APPROXIMATION TO BAYES RISK IN SEQUENCES OF NON-FINITE DECISION PROBLEMS

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ABSTRACT

APPROXIMATION TO BAYES RISK IN SEQUENCES OF
NON-FINITE DECISION PROBLEMS

by Dennis Crippen Gilliland

Consider a statistical decision problem where $X \sim P_\theta$, $\theta \in \Omega$, and an action $A \in \mathcal{A}$ is to be taken with A allowed to depend upon the random variable X. A loss function $L \geq 0$ is defined on $\Omega \times \mathcal{A}$, and a non-randomized procedure φ incurs a risk $R(\theta,\varphi) = \int L(\theta,\varphi)dP_\theta$. Suppose this decision problem occurs n times with $\underline{\theta}_n = (\theta_1,\ldots,\theta_n) \in \Omega^n$ and $\underline{X}_n = (X_1,\ldots,X_n) \sim P_{\theta_1} \times \ldots \times P_{\theta_n}$. A strongly sequential compound rule $\underline{\varphi} = (\varphi_1,\varphi_2,\ldots)$ is such that for each i, φ_i is the means by which the i th action is taken and $L(\theta,\varphi_i)$ is $\underline{X}_i$ measurable for each θ. The compound risk up to stage n is taken to be the average of the component risks, $R_n(\underline{\theta},\underline{\varphi}) = n^{-1}\Sigma_1^n R(\underline{\theta}_i,\varphi_i)$ where $\underline{\theta} = (\theta_1,\theta_2,\ldots)$. The modified regret $D_n(\underline{\theta},\underline{\varphi}) = R_n(\underline{\theta},\underline{\varphi}) - R(G_n)$ with G_n the empirical distribution of $\underline{\theta}_n$ and $R(G)$ the Bayes risk at G in the component problem has been used as a standard for compound procedures.

The results of the thesis show that $|D_n(\underline{\theta},\underline{\varphi})| = O(n^{-\frac{1}{2}})$ uniformly in $\underline{\theta} \in \Omega^\infty$ for squared error loss, certain discrete exponential families including the Poisson and negative binomial, and realizable sequential procedures $\underline{\varphi}$. Bounds like $o(1)$ and $o(1)$ uniform in $\underline{\theta}$ are obtained for larger classes of discrete exponential families. A problem involving normal $N(\theta,1)$ dis-

tributions is also investigated with a sequential compound procedure $\underline{\varphi}$ demonstrated such that $|D_n(\underline{\theta},\underline{\varphi})| = O(n^{-(1/5)})$ uniformly in $\underline{\theta}$.

In the last chapter sequence strategies are exhibited which at each stage i depend upon G_{i-1} and which achieve various rates of modified regret depending upon the exact structure of the component game. Sequence strategies were given by Hannan ((1957). Contributions to the Theory of Games, 3, 97-139. Ann. Math. Studies No. 39, Princeton University Press.) for various $M \times N$ games with M finite. These procedures result in an absolute modified regret $O(n^{-\frac{1}{2}})$ uniform in player I move sequences. We give a direct generalization to the countable M case. Also a theorem is proved with applications in the uncountable M case.

APPROXIMATION TO BAYES RISK IN SEQUENCES OF NON-FINITE DECISION PROBLEMS

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CHAPTER I

INTRODUCTION TO SEQUENTIAL COMPOUND DECISION PROBLEMS

The notation necessary to describe the mathematics of compound decision theory is necessarily cumbersome. To reduce the complexity we will find it convenient to suppress the display of some dependencies and to use operator notation. For example, PX, $P(X)$ or $P(X(\omega))$ denotes the integral $\int X(\omega)\, dP(\omega)$. Another reduction in notation is accomplished by letting square brackets denote the indicator function, $[A]$ being the indicator function of the set A.

1. The Component Problem.

The component problem has the structure of the usual statistical decision problem. Let Ω denote a parameter space indexing a set of probability measures $\{P_\theta | \theta \in \Omega\}$ and let $\mathcal{A}$ signify the action space. A real valued loss function $L \geq 0$ is defined on $\Omega \times \mathcal{A}$. The action A is allowed to depend upon the realization x in the measurable space $(\mathcal{X},\mathcal{B})$ of a random variable X distributed according to P_θ. A randomized decision function φ is for each x a probability measure $\nu_{\varphi(x)}$ over some σ-field of subsets of $\mathcal{A}$ such that $L(\theta,\varphi(x)) = \nu_{\varphi(x)}(L(\theta,A))$ is $(\mathcal{X},\mathcal{B})$ measurable. The risk or risk function associated with φ is the expected loss resulting from the use of φ: $R(\theta,\varphi) = P_\theta\ (L(\theta,\varphi(x)))$.

2. The Sequential Compound Decision Problem.

Suppose the decision problem described in §1 occurs n times with $\underline{\theta}_n = (\theta_1,\ldots,\theta_n) \in \Omega^n$ and $\underline{X}_n = (X_1,\ldots,X_n)$ distributed according to $\underline{P}_n = P_1 \times \ldots \times P_n$ where we have made the identification $P_i = P_{\theta_i}$. Using the terminology of Hannan (1957, p. 105) we make a distinction between weak and strong sequence games. In the weak sequence game n is known to player II at each component but not in the strong sequence game. Throughout this paper only the latter situation is considered. A strongly sequential compound rule $\varphi = (\varphi_1, \varphi_2, \ldots)$ is such that for each i, $\varphi_i = \varphi_i(x_i)$ is the means by which the ith action is taken and $L(\theta, \varphi_i)$ is $\underline{x}_i$ measurable for each θ. The risk in the ith component problem is $R(\underline{\theta}_i, \varphi_i) = \underline{P}_i(L(\theta_i, \varphi_i))$. The total risk in the sequential problem up to stage n is taken to be the average of the component risks, $R_n(\underline{\theta}, \varphi) = n^{-1} \Sigma_1^n R(\underline{\theta}_i, \varphi_i)$, where $\underline{\theta}$ denotes $(\theta_1, \theta_2, \theta_3, \ldots)$.

If $L(\theta, \varphi_i)$ is x_i measurable for each i we say φ is a simple rule. If in addition the φ_i are identical, $\varphi_i = \varphi_1$, we say φ is a simple symmetric rule. Simple symmetric rules are traditional and result in $R_n(\underline{\theta}, \varphi) = n^{-1} \Sigma_1^n R(\theta_i, \varphi_1)$. This is the same as the risk in the component problem resulting from the use of the procedure φ_1 if θ is assumed to be the realization of a random variable distributed according to G_n the empirical distribution of $\underline{\theta}_n$. With $R(G)$ denoting the Bayes risk versus G in the component problem the above observations lead to consideration of

$$D_n(\underline{\theta}, \varphi) = R_n(\underline{\theta}, \varphi) - R(G_n) \tag{1}$$

as a standard for evaluating compound procedures. Loosely speaking, if $D_n(\underline{\theta}, \varphi) \le B_n = o(1)$ as $n \to \infty$, then in the limit φ incurs a compound risk no greater than $R(G_n)$. $D_n(\underline{\theta}, \varphi)$ is called the modified regret.

Later we will prove theorems concerning rates of convergence of $|D_n(\underline{\theta}, \varphi)|$ for particular families of distributions and procedures. At this time we point out a necessary condition for the existence of any compound procedure φ satisfying $\overline{\lim}_n \sup \{D_n(\underline{\theta},\varphi) | \underline{\theta} \in \Omega^\infty\} < \infty$. It is the existence of a procedure φ_1 in the component problem satisfying $\sup \{P_\theta(L(\theta,\varphi_1)) - \inf_A L(\theta,A) | \theta \in \Omega\} < \infty$. This follows from

$$\sup \{D_n(\underline{\theta},\varphi) | \underline{\theta} \in \Omega^\infty\} \geq \sup \{D_n(\underline{\theta},\varphi) | \underline{\theta} \in \text{diagonal } \Omega^\infty\}$$

$$= \sup \{n^{-1} \Sigma_1^n \underline{P}_i(L(\theta_i,\varphi_i)) - \inf_A L(\theta_1,A) | \underline{\theta} \in \text{diagonal } \Omega^\infty\}$$

$$\geq n^{-1} \sup \{P_\theta(L(\theta,\varphi_1)) - \inf_A L(\theta,A) | \theta \in \Omega\}.$$

3. A Useful Theorem.

Let $\{P_\theta | \theta \in \Omega\}$ be a family of probability measures on a measurable space (X,B). Assume $\Omega \subset E_1$, the real numbers, and $P_\theta \ll \mu$ for all θ where μ is a σ-finite measure. With squared error loss and a priori G on Ω, a Bayes response versus G is provided by any determination of the conditional expectation of θ given $X = x$. With ψ_i generic for a Bayes response versus G_i, the empirical distribution of $\underline{\theta}_i$, $\psi_0 = 0$, we are interested in $P_i(|\psi_i(x) - \psi_{i-1}(x)|)$ and $n^{-1} \Sigma_1^n P_i(|\psi_i(x) - \psi_{i-1}(x)|)$; particularly, in conditions under which there is convergence to zero uniform in parameter sequences $\underline{\theta} \in \Omega^\infty$. To simplify the notation let $\Psi_i(x) = \psi_i(x) - \psi_{i-1}(x)$.

In what follows p_θ is a determination of $\frac{dP_\theta}{d\mu}$ and any ratios 0/0 will be interpreted as 0. A Bayes response versus G_i is

$$(2) \qquad \psi_i(x) = \frac{\Sigma_1^i \theta_j p_j(x)}{\Sigma_1^i p_j(x)} [\Sigma_1^i p_j(x) > 0].$$

The following example shows that for this response and a fixed bounded sequence $\underline{\theta}$, $n^{-1}\Sigma_1^n P_i(|\Psi_i(x)|)$ need not converge to zero.

<u>Example 1</u>. Let μ be counting measure on $\{2,3,\ldots\}$ and $\theta \in \Omega = \{1/2, 2/3, 3/4,\ldots\}$ index P_θ degenerate on $(1-\theta)^{-1}$. Then with $\theta_j = j(j+1)^{-1}$, we compute $\psi_i(x) = \Sigma_2^{i+1}(j-1)j^{-1}[x = j]$ so $\Psi_i(x) = i(i+1)^{-1}[x = i+1]$ and $P_i(|\Psi_i(x)|) = i(i+1)^{-1}$, $i \geq 1$. Hence, $n^{-1}\Sigma_1^n P_i(|\Psi_i(x)|) \to 1$ as $n \to \infty$.

However, in Theorem 1 are given conditions sufficient for convergence at the rate $O(n^{-1}\log n)$ uniform in $\underline{\theta}$.

<u>Theorem 1</u>. If $\Omega \subset [-A,A]$, $A < \infty$ and $M(x) = \sup\{p_\theta(x) | \theta \in \Omega\}$ is integrable $(X,\mathcal{B},\mu)$, then ψ_i given in (2) satisfies $n^{-1}\Sigma_1^n P_i(|\Psi_i(x)|) = O(n^{-1}\log n)$ uniformly in $\underline{\theta}$.

The following lemma is needed.

<u>Lemma 1</u>. $S_n(a_1,\ldots,a_n) = \Sigma_1^n a_i^2(\Sigma_1^i a_j)^{-1} \leq \Sigma_1^n i^{-1} = S_n(1,\ldots,1)$ for all $0 \leq a_i \leq 1$, $1 \leq i \leq n$, $n \geq 1$.

<u>Proof</u>. Let $A_i = \Sigma_1^i a_j$ so that $S_n = \Sigma_1^n a_i^2 A_i^{-1}$, $\dfrac{\partial S_n}{\partial a_n} = (2a_n A_{n-1} + a_n^2) A_n^{-2} \geq 0$. Therefore, $S_n(a_1,\ldots,a_n) \leq S_n(a_1,\ldots,a_{n-1},1)$ for all $0 \leq a_i \leq 1$, $1 \leq i \leq n$. Also, for $1 \leq k < n$

$$\frac{\partial S_n}{\partial a_k}(a_1,\ldots,a_k,1,\ldots,1) = (2a_k A_{k-1} + a_k^2)A_k^{-2} - \Sigma_{k+1}^n (A_k + i-k)^{-2}$$

$$\frac{\partial^2 S_n}{\partial a_k^2}(a_1,\ldots,a_k,1,\ldots,1) = 2A_{k-1}^2 A_k^{-3} + 2\Sigma_{k+1}^n (A_k + i-k)^{-3} \geq 0.$$

Since the second partial is non-negative and

$$S_n(a_1,\dots,a_{k-1},1,\dots,1) = \Sigma_1^{k-1} a_i^2 A_i^{-1} + \Sigma_k^n (A_{k-1} + i-k+1)^{-1}$$

$$\geq \Sigma_1^{k-1} a_i^2 A_i^{-1} + \Sigma_{k+1}^n (A_{k-1} + i - k)^{-1} = S_n(a_1,\dots,a_{k-1},0,1,\dots,1)$$

we have $S_n(a_1,\dots,a_k,1,\dots,1) \leq S_n(a_1,\dots,a_{k-1},1,\dots,1)$ for all $0 \leq a_i \leq 1$, $1 \leq i \leq k$. The proof is completed by backward induction on k.

Proof of Theorem 1. From (2) it follows that

$$\Psi_i(x) = \frac{p_i(x)\, \Sigma_1^{i-1}(\theta_i - \theta_j)\, p_j(x)}{\Sigma_1^{i-1} p_j(x)\, \Sigma_1^i p_j(x)} [\Sigma_1^{i-1} p_j(x) > 0] + \theta_i[\Sigma_1^{i-1} p_j(x) = 0,\ p_i(x) > 0] \tag{3}$$

so if $\Omega \subset [-A,A]$,

$$|\Psi_i(x)| \leq \frac{2A\ p_i(x)}{\Sigma_1^i p_j(x)} [\Sigma_1^{i-1} p_j(x) > 0] + A[\Sigma_1^{i-1} p_j(x) = 0,\ p_i(x) > 0]$$

and

$$P_i(|\Psi_i(x)|) \leq 2A\ \mu\Big(\frac{p_i^2(x)}{\Sigma_1^i p_j(x)} [\Sigma_1^{i-1} p_j(x) > 0]\Big) + A\ \mu(p_i(x)\ [\Sigma_1^{i-1} p_j(x) = 0,\ p_i(x) > 0]) \tag{4}$$

for $i \geq 1$. Therefore,

$$n^{-1} \Sigma_1^n P_i(|\Psi_i(x)|) \leq 2A\ n^{-1}\ \mu\Big(M(x)\ \Sigma_1^n \frac{(p_i(x)/M(x))^2}{\Sigma_1^i (p_j(x)/M(x))}\Big) + A\ n^{-1}\ \mu(\Sigma_1^n p_i(x)[\Sigma_1^{i-1} p_j(x) = 0,\ p_i(x) > 0]). \tag{5}$$

Lemma 1 implies that the first term on the right hand side is bounded by $2A\ \mu(M(x))\ n^{-1}\ \Sigma_1^n\ i^{-1}$. Since $M(x) \geq \Sigma_1^n\ p_i(x)[\Sigma_1^{i-1}\ p_j(x) = 0,\ p_i(x) > 0]$ the second term is bounded by $A\ \mu(M(x))\ n^{-1}$ and the theorem is proved.

We give an example to show that the bound on the rate of convergence indicated in Theorem 1 is tight.

Example 2. Consider the family of geometric distributions with densities $p_\theta(x) = \theta^x(1 - \theta)$, $x = 0, 1, 2, \ldots$; $\theta \in \Omega = [0,\frac{1}{2}]$ with respect to counting measure. At $\underline{\theta} = (0,\frac{1}{2},0,\frac{1}{2},\ldots)$, $\psi_{2i}(x) = (1/6)[x = 0] + \frac{1}{2}[x > 0]$, $\psi_{2i-1}(x) = \frac{1}{2}(i - 1)(3i - 1)^{-1}[x = 0] + \frac{1}{2}[x > 0]$ so $\Psi_{2i}(x) = (1/3)(3i - 1)^{-1}[x = 0]$.

