FORMATIONS AND RELATIVE
$\mathcal{F}$ NORMALIZERS

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Lee M. Sonneborn

Lee M. Sonneborn
Major professor

Date 1-13-70

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ABSTRACT

FORMATIONS AND RELATIVE $\mathfrak{F}$ NORMALIZERS

By

John David Gillam

In this thesis, G denotes a finite solvable group; $\mathfrak{F}$ is a formation locally defined by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for all primes p, and $\pi(\mathfrak{F}) = \{p\colon \mathfrak{F}(p) \neq \phi\}$.

Definition. Let $N \triangleleft G$. A p-chief factor H/K of G is N-$\mathfrak{F}$ central if $N/C_N(H/K) \in \mathfrak{F}(p)$, and is N-$\mathfrak{F}$ eccentric otherwise.

Definition. Let S^p be a Sylow p-complement of G, $N \triangleleft G$, and $C_p(N)$ the intersection of the centralizers of the $\mathfrak{F}$ central p-chief factors of N. Let $X(S^p,N) = N_G(S^p \cap C_p(N))$ if $\mathfrak{F}(p) \neq \phi$, and $X(S^p,N) = S^p$ if $\mathfrak{F}(p) = \phi$. Let Σ be a set of Sylow p-complements of G. A relative $\mathfrak{F}$ normalizer (with respect to N) is the subgroup $D_N(\Sigma) = \cap\{X(S^p,N)\colon S^p \in \Sigma\}$. Also let $T_N(\Sigma) = \cap\{X(S^p,N)\colon S^p \in \Sigma \text{ and } p \in \pi(\mathfrak{F})\}$.

The following statements are the main results obtained in this thesis.

Theorem. $D_N(\Sigma)$ covers N-$\mathfrak{F}$ central chief factors of G and avoids N-$\mathfrak{F}$ eccentric chief factors of G. $T_N(\Sigma)$ avoids N-$\mathfrak{F}$ eccentric $\pi(\mathfrak{F})$-chief factors of G and covers all other chief factors of G.

Theorem. If θ is an epimorphism of G, then $(D_N(\Sigma))\theta = D_{N\theta}(\Sigma\theta)$ and $(T_N(\Sigma))\theta = T_{N\theta}(\Sigma\theta)$.

Theorem. Let $N \triangleleft G$. Then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and all $\pi(\mathfrak{F})$-chief factors of G are N-$\mathfrak{F}$ central. Hence if Σ is a set of Sylow p-complements of G, $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and $T_N(\Sigma) = G$.

Theorem. Let N be a normal $\pi(\mathfrak{F})$ subgroup of G. If $N_{\mathfrak{F}}$ is complemented in N by an $\mathfrak{F}$ projector of N, then $N_{\mathfrak{F}}$ is complemented in G. Furthermore all complements of $N_{\mathfrak{F}}$ in G are conjugate and are precisely the various $T_N(\Sigma)$.

Example. Let G be a solvable Frobenius group with Frobenius Kernel K. Then K is complemented in any solvable extension of G, and all such complements are conjugate.

Theorem. Let $N \triangleleft G$; then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and $N\Phi(G)/\Phi(G) \in \mathfrak{F}$.

Corollary. Let $D, M \triangleleft G$ with M a $\pi(\mathfrak{F})$ group and $D \subseteq \Phi(G)$. If $M/D \in \mathfrak{F}$, then $M \in \mathfrak{F}$.

Theorem. Let $N \triangleleft G$. Then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and all $\pi(\mathfrak{F})$-chief factors of G between $\Phi(G)$ and $\mathrm{Fit}(G)$ are N-$\mathfrak{F}$ central.

Theorem. Let $N \triangleleft G$. Then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and for $p \in \pi(\mathfrak{F})$ with $\mathfrak{F}(p) \subset \mathfrak{F}$ $M \cap N/\mathrm{Core}(M) \cap N \in \mathfrak{F}(p)$ for all p-maximal subgroups M of G.

Example. Let $N \triangleleft G$ and π a set of primes. Then N is p-nilpotent for all $p \in \pi$ if and only if $M \cap N \subseteq \mathrm{Core}(M)$ for all π-maximal subgroups M of G.

Theorem. Let Σ be a set of Sylow p-complements of G. Then $D = D_G(\Sigma)$ is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})$.

Corollary. Let Σ be a set of Sylow p-complements of G. Then $D_G(\Sigma)$ is an $\mathfrak{F}$ projector and an $\mathfrak{F}$ normalizer of $N_G(\Sigma \cap G_{\mathfrak{F}})$. If $G \notin \mathfrak{F}$, then $D_G(\Sigma) \subset N_G(\Sigma \cap G_{\mathfrak{F}})$ and $G = \langle N_G(\Sigma \cap G_{\mathfrak{F}})^g : g \in G \rangle$. Hence every finite solvable group is generated by subgroups in which $\mathfrak{F}$ projectors coincide with $\mathfrak{F}$ normalizers.

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John David Gillam

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To Mary Lee, my wife.

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Chapter I

All groups in this thesis are finite and solvable. Unless otherwise indicated, G will denote an arbitrary finite solvable group. A chief series of G is a G-composition series. A chief factor of G is a composition factor of some chief series of G. If π is a set of primes, a π-chief factor of G is a chief factor whose order is a power of some prime belonging to π; a π-maximal subgroup of G is a maximal subgroup whose index in G is a power of some prime belonging to π. If H/K is a chief factor of G and M is a subgroup of G, then $C_M(\frac{H}{K}) = \{m \in M: h^m K = hK \text{ for all } h \in H\}$; M complements $\frac{H}{K}$ if $G = HM$ and $H \cap M = K$. A subgroup M of G covers a subgroup $\frac{A}{B}$ of $\frac{G}{B}$ if $A \subseteq MB$ (equivalently, $A = (M \cap A)B$); M avoids $\frac{A}{B}$ if $M \cap A \subseteq B$. A subgroup M of G is a cover-avoid subgroup if M covers or avoids every chief factor of G. Let $\Sigma = \{S^p : p \mid o(G)\}$ be a set of Sylow p-complements of G. If M is a subgroup of G, Σ is reducible to M if $S^p \cap M$ is a Sylow p-complement of M for all $S^p \in \Sigma$. Throughout, the intersection of a vacuous collection of subgroups of G will be interpreted as G.

A formation $\mathfrak{F}$ is a class of groups such that:

1). If $G \in \mathfrak{F}$, then every epimorphic image of G is in $\mathfrak{F}$.

2). If G/N_1 and G/N_2 are in $\mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

A non-empty formation $\mathfrak{F}$ is saturated if $G/\Phi(G)$ in $\mathfrak{F}$ implies $G \in \mathfrak{F}$. If $\mathfrak{F} \neq \phi$, the $\mathfrak{F}$ residual of G is the unique minimal normal subgroup $G_{\mathfrak{F}}$ such that $G/G_{\mathfrak{F}} \in \mathfrak{F}$. Note that if θ is an epimorphism of G, $(G_{\mathfrak{F}})\theta = (G\theta)_{\mathfrak{F}}$. A subgroup F of a group G is an $\mathfrak{F}$ projector of G if $F \in \mathfrak{F}$, and whenever $F \subseteq U \subseteq G$, $U = U_{\mathfrak{F}}F$. Let $\mathfrak{F}(p)$ be a formation for each prime p. We define $\mathfrak{F}$ by the following conditions: $G \in \mathfrak{F}$ if:

1). If $\mathfrak{F}(p) = \phi$, $p \nmid o(G)$.

2). If $\mathfrak{F}(p) \neq \phi$ and H/K is a p-chief factor of G, then $G/C_G(H/K) \in \mathfrak{F}(p)$.

If $\mathfrak{F}$ is defined as above, we say $\mathfrak{F}$ is locally defined by $\{\mathfrak{F}(p)\}$. Gaschütz and Lubeseder have shown [5; 7.5, 7.25 pp. 697, 715] that a formation $\mathfrak{F}$ of solvable groups is saturated if and only if $\mathfrak{F}$ can be locally defined. Carter and Hawkes [1, p. 177] have shown that every saturated formation $\mathfrak{F}$ can be locally defined by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for all primes p. Throughout, $\mathfrak{F}$ will denote a formation locally defined by $\mathfrak{F}(p)$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$. We denote by $\pi(\mathfrak{F})$ the set of primes $\{p\colon \mathfrak{F}(p) \neq \phi\}$.

