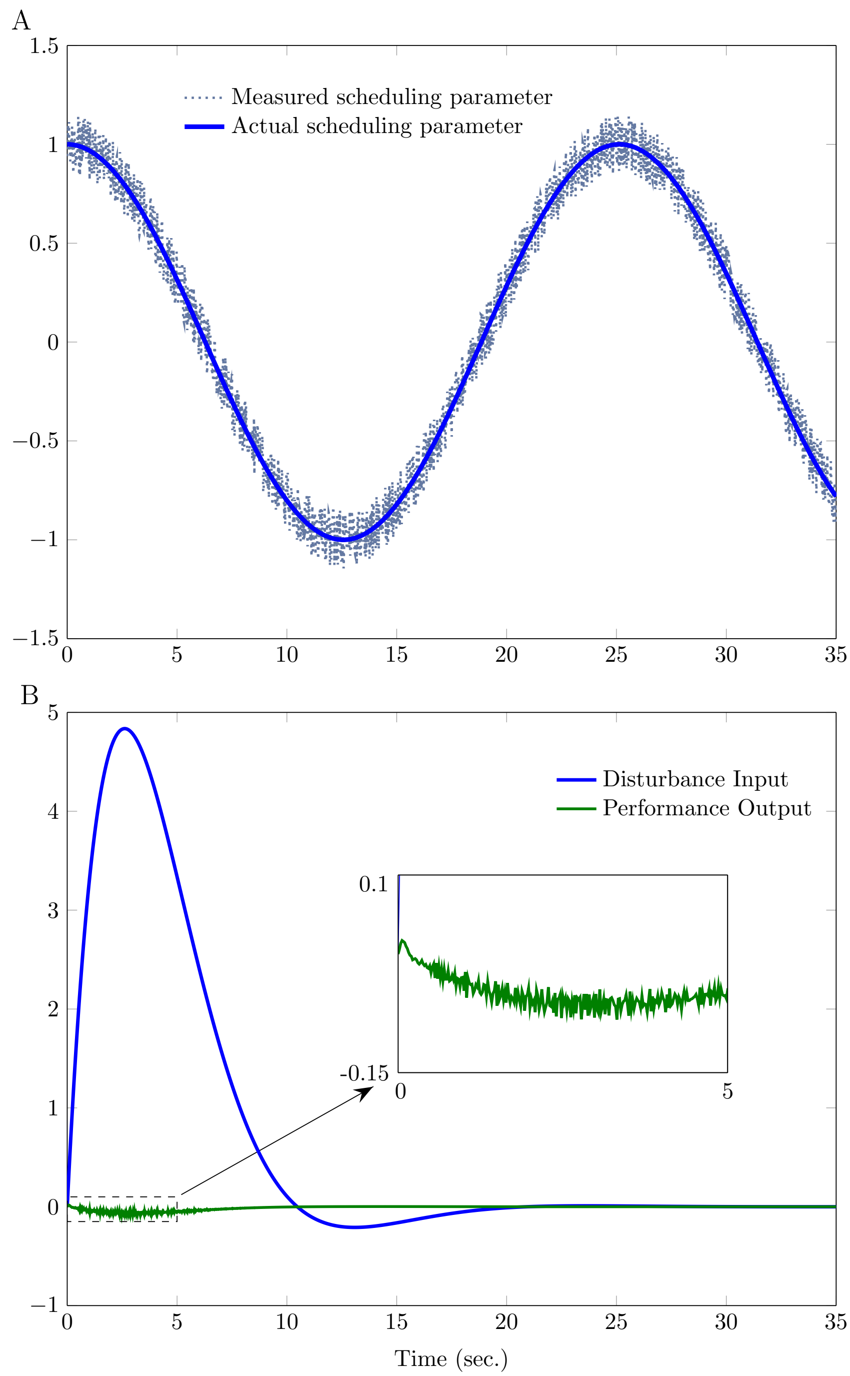


Figure 5.6: Simulation: A) Measured and actual scheduling parameters, B) Disturbance attenuation.

responds to the MSP. These simulations show not only achieved robustness to measurement noise in scheduling parameter but also good robustness to disturbance attenuation.

5.6 Summary

New synthesis conditions are derived to synthesize RGS DOF controllers with guaranteed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance in this chapter. The conditions are formulated in terms of PBMIs with scalar search. The synthesized controllers guarantee not only robust stability but also closed-loop performance against scheduling parameters uncertainties. The performance of the synthesized controllers are compared with existing design methods from literature via a realistic LPV system of a missile model. Comparisons results demonstrate the effectiveness of the developed conditions.

CHAPTER 6

RGS STATIC OUTPUT-FEEDBACK CONTROL

This chapter, characterizes novel synthesis conditions for RGS Static Output-Feedback (SOF) control with guaranteed performance. Both $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances are investigated. Two-stage design procedure is adopted to solve this control problem. The State-Feedback (SF) controller is designed first, then it is used as input to the second stage to synthesize RGS SOF controller. Numerical examples, simulations, and comparisons with other approaches from literature are included. A summary is given in the last section.

6.1 SOF Synthesis Problem

Consider the following open-loop system :

$$\mathscr{G}(\theta(t)) := \begin{cases} \dot{x}(t) & = A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t) \\ z(t) & = C_z(\theta(t))x(t) + D_{zu}(\theta(t))u(t) \\ y(t) & = C_y(\theta(t))x(t) + D_{yw}(\theta(t))w(t), \end{cases} \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{nu}$ is the control input, $w(t) \in \mathbb{R}^{nw}$ is the disturbance input, $z(t) \in \mathbb{R}^{nz}$ is the controlled output and $y(t) \in \mathbb{R}^{ny}$ is the measured output. The system matrices have the following dimensions $A(\theta(t)) \in \mathbb{R}^{n \times n}$, $B_u(\theta(t)) \in \mathbb{R}^{n \times nu}$, $B_w(\theta(t)) \in \mathbb{R}^{n \times nw}$, $C_z(\theta(t)) \in \mathbb{R}^{nz \times n}$, $D_{zu}(\theta(t)) \in \mathbb{R}^{nz \times nu}$, $C_y(\theta(t)) \in \mathbb{R}^{ny \times n}$, and $D_{yw}(\theta(t)) \in \mathbb{R}^{ny \times nw}$.

The aim is to synthesize a static output-feedback gain-scheduling controller of the form,

$$u(t) = \mathscr{K}(\tilde{\theta}(t))y(t) \tag{6.2}$$

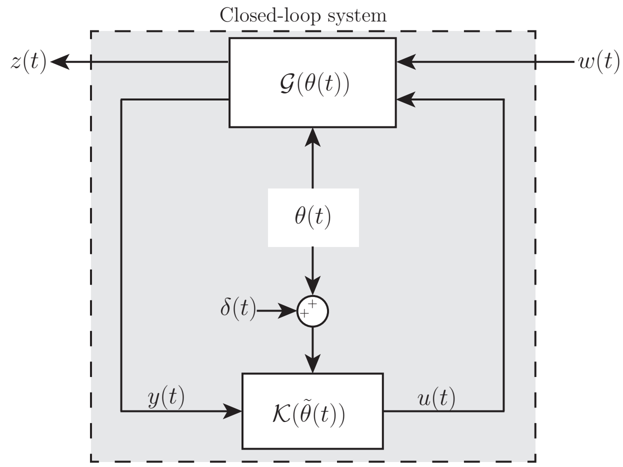


Figure 6.1: Closed-loop system with RGS control.

that robustly stabilizes the closed-loop system

$$\begin{aligned}
\dot{x}(t) &= \mathbb{A}(\theta,\tilde{\theta})x(t)+\mathbb{B}(\theta,\tilde{\theta})w(t)\\
z(t) &= \mathbb{C}(\theta,\tilde{\theta})x(t)+\mathbb{D}(\theta,\tilde{\theta})w(t)\\
\mathbb{A}(\theta,\tilde{\theta}) &:= A(\theta)+B_u(\theta)\mathscr{K}(\tilde{\theta})C_y(\theta)\\
\mathbb{B}(\theta,\tilde{\theta}) &:= B_w(\theta)+B_u(\theta)\mathscr{K}(\tilde{\theta})D_{yw}(\theta)\\
\mathbb{C}(\theta,\tilde{\theta}) &:= C_z(\theta)+D_{zu}(\theta)\mathscr{K}(\tilde{\theta})C_y(\theta)\\
\mathbb{D}(\theta,\tilde{\theta}) &:= D_{zw}(\theta)+D_{zu}(\theta)\mathscr{K}(\tilde{\theta})D_{yw}(\theta)
\end{aligned} \tag{6.3}$$

and guarantees a prescribed level of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances[1]. Furthermore, the synthesized controller should be robust to measurement uncertainties of the scheduling parameters. More specifically, the controller utilizes the measured (noisy) scheduling parameters for feedback control.

The controller matrix in (6.2) is assumed to have affine parametrization with respect to the MSPs. In other words, this matrix $\mathscr{K}(\tilde{\theta}(t))$ is parameterized as

$$\mathscr{K}(\tilde{\theta}(t)) = \mathscr{K}_0+\sum_{i=1}^{q}\tilde{\theta}_i(t)\mathscr{K}_i. \tag{6.4}$$

[1]Note that in the $\mathscr{H}_2$ control problem, the feedthrough matrix of the closed-loop system should be zero, i.e. $\mathbb{D}(\theta,\tilde{\theta}) = \mathbf{0}$.

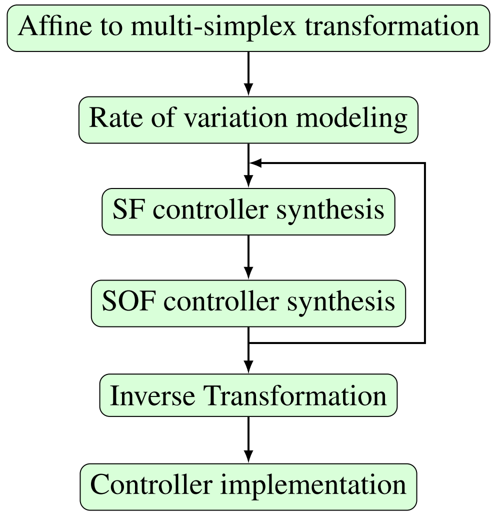


Figure 6.2: The developed synthesis approach.

Therefore, the goal is to obtain the controller coefficient matrices $\mathscr{K}_i$ for $i = 0, 1, 2, \cdots, q$, to implement the RGS controller by using only the MSPs $\tilde{\theta}_i$.

Following the lines given in [74, 75, 76, 77], two-stage design method has been adopted to solve this control problem. Gain-scheduling state-feedback controller should be designed in the first step, then, this controller is used as input parameter matrix at the second stage to synthesize RGS SOF controller in the form of (6.2) such that (6.10) satisfied.

6.2 Modeling Approach

The overall synthesis approach for RGS SOF design can be illustrated by Figure 6.2. It is a slightly modified version of Figure 3.3. An iterative procedure is developed for the two-stage design procedure.

Lemma 6.1. *[62] Let $u(t) = 0$ in* (6.1)*, for a given positive scalar ν, if there exist a continuously differentiable positive-definite matrix $P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$ and parameter-dependent matrix $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z\times n_z}$ such that the following PLMIs are satisfied*

$$\begin{bmatrix} \mathbb{A}(\alpha,\tilde{\alpha})'P(\alpha)+P(\alpha)\mathbb{A}(\alpha,\tilde{\alpha})+\dot{P}(\alpha) & \star \\ \mathbb{C}(\alpha,\tilde{\alpha}) & -I_{n_z} \end{bmatrix} < \mathbf{0}, \tag{6.5}$$

$$\begin{bmatrix} W(\alpha) & \star \\ P(\alpha)\mathbb{B}(\alpha,\tilde{\alpha}) & P(\alpha) \end{bmatrix} > \mathbf{0}, \tag{6.6}$$

$$trace(W(\alpha)) < \nu^2, \tag{6.7}$$

the open-loop system defined in (6.1) *is asymptotically stable for any pairs* $(\alpha,\dot{\alpha}) \in \Lambda\times\Omega$ *and* (6.10) *is satisfied, where* $\mathbb{A}(\alpha,\tilde{\alpha})$*,* $\mathbb{B}(\alpha,\tilde{\alpha})$*, and* $\mathbb{C}(\alpha,\tilde{\alpha})$ *are the closed-loop system matrices defined in* (6.3) *with* θ *and* $\tilde{\theta}$ *replaced by* α *and* $\tilde{\alpha}$ *using* (3.15) *and* (3.16)*.*

Lemma 6.2. *[78] (Projection Lemma) Given a symmetric matrix* $\Psi \in \mathbb{R}^{n\times n}$ *and two matrices* $\mathscr{U}$ *and* $\mathscr{V}$ *of column dimensions n, there exists an unstructured matrix* $\mathscr{X}$ *satisfying*

$$\Psi + \mathscr{V}\mathscr{X}\mathscr{U} + (\mathscr{V}\mathscr{X}\mathscr{U})' < 0 \tag{6.8}$$

if and only if the following projection inequalities with respect to $\mathscr{X}$ *are satisfied*

$$\mathscr{N}_v\Psi\mathscr{N}_v{}' < 0, \tag{6.9a}$$

$$\mathscr{N}_u{}'\Psi\mathscr{N}_u < 0 \tag{6.9b}$$

where $\mathscr{N}_u$ *and* $\mathscr{N}_v$ *any matrices whose columns form a bases of the null spaces of* $\mathscr{U}$ *and* $\mathscr{V}$*, respectively, such that* $\mathscr{N}_v\mathscr{V} = 0$ *and* $\mathscr{U}\mathscr{N}_u = 0$*.*

6.3 PLMIs Synthesis Conditions with $\mathscr{H}_2$ performance

Problem 6.1. *Suppose that the scheduling parameters* $\theta(t)$ *are provided as* $\tilde{\theta}(t)$ *with uncertainty* $\delta(t)$ *as defined in* (3.3)*. Suppose further that* $\mathscr{D}(\theta(t),\tilde{\theta}(t)) = 0$ *in* (6.3)*. Converting all the open-loop system matrices and controller variables to the multi-simplex variables* α *and* $\tilde{\alpha}$ *instead of* θ *and* $\tilde{\theta}$*, respectively, using* (3.15) *and* (3.16)*. For a given positive scalar* ν*, find a RGS static* *output-feedback controller in the form of* (6.11) *to stabilize the closed-loop system* (6.3) *for any pair* $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ *such that*

$$\sup_{(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega} \boldsymbol{E}\left\{ \int_0^T z(t)' z(t) dt \right\} < \nu^2, \tag{6.10}$$

for the disturbance input $w(t)$ *given by*

$$w(t) = w_0 \delta(t)$$

where $\delta(t)$ *is the Dirac's delta function and* w_0 *is a random variable satisfying*

$$\boldsymbol{E}\left\{ w_0 w_0' \right\} = I_{n_w}$$

and $\boldsymbol{E}\{\cdot\}$ *denotes the mathematical expectation.*

Two stage design procedure is presented in this section. In the the first stage, SF controller is designed. Then, this controller is used in the second stage to synthesize RGS SOF controller.

6.3.1 State-Feedback control

Theorem 6.1. *Let* $\mathscr{K}(\tilde{\alpha}) = \mathbf{0}$. *Given a scalar* $\nu > 0$ *and a sufficiently small scalar* $\varepsilon > 0$. *If there exist a continuously differentiable parameter-dependent matrix* $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n \times n}$, *parameter-dependent matrices* $W(\alpha) = W(\alpha)' \in \mathbb{R}^{n_z \times n_z}$, $Z(\alpha) \in \mathbb{R}^{n_u \times n}$, $G(\alpha) \in \mathbb{R}^{n \times n}$ *for any pair* $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ *such that the following PLMIs satisfied*

$$\begin{aligned}
\begin{bmatrix} \mathbb{A}(\alpha,\tilde{\alpha})G(\alpha) + B_u(\alpha)Z(\alpha) + (\bullet)' - \dot{P}(\alpha) & \star & \star \\ P(\alpha) - G(\alpha) + \varepsilon(\mathbb{A}(\alpha,\tilde{\alpha})G(\alpha) + B_u(\alpha)Z(\alpha))' & -\varepsilon(G(\alpha) + G(\alpha)') & \star \\ \mathbb{B}(\alpha,\tilde{\alpha})' & \mathbf{0}_{n_w \times n} & -I_{n_w} \end{bmatrix} &< \mathbf{0}_{2n+n_w}, \\
\begin{bmatrix} G(\alpha) + G(\alpha)' - P(\alpha) & \star \\ \mathbb{C}(\alpha,\tilde{\alpha})G(\alpha) + D_{zu}(\alpha)Z(\alpha) & W(\alpha) \end{bmatrix} &> \mathbf{0}_{n+n_z}, \\
trace(W(\alpha)) &< \nu^2,
\end{aligned}$$

Then, the gain-scheduling controller

$$K(\alpha) = Z(\alpha)G(\alpha)^{-1},$$

stabilizes the closed-loop system with guaranteed $\mathscr{H}_2$ bound ν defined in (6.10).

Proof. The proof is given in Chapter 4. □

Remark 6.1. *The state-feedback controller $K(\tilde{\alpha})$, obtained from Theorem 6.1, is used as input to Theorem 6.2 to synthesize the static output-feedback controller* (6.2).