Hence,

$$\Sigma_1^{2n}\ P_i(|\Psi_i(x)|) \geq \Sigma_1^n\ (1/6)(3i - 1)^{-1}$$

$$\geq (1/18)\ \Sigma_1^n\ i^{-1} \geq (1/18)\ \log n \geq (1/36)\ \log\ (2n)$$

for $n \geq 2$.

The hypothesis of Theorem 1 need not imply $P_i(|\Psi_i(x)|) \to 0$ as $i \to \infty$ uniformly in $\underline{\theta}$ no matter what determination of the conditional expectation ψ_i is used.

Example 3. Consider the family of distributions of the preceding example. For G degenerate on 0, $\psi_G(x) = a(x)[x > 0]$ for some function a. Let $b \in \Omega$ be any number such that $b \neq 0$ and $b \neq a(1)$. Define $\underline{\theta}^{(i)} = (\theta_1^{(i)}, \theta_2^{(i)},\ldots)$ where $\theta_j^{(i)} = b$ or 0 according as $j = i$ or $j \neq i$. Then at $\underline{\theta}^{(i)}$, $\psi_i(x) =$

$b(1 - b)(i - b)^{-1}[x = 0] + b[x > 0]$, $\psi_{i-1}(x) = a(x)[x > 0]$ so $P_i(|\Psi_i(x)|) > |b - a(1)|\ P_i[x = 1] = |b - a(1)|\ b(1 - b)$.

The next proposition shows that there is convergence as $i \to \infty$ uniform in $\underline{\theta}$ for a family of normal distributions. This result will be needed in Chapter III where a decision problem involving this family is discussed.

Proposition 1. Consider the family of $N(\theta,1)$ measures, $\Omega \subset [-A,A]$, $A < \infty$. Let p_θ be the continuous normal density with respect to Lebesgue measure μ. For the determination of ψ_i given in (2), $P_i(|\Psi_i(x)|) = O(i^{-1})$ as $i \to \infty$ uniformly in $\underline{\theta}$.

Proof. Inequality (4) implies

$$P_i(|\Psi_i(x)|) \le 2A\ i^{-1}\{\mu(p^2_{-A}(x)\ p_A^{-1}(x)\ [x < -A]) + (2\pi)^{-1}B^{-1}\mu[-A \le x \le A] + \mu(p_A^2(x)\ p_{-A}^{-1}(x)[x > A])\} \tag{6}$$

where $B = \min\{p_\theta(x) \mid (x,\theta) \in [-A,A]^2\}$. Since the factor of (6) appearing in curly brackets is finite and does not depend upon $\underline{\theta}$ the proof is complete.

4. A Bound for the Modified Regret under Squared Error Loss.

The loss function $L(\theta,A) = (\theta - A)^2$ is continuous and convex in A for each fixed θ so only non-randomized procedures need be considered. For the component problem this follows from an application of Jensen's inequality as is well known. It is true for the compound problem since the compound risk is monotone increasing in each component risk. A non-randomized procedure φ has modified regret

$$D_n(\underline{\boldsymbol{\theta}},\varphi) = n^{-1}\,\Sigma_1^n\,\underline{P}_i(\varphi_i - \theta_i)^2 - R(G_n). \tag{7}$$

Inequalities (8.8) and (8.11) of Hannan (1957) show that

$$n^{-1}\,\Sigma_1^n\,P_i(\psi_i - \theta_i)^2 \le R(G_n) \tag{8}$$

and

$$n^{-1}\,\Sigma_1^n\,P_i(\psi_{i-1} - \theta_i)^2 \ge R(G_n). \tag{9}$$

Inequalities (7) and (8) imply

$$D_n(\underline{\boldsymbol{\theta}},\varphi) \le n^{-1}\,\Sigma_1^n\,\underline{P}_i((\varphi_i - \psi_i)(\varphi_i + \psi_i - 2\theta_i)) \tag{10}$$

while subtracting and adding $n^{-1}\,\Sigma_1^n\,P_i(\psi_i - \theta_i)^2$ from the lower bound resulting from applying (9) to (7) proves

$$D_n(\underline{\boldsymbol{\theta}},\varphi) \ge n^{-1}\,\Sigma_1^n\,\underline{P}_i((\varphi_i - \psi_i)(\varphi_i + \psi_i - 2\boldsymbol{\theta}_i)) \tag{11}$$

$$+\, n^{-1}\,\Sigma_1^n\,P_i(\Psi_i(\psi_i + \psi_{i-1} - 2\theta_i)).$$

Assume $\Omega \subset [-A,A]$. Then ψ_i takes values in $[-A,A]$ and, therefore, if φ_i does, (10) and (11) imply

$$-4A\,n^{-1}\,\Sigma_1^n\,P_i(|\Psi_i|) - 4A\,n^{-1}\,\Sigma_1^n\,\underline{P}_i(|\varphi_i - \psi_i|) \le \tag{12}$$

$$D_n(\underline{\boldsymbol{\theta}},\varphi) \le 4A\,n^{-1}\,\Sigma_1^n\,\underline{P}_i(|\varphi_i - \psi_i|).$$

A corollary to Theorem 1 is immediate.

<u>Corollary 1</u>. Under the hypothesis of Theorem 1 if φ is a procedure taking values in $[-A,A]$, then

$$(13) \qquad |D_n(\underline{\theta},\varphi)| \le 4A\ n^{-1} \Sigma_1^n \underline{P}_i(|\varphi_i - \psi_i|) + O(n^{-1} \log n)$$

where $O(n^{-1} \log n)$ is uniform in $\underline{\theta}$.

CHAPTER II

SEQUENTIAL COMPOUND ESTIMATION FOR SQUARED ERROR LOSS AND SOME DISCRETE EXPONENTIAL FAMILIES

1. Introduction.

Consider the discrete exponential family of probability measures on the non-negative integers

$$p_\theta(x) = \theta^x\, h(\theta)\, g(x),\ x = 0, 1, 2, \ldots\ ;\ \theta \in \Omega$$

where $g(x) > 0$ and

(A1) $$\Omega = [0,\beta],\quad 0 < \beta < \infty$$

with β assumed known. (The condition $g(x) > 0$ is assumed by Samuel (1965) and is implied by assumptions of Swain (1965, p. 25).) Throughout this chapter we will be concerned with the Bayes response (1.2) which for this family takes the special form

(1) $$\psi_i(x) = \frac{g(x)\,\Sigma_1^i\, p_j(x+1)}{g(x+1)\,\Sigma_1^i\, p_j(x)} = \frac{g\,\Sigma_1^i\,\tilde{p}_j}{\tilde{g}\,\Sigma_1^i\, p_j}$$

where we have chosen to suppress the display of dependence upon x and $\tilde{f}$ is f evaluated at $x+1$. Any ratios $0/0$ should be interpreted as 0.

Since $p_\theta(x) \le \frac{h(0)}{h(\beta)}\, p_\beta(x)$ for all x and $0 \le \theta \le \beta$, the hypothesis of Theorem 1.1 is seen to be satisfied under (A1). Therefore, Corollary 1.1 implies that for any procedure taking values in $[0,\beta]$

(2) $$|D_n(\underline{\theta},\varphi)| \le 2\beta\, n^{-1}\, \Sigma_1^n\, \underline{P}_i(|\varphi_i - \psi_i|) + 0(n^{-1}\log n)$$

where the $O(n^{-1} \log n)$ is uniform in $\underline{\theta}$. In (2) we have made an adjustment in constant appropriate to the situation $\Omega \subset [0,\beta]$. However, for the most part we will be content to obtain bounds in terms of order and will not attempt to keep track of the constant. We will be explicit enough so that a constant can be recovered if anyone is so interested. In subsequent sections we introduce procedures and use (2) to obtain a bound on the rate of convergence of the absolute modified regret. Among other things it will be shown that if the family satisfies conditions in addition to (A1) then a bound of order $O(n^{-\frac{1}{2}})$ exists. Poisson and negative binomial families satisfy the conditions. Before doing this we review or reference what has been accomplished by others who have attacked the problem.

2. Review of Known Results and an Example.

Essentially, Samuel (1965) imposed the assumption

$$(A1^+) \qquad \Omega = [\alpha,\beta]$$

where $\alpha > 0$ and β are known and considered the procedure $\varphi_i(X_i)$ where

$$(3) \qquad \varphi_i(x) = \frac{g\,\{((i-1)^{-1}\,\Sigma_1^{i-1}\,\tilde{\delta}_j) \vee \tilde{m}\}}{\tilde{g}\,\{((i-1)^{-1}\,\Sigma_1^{i-1}\,\delta_j) \vee m\}} \wedge \beta$$

with $\delta_j = \delta(X_j,x)$, the Kronecker delta function, and $m(x) = \min\{p_\theta(x) \mid \theta \in [\alpha,\beta]\}$. (In Samuel (1965) the truncation at β was not made explicit but we assume this was intended since the indicated proof of Theorem 4 depends upon the boundedness of the procedure.) Samuel showed that under ($A1^+$) there exists a null sequence $K_n(\underline{\theta}) > 0$ such that $D_n(\underline{\theta},\varphi) \le K_n(\underline{\theta})$. (In §5 we point out that under ($A1^+$), $|D_n(\underline{\theta},\varphi)| = o(1)$ uniformly in $\underline{\theta}$ and under (A1), $|D_n(\underline{\theta},\varphi)| = o(1)$ at each fixed $\underline{\theta}$, for two procedures φ.) To motivate the introduction of other conditions in addition to (A1) or ($A1^+$) we give an example that illustrates that the null sequence $K_n(\underline{\theta})$ may be going to zero arbitrarily slowly. That is, given any null sequence $H_n > 0$, there exists an exponential family and a sequence $\underline{\theta} \in [\alpha,\beta]^\infty$ such that $D_n(\underline{\theta},\varphi) \ge H_n$ for all large n.

Example 1. Consider $p_\theta(x) = \theta^x h(\theta) g(x)$, $\Omega = [\alpha,\beta]$ with $\alpha < \beta = 1$ and where $g(0) = 1$ and $g(x) \le x^{-3}$, $x \ge 1$ is yet to be specified. We first point out that $m(x) = p_\alpha(x)$ for $x \ge 2$. The function $h(\theta) = (\Sigma_0^\infty \theta^x g(x))^{-1}$ is differentiable on $(0,1)$ with $h'(\theta) = -\Sigma_1^\infty x\, \theta^{x-1} g(x)\, h^2(\theta)$ so $P_\theta(X) = -\theta\, h'(\theta)/h(\theta)$. For fixed x, $\frac{\partial}{\partial\theta} p_\theta(x) = \theta^{x-1} g(x)\, [x\, h(\theta) + \theta\, h'(\theta)] = \theta^{x-1} g(x)\, h(\theta)\, [x - P_\theta(X)]$. Since $P_\theta(X) = \Sigma_1^\infty x\, \theta^x g(x)\, h(\theta) \le h(0)\, \Sigma_1^\infty x^{-2} = \Sigma_1^\infty x^{-2} = \pi^2/6$ we see that $\frac{\partial}{\partial\theta} p_\theta(x) > 0$ for all θ and $x \ge 2$. Hence, $m(x) = p_\alpha(x)$ for $x \ge 2$. We now consider the modified regret for the procedure φ at the fixed sequence $\underline{\theta} = \underline{1} = (1,1,1,\ldots)$. Here

$$
\begin{aligned}
D_n(\underline{1},\varphi) &= n^{-1} \Sigma_1^n \underline{P}_i((\varphi_i(x) - 1)^2) \\
&\ge n^{-1} \Sigma_1^n \underline{P}_i((\varphi_i(x) - 1)^2\, [\Sigma_1^{i-1} \tilde{\delta}_j = 0]) \\
&\ge n^{-1} \Sigma_1^n \sum_{x=0}^{\infty} p_1(x) \left\{ \frac{g(x)\ m(x+1)}{g(x+1)\ m(x)} \wedge \beta - 1 \right\}^2 \underline{P}_{i-1}[\Sigma_1^{i-1} \tilde{\delta}_j = 0] \\
&\ge n^{-1} \Sigma_1^n \Sigma_2^\infty p_1(x) \{\alpha - 1\}^2\, \underline{P}_{n-1}[\Sigma_1^{n-1} \tilde{\delta}_j = 0]
\end{aligned}
$$

$$= (\alpha - 1)^2 \Sigma_2^\infty p_1(x) (1 - p_1(x+1))^{n-1}$$

$$\geq \tfrac{1}{4} \Sigma_1^\infty p_1(2x+1) (1 - p_1(2x+2))^{n-1}$$

where we have taken $\alpha = \frac{1}{2}$. Let $g(x) = x^{-3}$, $x = 1, 3, 5, \ldots$ and $x^{-3} \geq g(x) = a(x)$, $x = 2, 4, 6, \ldots$ with $a(x)$ strictly decreasing. Then

$$D_n(\underline{1},\varphi) \geq \tfrac{1}{4} \Sigma_1^\infty h(1) (2x+1)^{-3} (1 - h(1)\ a(2x+2))^{n-1}$$

$$\geq \tfrac{1}{4} \Sigma_{A_n}^\infty h(1) (2x+1)^{-3} (1 - \frac{h(1)}{n-1})^{n-1}$$

where $A_n = \min\{x | a(2x+2) \leq (n-1)^{-1}\}$. Since $\Sigma_b^\infty (2x+1)^{-3} \geq \frac{1}{4}(2b+1)^{-2}$ and $h(1) = (\Sigma_0^\infty g(x))^{-1}$ implies $B = (1 + \Sigma_1^\infty x^{-3})^{-1} \leq h(1) \leq 1$, we can write

$$D_n(\underline{1},\varphi) \geq \frac{B}{16} (1 - \frac{1}{n-1})^{n-1} (2A_n + 1)^{-2}.$$

By choice of $a(x)$, A_n can be made to increase arbitrarily slowly. Since $(1 - \frac{1}{n-1})^{n-1} \to e^{-1}$ the example is complete.

Samuel's procedure (3) depends upon a knowledge of a positive lower bound for the parameter set Ω as well as an upper bound. The procedures we will introduce do not depend upon knowledge of a positive lower bound. In fact, the realizable procedures φ that we will be introducing are such that $\varphi_i(x) = 0$ on $[\Sigma_1^{i-1} \tilde{\delta}_j = 0]$ so that the lower bound for D_n developed in Example 1 applies to these procedures also.

Swain (1965, pp. 25-26) makes four assumptions concerning the discrete exponential family. Under these conditions he demonstrates a procedure φ such that $D_n(\underline{\theta},\varphi) \leq B_n$ for all $\underline{\theta}$ where $0 \leq B_n = O(n^{-\frac{1}{4}}\log n)$. His assumptions are satisfied by the negative binomial family for $\beta < 1$ and the Poisson family for $\beta < 1$. (A trivial modification of his assumption (iv) and a slight alteration of the proof of his Lemma 6 extends his result to cover the Poisson family for any β. However, this will not be detailed here since results to follow, in particular the theorem at the end of §5, show that assumption (iv) is not needed.) In the following sections we will derive bounds for the absolute modified regret under various sets of conditions on the underlying family of distributions.