The following three propositions are part of the folklore of finite solvable groups; however, the exact statements and proofs do not seem to appear in the literature.

Proposition 1.1. Let H/K be a chief factor of G and $M \subseteq G$ such that M covers H/K and $G/C_G(H/K)$. Then $H\cap M/K\cap M$ is a chief factor of M, $C_M(H\cap M/K\cap M) = M \cap C_G(\frac{H}{K})$, and $G/C_G(H/K) \cong M/C_M(H\cap M/K\cap M)$.

Proof. Since $H = (H\cap M)K$, $H\cap M/K\cap M \neq \langle 1\rangle$. Suppose $K\cap M \subseteq L \subset H\cap M$, where L is M invariant. Since $G = C_G(\frac{H}{K})M$,

$LK \lhd G$. If $LK = H$, then $L = L(M \cap K) = M \cap LK = M \cap H$. Therefore $LK = K$ and $L = K \cap M$. Clearly $M \cap C_G(H/K) \subseteq C_M(H \cap M/K \cap M)$, and since $H = (H \cap M)K$, we have equality. The last assertion is now clear.

Proposition 1.2. Suppose M is a cover-avoid subgroup of G and θ is an epimorphism of G. Then $M\theta$ is a cover-avoid subgroup of $G\theta$. If M is such that M covers a chief factor H/K of G if and only if M covers every chief factor G-isomorphic to H/K, and H/K is a chief factor of G with $H\theta \neq K\theta$, then $M\theta$ covers $H\theta/K\theta$ if and only if M covers H/K.

Proof. Let $N = \mathrm{Ker}(\theta)$ and $H/N/K/N$ a chief factor of G/N. If M avoids H/K, then $MN/N \cap H/N = MN \cap H/N = (M \cap H)N/N \subseteq K/N$. If M covers $\frac{H}{K}$, $H \subseteq KM \subseteq K(MN)$, and so MN/N covers $H/N/K/N$. Assume the additional hypothesis on M, and let H/K be a chief factor of G with $HN \neq KN$. Then HN/KN is G-isomorphic to H/K. Hence M covers H/K if and only if M covers HN/KN, if and only if MN/N covers $HN/N/KN/N$.

Proposition 1.3. Let θ be an epimorphism of G, N a normal subgroup of G, and H/K a chief factor of G such that $H\theta \neq K\theta$. Then $(C_N(H/K))\theta = C_{N\theta}(H\theta/K\theta)$, and $N/C_N(H/K) \cong N\theta/C_{N\theta}(H\theta/K\theta)$.

Proof. Let $M = \mathrm{Ker}(\theta)$. Since $HM \neq KM$, M does not cover H/K. Hence $(H \cap M)K$ is a normal subgroup of G contained properly in H. Therefore $(H \cap M)K = K$. Since $[H,M] \subseteq H \cap M$, $M \subseteq C_G(H/K)$. Clearly $C_N(H/K)M/M \subseteq C_{NM/M}(HM/M/KM/M)$. Let C be the pre-image of $C_{N\theta}(H\theta/K\theta)$. Then $[C,H] \subseteq KM \cap H = K(M \cap H) = K$, and so $C \subseteq C_{NM}(H/K) = C_G(H/K) \cap NM = C_N(H/K)M$. Hence $(C_N(H/K))\theta = C_{N\theta}(H\theta/K\theta)$. Also $N/C_N(H/K) = N/(N \cap M)C_N(H/K) \cong NM/C_N(H/K)M \cong N\theta/(C_N(H/K))\theta = N\theta/C_{N\theta}(H\theta/K\theta)$.

Chapter II

Definition 2.1. Let $N \lhd G$. A p-chief factor H/K of G is N-$\mathfrak{F}$ central if $N/C_N(H/K) \in \mathfrak{F}(p)$, and is N-$\mathfrak{F}$ eccentric otherwise. If $N = G$, we say $\mathfrak{F}$ central and $\mathfrak{F}$ eccentric, respectively.

Carter and Hawkes [1, 2.2, p. 178] have shown that if $\mathfrak{F}$ is locally defined by $\{\mathfrak{F}_1(p)\}$ and also by $\{\mathfrak{F}_2(p)\}$, then $\mathcal{O}\mathfrak{F}_1(p) = \mathcal{O}\mathfrak{F}_2(p)$ for all primes p. ($\mathcal{O}\mathfrak{F}_i(p) = \{G\colon G/O_p(G) \in \mathfrak{F}_i(p)\}$, $i = 1,2$.) If $N \lhd G$ and H/K is a chief factor of G, then H/K is G-isomorphic to a chief factor H_1/K_1 of G with $N \subseteq K_1$ or $H_1 \subseteq N$, and H/K is N-$\mathfrak{F}$ central if and only if H_1/K_1 is N-$\mathfrak{F}$ central. Hence to see that Definition 2.1 is independent of the local formations defining $\mathfrak{F}$, we need consider only the chief factors of G above and below N. Let $\mathfrak{F}$ be locally defined by $\{\mathfrak{F}_1(p)\}$ and by $\{\mathfrak{F}_2(p)\}$. Then $\mathfrak{F}_1(p) = \phi$ if and only if $\mathfrak{F}_2(p) = \phi$. Let $\mathfrak{F}_1(p) \neq \phi$ and H/K a p-chief factor of G above or below N. If $N \subseteq K$, then $N/C_N(H/K) = \langle 1\rangle \in \mathfrak{F}_1(p) \cap \mathfrak{F}_2(p)$. If $H \subseteq N$, then Proposition 2.2 below and the well-known fact [5, 5.17, p. 485] that $O_p(G) = \langle 1\rangle$ for any group G possessing a faithful irreducible representation over a field of characteristic p imply that $N/C_N(H/K) \in \mathfrak{F}_1(p)$ if and only if $N/C_N(H/K) \in \mathfrak{F}_2(p)$.

Proposition 2.2. Let $N \lhd G$ and H/K a p-chief factor of G with $H \subseteq N$. Then:

i). H/K is N-$\mathfrak{F}$ central if and only if all chief factors of N between H and K are $\mathfrak{F}$ central (in N).

ii). Chief factors of N between H and K are all $\mathfrak{F}$ central or all $\mathfrak{F}$ eccentric.

iii). $C_N(H/K) = \cap\{C_N(L/M)$: L/M is a chief factor of N between H and $K\}$.

Proof. i). Suppose H/K is N-$\mathfrak{F}$ central. Since $C_N(H/K) \subseteq C_N(L/M)$ for any chief factor L/M of N between H and K, $N/C_N(L/M)$ is an epimorphic image of $N/C_N(H/K)$. Hence L/M is $\mathfrak{F}$ central. Conversely, suppose all N chief factors between H and K are $\mathfrak{F}$ central. By Clifford's Theorem [5, 17.3, p. 565], $H/K = \oplus \sum_{(i)} H_i/K$ where the H_i/K are chief factors of N. Then $C_N(H/K) = \bigcap_{(i)} C_N(H_i/K)$, and so $N/C_N(H/K) \in \mathfrak{F}(p)$.

ii) and iii). As in i), let $H/K = \oplus \sum_{(i)} H_i/K$. The H_i/K are all conjugate in G, and hence one H_i/K is $\mathfrak{F}$ central if and only if all H_i/K are $\mathfrak{F}$ central. Also $C_N(H/K) = \bigcap_{(i)} C_N(H_i/K)$. The Jordan-Hölder Theorem applied to the N module H/K now yields both ii) and iii).

The following definition is due to Carter and Hawkes [1, p. 182]:

Definition 2.3. If $\mathfrak{F}(p) \neq \phi$, let $C_p(G) = \cap\{C_G(H/K)$: H/K is an $\mathfrak{F}$ central p-chief factor of $G\}$.

We note that $C_p(G)$ is a characteristic subgroup of G such that $G/C_p(G) \in \mathfrak{F}(p)$; hence $G_{\mathfrak{F}(p)} \subseteq C_p(G)$.

Proposition 2.4. Let $N \lhd G$ and $\mathfrak{F}(p) \neq \phi$. Then $C_p(N) = N \cap \cap\{C_G(H/K)$: H/K is an N-$\mathfrak{F}$ central p-chief factor of $G\}$.