6.3.2 Static Output-Feedback Control

Theorem 6.2. *Given $K(\tilde{\alpha})$, sufficiently small positive scalar η, and positive scalar $\nu > 0$. If there exist a continuously differentiable parameter-dependent matrix $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, parameter-dependent matrices $V(\alpha) \in \mathbb{R}^{n\times n}$, $F(\alpha) \in \mathbb{R}^{n\times n}$, $Q(\alpha) \in \mathbb{R}^{n_z\times n_z}$, $R(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n_u}$, and $L(\tilde{\alpha}) \in \mathbb{R}^{n_u\times n_y}$ for any pair $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ such that the following PLMI satisfied*

$$\begin{bmatrix} F(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) + (\bullet)' + \dot{P}(\alpha) & \star & \star & \star \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) & -\eta(V(\alpha)+V(\alpha)') & \star & \star \\ B_u(\alpha)'F(\alpha)' + L(\tilde{\alpha})C_y(\alpha) - R(\tilde{\alpha})K(\tilde{\alpha}) & \eta B_u(\alpha)'V(\alpha)' & -R(\tilde{\alpha}) - R(\tilde{\alpha})' & \star \\ Q(\alpha)'\mathscr{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'D_{zu}(\alpha) & I_{n_z} - Q(\alpha) - Q(\alpha)' \end{bmatrix} < \mathbf{0} \quad (6.11)$$

$$\begin{bmatrix} W(\alpha) & \star \\ P(\alpha)B_w(\alpha) & P(\alpha) \end{bmatrix} > \mathbf{0}, \qquad (6.12)$$

$$trace(W(\alpha)) < \nu^2, \qquad (6.13)$$

with

$$\mathscr{A}(\alpha,\tilde{\alpha}) := A(\alpha) + B_u(\alpha)K(\tilde{\alpha})$$

$$\mathscr{C}(\alpha,\tilde{\alpha}) := C_z(\alpha) + D_{zu}(\alpha)K(\tilde{\alpha}),$$

then, the static output-feedback gain-scheduling controller

$$\mathscr{K}(\tilde{\alpha}) = R(\tilde{\alpha})^{-1}L(\tilde{\alpha}), \qquad (6.14)$$

robustly stabilizes the closed-loop system (6.3) *and satisfies* (4.6).

Proof. Inequality (6.11) holds if (6.8) in Lemma 6.2 is satisfied with the following definitions

$$\mathscr{V} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ I \\ \mathbf{0} \end{bmatrix}, \qquad \mathscr{Z} = R(\tilde{\alpha}), \qquad \mathscr{U} = [\mathscr{X}(\alpha,\tilde{\alpha}) \quad \mathbf{0} \quad -I \quad \mathbf{0}],$$

$$\Psi = \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) + \dot{P}(\alpha) & \star & \star & \star \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) & -\eta(V(\alpha)+V(\alpha)') & \star & \star \\ B_u(\alpha)'F(\alpha)' & \eta B_u(\alpha)'V(\alpha)' & \mathbf{0}_{n_u \times n_w} & \star \\ Q(\alpha)'\mathscr{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z \times n} & Q(\alpha)'D_{zu}(\alpha) & -Q(\alpha)'Q(\alpha) \end{bmatrix}^{2}, \tag{6.15}$$

with the following null spaces of $\mathscr{V}$ and $\mathscr{U}$, i.e.

$$\mathscr{N}_v = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathscr{N}_u = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathscr{X}(\alpha,\tilde{\alpha}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}$$

and

$$\begin{aligned} \mathscr{A}(\alpha,\tilde{\alpha}) &:= A(\alpha) + B_u(\alpha)K(\tilde{\alpha}) \\ \mathscr{C}(\alpha,\tilde{\alpha}) &:= C_z(\alpha) + D_{zu}(\alpha)K(\tilde{\alpha}) \\ \mathscr{X}(\alpha,\tilde{\alpha}) &:= R(\tilde{\alpha})^{-1}L(\tilde{\alpha})C_y(\alpha) - K(\tilde{\alpha}). \end{aligned} \tag{6.16}$$

Thus,

$$\mathscr{U}\mathscr{N}_u = [\mathscr{X}(\alpha,\tilde{\alpha}) \quad \mathbf{0} \quad -I \quad \mathbf{0}] \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathscr{X}(\alpha,\tilde{\alpha}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} = \mathbf{0},$$

[2] This is because $(I_{n_z} - Q(\alpha))'(I_{n_z} - Q(\alpha)) > 0$ implies $-Q(\alpha)'Q(\alpha) < I_{n_z} - Q(\alpha) - Q(\alpha)'$. The same idea was used in [74].

$$\mathscr{N}_v\mathscr{V} = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ I \\ \mathbf{0} \end{bmatrix} = \mathbf{0},$$

$$\mathscr{V}\mathscr{Z}\mathscr{U} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ L(\tilde{\alpha})C_y(\alpha) - R(\tilde{\alpha})K(\tilde{\alpha}) & \mathbf{0} & -R(\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Consider now (6.9a),

$$\mathscr{N}_v\Psi\mathscr{N}_v{}' = \begin{bmatrix} \mathscr{A}(\alpha,\tilde{\alpha})'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) + \dot{P}(\alpha) & \star \\ P(\alpha) - F(\alpha)' + \varepsilon V(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) & -\varepsilon(V(\alpha) + V(\alpha)') \end{bmatrix} < 0. \tag{6.17}$$

Multiplying (6.17) by $\begin{bmatrix} I_n & \mathscr{A}(\alpha,\tilde{\alpha})' \end{bmatrix}$ from left and by its transpose from right to obtain

$$\mathscr{A}(\alpha,\tilde{\alpha})'P(\alpha) + P(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) + \dot{P}(\alpha) < 0. \tag{6.18}$$

In addition to $P(\alpha) > 0$, (6.18) represents Lyapunov stability condition for $\mathscr{A}(\alpha,\tilde{\alpha})$. Thus, (6.9a) is verified. On the other hand, (6.9b) is

$$\mathscr{N}_u{}'\Psi\mathscr{N}_u = \begin{bmatrix} I & \mathbf{0} & \mathscr{X}(\alpha,\tilde{\alpha})' & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \times$$

$$\begin{bmatrix} \Psi_{11} & \star & \star & \star \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) & \Psi_{22} & \star & \star \\ B_u(\alpha)'F(\alpha)' & \eta B_u(\alpha)'V(\alpha)' & \Psi_{33} & \star \\ Q(\alpha)'\mathscr{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'D_{zu}(\alpha) & \Psi_{44} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathscr{X}(\alpha,\tilde{\alpha}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}.$$

with

$$\Psi_{11} = \mathscr{A}(\alpha,\tilde{\alpha})'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) + \dot{P}(\alpha),$$

$$\Psi_{22} = -\eta(V(\alpha) + V(\alpha)'),$$

$$\Psi_{33} = \mathbf{0}_{n_u \times n_w},$$

$$\Psi_{44} = -Q(\alpha)'Q(\alpha).$$

Therefore,

$$\mathscr{N}_u{}'\Psi\mathscr{N}_u = \begin{bmatrix} \Psi_{11} + \mathscr{X}(\alpha,\tilde{\alpha})'B_u(\alpha)'F(\alpha)' & \bar{\Psi}_{12} & F(\alpha)B_u(\alpha) & \bar{\Psi}_{14} \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\alpha,\tilde{\alpha}) & \Psi_{22} & \eta V(\alpha)B_u(\alpha) & \mathbf{0}_{n\times n_z} \\ Q(\alpha)'\mathscr{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'D_{zu}(\alpha) & \Psi_{44} \end{bmatrix} \times \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathscr{X}(\alpha,\tilde{\alpha}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix},$$

with

$$\bar{\Psi}_{12} = P(\alpha) - F(\alpha) + \eta\mathscr{A}(\alpha,\tilde{\alpha})'V(\alpha)' + \eta\mathscr{X}(\alpha,\tilde{\alpha})'B_u(\alpha)'V(\alpha)',$$

$$\bar{\Psi}_{14} = \mathscr{C}(\alpha,\tilde{\alpha})'Q(\alpha) + \mathscr{X}(\alpha,\tilde{\alpha})'D_{zu}(\alpha)'Q(\tilde{\alpha}),$$

leads to

$$\mathscr{N}_u{}'\Psi\mathscr{N}_u = \begin{bmatrix} F(\alpha)\left[\mathscr{A}(\alpha,\tilde{\alpha}) + B_u(\alpha)\mathscr{X}(\alpha,\tilde{\alpha})\right] + (\bullet)' + \dot{P}(\alpha) & \star & \star \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\left[\mathscr{A}(\alpha,\tilde{\alpha}) + B_u(\alpha)\mathscr{X}(\alpha,\tilde{\alpha})\right] & -\eta(V(\alpha)+V(\alpha)') & \star \\ Q(\alpha)'\left[\mathscr{C}(\alpha,\tilde{\alpha}) + D_{zu}(\alpha)\mathscr{X}(\alpha,\tilde{\alpha})\right] & \mathbf{0}_{n_z\times n} & -Q(\alpha)'Q(\alpha) \end{bmatrix}.$$

Considering (6.16) with

$$\mathbb{A}(\alpha,\tilde{\alpha}) := \mathscr{A}(\alpha,\tilde{\alpha}) + B_u(\alpha)\mathscr{X}(\alpha,\tilde{\alpha}) = A(\alpha) + B_u(\alpha)\mathscr{K}(\tilde{\alpha})C_y(\alpha),$$

$$\mathbb{C}(\alpha,\tilde{\alpha}) := \mathscr{C}(\alpha,\tilde{\alpha}) + D_{zu}(\alpha)\mathscr{X}(\alpha,\tilde{\alpha}) = C_z(\alpha) + D_{zu}(\alpha)\mathscr{K}(\tilde{\alpha})C_y(\alpha),$$

$$\mathscr{K}(\tilde{\alpha}) := R(\tilde{\alpha})^{-1}L(\tilde{\alpha}),$$

yields

$$\mathcal{N}_u{}'\Psi\mathcal{N}_u = \begin{bmatrix} F(\alpha)\mathbb{A}(\alpha,\tilde{\alpha})+(\bullet)'+\dot{P}(\alpha) & P(\alpha)-F(\alpha)+\eta\mathbb{A}(\alpha,\tilde{\alpha})'V(\alpha)' & \mathbb{C}(\alpha,\tilde{\alpha})'Q(\tilde{\alpha}) \\ P(\alpha)-F(\alpha)'+\eta V(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) & -\eta(V(\alpha)+V(\alpha)') & \mathbf{0}_{n\times n_z} \\ Q(\alpha)'\mathbb{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & -Q(\alpha)'Q(\alpha) \end{bmatrix}. \tag{6.19}$$

Multiplying (6.19) by $\mathscr{T}_2$ from left and by its transpose from right with

$$\mathscr{T}_2 = \begin{bmatrix} I & \mathbb{A}(\alpha,\tilde{\alpha})' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (Q(\alpha)^{-1})' \end{bmatrix}$$

leads to (6.5), i.e.

$$\mathscr{T}_2\ (6.19)\ \mathscr{T}_2' = \begin{bmatrix} \mathbb{A}(\alpha,\tilde{\alpha})'P(\alpha)+P(\alpha)\mathbb{A}(\alpha,\tilde{\alpha})+\dot{P}(\alpha) & \mathbb{C}(\alpha,\tilde{\alpha})' \\ \mathbb{C}(\alpha,\tilde{\alpha}) & -I_{n_z} \end{bmatrix} < 0.$$

□

6.4 Synthesis Conditions with $\mathscr{H}_\infty$ Performance

Problem 6.2. *Suppose that the scheduling parameters $\theta(t)$ are provided as $\tilde{\theta}(t)$ with uncertainty $\delta(t)$. Converting all the open-loop system matrices and synthesis variables to the multi-simplex variables α or $\tilde{\alpha}$ instead of θ and $\tilde{\theta}$, respectively, using* (3.15) *and* (3.16)*. For a given positive scalar γ_∞, find a RGS static output-feedback controller in the form of* (6.2) *to stabilize the closed-loop system* (6.3) *for any pair $(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t)) \in \Lambda\times\Omega$ such that*

$$\sup_{(\tilde{\alpha}(t),\dot{\tilde{\alpha}}(t))\in\Lambda\times\Omega}\ \sup_{w\in\mathscr{L}_2, w\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma_\infty. \tag{6.20}$$

Similarly, the SF controller is synthesized first, and then it is used in the second stage to synthesize RGS SOF controller.

6.4.1 State-Feedback $\mathscr{H}_\infty$ control

Theorem 6.3. *Given a scalar $\gamma_\infty > 0$ and a sufficiently small scalar $\varepsilon > 0$. If there exist a continuously differentiable parameter-dependent matrix $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, parameter-dependent* *matrices* $Z(\alpha) \in \mathbb{R}^{nu \times n}$*,* $G(\alpha) \in \mathbb{R}^{n \times n}$ *for any pair* $(\alpha(t), \dot{\alpha}(t)) \in \Lambda \times \Omega$ *such that the following PLMI satisfied*

$$\begin{bmatrix} \mathbb{A}(\alpha)G(\alpha) + B_u(\alpha)Z(\alpha) + (\bullet)' - \dot{P}(\alpha) & \star & \star & \star \\ P(\alpha) - G(\alpha) + \varepsilon(\mathbb{A}(\alpha)G(\alpha) + B_u(\alpha)Z(\alpha))' & -\varepsilon(G(\alpha) + G(\alpha)') & \star & \star \\ \mathbb{C}(\alpha)G(\alpha) + D_{zu}(\alpha)Z(\alpha) & \varepsilon\mathbb{C}(\alpha)G(\tilde{\alpha}) + \varepsilon D_{zu}(\alpha)Z(\alpha) & -I_{n_z} & \star \\ \mathbb{B}(\alpha)' & \mathbf{0}_{n_w \times n} & \mathbb{D}(\alpha)' & -\gamma_\infty^2 I_{n_w} \end{bmatrix} < \mathbf{0},$$

then, the gain-scheduling controller

$$K(\alpha) = Z(\alpha)G(\alpha)^{-1},$$

stabilizes the closed-loop system with guaranteed $\mathscr{H}_\infty$ *performance bound* γ_∞ *satisfying* (6.20).

Proof. The proof can be found in Chapter 5 □

6.4.2 Static Output-Feedback $\mathscr{H}_\infty$ control

Theorem 6.4. *Given* $K(\alpha)$*, sufficiently small positive scalar* η*, and positive scalar* $\gamma_\infty > 0$*. If there exist a continuously differentiable parameter-dependent matrix* $0 < P(\alpha) = P(\alpha)' \in \mathbb{R}^{n \times n}$*, parameter-dependent matrices* $V(\tilde{\alpha}) \in \mathbb{R}^{n \times n}$*,* $F(\tilde{\alpha}) \in \mathbb{R}^{n \times n}$*,* $Q(\tilde{\alpha}) \in \mathbb{R}^{nz \times nz}$*,* $R(\tilde{\alpha}) \in \mathbb{R}^{nu \times nu}$*, and* $L(\tilde{\alpha}) \in \mathbb{R}^{nu \times ny}$ *for any pair* $(\tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) \in \Lambda \times \Omega$ *such that the following PLMI satisfied*

$$\begin{bmatrix} \mathscr{A}(\alpha)'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha) + \dot{P}(\alpha) & \star & \star & \star & \star \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\alpha) & -\eta(V(\alpha) + V(\alpha)') & \star & \star & \star \\ B_w(\alpha)'F(\alpha)' & \eta B_w(\alpha)'V(\alpha)' & -\gamma_\infty^2 I_{n_w} & \star & \star \\ Q(\alpha)'\mathscr{C}(\alpha) & \mathbf{0}_{n_z \times n} & Q(\alpha)'D_{zw}(\alpha) & I_{n_z} - Q(\alpha) - Q(\alpha)' & \star \\ B_u(\alpha)'F(\alpha)' + L(\tilde{\alpha})C_y(\alpha) - R(\tilde{\alpha})K(\alpha) & \eta B_u(\alpha)'V(\alpha)' & L(\tilde{\alpha})D_{yw}(\alpha) & D_{zu}(\alpha)'Q(\alpha) & -R(\tilde{\alpha}) - R(\tilde{\alpha})' \end{bmatrix} < \mathbf{0} \tag{6.21}$$

with

$$\mathscr{A}(\alpha) := A(\alpha) + B_u(\alpha)K(\alpha)$$

$$\mathscr{C}(\alpha) := C_z(\alpha) + D_{zu}(\alpha)K(\alpha).$$

Then, the static output-feedback gain-scheduling controller

$$\mathscr{K}(\tilde{\alpha}) = R(\tilde{\alpha})^{-1}L(\tilde{\alpha}) \tag{6.22}$$

is the stabilizing controller for the closed-loop system (6.3) *satisfying* (6.20).