3. Bounds for $n^{-1} \Sigma_1^n E(|\varphi_i' - \psi_i|)$.

We first introduce a procedure φ' which is not realizable under the sequential problem as originally stated. Define $\varphi_i'(X_i)$ by

$$\varphi_i'(x) = \frac{g\,\{\Sigma_1^{i-1}\tilde{\delta}_j + \tilde{\delta}_i'\}}{\tilde{g}\,\{\Sigma_1^{i-1}\delta_j + \delta_i'\}} \wedge \beta \tag{4}$$

where $\delta_j = \delta(X_j, x)$ and $\delta_i' = \delta(X_i', x)$; X_i, X_i' are identically distributed P_i and X_i' is independent of $(X_1, \ldots, X_i)$ for all $i \geq 1$. (Recall that all ratios $0/0$ are to be interpreted as 0.) For each fixed $X_i = x$ write

$$\begin{aligned} E_i'(|\varphi_i'(x) - \psi_i(x)|) = &\int_0^{\psi_i(x)} E_i'[\varphi_i'(x) - \psi_i(x) \leq -t]dt \\ &+ \int_0^{\beta} E_i'[\varphi_i'(x) - \psi_i(x) \geq t]dt \end{aligned} \tag{5}$$

where E_i' is expectation with respect to the measure induced by $(X_1, \ldots, X_{i-1}, X_i')$. We use exponential bounds of Hoeffding (1963) to approximate each integrand displayed on the right hand side of

(5). Temporarily we completely suppress the display of dependence on x and delete the primes on δ_i' and E_i'. Also dropping the subscript from E_i we have for $0 \le t \le \beta$

$$E[\varphi_i' - \psi_i \ge t] \le E[\frac{g\,\Sigma_1^i \tilde{\delta}_j}{\tilde{g}\,\Sigma_1^i \delta_j} - (\psi_i + t) \ge 0]$$

$$= E[\Sigma_1^i Y_j \ge 0] = E[\Sigma_1^i (Y_j - EY_j) \ge -\Sigma_1^i EY_j]$$

with $Y_j = \tilde{\delta}_j - R_i(t)\,\delta_j$, $R_i(t) = \frac{\tilde{g}}{g}(\psi_i + t)$. Since $-\Sigma_1^i EY_j = t\,\frac{\tilde{g}}{g}\,\Sigma_1^i p_j \ge 0$ and $-R_i(t) - EY_j \le Y_j - EY_j \le 1 - EY_j$, Hoeffding's Theorem 2 (1963) yields

$$\text{(6)} \qquad E[\varphi_i' - \psi_i \ge t] \le \exp\{-\frac{2\,\tilde{g}^2(\Sigma_1^i p_j)^2}{i\,g^2\,(1+R_i(t))^2}\,t^2\}.$$

Similarly, we treat the other tail writing for $0 \le t \le \psi_i$

$$E[\varphi_i' - \psi_i \le -t] = E[\frac{g\,\Sigma_1^i \tilde{\delta}_j}{\tilde{g}\,\Sigma_1^i \delta_j} - (\psi_i - t) \le 0]$$

$$= E[\Sigma_1^i (Z_j - EZ_j) \le -\Sigma_1^i EZ_j]$$

with $Z_j = \tilde{\delta}_j - S_i(t)\,\delta_j$, $S_i(t) = \frac{\tilde{g}}{g}(\psi_i - t)$. The inequalities $-\Sigma_1^i EZ_j = -t\,\frac{\tilde{g}}{g}\,\Sigma_1^i p_j \le 0$ and $-S_i(t) - EZ_j \le Z_j - EZ_j \le 1 - EZ_j$ imply

$$\text{(7)} \qquad E[\varphi_i' - \psi_i \le -t] \le \exp\{-\frac{2\,\tilde{g}2\;(\Sigma_1^i p_j)^2}{i\,g^2\,(1+S_i(t))^2}\,t^2\}.$$

Noting upper bounds for R_i and S_i we combine (6) and (7) obtaining for all $0 \le t \le \beta$

$$\text{(8)} \qquad E[|\varphi_i' - \psi_i| \geq t] \leq 2 \exp\{-\frac{2\, g^2\, (\Sigma_1^i\, p_j)^2}{i\, g^2\, (1+2\beta\frac{\tilde{g}}{g})^2}\, t^2\}.$$

Since $\int_0^\beta \exp\{-ct^2\}dt \leq \frac{1}{2}(\pi/c)^{\frac{1}{2}}$, (5) and (8) imply

$$\text{(9)} \qquad E(|\varphi_i' - \psi_i|) \leq B\,(1 + \frac{g}{\tilde{g}})(\Sigma_1^i\, p_j)^{-1}\, i^{\frac{1}{2}}$$

for a constant B not depending upon $\underline{\theta}$, i and x. Inequality (9) provides a high rate of convergence at $x = 0$ because $\Sigma_1^i\, p_j(0) \geq h(\beta)\, g(0)\, i$; and, therefore,

$$\text{(10)} \qquad E(|\varphi_i'(0) - \psi_i(0)|) \leq C\, i^{-\frac{1}{2}}$$

for a constant C not depending upon $\underline{\theta}$ and i.

<u>Proposition 1</u>. If the family of distributions satisfies $(A1^+)$, then

$$\text{(11)} \qquad P_i E(|\varphi_i' - \psi_i|) = o(1) \quad \text{as } i \to \infty$$

uniformly in $\underline{\theta}$; and, consequently,

$$\text{(12)} \qquad n^{-1}\, \Sigma_1^n\, P_i E(|\varphi_i' - \psi_i|) = o(1) \quad \text{as } n \to \infty$$

uniformly in $\underline{\theta}$.

<u>Proof</u>. It follows from (9) and the fact that φ_i' and ψ_i take values in $[0,\beta]$ that for each fixed x, $E(|\varphi_i'(x) - \psi_i(x)|) \leq f_i(x) \wedge \beta$ where $f_i(x) = B(1 + \frac{g(x)}{g(x+1)})\, m^{-1}(x)\, i^{-\frac{1}{2}}$, $m(x) = \inf\{p_\theta(x) | \theta \in \Omega\}$

and $m(x) > 0$ under $(A1^+)$. Since $P_{\theta} \le \frac{h(0)}{h(\beta)} P_\beta$, $P_i E(|\varphi_i' - \psi_i|) \le \frac{h(0)}{h(\beta)} P_\beta (f_i(x) \wedge \beta)$. The right hand side does not depend upon $\underline{\theta}$ and goes to zero by the dominated convergence theorem.

We now use the Berry-Esseen normal approximation (Loève, 1963, p. 288) to bound the integrands displayed on the right hand side of (5) for each $x \ge 1$. The resulting upper bound for $E(|\varphi_i' - \psi_i|)$ is more tractable than that provided by (9).

As before consider $Y_j = \tilde{\delta}_j - R_i(t)\,\delta_j$ and let $W_j = (Y_j - EY_j)/(1 + R_i(t))$. Since $|W_j| \le 1$, $E(|W_j|^3) \le E(W_j^2)$ and it follows with $r_i^2(t) = \mathrm{Var}(\Sigma_1^i Y_j)$ that

$$
\begin{aligned}
(13)\qquad L_i(t) &= r_i^{-3}(t)\, \Sigma_1^i E(|Y_j - EY_j|^3) \\
&\le (\Sigma_1^i E(W_j^2))^{-\frac{1}{2}} = r_i^{-1}(t)(1 + R_i(t)).
\end{aligned}
$$

Explicitly, $r_i^2(t) = \Sigma_1^i\{\tilde{p}_j + R_i^2(t)p_j - (\tilde{p}_j - R_i(t)p_j)^2\}$. Note that $q = \inf\{1 - p_{\theta}(x) \mid x \ge 1,\ \theta \in \Omega\} > 0$, $R_i^2(t) \le (2\beta\frac{\tilde{g}}{g})^2$, $\tilde{p}_j \le \beta\frac{\tilde{g}}{g} p_j$ so with $T^2 = \beta\frac{\tilde{g}}{g} + 4\beta^2\frac{\tilde{g}^2}{g^2}$, $x \ge 1$,

$$
(14)\qquad q\, \Sigma_1^i \tilde{p}_j + q\, R_i^2(t)\, \Sigma_1^i p_j \le r_i^2(t) \le T^2\, \Sigma_1^i p_j .
$$

Hence,

$$
(15)\qquad L_i(t) \le (q\, \Sigma_1^i \tilde{p}_j)^{-\frac{1}{2}} + (q\, \Sigma_1^i p_j)^{-\frac{1}{2}}.
$$

The Berry-Esseen theorem implies that for a constant c

$$
(16)\qquad E[\varphi_i' - \psi_i \ge t] \le \Phi\Big(-\frac{\tilde{g}\, \Sigma_1^i p_j}{g\, r_i(t)}\, t\Big) + c\, L_i(t)
$$

where Φ denotes the distribution function of $N(0,1)$. We can write for $a > 0$, $\int_0^\beta \Phi(-at)dt \le a^{-1} \int_{-\infty}^0 \Phi(\tau)d\tau = 2^{-1} a^{-1} \int_0^\infty P[|X| \ge \tau]d\tau = 2^{-1} a^{-1} P(|X|)$ where $X \sim N(0,1)$. Therefore, $\int_0^\beta \Phi(-at)dt \le (2\pi)^{-\frac{1}{2}} a^{-1}$. Using this result and the upper bound in (14) yields

$$\text{(17)} \qquad \int_0^\beta \Phi\left(-\frac{\tilde{g}\, \Sigma_1^i p_j}{g\, r_i(t)} t\right)dt \le \frac{g\, T}{(2\pi)^{\frac{1}{2}}\, \tilde{g}(\Sigma_1^i p_j)^{\frac{1}{2}}}\ .$$

We combine (15)- (17) to prove

$$\text{(18)} \qquad \int_0^\beta E[\varphi_i' - \psi_i \ge t]dt \le D\{(\frac{g}{\tilde{g}} T + 1)(\Sigma_1^i p_j)^{-\frac{1}{2}} + (\Sigma_1^i \tilde{p}_j)^{-\frac{1}{2}}\}$$

for a constant D.

Unfortunately, the problem is not symmetric and the other tail must be treated separately. As before let $Z_j = \tilde{\delta}_j - S_i(t)\, \delta_j$. For $0 \le t \le \psi_i$, $s_i^2(t) = \mathrm{Var}(\Sigma_1^i Z_j)$, and $U^2 = \beta \frac{\tilde{g}}{g} + \beta^2 \frac{\tilde{g}^2}{g^2}$ we have

$$\text{(19)} \qquad q\, \Sigma_1^i \tilde{p}_j + q\, S_i^2(t)\, \Sigma_1^i p_j \le s_i^2(t) \le U^2\, \Sigma_1^i p_j .$$

With $L_i'(t) = s_i^{-3}(t)\, \Sigma_1^i E(|Z_j - EZ_j|^3)$ we have $L_i'(t) \le s_i^{-1}(t)$ $(1 + S_i(t))$ so

$$\text{(20)} \qquad L_i'(t) \le (q\, \Sigma_1^i \tilde{p}_j)^{-\frac{1}{2}} + (q\, \Sigma_1^i p_j)^{-\frac{1}{2}}$$

and proceeding as in the derivation of (18)

$$\text{(21)} \qquad \int_0^{\psi_i} E[\varphi_i' - \psi_i \le -t]dt \le D\{(\frac{g}{g} U + 1)(\Sigma_1^i p_j)^{-\frac{1}{2}} + (\Sigma_1^i \tilde{p}_j)^{-\frac{1}{2}}\}.$$

Since $U \le T \le \beta^{\frac{1}{2}} \frac{\tilde{g}^{\frac{1}{2}}}{g^{\frac{1}{2}}} + 2\beta \frac{\tilde{g}}{g}$, (18) and (21) combine to yield

$$(22) \qquad E(|\varphi_i' - \psi_i|) \le D\{(\frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}} + 1)(\Sigma_1^i p_j)^{-\frac{1}{2}} + (\Sigma_1^i \tilde{p}_j)^{-\frac{1}{2}}\}$$

for each fixed $x \ge 1$ and a constant D.

Proposition 2. If the family of distributions satisfies (A1), then at each $\underline{\theta} \in \Omega^{\infty}$

$$(23) \qquad P_i E(|\varphi_i' - \psi_i|) = o(1) \quad \text{as } i \to \infty ;$$

and, consequently,

$$(24) \qquad n^{-1} \Sigma_1^n P_i E(|\varphi_i' - \psi_i|) = o(1) \quad \text{as } n \to \infty.$$

Proof. We can write $P_i E(|\varphi_i' - \psi_i|) \le E(|\varphi_i'(0) - \psi_i(0)|) + \beta\, P_i[x > 0] \le C\, i^{-\frac{1}{2}} + \beta\, P_i[x > 0]$ using (10). Therefore, if $\underline{\theta}$ is such that $\theta_i \to 0$ as $i \to \infty$ then $P_i E(|\varphi_i' - \psi_i|) = o(1)$. Otherwise, $\overline{\lim}_i\, \theta_i > 0$ and for each x, $\Sigma_1^i p_j(x) \to \infty$. Thus, (22) applied for $x \ge 1$ and (10) for $x = 0$ shows that at each fixed x, $E(|\varphi_i'(x) - \psi_i(x)|) \to 0$. Noting $P_i \le \frac{h(0)}{h(\beta)} P_\beta$ the proof is completed by application of the dominated convergence theorem.

Under the assumption (A1$^+$) the bound (22) takes the form

$$(25) \qquad E(|\varphi_i' - \psi_i|) \le D(1 + \frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}})(\Sigma_1^i p_j)^{-\frac{1}{2}}$$

where we have used the relation $\tilde{p}_j \ge \alpha \frac{\tilde{g}}{g} p_j$ and D is some constant independent of $\underline{\theta}$, i and $x \ge 1$. Writing $\Sigma_1^n P_i E(|\varphi_i' - \psi_i|) = \sum_{x=0}^{\infty} \Sigma_1^n p_i(x)\, E(|\varphi_i'(x) - \psi_i(x)|)$, (10) and (25) yield

$$n^{-1}\,\Sigma_1^n\, P_i E(|\varphi_i' - \psi_i|) \le C\, n^{-1}\,\Sigma_1^n\, i^{-\frac{1}{2}} \tag{26}$$

$$+ D\, n^{-1} \sum_{x=1}^{\infty} \Big(1 + \frac{g^{\frac{1}{2}}(x)}{g^{\frac{1}{2}}(x+1)}\Big)\, \Sigma_1^n\, p_i(x) (\Sigma_1^i\, p_j(x))^{-\frac{1}{2}}.$$

<u>Lemma 1</u>. $\Sigma_1^n\, a_i(\Sigma_1^i\, a_j)^{-\frac{1}{2}} \le 2\, (\Sigma_1^n\, a_i)^{\frac{1}{2}}$ for all $a_i \ge 0$, $1 \le i \le n$, $n \ge 1$.

<u>Proof</u>. Let $A_i = \Sigma_1^i\, a_j$. The inequality $0 \le A_i - 2\, A_i^{\frac{1}{2}}\, A_{i-1}^{\frac{1}{2}} + A_{i-1}$ implies $a_i \le 2(A_i - A_i^{\frac{1}{2}}\, A_{i-1}^{\frac{1}{2}})$ from which $a_i\, A_i^{-\frac{1}{2}} \le 2(A_i^{\frac{1}{2}} - A_{i-1}^{\frac{1}{2}})$. Summing from 1 to n proves the lemma.

Let $M(x) = \sup\,\{p_\theta(x) | \theta \in \Omega\}$. Dividing the $p_i(x)$ by $M(x)$ we use the lemma to bound the last term in (26) by $2D\, n^{-\frac{1}{2}}$ $\sum_{x=1}^{\infty}(1 + g^{\frac{1}{2}}(x)/g^{\frac{1}{2}}(x+1))\, M^{\frac{1}{2}}(x)$. Since $M(x) \le \frac{h(0)}{h(\beta)}\, p_\beta(x)$ (26) becomes

$$n^{-1}\,\Sigma_1^n\, P_i E(|\varphi_i' - \psi_i|) \le C' n^{-\frac{1}{2}} + D' n^{-\frac{1}{2}} \sum_{x=1}^{\infty}\Big(1 + \frac{g^{\frac{1}{2}}(x)}{g^{\frac{1}{2}}(x+1)}\Big)\, p_\beta^{\frac{1}{2}}(x) \tag{27}$$

for appropriate constants C' and D'. This motivates the following assumptions concerning the family of distributions:

$$\sum_x p_\beta^{\frac{1}{2}}(x) < \infty \tag{A2}$$

and

$$\sum_x \Big(\frac{g(x)\; p_\beta(x)}{g(x+1)}\Big)^{\frac{1}{2}} < \infty. \tag{A3}$$

<u>Proposition 3</u>. If the family of distributions satisfies $(A1^+)$, (A2) and (A3), then

$$n^{-1}\,\Sigma_1^n\, P_i E(|\varphi_i' - \psi_i|) = 0(n^{-\frac{1}{2}}) \tag{28}$$

uniformly in $\underline{\theta}$.