Proof. Let $\langle 1\rangle = H_o \subset H_1 \subset \ldots \subset H_s = N \subset \ldots \subset H_t = G$ be a chief series of G. (If $N = \langle 1\rangle$ or G, the result is trivial.) Refining $\langle 1\rangle = H_o \subset H_1 \subset \ldots \subset H_s = N$ to a chief series of N, we see from 2.2 that $C_p(N) = \cap\{C_N(H_{i+1}/H_i)\colon 0 \le i \le s-1$ and H_{i+1}/H_i is an N-$\mathfrak{F}$ central p-chief factor$\}$. Since N centralizes any chief factor above N, $C_p(N) = \cap\{C_N(H_{i+1}/H_i)\colon 0 \le i \le t-1$ and H_{i+1}/H_i is an N-$\mathfrak{F}$ central p-chief factor$\}$. The desired conclusion now follows from the Jordan-Hölder Theorem.

Definition 2.5. Let S^p be a Sylow p-complement of G and N a normal subgroup of G. Let $X(S^p,N) = N_G(S^p \cap C_p(N))$ if $\mathfrak{F}(p) \neq \phi$, and $X(S^p,N) = S^p$ if $\mathfrak{F}(p) = \phi$. Let Σ be a set of Sylow p-complements of G. A relative $\mathfrak{F}$ normalizer (with respect to N) is the subgroup $D_N(\Sigma) = \cap\{X(S^p,N)\colon S^p \in \Sigma\}$. Also let $T_N(\Sigma) = \cap\{X(S^p,N)\colon S^p \in \Sigma$ and $p \in \pi(\mathfrak{F})\}$.

We note that the various $D_N(\Sigma)$ form a conjugate class of subgroups of G, and the same is true for the various $T_N(\Sigma)$. If $N = G$, we write $D(\Sigma)$ in place of $D_G(\Sigma)$.

Proposition 2.6. Let $\mathfrak{F}(p) \neq \phi$, $N \triangleleft G$, and S^p a Sylow p-complement of G. Then $X(S^p,N)$ covers the N-$\mathfrak{F}$ central p-chief factors of G and avoids the N-$\mathfrak{F}$ eccentric p-chief factors of G.

Proof. Since $C_p(N) \triangleleft G$, $N_G(S^p \cap C_p(N))$ is a cover-avoid subgroup which covers the p-chief factor H/K if and only if $C_p(N) \subseteq C_G(H/K)$ (Appendix, Theorem A). Every chief factor of G is G-isomorphic to a chief factor above or below N. Hence it is sufficient to prove the statement for chief factors above and below N. If H/K is a p-chief factor with $K \supseteq N$,

$N/C_N(H/K) = \langle 1\rangle \in \mathfrak{F}(p)$; i.e. H/K is N-$\mathfrak{F}$ central. Since $C_p(N) \subseteq N \subseteq C_G(H/K)$, $N_G(S^p \cap C_p(N))$ covers H/K. Suppose H/K is a p-chief factor of G with $H \subseteq N$. Then $N_G(S^p \cap C_p(N))$ covers H/K if and only if $C_p(N) \subseteq C_N(H/K)$. If $C_p(N) \subseteq C_N(H/K)$, H/K is N-$\mathfrak{F}$ central since $N/C_p(N) \in \mathfrak{F}(p)$. If H/K is N-$\mathfrak{F}$ central, 2.2 implies $C_p(N) \subseteq C_N(H/K)$. Hence $N_G(S^p \cap C_p(N))$ covers the p-chief factor H/K if and only if H/K is N-$\mathfrak{F}$ central.

Corollary 2.6.1. With the same hypothesis as Proposition 2.6, $N_G(S^p \cap C_p(N)) = N_G(S^p \cap N_{\mathfrak{F}(p)})$.

Proof. Since $N_{\mathfrak{F}(p)} \lhd G$ and $N_{\mathfrak{F}(p)} \subseteq C_p(N)$, $N_G(S^p \cap C_p(N)) \subseteq N_G(S^p \cap N_{\mathfrak{F}(p)})$. Each subgroup clearly covers all q-chief factors of G with $q \neq p$, and $N_G(S^p \cap N_{\mathfrak{F}(p)})$ covers the p-chief factor H/K if and only if $N_{\mathfrak{F}(p)} \subseteq C_N(H/K)$, if and only if H/K is N-$\mathfrak{F}$ central. Hence the two subgroups have the same order and are equal.

If $\mathfrak{F} = \mathfrak{N}$, the formation of nilpotent groups, then $\mathfrak{F} = \{\mathfrak{F}(p)\}$ where $\mathfrak{F}(p) = \{\langle 1\rangle\}$ for all primes p. Hence if $N \lhd G$, then $N_{\mathfrak{F}(p)} = N$. By 2.6.1 $D_N(\Sigma) = \bigcap\{N_G(S^p \cap N)$: $p|o(G)$ and $S^p \in \Sigma\}$ is the relative system normalizer as defined by P. Hall in [3].

Proposition 2.7. Let $N \lhd G$, $\Sigma = \{S^p\colon p|o(G)\}$ a set of Sylow p-complements of G, and π a set of primes. Then $S = \bigcap\{X(S^p,N)\colon S^p \in \Sigma$ and $p \in \pi\}$ avoids N-$\mathfrak{F}$ eccentric π-chief factors of G, and covers all other chief factors of G.

Proof. If $\mathfrak{F}(p) = \phi$ with $p \in \pi$, $S \subseteq X(S^p,N) = S^p$, which avoids all p-chief factors of G. (If $\mathfrak{F}(p) = \phi$, all p-chief factors are N-$\mathfrak{F}$ eccentric.) If $\mathfrak{F}(p) \neq \phi$ and $p \in \pi$, $S \subseteq X(S^p,N)$, which avoids all N-$\mathfrak{F}$ eccentric p-chief factors by 2.6. Therefore S avoids all N-$\mathfrak{F}$ eccentric π-chief factors. Since $[G: X(S^p,N)]$ is a power of p and equals the product of the orders of the N-$\mathfrak{F}$ eccentric p-chief factors in a given chief series of G, $[G:S] = \pi\{[G: X(S^p,N)]: p \in \pi\}$ equals the product of the orders of the N-$\mathfrak{F}$ eccentric π-chief factors in a given chief series of G. By considering the order of S, we see that S must cover all other chief factors of G.

Corollary 2.7.1. Let $N \triangleleft G$ and Σ a set of Sylow p-complements of G. Then $D_N(\Sigma)$ covers N-$\mathfrak{F}$ central chief factors of G and avoids N-$\mathfrak{F}$ eccentric chief factors of G. $T_N(\Sigma)$ avoids all N-$\mathfrak{F}$ eccentric $\pi(\mathfrak{F})$-chief factors of G and covers all other chief factors of G.

Proof. In Proposition 2.7, let π be the set of all primes and $\pi(\mathfrak{F})$, respectively.

Proposition 2.8. Let $N \triangleleft G$ and θ an epimorphism of G. Then an N-$\mathfrak{F}$ central (N-$\mathfrak{F}$ eccentric) chief factor of G maps to an $N\theta$-$\mathfrak{F}$ central ($N\theta$-$\mathfrak{F}$ eccentric) chief factor of $G\theta$ or $\langle 1 \rangle$. Conversely, any $N\theta$-$\mathfrak{F}$ central ($N\theta$-$\mathfrak{F}$ eccentric) chief factor of $G\theta$ is the image of an N-$\mathfrak{F}$ central (N-$\mathfrak{F}$ eccentric) chief factor of G.

Proof. Both statements follow immediately from 1.3.

Proposition 2.9. Let $N \triangleleft G$, $\Sigma = \{S^p: p|o(G)\}$ a set of Sylow p-complements of G, π a set of primes, and θ an epimorphism of G. Then $(\bigcap\{X(S^p,N): S^p \in \Sigma$ and $p \in \pi\})\theta$ $= \bigcap\{X(S^p\theta,N\theta): S^p \in \Sigma$ and $p \in \pi\}$.