Proof. Inequality (6.21) holds if (6.8) in Lemma 6.2 is satisfied with the following definitions

$$\mathscr{V} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ I \end{bmatrix}, \qquad \mathscr{Z} = R(\tilde{\alpha}), \qquad \mathscr{U} = [\mathscr{X}(\tilde{\alpha}) \;\; \mathbf{0} \;\; \mathscr{Y}(\tilde{\alpha}) \;\; \mathbf{0} \;\; -I],$$

$$\Psi = \begin{bmatrix} \mathscr{A}(\alpha)'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha) + \dot{P}(\alpha) & \star & \star & \star & \star \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\tilde{\alpha}) & -\eta(V(\alpha) + V(\alpha)') & \star & \star & \star \\ B_w(\alpha)'F(\alpha)' & \eta B_w(\alpha)'V(\alpha)' & -\gamma_\infty^2 I_{n_w} & \star & \star \\ Q(\alpha)'\mathscr{C}(\alpha) & \mathbf{0}_{n_z \times n} & Q(\alpha)'D_{zw}(\alpha) & -Q(\alpha)'Q(\alpha) & \star \\ B_u(\alpha)'F(\alpha)' & \eta B_u(\alpha)'V(\alpha)' & \mathbf{0}_{n_u \times n_w} & D_{zu}(\alpha)'Q(\alpha) & \mathbf{0}_{n_u} \end{bmatrix},$$

with the following null spaces

$$\mathscr{N}_v = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathscr{N}_u = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \\ \mathscr{X}(\tilde{\alpha}) & \mathbf{0} & \mathscr{Y}(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}$$

and

$$\begin{aligned} \mathscr{A}(\alpha) &:= A(\alpha) + B_u(\alpha)K(\tilde{\alpha}) \\ \mathscr{C}(\alpha) &:= C_z(\alpha) + D_{zu}(\alpha)K(\tilde{\alpha}) \\ \mathscr{X}(\tilde{\alpha}) &:= R(\tilde{\alpha})^{-1}L(\tilde{\alpha})C_y(\alpha) - K(\tilde{\alpha}) \\ \mathscr{Y}(\tilde{\alpha}) &:= R(\tilde{\alpha})^{-1}L(\tilde{\alpha})D_{yw}(\alpha). \end{aligned}$$

Thus,

$$\mathscr{V}\mathscr{Z}\mathscr{U} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ L(\tilde{\alpha})C_y(\alpha) - R(\tilde{\alpha})K(\alpha) & \mathbf{0} & L(\tilde{\alpha})D_{yw}(\alpha) & \mathbf{0} & -R(\tilde{\alpha}) \end{bmatrix}.$$

Consider now (6.9a),

$$\mathscr{N}_v \Psi \mathscr{N}_v' = \begin{bmatrix} \mathscr{A}(\alpha)'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha) + \dot{P}(\alpha) & \star \\ P(\alpha) - F(\alpha)' + \varepsilon V(\alpha)\mathscr{A}(\alpha) & -\varepsilon(V(\alpha) + V(\alpha)') \end{bmatrix} < 0. \tag{6.23}$$

Multiplying (6.23) by $\begin{bmatrix} I_n & \mathscr{A}(\alpha)' \end{bmatrix}$ from left and its transpose from right to obtain

$$\mathscr{A}(\alpha)'P(\alpha) + P(\alpha)\mathscr{A}(\alpha) + \dot{P}(\alpha) < 0,$$

which is in addition to $P(\alpha) > 0$, represents Lyapunov stability condition for $\mathscr{A}(\alpha)$. This proves (6.9a). On the other hand, (6.9b) is

$$\mathscr{N}_u{}'\Psi\mathscr{N}_u = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathscr{X}(\tilde{\alpha})' \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathscr{Y}(\tilde{\alpha})' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \end{bmatrix} \times$$

$$\begin{bmatrix} \mathscr{A}(\alpha)'F(\alpha)' + F(\alpha)\mathscr{A}(\alpha) + \dot{P}(\alpha) & P(\alpha) - F(\alpha) + \varepsilon\mathscr{A}(\alpha)'V(\alpha)' & F(\alpha)B_w(\tilde{\alpha}) & \mathscr{C}(\alpha)'Q(\alpha) & F(\alpha)B_u(\alpha) \\ P(\alpha) - F(\alpha)' + \eta V(\alpha)\mathscr{A}(\alpha) & -\eta(V(\alpha) + V(\alpha)') & \eta V(\alpha)B_w(\alpha) & \mathbf{0}_{n\times n_z} & \eta V(\alpha)B_u(\alpha) \\ B_w(\alpha)'F(\alpha)' & \eta B_w(\alpha)'V(\alpha)' & -\gamma_\infty^2 I_{n_w} & D_{zw}(\alpha)'Q(\alpha) & \mathbf{0}_{n_w\times n_u} \\ Q(\alpha)'\mathscr{C}(\alpha) & \mathbf{0}_{n_z\times n} & Q(\alpha)'D_{zw}(\alpha) & -Q(\alpha)'Q(\alpha) & Q(\alpha)'D_{zu}(\alpha) \\ B_u(\alpha)'F(\alpha)' & \eta B_u(\alpha)'V(\alpha)' & \mathbf{0}_{n_u\times n_w} & D_{zu}(\alpha)'Q(\alpha) & \mathbf{0}_{n_u} \end{bmatrix}$$

$$\times \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \\ \mathscr{X}(\tilde{\alpha}) & \mathbf{0} & \mathscr{Y}(\tilde{\alpha}) & \mathbf{0} \end{bmatrix}.$$

results

$$
\mathcal{N}_u{}'\Psi\mathcal{N}_u = \begin{bmatrix} F(\alpha)\mathscr{A}(\alpha)+F(\alpha)B_u(\alpha)\mathscr{X}(\tilde{\alpha})+(\bullet)'+\dot{P}(\alpha) & \star & \star & \star \\ P(\alpha)-F(\alpha)'+\eta V(\alpha)\mathscr{A}(\alpha)+\eta V(\alpha)B_u(\alpha)\mathscr{X}(\alpha) & -\eta(V(\alpha)+V(\alpha)') & \star & \star \\ B_w(\alpha)'F(\alpha)'+\mathscr{Y}(\tilde{\alpha})'B_u(\alpha)'F(\alpha)' & \eta B_w(\alpha)'V(\alpha)'+\eta\mathscr{Y}(\alpha)'B_u(\alpha)'V(\alpha)' & -\gamma_\infty^2 I_{n_w} & \star \\ Q(\alpha)'\mathscr{C}(\alpha)+Q(\alpha)'D_{zu}(\alpha)\mathscr{X}(\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'D_{zw}(\alpha)+Q(\alpha)'D_{zu}(\alpha)\mathscr{Y}(\tilde{\alpha}) & -Q(\alpha)'Q(\alpha) \end{bmatrix}.
$$

and

$$
\mathcal{N}_u{}'\Psi\mathcal{N}_u = \begin{bmatrix} F(\alpha)\mathbb{A}(\alpha,\tilde{\alpha})+\mathbb{A}(\alpha,\tilde{\alpha})'F(\alpha)'+\dot{P}(\alpha) & \star & \star & \star \\ P(\alpha)-F(\alpha)'+\eta V(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) & -\eta(V(\alpha)+V(\alpha)') & \star & \star \\ \mathbb{B}(\alpha,\tilde{\alpha})'F(\alpha)' & \eta\mathbb{B}(\alpha,\tilde{\alpha})'V(\alpha)' & -\gamma_\infty^2 I_{n_w} & \star \\ Q(\alpha)'\mathbb{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'\mathbb{D}(\alpha,\tilde{\alpha}) & -Q(\alpha)'Q(\alpha) \end{bmatrix}. \tag{6.24}
$$

with the following closed-loop system relationships

$$
\begin{aligned}
\mathbb{A}(\alpha,\tilde{\alpha}) &= A(\alpha)+B_u(\alpha)\mathscr{K}(\tilde{\alpha})C_y(\alpha) \\
\mathbb{B}(\alpha,\tilde{\alpha}) &= B_w(\alpha)+B_u(\alpha)\mathscr{K}(\tilde{\alpha})D_{yw}(\alpha) \\
\mathbb{C}(\alpha,\tilde{\alpha}) &= C_z(\alpha)+D_{zu}(\alpha)\mathscr{K}(\tilde{\alpha})C_y(\alpha) \\
\mathbb{D}(\alpha,\tilde{\alpha}) &= D_{zw}(\alpha)+D_{zu}(\alpha)\mathscr{K}(\tilde{\alpha})D_{yw}(\alpha)
\end{aligned}
$$

where $\mathscr{K}(\tilde{\alpha})$ is defined in (6.14).

In order to show the negative definiteness of (6.24), the steps depicted in [74] is followed and may produce some conservativeness to the conditions. Note that if inequality (6.21) is negative-definite, then it is also satisfied with $-Q(\alpha)'Q(\alpha)$ replacing $I_{n_z}-Q(\alpha)-Q(\alpha)'$, since $(I_{n_z}-Q(\alpha))'(I_{n_z}-Q(\alpha))>0$ implies $-Q(\alpha)'Q(\alpha)<I_{n_z}-Q(\alpha)-Q(\alpha)'$. In other words,

$$
\mathcal{N}_u{}'\Psi\mathcal{N}_u < \begin{bmatrix} F(\alpha)\mathbb{A}(\alpha,\tilde{\alpha})+\mathbb{A}(\alpha,\tilde{\alpha})'F(\alpha)'+\dot{P}(\alpha) & \star & \star & \star \\ P(\alpha)-F(\alpha)'+\eta V(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) & -\eta(V(\alpha)+V(\alpha)') & \star & \star \\ \mathbb{B}(\alpha,\tilde{\alpha})'F(\alpha)' & \eta\mathbb{B}(\alpha,\tilde{\alpha})'V(\alpha)' & -\gamma_\infty^2 I_{n_w} & \star \\ Q(\alpha)'\mathbb{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'\mathbb{D}(\alpha,\tilde{\alpha}) & I_{n_z}-Q(\alpha)-Q(\alpha)' \end{bmatrix} < 0 \tag{6.25}
$$

Setting $\eta V(\alpha) := U(\alpha)$ and multiplying the middle term of (6.25) by $\mathscr{T}(\tilde{\alpha})'$ on the left and by

$\mathscr{T}(\tilde{\alpha})$ on the right, with

$$\mathscr{T}(\alpha,\tilde{\alpha}) = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbb{A}(\alpha,\tilde{\alpha}) & \mathbb{B}(\alpha,\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q'(\alpha)^{-1} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} I & \mathbb{A}(\alpha,\tilde{\alpha})' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{B}(\alpha,\tilde{\alpha})' & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Q'(\alpha)^{-1} \end{bmatrix} \begin{bmatrix} F(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) + \mathbb{A}(\alpha,\tilde{\alpha})'F(\alpha)' + \dot{P}(\alpha) & \star & \star & \star \\ P(\alpha) - F(\alpha)' + U(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) & -U(\alpha) - U(\alpha)' & \star & \star \\ \mathbb{B}(\alpha,\tilde{\alpha})'F(\alpha)' & \mathbb{B}(\alpha,\tilde{\alpha})'U(\alpha)' & -\gamma_\infty^2 I_{n_w} & \star \\ Q(\alpha)'\mathbb{C}(\alpha,\tilde{\alpha}) & \mathbf{0}_{n_z\times n} & Q(\alpha)'\mathbb{D}(\alpha,\tilde{\alpha}) & -Q(\alpha)'Q(\alpha)' \end{bmatrix} \times$$

$$\begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbb{A}(\alpha,\tilde{\alpha}) & \mathbb{B}(\alpha,\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q(\alpha)^{-1} \end{bmatrix} < 0$$

that leads to

$$\begin{bmatrix} F(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) + \dot{P}(\alpha) + \mathbb{A}(\alpha,\tilde{\alpha})'P(\alpha) + \mathbb{A}(\alpha,\tilde{\alpha})'U(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) & P(\alpha) - F(\alpha) - \mathbb{A}(\alpha,\tilde{\alpha})'U(\alpha) & F(\alpha)\mathbb{B}(\alpha,\tilde{\alpha}) + \mathbb{A}(\alpha,\tilde{\alpha})'U(\alpha)\mathbb{B}(\alpha,\tilde{\alpha}) & \mathbb{C}(\alpha,\tilde{\alpha})'Q(\alpha) \\ \mathbb{B}(\alpha,\tilde{\alpha})'P(\alpha) + \mathbb{B}(\alpha,\tilde{\alpha})'U(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) & -\mathbb{B}(\alpha,\tilde{\alpha})'U(\alpha) & \mathbb{B}(\alpha,\tilde{\alpha})'U(\alpha)\mathbb{B}(\alpha,\tilde{\alpha}) - \gamma_\infty^2 I_{n_w} & \mathbb{D}(\alpha,\tilde{\alpha})'Q(\alpha) \\ \mathbb{C}(\alpha,\tilde{\alpha}) & \mathbf{0} & \mathbb{D}(\alpha,\tilde{\alpha}) & -Q(\alpha) \end{bmatrix}$$

$$\times \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbb{A}(\alpha,\tilde{\alpha}) & \mathbb{B}(\alpha,\tilde{\alpha}) & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q(\alpha)^{-1} \end{bmatrix} < 0$$

and

$$\begin{bmatrix} \mathbb{A}(\alpha,\tilde{\alpha})'P(\alpha) + P(\alpha)\mathbb{A}(\alpha,\tilde{\alpha}) + \dot{P}(\alpha) & P(\alpha)\mathbb{B}(\alpha,\tilde{\alpha}) & \mathbb{C}(\alpha,\tilde{\alpha})' \\ \mathbb{B}(\alpha,\tilde{\alpha})'P(\alpha) & -\gamma_\infty^2 I_{n_w} & \mathbb{D}(\alpha,\tilde{\alpha})' \\ \mathbb{C}(\alpha,\tilde{\alpha}) & \mathbb{D}(\alpha,\tilde{\alpha}) & -I_{n_z} \end{bmatrix} < 0$$

which represents a parameter-dependent version of the bounded-real lemma for LPV systems [79]. Thus, (6.9b) is verified. □

Remark 6.2. *Once a feasible solution exists for Theorem 6.2 and Theorem 6.4, the controller coefficients $\mathscr{K}_i$ for $i = 0, 1, \cdots, q$ in* (6.4) *can be obtained from the multi-simplex coefficients of* *the controller matrix* $\mathscr{K}_{j_1,j_2,\cdots,j_q,k_1,k_2,\cdots,k_q}$ *in* (6.14) *and* (6.22) *using the inverse transformation (multi-simplex to affine transformation)* (3.40) *and* (3.41).

Remark 6.3. *In contrast to the existing literature of RGS control [2, 36, 80, 38] that only allows the state matrix* $A(\theta)$ *to be affected by the varying parameters, the synthesis conditions of Theorem 6.2 and Theorem 6.4 deal with the general case that all the open-loop matrices are functions of varying parameters. This is the main advantage of the developed conditions over the existing ones.*

Remark 6.4. *The conditions of Theorem 6.2 and Theorem 6.4 associated with some conservativeness due to over-bounding the* $(4,4)$ *block of* (6.15) *and* (6.24). *Therefore, Iterative Static Output-Feedback Design (ISOFD) algorithm was developed to reduce conservativeness. This algorithm is shown for the* $\mathscr{H}_2$ *performance but it applies for* $\mathscr{H}_\infty$ *performance as well. Initially, SF controller* $K(\alpha)$ *is obtained via Theorem 6.1 or Theorem 6.3. This controller is used in Theorem 6.2 or Theorem 6.4 to synthesize the RGS SOF controller. Each iteration of the ISOFD algorithm assures a feasible solution with at least* $\nu_i \leq \nu_{i-1}$. *Note that the algorithm convergence is guaranteed since* ν *is monotonically decreasing and bounded from below.*

Remark 6.5. *As a special case of Theorem 6.2 and Theorem 6.4, robust SOF controller can also be synthesized. More specifically, constraining the synthesis variables* $R(\tilde{\alpha})$ *and* $L(\tilde{\alpha})$ *to be parameter independent (i.e. constant matrices) leads to a robust SOF controller.*

Remark 6.6. *As shown above, the synthesis conditions of Theorem 6.2 and Theorem 6.4 are formulated as PLMIs (for a fixed* ε *and* η*) in terms of time-varying parameters inside the multi-simplex domain. ROLMIP [56] is used to implement the ISOFD algorithm to obtain the RGS SOF optimal controller.*

6.5 Illustrative Examples

In this section, two examples are presented to demonstrate the effectiveness of the developed approach. First, an academic example is given to illustrate the achieved performance with different

Algorithm 2: Iterative Static Output-Feedback Design (ISOFD) Algorithm.

Initialization:

- Set $i = 0$, $\mathscr{K}_0(\tilde{\alpha}) = \mathbf{0}$.
- Using Theorem 6.1, compute initial state-feedback controller $K_0(\alpha)$.
- Set i_{max} and $Tolarance$.

repeat

- Set $i = i + 1$.
- Given $K_{i-1}(\alpha)$, solve the conditions of Theorem 6.2 to obtain SOF controller $\mathscr{K}_i(\tilde{\alpha})$ with a minimal achievable bound ν_i.
- Given $\mathscr{K}_i(\tilde{\alpha})$, solve the conditions of Theorem 6.1 to obtain $K_i(\alpha)$.

until $i \geqslant i_{max}$ OR $|\nu_i - \nu_{i-1}| < Tolarance$;

bounds of measurement noise level using Algorithm 2. Second, the developed method is applied to a realistic EVVT actuator obtained from experimental study [81]. This example serves to validate the developed synthesis approach through realistic LPV model from engineering application point of view. The synthesis conditions are implemented in MATLAB environment (R2013a). The computer used for control design is an Intel Core $i7$ (2.4 GHz) processor, 6GB RAM with Windows 10.