Proof. The proof follows directly from (27).

It follows from (2) that the bounds on the rates of convergence indicated in Propositions 1, 2, and 3 also apply to $|D_n(\underline{\theta},\varphi')|$. In §5 we will bound the absolute modified regret of realizable procedures φ by bounding $n^{-1}\Sigma_1^n P_i E(|\varphi' - \varphi|)$ and applying a triangle inequality. Before doing that we show that under (A1) and assumptions stronger than (A2) and (A3) the bound $O(n^{-\frac{1}{2}})$ holds for a procedure φ''.

4. Bounds for $n^{-1}\Sigma_1^n E(|\varphi_i'' - \psi_i|)$.

We introduce the procedure $\varphi_i''(X_i)$ where

$$\varphi_i''(x) = \frac{g\{\Sigma_1^{i-1}\tilde{\delta}_j + \tilde{\delta}_i' + \Sigma_1^i \zeta_j\}}{\tilde{g}\{\Sigma_1^{i-1}\delta_j + \delta_i'\}} \wedge \beta. \tag{29}$$

Here $\zeta_1, \zeta_2, \ldots$ is any sequence of independent random variables satisfying $|\zeta_j| \le 1$, $E\,\zeta_j = 0$, with $(\zeta_1,\ldots,\zeta_j)$ independent of $(X_1,\ldots,X_j,X_j')$ for all $j \ge 1$ and $\sigma_i^2 = \Sigma_1^i E\,\zeta_j^2 \to \infty$. Henceforth, the prime on δ_i' is omitted and E denotes expectation with respect to the measure induced by $(X_i,\ldots,X_{i-1},X_i',\zeta_1,\ldots,\zeta_i)$. Bounds for $n^{-1}\Sigma_1^n P_i E(|\varphi_i''(x) - \psi_i(x)|)$ will be evolved by bounding for fixed x and then completing the Cesaro expectation. The artificial randomization makes the variance of a sum divergent and permits the use of Survila's strengthening of the Berry-Esseen theorem (Survila, 1962) to develop the bound for fixed $x \ge 1$.

Theorem (Survila). Let $U_1, U_2, \ldots$ be a sequence of independent random variables with $PU_i = 0$ and $\xi_i = P|U_i|^3 < \infty$. Define $s_n^2 =$

$\text{Var}(\Sigma_1^n U_i)$ and $L_n = s_n^{-3}\Sigma_1^n \xi_i$. If $L_n \to 0$ as $n \to \infty$ then $|P[\Sigma_1^n U_i \le ts_n] - \Phi(t)| \le c\,(1+t^2)^{-1}L_n$ for all n and t where c is a constant.

Let $x \ge 1$ and $0 \le t \le \beta$ be fixed and the display of dependence upon x suppressed. Then we can write

$$E[\varphi_i'' - \psi_i \ge t] \le E[\Sigma_1^i(Y_j - EY_j) \le -\Sigma_1^i EY_j] \tag{30}$$

where $Y_j = \tilde{\delta}_j + \zeta_j - R_i(t)\delta_j$, $R_i(t) = \frac{g}{\tilde{g}}(\psi_i + t)$, $EY_j = \tilde{p}_j - R_i(t)p_j$, and $\Sigma_1^i EY_j = -\frac{\tilde{g}}{g}\Sigma_1^i p_j\,t$. With $L_i(t)$ and $r_i(t)$ defined as in (13) it follows from $|Y_j - EY_j| \le 2(1+R_i(t))$ that

$$L_i(t) \le 2\, r_i^{-1}(t)\,(1+R_i(t)). \tag{31}$$

Explicitly, $r_i^2(t) = \sigma_i^2 + \Sigma_1^i\{\tilde{p}_j + R_i^2(t)p_j - (\tilde{p}_j - R_i(t)p_j)^2\}$ so

$$\sigma_i^2 + q\,R_i^2(t)\,\Sigma_1^i p_j \le r_i^2(t) \le \sigma_i^2 + T^2\,\Sigma_1^i p_j \tag{32}$$

where $T^2 = \beta\frac{\tilde{g}}{g} + 4\beta^2\frac{\tilde{g}^2}{g^2}$. Since $\sigma_i^2 \to \infty$ and $R_i(t) \le 2\beta\frac{\tilde{g}}{g}$, $L_i(t) \to 0$ as $i \to \infty$ for fixed $x \ge 1$, $0 \le t \le \beta$. Survila's theorem and (30) imply

$$E[\varphi_i'' - \psi_i \ge t] \le \Phi\left(-\frac{\tilde{g}\,\Sigma_1^i p_j}{g\,r_i(t)}t\right) + c\,L_i(t)\left\{1 + \left(\frac{\tilde{g}\,\Sigma_1^i p_j}{g\,r_i(t)}t\right)^2\right\}^{-1}. \tag{33}$$

The inequality $r_i(t) \le \sigma_i + T(\Sigma_1^i p_j)^{\frac{1}{2}}$ together with the technique culminating in (17) imply

$$\int_0^\beta \Phi\left(-\frac{\tilde{g}\,\Sigma_1^i p_j}{g\,r_i(t)}t\right)dt \le \frac{g\{\sigma_i + T(\Sigma_1^i p_j)^{\frac{1}{2}}\}}{\tilde{g}\,\Sigma_1^i p_j}$$

and another application of $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + b^{\frac{1}{2}}$, $a, b > 0$ on T^2 shows

$$(34) \qquad \int_0^{\beta} \Phi\left(-\frac{\tilde{g}\,\Sigma_1^i\, p_j}{g\, r_i(t)}t\right) dt \le \frac{g\,\sigma_i}{\tilde{g}\,\Sigma_1^i\, p_j} + \frac{2\beta}{(\Sigma_1^i\, p_j)^{\frac{1}{2}}} + \left(\frac{\beta\, g}{\tilde{g}\,\Sigma_1^i\, p_j}\right)^{\frac{1}{2}}.$$

The inequalities $2\pi^{-1}\int_0^{\beta}(1+a^2t^2)^{-1}dt \le a^{-1}$, (32), $L_i(t) \le 2\{\sigma_i^{-1} + (q\,\Sigma_1^i\, p_j)^{-\frac{1}{2}}\}$ and an upper bound for T show that the 0 to β t-integral of last term of (33) is bounded by

$$(35) \qquad B\left\{\frac{\sigma_i\, g}{\tilde{g}}(\Sigma_1^i\, p_j)^{-\frac{3}{2}} + \left(\frac{g}{\tilde{g}} + \frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}} + 1\right)(\Sigma_1^i\, p_j)^{-1} + \left(\frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}} + 1\right)(\Sigma_1^i\, p_j)^{-\frac{1}{2}}\right\}$$

for some constant B. (In deriving (35) we take $\sigma_1 = 1$ so that $\sigma_i^{-1} \le 1$ for all i.)

The other tail can be treated analogously and the integral $\int_0^{\psi_i} E[\varphi_i'' - \psi_i \le -t]dt$ has a bound with the same structure as the bound (34) and (35) provide for the t-integral of the left hand side of (33). Therefore, the following inequality is proved:

$$(36) \qquad E(|\varphi_i'' - \psi_i|) \le B\left\{\frac{\sigma_i\, g}{\tilde{g}}(\Sigma_1^i\, p_j)^{-\frac{3}{2}} + \left(\frac{\sigma_i\, g}{\tilde{g}} + \frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}} + 1\right)(\Sigma_1^i\, p_j)^{-1} + \left(\frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}} + 1\right)(\Sigma_1^i\, p_j)^{-\frac{1}{2}}\right\}$$

for some constant B independent of $\underline{\theta}$, i and $x \ge 1$.

Inequality (36) is analogous to (22). Recall that Lemma 1 and assumptions ($A1^+$), (A2) and (A3) were needed to deduce the consequence of Proposition 3 from (22). At this point we prove a lemma which is useful in applying (36).

Let $\Omega^* = \{\theta | 0 \le \theta < \infty, \Sigma_x\, \theta^x\, g(x) < \infty\}$. If $\beta > 0$ is in the interior of Ω^* then each P_θ, $\theta \le \beta$, has an exponential rate on its tail probability. Write

$$(A2^+) \qquad \beta \in \text{interior } \Omega^*.$$

(It is clear that $(A2^+)$ implies (A2) and the exponential rate on the tail probability. For if $\beta < d$, $d \in \Omega^*$, then $p_\beta(m) \le P_\beta[x \ge m] = \Sigma_m^\infty \beta^x h(\beta) g(x) \le \frac{h(\beta)}{h(d)}(\frac{\beta}{d})^m P_d[x \ge m] < \frac{h(\beta)}{h(d)}(\frac{\beta}{d})^m$ for each m.)

Lemma 2. Let $1 \le B_i \le i$ be an increasing sequence. If (A1) and $(A2^+)$ obtain, then

$$(37) \qquad \Sigma_1^n P_i[\Sigma_1^i p_j(x) \le B_i] = 0(B_n \log n)$$

uniformly in $\underline{\theta}$.

Proof. For fixed x, $\Sigma_1^n p_i(x)[\Sigma_1^i p_j(x) \le B_i] \le B_n$ so $\Sigma_1^n P_i[\Sigma_1^i p_j(x) \le B_i] \le mB_n + \Sigma_1^n P_i[x \ge m]$. Since the family has monotone increasing likelihood ratio and $[x \ge m]$ is increasing in x, $P_i[x \ge m] \le P_\beta[x \ge m]$. But under $(A2^+)$ $P_\beta[x \ge m] \le c\, r^m$ where $c = \frac{h(\beta)}{h(d)}$, $r = \frac{\beta}{d}$ for some $\beta < d$, $d \in \Omega^*$. Therefore, for any $m \ge 0$

$$(38) \qquad \Sigma_1^n P_i[\Sigma_1^i p_j(x) \le B_i] \le m\, B_n + c\, n\, r^m.$$

Letting m be such that $n\, r^m = B_n$, it follows that $m = (\log \frac{B_n}{n})/\log r \le (\log n)/\log r^{-1}$; and, hence, the result follows from (38).

Before proceeding we digress to compare this lemma with Swain's Lemma 6 (1965, p. 31) specialized to $k = 1$. To prove (37) for the case $B_i = i^\epsilon$, $0 < \epsilon < 1$, assumptions (i), (iii) and (iv) were used, (iii) being

(iii) $\qquad \dfrac{g(x+y)}{g(x)g(y)} \le M' < \infty$ for all x and y.

<u>Remark</u>. (iii) implies Ω^* is open on the right; consequently, (iii) implies $(A2^+)$.

<u>Proof</u>. We have $p_\theta^{-1}(m)\, P_\theta[x \ge m] = h^{-1}(\theta)\, \Sigma_m^\infty\, \theta^{x-m}\, h(\theta)\, g(x-m)$ $\dfrac{g(x)}{g(m)g(x-m)} \le h^{-1}(\beta)\, M' = M$ for all $\theta \in \Omega^*$ and every non-negative integer m. Thus, $P_\theta[x \ge m] = \pi_0^{m-1}\left(1 - \dfrac{p_\theta(j)}{P_\theta[x \ge j]}\right)$ implies $P_\theta[x \ge m] \le r^m$ where $r = 1 - \frac{1}{M} < 1$. From the exponential rate on the tail probability it follows for any $\beta \in \Omega^*$ there exists a $d > \beta$ such that $d \in \Omega^*$. For with $\frac{d}{\beta} = s > 1$, $h(\beta)\, \Sigma_0^\infty\, d^x g(x) = P_\beta(s^x) = \Sigma_0^\infty\, s^x\, p_\beta(x) \le$ $\Sigma_0^\infty\, (sr)^x$ and the last series converges with $s < r^{-1}$.

Preparing (36) for application of Lemma 2 the trivial bound $|\varphi_i'' - \psi_i| \le \beta$ and partitioning by $[\Sigma_1^i\, p_j \le i^{\frac{1}{4}}]$ lead to

$$(39) \quad E(|\varphi_i'' - \psi_i|) \le \beta[\Sigma_1^i\, p_j \le i^{\frac{1}{4}}] + B\left\{\frac{\sigma_i\, i^{-\frac{1}{8}}\, g}{\tilde{g}} + \frac{g^{\frac{1}{2}}}{\tilde{g}^{\frac{1}{2}}} + 1\right\}(\Sigma_1^i\, p_j)^{-\frac{1}{2}}$$

for some constant B and all $x \ge 1$. On $[x = 0]$ a bound like (10) holds for φ_i''. (The artificial randomization did not change the expectation of the Y_j and increased the range of $Y_j - EY_j$, by a factor of no more than three. Then Hoeffding's Theorem 2 (1963) implies the result for φ_i'' as it did for φ_i'). Therefore, (39) leads to

$$(40) \quad n^{-1}\, \Sigma_1^n\, P_i E(|\varphi_i'' - \psi_i|) \le 0(n^{-\frac{1}{2}}) + \beta\, n^{-1}\, \Sigma_1^n\, P_i[\Sigma_1^i\, p_j \le i^{\frac{1}{4}}]$$
$$+ B\, n^{-1} \sum_{x=1}^{\infty}\left\{\frac{\sigma_i\, i^{-\frac{1}{8}}\, g(x)}{g(x+1)} + \frac{g^{\frac{1}{2}}(x)}{g^{\frac{1}{2}}(x+1)} + 1\right\} \Sigma_1^n\, p_i(x)(\Sigma_1^i\, p_j(x))^{-\frac{1}{2}}.$$

Write

$$\text{(A3}'\text{)} \qquad \sum_x \frac{g(x)}{g(x+1)}\, p_\beta^{\frac{1}{2}}(x) < \infty.$$

Proposition 4. If the family of distributions satisfies (A1), (A2$^+$) and (A3$'$) and φ_i'' is such that $i \geq \sigma_i^8 \uparrow \infty$, then

$$n^{-1} \Sigma_1^n \; P_i E(|\varphi_i'' - \psi_i|) = O(n^{-\frac{1}{2}}) \tag{41}$$

uniformly in $\underline{\theta}$.

Proof. The proof follows from application of Lemmas 1 and 2 to (40) and the fact (A2$^+$) implies (A2) which together with (A3$'$) implies (A3).

In the next section we introduce some realizable procedures and bound their corresponding modified regrets.

5. Bounds for Modified Regret.

Define two procedures $\varphi_i^*(X_i)$ and $\varphi_i^{**}(X_i)$ by

$$\varphi_i^*(x) = \frac{g\, \Sigma_1^{i-1}\, \tilde{\delta}_j}{\tilde{g}\, \Sigma_1^{i-1}\, \delta_j} \wedge \beta \; , \tag{42}$$

$$\varphi_i^{**}(x) = \frac{g\, \Sigma_1^{i-1}\, \tilde{\delta}_j}{\tilde{g}\, \{1 + \Sigma_1^{i-1}\, \delta_j\}} \wedge \beta \; , \tag{43}$$

and abbreviate $\Sigma_1^{i-1}\, \delta_j = S_{i-1}$. Since $\varphi_i' = \varphi_i^* = \varphi_i^{**} = \beta$ on $1 - I = [g\, \tilde{S}_{i-1} \geq \beta\, \tilde{g}\, (S_{i-1} + 1)]$ we can write $|\varphi_i' - \varphi_i^*| = |\varphi_i' - \varphi_i^*| I$, $|\varphi_i' - \varphi_i^{**}| = |\varphi_i' - \varphi_i^{**}| I$. It then follows by consideration of the values taken on by φ_i' on $[X_i' = x]$, $[X_i' = x + 1]$, and $[X_i' \neq x, X_i' \neq x + 1]$ that

$$|\varphi_i' - \varphi_i^*| \le \beta[S_{i-1} = 0] + \frac{\beta}{S_{i-1}}[X_i' = x][S_{i-1} > 0] + \frac{g}{\tilde{g}\, S_{i-1}}[X_i' = x + 1][S_{i-1} > 0]$$

and

$$|\varphi_i' - \varphi_i^{**}| \le \beta[S_{i-1} = 0] + \{\frac{g}{\tilde{g}\, S_{i-1}} + \frac{\beta}{S_{i-1}}\}[X_i' = x + 1][S_{i-1} > 0]$$
$$+ \frac{\beta}{S_{i-1}}[X_i' \ne x, X_i' \ne x + 1][S_{i-1} > 0].$$

From these two inequalities we have

$$(44) \qquad |\varphi_i' - \varphi_i^*| \vee |\varphi_i' - \varphi_i^{**}| \le \beta[S_{i-1} = 0] + \frac{\beta}{S_{i-1}}[S_{i-1} > 0]$$
$$+ \frac{g}{\tilde{g}\, S_{i-1}}[X_i' = x + 1][S_{i-1} > 0].$$

Lemma 3. For each fixed x and $i \ge 2$

$$(45) \qquad \underline{P}_{i-1}\left(\frac{[S_{i-1} > 0]}{S_{i-1}}\right) \le 4\, (\Sigma_1^i\, p_j)^{-1}.$$

Proof. For each x, $[S_{i-1} > 0]\, S_{i-1}^{-1} \le 4\, (S_{i-1} + \delta_i' + 1)^{-1}$. Since $(S_{i-1} + \delta_i' + 1)^{-1}$ is convex in $S_{i-1} + \delta_i'$, Hoeffding's Theorem 3 (Hoeffding, 1956) implies $E((S_{i-1} + \delta_i' + 1)^{-1}) \le \Sigma_0^i\, (j + 1)^{-1}\binom{i}{j}\, p^j(1 - p)^{i-j}$ where $i\, p = \Sigma_1^i\, p_j$. This last inequality implies $E((S_{i-1} + \delta_i' + 1)^{-1}) \le i/((i + 1)\, \Sigma_1^i\, p_j) \le (\Sigma_1^i\, p_j)^{-1}$ and completes the proof.