Proof. Let $S = \bigcap\{X(S^p,N): S^p \in \Sigma$ and $p \in \pi\}$ and $S_1 = \bigcap\{X(S^p\theta,N\theta): S^p \in \Sigma$ and $p \in \pi\}$. If $p \in \pi$ and $\mathfrak{F}(p) = \phi$, then $S\theta \subseteq X(S^p,N)\theta = S^p\theta = X(S^p\theta,N\theta)$. If $p \in \pi$ and $\mathfrak{F}(p) \neq \phi$, then $S\theta \subseteq X(S^p,N)\theta = N_G(S^p \cap N_{\mathfrak{F}(p)})\theta \subseteq N_{G\theta}((S^p \cap N_{\mathfrak{F}(p)})\theta)$. Since $N_{\mathfrak{F}(p)} \lhd G$, $S^p \cap N_{\mathfrak{F}(p)}$ is a Sylow p-complement of $N_{\mathfrak{F}(p)}$. Hence $(S^p \cap N_{\mathfrak{F}(p)})\theta = S^p\theta \cap (N_{\mathfrak{F}(p)})\theta = S^p\theta \cap (N\theta)_{\mathfrak{F}(p)}$. Therefore $S\theta \subseteq N_{G\theta}(S^p\theta \cap (N\theta)_{\mathfrak{F}(p)}) = X(S^p\theta,N\theta)$, and $S\theta \subseteq S_1$. But by 1.2, 2.7, and 2.8, $S\theta$ and S_1 cover and avoid the same chief factors of $G\theta$. Hence $S\theta = S_1$.

Corollary 2.9.1. Let $N \lhd G$, Σ a set of Sylow p-complements of G, and θ an epimorphism of G. Then $(D_N(\Sigma))\theta = D_{N\theta}(\Sigma\theta)$ and $(T_N(\Sigma))\theta = T_{N\theta}(\Sigma\theta)$ where $\Sigma\theta = \{S^p\theta: S^p \in \Sigma\}$.

Proof. In Proposition 2.9, let π be the set of all primes and $\pi(\mathfrak{F})$, respectively.

The cover-avoidance property of 2.7.1 does not in general characterize $D_N(\Sigma)$. If $N = G$ and $\mathfrak{F}$ is the formation of nilpotent groups, T.O. Hawkes has given an example [5, 11.12, p. 730] of a group G and a subgroup S of G with the cover-avoidance property, but S is not a system normalizer of G.

Proposition 2.10. Let $N \lhd G$ and $M \subseteq G$ such that M covers all q-chief factors of G with $q \neq p$, all N-$\mathfrak{F}$ central p-chief factors of G, and avoids all N-$\mathfrak{F}$ eccentric p-chief factors of G. Then M contains a Sylow p-complement of G, and if S^p is any such Sylow p-complement, then $M = X(S^p,N)$.

Proof. It is clear that M contains a Sylow p-complement of G. If G has no N-$\mathfrak{F}$ central p-chief factors, M is a Sylow p-complement of G, and so the statement is true in this case. Assume H/K is an N-$\mathfrak{F}$ central p-chief factor of G. Since $N/N \cap C_G(H/K) \in \mathfrak{F}(p)$, any p-chief factor above $C_G(H/K)$ is N-$\mathfrak{F}$ central. By hypothesis, M covers H/K and $G/C_G(H/K)$. Hence by 1.1, $H \cap M/K \cap M$ is a p-chief factor of M and $M \cap C_G(H/K) = C_M(H \cap M/K \cap M)$. Intersecting the terms of a chief series of G with M, we obtain

$M \cap \cap\{C_G(H/K)$: H/K is an N-$\mathfrak{F}$ central p-chief factor of $G\}$

$= \cap\{C_M(H \cap M/K \cap M)$: H/K is an N-$\mathfrak{F}$ central p-chief factor of $G\}$

$= \cap\{C_M(A/B$: A/B is a p-chief factor of $M\}$

$= O_{p'p}(M)$ [5, 5.4, p. 686]. Now let S^p be a Sylow p-complement of G contained in M. Then $M \subseteq N_G(S^p \cap O_{p'p}(M))$

$\subseteq N_G(S^p \cap O_{p'p}(M) \cap N) = N_G(S^p \cap M \cap C_p(N))$ (by 2.4) $= X(S^p,N)$.

Since M and $X(S^p,N)$ have the same order, $M = X(S^p,N)$.

Corollary 2.10.1. Suppose $D_N(\Sigma)$, a relative $\mathfrak{F}$ normalizer of G, has index a power of p in G. Then if M is any subgroup of G which covers N-$\mathfrak{F}$ central chief factors and avoids N-$\mathfrak{F}$ eccentric chief factors, M is a relative $\mathfrak{F}$ normalizer.

Proof. Since $D_N(\Sigma)$ has index a power of p in G, $D_N(\Sigma) = X(S^p,N)$ where $S^p \in \Sigma$. By 2.7.1, all q-chief factors of G, $q \neq p$, are N-$\mathfrak{F}$ central. Proposition 2.10 now implies the statement.

Proposition 2.11. Let G be a group such that $\mathfrak{F}$ normalizers coincide with $\mathfrak{F}$ projectors. Then if M is any subgroup which covers $\mathfrak{F}$ central chief factors and avoids $\mathfrak{F}$

eccentric chief factors, M is an $\mathfrak{F}$ normalizer of G.

Proof. Induct on o(G) and let A be a minimal normal subgroup of G. By 1.2 and 2.8, MA/A has the same cover-avoid property in G/A. Also by 2.9.1 and Theorem B, $\mathfrak{F}$ projectors of G/A coincide with $\mathfrak{F}$ normalizers of G/A. Hence MA/A = FA/A where F is an $\mathfrak{F}$ projector of G. Suppose A is $\mathfrak{F}$ central. Then $M \supseteq A$, $F \supseteq A$, and F = M. If A is $\mathfrak{F}$ eccentric, F is an $\mathfrak{F}$ projector of FA and $F \cap A = \langle 1 \rangle$. Hence $(FA)_{\mathfrak{F}} = A$, and since A is abelian, F is conjugate to M in FA. (Theorem E.) Therefore M is an $\mathfrak{F}$ projector of G.

Proposition 2.12. Let $N \triangleleft G$. Then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and all $\pi(\mathfrak{F})$-chief factors of G are N-$\mathfrak{F}$ central. Hence if Σ is a set of Sylow p-complements of G, $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and $T_N(\Sigma) = G$.

Proof. Suppose $N \in \mathfrak{F}$; then N is a $\pi(\mathfrak{F})$ group. By 2.2, all chief factors of G below N are N-$\mathfrak{F}$ central, and clearly all $\pi(\mathfrak{F})$-chief factors above N are N-$\mathfrak{F}$ central. Conversely, suppose N is a $\pi(\mathfrak{F})$ group and all $\pi(\mathfrak{F})$-chief factors of G are N-$\mathfrak{F}$ central. Again by 2.2, all chief factors of N are $\mathfrak{F}$ central. Therefore $N \in \mathfrak{F}$. The second statement now follows readily from 2.7.1.

Corollary 2.12.1. Let $N \triangleleft G$ and Σ a set of Sylow p-complements of G. Then $G = T_N(\Sigma)N_{\mathfrak{F}}$.

Proof. $T_N(\Sigma)N_{\mathfrak{F}}/N_{\mathfrak{F}} = T_{N/N_{\mathfrak{F}}}(\Sigma N_{\mathfrak{F}}/N_{\mathfrak{F}}) = G/N_{\mathfrak{F}}$.

Proposition 2.13. Let $\Sigma = \{S^p \colon p \mid o(G)\}$ be a set of Sylow p-complements of G, and for each $S^p \in \Sigma$, let $S^p \subseteq T^p \subseteq G$. Then Σ reduces to $T = \bigcap\{T^p \colon p \mid o(G)\}$. In particular, Σ reduces to $D_N(\Sigma)$.

Proof. Let $S^p \in \Sigma$; then $[T: S^p \cap T] = [T^p: S^p]$, a power of p. Hence Σ reduces to T. The second statement follows since $X(S^p,N) \supseteq S^p$ for all primes p.

Proposition 2.14. Let Σ be a set of Sylow p-complements of G. If $D(\Sigma) \subseteq M \subseteq G$ and Σ reduces to M, then $D(\Sigma) \subseteq D(\Sigma \cap M)$.