6.5.1 Academic Example

Consider the following LPV system [1],

$$\left[\begin{array}{c|c:c} A(\theta) & B_u(\theta) & B_w(\theta) \\ \hline C_z(\theta) & D_{zu}(\theta) & \\ \hdashline C_y(\theta) & & D_{yw}(\theta) \end{array}\right] = \left[\begin{array}{cc|c:c} 25.9-60\theta(t) & 1 & 3 & -0.03 \\ 20-40\theta(t) & 34-64\theta(t) & 2 & -0.47 \\ \hline 1 & 1 & 0 & \\ 0 & 0 & 1 & \\ \hdashline 1 & 0 & & 0 \end{array}\right].$$

The varying parameter $\theta(t)$ has the following bounds $0 \leq \theta(t) \leq 1$, $|\dot{\theta}(t)| \leq 1$, with measurement uncertainty bound $|\delta(t)| \leq \zeta$ and $|\dot{\delta}(t)| \leq 10 \times \zeta$. The scalars ε and η in Theorem 6.1 and Theorem 6.2 are introduced to provide additional degrees of freedom in the synthesis conditions. To reduce computational burdens, these scalars are chosen (in a logarithmic scale) between 10^{-1} and 10^{-5}. In this example, $\varepsilon = 0.01$ and $\eta = 0.1$ are found to achieve best performance bound. For the iterative algorithm, a tolerance and maximum number of iterations are set to 10^{-4} and four, respectively.

Table 6.1 shows the guaranteed bound ν for the conditions of Theorem 6.1 and Theorem 6.2 with the iterative algorithm. In the first stage, the conditions of Theorem 6.1 is used to obtain GS state-feedback controller, and then, this controller is fed to the second stage to synthesize the RGS SOF controller. Note that the achievable performance using the first stage is slightly lower than the one associated with the second stage (as expected) since in the case of state-feedback full state measurement is assumed to be available in real-time for controller implementation while in the SOF case only the output feedback is used. Algorithm 2 is used to reduce conservativeness gradually as shown in Table 6.1 for the first four iterations. Figure 6.3 illustrates the convergence of the iterative algorithm with $\zeta = 0.1$. To the best authors' knowledge there is no SOF synthesis method that is able to handle the measurement noises in scheduling parameters. Comparing with full-order dynamic output-feedback results in [2], it is clear that the method in [2] is very sensitive to small values of ζ, while the developed synthesis conditions presented in Table 6.1 shows good robustness against uncertainties in scheduling parameters.

6.5.2 EVVT Actuator

To validate the result of the developed approach for practical engineering applications, the Electric Variable Valve Timing (EVVT) cam phaser system is investigated in this section. The EVVT actuator consists of two main components: an electric motor and a planetary gear set. The planetary gear set consists of an outer ring gear, a planet gear carrier with planet gears attached, and a sun

Table 6.1: $\mathscr{H}_2$ performance with different bounds of measurement noise. $\varepsilon = 0.01$, $\eta = 0.1$.

		ζ						
		0	0.001	0.01	0.1	0.2	0.5	1
Method of [2]		9.087	9.15	9.706	39.57	75.406	173.92	324.33
1st stage		3.120	3.131	3.230	3.423	3.738	4.053	4.081
2nd stage		3.481	3.495	3.490	3.512	3.962	4.213	4.311
Iterations	1	3.463	3.474	3.480	3.500	3.732	3.816	4.115
	2	3.438	3.458	3.450	3.480	3.609	3.775	3.979
	3	3.380	3.402	3.424	3.462	3.588	3.649	3.888
	4	3.356	3.371	3.391	3.437	3.502	3.620	3.820

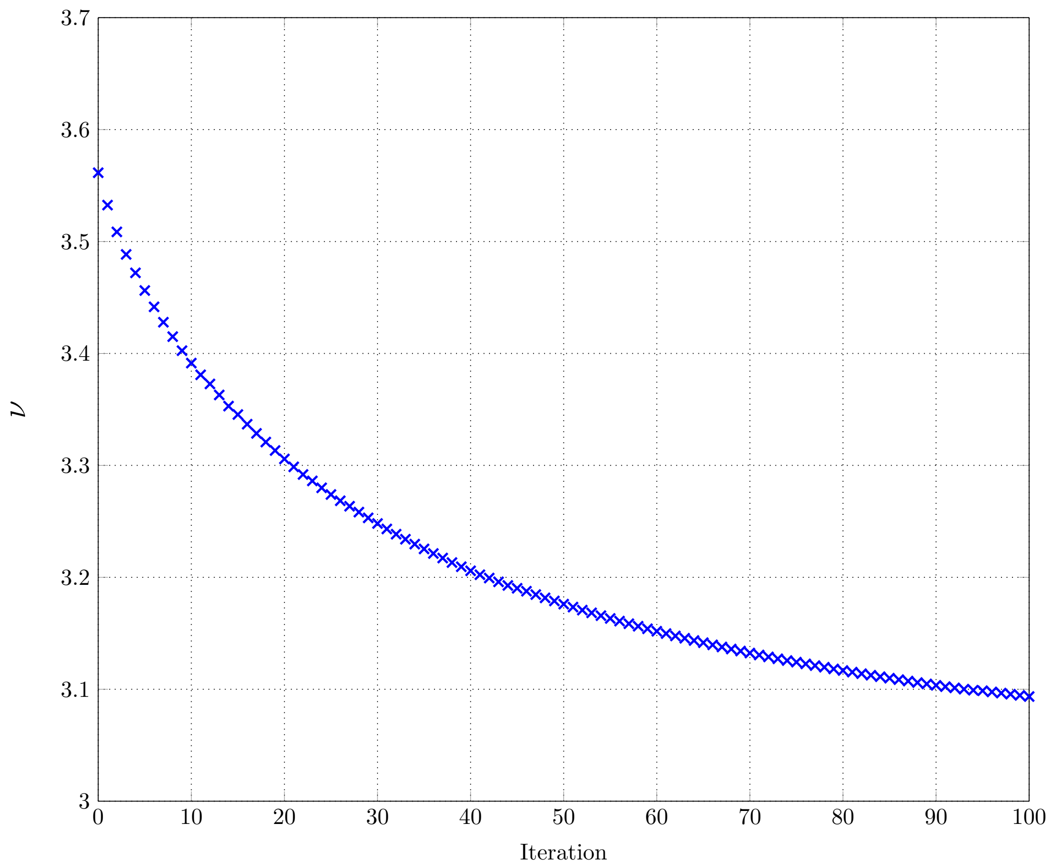


Figure 6.3: Algorithm Convergence

gear. The ring gear, that is running at the half speed of the crankshaft, is driven by the crankshaft through the engine timing belt. Details of the EVVT modeling work can be found in [82]. The ring gear, which is running at the half speed of the crankshaft, is driven by the crankshaft through the engine timing belt (see Figure 6.4). The planetary gear carrier is driven by an electric motor

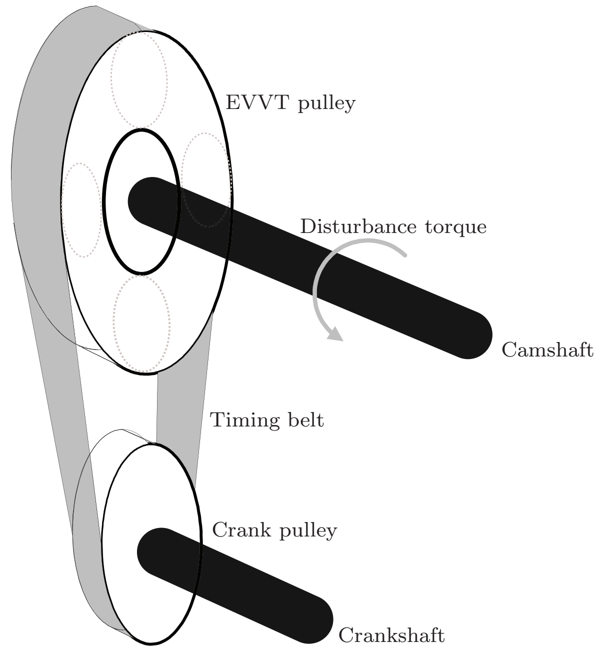


Figure 6.4: EVVT cam-phase actuator schematic diagram.

and four planet gears engaging both ring and sun gears at the same time, where the sun gear is connected to the camshaft. The speed of the camshaft is determined by the ring gear speed together with the EVVT motor speed, which provides the engine with flexible valve opening timing. Therefore, the cam-phase can be adjusted by controlling the EVVT motor speed with respect to the engine speed.

A series of system identification experimental tests were conducted at a range of fixed values of engine speed (N) and battery voltage (V) at the Energy and Automotive Research Lab of the Michigan State University (see [81] for the authors). It was found that the identified model of the EVVT actuator is in the following form,

$$G(N(t),V(t),s) = \frac{\theta_1(N(t),V(t))}{s(s+\theta_2(N(t),V(t)))}, \tag{6.26}$$

where $\theta_1(N(t),V(t))$ and $\theta_2(N(t),V(t))$ are time-varying coefficients as functions of engine speed and battery voltage. For notational simplicity, $\theta_1(t)$ and $\theta_2(t)$ are used to refer to $\theta_1(N(t),V(t))$ and $\theta_2(N(t),V(t))$, respectively. Note that $\theta_1(t)$ is associated with the DC gain of the transfer function (6.26), and $\theta_2(t)$ is the location of the open-loop pole of the 2nd order system. In other

Table 6.2: Range of the time-varying parameters.

$\theta_1(t) \in$	$[0.2529 \quad 0.6472]$
$\theta_2(t) \in$	$[6.975 \quad 14.540]$

words, the DC gain and pole location of the transfer function (6.26) are time-varying coefficients and functions of engine speed and battery voltage. It is worth mentioning that the values of $\theta_1(t)$ and $\theta_2(t)$ were obtained experimentally over specified fixed values of battery voltage and engine speed that cover the entire range of engine operating conditions. The ranges of the varying coefficients $\theta_1(t)$ and $\theta_2(t)$ are given in Table 6.2. In order to perform controller design using the conditions developed in Section 6.4, the EVVT plant model (6.26) needs to be realized in the state-space form as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\theta_2(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_1(t) \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where x_1 represents the cam-phase angle and u is the EVVT motor speed command. In order to accommodate for performance output, control energy, and disturbance input,

$$C_z(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{zu}(\theta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w(\theta) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},$$

are defined and the state-space realization of the overall LPV system corresponding to the matrices defined in (6.1) are

$$\left[\begin{array}{c|c:c} A(\theta) & B_u(\theta) & B_w(\theta) \\ \hline C_z(\theta) & D_{zu}(\theta) & \\ \hdashline C_y(\theta) & & D_{yw}(\theta) \end{array}\right] = \left[\begin{array}{cc|c:c} 0 & 1 & 0 & 0.1 \\ 0 & -\theta_2(t) & \theta_1(t) & 0 \\ \hline 1 & 0 & 0 & \\ 0 & 0 & 1 & \\ \hdashline 1 & 0 & & 0 \end{array}\right].$$

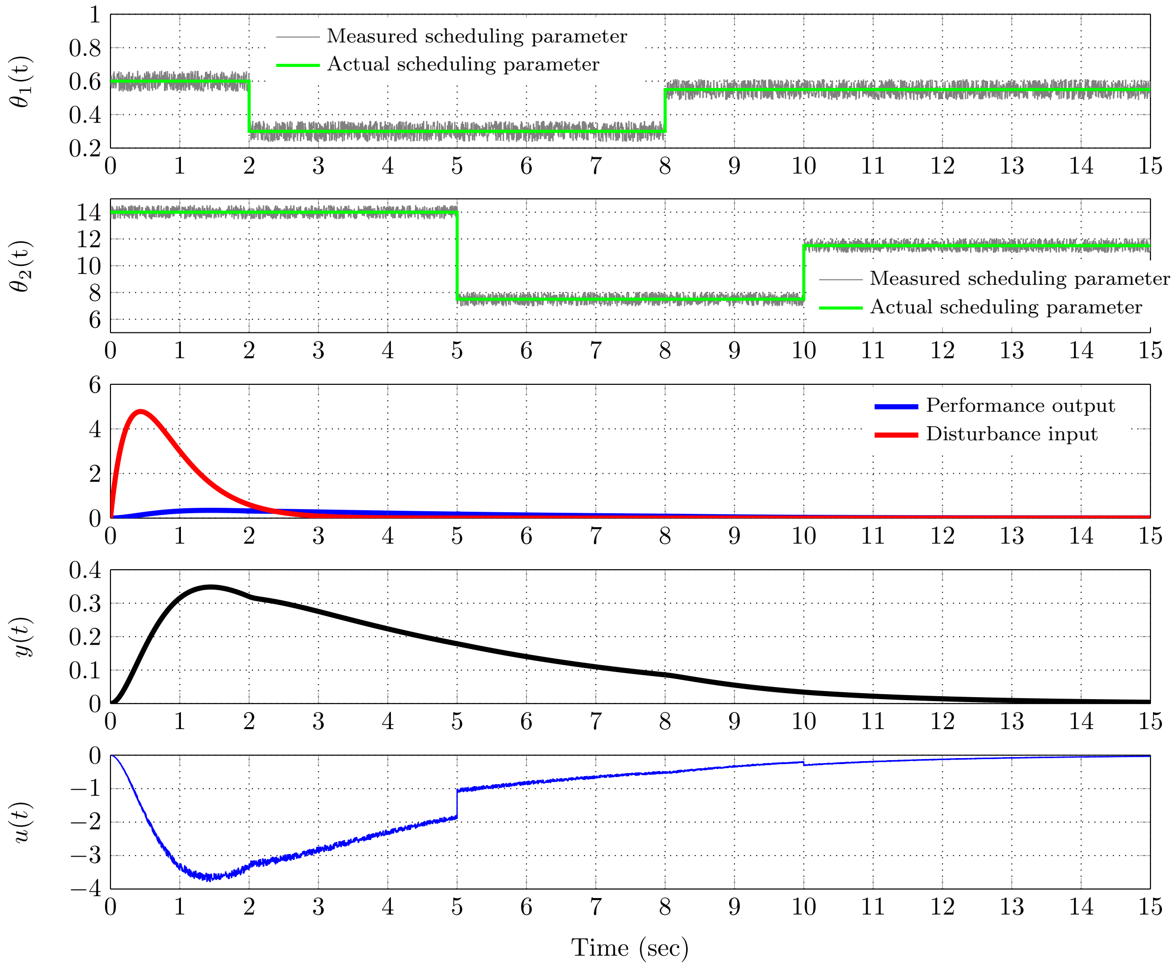


Figure 6.5: Time-domain simulations for the EVVT actuator (piecewise-constant scheduling signals).

First, the open-loop system matrices are converted from the $\theta(t)$-space into the $\alpha(t)$-space using the procedure presented in Section 6.2. Then, the conditions of Theorem 6.1 and Theorem 6.2 are used to synthesize RGS SOF controller. Table 6.3 shows the achieved $\mathscr{H}_2$ bound ν for both controllers (state-feedback and static output-feedback) as well as the number of LMI variables and LMI constraints associated with each theorem. After a feasible solution is obtained, the controller matrix in (6.14) is calculated, than the inverse transformation (3.40) and (3.41) is used to calculate controller coefficients in the original parameter space, i.e. the coefficient in (6.4) to implement the controller in real-time using the MSPs. These coefficients are obtained as

$$\mathscr{K}_0 = -0.7126, \qquad \mathscr{K}_1 = -0.5879, \qquad \mathscr{K}_2 = -0.6736.$$

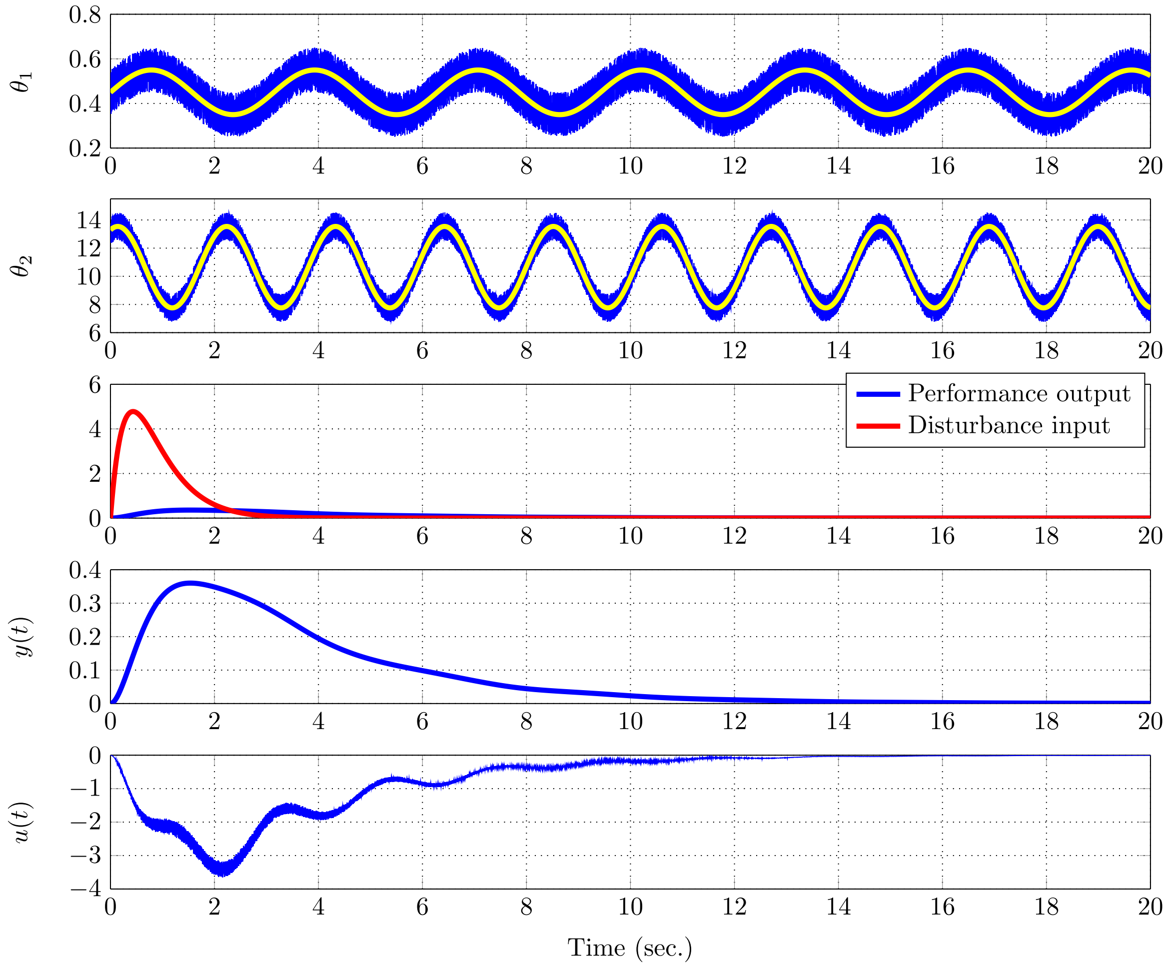


Figure 6.6: Time-domain simulations for the EVVT actuator (sinusoidal scheduling signals).