Proposition 5. If the family of distributions satisfies $(A1^+)$, then

$$(46) \qquad P_i E(|\varphi_i' - \varphi_i^*|) = o(1), \quad P_i E(|\varphi_i' - \varphi_i^{**}|) = o(1)$$

uniformly in $\underline{\theta}$; and, consequently,

$$(47)\quad n^{-1} \Sigma_1^n P_i E(|\varphi_i' - \varphi_i^*|) = o(1), \quad n^{-1} \Sigma_1^n P_i E(|\varphi_i' - \varphi_i^{**}|) = o(1)$$

uniformly in $\underline{\theta}$.

Proof. For fixed x the E expectation of the right hand side of (44) is bounded by $\beta \exp\{-\Sigma_1^{i-1} p_j\} + 4\beta(\Sigma_1^i p_j)^{-1} + 4\beta\, p_i\, (\Sigma_1^i p_j)^{-1} = g_i$ where Lemma 3 and $\frac{g}{g} \tilde{p}_i \le \beta p_i$ have been used. Weakening the bound to $\beta \exp\{-(i-1)\, m\} + 8\beta\, i^{-1} m^{-1} = f_i$ where as before $m(x) = \inf\{p_\theta(x) | \theta \in \Omega\} > 0$, we have $P_i E(|\varphi_i' - \varphi_i^*|) \vee P_i E(|\varphi_i' - \varphi_i^{**}|) \le \frac{h(0)}{h(\beta)} P_\beta(f_i \wedge \beta)$. The right hand side is independent of $\underline{\theta}$ and converges to zero by the dominated convergence theorem.

Theorem 1. If the family of distributions satisfies (A1$^+$), then

$$(48)\qquad |D_n(\underline{\theta}, \varphi^*)| = o(1), \quad |D_n(\underline{\theta}, \varphi^{**})| = o(1)$$

uniformly in $\underline{\theta}$.

Proof. The proof follows directly from (2), Propositions 1 and 5, and a triangle inequality.

Proposition 6. If the family of distributions satisfies (A1), then at each fixed $\underline{\theta} \in \Omega^\infty$

$$(49)\qquad P_i E(|\varphi_i' - \varphi_i^*|) = o(1), \quad P_i E(|\varphi_i' - \varphi_i^{**}|) = o(1);$$

and, consequently,

(50) $n^{-1}\,\Sigma_1^n\, P_i E(|\varphi_i' - \varphi_i^*|) = o(1),\quad n^{-1}\,\Sigma_1^n\, P_i E(|\varphi_i' - \varphi_i^{**}|) = o(1).$

Proof. Consider the bound g_i of the proof of Proposition 5. If $\underline{\theta}$ is such that $\overline{\lim}_i\, \theta_i > 0$ then $\Sigma_1^i\, p_j \to \infty$ and $g_i \to 0$ so the dominated convergence theorem implies the result. Write $P_i E(|\varphi_i' - \varphi_i^*|) \le E(|\varphi_i'(0) - \varphi_i^*(0)|) + \beta\, P_i[x > 0]$. The term $E(|\varphi_i'(0) - \varphi_i^*(0)|) \to 0$ at any $\underline{\theta}$ since $m(0) > 0$, and, therefore, $g_i(0) \to 0$. For $\theta_i \to 0$, $P_i[x > 0] \to 0$ so the result is proved for the procedure φ_i^*. The same proof works for φ_i^{**}.

Theorem 2. If the family of distributions satisfies (A1), then at each fixed $\underline{\theta} \in \Omega^\infty$

(51) $$|D_n(\underline{\theta},\varphi^*)| = o(1),\quad |D_n(\underline{\theta},\varphi^{**})| = o(1).$$

Proof. The proof follows from (2) and Propositions 2 and 6.

Example 1 illustrates that in order to deduce rates of convergence it will be necessary to examine the bound (44) in the case the family satisfies conditions in addition to (A1) or (A1$^+$).

One such condition is

(A2$'$) $$P_\beta(X) < \infty.$$

and notice that (A2$^+$) implies (A2$'$).

Lemma 4. (a) Under (A1) and (A2$^+$), $n^{-1}\,\Sigma_1^n\, \underline{P}_i[S_{i-1} = 0] = O(n^{-1}\log n)$ uniformly in $\underline{\theta}$.

(b) Under (A1) and (A2$'$), $n^{-1}\,\Sigma_1^n\, \underline{P}_i[S_{i-1} = 0] = O(n^{-\frac{1}{2}})$ uniformly in $\underline{\theta}$.

Proof. We write $\Sigma_1^n \underline{P}_i[S_{i-1} = 0] \le \sum_{x<m} \Sigma_1^n p_i(x) \underline{P}_{i-1}[S_{i-1} = 0] +$ $\Sigma_1^n P_i[x \ge m]$. The monotone likelihood ratio property and the fact that the $p_i(x) \underline{P}_{i-1}[S_{i-1} = 0] = \underline{P}_i[X_1 \ne x,\ldots,X_{i-1} \ne x, X_i = x]$ are probabilities of disjoint events together imply

$$\Sigma_1^n \underline{P}_i[S_{i-1} = 0] \le m + n\, P_\beta[x \ge m] \tag{52}$$

for any $m \ge 0$. Under (A2$^+$), $P_\beta[x \ge m] \le c\, r^m$ for some constants c, $0 < r < 1$. The choice $m = -\log n/\log r$ proves part (a). Under (A2$'$), $P_\beta[x \ge m] \le P_\beta(X)\, m^{-1}$ and the choice $m = n^{\frac{1}{2}}$ proves (b).

This lemma allows us to treat the expectation of the Cesaro mean of the leading term in the bound (44). The second and third terms are now investigated.

Lemma 5. Under (A1) and (A2), $n^{-1} \Sigma_1^n \underline{P}_i(\frac{[S_{i-1} > 0]}{S_{i-1}}) = O(n^{-\frac{1}{2}})$ uniformly in $\underline{\theta}$.

Proof. From $[S_{i-1} > 0]\, S_{i-1}^{-1} \le ([S_{i-1} > 0]\, S_{i-1}^{-1})^{\frac{1}{2}}$, Jensen's inequality and Lemma 3 it follows that $\underline{P}_{i-1}([S_{i-1} > 0]\, S_{i-1}^{-1}) \le$ $2\,(\Sigma_1^i p_j)^{-\frac{1}{2}}$. The proof is completed by application of Lemma 1.

Lemma 6. Under (A1), $n^{-1} \Sigma_1^n P_i E(\frac{g[X_i' = x+1][S_{i-1} > 0]}{\tilde{g}\, S_{i-1}}) =$ $O(n^{-1} \log n)$ uniformly in $\underline{\theta}$.

Proof. The left hand side can be written $n^{-1} \Sigma_1^n \Sigma_0^\infty p_i(x)$ $p_i(x+1)\, g(x)\, g^{-1}(x+1)\, \underline{P}_{i-1}([S_{i-1} > 0]\, S_{i-1}^{-1}) \le 4\beta\, n^{-1} \Sigma_0^\infty \Sigma_1^n p_i^2(x)$ $(\Sigma_1^i p_j(x))^{-1}$ where use has been made of Lemma 3. The rate $O(n^{-1} \log n)$ follows from Lemma 1.1.

Proposition 7. If the family of distributions satisfies

(A1), (A2) and (A2$'$), then

$$(53)\quad n^{-1}\Sigma_1^n p_i E(|\varphi_i' - \varphi_i^*|) = O(n^{-\frac{1}{2}}),\ n^{-1}\Sigma_1^n p_i E(|\varphi_i' - \varphi_i^{**}|) = O(n^{-\frac{1}{2}})$$

uniformly in $\underline{\theta}$.

Proof. The proof follows from (44) in view of Lemmas 4(b), 5 and 6.

Theorem 3. If the family of distributions satisfies (A1$^+$), (A2), (A2$'$) and (A3), then

$$(54)\qquad |D_n(\underline{\theta},\varphi^*)| = O(n^{-\frac{1}{2}}),\qquad |D_n(\underline{\theta},\varphi^{**})| = O(n^{-\frac{1}{2}})$$

uniformly in $\underline{\theta}$.

Proof. The proof follows from (2) and Propositions 3 and 7.

In order to prove rates for φ^* and φ^{**} in the case (A1) rather than (A1$^+$) is assumed, we investigate $n^{-1}\Sigma_1^n p_i E(|\varphi_i' - \varphi_i''|)$. A straight forward computation shows

$$(55)\quad |\varphi_i' - \varphi_i''| \le \beta[S_{i-1} + \delta_i' = 0] + \frac{g|T_i|}{\bar{g}\,(S_{i-1} + \delta_i')}[S_{i-1} + \delta_i' > 0]$$

where $T_i = \Sigma_1^i \zeta_j$. Since $|\varphi_i' - \varphi_i''| \le \beta$ we can partition by $[\Sigma_1^i p_j \le i^{\frac{1}{4}}]$ and weaken the bound of (55) to

$$\beta[\Sigma_1^i p_j \le i^{\frac{1}{4}}] + \beta[S_{i-1} = 0] + \frac{2\,g\,|T_i|}{\bar{g}\,(S_{i-1} + \delta_i' + 1)}[\Sigma_1^i p_j > i^{\frac{1}{4}}].$$

The first two terms can be disposed of by Lemmas 2 and 4. As indicated in the proof of Lemma 3, $E(S_{i-1} + \delta_i' + 1)^{-1} \le (\Sigma_1^i p_j)^{-1}$. Hence, the independence of T_i and $(X_1,\ldots X_i,X_i')$ together with fact $E|T_i| \le \sigma_i$ show that for each fixed $X_i = x$

$$E\left(\frac{|T_i|}{S_{i-1} + \delta_i' + 1}\,[\Sigma_1^i\, p_j > i^{\frac{1}{4}}]\right) \le (\Sigma_1^i\, p_j)^{-\frac{1}{2}}$$

if $\sigma_i^8 \le i$. Therefore, under (A3$'$) the Cesaro mean of the overall expectation of this term is $0(n^{-\frac{1}{2}})$ in view of Lemma 1. We have proved:

<u>Proposition 8</u>. If the family of distributions satisfies (A1), (A2$^+$), (A3$'$) and $\zeta_1, \zeta_2, \ldots$ is such that $\sigma_i^8 \le i$, then

(56) $$n^{-1}\,\Sigma_1^n\, p_i\, E(|\varphi_i' - \varphi_i''|) = 0(n^{-\frac{1}{2}})$$

uniformly in $\underline{\theta}$.

<u>Theorem 4</u>. If the family of distributions satisfies (A1), (A2$^+$) and (A3$'$), then

(57) $$|D_n(\underline{\theta},\varphi^*)| = 0(n^{-\frac{1}{2}}), \quad |D_n(\underline{\theta},\varphi^{**})| = 0(n^{-\frac{1}{2}})$$

uniformly in $\underline{\theta}$.

<u>Proof</u>. The proof follows from (2) and the Propositions 4, 7 and 8.

Theorem 4 can be compared with another result. In view of Lemma 2 and the remark that follows it, we can state a strengthened version of Swain's Theorem 3 (Swain, 1965) specialized to $k = 1$.

<u>Theorem</u>. If the family of distributions satisfies (A1), (A2$^+$) and

$$(A3^-) \qquad \sum_x \frac{g(x)}{g(x+1)} p_\beta(x) < \infty$$

then $\sup\{D_n(\underline{\theta},\varphi^{**}) | \underline{\theta} \in \Omega^\infty\} \le B_n = O(n^{-\frac{1}{4}} \log n)$.
The higher rate of convergence indicated in Theorem 4 is obtained under an assumption $(A3')$ stronger than $(A3^-)$.

The construction of the procedures φ^* and φ^{**} depends upon knowledge of β. If β were not known, then the natural estimates of ψ_i could be truncated back to a_i where $a_i \uparrow \infty$. It can be shown that these procedures result in bounds of order $O(a_n^2 n^{-\frac{1}{2}})$ replacing the $O(n^{-\frac{1}{2}})$ given in both Theorem 3 and Theorem 4.

Example 1 demonstrates if all that is assumed is (A1) or $(A1^+)$ then $o(1)$ is the best available uniform bound on $|D_n(\underline{\theta},\varphi)|$ when φ is either φ^*,φ^{**} or the procedure suggested by Samuel. In the next section we give an example which satisfies the hypothesis of Theorem 3 and shows that the bound $O(n^{-\frac{1}{2}})$ is fairly tight.

6. Examples.

Examples of exponential families which satisfy conditions (A1), $(A2^+)$, (A3) and $(A3')$ are provided by the Poisson family for $\beta < \infty$ and the negative binomial family $p_\theta(x) = \binom{a+x-1}{x}\theta^x (1-\theta)^a$, $\theta \le \beta < 1$, $a > 0$ fixed.

In §1.2 a necessary condition was stated for the existence of a compound procedure φ satisfying

$$\overline{\lim}_n \sup\{D_n(\underline{\theta},\varphi) | \underline{\theta} \in \Omega^\infty\} < \infty.$$

For squared error loss the necessary condition reduces to the existence of a non-randomized procedure φ_1 in the component problem satisfying

$$(58) \qquad \sup \{P_\theta((\theta - \varphi_1)^2) | \theta \in \Omega\} < \infty.$$

With the negative binomial family all $[0,1]$-valued procedures φ_1 have risk functions uniformly bounded by unity. However, (58) illustrates the necessity of assumption (A1) or (A1$^+$) in the Theorems 1, 3 and 4 for the Poisson family. For with this family and unbounded parameter set, (58) fails for any procedure φ_1 (Lehmann, 1950, p. 4-13). We sketch the proof. Letting $P_\theta(\varphi_1) = \theta + b(\theta)$ the Cramér-Rao inequality proves $P_\theta((\theta - \varphi_1)^2) \geq \{(1 + b'(\theta))^2 / P_\theta((\frac{\partial}{\partial\theta} \log p_\theta)^2)\} + b^2(\theta) = \theta(1 + b'(\theta))^2 + b^2(\theta)$ from which it follows that $\overline{\lim}_{\theta\to\infty} P_\theta((\theta - \varphi_1)^2) = +\infty$.