Proof. If $\mathfrak{F}(p) = \phi$, $D(\Sigma) \subseteq S^p \cap M$ for $S^p \in \Sigma$. If $\mathfrak{F}(p) \neq \phi$, let H/K be an $\mathfrak{F}$ central p-chief factor of G. Then $G/C_G(H/K) \in \mathfrak{F}(p) \subseteq \mathfrak{F}$. Since $D(\Sigma) \subseteq M$, M covers $G/C_G(H/K)$ and H/K. By 1.1, $H\cap M/K\cap M$ is an $\mathfrak{F}$ central p-chief factor of M, and $C_M(H\cap M/K\cap M) = M \cap C_G(H/K)$. Therefore $C_p(M) \subseteq M \cap C_p(G)$. Since $D(\Sigma) \subseteq M$, $D(\Sigma)$ normalizes $C_p(M)$ and $S^p \cap C_p(G)$, where $S^p \in \Sigma$. Therefore $D(\Sigma) \subseteq N_M(S^p \cap C_p(M))$, and so $D(\Sigma) \subseteq D(\Sigma \cap M)$.

The following proposition was proved by T.O. Hawkes [4, 3.3, p. 244] in the case that $\mathfrak{F} \supseteq \mathfrak{N}$, the formation of nilpotent groups. Trivial modifications can easily be given to include the case $\mathfrak{F} \not\supseteq \mathfrak{N}$; however, for completeness we give an alternate proof.

Proposition 2.15. Let $D(\Sigma)$ be an $\mathfrak{F}$ normalizer of G. Then there exists an $\mathfrak{F}$ projector F of G such that $D(\Sigma) \subseteq F$ and Σ reduces to F.

Proof. We may suppose $G \notin \mathfrak{F}$. Let A be minimal normal in G. Then by induction $D(\Sigma)A/A = D(\Sigma A/A) \subseteq F_1/A$ where F_1/A is an $\mathfrak{F}$ projector of G/A and $\Sigma A/A$ reduces to F_1/A. Suppose o(A) is a power of p. If $q \neq p$ and $S^q \in \Sigma$, $A \subseteq S^q$ and $[F_1: S^q \cap F_1] = [F_1/A: S^q/A \cap F_1/A]$, a power of q. For $S^p \in \Sigma$, $[F_1: S^p \cap F_1] = [F_1: S^pA \cap F_1][(S^p \cap F_1)A: S^p \cap F_1]$, a power of p.

Hence Σ reduces to F_1, and by 2.14 $D(\Sigma) \subseteq D(\Sigma \cap F_1)$. If $F_1 < G$, by induction $D(\Sigma \cap F_1) \subseteq F$ where F is an $\mathfrak{F}$ projector of F_1 and $\Sigma \cap F_1$ reduces to F. Hence Σ reduces to F, F is an $\mathfrak{F}$ projector of G (Theorem B), and $D(\Sigma) \subseteq F$. Suppose $F_1 = G$. Since $G \notin \mathfrak{F}$ and $G/A \in \mathfrak{F}$, every complement of A in G is an $\mathfrak{F}$ projector of G (Theorem E). But $G = AD(\Sigma)$ and A is $\mathfrak{F}$ eccentric. Hence $D(\Sigma) \cap A = \langle 1 \rangle$ and $D(\Sigma)$ is an $\mathfrak{F}$ projector of G. Σ reduces to $D(\Sigma)$ by 2.13.

Proposition 2.16. Suppose $G_{\mathfrak{F}}$ is complemented in G by an $\mathfrak{F}$ projector of G. Then $\mathfrak{F}$ projectors of G coincide with $\mathfrak{F}$ normalizers of G, and all complements of $G_{\mathfrak{F}}$ in G are conjugate in G.

Proof. Suppose $G = G_{\mathfrak{F}}F$, $F \cap G_{\mathfrak{F}} = \langle 1 \rangle$, and F is an $\mathfrak{F}$ projector of G. Let $D \subseteq F$ be an $\mathfrak{F}$ normalizer of G; then $G = G_{\mathfrak{F}}D$ and $D = F$. Now suppose H is a complement of $G_{\mathfrak{F}}$ in G. Let $A \subseteq G_{\mathfrak{F}}$ be minimal normal in G. (If $G_{\mathfrak{F}} = \langle 1 \rangle$, the statement is trivial.) Then HA/A complements $G_{\mathfrak{F}}/A$ in G/A. By induction, $HA = FA$ where F is an $\mathfrak{F}$ projector of G. If A is $\mathfrak{F}$ central, $A \subseteq F$ and so $H \subseteq F$. Therefore $H = F$ since both are complements of $G_{\mathfrak{F}}$ in G. Suppose A is $\mathfrak{F}$ eccentric, then as in the proof of 2.11, $(FA)_{\mathfrak{F}} = A$ and F is conjugate to H.

Proposition 2.17. Let N be a normal $\pi(\mathfrak{F})$ subgroup of G. If $N_{\mathfrak{F}}$ is complemented in N by an $\mathfrak{F}$ projector of N, then $N_{\mathfrak{F}}$ is complemented in G. Furthermore all complements of $N_{\mathfrak{F}}$ in G are conjugate and are precisely the various $T_N(\Sigma)$.

Proof. By 2.12.1, $G = T_N(\Sigma)N_{\mathfrak{F}}$ where Σ is a set of Sylow p-complements of G. Now $D_N(\Sigma) \cap N_{\mathfrak{F}} \subseteq T_N(\Sigma) \cap N_{\mathfrak{F}}$, but since N is a normal $\pi(\mathfrak{F})$ subgroup, $N \subseteq S^p$ for all $S^p \in \Sigma$ such that $p \in \pi(\mathfrak{F})'$. Therefore $N \cap T_N(\Sigma) \subseteq D_N(\Sigma)$ and $T_N(\Sigma) \cap N_{\mathfrak{F}} = D_N(\Sigma) \cap N_{\mathfrak{F}} = D(\Sigma \cap N) \cap N_{\mathfrak{F}} = \langle 1 \rangle$ by hypothesis and 2.16. If H is a complement of $N_{\mathfrak{F}}$ in G, $N = N_{\mathfrak{F}}(H \cap N)$, and $H \cap N$ is an $\mathfrak{F}$ projector of N by 2.16. Clearly $N_G(H \cap N) \supseteq H$, and so $N_G(H \cap N) = HN_N(H \cap N) = H(H \cap N)$ (Theorem D) $= H$. Hence if H_1 and H_2 are complements of $N_{\mathfrak{F}}$ in G, $H_1 = N_G(H_1 \cap N)$ is conjugate to $H_2 = N_G(H_2 \cap N)$. Since $T_N(\Sigma)$ is a complement of $N_{\mathfrak{F}}$ in G, and the various $T_N(\Sigma)$ form a conjugate class, the statement is proved.

Corollary 2.17.1. Let $N \triangleleft G$. If $N_{\mathfrak{F}}$ is an abelian $\pi(\mathfrak{F})$ group, the conclusions of Proposition 2.17 hold.

Proof. Theorem E states that $N_{\mathfrak{F}}$ is complemented in N by an $\mathfrak{F}$ projector of N.

Example 2.17.2. Let G be a solvable Frobenius group with Frobenius Kernel K. Then K is complemented in any solvable extension of G, and all such complements are conjugate.

Proof. Let $\pi = \{p: p \mid o(K)\}$ and $\mathfrak{F}$ the saturated formation of groups which are p-nilpotent for all $p \in \pi$. $\mathfrak{F}$ is locally defined by: $\mathfrak{F}(p) = \{\langle 1 \rangle\}$ if $p \in \pi$ and $\mathfrak{F}(p) = \mathfrak{F}$ if $p \notin \pi$. Hence $\pi(\mathfrak{F})$ is the set of all primes. We now show that if H is any solvable Frobenius group, and the Frobenius Kernel L of H is a π Hall subgroup of H, then $H_{\mathfrak{F}} = L$. Since H/L is a π' group, $H_{\mathfrak{F}} \subseteq L$. If K_1/K_2 is any chief factor of H with $K_1 \subseteq L$, then $C_H(K_1/K_2) = L$. Hence $H_{\mathfrak{F}} = L$. Now let M

be a complement of K in G. We show that M is an $\mathfrak{F}$ projector of G. $M \in \mathfrak{F}$ since M is a π' group. Suppose $M \subset U$; then $U = (K \cap U)M$, a solvable Frobenius group with kernel $K \cap U$. Therefore $U_{\mathfrak{F}} = (K \cap U)$ and $U = U_{\mathfrak{F}}M$. Proposition 2.17 now implies the conclusion.