Table 6.3: $\mathscr{H}_2$ performance for stage 1 and stage 2. $\varepsilon = 10^{-5}$, $\eta = 0.01$.

	# LMI variables	# LMI constraints	ν
1st stage	25	60	0.759
2nd stage	68	487	0.767

It is clear from Table 6.3 that the guaranteed performance using SOF controller is very close to the performance of the state-feedback one.

Time-domain simulation results of the EVVT actuator are shown in Figure 6.5 and Figure 6.6 with different scheduling trajectories. Table look-up method is used to map the engine speed and battery voltage into the varying parameters $\theta_1(t)$ and $\theta_2(t)$. The simulation results demonstrate not only robustness of the synthesized controller against uncertainties in scheduling parameters but also the ability to attenuate disturbance torque on the cam-phase angle (performance output).

6.6 Summary

PLMIs conditions for synthesizing RGS SOF controller subject to inexactly MSPs are developed in this chapter. The foundation of the approach is based on a two-stage design method by designing state-feedback controller in the first stage and using the resulting controller for synthesizing RGS SOF controller in the second stage. Both performances, $\mathscr{H}_2$ and $\mathscr{H}_\infty$, are considered. An iterative procedure (Iterative Static Output-Feedback Design (ISOFD) algorithm) is developed to reduce the upper bounds of the $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performances. The main novelty of the developed method is that it can handle the most general Linear Parameter-Varying (LPV) systems, where the varying parameters affect all the open-loop system matrices. Note that robust SOF controller can be handled as a special case of developed conditions. Two examples are presented, one is an academic example and the second is a practical Electric Variable Valve Timing (EVVT) actuator. Numerical results and simulations of these examples illustrate the effectiveness of the developed approach.

CHAPTER 7

EXPERIMENTAL VALIDATION ON EVVT ACTUATOR

This chapter presents experimental demonstration of the developed synthesis conditions for the Electric Variable Valve Timing (EVVT) actuator. The fuel economy, emissions, and performance of an internal combustion (IC) engine are heavily influenced by the intake and exhaust valve timings. With a conventional valve-train system, the intake and exhaust valve timing can only be optimized for a single operating condition. That is, the optimized valve timing can either improve fuel economy and reduce emissions at low engine speeds or maximize engine power and torque outputs at high engine speeds. Due to the growing fuel economy demands and emission regulations, electric variable valve timing (EVVT) systems [83] were developed. The challenging problem of improving fuel economy and reducing emissions at low engine speed while maintaining engine performance at high engine speed can be addressed with the help of EVVT systems. The EVVT system is an excellent candidate to verify the effectiveness of the RGS control approach developed in this dissertation. Since engine speed and vehicle battery voltage have significant impact on the engine performance, these two time-varying parameters have been selected for feedback control as noisy scheduling parameters.

Therefore, the goal is to design and implement RGS Static Output-Feedback (SOF) controller experimentally on EVVT actuator subject to noisy scheduling parameter. To do this, the procedure described in Fig. 7.1 is followed. First, a series of standard system identification tests are conducted on the EVVT system bench to obtain a family of Linear Time-Invariant (LTI) models. Using engine speed and battery voltage as time-varying parameters, a family of linear models of the EVVT system are obtained by performing multiple tests while maintaining engine speed and battery voltage at fixed predefined values. With the family of local LTI models, the LPV model of the EVVT system is formulated. Since the obtained LPV model has affine parametrization structure in terms of the varying parameters, this model is converted into a polytopic (multi-simplex) domain so that the $\mathscr{H}_\infty$ RGS control synthesis method presented in Chapter 5 can be applied to obtain the RGS

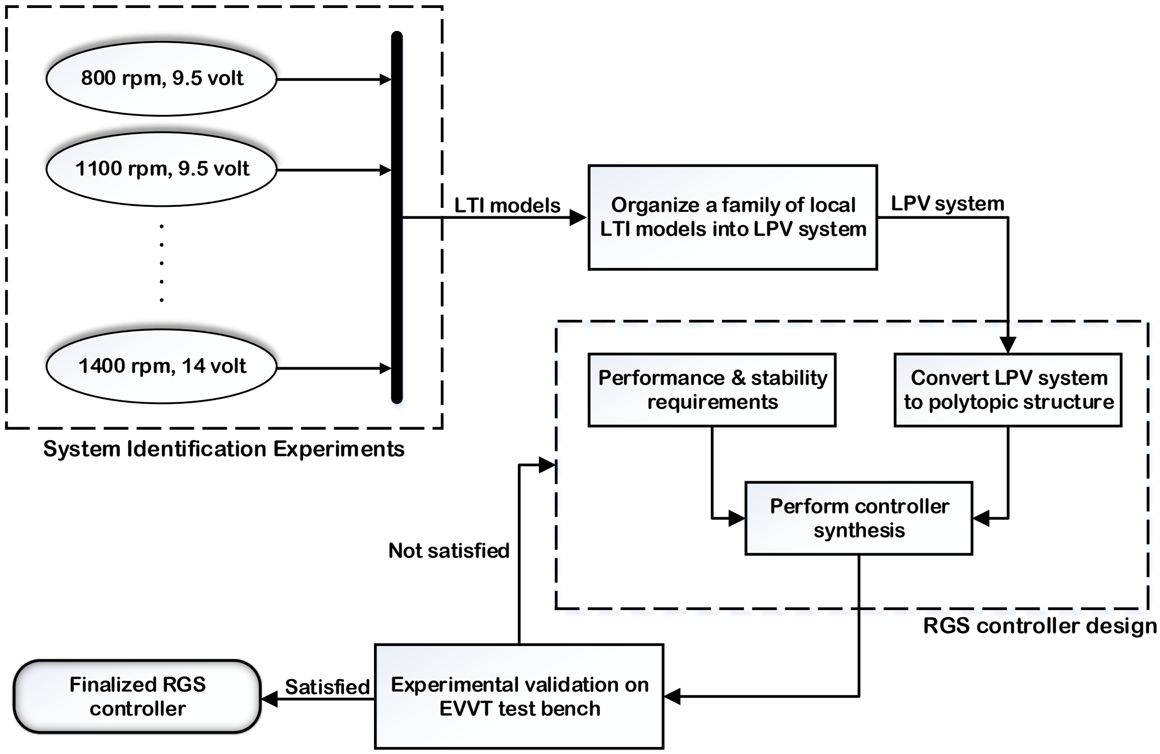


Figure 7.1: Flow chart for the design and implementation of a RGS controller on EVVT system.

controller. Once a controller is obtained, its performance is experimentally evaluated on the same test bench that was used to perform the system identification tests. If the performance and stability requirements of the EVVT system are not satisfied, another controller can be synthesized again by tuning design parameters in the synthesis conditions. This loop is performed until the stability and satisfactory performance are obtained on the test bench.

7.1 EVVT Engine Cam-Phasing Actuator

The EVVT system components are illustrated first and then experimental results of the system identification tests are given for fixed values of engine speed and battery voltage that cover the entire range of the parameters variations. In the third subsection, LPV modeling in the state-space representation will be explained for the EVVT actuator.

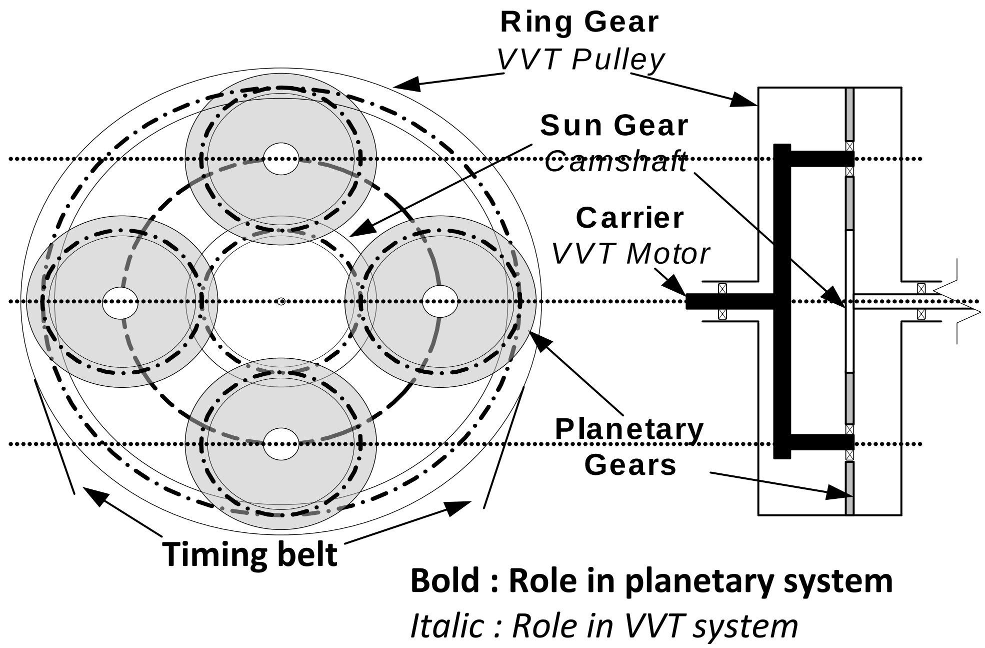


Figure 7.2: Electric planetary gear EVVT system.

7.1.1 Actuator Components

The EVVT system studied in this chapter operates with two main components: an electric motor and a planetary gear set (see Figure 7.2). The planetary gear set consists of an outer ring gear, a planet gear carrier with planet gears attached, and a sun gear. The ring gear, that is running at the half speed of the crankshaft, is driven by the crankshaft through the engine timing belt. Details of the modeling work for the EVVT can be found in [82]. Note that for the traditional valve-train system the ring gear is connected to the camshaft directly, resulting in a fixed valve timing, while for the EVVT system the ring gear and the camshaft is connected through the planetary gear set.

The planetary gear carrier is driven by an electric motor and four planet gears engaging both ring and sun gears at the same time, where the sun gear is connected to the camshaft. The speed of the camshaft is determined by the ring gear speed together with the EVVT motor speed, which provides the engine with flexible valve opening timing. The cam-phase angle can be expressed as the integration of the speed difference between the EVVT motor and ring gear. Therefore, the cam-phase can be adjusted by controlling the EVVT motor speed with respect to the engine speed. To hold the cam-phase at a constant value, the EVVT motor speed should match the ring gear speed. To advance the cam-phase, the EVVT motor speed should be faster than the ring gear speed; and to retard the cam-phase, the EVVT motor speed should be slower than the ring gear speed.

7.1.2 Test bench set-up

The EVVT experimental test bench is shown in Fig. 7.3. This bench is used for both system identification tests and control validation. Figure 7.3 shows the engine body, the EVVT actuator, and the cam-phase position sensor. A Ford 5.4L V8 engine head is used in these experiments. The cylinder head has a single camshaft driving two intake valves and one exhaust valve. An electric motor is used to simulate the motion of the engine crankshaft. Due to speed limitation of this electric motor, the maximum engine speed that can be achieved on the test bench is 1750 rpm. An encoder is installed on the motor shaft that generates crank angle signal with $1/64$ crank degree resolution and a gate signal (one pulse per revolution). Both encoder signals are used to obtain engine position and speed. A cam position sensor with four pulses per engine cycle, is installed on the back of the camshaft to detect the engine firing Top Dead Center (TDC) and to calculate the cam phase angle. Thus, the cam-phase angle is updated four times every engine cycle. An electrical oil pump is used to supply pressurized oil for lubricating both EVVT actuator and cylinder head assembly. Intake and exhaust valves are installed to provide the cyclic camshaft torque load. Opal-RT real-time engine prototype controller is used to control the EVVT and collect the experimental data for the test bench. The cam-phase position sensor signal is sampled by the Opal-RT prototype controller and the corresponding cam-phase angle is calculated within the Opal-RT real-time controller. The experiments are performed at MSU Automotive Controls Lab at a room temperature of 25°.

7.2 System Identification and LPV Modeling

This section presents detailed work of the system identification tests, construction of the LPV model, and state-space representation of the EVVT actuator.

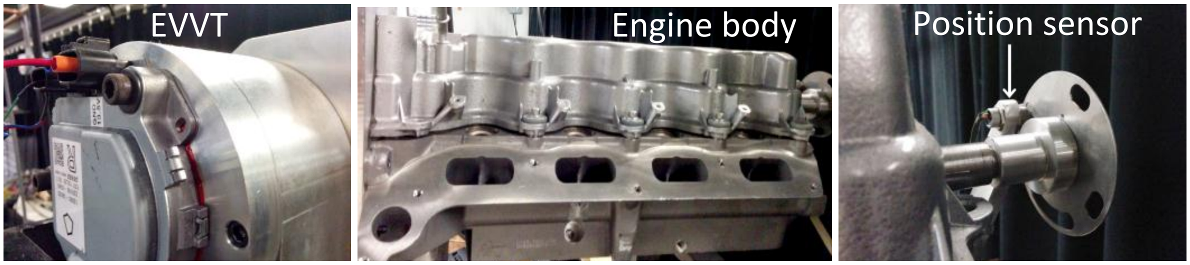


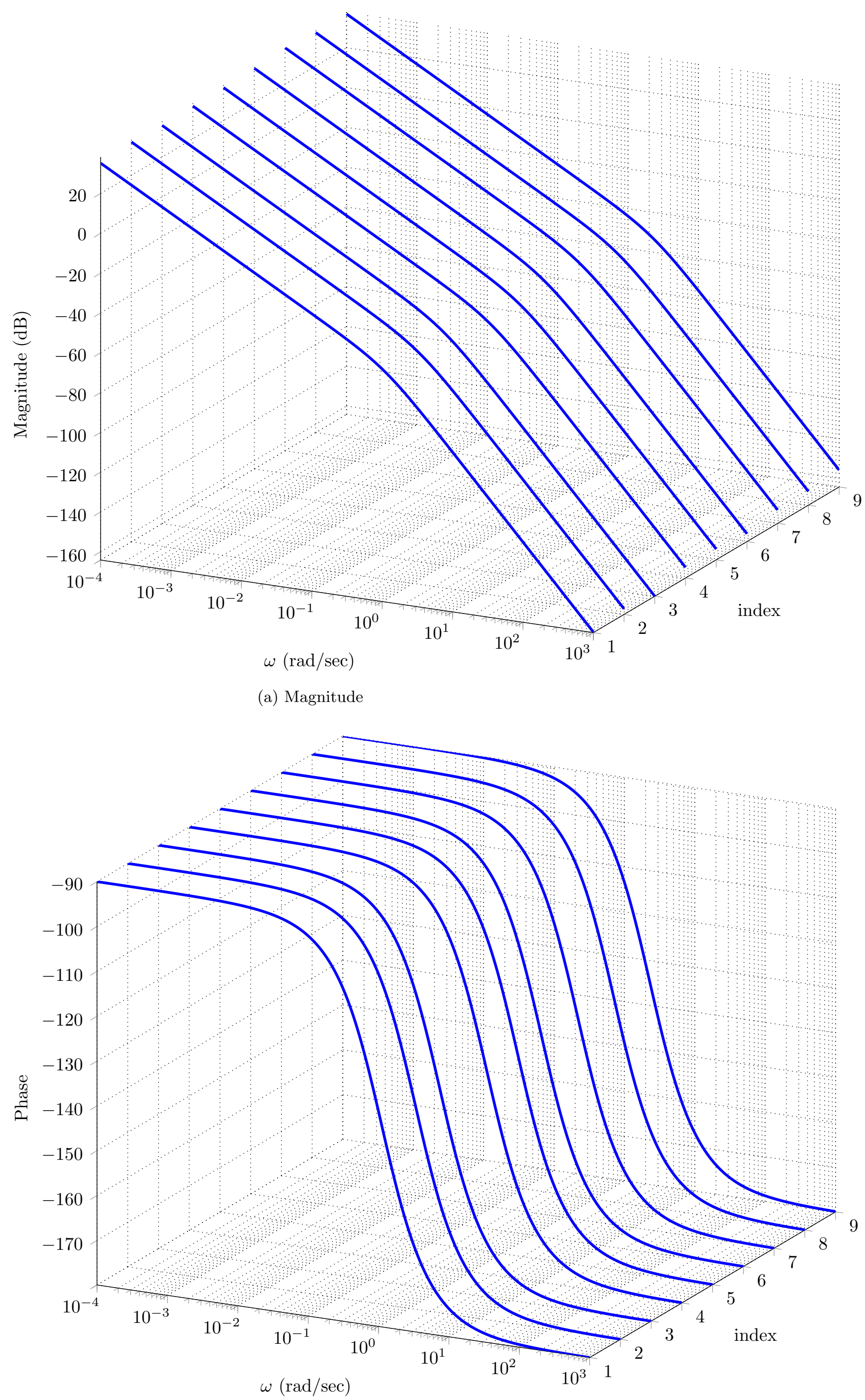
Figure 7.3: Engine Experiment setup.