As a matter of possible independent interest we give an example which illustrates that this property exhibited by the Poisson family need not hold for other exponential distributions on the non-negative integers with unbounded natural parameter spaces. Consider $p_\theta(x) = \theta^x(x!)^{-\frac{1}{2}}h(\theta)$, $0 < \theta < \infty$, and $\varphi(x) = x^{\frac{1}{2}}$. Then $P_\theta(\varphi) = \theta$ and $P_\theta((\theta - \varphi)^2) = h(\theta)\ \{\Sigma_1^\infty\ x\ \theta^x(x!)^{-\frac{1}{2}} - \theta^2\ h^{-1}(\theta)\} = h(\theta)\{\theta + \Sigma_2^\infty[x\ (x!)^{-\frac{1}{2}} - ((x-2)!)^{-\frac{1}{2}}]\theta^x\} = h(\theta)\{\theta + \Sigma_2^\infty\ (x^{\frac{1}{2}}(x-1)^{-\frac{1}{2}} - 1)((x-2)!)^{-\frac{1}{2}}\ \theta^x\}$. It is easy to verify that $x^{\frac{1}{2}}(x-1)^{-\frac{1}{2}} - 1 \leq (x(x-1))^{-\frac{1}{2}}$ for $x \geq 1$ so $P_\theta((\theta - \varphi)^2) \leq h(\theta)\{\theta + \Sigma_2^\infty\ (x!)^{-\frac{1}{2}}\ \theta^x\} = h(\theta)(h^{-1}(\theta) - 1) < 1$ for all θ.

The family of distributions in Example 1 satisfies the assumptions (A1$^+$), (A2) and (A2$'$) and illustrates the necessity of adding another condition in order that the procedures φ^* or φ^{**} attain any rate $O(a_n)$ uniformly in $\underline{\theta}$, a_n a null sequence.

We now give an example of a family satisfying the hypothesis of Theorem 3 which shows that the bound is fairly tight.

Example 2. Consider the exponential family with $g(0) = g(1) = 1$; $g(x) = x^{-4}$, $x = 2, 4, 6, \ldots$; $g(x) = x^{-3}\log^{-a-1}x$, $x = 3, 5, 7, \ldots$ and $\Omega = [\alpha, 1]$, $0 < \alpha < 1$, where $a > 0$ is fixed but otherwise arbitrary. It is not difficult to verify that (A2), (A2$'$) and (A3) are satisfied. Then at $\underline{\theta} = \underline{1}$ with $n \geq 2$ and any procedure φ such that $[\varphi_i(x) \leq \alpha] \geq [\Sigma_1^{i-1} \tilde{\delta}_j = 0]$,

$$
\begin{aligned}
D_n(\underline{1},\varphi) &= n^{-1} \Sigma_1^n \underline{P}_i((1 - \varphi_i)^2) \\
&\geq n^{-1} \Sigma_1^n \underline{P}_i((1 - \varphi_i)^2 [\Sigma_1^{i-1} \tilde{\delta}_j = 0]) \\
&\geq (1 - \alpha)^2 \ \underline{P}_n[\Sigma_1^{n-1} \tilde{\delta}_j = 0] \\
&\geq (1 - \alpha)^2 \ \Sigma_1^{\infty} p_1(2x + 1)(1 - p_1(2x + 2))^{n-1} \\
&\geq (1 - \alpha)^2 \ \left(1 - \frac{h(1)}{n-1}\right)^{n-1} \Sigma_{1 \vee A_n}^{\infty} p_1(2x + 1)
\end{aligned}
$$

where $A_n = \min[x \mid (2x + 4)^4 \geq n-1\} \leq \frac{1}{2}(n-1)^{\frac{1}{4}}$. Comparing the series $\Sigma_A^{\infty} p_1(2x + 1)$ with the integral $\int_{2A+1}^{\infty} x^{-3} \log^{-a-1}x \, dx$ shows via integration by parts that for $A \geq 1$, $\Sigma_A^{\infty} p_1(2x + 1) \geq \frac{1}{2} h(1) (a + 3)^{-1}(2A + 1)^{-2} \log^{-a-1}(2A + 1)$. Therefore,

$$
D_n(\underline{1},\varphi) \geq K\, n^{-\frac{1}{2}} \log^{-a-1} n
$$

for some constant $K > 0$ and all large n.

CHAPTER III

SEQUENTIAL COMPOUND ESTIMATION FOR SQUARED ERROR LOSS AND A FAMILY OF NORMAL DISTRIBUTIONS

1. Introduction.

Let $\mathcal{P}$ be a family of probability measures on the measurable space $(\mathcal{X},\mathcal{B})$ where $\mathcal{X}$ is the reals and $\mathcal{B}$ the Borel σ-field. With μ denoting Lebesgue measure suppose that for each $P \in \mathcal{P}$, $P << \mu$ and $p = \frac{dP}{d\mu}$ is a differentiable determination of the Radon-Nikodym derivative. (Consequently, p is the ordinary derivative of the cumulative distribution function.) Let $\underline{P} = (P_1,P_2,\ldots)$ where $P_i \in \mathcal{P}$ and let $X_i \sim P_i$ be a sequence of independent random variables. For each $x \in \mathcal{X}$ we are interested in estimating

$$(1) \qquad u_i(x) = \frac{d}{dx}(\log \Sigma_1^i\, p_j(x)).$$

For example, when $\mathcal{P} = \{P_\theta | \theta \in \Omega\}$, $\Omega \subset \mathcal{X}$, and $p_\theta(x) = k(\theta)\,\ell(x)\, e^{\theta x}$, a Bayes response versus G_i the empirical distribution of $\underline{\theta}_i$ takes the form

$$(2) \qquad \psi_i(x) = u_i(x) - \frac{\ell'(x)}{\ell(x)}.$$

Therefore, if $u_i(x)$ can be estimated then $\psi_i(x)$ can be estimated through (2).

2. Estimation of $u_i(x)$.

Define the distribution function

$$(3) \qquad F_i(x) = i^{-1} \int_{-\infty}^{x} \Sigma_1^i\, p_j(v)\, dv$$

where we have abbreviated $d\mu(v)$ to dv. Also for $1 \geq h_i \downarrow 0$

and any distribution function F define

$$(4) \qquad t_i(F)(x) = h_i^{-1} \log \frac{F]_x^{x+h_i}}{F]_{x-h_i}^{x}} .$$

The mean value theorem for integrals implies that under suitable conditions $t_i(F_i)(x) - u_i(x) \to 0$ as $i \to \infty$. This will be discussed later after investigation of the problem of estimating $t_i(F_i)(x)$ from the sample $(X_1,\ldots,X_i)$. We make the following assumption: there exist functions M_i such that

$$(A1) \qquad \sup \{|t_i(F_i)(x)| \mid \underline{P} \in P^{\infty}\} \le M_i(x) < \infty.$$

Consider the estimator

$$(5) \qquad t_i^* = (t_i(F_i^*) \wedge M_i) \vee (-M_i)$$

where we have no longer displayed the dependence on x and F_i^* is the empirical distribution function of $(X_1,\ldots,X_i)$. In (5) any undefined ratios are to be taken as zero. We use the method of Chapter II in writing

$$(6) \qquad \underline{P}_i(|t_i^* - t_i(F_i)|) = \int_0^{2M_i} \underline{P}_i[|t_i^* - t_i(F_i)| \ge v] \, dv$$

and bounding the integrand by use of the Berry-Esseen theorem.

Let $0 \le v \le 2M_i$ and write

$$[t_i^* - t_i(F_i) \ge v] \le [\Sigma_1^i Y_j \ge 0]$$

where $Y_j = \tilde{\delta}_j - \delta_j \exp \{h_i(t_i(F_i) + v)\}$ with $\tilde{\delta}_j = 1$ or 0 as $X_j \in (x, x + h_i]$ or not and $\delta_j = 1$ or 0 as $X_j \in (x - h_i, x]$

or not. The Berry-Esseen theorem implies

$$(7) \qquad \underline{P}_i[t_i^* - t_i(F_i) \geq v] \leq \Phi(r_i^{-1}(v)\ \Sigma_1^i\ P_j(Y_j)) + c\ L_i(v)$$

for $r_i^2(v) = \mathrm{Var}(\Sigma_1^i\ Y_j)$ and $L_i(v) = r_i^{-3}(v)\ \Sigma_1^i\ P_j(|Y_j - P_j(Y_j)|^3)$.
We have

$$\Sigma_1^i\ P_j(Y_j) = i\ F_i]_x^{x+h_i}(1 - e^{h_i v}) \leq -i\ h_i\ F_i]_x^{x+h_i}\ v$$

and by consideration of the variances of the three point distributions of the Y_j,

$$q^2\ i\ F_i]_x^{x+h_i} \leq r_i^2(v) \leq i\ e^{6M_i}\ F_i]_{x-h_i}^{x+h_i}$$

with

$$1 - q^2 = \sup\ \{P[x, x+h_1]\ |\ x \in X,\ P \in \mathcal{P}\}.$$

Since $|Y_j - P_j(Y_j)| \leq 1 + e^{3M_i}$ implies $L_i(v) \leq (1 + e^{3M_i})\ r_i^{-1}(v)$ and $\int_0^\infty \Phi\ (-av)\ dv \leq (2\pi)^{-\frac{1}{2}}\ a^{-1}$, we see that (7) implies

$$(8) \qquad \int_0^{2M_i} \underline{P}_i[t_i^* - t_i(F_i) \geq v]\ dv \leq \frac{e^{3M_i}\ (F_i]_{x-h_i}^{x+h_i})^{\frac{1}{2}}}{(2\pi\ i)^{\frac{1}{2}}\ h_i\ F_i]_x^{x+h_i}} + \frac{2c\ M_i\ (1 + e^{3M_i})}{q\ (i\ F_i]_x^{x+h_i})^{\frac{1}{2}}}\ .$$

To treat the other tail let $0 \leq v \leq 2M_i$ and write

$$[t_i^* - t_i(F_i) \leq -v] \leq [\Sigma_1^i\ Z_j \leq 0]$$

where $Z_j = \tilde{\delta}_j - \delta_j \exp\{h_i(t_i(F_i) - v)\}$. The Berry-Esseen theorem yields

$$(9) \qquad \underline{P}_i[t_i^* - t_i(F_i) \le -v] \le \Phi(-s_i^{-1}(v)\, \Sigma_1^i\, P_j(Z_j)) + c\, L_i'(v)$$

with $s_i^2(v) = \mathrm{Var}\,(\Sigma_1^i Z_j)$ and $L_i'(v) = s_i^{-3}(v)\, \Sigma_1^i\, P_j(|Z_j - P_j(Z_j)|^3)$. We note

$$\Sigma_1^i\, P_j(Z_j) = i\, F_i]_x^{x + h_i}(1 - e^{-h_i v}) \ge \tfrac{1}{2} i\, h_i\, F_i]_x^{x + h_i}\, v,\ 0 \le v \le h_i^{-1}$$

and that the variance satisfies

$$q^2\, i\, F_i]_x^{x + h_i} \le s_i^2(v) \le i\, e^{2M_i}\, F_i]_{x - h_i}^{x + h_i}\ .$$

Since $L_i'(v) \le (1 + e^{M_i})\, s_i^{-1}(v)$, it follows from (9) that

$$(10) \qquad \int_0^{(2M_i) \wedge h_i^{-1}} \underline{P}_i[t_i^* - t_i(F_i) \le -v]\, dv \le \frac{2\, e^{M_i}\, (F_i]_{x - h_i}^{x + h_i})^{\frac{1}{2}}}{(2\, \pi\, i)^{\frac{1}{2}}\, h_i\, F_i]_x^{x + h_i}} + \frac{c((2M_i) \wedge h_i^{-1})(1 + e^{M_i})}{q\, (i\, F_i]_x^{x + h_i})^{\frac{1}{2}}}\ .$$

For any non-increasing function g and $0 < a \le b$, $\int_0^b g(v)\, dv \le a^{-1}\, b \int_0^a g(v)\, dv$. Applying this observation with $g(v) = \underline{P}_i[t_i^* - t_i(F_i) \le -v]$, $a = (2M_i) \wedge h_i^{-1} \ge 1$, $b = 2M_i$ and the bound (10) we obtain

$$(11) \qquad \int_0^{2M_i} \underline{P}_i[t_i^* - t_i(F_i) \le -v]\, dv \le \frac{4M_i\, e^{M_i}\, (F_i]_{x - h_i}^{x + h_i})^{\frac{1}{2}}}{(2\, \pi\, i)^{\frac{1}{2}} h_i\, F_i]_x^{x + h_i}} + \frac{2c\, M_i\, (1 + e^{M_i})}{q\, (i\, F_i]_x^{x + h_i})^{\frac{1}{2}}}$$

which when combined with (8) proves via (6) that

$$(12) \qquad \underline{P}_i(|t_i^* - t_i(F_i)|) \le \frac{(1 + 4M_i)\, e^{3M_i}\, (F_i]_{x - h_i}^{x + h_i})^{\frac{1}{2}}}{(2\pi i)^{\frac{1}{2}}\, h_i\, F_i]_x^{x + h_i}} + \frac{4c\, M_i\, (1 + e^{2M_i})}{q\, (i\, F_i]_x^{x + h_i})^{\frac{1}{2}}} .$$

Now we introduce conditions that will ensure that $t_i(F_i) - u_i \to 0$ as $i \to \infty$. The conditions seem arbitrary but they are convenient for reducing the notation in the bounds and are easily verified for a family of normal $N(\theta,1)$ distributions. We introduce the notation

$$f_{(\delta)}(x) = \sup \{|f(y)| \mid |y - x| \le \delta\}.$$

A basic assumption is that each p is twice differentiable at each x. We also assume that there exist functions A and B such that for all P

$$(A2) \qquad p'_{(1)}(x) \le A(x) < \infty$$

and

$$(A3) \qquad p''_{(1)}(x) \le B(x) < \infty.$$

As immediate consequences we have

$$(13) \qquad |p(x + \Delta) - p(x)| \le A(x)\, |\Delta| \qquad \text{for all } p,\ |\Delta| \le 1$$

and

(14) $\quad |p'(x + \Delta) - p'(x)| \le B(x)\, |\Delta| \quad$ for all p, $|\Delta| \le 1$.

Also, $\int_{x-\Delta}^{x+\Delta} p(v)\, dv = \int_{x-\Delta}^{x+\Delta} \{p(x) + (v - x)\, p'(x) + \tfrac{1}{2}(v - x)^2\, p''(z)\}\, dv$

for some $z \in \{v,x\}$ proves by way of (A3) that

(15) $\quad \left|(2\,\Delta)^{-1} \int_{x-\Delta}^{x+\Delta} p(v)\, dv - p(x)\right| \le \frac{B(x)\, \Delta^2}{6} \quad$ for all p, $0 < \Delta \le 1$.