Proposition 2.18. Let $N \lhd G$; then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and $N\Phi(G)/\Phi(G) \in \mathfrak{F}$.

Proof. Suppose N is a $\pi(\mathfrak{F})$ group and $N\Phi(G)/\Phi(G) \in \mathfrak{F}$. Let Σ be a set of Sylow p-complements of G. Then $T_N(\Sigma)\Phi(G)/\Phi(G) = T_{N\Phi(G)/\Phi(G)}(\Sigma\Phi(G)/\Phi(G)) = G/\Phi(G)$. Therefore $T_N(\Sigma)\Phi(G) = G$, and so $T_N(\Sigma) = G$. By 2.12, $N \in \mathfrak{F}$. The converse is clear.

The following corollary is due to Gaschütz [5, 3.5, p. 270] in the case that $\mathfrak{F} = \mathfrak{N}$; furthermore, the solvability of G is not necessary in Gaschütz result. The case $\mathfrak{F}$ the formation of supersolvable groups, for example, does not seem to appear in the literature.

Corollary 2.18.1. Let $D, M \lhd G$ with M a $\pi(\mathfrak{F})$ group and $D \subseteq \Phi(G)$. If $M/D \in \mathfrak{F}$, then $M \in \mathfrak{F}$.

Proof. Suppose $M/D \in \mathfrak{F}$. Then $M\Phi(G)/\Phi(G) = M\Phi(G)/D\Phi(G) \cong M/D(M \cap \Phi(G)) \in \mathfrak{F}$. Hence $M \in \mathfrak{F}$ by 2.18.

Corollary 2.18.2. Suppose $\mathfrak{F} \supseteq \mathfrak{N}$, and $\mathfrak{F}$ is normal subgroup closed. If $\mathfrak{F}^*$ is any non-empty formation, $\mathfrak{F}\mathfrak{F}^* = \{G : \exists\, K \lhd G$ with $K \in \mathfrak{F}$ and $G/K \in \mathfrak{F}^*\}$ is a saturated formation.

Proof. Since $\mathfrak{F}$ is normal subgroup closed, $G \in \mathfrak{F}\mathfrak{F}^*$ if and only if $G_{\mathfrak{F}^*} \in \mathfrak{F}$. Let $G \in \mathfrak{F}\mathfrak{F}^*$ and $N \lhd G$; $(G/N)_{\mathfrak{F}^*}$

$= G_{\mathfrak{F}^*}N/N \in \mathfrak{F}$. If G/N_1 and $G/N_2 \in \mathfrak{F}\mathfrak{F}^*$, $G_{\mathfrak{F}^*}N_i/N_i \in \mathfrak{F}$, $i = 1,2$. Hence $(G/N_1 \cap N_2)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*}(N_1 \cap N_2)/N_1 \cap N_2 \cong G_{\mathfrak{F}^*}/(N_1 \cap N_2) \cap G_{\mathfrak{F}^*} \in \mathfrak{F}$. If $G/\Phi(G) \in \mathfrak{F}\mathfrak{F}^*$, then $G_{\mathfrak{F}^*}\Phi(G)/\Phi(G) \in \mathfrak{F}$. By 2.18, $G_{\mathfrak{F}^*} \in \mathfrak{F}$.

Proposition 2.19. Let $N \triangleleft G$. Then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and all $\pi(\mathfrak{F})$-chief factors of G between $\Phi(G)$ and $\mathrm{Fit}(G)$ are N-$\mathfrak{F}$ central.

Proof. Assume the criteria. By 2.8 and 2.18, we may assume $\Phi(G) = \langle 1 \rangle$. $\mathrm{Fit}(G)$ is a completely reducible G-module [5, 4.5, p. 279], say $\mathrm{Fit}(G) = \oplus\Sigma\{H_i\colon 1 \le i \le n\}$ where the H_i are minimal normal subgroups of G. We may suppose $\oplus\Sigma\{H_i\colon 1 \le i \le s\}$; $s \ge 1$, is a $\pi(\mathfrak{F})$ group and $\oplus\Sigma\{H_i\colon s+1 \le i \le n\}$ is a $\pi(\mathfrak{F})'$ group, for if $N = \langle 1 \rangle$, there is nothing to prove. By hypothesis, $N/C_N(H_i) \in \mathfrak{F}(p) \subseteq \mathfrak{F}$ for $1 \le i \le s$ and appropriate primes p. Therefore $N/\cap\{C_N(H_i)\colon 1 \le i \le s\} \in \mathfrak{F}$. Since N is a normal $\pi(\mathfrak{F})$ group, N centralizes $\oplus\Sigma\{H_i\colon s+1 \le i \le n\}$. Hence $\cap\{C_N(H_i)\colon 1 \le i \le s\} \subseteq N \cap C_G(\mathrm{Fit}(G)) = N \cap \mathrm{Fit}(G)$, and so $N/N \cap \mathrm{Fit}(G) \in \mathfrak{F}$. Let H/K be a $\pi(\mathfrak{F})$-chief factor above $\mathrm{Fit}(G)$. If $H \cap N = K \cap N$, $N \subseteq C_G(H/K)$ and H/K is N-$\mathfrak{F}$ central. If $H \cap N \ne K \cap N$, H/K is G-isomorphic to $H \cap N/K \cap N$. By 2.2, H/K is N-$\mathfrak{F}$ central if and only if every chief factor of N between $K \cap N$ and $H \cap N$ is $\mathfrak{F}$ central. Since $K \cap N \supseteq \mathrm{Fit}(G) \cap N$ and $N/N \cap \mathrm{Fit}(G) \in \mathfrak{F}$, H/K is N-$\mathfrak{F}$ central. Therefore all $\pi(\mathfrak{F})$-chief factors of G are N-$\mathfrak{F}$ central, and $N \in \mathfrak{F}$ by 2.12. Also the converse is clear by 2.12.

Definition 2.20. Let $N \triangleleft G$. A p-maximal subgroup M of G is N-$\mathfrak{F}$ normal if $M \cap N/\mathrm{Core}(M) \cap N \in \mathfrak{F}(p)$, and is N-$\mathfrak{F}$ abnormal otherwise.

Proposition 2.21. Let $N \triangleleft G$ and M a p-maximal subgroup of G. Then M is N-$\mathfrak{F}$ normal if and only if M complements an N-$\mathfrak{F}$ central p-chief factor of G.

Proof. Suppose M is N-$\mathfrak{F}$ normal; then $\mathfrak{F}(p) \neq \phi$. Let $K = \mathrm{Core}(M)$. Then G/K has a unique minimal normal subgroup H/K, $C_G(H/K) = H$, and M complements H/K [5, 3.2, p. 159]. If $N \cap H \subseteq M$, $N \cap H \subseteq K$, and H/K is N-$\mathfrak{F}$ central. Suppose $N \cap H \not\subseteq M$, then $G = (N \cap H)M$ and $N = (N \cap H)(M \cap N)$. Therefore $N/C_N(H/K) = N/N \cap H \cong M \cap N/M \cap H \cap N = M \cap N/K \cap N \in \mathfrak{F}(p)$. Suppose M complements the N-$\mathfrak{F}$ central p-chief factor H/K. Let $C = C_G(H/K)$. Then $\mathrm{Core}(M) = C \cap M$ and $C = H(C \cap M)$. Hence $C/C \cap M$ is G-isomorphic to H/K, and M complements $C/C \cap M$. Therefore we may suppose $C_G(H/K) = H$ and $K = \mathrm{Core}(M)$. If $N \cap H \subseteq M$, then $N \cap H \subseteq K$, and so $N \subseteq C_G(H/K) = H$. Therefore $N = N \cap H \subseteq M$ and $M \cap N/\mathrm{Core}(M) \cap N = \langle 1 \rangle \in \mathfrak{F}(p)$. ($\mathfrak{F}(p) \neq \phi$ since H/K is N-$\mathfrak{F}$ central.) If $N \cap H \not\subseteq M$, $G = (N \cap H)M$ and $N = (N \cap H)(M \cap N)$. Hence $M \cap N/K \cap N \cong N/N \cap H = N/C_N(H/K) \in \mathfrak{F}(p)$.