Table 7.1: Fixed values of engine speed and battery voltage used in the system ID tests.

Engine speed $= [800\ ,\ 1100\ ,\ 1400](rpm)$
Battery voltage $= [9.5\ ,\ 11.5\ ,\ 14](volt)$

7.2.1 System Identification Tests of EVVT Actuator

Using the test bench described above, a series of tests at fixed values of engine speeds and battery voltage have been conducted. Standard tests [84, 85] are carried out to investigate step response deviations under various values of engine speed and vehicle battery voltage. These tests showed that when the vehicle battery voltage, supplied to the EVVT motor, drops from $14V$ to $9.5V$ at 800 *rpm* engine speed, the settling time of step response is almost doubled due to the reduced torque available for the EVVT motor to drive the ring gear. However, when the engine speed is reduced from 1400 *rpm* down to 800 *rpm* with a fixed battery voltage at $9.5V$, the settling time is cut by half because at low motor speed the available motor torque increases. Thus, these tests showed that the response time of the EVVT is strongly dependent on both engine speed and battery voltage. Therefore, these two varying parameters have been selected to be the time-varying parameters of the LPV EVVT model. Table 7.1 shows the fixed values of engine speed (N) and battery voltage (V) that have been used in the system identification tests. Therefore, to obtain a family of local LTI models, closed-loop system identification tests have been conducted at the fixed values of engine speed and battery voltage shown in Table 7.1 and nine LTI models are obtained (see the conference version for the authors [86]).



(a) Magnitude

(b) Phase

Figure 7.4: Bode plot of the 9 local LTI models.

7.2.2 LPV model construction

In this subsection the identified local LTI models are used to construct a single LPV model with engine speed and vehicle battery voltage as the time-varying parameters. It is found that the LPV

Table 7.2: Identified Coefficients $\theta_1(t)$ & $\theta_2(t)$

Battery voltage V (volt)	Engine Speed N (rpm)	$\theta_1(t)$	$\theta_2(t)$
9.5	800	0.2529	6.975
	1100	0.3371	7.809
	1400	0.2441	6.233
11.5	800	0.3835	8.470
	1100	0.4533	10.710
	1400	0.4520	10.780
14	800	0.5304	10.260
	1100	0.5390	12.420
	1400	0.6472	14.540

model of the EVVT actuator is in the following second order form,

$$G(N(t),V(t),s) = \frac{\theta_1(N(t),V(t))}{s(s+\theta_2(N(t),V(t)))}, \tag{7.1}$$

where $\theta_1(N(t),V(t))$ and $\theta_2(N(t),V(t))$ are time-varying coefficients functions of engine speed and battery voltage. For notational simplicity, $\theta_1(t)$ and $\theta_2(t)$ will be used to refer to $\theta_1(N(t),V(t))$ and $\theta_2(N(t),V(t))$[1], respectively. Table 7.2 shows the values of θ_1 and θ_2 that have been identified experimentally at the nine fixed operating conditions. Based on these identified values, the Bode plot corresponding to the nine local LTI models is shown in Fig. 7.4.

Note that $\theta_1(t)$ is associated with the DC gain of the transfer function (7.1), while $\theta_2(t)$ is the location of the open-loop pole of the 2nd order system. In other words, the DC gain and pole location of the transfer function (7.1) are time-varying coefficients and functions of engine speed and battery voltage. The range of the varying coefficients $\theta_1(t)$ and $\theta_2(t)$ are given in Table 7.3 corresponding to the entire range of the scheduling parameters (engine speeds and battery voltage).

θ_1 and θ_2 are plotted as function of engine speed and battery voltage in Fig. 7.5. From this figure, it can be observed that the vehicle battery voltage has substantial influence on both coefficients θ_1 and θ_2. The higher the battery voltage is, the larger both θ_1 and θ_2. This is mainly due to the fact that motor torque increases as the battery voltage increases. On the other hand, the

[1]Sometimes the time dependency will be ignored for simplified notations.

effects of the engine speed to both coefficients (for a given voltage) are mixed due to the following facts. At high voltage ($14\ volt$), the motor torque is saturated and as the engine speed increases the output power increases leading to increased values of θ_1 and θ_2. However, when the battery voltage drops to ($9.5\ volt$), the motor output power saturates at high engine speed leading to slower response (small value of θ_2) and reduced DC gain θ_1. To sum up, the engine speed has less influence on θ_1 and θ_2 than the vehicle battery voltage. It is worth mentioning that table look-up method has been used to interpolate the values of θ_1 and θ_2 in real-time implementation.

7.2.3 State-Space Representation

The state-space representation of any LPV systems can be given by

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t) \\ z(t) &= C_z(\theta(t))x(t) + D_{zu}(\theta(t))u(t) \\ y(t) &= C_y x(t) + D_{yw}(\theta(t))w(t), \end{aligned} \tag{7.2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{nu}$ is the control input, $w(t) \in \mathbb{R}^{nw}$ is the disturbance input, $z(t) \in \mathbb{R}^{nz}$ is the controlled output, and $y(t) \in \mathbb{R}^{ny}$ is the measured output. The system matrices have the following compatible dimensions $A(\theta(t)) \in \mathbb{R}^{n\times n}$, $B_u(\theta(t)) \in \mathbb{R}^{n\times nu}$, $B_w(\theta(t)) \in \mathbb{R}^{n\times nw}$, $C_z(\theta(t)) \in \mathbb{R}^{nz\times n}$, $D_{zu}(\theta(t)) \in \mathbb{R}^{nz\times nu}$, $C_y \in \mathbb{R}^{ny\times n}$, and $D_{yw}(\theta(t)) \in \mathbb{R}^{ny\times nw}$. $\theta(t)$ is a real vector containing the time-varying scheduling parameters defined as

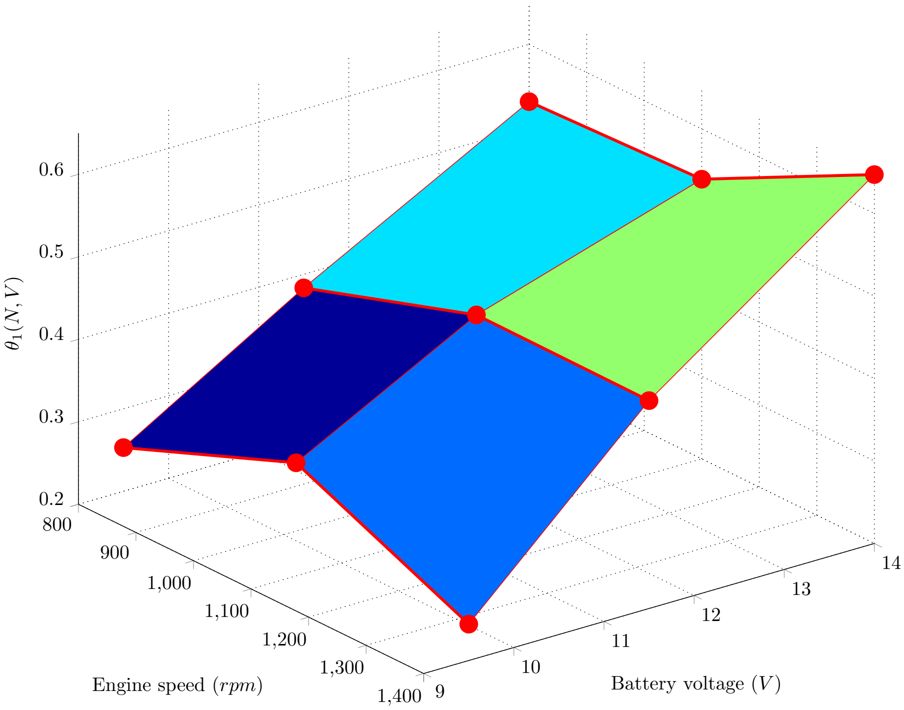
$$\theta(t) = \left[\theta_1(t), \theta_2(t), \cdots, \theta_q(t)\right]',$$

where q represents the number of scheduling parameters.

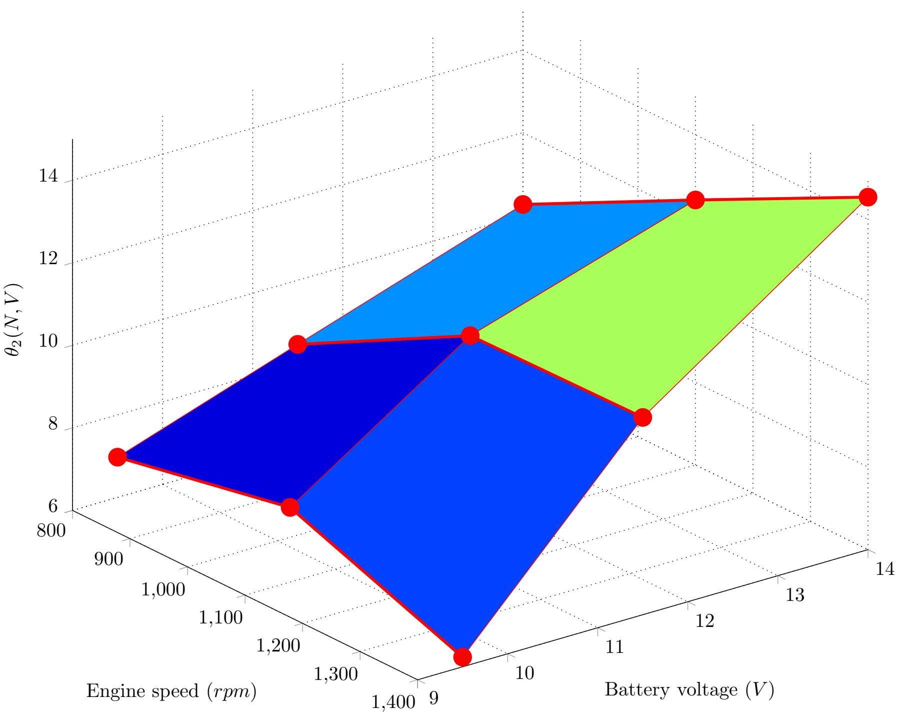
Therefore, the EVVT plant model (7.1) can be realized in state-space form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\theta_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_1(t) \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$



(a)



(b)

Figure 7.5: The varying parameters as function of engine speed and battery voltage. (a) $\theta_1(N,V)$; (b) $\theta_2(N,V)$

Table 7.3: Range of the time-varying parameters.

$\theta_1(t) \in [0.2529 \quad 0.6472]$
$\theta_2(t) \in [6.975 \quad 14.540]$

where $x_1(t)$ represents the cam-phase angle and $u(t)$ is the EVVT motor speed command. In order to accommodate for disturbance input, measurement noise, and control signal,

$$B_w(\theta) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{yw}(\theta) = \begin{bmatrix} 0 & 0.01 \end{bmatrix},$$
$$C_z(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{zu}(\theta) = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}.$$

are defined, thus, the state-space realization of the overall system corresponding to the matrices defined in (7.2) are

$$\left[\begin{array}{c|c|c} A(\theta) & B_u(\theta) & B_w(\theta) \\ \hline C_z(\theta) & D_{zu}(\theta) & \\ \hline C_y & & D_{yw}(\theta) \end{array}\right] = \left[\begin{array}{cc|c|cc} 0 & 1 & 0 & 0.2 & 0 \\ 0 & -\theta_2(t) & \theta_1(t) & 0 & 0 \\ \hline 1 & 0 & 0 & & \\ 0 & 0 & 0.01 & & \\ \hline 1 & 0 & & 0 & 0.01 \end{array}\right].$$

7.3 RGS Controller Design

The objective of the control problem is to regulate the cam phase angle to a reference phase using RGS dynamic output-feedback control for any trajectories of engine speed and battery voltage. More specifically, the goal is to guarantee the robust stability and $\mathscr{H}_\infty$ performance of the closed-loop system against uncertainties in scheduling parameters. Following the lines presented in Chapter 5, the RGS synthesis approach for the EVVT model is summarized by the following points,

1. Set the bounds on scheduling parameters and measurement noises associated with each scheduling parameter, $N \in [800\ ,1400] rpm$, $V \in [9.5\ ,14] volt$, $|\delta_1(t)| \leqslant 10\ rpm$, and $|\delta_2(t)| \leqslant 1.2\ volt$.

2. Set the bounds on the rates of change of the time-varying parameters, i.e. $|\dot{N}| \leqslant 100$, $|\dot{V}| \leqslant 12$, $|\dot{\delta}_1(t)| \leqslant 10 \times \delta_1(t)$, and $|\dot{\delta}_2(t)| \leqslant 10 \times \delta_2(t)$.

3. Convert the LPV model of the EVVT actuator from the original (affine) parameter space into multi-simplex parameters [39, 87] to define the multi-simplex domain Λ.

4. Model the rate of change of the scheduling parameters in a convex set Ω [45].

5. Solve the conditions of Theorem 5.2 using the iterative algorithm presented in Section 5.4 to obtain controller matrices $A_c(\tilde{\theta})$, $B_c(\tilde{\theta})$, and $C_c(\tilde{\theta})$. With $\varepsilon = 0.05$, the achieved upper bound on the $\mathscr{H}_\infty$ performance is $\gamma_\infty = 0.1312$ and the obtained controller matrices at the vertices of the multi-simplex domain are given by,

$$A_c^{(11)} = \begin{bmatrix} -28.998 & 37.356 \\ -12.157 & 13.487 \end{bmatrix},\ A_c^{(12)} = \begin{bmatrix} -30.446 & 40.085 \\ -12.795 & 15.079 \end{bmatrix},$$

$$A_c^{(21)} = \begin{bmatrix} -31.049 & 40.249 \\ -12.828 & 14.462 \end{bmatrix},\ A_c^{(22)} = \begin{bmatrix} -32.032 & 42.508 \\ -13.246 & 15.817 \end{bmatrix},$$

$$B_c^{(11)} = \begin{bmatrix} 16.245 \\ 7.0632 \end{bmatrix}, \qquad B_c^{(12)} = \begin{bmatrix} 16.992 \\ 7.3396 \end{bmatrix}$$

$$B_c^{(21)} = \begin{bmatrix} 17.435 \\ 7.459 \end{bmatrix}, \qquad B_c^{(22)} = \begin{bmatrix} 17.887 \\ 7.5975 \end{bmatrix},$$

$$C_c^{(11)} = \begin{bmatrix} 122.27 & -384.5 \end{bmatrix},\quad C_c^{(12)} = \begin{bmatrix} 19.087 & -53.672 \end{bmatrix},$$

$$C_c^{(21)} = \begin{bmatrix} 124.09 & -394.94 \end{bmatrix},\quad C_c^{(22)} = \begin{bmatrix} 19.336 & -55.089 \end{bmatrix},$$

6. The controller coefficients in (3.14) are obtained by applying the inverse transformation (3.40) and (3.40). Thus, these coefficients have been obtained

$$A_{c_0} = \begin{bmatrix} -122.52 & 160.2 \\ -51.026 & 58.845 \end{bmatrix}, A_{c_1} = \begin{bmatrix} 3.6366 & -5.3166 \\ 1.1223 & -1.7136 \end{bmatrix}$$

$$A_{c_2} = \begin{bmatrix} 2.4315 & -4.9883 \\ 1.0547 & -2.9481 \end{bmatrix},$$

$$B_{c_0} = \begin{bmatrix} 68.558 \\ 29.459 \end{bmatrix}, \quad B_{c_1} = \begin{bmatrix} -2.0852 \\ -0.65373 \end{bmatrix}, \quad B_{c_2} = \begin{bmatrix} -1.1992 \\ -0.41481 \end{bmatrix},$$

$$C_{c_0} = \begin{bmatrix} 284.78 & -888.21 \end{bmatrix}, \quad C_{c_1} = \begin{bmatrix} -2.0693 & 11.863 \end{bmatrix},$$

$$C_{c_2} = \begin{bmatrix} 207.93 & -670.68 \end{bmatrix}.$$

7.4 Experimental Results

This section presents the experimental implementation of the RGS controller, obtained in the previous chapters, to the EVVT test bench. Note that the same test bench used to perform the system identification experiments are also used for controller validation. Euler's forward (rectangular) method that is given in [88] has been utilized to discretize the controller with a sampling period of $5ms$ on the Opal-RT prototype controller.

For the purpose of experimental demonstration, different operational conditions are investigated in this experiment with various engine speed and battery voltage profiles. Figure 7.6 shows one of these trajectories that are used in the experimental study. It is worth mentioning that this trajectory is corresponding to the best possible measurements that can be obtained on the test bench. It is clear from Figure 7.6 that perfect measurement is very difficult to obtain experimentally. This supports the core idea that noisy scheduling parameter is a reasonable assumption in practice.