We write

(16) $$|t_i(F_i)(x) - u_i(x)| \le I(x) + J(x) + K(x)$$

where

$$I(x) = \left|\log(h_i^{-1}\, F_i]_x^{x + h_i}) - \log F_i'(x + \tfrac{1}{2}h_i)\right|\, h_i^{-1}$$

$$J(x) = \left|\log(h_i^{-1}\, F_i]_{x - h_i}^{x}) - \log F_i'(x - \tfrac{1}{2}h_i)\right|\, h_i^{-1}$$

and

$$K(x) = \left|h_i^{-1}(\log F_i')]_{x - \frac{1}{2}h_i}^{x + \frac{1}{2}h_i} - \frac{d}{dx} \log F_i'(x)\right| .$$

The inequality $|\log a - \log b| \le (a \wedge b)^{-1}|a - b|$ and the technique used to prove (15) result in

(17) $$I \vee J \le \frac{B\, C\, h_i}{24\, F_i'}$$

where the display of dependence on x is suppressed and the function C is defined by

$$\inf\{p(x + \Delta) \mid |\Delta| \le 1\} = \frac{p(x)}{C(x)} .$$

In order to put K in a more tractable form let $\epsilon_i \in (x, x + \frac{1}{2}h_i)$ and $\delta_i \in (x - \frac{1}{2}h_i, x)$ be such that $\log F_i'(x + \frac{1}{2}h_i) = \frac{1}{2}h_i (F_i''(\epsilon_i)/F_i'(\epsilon_i)) + \log F_i'(x)$ and $-\log F_i'(x - \frac{1}{2}h_i) = \frac{1}{2}h_i(F_i''(\delta_i)/F_i'(\delta_i)) - \log F_i'(x)$. It follows that

$$2K(x) \le \frac{|F_i''(x)|}{F_i'(x)} \left\{\frac{|F_i'(\epsilon_i) - F_i'(x)|}{F_i'(\epsilon_i)} + \frac{|F_i'(\delta_i) - F_i'(x)|}{F_i'(\delta_i)}\right\}$$

$$+ \frac{|F_i''(\epsilon_i) - F_i''(x)|}{F_i'(\epsilon_i)} + \frac{|F_i''(\delta_i) - F_i''(x)|}{F_i'(\delta_i)} .$$

Assuming there exists a function D such that for all $\underline{P}_i \in \mathcal{P}^i$, $i \ge 1$

(A4) $$\frac{|F_i''(x)|}{F_i'(x)} \le D(x) < \infty$$

we have

(18) $$K \le \frac{(AD + B)\, C\, h_i}{2\, F_i'}$$

Combining (16) - (18) yields

(19) $$|t_i(F_i) - u_i| \le \frac{(AD + B)\, C\, h_i}{F_i'} .$$

Inequality (15) implies $F_i\Big]_{x - h_i}^{x + h_i} \le 2h_i\, F_i'(x) + \dfrac{B(x)\, h_i^3}{3}$ and the bound (12) can be weakened to

(20) $$\underline{P}_i(|t_i^* - t_i(F_i)|) \le k'\, e^{3M_i} \left\{\frac{C}{h_i^{3/2}(F_i')^{\frac{1}{2}}} + \frac{B^{\frac{1}{2}}\, C}{h_i^{\frac{1}{2}}\, F_i'} + \left(\frac{C}{h_i\, F_i'}\right)^{\frac{1}{2}}\right\} i^{-\frac{1}{2}}$$

for some constant k' not depending on $\underline{P}$, i and x. Combining

(19) and (20) with the choice $h_i = i^{-1/5}$ shows that for each fixed x,

$$(21) \quad \underline{P}_i(|t_i^* - u_i|) \leq k \left\{ \frac{(AD + B + B^{\frac{1}{2}} e^{3M_i}) C}{F_i'} + \frac{(C + C^{\frac{1}{2}}) e^{3M_i}}{(F_i')^{\frac{1}{2}}} \right\} i^{-1/5}$$

for some constant k.

The following observation is of use in finding the functions M_i. Repeated application of the law of the mean proves that $t_i(F_i)(x) = (F_i''(\beta_i)/F_i'(\beta_i))(\epsilon_i - \delta_i) h_i^{-1}$ for some $\beta_i \in (\delta_i, \epsilon_i)$, $\delta_i \in (x - h_i, x)$ and ϵ_i $(x, x + h_i)$; hence, $|t_i(F_i)(x)| \leq 2\, D(\beta_i)$ and we may take $M_i = 2\, D_{(1)}$. In the case of a family of probability measures $\mathcal{P} = \{P_\theta | \theta \in \Omega\}$, $\Omega = [-\alpha, \alpha]$, where $p_\theta(x) = p(x - \theta)$ we may take $A = p'_{(\alpha + 1)}$ and $B = p''_{(\alpha + 1)}$.

3. A Decision Problem.

Consider the family of normal distributions $\mathcal{P} = \{P_\theta | \theta \in \Omega\}$, $\Omega = [-\alpha, \alpha]$, $\alpha < \infty$, with P_θ denoting the $N(\theta, 1)$ law. Let $X_j \sim P_{\theta_j}$ be a sequence of independent random variables and p denote the usual $N(0,1)$ density. The Bayes response (2) takes the form

$$\psi_i(x) = u_i(x) + x.$$

Theorem 1. The sequential compound procedure

$$(22) \quad \varphi_i(X_i) = ((t_{i-1}^*(X_i) + X_i) \wedge \alpha) \vee (-\alpha)$$

is such that

(23) $$\underline{P}_i(|\varphi_i - \psi_i|) = O(i^{-1/5})$$

and

(24) $$|D_n(\underline{\theta},\varphi)| = O(n^{-1/5})$$

uniformly in $\underline{\theta}$.

Proof. For each fixed $X_{i+1} = x$ the bound (21) applies to $\underline{P}_i(|t_i^*(x) + x - \psi_i(x)|)$ with $A(x) = p'_{(\alpha+1)}(x)$, $B(x) = p''_{(\alpha+1)}(x)$, $C(x) \le \exp\{|x| + \alpha + \frac{1}{2}\}$, $D(x) = |x| + \alpha$ and $M_i(x) = 2(|x| + \alpha + 1)$. If we weaken the bound with $F'_i(x) \ge p_\alpha(x)\,[x \le 0] + p_{-\alpha}(x)\,[x > 0]$ and use $p_{i+1}(x) \le p_{-\alpha}(x)\,[x < -\alpha] + [|x| \le \alpha] + p_\alpha(x)\,[x > \alpha]$, it follows that $\underline{P}_{i+1}(|\varphi_{i+1} - \psi_i|) = O(i^{-1/5})$ uniformly in $\underline{\theta}$. Since Proposition 1.1 implies $P_{i+1}(|\psi_{i+1}(x) - \psi_i(x)|) = O(i^{-1})$ uniformly in $\underline{\theta}$, (23) is proved. The bound (24) is now an immediate consequence of Corollary 1.1.

Swain (1965) suggests a different sequential compound procedure and proves that the modified regret is bounded from above by a bound o(1) uniform in parameter sequences.

CHAPTER IV

ON THE BAYES RESPONSE TO THE EMPIRICAL DISTRIBUTION OF OPPONENTS PAST IN SEQUENTIAL DECISION PROBLEMS

1. Introduction.

In the preceding chapters sequential compound procedures were suggested naturally by the structure of a Bayes response ψ_i versus G_i, the empirical distribution of player I's (nature's) past through present moves. The problem of bounding the modified regret associated with certain natural procedures φ was reduced to bounding a Cesaro mean difference $n^{-1} \Sigma_1^n \underline{P}_i (|\varphi_i - \psi_i|)$. Inequality (8.8) of Hannan (1957) shows that the simple procedure $\psi_i(X_i)$ achieves a non-positive modified regret and illustrates why one would try procedures that approximate $\psi_i(X_i)$ in order to obtain small modified regret. In fact, with the sequence (set) compound problem investigators have used procedures which at stage i are Bayes versus estimates of G_i or $G_{i-1}(G_n)$ where estimable, as in the finite problem, or procedures which approximate ψ_i or $\psi_{i-1}(\psi_n)$ in case the distribution functions are not estimable directly.

The question arises as to what can be achieved in the sequential problem when an estimation problem is removed by assuming that G_{i-1} is known at stage i. Usable sequence strategies were exhibited by Hannan (1956) and (1957) for various $M \times N$ games (M finite) which achieve uniform rates of $0 (n^{-\frac{1}{2}})$ in absolute modified regret. The strategy is to play Bayes versus $\bar{h}_{i-1} + Z$ where $\bar{h}_{i-1}$ is an unbiased estimate of G_{i-1} (possibly G_{i-1} itself) and Z is a suitable random vector; Theorem 1 of Van Ryzin (1965) demonstrates that in the finite problem direct play against $\bar{h}_{i-1}$ at stage i results in a uniform bound $O(n^{-\frac{1}{2}})$ if there is a certain non-degeneracy in the estimate.

In some finite problems a bound of the same order obtains whether the procedure used is either Bayes versus an unperturbed G_{i-1} or an estimate $\bar{h}_{i-1}$. For consider a 2×2 statistical decision problem where player I's set of pure strategies is $\Omega = \{0,1\}$ with $P_\theta = N(\theta,1)$, and the loss matrix is $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$, $a, b > 0$. A Bayes rule versus $(1-p, p)$ is

$$t_p(x) = [\frac{b\, p_0(x)}{bp_0(x) + ap_1(x)} < p] = [x > \tfrac{1}{2} + \log (\frac{b(1-p)}{a\, p})]$$

where $t_p(x) = 1$ means decide $\theta = 1$. Let $\sigma(p)$ denote the Bayes risk vector. We claim that the Bayes response satisfies a Lipschitz condition of order $\alpha = 1$, (8.13) of Hannan (1957). Condition (8.13) reduces to showing $\sigma_0(p) - \sigma_0(\frac{p}{1+t}) = O(t)$ and $\sigma_1(p) - \sigma_1(\frac{p+t}{1+t}) = O(t)$ as $t \to 0$ uniformly in p where $\sigma_i(p)$ is the risk function $\sigma(p)$ evaluated at $\theta = i$. These orders can be verified and then via Hannan's Theorem 5, direct play versus G_{i-1} results in a non-negative modified regret of order $O(n^{-1}\log n)$ uniform in player I move sequences. Van Ryzin's Theorem 5 (1965) implies that a bound of the same order obtains if the procedure plays Bayes versus any unbiased estimate $\bar{h}_{i-1}$ that has a bounded kernel.

Throughout the remainder of this chapter we assume that

G_{i-1} is available to player II at each stage i. In the next section we point out some results which apply to statistical decision problems having a certain structure. The problems of Chapters II and III possess this structure. In subsequent sections we exhibit sequence strategies for non-finite games and prove rates of convergence for the modified regret.

2. Games Involving Squared Error Loss.

For each $\theta \in \Omega$ let there correspond a probability measure $P_\theta \ll \mu$ on the measurable space $(X,\mathcal{B})$. Let the loss function be squared error and consider the simple procedure $\bar{\varphi}$ where $\bar{\varphi}_i = \psi_{i-1}(X_i)$, ψ_i being the Bayes procedure (1.2). Inequalities (1.9) and (1.10) imply

$$(1)\quad 0 \le D_n(\underline{\theta},\bar{\varphi}) \le n^{-1}\Sigma_1^n P_i\,((\psi_{i-1} - \psi_i)(\psi_{i-1} + \psi_i - 2\theta_i)).$$

In view of Theorem 1.1 and (1) the following proposition is immediate.

Proposition 1. If $\Omega \subset [-A, A]$, $A < \infty$ and $M(x) = \sup\{p_\theta(x) | \theta \in \Omega\}$ is integrable $(X,\mathcal{B},\mu)$, then

$$(2)\qquad 0 \le D_n(\underline{\theta},\bar{\varphi}) = O(n^{-1}\log n)$$

uniformly in $\underline{\theta}$.

Hannan (1957) uses symbolism $\sigma(x)$ to represent a risk function Bayes versus x. Then $\epsilon\,\sigma(x)$ denotes that function evaluated at ϵ, a pure strategy for player I in the component game. Specialization of Hannan's Theorem 5 to $\alpha = 1$ shows that

(3) $$\varepsilon[\sigma(x) - \sigma(x+t\varepsilon)] \le L\left(\frac{t}{1+t}\right)$$

for a constant L, all ε, all $t > 0$ and all probability measures x with finite support, is sufficient for (2) to obtain. We give an example to show that (3) is not a necessary condition for (2) to obtain.

Example 1. Consider the discrete exponential family of Chapter II with $\Omega = [a, b]$. For squared error loss, and in Hannan's notation, x degenerate on a and ε a probability measure degenerate on b,

$$\varepsilon[\sigma(x) - \sigma(x+t\varepsilon)] = P_b((a-b)^2) - P_b\left(\left(\frac{ap_a+bt\ p_b}{p_a+tp_b} - b\right)^2\right)$$

$$= (a-b)^2 \,\ell(t)\, t$$

where $\ell(t) = P_b((2p_a\ p_b + t\ p_b^2)/(p_a + tp_b)^2)$. By Fatou's lemma, $\underline{\lim}_{\,t \to 0}\,\ell(t) \ge 2\ P_b(p_b/p_a)$. For the geometric family and $0 < a \le b^2 < 1$, $P_b(p_b/p_a) = +\infty$; and, therefore, (3) fails. Since the hypothesis of Proposition 1 is satisfied, (2) obtains.

Another application of Proposition 1 is now given. Consider the problem where the component game has the following structure. Player I picks an $\varepsilon \in [0,1]$ and II picks a $\delta \in [0,1]$ with loss $(\varepsilon - \delta)^2$. This is Hannan's Example 2 (1957) where it is observed that the Lipschitz condition (3) is satisfied for $\sigma(\cdot)$ with $\varepsilon\,\sigma(x) = (\varepsilon - \delta(x))^2$, $\delta(x)$ being the mean of the distribution x. Therefore, $0 \le n^{-1}\Sigma_1^n \varepsilon_i \sigma(G_{i-1}) - R(G_n) = O(n^{-1}\log n)$ uniformly in $\underline{\varepsilon}$. This result can be deduced from Proposition 1 in the following way. Associate with each

$\varepsilon \in [0,1]$ the same probability measure P degenerate on 0. If $\mu = P$, then $p(\varepsilon) = \frac{dP}{d\mu}(\varepsilon) = [\varepsilon = 0]$ and $\psi_{i-1}(\varepsilon) = (i-1)^{-1}\Sigma_1^{i-1}\varepsilon_j[\varepsilon = 0] = \delta(G_{i-1})$ a.s. P. Since the hypothesis of Proposition 1 is satisfied, (2) obtains with $D_n(\underline{\theta},\underline{\varphi}) = n^{-1}\Sigma_1^n \varepsilon_i\sigma(G_{i-1}) - R(G_n)$.

We now proceed to the main results of this chapter. An effort will be made to keep the presentation self-contained although the basis for the results is found in Hannan (1957).

3. Games With Countable M.

Let player I's set of pure strategies be $M = \{1, 2,\ldots\}$, possibly finite. We identify each δ of N, the set of pure strategies for II, with the risk vector $\sigma = (\sigma^1,\sigma^2,\ldots)$ where $L(j,\delta) = \sigma^j \geq 0$ is the loss to II when I's choice is j and II's choice is δ. We identify each $j \in M$ with the vector of 0's and 1's with 0's in all but the jth position and let ε be generic for such a vector. Then $\varepsilon\sigma$ denotes the inner product of ε and σ for all $\varepsilon \in M$, $\sigma \in N$. For $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2,\ldots) \in M^\infty$ and $\underline{\sigma} = (\sigma_1, \sigma_2,\ldots) \in N^\infty$ the following identity ((6.5) of Hannan (1957)) is basic:

(4) $$\Sigma_1^n \varepsilon_i\sigma_i = E_n\sigma_{n+1} + \Sigma_1^n E_{i-1}(\sigma_i - \sigma_{i+1}) + \Sigma_1^n \varepsilon_i(\sigma_i - \sigma_{i+1})$$

where $E_i = \varepsilon_1+\ldots+\varepsilon_i = i\, G_i$, $E_0 = 0$. We assume

(A1) N is sequentially compact under pointwise convergence

and

(A2) $$\sup \{\|\sigma\|_1 \mid \sigma \in N\} = B < \infty$$

where $\| \ \|_1$ denotes the ℓ_1 sequence norm. It follows from (A1) that for all $w \in m^+$, the set of bounded sequences with non-negative components, $\inf \{w\, \sigma^* | \sigma^* \in N\}$ is attained. For let $w\, \sigma_i \to \inf \{w\, \sigma^* | \sigma^* \in N\}$ and for each j, $\sigma^j_{i_k} \to \sigma^j$ as $k \to \infty$ where $\sigma = (\sigma^1, \sigma^2, \ldots) \in N$. By Fatou's lemma, $w\sigma = \Sigma_1^\infty w^j \sigma^j \leq \underline{\lim}_k \Sigma_1^\infty w^j \sigma^j_{i_k} = \lim_i w\, \sigma_i = \inf \{w\, \sigma^* | \sigma^* \in N\}$ so the infimum is attained. Denote an infimizing σ by $\sigma(w)$. We may take $\sigma(\cdot)$ to be positive homogeneous and such that each component $\sigma^j(\cdot)$ is a measurable function. (Let $\mathcal{B}_i$ be the Borel subsets of $\mathcal{X}_i = [0,1]$ and consider $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X} = \overset{\infty}{\underset{1}{\times}} \mathcal{X}_i$ and $\mathcal{B}$ is the product σ-field. Then the $\sigma^j(\cdot)$ may be taken to be measurable functions from $(\mathcal{X}, \mathcal{B})$ to the reals.)