Proposition 2.22. Let $N \triangleleft G$. Then $N \in \mathfrak{F}$ if and only if N is a $\pi(\mathfrak{F})$ group and for $p \in \pi(\mathfrak{F})$ with $\mathfrak{F}(p) \subset \mathfrak{F}$ all p-maximal subgroups of G are N-$\mathfrak{F}$ normal.

Proof. Assume the criteria and induct on $o(G)$. We may assume $N \neq \langle 1 \rangle$. Let A be a minimal normal subgroup of G contained in N. By 2.8, 2.21, and the induction hypothesis, $N/A \in \mathfrak{F}$. Then $N_{\mathfrak{F}} \subseteq A$. Suppose $N_{\mathfrak{F}} = A$ and A is a p-chief factor of G. By 2.17.1, $T_N(\Sigma)$ is a p-maximal subgroup of G complementing A, where Σ is a set of Sylow p-complements of G. If $\mathfrak{F}(p) \subset \mathfrak{F}$, $T_N(\Sigma)$ is N-$\mathfrak{F}$ normal by hypothesis. If $\mathfrak{F}(p) = \mathfrak{F}$,

$T_N(\Sigma) \cap N = D(\Sigma \cap N) \in \mathfrak{F}$ (Theorem F). Hence, in any case $T_N(\Sigma)$ is N-$\mathfrak{F}$ normal. By 2.21, A is an N-$\mathfrak{F}$ central $\pi(\mathfrak{F})$-chief factor. But $A \cap T_N(\Sigma) = \langle 1 \rangle$, a contradiction to 2.7.1. Hence $N_{\mathfrak{F}} = \langle 1 \rangle$ and $N \in \mathfrak{F}$. Conversely, if $N \in \mathfrak{F}$, all $\pi(\mathfrak{F})$-chief factors of G are N-$\mathfrak{F}$ central. Hence all $\pi(\mathfrak{F})$-maximal subgroups of G are N-$\mathfrak{F}$ normal by 2.21.

Example 2.22.1. Let $N \triangleleft G$ and π a set of primes. Then N is p-nilpotent for all $p \in \pi$ if and only if $N \cap M \subseteq \mathrm{Core}(M)$ for all π-maximal subgroups M of G.

Proof. Let $\mathfrak{F}$ be the formation of groups which are p-nilpotent for all $p \in \pi$. Using the local definition of $\mathfrak{F}$ given in 2.17.2, we obtain the statement as an immediate consequence of 2.22.

Proposition 2.23. Let Σ be a set of Sylow p-complements of G. Then $D = D(\Sigma)$ is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})$.

B. Fischer has defined $\mathfrak{F}$ normalizers of G to be $\mathfrak{F}$ projectors of $N_G(\Sigma \cap G_{\mathfrak{F}})$ in the case that $\mathfrak{F} \supseteq \mathfrak{N}$ [2, 8.4, p. 63].

Proof. First note that $D \subseteq N_G(\Sigma \cap G_{\mathfrak{F}})$: If $\mathfrak{F}(p) = \phi$, $S^p \subseteq N_G(S^p \cap G_{\mathfrak{F}})$ for $S^p \in \Sigma$; if $\mathfrak{F}(p) \neq \phi$, $N_G(S^p \cap G_{\mathfrak{F}(p)}) \subseteq N_G(S^p \cap G_{\mathfrak{F}})$ since $G_{\mathfrak{F}} \subseteq G_{\mathfrak{F}(p)}$. Now induct on $o(G)$ and let A be minimal normal in G. Then DA/A is an $\mathfrak{F}$ projector of $N_{G/A}(\Sigma A/A \cap G_{\mathfrak{F}}A/A)$. But $N_G(\Sigma \cap G_{\mathfrak{F}})A/A \subseteq N_{G/A}((S^p \cap G_{\mathfrak{F}})A/A) = N_{G/A}(S^pA/A \cap G_{\mathfrak{F}}A/A)$ for all $S^p \in \Sigma$. Therefore DA/A is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})A/A$. If A is $\mathfrak{F}$ central, then $D \supseteq A$, and D is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})$ by Theorems B and F. If $G_{\mathfrak{F}} \not\subseteq C_G(A)$, $N_G(\Sigma \cap G_{\mathfrak{F}})$ avoids A, and the natural isomorphism from $N_G(\Sigma \cap G_{\mathfrak{F}})A/A$ to $N_G(\Sigma \cap G_{\mathfrak{F}})$ maps DA/A onto

D since $D \subseteq N_G(\Sigma \cap G_{\mathfrak{F}})$. Hence we may assume A is $\mathfrak{F}$ eccentric and $G_{\mathfrak{F}} \subseteq C_G(A)$. Therefore $G = C_G(A)D$, and so A is minimal normal in AD. Now $G/C_G(A) \cong AD/C_{AD}(A) \notin \mathfrak{F}(p)$, but $AD/A \cong D \in \mathfrak{F}$. Hence D is an $\mathfrak{F}$ projector of AD. Again by Theorem B, we conclude that D is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})$.

Corollary 2.23.1. Let Σ be a set of Sylow p-complements of G. Then $D(\Sigma)$ is an $\mathfrak{F}$ projector and an $\mathfrak{F}$ normalizer of $N_G(\Sigma \cap G_{\mathfrak{F}})$. If $G \notin \mathfrak{F}$, $D(\Sigma) \subset N_G(\Sigma \cap G_{\mathfrak{F}})$ and $G = \langle N_G(\Sigma \cap G_{\mathfrak{F}})^g : g \in G\rangle$. Hence every finite solvable group is generated by subgroups in which $\mathfrak{F}$ projectors coincide with $\mathfrak{F}$ normalizers.

Proof. By 2.23, $D(\Sigma)$ is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})$. By 2.13 Σ reduces to $N_G(\Sigma \cap G_{\mathfrak{F}})$, and by 2.14 $D(\Sigma) \subseteq D(\Sigma \cap N_G(\Sigma \cap G_{\mathfrak{F}}))$. But since $D(\Sigma)$ is a maximal $\mathfrak{F}$ subgroup, we must have equality (Theorem F). Now if $D(\Sigma)$ covers the p-chief factor H/K of G, $G_{\mathfrak{F}} \subseteq G_{\mathfrak{F}(p)} \subseteq C_G(H/K)$, and so $N_G(\Sigma \cap G_{\mathfrak{F}})$ covers H/K. If $G \notin \mathfrak{F}$, $\langle 1\rangle \subset G_{\mathfrak{F}}$ and $D(\Sigma)$ avoids $G_{\mathfrak{F}}/G_{\mathfrak{F}}'$. Hence $o(D(\Sigma)) < o(N_G(\Sigma \cap G_{\mathfrak{F}}))$. If $N_G(\Sigma \cap G_{\mathfrak{F}})$ is contained in a proper normal subgroup M of G, we may suppose M is maximal normal. But $G_{\mathfrak{F}} \subseteq G = C_G(G/M)$, and so $G = MN_G(\Sigma \cap G_{\mathfrak{F}}) = M$, a contradiction. Hence $G = \langle N_G(\Sigma \cap G_{\mathfrak{F}})^g : g \in G\rangle$.

Corollary 2.23.2. Let F be an $\mathfrak{F}$ projector of G. Then F is an $\mathfrak{F}$ normalizer of G if and only if F normalizes a Sylow p-complement of $G_{\mathfrak{F}}$ for all primes p.

Proof. Suppose $F = D(\Sigma)$; then $F \subseteq N_G(\Sigma \cap G_{\mathfrak{F}})$. Conversely, if $F \subseteq N_G(\Sigma \cap G_{\mathfrak{F}})$ for a set Σ of Sylow p-complements, then F

is an $\mathfrak{F}$ projector of $N_G(\Sigma \cap G_{\mathfrak{F}})$ and so is conjugate to $D(\Sigma)$.

Proposition 2.24. Let $\mathfrak{N} \subseteq \mathfrak{F}$. If $G = NF$, $N \lhd G$, $F \lhd G$, $N \in \mathfrak{N}$ and $F \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

Proof. Induct on $o(G)$. The hypothesis is preserved under epimorphic images. Hence there exists a unique minimal normal subgroup A and $G/A \in \mathfrak{F}$. If $G \notin \mathfrak{F}$, A is complemented in G by Theorem E. Hence A is self-centralizing. Since $A \subseteq N$, $A \subseteq Z(N)$ and so $A = N$. ($N \neq \langle 1 \rangle$ since $G \notin \mathfrak{F}$.) Therefore $F \cap N = N$ and $G = F \in \mathfrak{F}$, or $F \cap N = \langle 1 \rangle$ and $G = N \in \mathfrak{F}$.