Figure 7.7 illustrates the cam-phase angle tracking to a reference step input of 40°. During this experiment, engine speed and battery voltage are varying to cover the entire parameter space and

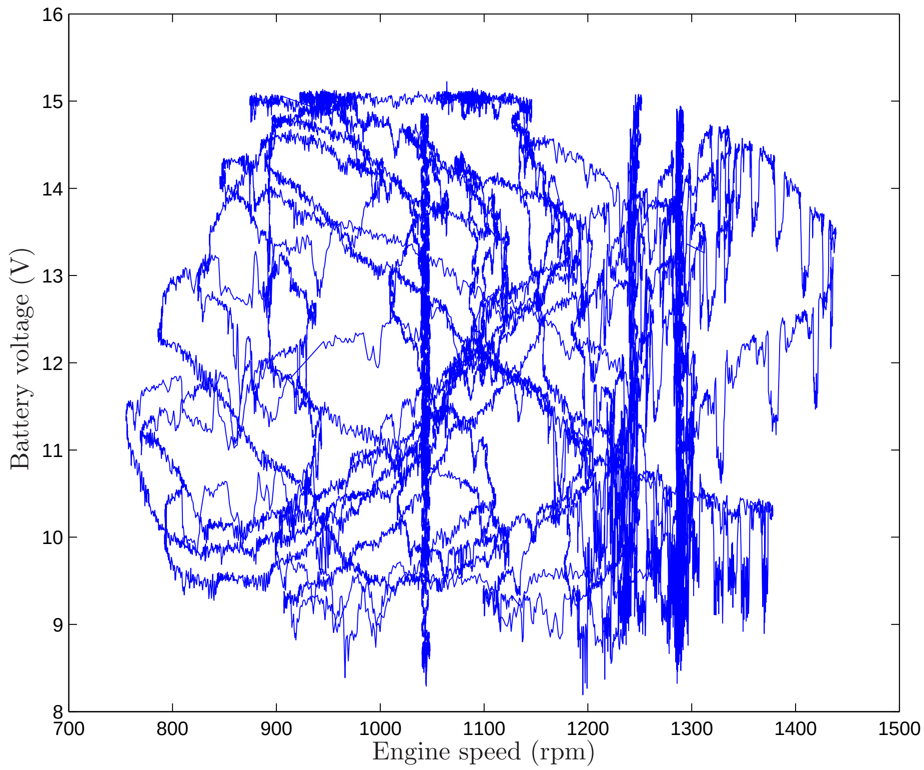


Figure 7.6: Engine experimental operating trajectory in parameter space.

to encompass various vehicle operating conditions. It is clear from this figure that, although sever variations are considered with engine speed and battery voltage, fairly good tracking response is achieved for the cam-phase angle. Again, note that the measured battery voltage is quite noisy indicating the need for gain-scheduling controllers to be robust against these measurement noises. Thus, the controlled response is robust against measurement noise associated with the battery voltage signal.

In Figure 7.8, step-change from 40° to 5° and then back to 40° is applied as a reference input for the cam-phase angle. This figure demonstrates very good tracking performance of the synthesized controllers for a step-change reference. Similarly, since the battery voltage trajectory is not perfect, the synthesized controller shows robustness to measurement inaccuracies in scheduling parameters.

Furthermore, measurement noises are intentionally inserted into the measured battery voltage and engine speed to illustrate robustness of the designed RGS controller. bounds of the noise are

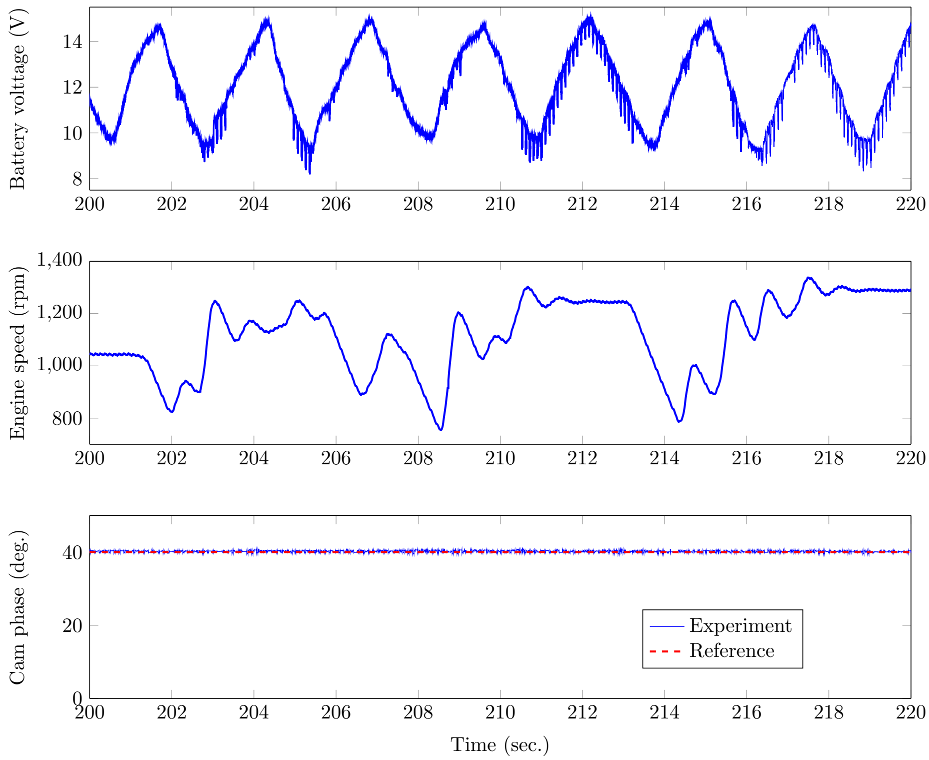


Figure 7.7: Measured engine speed and battery voltage, and cam-phasing angle tracking with step reference of 40 degree.

given by $\pm 100\, rpm$ and $\pm 1.2\, Volt$, respectively. In terms of the measurement noise bounds defined earlier in (3.4), these noise bounds can be expressed as $|\bar{\delta}_1| \leq 100\, rpm$ (for engine speed) and $|\bar{\delta}_2| \leq 1.2\, Volt$ (for battery voltage). Extensive experimental tests are conducted for various measurement error bounds but only tests for the above mentioned bounds are presented here since the results of other tests are quite similar. Figure 7.9, Figure 7.10 and Figure 7.11 illustrate experimental results for the noisy scheduling parameters (right-side of the figures). Again, these figures demonstrate robustness of the synthesized controller.

7.5 Summary

Experimental study of Robust Gain Scheduling (RGS) controller synthesis for the Electric Variable Valve Timing (EVVT) actuator is considered in this chapter. Since engine speed and battery voltage have substantial impact on engine performance, fuel economy, and emissions; these two

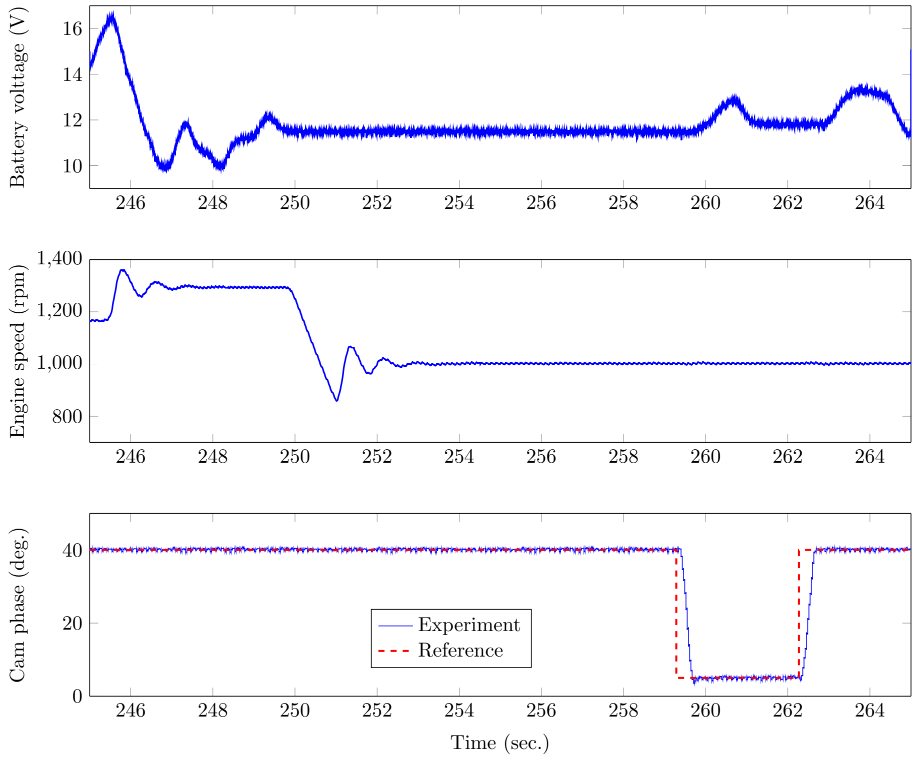


Figure 7.8: Measured engine speed and battery voltage, and cam-phasing angle tracking with step change reference.

time-varying parameters are chosen as scheduling parameters to model the EVVT system in LPV framework. First, a series of system identification tests were conducted to obtain a family of local Linear Time-Invariant (LTI) models. Then, these models were grouped together through linear interpolation to obtain the LPV model for the EVVT actuator for the entire range of operational conditions. Following the theoretical approach presented in Chapter 5, RGS controller was designed to regulate the cam-phasing angle to a specified reference trajectory. The designed controller was implemented and validated experimentally on the test bench utilizing the measured (noisy) scheduling signals. Experimental results confirmed robustness of the developed controller against measurement noises in the scheduling parameters.

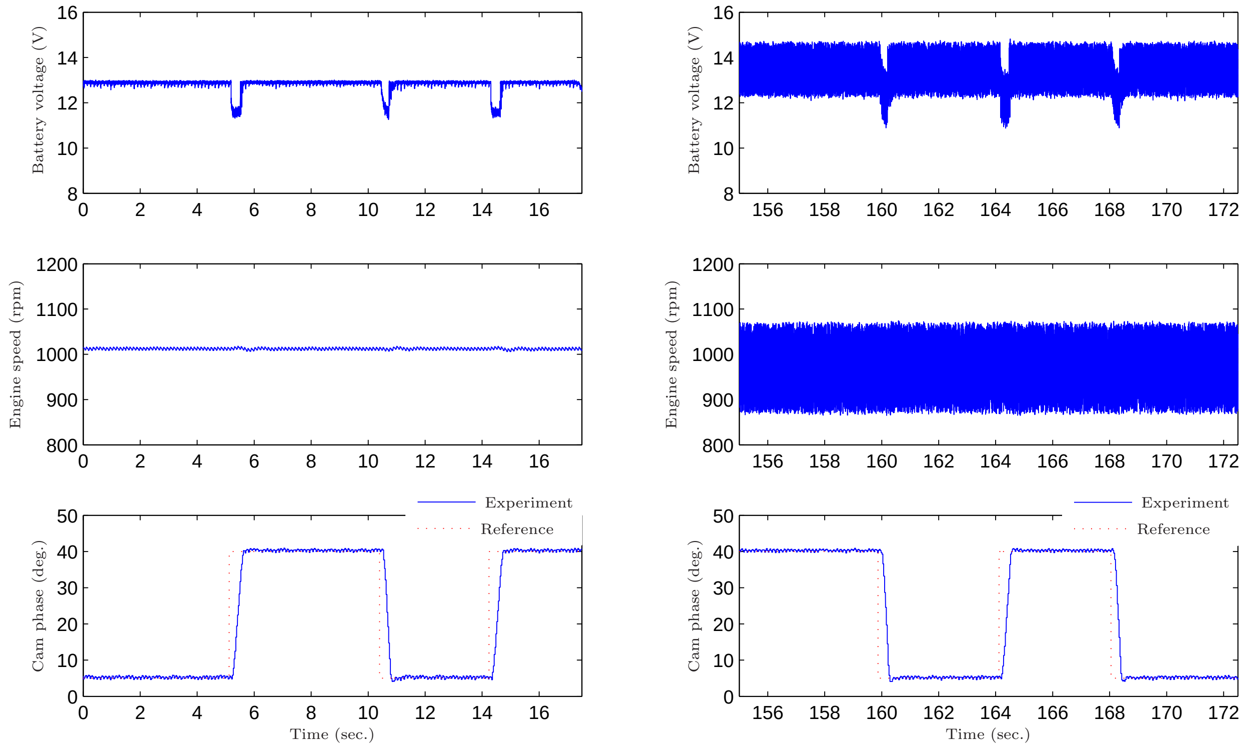


Figure 7.9: Speed and voltage variations with perfect measurement (left) and noisy measurement (right) and corresponding cam-phase responses.

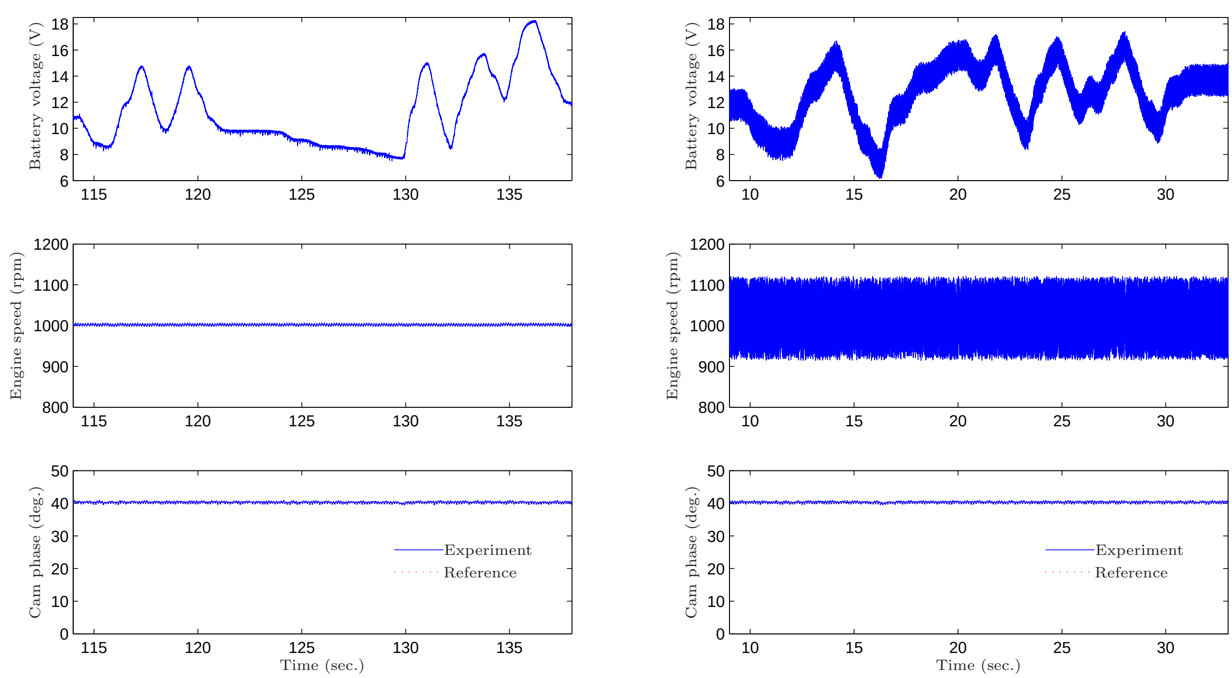


Figure 7.10: Speed and voltage variations with perfect measurement (left) and noisy measurement (right) and corresponding cam-phase responses.

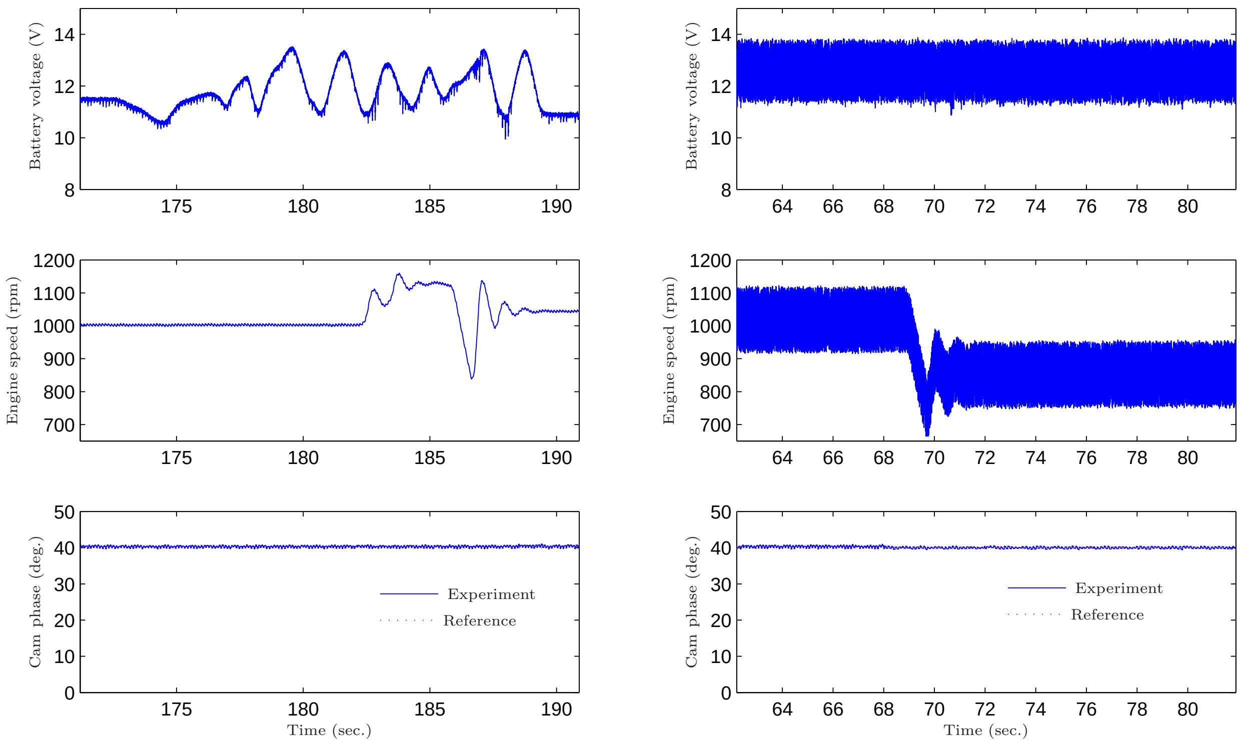


Figure 7.11: Speed and voltage variations with perfect measurement (left) and noisy measurement (right) and corresponding cam-phase responses.