Letting μ denote the distribution of $Z = (Z^1, Z^2, \ldots)$ where the Z^k are independent and identically distributed uniform $[0,1]$ random variables, we investigate the randomized sequential procedures

(5) $$\sigma_i = \sigma(E_{i-1} + H_{i-1}\, Z)$$

with the sequence of constants H_i satisfying

(6) $$0 < H_i \uparrow \qquad \text{and} \qquad i\, H_i^{-1} \uparrow \qquad \text{with respect to } i.$$

<u>Theorem 1</u>. Assuming (A1) and (A2) and with $H_i = i^{\frac{1}{2}}$, the procedure (5) results in

(7) $$|n^{-1}\, \mu(\Sigma_1^n\, \epsilon_i \sigma_i) - R(G_n)| = O(n^{-\frac{1}{2}})$$

uniformly in $\underline{\epsilon}$.

<u>Proof</u>. Writing $E_i(\sigma_i - \sigma_{i+1}) \geq (E_i - (E_i + H_i Z))(\sigma_i - \sigma_{i+1})$ we have

$$(8) \qquad \Sigma_1^n E_i(\sigma_i - \sigma_{i+1}) \geq -\Sigma_1^n H_i Z(\sigma_i - \sigma_{i+1})$$

$$= H_n Z \sigma_{n+1} - H_0 Z \sigma_1 - \Sigma_1^n(H_i - H_{i-1}) Z \sigma_i$$

$$\geq -B H_0 - \Sigma_1^n(H_i - H_{i-1})B = -B H_n.$$

Also, $E_n \sigma_{n+1} - n R(G_n) = E_n(\sigma_{n+1} - \sigma(E_n)) \geq 0$ so (4) yields

$$(9) \qquad \Sigma_1^n \epsilon_i \sigma_i - n R(G_n) \geq -B H_n.$$

In order to obtain an upper bound we note that $E_n \sigma_{n+1} - n R(G_n) \leq -H_n Z(\sigma_{n+1} - \sigma(E_n))$; and, similarly, $E_{i-1}(\sigma_i - \sigma_{i+1}) \leq -H_{i-1} Z(\sigma_i - \sigma_{i+1})$ so that summation by parts and (4) imply

$$(10) \qquad \Sigma_1^n \epsilon_i \sigma_i - n R(G_n) \leq B H_n + \Sigma_1^n \epsilon_i(\sigma_i - \sigma_{i+1}).$$

The expectation of the last term is bounded by a direct extension of Hannan's Lemma 2 (1957). Here we state and prove the needed specialization of that extension.

<u>Lemma 1</u>. Under the assumptions of Theorem 1,

$$(11) \qquad |\mu\sigma^i(w + Z) - \mu\sigma^i(w' + Z)| \leq B\|w - w'\|_1$$

for each i and all w, w' in m^+.

<u>Proof</u>. With $X = [0,1]^\infty$ we write

$$\mu\sigma^i(w' + Z) = \int_X \sigma^i(w' + z)d\mu(z) = \int_{TX} \sigma^i(w + v)d\nu(v)$$

where $\nu = \mu T^{-1}$ is the measure induced by the transformation T defined by $v = Tz = w' - w + z$. Therefore,

$$\mu\sigma^i(w + Z) - \mu\sigma^i(w' + Z) \le \int_X \sigma^i(w + z)d\mu(z) - \int_{X \cap TX} \sigma^i(w + v)d\nu(v)$$

and, since the restrictions of μ and ν to $X \cap TX$ are equal,

$$\mu\sigma^i(w + Z) - \mu\sigma^i(w' + Z) \le B\ \mu(X - TX).$$

Since $z \in X - TX$ if and only if $z^j \in [0,1]$ for all j and $z^j - w'^j + w^j \notin [0,1]$ for some j, it follows that

$$\begin{aligned}\mu(X - TX) &\le \Sigma_1^\infty \mu[z^j < w'^j - w^j \text{ or } z^j > 1 + w'^j - w^j] \\ &\le \Sigma_1^\infty \{(w'^j - w^j)^+ + (w^j - w'^j)^+\} \\ &= \|w - w'\|_1.\end{aligned}$$

The proof is completed by interchanging the roles of w and w'.

We apply the lemma with $w = H_{i-1}^{-1} E_{i-1}$ and $w' = H_i^{-1} E_i$. With this specification

$$\begin{aligned}\|w - w'\|_1 &= H_i^{-1} E_i^{\epsilon_i} - H_{i-1}^{-1} E_{i-1}^{\epsilon_i} + \sum_{j \ne \epsilon_i} (H_{i-1}^{-1} E_{i-1}^j - H_i^{-1} E_i^j) \\ &= H_i^{-1} + (H_i^{-1} - H_{i-1}^{-1})E_{i-1}^{\epsilon_i} + (i - 1 - E_{i-1}^{\epsilon_i})(H_{i-1}^{-1} - H_i^{-1}) \\ &\le H_i^{-1} + (i - 1)(H_{i-1}^{-1} - H_i^{-1}) \le 2\ H_i^{-1}.\end{aligned}$$

Therefore, the lemma implies $|\mu(\epsilon_i(\sigma_i - \sigma_{i+1}))| \le 2\ B\ H_i^{-1}$ which proves via (9) and (10) that

(12) $$-B\,H_n \le \mu(\Sigma_1^n \epsilon_i\sigma_i) - n\,R(G_n) \le B\{H_n + 2\,\Sigma_1^n H_i^{-1}\}.$$

The choice $H_i = i^{\frac{1}{2}}$ proves the theorem.

4. A Theorem for Non-Finite Games.

Let $L \ge 0$ be a loss function over the cartesian product $M \times N^*$ of arbitrary spaces M and N^*. In the component game player I chooses an element $\epsilon \in M$, II picks a $\sigma \in N = \{L(\cdot,\delta)|\delta \in N^*\}$ and the loss to II is $\epsilon\sigma$, $\epsilon\sigma$ denoting σ evaluated at ϵ. If w is a discrete measure putting mass w_i on $\epsilon_i \in M$ then $w\sigma$ denotes $\Sigma_i\, w_i(\epsilon_i\sigma)$. Assume that for all discrete measure w with finite support in M,

(A1') $$\inf\{w\,\sigma|\sigma \in N\} \text{ is attained.}$$

As before we denote an infimizing σ by $\sigma(w)$ and take $\sigma(\cdot)$ to be positive homogeneous. (If N^* is a compact topological space and each section $L(\epsilon,\cdot)$ is continuous on N^* then (A1') is satisfied.)

Consider the game where $M = N^* = [0,1]$ and $L(\epsilon,\delta) = |\epsilon - \delta|$. In the sequential version a strategy $\overline{\varphi}$ with $\overline{\varphi}_i = \sigma(E_{i-1})$, $E_i = i\,G_i$, where G_i is the empirical distribution of $(\epsilon_1,\ldots,\epsilon_i)$, is such that $\sup\{D_n(\underline{\epsilon},\overline{\varphi})|\underline{\epsilon} \in [0,1]^\infty\} \ge \frac{1}{4}$ (Hannan, 1957, p. 130). This exemplifies the need for artificial randomization as was the case in the preceding section. However, here the measures E_{i-1} have support changing with $\underline{\epsilon}$ and i; since the union of all possible supports is M, uncountable, it is not clear how to add the randomization in order to perturb the measure E_{i-1}. This difficulty will be overcome by embedding a countable set in M.

A basic assumption is that

$$\text{(A2')} \qquad \sup\{\epsilon\sigma \mid \epsilon \in M,\ \sigma \in N\} = B < \infty.$$

We define the real valued function

$$\text{(13)} \qquad d(\epsilon,\epsilon') = \sup\{|\epsilon\sigma - \epsilon'\sigma| \mid \sigma \in N\}.$$

Clearly (M,d) is a pseudo-metric space; and, if loss equivalent player I moves are identified, it constitutes a metric space. Let $J_i \geq 1$ be a non-decreasing integer valued sequence, $A = \{a_1, a_2, \ldots\} \subset M$ and $A_i = \{a_1, \ldots, a_{J_i}\}$. Corresponding to each sequence $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \ldots) \in M^\infty$ there is a sequence $\underline{\epsilon}' = (\epsilon_1', \epsilon_2', \ldots)$ where ϵ_i' is an element of A_i closest to ϵ_i in the metric d. We let G_i' be the empirical distribution of $(\epsilon_1', \ldots, \epsilon_i')$ and $Z_0, Z_1, Z_2, \ldots$ be independent and identically distributed uniform $[0,1]$ random variables. We investigate the sequential procedure

$$\text{(14)} \qquad \sigma_i = \sigma(E_{i-1}' + H_{i-1}\,\underline{Z}_{i-1})$$

where H_i satisfies (6) and $\underline{Z}_i$ is to be interpreted as the measure placing mass Z_j on a_j, $j = 1, \ldots J_i$. Let μ denote the distribution of $(Z_0, Z_1, Z_2, \ldots)$.

<u>Remark</u>. Let $\{a_1, \ldots, a_J\} \subset M$ be fixed but otherwise arbitrary and let $W = \{(w_1, \ldots, w_J) \mid 0 \leq w_i < \infty,\ i = 1, \ldots, J\}$

be the set of finite measures with support in $\{a_1,\ldots,a_J\}$. Then $\sigma^j(\cdot)$ is a function from W, considered as a subset of Euclidean J-space, to the reals. If $\{(a_1\sigma,\ldots,a_J\sigma)|\sigma \in N\}$ is a compact subset of Euclidean J-space, there is a determination $\sigma(\cdot)$ such that each component $\sigma^j(\cdot)$ is measurable. In the case of countably infinite support, we noted in the preceding section that sequential compactness under coordinatewise convergence and ℓ_1 boundedness for the set of risk vectors is sufficient for measurability.

<u>Theorem</u> <u>2</u>. Under assumptions (A1') and (A2'), and with each component of σ_i measurable,

$$(15)\qquad |\mu(\Sigma_1^n \epsilon_i\sigma_i) - n\,R(G_n)| \le B\{H_n J_n + J_n + 2\,\Sigma_1^n H_i^{-1}\} + 2\,\Sigma_1^n d(\epsilon_i,\epsilon_i').$$

<u>Proof</u>. A bound for the left hand side is provided by $|B_n| + |C_n| + |D_n|$ where

$$B_n = \mu(\Sigma_1^n \epsilon_i'\sigma_i) - n\,R(G_n')$$

$$C_n = \mu(\Sigma_1^n (\epsilon_i'\sigma_i - \epsilon_i\sigma_i))$$

$$D_n = n(R(G_n) - R(G_n')).$$

Clearly $\Sigma_1^n(\epsilon_i'\sigma_i - \epsilon_i\sigma_i) \le \Sigma_1^n d(\epsilon_i, \epsilon_i')$ and $n(R(G_n) - R(G_n')) = \inf_\sigma \Sigma_1^n \epsilon_i\sigma - \inf_\sigma \Sigma_1^n \epsilon_i'\sigma \le \inf_\sigma \Sigma_1^n(\epsilon_i'\sigma + d(\epsilon_i, \epsilon_i')) - \inf_\sigma \Sigma_1^n \epsilon_i'\sigma = \Sigma_1^n d(\epsilon_i,\epsilon_i')$ so

$$|C_n| \vee |D_n| \le \Sigma_1^n d(\epsilon_i, \epsilon_i'). \tag{16}$$

The identity (4) yields

$$B_n = \mu(E_n'\sigma_{n+1}) - E_n'\sigma(E_n') + \mu(\Sigma_1^n E_{i-1}'(\sigma_i - \sigma_{i+1})) + \mu(\Sigma_1^n \epsilon_i'(\sigma_i - \sigma_{i+1})). \tag{17}$$

Since

$$\begin{aligned}\Sigma_1^n E_i'(\sigma_i - \sigma_{i+1}) &\ge -\Sigma_1^n H_i \underline{Z}_i(\sigma_i - \sigma_{i+1})\\ &= H_n \underline{Z}_n \sigma_{n+1} - H_0 \underline{Z}_0 \sigma_1 - \Sigma_1^n (H_i \underline{Z}_i - H_{i-1} \underline{Z}_{i-1})\sigma_i\\ &\ge -B\, H_0 J_0 - \Sigma_1^n (H_i J_i - H_{i-1} J_{i-1})\, B\\ &= -B\, H_n J_n\end{aligned}$$

and $E_n'(\sigma_{n+1} - \sigma(E_n')) \ge 0$, it follows that

$$\Sigma_1^n \epsilon_i'\sigma_i - n\, R(G_n') \ge -B\, H_n J_n. \tag{18}$$

As in the derivation of (10),

$$\Sigma_1^n \epsilon_i'\sigma_i - n\, R(G_n) \le B\, H_n J_n + \Sigma_1^n \epsilon_i'(\sigma_i - \sigma_{i+1}). \tag{19}$$

For those i such that $J_i = J_{i-1}$ we have $|\mu(\epsilon_i'(\sigma_i - \sigma_{i+1}))| \le 2B\, H_i^{-1}$ by application of the specialization (9.8) of Hannan's Lemma 2 (1957) with $w = H_{i-1}^{-1} E_{i-1}'$, $w' = H_i^{-1} E_i'$. Therefore,

$$\begin{aligned}|\mu(\Sigma_1^n \epsilon_i'(\sigma_i - \sigma_{i+1}))| &\le \Sigma^n_{i|J_i = J_{i-1}} 2B\, H_i^{-1} + \Sigma^n_{i|J_i > J_{i-1}} B\\ &\le 2B\, \Sigma_1^n H_i^{-1} + B\, J_n,\end{aligned} \tag{20}$$

and (17) - (20) combine to yield

$$(21) \qquad -B\, H_n J_n \le B_n \le B\{H_n J_n + J_n + 2\, \Sigma_1^n H_i^{-1}\}.$$

In view of (16) and (21) the theorem is proved.

If the set A is dense in the metric space (M,d) with $d(\epsilon_i, \epsilon_i') \downarrow 0$ as $J_i \uparrow \infty$ uniformly in $\underline{\epsilon}$, then a balance in the bound of (15) can be obtained by choice of the sequences H_i and J_i. We note that in the finite M problem with J_i equal cardinality of M for large i and $H_i = i^{\frac{1}{2}}$, (15) shows that the expected modified regret resulting from the procedure (14) is $0(n^{-\frac{1}{2}})$ uniform in $\underline{\epsilon}$. We now apply Theorem 2 to a problem involving uncountable M.

<u>Example 2</u>. Consider the game of absolute deviation on the unit square. Here $d(\epsilon, \epsilon') = \sup\{\big|\, |\epsilon - \delta| - |\epsilon' - \delta|\, \big| \; | \delta \in [0,1]\} = |\epsilon - \epsilon'|$, and we let $H_i = i^a$, J_i equal the greatest integer in i^b, $J_0 = 1$, where $a, b \in (0,1)$ are yet to be specified. For the set $A = \{a_1, a_2, \ldots\}$ we take the points 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8,...; that is, $a_i = b_{kj}$ where $i = 2^{k-1} + j - 1$, $b_{kj} = (2j - 1)\, 2^{-k}$, $1 \le j \le 2^{k-1}$, $k \ge 1$. With this choice $d(\epsilon_i, \epsilon_i') \le 2\, J_i^{-1} \le 2(i^b - 1)^{-1}$; and, hence, it follows from (15) that

$$\begin{aligned} |\mu(\Sigma_1^n \epsilon_i \sigma_i) - n\, R(G_n)| &\le \{n^{a+b} + n^b + 2\, \Sigma_1^n i^{-a}\} \\ &\quad + 2\, \{\Sigma_{i | i^b < 2}^n 1 + \Sigma_{i | i^b \ge 2}^n 4\, i^{-b}\} \\ &\le c\{n^{a+b} + n^{1-a} + n^{1-b}\} \end{aligned}$$

for some constant c. The choice $a = b = 1/3$ results in an expected modified regret $0(n^{-(1/3)})$ uniform in $\underline{\epsilon}$.

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