Proposition 2.25. Let $\mathfrak{N} \subseteq \mathfrak{F}$. Suppose $G = AB = AC = BC$ with A,B abelian subgroups of G and $C \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

Proof. Induct on $o(G)$. The hypothesis is preserved under epimorphic images; hence without loss of generality, there exists a unique minimal normal subgroup M of G and $G/M \in \mathfrak{F}$. If $G \notin \mathfrak{F}$, M is complemented in G and is thus self-centralizing. Since $G = AB$ with A,B abelian, $\mathrm{Core}(A) \neq \langle 1 \rangle$ or $\mathrm{Core}(B) \neq \langle 1 \rangle$ [7, 13.3.3, p. 384]. Hence we may suppose $M \subseteq A$. Therefore $M = A = G_{\mathfrak{F}}$. If B and C are both proper subgroups of G, they are complements of $M = G_{\mathfrak{F}}$. Hence B is conjugate to C (Theorem E), but $G = BC$, a contradiction. Therefore $G = B$ or $G = C$. In either case, $G \in \mathfrak{F}$.

O.H. Kegel proved the following proposition in [6] for the case $\mathfrak{F}$ the formation of supersolvable groups. The proof given here is very similar to Kegel's proof.

Proposition 2.26. Let $\mathfrak{N} \subseteq \mathfrak{F}$. Suppose $G = AB = AC = BC$ with $A,B \in \mathfrak{N}$, C a Sylow tower group, and $C \in \mathfrak{F}$. Then $G \in \mathfrak{F}$.

Proof. Suppose not, and let G be a counterexample of minimal order. As in 2.25 $G_{\mathfrak{F}}$ is the unique minimal normal subgroup of G, and $G_{\mathfrak{F}}$ is self-centralizing. Let p be a prime such that C has a non-trivial normal Sylow p-subgroup. Let A_p, B_p, and C_p be the normal Sylow p-subgroups of A, B, C, respectively. Then $P = A_pB_p = A_pC_p = B_pC_p$ is a normal Sylow p-subgroup of G [6, 1, p. 43]. Hence $G_{\mathfrak{F}} \subseteq P$. Since $C_G(G_{\mathfrak{F}}) = G_{\mathfrak{F}}$, $G_{\mathfrak{F}} = P$. Now $A_p \cap B_p \subseteq Z(G)$ and so $A_p \cap B_p = P$ or $A_p \cap B_p = \langle 1 \rangle$. If $A_p \cap B_p = P = G_{\mathfrak{F}}$, $G_{\mathfrak{F}}$ is an $\mathfrak{F}$ central chief factor, which is impossible. Hence $A_p \cap B_p = \langle 1 \rangle$. Suppose $A_p = \langle 1 \rangle$, then $B_p = G_{\mathfrak{F}}$ and so $B_p = B = G_{\mathfrak{F}}$. Since $A, C \subset G$, both A and C are complements of $B = G_{\mathfrak{F}}$. By Theorem E, A is conjugate to C, an impossibility since $G = AC$. Therefore $A_p \neq \langle 1 \rangle$, and likewise $B_p \neq \langle 1 \rangle$. Let K be an A-invariant complement of A_p in $G_{\mathfrak{F}}$. Then there exists an A-invariant subgroup T of $G_{\mathfrak{F}}$ of index p in $G_{\mathfrak{F}}$. (T can be taken as the product of K and a subgroup of index p in A_p.) Suppose $o(B_p) \geq p^2$. Then $T \cap B_p \neq \langle 1 \rangle$ and $T \cap B_p \subset G_{\mathfrak{F}}$. But $(T \cap B_p)^G = (T \cap B_p)^A \subseteq T \subset G_{\mathfrak{F}}$. Hence $o(B_p) = p$ and likewise $o(A_p) = p$. Let $A_{p'}$ be the complement of A_p in A. Then $C_{A_{p'}}(K) = \langle 1 \rangle$, for otherwise $G_{\mathfrak{F}} = A_pK$ is not self-centralizing. Hence $A_{p'}$ is represented faithfully on K, and so $A_{p'}$ is cyclic. Therefore A is cyclic, and likewise B is cyclic. Hence $G \in \mathfrak{F}$ by 2.25.

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APPENDIX

APPENDIX

For the convenience of the reader, we state and prove a well known result of P. Hall, which is contained in Theorem 7.2 of [3].

Theorem A. Let $N \triangleleft G$, S^p a Sylow p-complement of G, and H/K a p-chief factor of G. Then:

i). If $N_G(S^p)$ does not avoid H/K, $C_G(H/K) = G$.

ii). If $N \not\subseteq C_G(H/K)$, $N_G(S^p \cap N)$ avoids H/K.

iii). If $N \subseteq C_G(H/K)$, $N_G(S^p \cap N)$ covers H/K.

Proof. i). Suppose $N_G(S^p)$ does not avoid H/K. Then in G/K, $N_{G/K}(S^pK/K)$ does not avoid H/K. Hence we may assume $K = \langle 1 \rangle$. Let $h \in H \cap N_G(S^p)$, then $S^p \subseteq C_G(h)$. Let $H_1 = [h,G]$ and P a Sylow p-subgroup of G. Then $H_1 \triangleleft G$ and $H_1 = [h,P] \subset H$. Therefore $H_1 = \langle 1 \rangle$ and $h \in Z(G)$. Hence $\langle h \rangle = H \subseteq Z(G)$.

ii). Suppose $N_G(S^p \cap N)$ does not avoid H/K. Again we may suppose $K = \langle 1 \rangle$. If $H \cap N = \langle 1 \rangle$, then $[H,N] = \langle 1 \rangle$ and $N \subseteq C_G(H)$. Suppose $H \subseteq N$. Then $H \cap N_N(S^p \cap N) \neq \langle 1 \rangle$, and so $N_N(S^p \cap N)$ does not avoid some p-chief factor of N below H. By i), this chief factor is central in N, and by Proposition 2.2 (with $\mathfrak{Z}(p) = \{\langle 1 \rangle\}$), $N \subseteq C_G(H)$.

iii). Suppose $N \subseteq C_G(H/K)$.. Let $G_1 = HN$. Then $N_G(S^p \cap N) \supseteq N_{G_1}(S^p \cap G_1)$, and $H/K \subseteq Z(G_1/K) \subseteq N_{G_1/K}((S^p \cap G_1)K/K) = N_{G_1}(S^p \cap G_1)K/K$. Therefore $H \subseteq N_{G_1}(S^p \cap G_1)K \subseteq N_{G_1}(S^p \cap N)K \subseteq N_G(S^p \cap N)K$.

Theorem B [5, 7.9, p. 699]. a). If $U \subseteq G$ and F is an $\mathfrak{F}$ projector of G with $F \subseteq U$, then F is an $\mathfrak{F}$ projector of U.

b). If $N \triangleleft G$ and F is an $\mathfrak{F}$ projector of G, then FN/N is an $\mathfrak{F}$ projector of G/N.

c). If F_1/N is an $\mathfrak{F}$ projector of G/N and F is an $\mathfrak{F}$ projector of F_1, then F is an $\mathfrak{F}$ projector of G.

Theorem C [5, 7.10-b, p. 700]. Every solvable group G possesses $\mathfrak{F}$ projectors, and all $\mathfrak{F}$ projectors of G are conjugate in G.

The following theorem is contained in the proof of [5, 7.11, p. 701]:

Theorem D. Let G be a $\pi(\mathfrak{F})$ group. If F is an $\mathfrak{F}$ projector of G and $F \subseteq U \subseteq G$, then $N_G(U) = U$. In particular, F is self normalizing in G.

Theorem E [5, 7.15, p. 703]. If $G_{\mathfrak{F}}$ is abelian, then the $\mathfrak{F}$ projectors of G are precisely the complements of $G_{\mathfrak{F}}$ in G.

The following theorem is contained in the proof of [1, 4.1, p. 185]:

Theorem F. Let Σ be a set of Sylow p-complements of G and $D(\Sigma)$ an $\mathfrak{F}$ normalizer of G. Then $D(\Sigma) \in \mathfrak{F}$.