CHAPTER 8

CONCLUSIONS AND FUTURE RESEARCH

In this dissertation, a unifying approach for synthesizing Robust Gain-Scheduling (RGS) controllers of Linear Parameter-Varying (LPV) systems subject to noisy scheduling parameters was developed. This chapter provides concluding remarks and suggestions for future research directions.

8.1 Conclusions

Since stability and performance of dynamical systems are of most importance for a successful control design, significant efforts have been made by researchers to develop new control strategies to achieve these two objectives. In this dissertation, a general approach has been developed to improve these objectives in the LPV framework when the time-varying parameters are polluted by measurement noise.

In the first chapter, different LPV control methods were reviewed and the overall organization of the dissertation was given. Multi-simplex modeling approach was used throughout this dissertation to model the time-varying scheduling parameters and associated uncertainties in a convex domain. Chapter 2 introduces the necessary notations, definitions of the multi-simplex modeling approach, the homogeneous polynomials, and the technical machinery used throughout this dissertation.

In Chapter 3, the control problem formulation of the RGS controller synthesis was presented. Then, the general solution approach for this synthesis problem was introduced in the same chapter. This solution approach includes transformation of the affine LPV systems into multi-simplex domain first. Then, the rates of change of the scheduling parameters and associated uncertainties were modeled in a convex domain as well. Third, the controller synthesis conditions were derived in terms of Parametrized Linear/Bilinear Matrix Inequalities (PLMIs/PBMIs). Finally, matrix coefficient check method was utilized to relax the PLMIs/PBMIs at the vertices of the multi-simplex domain to obtain the optimal controller.

Synthesis conditions of RGS state-feedback controller with guaranteed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances were presented in Chapter 4. These conditions were formulated in terms of PLMIs with scalar search as a tuning parameter. Numerical examples and comparisons with other existing methods were given in this chapter. The comparison results show that the developed approach achieve a better performance in the presence of measurement noises. As a special case of the these synthesis conditions, robust state-feedback controller can be synthesized by constraining synthesis variables to be parameter independent.

Chapter 5 introduces synthesis conditions for RGS Dynamic Output Feedback (DOF) controller with guaranteed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances. These conditions were formulated in terms of PBMIs with scalar search for both performances. Since PBMIs are non-tractable, a numerical algorithm was developed to solve these conditions. Demonstrating examples and comparisons were presented to illustrate the effectiveness of the developed approach.

Synthesis conditions for Static Output-Feedback (SOF) robust gain-scheduling controller were developed in Chapter 6. Two-stage design method was used to synthesize the SOF RGS controller. In the first stage, state-feedback gain-scheduling controller was synthesized. Then, this controller was used as input to the second stage to synthesize the SOF RGS controller. Numerical algorithm was developed to reduce conservativeness iteratively. Simulation results of the SOF RGS controller for Electric Variable Valve Timing (EVVT) actuator was presented to demonstrate the effectiveness of the designed controller.

Experimental validation of the developed approach was demonstrated in Chapter 7 on the EVVT bench. Engine speed and battery voltage were chosen as time-varying (noisy) scheduling parameters. RGS controller has been designed and implemented experimentally on the test bench of the EVVT actuator. Experimental results showed very satisfactory tracking performance of the cam-phasing angle.

8.2 Recommendations for Future Research

This section presents recommended directions for future research.

1. In view of the developed synthesis conditions, it would also be interesting to study the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ synthesis problem. Since the developed conditions utilize slack variable approach and the controller construction was independent on Lyapunov variables, the developed synthesis conditions lend itself as an excellent candidate to achieve multi-objective control. In other words, each performance objective has its own Lyapunov matrix allowing to achieve multiple objectives without introducing additional conservativeness.

2. In the RGS state-feedback conditions, it would also be interesting to avoid line search by replacing the scalar (ε) with a full matrix and solving the resulting conditions iteratively as PBMIs problem. In this case, considerable performance improvement can be expected on the expense of additional numerical burden.

3. Since matrix coefficient check relaxation method [41] was used to solve the developed synthesis conditions, it is interesting to use different relaxation method such as matrix Sum-Of-Squares (SOS) [52] to solve these conditions and performing comparative study in terms of the achieved performance and numerical complexity.

4. An interesting topic to study is to extend the developed approach to design gain-scheduling observers subject to noisy scheduling parameters. By using duality of the RGS state-feedback synthesis conditions it is not difficult to derive PLMI conditions to synthesize observer gain that guarantees asymptotic convergence of the estimation error while achieving robustness against scheduling parameters uncertainties.

5. In this dissertation $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performances have been considered. It is very encouraging to study other performance such as $\mathscr{L}_2$-to-$\mathscr{L}_\infty$ performance in the RGS framework. With the $\mathscr{L}_2$-to-$\mathscr{L}_\infty$ performance, the control objective is to minimize the control energy while

satisfying predefined constraints on the performance output. This performance is very powerful for many practical control problems in automotive and aerospace applications when hard constraints are imposed on the performance outputs.

6. It would also be interesting to extend the developed approach to handle RGS controllers that minimize the output performance while satisfying constraints on the available control energy. This is a very interesting problem in practice when certain constraints on the available actuators should be met while achieving optimal output performance.

APPENDIX

APPENDIX

FUNDAMENTALS OF LINEAR MATRIX INEQUALITIES

This appendix provides an introduction to the application of Linear Matrix Inequalities (LMIs) in control systems. Basic concepts of LMI problems with essential manipulation tools are introduced.

LMI Basics

The central notion to understand matrix inequalities is *definiteness*. In particular, a matrix Q is defined to be *positive definite* if

$$x'Qx > 0, \quad \forall x \neq 0 \tag{.1}$$

Likewise, Q is said to be *positive semi-definite* if

$$x'Qx \geq 0, \quad \forall x \tag{.2}$$

It is common to write $Q > 0$ $(Q \geq 0)$ to indicating positive (semi-) definite matrix. In particular, the interest is to find positive definite matrices that are also symmetric, i.e., $Q = Q'$. A symmetric, positive definite matrix has two key features: it is square and all of its eigenvalues are positive real. A symmetric, positive semi-definite matrix shares the first attribute, but the last is relaxed to the requirement that all of its eigenvalues are positive real or zero. A matrix $P = -Q$ is said to be negative (semi-) definite if Q is positive (semi-) definite. $P < 0$ $(P \leq 0)$ is used to indicate negative (semi-) definiteness.

The most general form of an LMI is

$$F(x) = F_0 + x_1F_1 + x_2F_2 + \cdots + x_mF_m = F_0 + \sum_{i=1}^{m} x_iF_i > 0 \tag{.3}$$

where x_i are real, scalar optimization variables, $\mathbf{x} = [x_1\ x_2\ \cdots x_m]' \in \mathbb{R}^m$ and $F_0, F_i \in \mathbb{R}^{n\times n}$ are given constant symmetric matrices. The above LMI is *feasible*, if a vector $\mathbf{x}$ exists which satisfies (.3). Note that $F(x) > 0$ describes an *affine* relationship in terms of optimization variables $\mathbf{x}$.

In most control problem, it is more convenient to be formulated as the following LMI:

$$F(X_1, X_2, \cdots, X_k) = F_0 + \sum_{i=1}^{k} G_i X_i H_i > 0 \tag{.4}$$

where $X_i \in \mathbb{R}^{p_i \times q_i}$ are matrix variables to be obtained and $G_i \in \mathbb{R}^{n \times p_i}$ and $H_i \in \mathbb{R}^{q_i \times n}$ are known matrices. It is easy to see that the vector variable $\mathbf{x}$ in (.3) can be formed by stacking the columns of X_i in (.4).

Mathematical Tools for LMI Manipulation

Althoough many control problem can be formulated as LMI problems, a substantial number of these need to be manipulated before they are in a suitable LMI framework. Fortunately, there are a number of common mathematical tools that can be used to transform problems into suitable LMI forms. Some of these useful tools are described below.

Schur Complement

The usefulness of the Schur complement is to transform quadratic matrix inequalities into linear matrix inequalities, or at least as a step in this direction. Schur's formula states that the following two statments are equivalent

-
$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} > 0 \Longleftrightarrow \begin{cases} Q > 0 \\ R > 0 \\ Q - SR^{-1}S' > 0 \end{cases} \tag{.5}$$

-
$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} > 0 \Longleftrightarrow \begin{cases} Q > 0 \\ R > 0 \\ R - S'Q^{-1}S > 0 \end{cases} \tag{.6}$$

For non-strict inequalities, the Moore-Penrose pseudo inverse of constant matrix will be used [89].

Congruence Transformation

For a given positive definite matrix $Q \in \mathbb{R}^{n \times n}$, the following inequality holds

$$WQW' > 0, \tag{.7}$$

for a (real) full rank matrix $W \in \mathbb{R}^{n \times n}$. Therefore, *definiteness* of a matrix is invariant under pre- and post-multiplication by a full rank real matrix, and its transpose, respectively. The process of transforming $Q > 0$ into (.7) using a real full rank matrix is called a *congruence transformation*. It is very useful for linearizing nonlinear matrix inequalities with a suitable change of variables. Often, the congruence transformation matrix W is chosen to be diagonal.

The S-Procedure

The S-procedure is essentially a method used to combine several quadratic inequalities into a single one (generally with some conservatism). More specifically, It is preferred to guarantee that a single quadratic function of $x \in \mathbb{R}^n$ such that

$$F_0(x) \leq 0, \quad F_0(x) := x'A_0x + 2b_0x + c_0$$

whenever certain other quadratic functions are positive semi-definite

$$F_i(x) \geq 0, \quad F_i(x) := x'A_ix + 2b_0x + c_0, \quad i \in \{1, 2, \cdots, q\}$$

To illustrate, consider $i = 1$, if there exist a scalar constant $\tau > 0$, such that

$$F_{aug}(x) := F_0(x) + \tau F_1(x) \leq 0, \quad \forall x, \quad s.t.\ F_1(x) \geq 0$$

then $F_0(x) \leq 0$. In other words, $F_{aug}(x) \leq 0$ implies that $F_0(x) \leq 0$ if $\tau F_1(x) \geq 0$ because $F_0(x) \leq F_{aug}(x)$ if $F_1(x) \geq 0$. Extending the idea to q inequalities constraint

$$F_0(x) \leq 0, \quad whenever\ F_i(x) \geq 0 \tag{.8}$$

hold if

$$F_0(x) + \sum_{i=1}^{q} \tau_i F_i(x) \leq 0, \quad \tau_i \geq 0. \tag{.9}$$

Thus, the S-procedure is a method of verifying (.8) using (.9). This is quit useful when $F_0(x)$ is not a convex function. It is worth noting that these conditions are sufficient for $i > 1$, it provides sufficient and necessary conditions only when $i = 1$. Usually, τ_i's are considered as additional LMI optimization variables.

Projection Lemma

The projection Lemma is useful for eliminating decision variables from LMIs.It is also has a convexifying effect on certain nonlinear matrix inequalities [90] .

Lemma .1. *Let $\Psi = \Psi' \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $R \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{m\times n}$ be given matrices. Then, the following statements are equivalent*

1. *There exists a matrix $\Omega \in \mathbb{R}^{p\times m}$ such that*

$$\Psi + R'\Omega Q + Q'\Omega' R < 0. \tag{.10}$$

2. *The LMIs*

$$R'_\perp \Psi R_\perp < 0$$

$$Q'_\perp \Psi Q_\perp < 0$$

   hold, where $R_\perp$ and $P_\perp$ are bases of the null-space of P and Q, respectively.

3. *There exists scalars τ_1, $\tau_2 \in \mathbb{R}$ such that the LMIs*

$$\Psi - \tau_1 R' R < 0$$

$$\Psi - \tau_2 Q' Q < 0$$

   hold.

Common LMI control Problem

This section presents common control problems that uses LMI formulations.

Lyapunov Stability Criteria

Consider the continuous-time linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{.11}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the control (input) vector, and $y(t) \in \mathbb{R}^p$ the output (measured) vector, the system is asymptotically stable if the following *feasibility problem* satisfied

$$\begin{cases} P > 0, \\ A'P + PA < 0. \end{cases} \tag{.12}$$

Stabilization by State-Feedback

Assuming that the system (.11) is not asymptotically stable, state-feedback controller K, that is $u = w + Kx$, may be sought to form the following closed-loop system

$$\begin{aligned} \dot{x} &= (A + BK)x(t) + Bw(t), \\ y(t) &= Cx(t) + Dw(t), \end{aligned} \tag{.13}$$

where $w(t)$ denotes the external input. In order to stabilize the closed-loop system, the state-feedback matrix K required to satisfy (.12),

$$\begin{cases} P > 0, \\ (A + BK)'P + P(A + BF) < 0 \end{cases} \tag{.14}$$

or

$$\begin{cases} P > 0, \\ A'P + PA + K'B'P + PBK < 0 \end{cases} \tag{.15}$$

The second inequality in (.15) is *bilinear*. It can be converted back to a linear inequality by simple *change of variables*. Pre- and post-multiply (.15) by P^{-1} to get

$$P^{-1}A' + AP^{-1} + P^{-1}K'B' + BKP^{-1} < 0. \tag{.16}$$

Letting $Q := P^{-1}$ and defining $L = KQ$,

$$QA' + AQ + L'B' + BL < 0. \tag{.17}$$

Hence, inequality (.17) is linear with respect to the new variables Q and L. Once these variables are obtained, the original variables P and K can be easily recovered.

$\mathscr{H}_\infty$ Norm

Consider the problem of obtaining the $\mathscr{H}_\infty$ norm for the following system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \\ y(t) &= Cx(t) + Dw(t). \end{aligned} \tag{.18}$$

The $\mathscr{L}_2$ gain (γ_∞) is defined by

$$\int_0^\infty y(t)'y(t)dt \le \gamma_\infty^2 \int_0^\infty w(t)'w(t)dt. \tag{.19}$$

Let's define quadratic Lyapunov function $V(x) = x(t)'Px(t)$, where $P > 0$, then

$$\mathscr{T} := \dot{V}(t) + y(t)'y(t) - \gamma_\infty^2 w(t)'w(t) \le 0. \tag{.20}$$

Substituting for $\dot{V}(t)$ and $y(t)$, (.20) can be written as

$$\mathscr{T} = x'PAx + x'A'Px + x'PBw + w'B'Px + (Cx+Dw)'(Cx+Dw) - \gamma_\infty^2 w(t)'w(t) \le 0, \tag{.21}$$

that can also be written in the following form,

$$\mathscr{T} = \begin{bmatrix} x' & w' \end{bmatrix} \begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma_\infty^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0. \tag{.22}$$

A sufficient condition of (.20) is

$$\begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'C & D'D-\gamma_\infty^2 I \end{bmatrix} < 0. \tag{.23}$$

Inequality (.23) can be written in different (equivalent) ways,

$$\begin{bmatrix} A'P+PA & PB \\ B'P & -\gamma_\infty^2 I \end{bmatrix} + \begin{bmatrix} C' \\ D' \end{bmatrix} I \begin{bmatrix} C & D \end{bmatrix} < 0, \tag{.24}$$

or

$$\begin{bmatrix} A'P+PA & PB & C' \\ B'P & -\gamma_\infty^2 I & D' \\ C & D & -I \end{bmatrix} < 0. \tag{.25}$$

Pre- and post-multiply the previous inequality by

$$\begin{bmatrix} \frac{1}{\sqrt{\gamma_\infty}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\gamma_\infty}} & 0 \\ 0 & 0 & \sqrt{\gamma_\infty} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} A'\hat{P}+\hat{P}A & \hat{P}B & C' \\ B'\hat{P} & -\gamma_\infty I & D' \\ C & D & -\gamma_\infty I \end{bmatrix} < 0, \tag{.26}$$

with $\hat{P} = P/\gamma_\infty$. There is another form that is usually used in synthesis problem that can be obtained by multiplying (.26) from left and right by

$$\begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

with $Q = P^{-1}$, to obtain

$$\begin{bmatrix} AQ+QA' & B & QC' \\ B' & -\gamma_\infty I & D' \\ CQ & D & -\gamma_\infty I \end{bmatrix} < 0. \tag{.27}$$

Thus, the *bounded real lemma* [91] could be any of the above inequalities (.25), (.26), and (.27). In all of the previous inequalities, the goal is to seek a positive definite matrix (P or Q) that minimizes the $\mathscr{H}_\infty$ bound γ_∞.

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